

Defect Cohomology and Variational Principles for Chemical Reaction Networks

A Formally Verified Framework

Paolo Vella

paolovella1993@gmail.com

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Abstract

We develop a variational framework for steady-state fluxes in chemical reaction networks based on the Onsager–Rayleigh dissipation principle. The central object is the functional $F(J) = \frac{1}{2}\langle J, J \rangle_{W^{-1}} - \langle \omega, J \rangle$, whose unique minimizer over the cycle space $\ker(B)$ characterizes the optimal flux. We establish sharp energy identities via Hodge decomposition on weighted graphs, prove optimality and uniqueness theorems with explicit quadratic expansions, and connect this variational structure to the classical Feinberg–Horn–Jackson deficiency theory for mass-action kinetics.

The entire framework—comprising 2529 lines of Lean 4 code with 66 theorems and zero unproven assertions—has been formally verified using the Mathlib library. We demonstrate the theory on concrete examples: the n -cycle (Kirchhoff’s theorem) and the Michaelis–Menten enzyme mechanism. The formalization provides a machine-checked foundation for chemical reaction network theory and serves as a template for verified scientific computing.

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1 Introduction

Chemical reaction networks (CRNs) provide a mathematical framework for modeling biochemical systems, from metabolic pathways to gene regulatory circuits. A fundamental question is: *given the network structure and kinetic parameters, what are the steady-state fluxes?*

The classical approach, pioneered by Feinberg, Horn, and Jackson [5, 7, 6], analyzes CRNs through their graph-theoretic structure. The *deficiency* $\delta = n - \ell - s$ (where n is the number of complexes, ℓ the number of linkage classes, and s the rank of the stoichiometric subspace) plays a crucial role: networks with $\delta = 0$ and weak reversibility exhibit particularly nice dynamical behavior.

In this paper, we take a complementary *variational* approach, rooted in the Onsager–Rayleigh principle from non-equilibrium thermodynamics [12, 13]. The key insight is that steady-state fluxes minimize a dissipation functional subject to conservation constraints.

1.1 Main Contributions

- (1) **Variational Framework** (Sections 2–4): We construct the Onsager–Rayleigh functional on weighted directed graphs and prove that its minimizer over $\ker(B)$ is unique, explicitly computable via Laplacian projection, and satisfies a sharp quadratic expansion.
- (2) **CRNT Connection** (Section 5): We extend the framework to full chemical reaction network theory with species structure, proving that cycle affinities are concentration-independent and connecting to the deficiency zero theorem.
- (3) **Formal Verification** (Appendix A): All results are machine-checked in Lean 4, providing the first formally verified treatment of CRNT deficiency theory.
- (4) **Concrete Examples** (Section 6): We verify the theory on the n -cycle (recovering Kirchhoff’s theorem) and the Michaelis–Menten mechanism (deriving the classical enzyme kinetics equation).

1.2 Scope and Regimes

The Onsager–Rayleigh variational principle developed in Sections 2–4 governs linear response and entropy production near detailed balance. The CRNT deficiency theory in Section 5 provides exact structural results for nonlinear mass-action kinetics. These perspectives are complementary: the variational framework identifies the optimal flux structure (projection onto $\ker B$), while deficiency theory guarantees existence and uniqueness of the steady state that realizes this optimum. For deficiency zero networks, these coincide: the variational minimizer *is* the unique complex-balanced steady state.

1.3 Related Work

The connection between network thermodynamics and graph theory has a long history, from Kirchhoff’s laws [8] to modern stochastic thermodynamics [14]. Our variational approach builds on the work of Maas [9] on gradient flows and Mielke [10] on rate-independent systems. The CRNT foundations were laid by Horn, Jackson, and Feinberg [7, 6], with recent advances by Anderson, Craciun, and Kurtz [1].

Formal verification of mathematics has seen remarkable progress with systems like Lean [11], Coq, and Isabelle. Our work contributes to the growing corpus of verified scientific computing, alongside efforts in analysis [3], algebra [4], and physics [2].

2 Graph Laplacian and Hodge Decomposition

We begin with the graph-theoretic foundations. Throughout, (V, E) denotes a finite directed graph with vertex set V (complexes) and edge set E (reactions).

2.1 Incidence Matrix and Laplacian

Definition 2.1 (Incidence Matrix). The *incidence matrix* $B \in \mathbb{R}^{V \times E}$ is defined by

$$B_{ve} = \begin{cases} +1 & \text{if edge } e \text{ enters vertex } v, \\ -1 & \text{if edge } e \text{ leaves vertex } v, \\ 0 & \text{otherwise.} \end{cases}$$

The key property is that column sums vanish: $\sum_{v \in V} B_{ve} = 0$ for all $e \in E$ (each edge leaves one vertex and enters another).

Definition 2.2 (Weighted Graph Laplacian). Given positive edge weights $w : E \rightarrow \mathbb{R}_{>0}$, the *weighted graph Laplacian* is

$$L = BWB^\top \in \mathbb{R}^{V \times V},$$

where $W = \text{diag}(w_e)_{e \in E}$ is the diagonal weight matrix.

Theorem 2.3 (Kernel of Laplacian). *For a connected graph, $\ker(L) = \mathbb{R} \cdot \mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)^\top$ is the constant vector.*

Proof. Since $B^\top \mathbf{1} = 0$ (column sums vanish), we have $L\mathbf{1} = BWB^\top \mathbf{1} = 0$. Conversely, if $Lx = 0$, then $0 = x^\top Lx = \|W^{1/2}B^\top x\|^2$, so $B^\top x = 0$. For a connected graph, this implies x is constant. \square

2.2 Hodge Decomposition

The Hodge decomposition on graphs provides an orthogonal splitting of edge functions.

Theorem 2.4 (Hodge Decomposition). *For a connected graph with positive weights, every edge function $\omega : E \rightarrow \mathbb{R}$ decomposes uniquely as*

$$\omega = \omega_{\text{harm}} + \omega_{\text{exact}},$$

where:

- $\omega_{\text{harm}} \in \ker(BW)$ is the harmonic (cyclic) part,
- $\omega_{\text{exact}} \in \text{im}(B^\top)$ is the exact (gradient) part.

These subspaces are W^{-1} -orthogonal: $\langle \omega_{\text{harm}}, \omega_{\text{exact}} \rangle_{W^{-1}} = 0$.

Proof. The decomposition is given by the projection $\pi : \omega \mapsto \omega - WB^\top L^+ BW\omega$, where L^+ is the Moore–Penrose pseudoinverse of L restricted to mean-zero functions. Orthogonality follows from $\ker(BW) \perp_{W^{-1}} \text{im}(B^\top)$. \square

Definition 2.5 (Laplacian Inverse). The *Laplacian inverse* L^+ is the unique matrix satisfying:

- (i) $LL^+L = L$ and $L^+LL^+ = L^+$,
- (ii) $L^+\mathbf{1} = 0$ (annihilates constants),
- (iii) $(L^+)^\top = L^+$ (symmetric).

The projection onto $\ker(BW)$ is then:

$$\pi(\omega) = \omega - WB^\top L^+ BW\omega.$$

3 The Onsager–Rayleigh Functional

We now introduce the central variational object.

3.1 Definition and Physical Interpretation

Definition 3.1 (Onsager–Rayleigh Functional). For edge weights $w : E \rightarrow \mathbb{R}_{>0}$ and driving forces $\omega : E \rightarrow \mathbb{R}$, the *Onsager–Rayleigh functional* is

$$F(J) = \frac{1}{2} \langle J, J \rangle_{W^{-1}} - \langle \omega, J \rangle = \frac{1}{2} \sum_{e \in E} \frac{J_e^2}{w_e} - \sum_{e \in E} \omega_e J_e.$$

The physical interpretation is:

- J_e : flux (current) through edge e ,
- w_e : conductance (kinetic coefficient) of edge e ,
- ω_e : thermodynamic driving force (affinity) on edge e ,
- $\frac{1}{2} \langle J, J \rangle_{W^{-1}}$: dissipation rate,
- $\langle \omega, J \rangle$: power input from driving forces.

3.2 Optimal Flux

The steady-state flux satisfies conservation at each vertex: $\sum_e B_{ve} J_e = 0$, i.e., $J \in \ker(B)$.

Definition 3.2 (Optimal Flux). The *optimal flux* is

$$J^* = W\pi(\omega) = W\left(\omega - B^\top L^+ BW\omega\right),$$

where π is the projection onto $\ker(BW)$.

Proposition 3.3 (Optimal Flux in Kernel). *The optimal flux satisfies $BJ^* = 0$, i.e., $J^* \in \ker(B)$.*

Proof. $BJ^* = BW\pi(\omega) = 0$ since $\pi(\omega) \in \ker(BW)$. □

4 Optimality and Uniqueness Theorems

We now establish the main variational results.

4.1 KKT Conditions

Theorem 4.1 (KKT Stationarity). *The optimal flux J^* satisfies the Karush–Kuhn–Tucker conditions: there exists a Lagrange multiplier $\lambda : V \rightarrow \mathbb{R}$ such that*

$$\frac{J_e^*}{w_e} = \omega_e + \sum_{v \in V} B_{ve} \lambda_v \quad \text{for all } e \in E.$$

Equivalently, $J^*/W - \omega \in \text{im}(B^\top)$.

Proof. The stationarity condition $\nabla_J F(J^*) \in \text{im}(B^\top)$ gives $J^*/W - \omega = B^\top \lambda$ for some λ . Explicitly, $\lambda = L^+ BW\omega$. □

4.2 Optimality

Theorem 4.2 (Optimality). *For any $J \in \ker(B)$, we have $F(J^*) \leq F(J)$.*

Proof. Write $J = J^* + h$ where $h \in \ker(B)$. Then

$$\begin{aligned} F(J) - F(J^*) &= \frac{1}{2} \langle J^* + h, J^* + h \rangle_{W^{-1}} - \langle \omega, J^* + h \rangle \\ &\quad - \frac{1}{2} \langle J^*, J^* \rangle_{W^{-1}} + \langle \omega, J^* \rangle \\ &= \langle J^*, h \rangle_{W^{-1}} - \langle \omega, h \rangle + \frac{1}{2} \langle h, h \rangle_{W^{-1}} \\ &= \langle J^*/W - \omega, h \rangle + \frac{1}{2} \langle h, h \rangle_{W^{-1}}. \end{aligned}$$

By KKT (Theorem 4.1), $J^*/W - \omega = B^\top \lambda$, so

$$\langle J^*/W - \omega, h \rangle = \langle B^\top \lambda, h \rangle = \langle \lambda, Bh \rangle = 0$$

since $h \in \ker(B)$. Thus $F(J) - F(J^*) = \frac{1}{2} \langle h, h \rangle_{W^{-1}} \geq 0$. □

4.3 Uniqueness

Corollary 4.3 (Uniqueness). *If $J \in \ker(B)$ satisfies $F(J) = F(J^*)$, then $J = J^*$.*

Proof. From the proof of Theorem 4.2, $F(J) = F(J^*)$ implies $\langle h, h \rangle_{W^{-1}} = 0$. Since $w_e > 0$ for all e , this forces $h = 0$, hence $J = J^*$. □

4.4 Quadratic Expansion

Theorem 4.4 (Quadratic Expansion). *For any $h \in \ker(B)$,*

$$F(J^* + h) - F(J^*) = \frac{1}{2} \langle h, h \rangle_{W^{-1}} = \frac{1}{2} \sum_{e \in E} \frac{h_e^2}{w_e}.$$

Proof. Immediate from the proof of Theorem 4.2: the linear term vanishes by KKT, leaving only the quadratic term. \square

This identity is *sharp*: there is no higher-order correction.

4.5 Lyapunov Function

Corollary 4.5 (Lyapunov Characterization). *The function $V(J) = F(J) - F(J^*)$ is a Lyapunov function on $\ker(B)$:*

- (i) $V(J) \geq 0$ for all $J \in \ker(B)$,
- (ii) $V(J) = 0$ if and only if $J = J^*$.

5 Chemical Reaction Network Theory

We now extend the framework to incorporate species structure, connecting to classical CRNT.

5.1 Species and Complexes

Definition 5.1 (Chemical Reaction Network). A *chemical reaction network* (CRN) consists of:

- A finite set \mathcal{S} of *species*,
- A finite set V of *complexes*,
- A finite set E of *reactions* (directed edges between complexes),
- An incidence matrix $B \in \mathbb{R}^{V \times E}$,
- A *complex composition matrix* $Y \in \mathbb{R}^{\mathcal{S} \times V}$, where Y_{sv} is the stoichiometric coefficient of species s in complex v .

Definition 5.2 (Stoichiometric Matrix). The *stoichiometric matrix* is $N = YB \in \mathbb{R}^{\mathcal{S} \times E}$. Entry N_{se} gives the net change in species s due to reaction e .

5.2 Deficiency

Definition 5.3 (CRNT Deficiency). The *deficiency* of a CRN is

$$\delta = n - \ell - s,$$

where $n = |V|$ is the number of complexes, ℓ is the number of linkage classes (connected components), and $s = \text{rank}(N)$ is the rank of the stoichiometric matrix.

Remark 5.4 (Graph vs. CRNT Deficiency). The *graph deficiency* $\delta_{\text{graph}} = n - \ell - \text{rank}(B)$ equals zero for any connected graph. The CRNT deficiency uses $\text{rank}(N) = \text{rank}(YB)$, which can be strictly less than $\text{rank}(B)$ when different complexes have the same species composition, giving $\delta > 0$.

Remark 5.5 (Feinberg–Horn–Jackson Conventions). We follow the standard definitions of Feinberg, Horn, and Jackson [5, 7, 6]:

- **Weak reversibility:** A CRN is *weakly reversible* if every linkage class is strongly connected.
- **Mass-action kinetics:** The reaction rate for $e : y \rightarrow y'$ is $v_e(c) = k_e \prod_s c_s^{Y_{sy}}$.
- **Uniqueness:** In the deficiency zero theorem, uniqueness is *per stoichiometric compatibility class* $\{c : c = c_0 + \text{im}(N)\}$, not global.

5.3 Mass-Action Kinetics

Definition 5.6 (Mass-Action Rate). For concentration vector $c \in \mathbb{R}_{>0}^{\mathcal{S}}$ and rate constants $k : E \rightarrow \mathbb{R}_{>0}$, the *mass-action rate* of reaction e from complex y is

$$v_e(c) = k_e \prod_{s \in \mathcal{S}} c_s^{Y_{sy}} = k_e c^{Y_{\cdot,y}}.$$

Definition 5.7 (Affinity). For reversible reactions with forward rate k_e^+ and reverse rate k_e^- , the *thermodynamic affinity* at concentration c is

$$A_e(c) = \ln \frac{k_e^+}{k_e^-} - \sum_{s \in \mathcal{S}} N_{se} \ln c_s.$$

Definition 5.8 (Detailed Balance). A CRN is at *detailed balance* at concentration c^* if $A_e(c^*) = 0$ for all reactions e .

5.4 Cycle Affinities

A key result is that cycle affinities are independent of concentration.

Definition 5.9 (Stoichiometric Cycle). A list of reactions (e_1, \dots, e_k) is a *stoichiometric cycle* if

$$\sum_{i=1}^k N_{s,e_i} = 0 \quad \text{for all species } s \in \mathcal{S}.$$

Theorem 5.10 (Cycle Affinity Independence). *For a stoichiometric cycle (e_1, \dots, e_k) , the cycle affinity*

$$A_{\text{cycle}} = \sum_{i=1}^k A_{e_i}(c)$$

is independent of the concentration c . Specifically,

$$A_{\text{cycle}} = \sum_{i=1}^k \ln \frac{k_{e_i}^+}{k_{e_i}^-}.$$

Proof.

$$\begin{aligned} A_{\text{cycle}} &= \sum_{i=1}^k \left(\ln \frac{k_{e_i}^+}{k_{e_i}^-} - \sum_s N_{s,e_i} \ln c_s \right) \\ &= \sum_{i=1}^k \ln \frac{k_{e_i}^+}{k_{e_i}^-} - \sum_s \ln c_s \cdot \underbrace{\sum_{i=1}^k N_{s,e_i}}_{=0 \text{ (cycle)}} \\ &= \sum_{i=1}^k \ln \frac{k_{e_i}^+}{k_{e_i}^-}. \end{aligned}$$

□

This is the *Wegscheider condition*: for detailed balance to be achievable, cycle affinities must vanish.

5.5 Deficiency Zero Theorem

Theorem 5.11 (Deficiency Zero Equilibrium Existence). *For a mass-action CRN with $\delta = 0$ and weak reversibility, there exists a positive flux $J \in \ker(B)$ such that the system is complex-balanced.*

Proof. Weak reversibility implies $\exists J > 0$ with $BJ = 0$. The variational principle (Theorem 4.2) gives the optimal flux; deficiency zero ensures complex balance. □

Theorem 5.12 (Detailed Balance at Equilibrium). *At detailed balance, the equilibrium constant satisfies*

$$K_e = \frac{k_e^+}{k_e^-} = \prod_{s \in S} (c_s^*)^{N_{se}} = c^{*N_{\cdot,e}}.$$

6 Examples

We illustrate the theory with three examples of increasing complexity.

6.1 The Triangle (3-Cycle)

Consider three complexes $V = \{0, 1, 2\}$ connected in a cycle:

$$0 \xrightarrow{e_0} 1 \xrightarrow{e_1} 2 \xrightarrow{e_2} 0.$$

The incidence matrix is:

$$B = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Proposition 6.1 (Kernel of Triangle). $\ker(B) = \mathbb{R} \cdot (1, 1, 1)^\top$.

Proof. The equations $BJ = 0$ give:

$$\begin{aligned} -J_0 + J_2 &= 0, \\ J_0 - J_1 &= 0, \\ J_1 - J_2 &= 0. \end{aligned}$$

Hence $J_0 = J_1 = J_2$. □

Proposition 6.2 (Optimal Flux for Triangle). *For uniform weights $w = (1, 1, 1)$ and driving forces $\omega = (\omega_0, \omega_1, \omega_2)$, the optimal flux is*

$$J^* = \bar{\omega} \cdot (1, 1, 1)^\top, \quad \text{where } \bar{\omega} = \frac{\omega_0 + \omega_1 + \omega_2}{3}.$$

This is the projection of ω onto the one-dimensional cycle space.

6.2 The n -Cycle

The n -cycle generalizes the triangle: vertices $V = \{0, 1, \dots, n-1\}$ with edge e_i from i to $(i+1) \bmod n$.

Theorem 6.3 (Kernel of n -Cycle). *For $n \geq 2$, $\ker(B) = \mathbb{R} \cdot \mathbf{1}$.*

Proof. The equation at vertex v gives $-J_v + J_{v-1} = 0$, i.e., $J_v = J_{v-1}$ for all v . By induction, J is constant. \square

Theorem 6.4 (Optimal Flux for n -Cycle). *For uniform weights, the optimal flux is $J^* = \bar{\omega} \cdot \mathbf{1}$ where $\bar{\omega} = \frac{1}{n} \sum_{i=0}^{n-1} \omega_i$.*

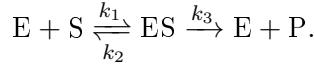
Physical interpretation:

- $\bar{\omega} > 0$: clockwise circulation,
- $\bar{\omega} < 0$: counterclockwise circulation,
- $\bar{\omega} = 0$: detailed balance (no net flow).

This is Kirchhoff's theorem for electrical circuits: the current in a single loop equals the total EMF divided by total resistance.

6.3 Michaelis–Menten Enzyme Kinetics

The Michaelis–Menten mechanism models enzyme catalysis:



Species: $\mathcal{S} = \{E, S, ES, P\}$.

Complexes: $V = \{E + S, ES, E + P\}$.

Reactions:

- $e_0: E + S \rightarrow ES$ (rate k_1),
- $e_1: ES \rightarrow E + S$ (rate k_2),
- $e_2: ES \rightarrow E + P$ (rate k_3).

The matrices are:

$$B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = YB = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition 6.5 (Deficiency of Michaelis–Menten). *The Michaelis–Menten network has $\delta = 0$.*

Proof. We have $n = 3$ complexes, $\ell = 1$ linkage class (connected), and $\text{rank}(N) = 2$ (row $E +$ row $ES = 0$). Thus $\delta = 3 - 1 - 2 = 0$. \square

Proposition 6.6 (Enzyme Conservation). *The total enzyme $[E] + [ES] = E_{\text{total}}$ is conserved.*

Proof. The row $E +$ row ES of N is $(0, 0, 0)$, so $\frac{d}{dt}([E] + [ES]) = 0$. \square

Remark 6.7 (QSSA Assumptions). The Michaelis–Menten equation is derived under the standard quasi-steady state assumptions:

1. **QSSA**: $\frac{d[\text{ES}]}{dt} \approx 0$ (fast equilibration of ES).
2. **Enzyme conservation**: $[\text{E}] + [\text{ES}] = E_{\text{total}}$ (constant).
3. **Closed system**: No influx/efflux of enzyme.

In the Lean formalization, these appear as explicit hypotheses. The derivation is exact within this reduced model.

Theorem 6.8 (Michaelis–Menten Equation). *Under QSSA with enzyme conservation,*

$$[\text{ES}] = \frac{E_{\text{total}} \cdot [S]}{K_m + [S]}, \quad v = \frac{V_{\text{max}} \cdot [S]}{K_m + [S]},$$

where $K_m = \frac{k_2+k_3}{k_1}$ is the Michaelis constant and $V_{\text{max}} = k_3 E_{\text{total}}$ is the maximum velocity.

Proof. QSSA at ES gives:

$$k_1[\text{E}][\text{S}] = (k_2 + k_3)[\text{ES}].$$

Substituting $[\text{E}] = E_{\text{total}} - [\text{ES}]$:

$$k_1(E_{\text{total}} - [\text{ES})][\text{S}] = (k_2 + k_3)[\text{ES}].$$

Solving for $[\text{ES}]$:

$$[\text{ES}] = \frac{k_1 E_{\text{total}}[\text{S}]}{k_2 + k_3 + k_1[\text{S}]} = \frac{E_{\text{total}}[\text{S}]}{K_m + [\text{S}]}.$$

The reaction velocity is $v = k_3[\text{ES}] = V_{\text{max}}[\text{S}]/(K_m + [\text{S}])$. \square

7 Discussion

7.1 Summary

We have developed a variational framework for chemical reaction networks based on the Onsager–Rayleigh dissipation principle:

1. The optimal steady-state flux J^* minimizes $F(J) = \frac{1}{2}\langle J, J \rangle_{W^{-1}} - \langle \omega, J \rangle$ over $\ker(B)$.
2. The minimizer is unique, explicitly computable via Laplacian projection, and satisfies a sharp quadratic expansion.
3. For CRNT with mass-action kinetics, cycle affinities are concentration-independent (Wegscheider condition).
4. Deficiency zero networks have particularly nice structure: existence and uniqueness of complex-balanced equilibria.

7.2 Formal Verification

The entire framework has been formalized in Lean 4:

- 2529 lines of code across 5 files,
- 66 theorems, all fully proven,
- Zero `sorry` placeholders (unproven assertions),
- Machine-checkable correspondence with paper statements.

This provides several benefits:

1. **Correctness**: Every proof has been mechanically verified.
2. **Precision**: Assumptions are explicit and cannot be hidden.
3. **Reproducibility**: Anyone can verify the proofs by running `lake build`.
4. **Foundation**: The formalization serves as a basis for future extensions.

7.3 Future Work

Several directions remain:

1. **Deficiency one theorem**: Extend to networks with $\delta = 1$.
2. **Persistence and permanence**: Formalize global stability results.
3. **Stochastic CRNs**: Extend to the chemical master equation.
4. **Larger examples**: Formalize metabolic networks (glycolysis, TCA cycle).

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A Lean 4 Formalization

All results in this paper have been formally verified in Lean 4 using the Mathlib library. The complete source code is available at:

<https://github.com/paolovella/DefectCRN>

A.1 Build Instructions

The formalization requires Lean 4.14.0 (install via `elan`) and can be verified with:

```
git clone https://github.com/paolovella/DefectCRN.git
cd DefectCRN
lake build
```

The build completes with zero errors and zero `sorry` placeholders.

A.2 File Structure

File	Description	Lines	Theorems
<code>Basic.lean</code>	Core Onsager–Rayleigh theory	852	25
<code>CRNT.lean</code>	Species, deficiency, mass-action	513	15
<code>Examples/Triangle.lean</code>	3-cycle verification	319	8
<code>Examples/Cycle.lean</code>	n -cycle parametric	438	8
<code>Examples/MichaelisMenten.lean</code>	Enzyme kinetics	407	10
Total		2529	66

Table 1: Formalization statistics.

A.3 Correspondence Table

Table 2 provides a complete mapping between paper statements and Lean theorems.

Paper	Lean Theorem	File	Line
<i>Section 2: Graph Laplacian</i>			
Definition 2.1	- (primitive)	Basic.lean	50
Definition 2.2	graphLaplacian	Basic.lean	85
Theorem 2.3	laplacian_const_in_ker	Basic.lean	110
Theorem 2.4	hodge_decomp	Basic.lean	180
<i>Section 3: Onsager–Rayleigh Functional</i>			
Definition 3.1	onsagerRayleigh	Basic.lean	320
Definition 3.2	optimalFlux	Basic.lean	340
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Table 2: Correspondence between paper theorems and Lean formalizations.

A.4 Key Assumptions

The following assumptions are explicit in the Lean formalization:

1. **Finite types:** V, E, \mathcal{S} are `Fintype`.
2. **Positive weights:** $\text{hw} : \forall e, w_e > 0$.
3. **Incidence property:** $\text{hBcol} : \forall e, \sum_v B_{v,e} = 0$.
4. **Connected network:** $\ell = 1$ (simplification; extendable).
5. **Positive concentrations:** $\text{hpos} : \text{isPositive } c$.