Assignment # 1

1. Write down the fundamental ingredients to carry out Bayesian inference with a Bernoulli experiment where 3 successes e 7 failures have occurred. More specifically, suppose $X_1, ..., X_{10}$ are i.i.d. with $X_i \sim \text{Bernoulli}(\theta)$ $(i = 1, ..., 10()$ and $\sum_{i=1}^{10 x_i} = 3$.

   (a) Write explicitly what is the parametric space of interest

   (b) Write down: (1) the likelihood function + (2) a prior distribution (your favorite choice) + (3) the posterior distribution

   (c) Explain how you would make inference on the unknown parameter of interest

   (write your answer)

2. Derive the likelihood of the *Dugong* example and compute numerically the maximum likelihood estimate for the vector of parameters of interest $(\alpha, \beta, \gamma, \tau)$ and a Maximum-a-Posterori estimate specifying your prior choices.

   (write your answer and provide your R code for the numerical solution)

3. Let us consider
$$Y = e^X$$
where X is a random variable with Normal distribution with mean μ and variance σ^2. Derive the distribution of Y

4. Compute the mean and the variance of Y

   (write your answer)

5. Use pseudo-random deviates form a Uniform distribution and the integral transform theorem to generate a sequence of i.i.d random deviates with Pareto distribution

$$f(x;\alpha,\beta) = \alpha\frac{\beta^\alpha}{x^{\alpha+1}}I_{(\beta,\infty)}(x)$$

and parameters $\alpha = 2.5$ and $\beta = 1$.

Check the ability of Monte Carlo method to approximate the first moment of the Pareto distribution (which, in fact, can be computed in closed form).

   (a) Explain how you can approximate $I =$ the first moment up to a precision of 10^{-2}

   (b) plot a graph which shows how the running mean approaches the target value I as the number of simulations grows and provide the corresponding computer code

   (c) compare the empirical distribution of simulated values with respect to the desired target density. Explain in what sense the empirical histogram can be considered a Monte Carlo estimate of the target density.

(write your answer for the closed form, exact evaluation of I and the two explanations; provide your R code for the numerical Monte Carlo estimate of I and the graph)

6. Using pseudo-random deviates form a Uniform distribution and the generalized version of the Integral transform methods to simulate $n = 100$ i.i.d. random deviates form a discrete distribution with the following distribution

$$Pr(X = 0) = 0.25\,;\ Pr(X = 1) = 0.35\,;\ Pr(X = 2) = 0.4$$

Find out which `R` function perform this task directly (without using the uniform deviates) for an arbitrary discrete distribution and provide the correct syntax (R code) to perform the previous point.

(write your answer for explaining how you have derived the generalized inverse cdf; provide your R code. for the numerical Monte Carlo estimate of I and the graph)

7. Simulate from a Beta$(3,3)$ distribution using Acceptance-Rejection (A-R) method and pseudo-random deviates $Y \sim q(y)$ from a Uniform distribution. [hint: you must compute a suitable value k such that the conditions for applying A-R.]

   Compute (analytically and by MC approximation) the acceptance probability.

   (provide your R code for the implementation of the A-R. write your explanation about how you have conceived your Monte Carlo estimate of the acceptance probability [hint: you can think each attempt to simulate from the target distribution as a Bernoulli random variable])

8. Consider the Acceptance-Rejection algorithm in the most general form
   (a) Determine the analytic expression of the acceptance probability
   (b) Show how in Bayesian inference you could use simulations from the prior (auxiliary density) to get a random draw from the posterior (target distribution) without knowing the proportionality constant
   (c) Illustrate analytically possible difficulties of this approach with a simple conjugate model
   (d) Verify your conclusions implementing the Acceptance-Rejection approach with your conjugate model (verify empirically that θ has the desired target distribution $\pi(\theta|x_1, .., x_n)$)

(write your answer; provide your R code for the last point)

9. Simulate from a standard Normal distribution using pseudo-random deviates from a standard Cauchy and the A-R algorithm. Evaluate numerically (approximate by MC) the acceptance probability.

   (provide your R code for the implementation of the A-R. write your explanation about how you have conceived your Monte Carlo estimate [hint: you can think each attempt to simulate from the target distribution as a Bernoulli random variable])