Slowly, then all at once: Uncovering the dynamics of a catastrophe

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1. The Question

What does the desertification of the Sahara, 6000 years ago, and the Great Recession of 2008 have in common?

They both unfolded in two phases: first **gradually** and then **suddenly**!

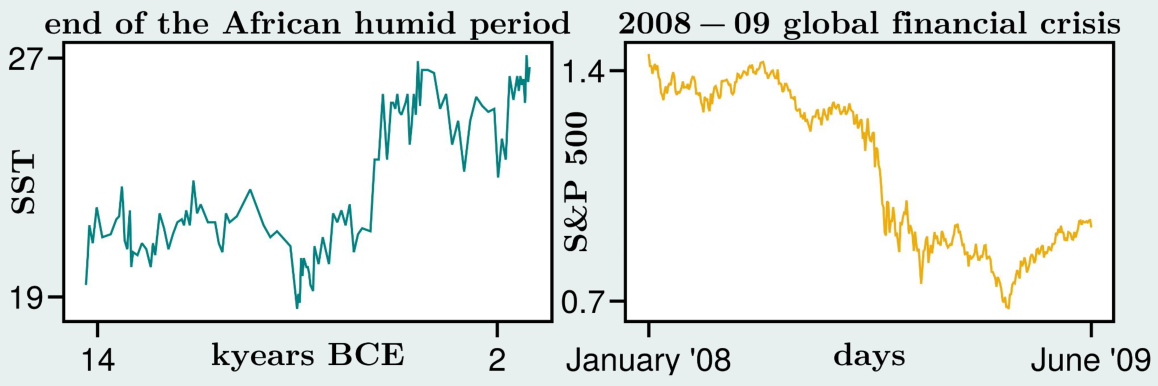


Fig. 1: Examples of real-world catastrophic transitions in dynamical systems.

Is this a generic trait of catastrophic collapses in nature?
Are there early-warning signs that can forecast these events?

We consider a minimal working model that can reproduce tipping phenomena between alternative stable states

$$dx = f(x,\mu)\,dt + \sigma(x)\,dW\,, \qquad d\mu = \varepsilon(t)\,dt\,, \tag{1}$$

- Cubic drift: $f(x,\mu) = -\mu - 2x + 3x^2 - 0.8x^3$
- Additive stochastic diffusion: $\sigma(x) = \sigma > 0$
- Linear ramp of the bifurcation parameter μ: $\varepsilon(t) = \varepsilon > 0$
- Pair of saddle-node bifurcations: $\mu \approx -0.371$ and $\mu \approx 1.621$
- Analogy with Langevin dynamics: overdamped particle in a quartic potential landscape $V(x,\mu)$ s.t. $f(x,\mu) = -\partial x V$

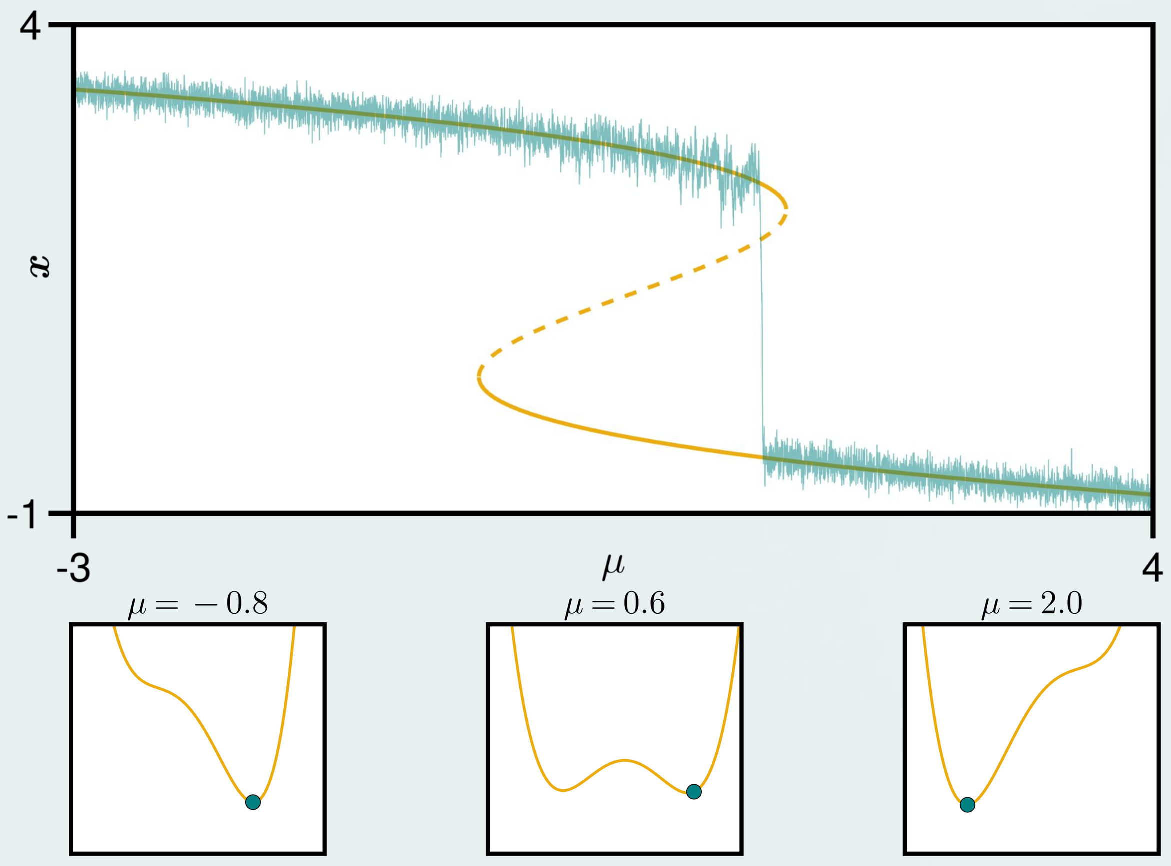


Fig. 2: Timeseries of a solution of (1) and its potential ($\sigma = 0.3$, $\varepsilon = 0.01$).

2. The Problem

Tipping points

Mechanisms to escape the basin of attraction:

- A bifurcation is reached (B-tipping)
- The parameter drift is too fast (R-tipping)
- The Brownian noise dominates (N-tipping)

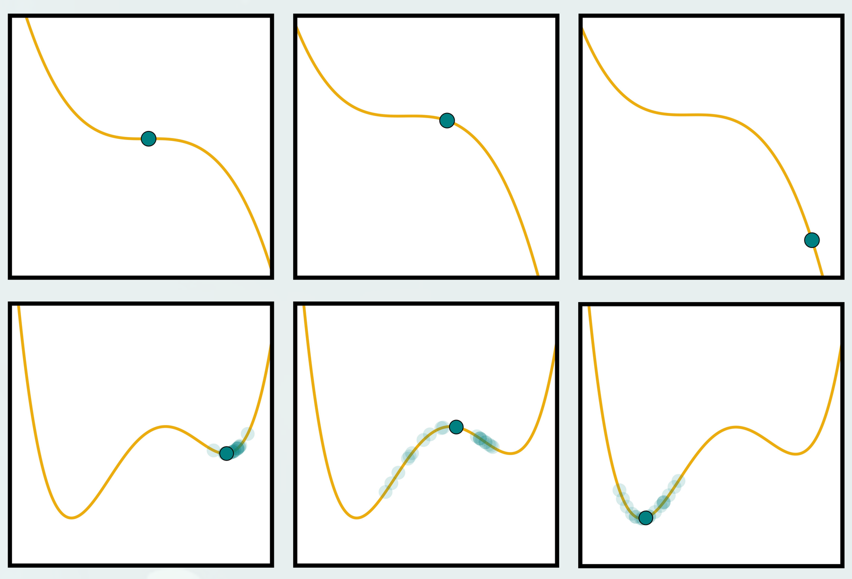


Fig. 3: Tipping mechanism visualised: R-tipping (top) in a deterministic ($\sigma = 0$) cubic [1]; N-tipping (bottom) in a stationary ($\varepsilon = 0$) quartic; B-tipping shown in **Fig. 2**.

Critical slowing down

As the bifurcation is approached the stable equilibria become increasingly weaker in attracting nearby solutions.

Increases in variance precede B-tipping [2].

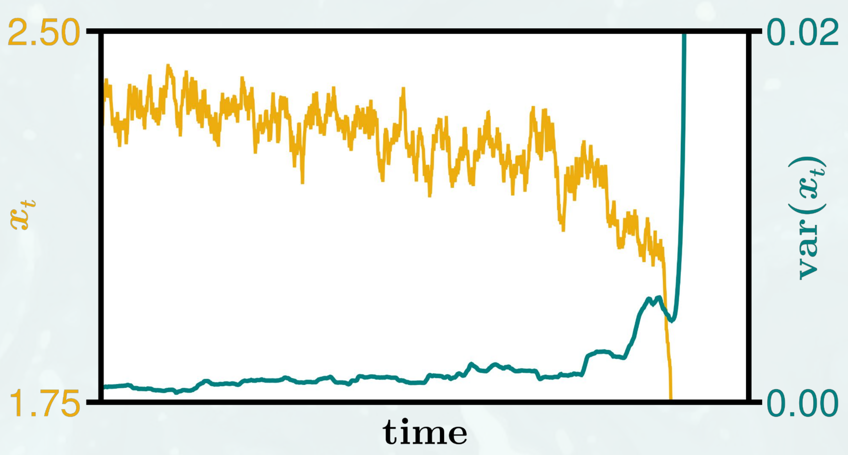
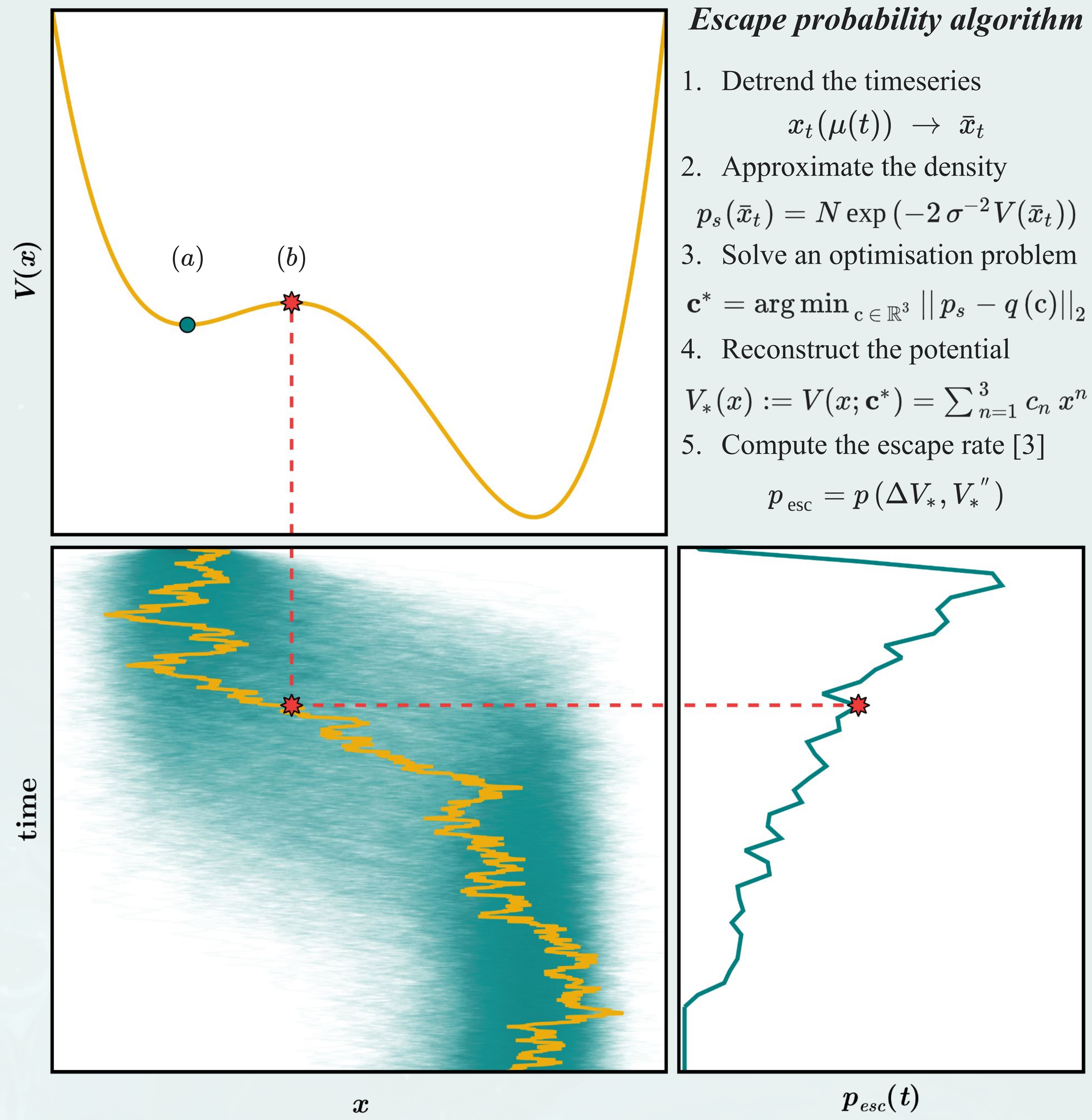


Fig. 4: B-tipping of (1) in a fast-slow ($\varepsilon = 10^{-3}$), stochastic ($\sigma = 5{\cdot}10^{-2}$) regime (yellow) and its variance (teal) computed on a sliding window.

3. The Method

An alternative measure of imminent catastrophe

Far from the bifurcation and at equilibrium (1) is topologically equivalent to an Ornstein-Uhlenbeck process $f(x) = -\theta(x-a) + \sigma dW$, where $\theta = \partial x V(a)$.
A solution starting at $x = a$ will thus tip, with some probability $p_{esc} > 0$, over the local maxima $x = b$ (boundary of the basin of attraction) of the scalar potential.



Escape probability algorithm

1. Detrend the timeseries
$$x_t(\mu(t)) \;\rightarrow\; \bar{x}_t$$
2. Approximate the density
$$p_s(\bar{x}_t) = N\exp\left(-2\sigma^{-2}V(\bar{x}_t)\right)$$
3. Solve an optimisation problem
$$\mathbf{c}^* = \arg\min_{\,\mathbf{c}\in\mathbb{R}^3} ||p_s - q(\mathbf{c})||_2$$
4. Reconstruct the potential
$$V_*(x) := V(x;\mathbf{c}^*) = \textstyle\sum_{n=1}^{3} c_n\, x^n$$
5. Compute the escape rate [3]
$$p_{\,esc} = p\,(\Delta V_*, V_*'')$$

Fig. 5: Stationary ($\varepsilon = 0$) escape time distribution (bottom right) of an ensemble of N=3000 overdamped particles (bottom left) out of the basin of attraction of the potential (top) at fixed parameter value μ=0. The histogram $p_{esc}(t)$ approximates an inverse Gaussian distribution at noise level $\sigma = 1.0$.

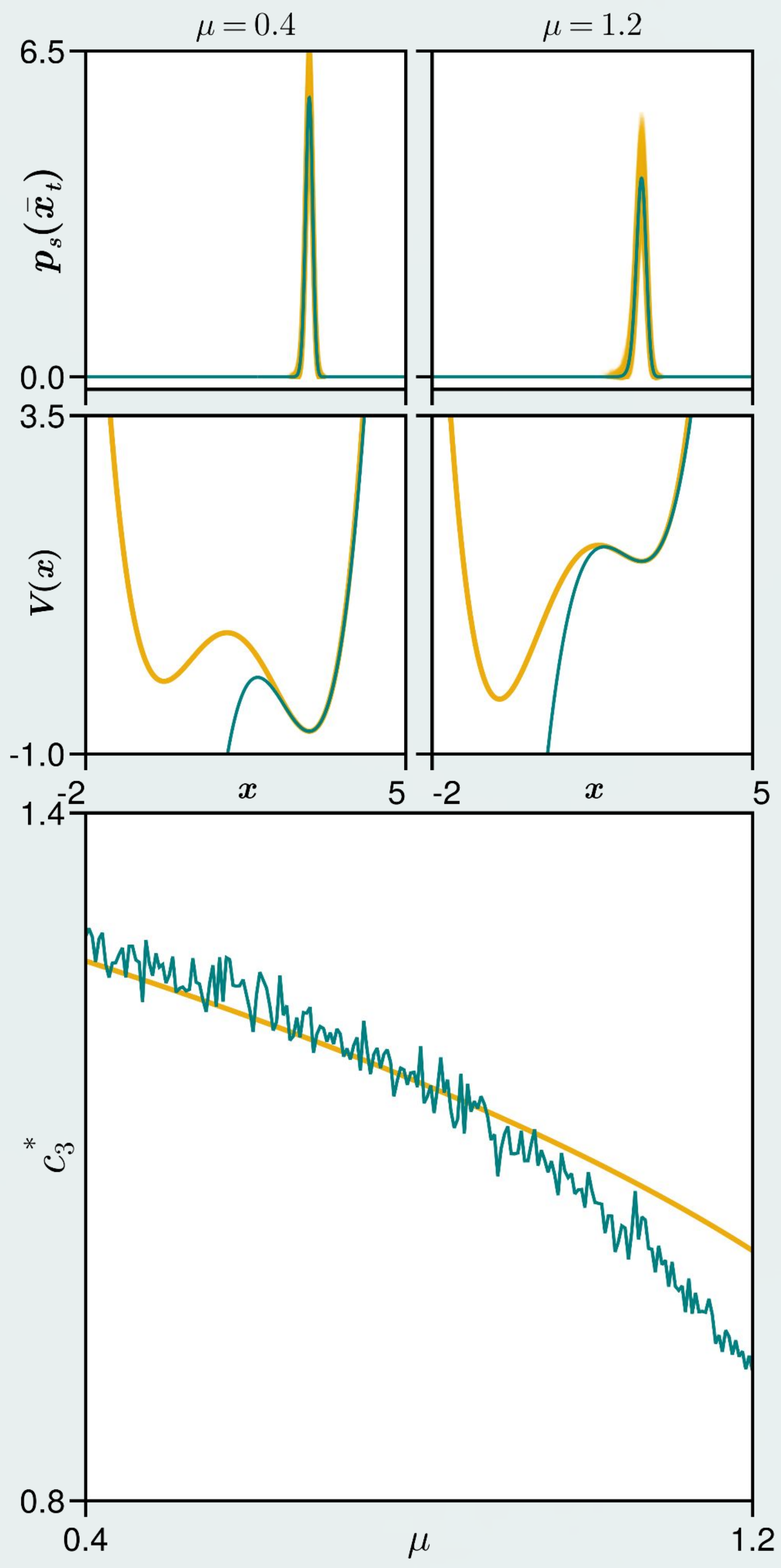


Fig. 6: Steps of the escape probability algorithm illustrated: parameter sweep simulation ($\varepsilon = 0$, $\sigma = 0.2$) of an ensemble of N=3000 particles evolving according to (1).
(Top) Ensemble histograms of the stationary density (yellow) and the solution of the (non-linear) optimisation problem (teal).
(Middle) True potential function $V(x)$ (yellow) and its local reconstruction $V_*(x)$ (teal) from the solution of the optimisation problem.
(Bottom) Cubic term of the Taylor expansion of the potential around $x = a$ (yellow) and its approximation (teal).

4. The Results

- The error improves as we approach the bifurcation, i.e.
$||V - V_*||_2 \to 0$ as $\mu \to \mu_c$
- We find a probabilistic early-warning signal for (1) in stationary ($\varepsilon = 0$) and low-noise regimes ($\sigma \sim 10^{-1}$)
- **What's next?** Generalisation of the method to systems in quasi-stationary ($0 < \varepsilon \ll 1$) regimes [4]

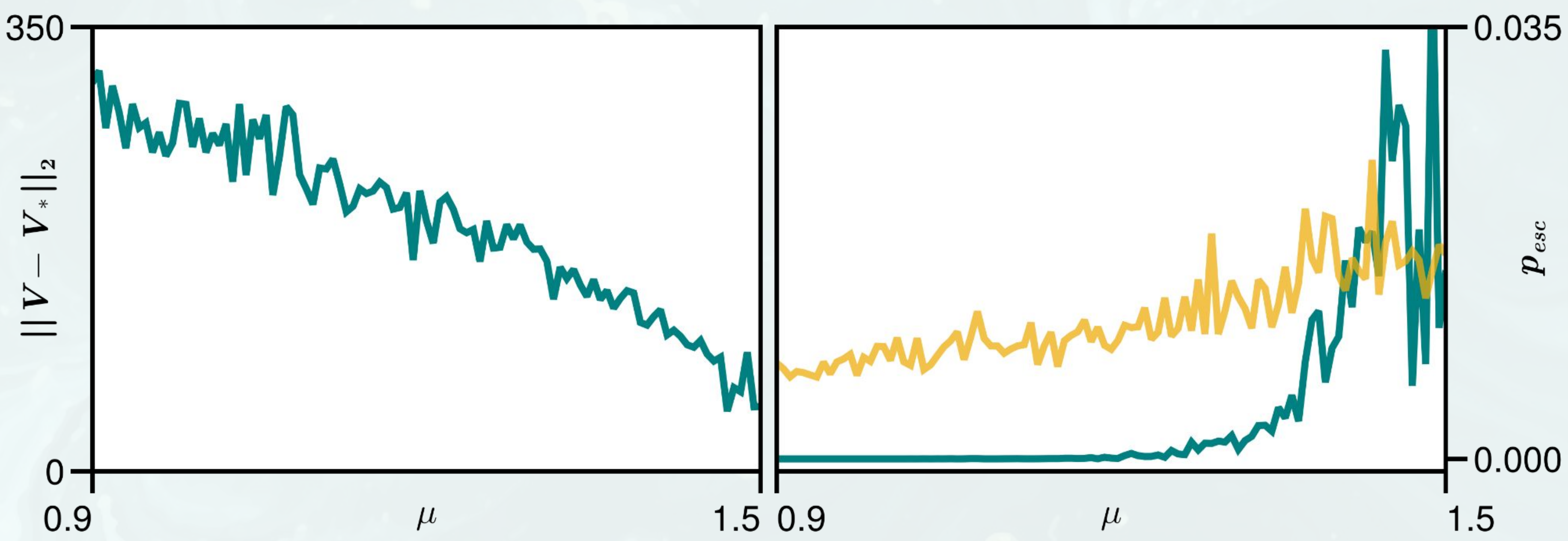


Fig. 7: (Left) Approximation error of the reconstruction of the potential landscape $V(x)$ for the ensemble simulation performed in **Fig. 6**. (Right) Probability-to-tip early-warning signal (teal), computed out of the reconstructed potential $V_*(x)$, compared to the stationary variance (yellow) for an indicative timeseries in the ensemble.

5. The Literature

[1] Ashwin, P. et al., Philos. Trans. R. Soc. A, Vol. 370 (2012)
[2] Kuehn, C., Physica D, Vol. 240 (2011)
[3] Kramers, H., Physica, Vol. 7 (1940)
[4] Donovan, G., Physica A, Vol. 646 (2024)

6. Bella ciao! 👋😀

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