Functional characterisation of parameter shifts for rate-induced tipping

1 Dependence on the shift rate

sec:critical_rate

Here the parameter shift $\Lambda(s) \in \mathcal{P}(\lambda_-, \lambda_+)$ will be a monotonically increasing, smooth and bounded ramp; in particular we will choose

$$\lambda(t) = \tanh(\varepsilon t) + C\,, \qquad \text{\{eq:tanh\_shift\}}$$

where $C \in \mathbb{R}$ is to be determined.

1.1 Setting

subsec:setting

Differentiating (1) w.r.t. t gives us a non-autonomous differential equation for λ

$$\dot{\lambda} = \varepsilon\,\mathrm{sech}^2(\varepsilon t)\,. \qquad \text{\{eq:tanh\_shift\_diff\_eq\}}$$

Integrating (2) by separation of variables, and changing the timescale $t \to s = \varepsilon t$, will obviously give us back (1)

$$\int d\lambda = \int \varepsilon\,\mathrm{sech}^2(\varepsilon t)dt \;\Rightarrow\; \lambda + C_1 = \varepsilon \int \mathrm{sech}^2(s)\left(\frac{1}{\varepsilon}\,ds\right) = \int \mathrm{sech}^2(s)ds = \tanh(s) + C_2 =$$
$$= \tanh(\varepsilon t) + C_2 \;\Rightarrow\; \lambda(t) = \tanh(\varepsilon t) + C\,.$$

From the above it becomes evident that the choice of C is tied to fixing a value for the initial condition $\lambda(t_0) = \lambda_0$ of the parameter shift.

1.1.1 How to choose C

subsubsec:integration_constant

Since the parameter shift must connect two asymptotic values λ_-, λ_+ at $t \to \pm\infty$, to choose the initial condition which will uniquely fix C we impose that

$$\lim_{t\to-\infty} \lambda(t) = \lambda_-\,.$$

In simulations we must fix a finite time interval in which the shift occurs, therefore we fix $T > 0$ and therefore set $[-T, T] \subset (-\infty, +\infty)$. The above constraint thus becomes

$$\lambda(-T) = \lambda_- \;\Rightarrow\; C = \lambda_- - \tanh(-\varepsilon T) = \lambda_- + \tanh(\varepsilon T)\,,$$ {eq:integration_constant}

where the last equality comes from $\tanh(x)$ being an odd function of x in the symmetric domain $[-X, X]$, $X > 0$.

Compactification of time

par:compactification

Notice that $\forall \varepsilon > 0$ there is a sufficiently large T s.t. $\tanh(\varepsilon T) \approx 0$.

1.2 Critical range

subsec:critical_range

We will now empirically determine the range of values of $\varepsilon \in (0, 1]$ for which irreversible R-tipping is observed. We consider the following non-autonomous dynamical system [1, Example 3.1, p.2200]

$$\dot{x} = f(x, \lambda(t)) = -\Big((x + a + b\lambda)^2 + c\tanh(\lambda - d)\Big)\Big(x - \frac{k}{\cosh(e\lambda)}\Big)\,,$$ {eq:dyn_sys}

with $a = -\frac{1}{4}$, $b = \frac{6}{5}$, $c = -\frac{2}{5}$, $d = -0.3$, $e = 3$ and $k = 2$ fixed. Our choice of parameter shift will be (1) with varying values of ε. As outlined in 1.1.1, varying ε implies we either choose different initial conditions λ_- or different truncations for the asymptotic time horizon T. Regardless of the case, the integration constant will be updated for each value of ε i.e. we should substitute $C \to C(\varepsilon)$ in (1).

1.2.1 Results

subsubsec:results_tanh_shift

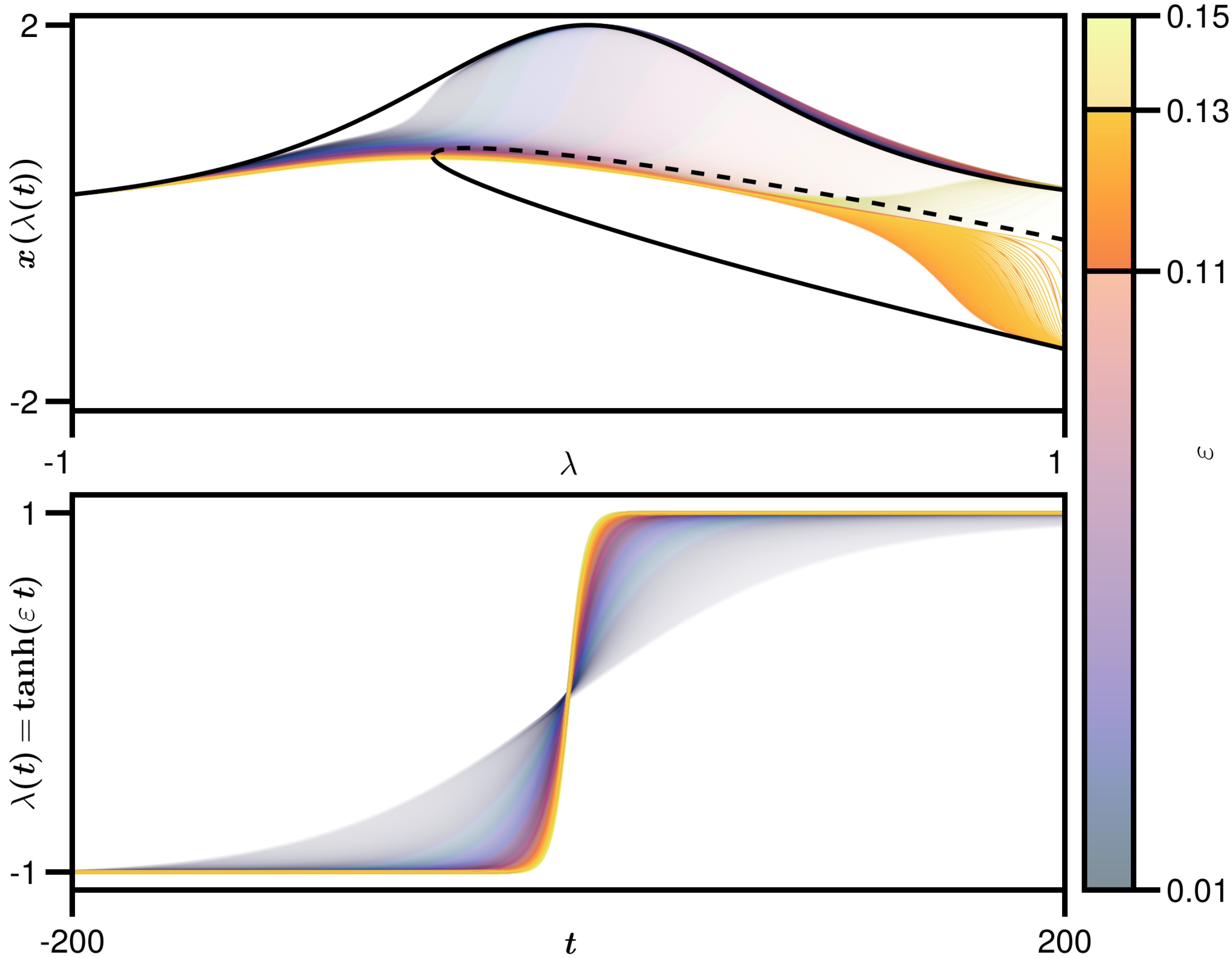

Figure 1: Family of solutions of (4) with parameter shift (1) for different values of rate $\varepsilon \in [0.01, 0.15]$. The range of values for which the solution undergoes irreversible R-tipping (critical tange) is highlighted in the colorbar and roughly corresponds to the set [0.11,0.13].

fig:rate_sweep

2 Dependence on the growth rate

sec:critical_growth

Here we consider a family of $\mathcal{C}^2$ parameter shifts $\Lambda(t) \in \mathcal{P}(\lambda_-, \lambda_+)$ of the form of n^{th}-degree polynomials in $s := \sigma(t)$, where $\sigma(x)$ denotes any sigmoid function of x. We immediately notice that differently from the previous case 1, here we do not investigate the dependence of R-tipping from the perspective of the critical rate ε but rather fix a new timescale $t \to s = \sigma(t)$ (so that $s \to \pm 1$ as $t \to \pm\infty$) and investigate the properties of the polynomial parameter shift $g(s) = \sum_{k=0}^{n} c_k\, s^k \in \mathbb{P}_n[s] \subset \Lambda(s)$ that generate irreversible R-tipping. In particular for a fixed degree $n > 2$ we will choose a set $\{c_k\}_{k=0,\dots,n}$ of coefficients s.t. $g(s)$ is non-monotonic.

2.1 Lipschitz continuity

sub def:lipschitz_fun

Definition 2.1 (Lipschitz function)**.** A continuous and differentiable function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz is there is a $L > 0$ s.t.

$$|f(x) - f(y)| < L\,|x - y|\,, \quad \forall x \in \mathbb{R} \setminus \{y\}\,.$$

Theorem 2.1: Mean Value Theorem

If f is differentiable then $\forall x, y \in \mathbb{R}$, $x \neq y$ there exist $z \in (x, y)$ s.t.

$$\frac{f(x) - f(y)}{x - y} = f'(z)\,.$$

Corollary 2.1. *By Definition 2.1 and Theorem 2.1, if there exists $L > 0$ s.t. $|f'(z)| < L$, $\forall z \in \mathbb{R}$ then f is Lipschitz. This further implies that any f whose first derivative f' is bounded is necessarly Lipschitz.*

2.1.1 Computing Lipschitz constant

subsubsec:lipschitz_const

We can use Theorem 2.1 to determine L for a given function f. Since we consider parameter shifts of the form of polynomials of sigmoids we automatically restrict ourselves to functions whose derivatives f' are bounded and therefore Lipschitz as per Corollary 2.1. The algorithm proceeds as follows:

1. given f compute f';
2. find the set Γ of critical points of f', i.e. $\Gamma = \{x \in \mathbb{R} \;:\; f''(x) = 0\}$;
3. find the supremum of the set $|f'(\Gamma)|$.

In other words

$$L = \sup\left\{|f'(x)|\,,\ \forall x \in \mathbb{R} \;:\; f''(x) = 0\right\}. \qquad \text{\{eq:lipschitz\_constant\}}$$

2.2 Simulations

subsec:simulations

In the following we will consider diffent timescale transformations (although all sigmoids $\sigma(t) \in (-1, +1)$), in particular:

- $s = \tanh(t)$;
- $s = \frac{t}{1+|t|}$;
- $s = \frac{t}{\sqrt{1+t^2}}$.

From 2 we emphasize that the 3 time transformations listed above are sorted in descending order from faster ($\tanh(t)$) to slower ($t\left(\sqrt{1+t^2}\right)^{-1}$). We remark that choosing timescale transformations with different growth rates is in some sense equivalent to the tuning of the rate value ε in the previous case 1.

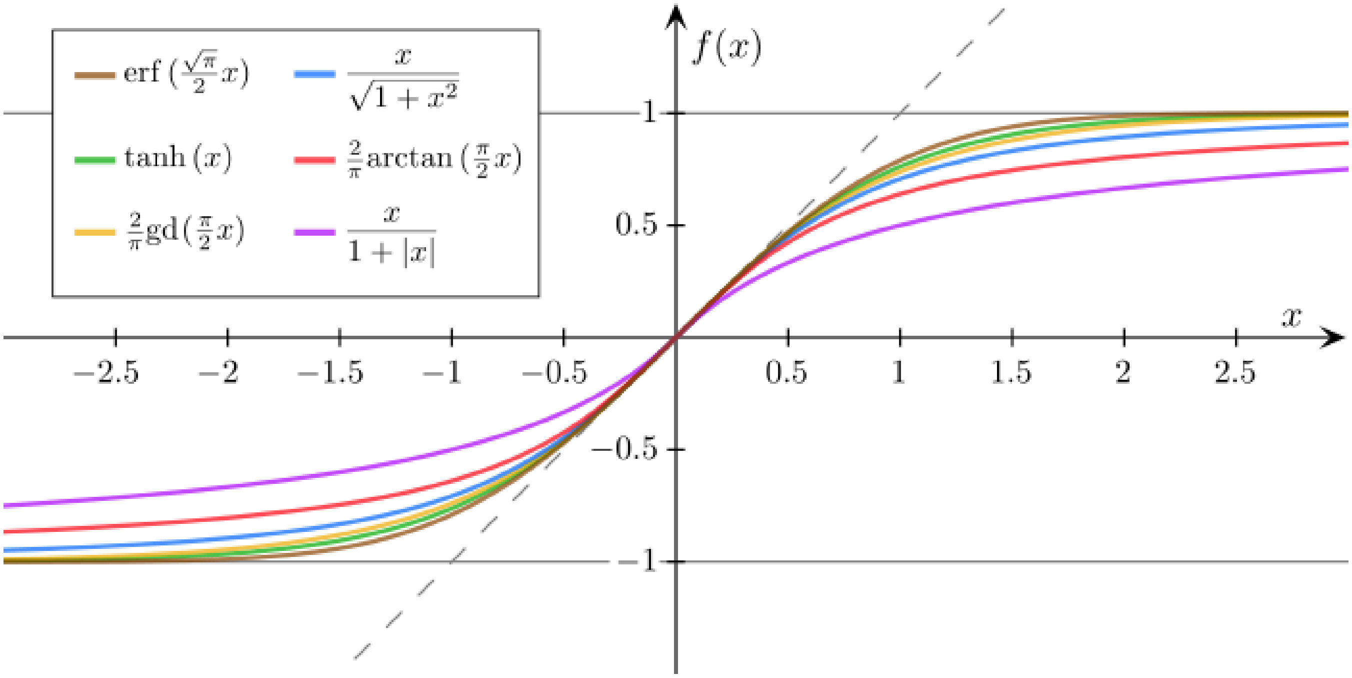

Figure 2: Visualisation of different sigmoid functions $\sigma(t) \in (-1, +1)$. Image taken from Wikipedia and to be later replaced by one of your own. fig:sigmoids

Finally we hereby provide a closed form expression for the non-autonomous differential equation governing the parameter shift, similar to (2), for the case of a generic polynomial ramp in the transformed timescale $s = \sigma(t)$

$$g(s) = (g \circ \sigma)(t) = \sum_{k=0}^{n} c_k \, \sigma^k(t)$$

$$\dot{\lambda}(t) = \frac{dg}{d\sigma}\frac{d\sigma}{dt} = \sum_{k=1}^{n} c_k \frac{d}{dt}\sigma^k(t) = \left(\sum_{k=1}^{n} k \, c_k \, \sigma^{k-1}(t)\right)\dot{\sigma}(t) \; .$$

{eq:sigm_shift_diff_eq}

2.2.1 $\sigma = \tanh(t) \;\Rightarrow\; \dot{\sigma} = \mathrm{sech}^2(t)$

subsubsec:timescale_1

Simulation 2

par:sim_2

$$g(s) \in \mathbb{P}_4[s] \, , \quad \{c_1 = -1.6, \, c_2 = -2, \, c_3 = 2.6, \, c_4 = 2\} \, .$$

{eq:sim_2_shift}

The critical points of g' are $t_1 =? \;\Rightarrow\; f'(t_1) = 0.33796$ and $t_2 =? \;\Rightarrow\; f'(t_2) = -1.6155$. As such we have

$$L = \sup\,(-\infty, 1.61155] = \inf\,[1.61155, +\infty) \, .$$

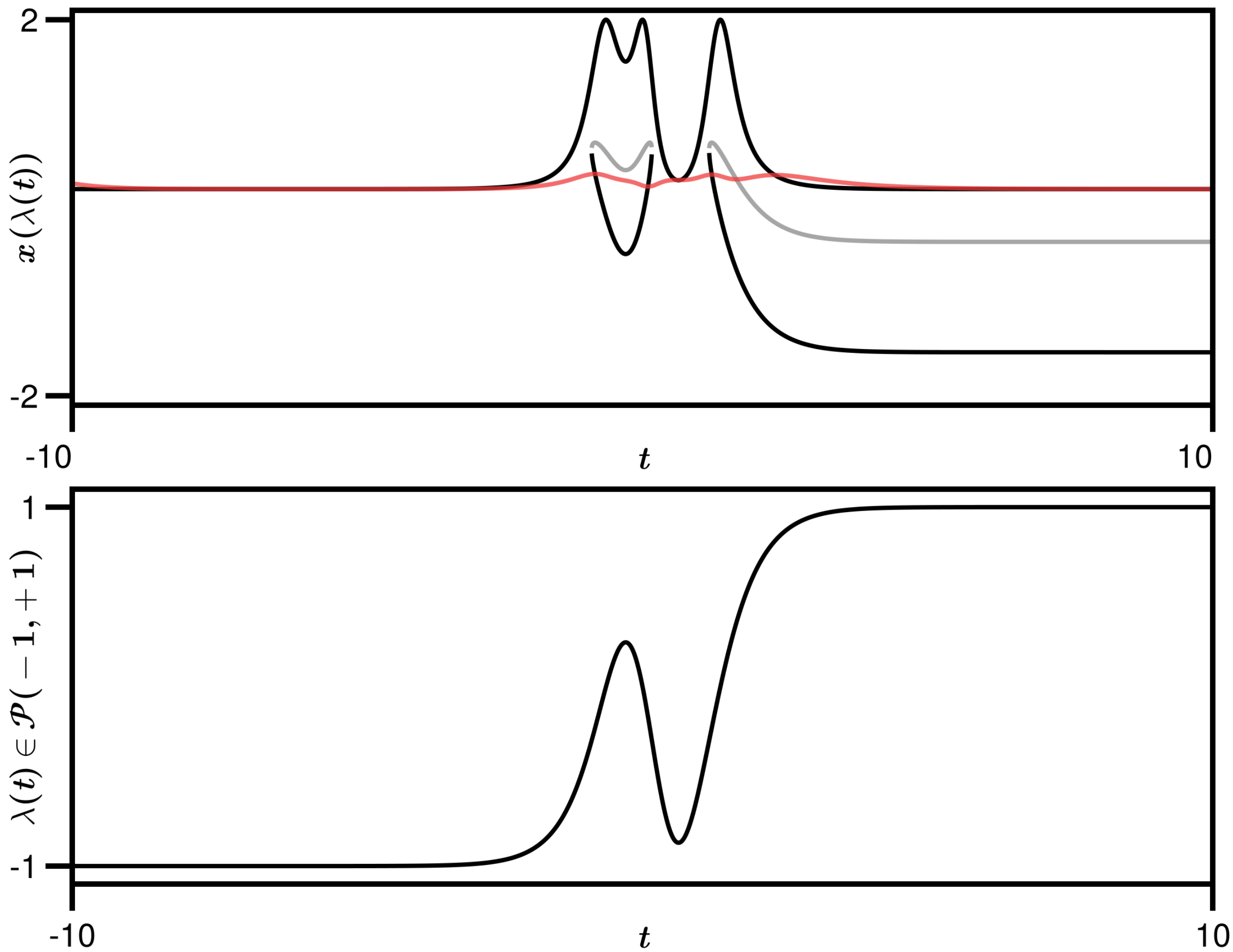

Figure 3: Solution of (4) (top) with the parameter shift (7) (bottom). No R-tipping. fig:sim_2

Simulation 3

par:sim_3

$$g(s) \in \mathbb{P}_9[s]\,, \quad \{c_1 = 1.9,\, c_2 = -0.6,\, c_3 = -7,\, c_4 = -2,\, c_5 = 1.7,\, c_6 = 2.4,\, c_7 = 3.2,\, c_8 = 0.4,\, c_9 = 2.2\}\,.$$

{eq:sim_3_shift}

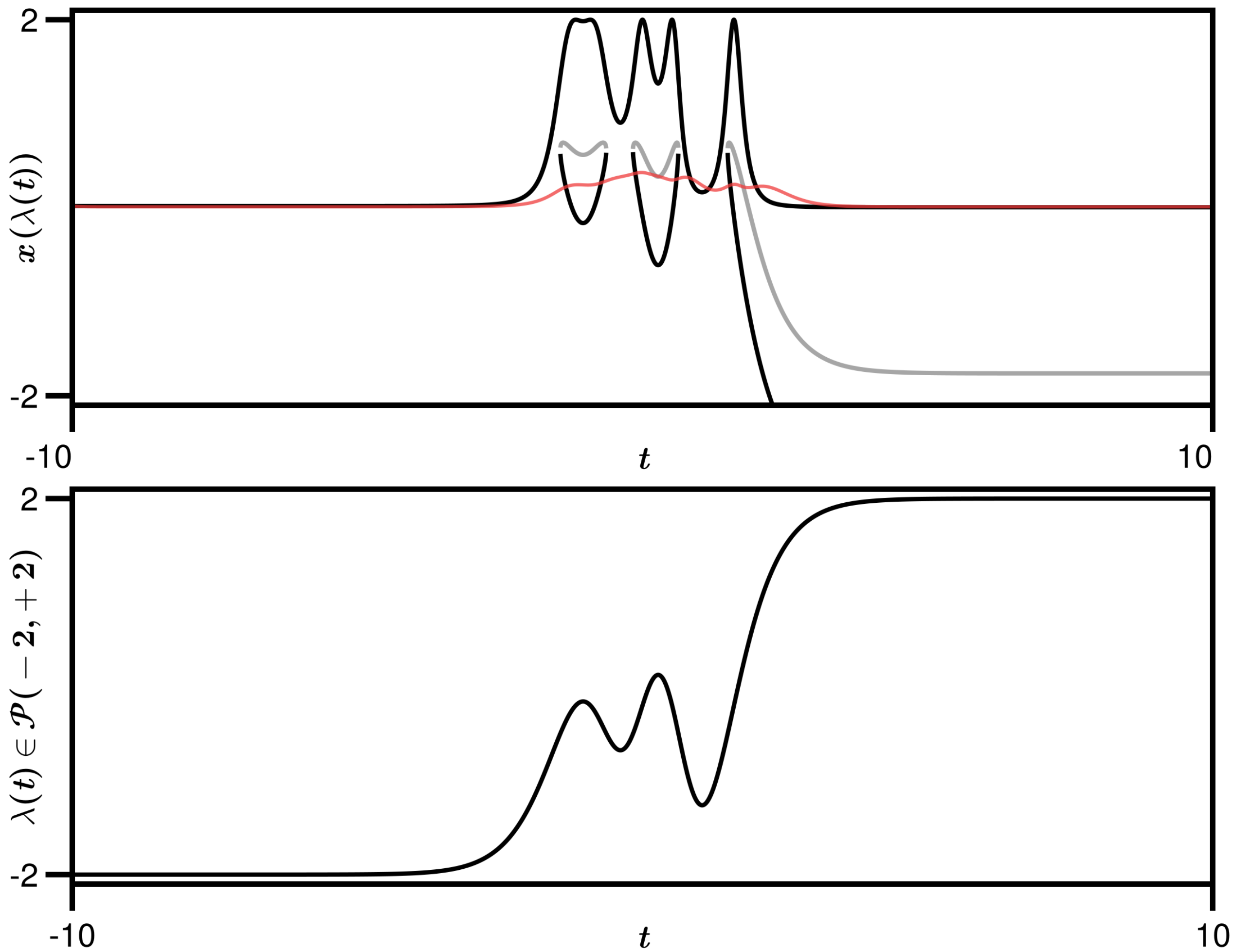

Figure 4: Solution of (4) (top) with the parameter shift (8) (bottom). No R-tipping. fig:sim_3

Simulation 6

par:sim_6

We will fix the following (odd) coefficients

$$g(s) \in \mathbb{P}_{11}[s] \,, \quad \{c_5 = -1.6,\, c_7 = 2,\, c_9 = \text{undef},\, c_{11} = -0.5\} \,, \tag{eq:sim\_6\_shift}$$

and let the coefficient of the 9$^{\text{th}}-$degree term to vary.

Notice that g is odd and thus symmetric in the interval $[-T, T]$ for any $T > 0$. We chose the coefficients s.t. g' has 5 critical points $\{t_j < j_{j+1}\}_{j=1,\ldots,4}$, with middle one being always $t_3 = 0$. The Lipschitz constant will always correspond to the values of g' at the end values of the critical points, i.e.

$$L = \sup\left(-\infty, g(t_{\text{end}})\right] ,$$

where $t_{end} = t_1 = -t_5$ since $|g'(t_1)| = |g'(t_5)|$.

subpar:sim_6_b

Simulation 6.b $c_9 = 1.1$ and $|g(t_{\text{end}})| = 0.8617$.

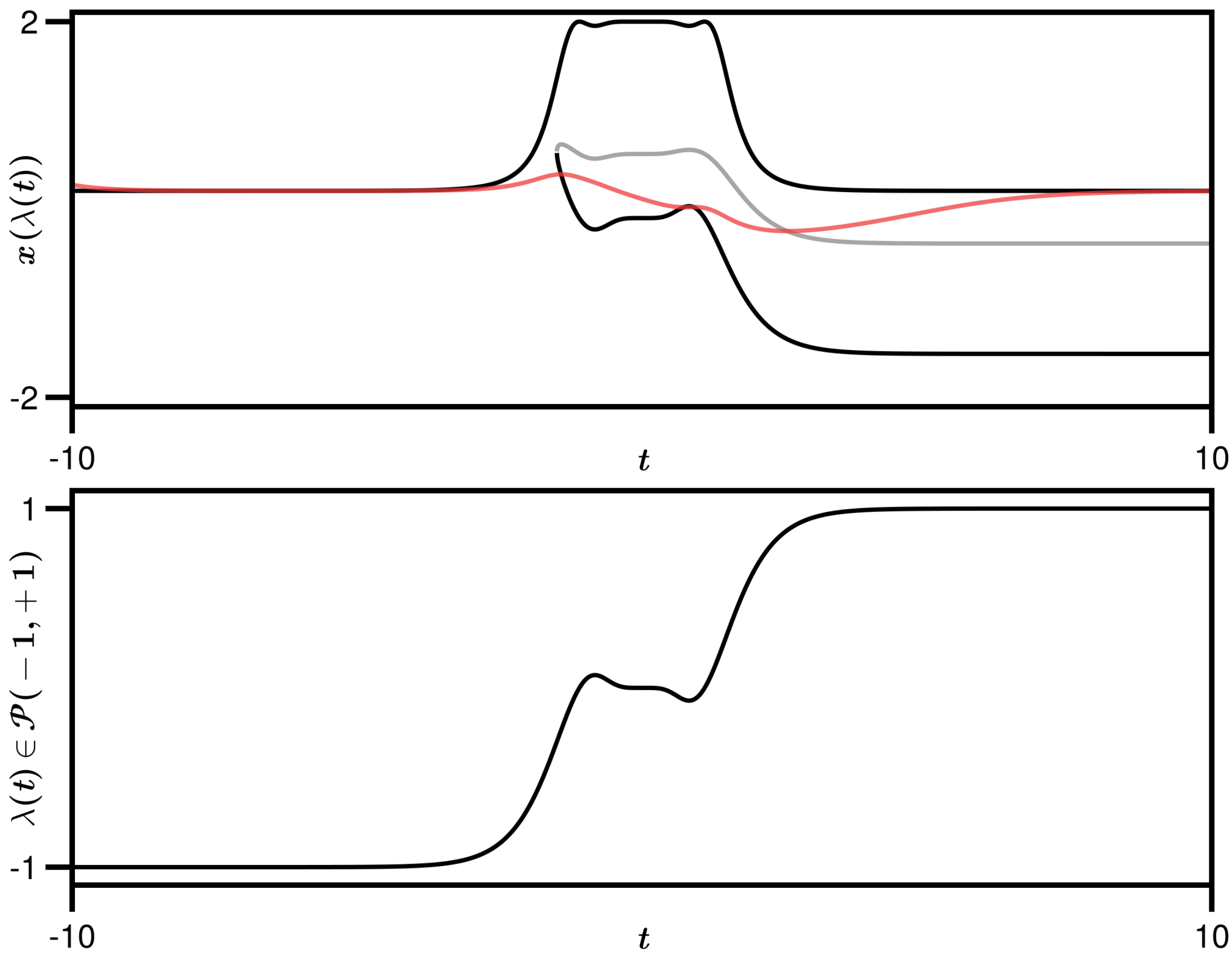

Figure 5: Solution of (4) (top) with the parameter shift (9) (bottom). No R-tipping. fig:sim_6_b

subpar:sim_6_c

Simulation 6.c $c_9 = 0.9$ and $|g(t_{\text{end}})| = 0.7145$.

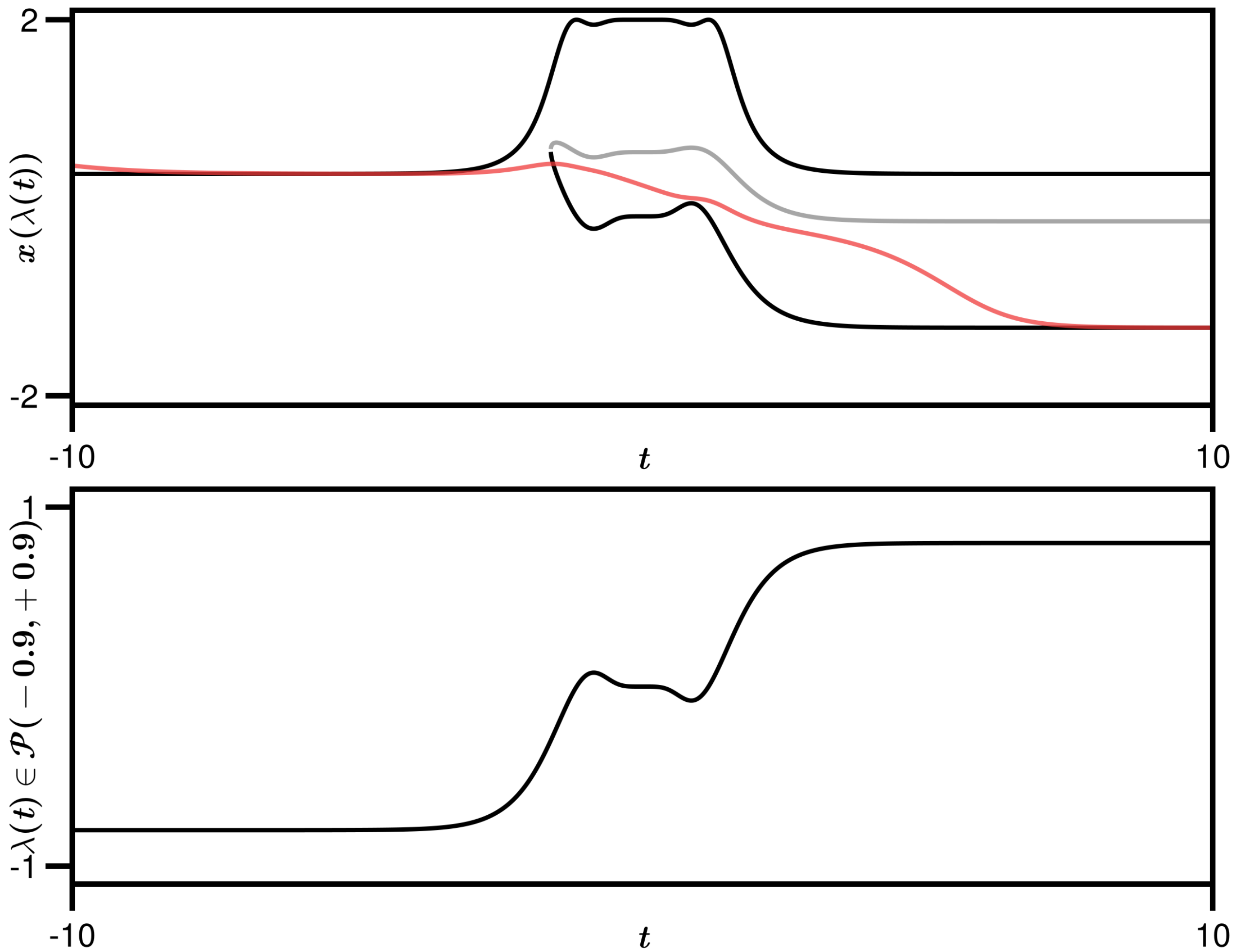

Figure 6: Solution of (4) (top) with the parameter shift (9) (bottom).

fig:sim_6_c

subpar:sim_6_d

Simulation 6.d $c_9 = 0.8$ and $|g(t_{\text{end}})| = 0.6409$.

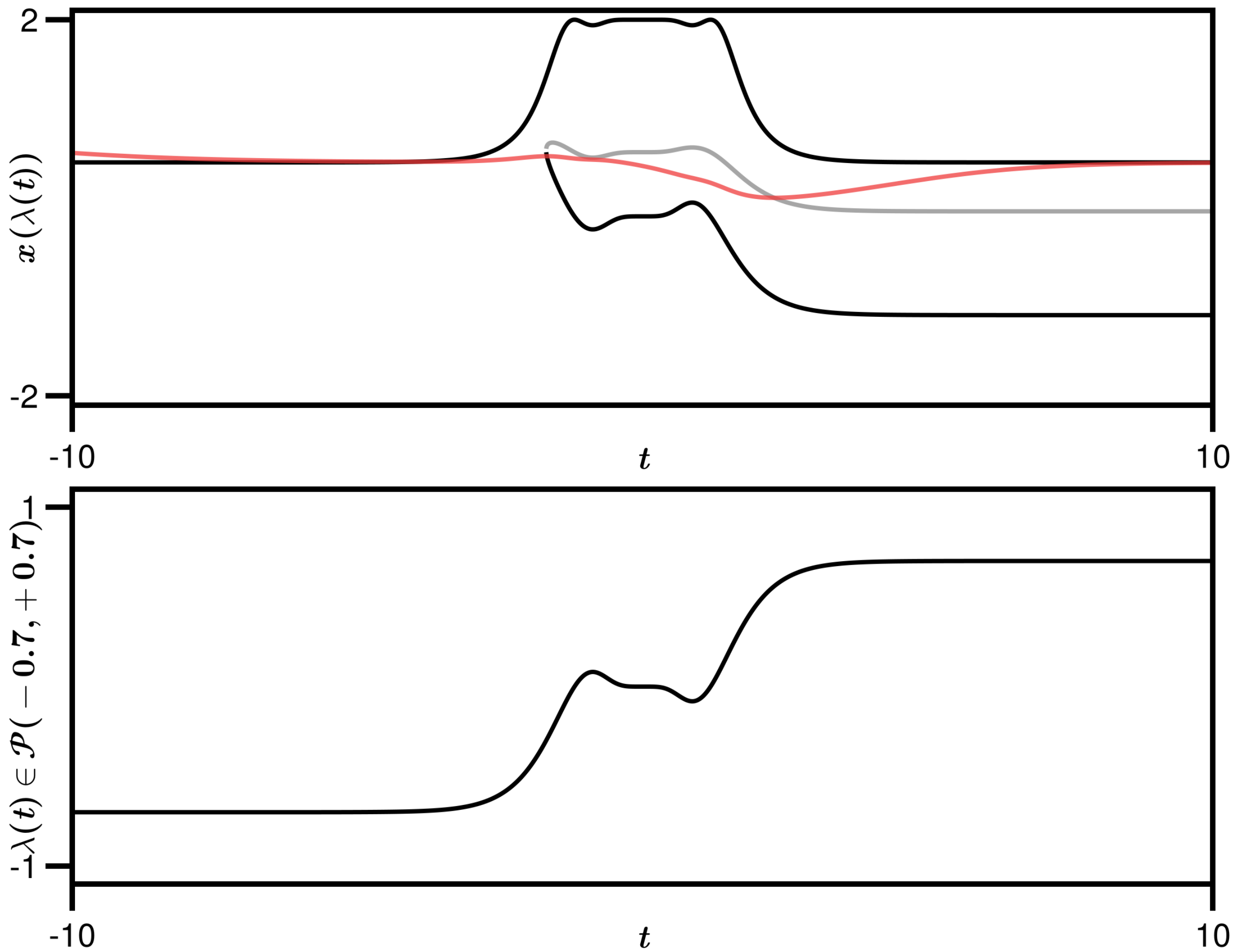

Figure 7: Solution of (4) (top) with the parameter shift (9) (bottom). No R-tipping. fig:sim_6_d

References

[1] Ashwin, P., Perryman, C., and Wieczorek, S. "Parameter shifts for nonautonomous systems in low dimension: bifurcation- and rate-induced tipping". *Nonlinearity*, 30 (2017). DOI.