Rate-induced tipping in a game-theoretic system

1 Autonomous replicator equation

sec:aut_replicator

We consider the following $1-$dimensional dynamical system

$$\dot{x} = f(x) = x(1-x)(r_1(x) - r_2(x))\,,$$

{eq:replicator_implicit}

where $r_1(x) = ax + b(1-x)$, $r_2(x) = cx + d(1-x)$ and $a, b, c, d \in \mathbb{R}$. Rearranging the terms of the last factor in (1) yields

$$\begin{aligned}\dot{x} =& x(1-x)(ax + b(1-x) - cx - d(1-x)) = \\ =& x(1-x)((a-c)x - (d-b)(1-x))\,.\end{aligned}$$

{eq:replicator_full}

This system's parameter space is therefore $\mathbb{R}^4$. We can define $\mathbb{R} \ni \alpha := a - c$ and $\mathbb{R} \ni \beta := d - b$ so that we can write (2) in a compactified form

$$\dot{x} = x(1-x)(\alpha x - \beta(1-x))\,.$$

{eq:replicator_reduced}

In the following we will use (2) or (3) depending on the conditions we want to derive.

1.1 Game-theoretic setting

subsec:aut_setting

The replicator equation in its most abstract form (1) models the [3]. The dynamic variable x models a fraction of a population of players adopting a particular strategy. The $4-$dimensional parameter $(a, b, c, d) \in \mathbb{R}^4$ models the payoff of adopting said strategy and is encoded as entries of a 2×2 matrix A. As such we restrict the parameter space to reflect such property.

def:admissable_set

Definition 1.1 (Admissable set)**.** Let (1) be our dynamical system, $\Omega = [0, 1] \subset \mathbb{R}$ be a target subset of its phase space, $x_0 := x(0) \in \Omega$ be an initial condition for (1) and $\phi_t(x_0)$ be the (forward) flow of the initial condition x_0 under (1). Then we denote

$$\Gamma := \{(a, b, c, d) \in \mathbb{R}^4 \ : \ \phi_t(x_0) \in \Omega \quad \forall x_0 \in \Omega,\, t > 0\}\,,$$

as the subset of admissable values in the parameter space of (1).

We will further characterise the admissable set Γ by looking at equilibria of the replicator equation.

1.1.1 Equilibria and stability

subsubsec:equilibria

The system has 3 equilibria: 2 *fixed* ones (meaning that they do not depend on the parameter vector) at $x = 0 =: x_1$ and $x = 1 =: x_2$ and 1 parametrised equilibrium

$$x_* = \frac{d-b}{a-b-c+d} = \frac{\beta}{\beta+\alpha}\,,$$ {eq:unstable_eq}

which we expressed both in terms of the full (2) and reduced (3) formulations of the system (1). Notice that x_* only exists in $\mathbb{R}$ for parameter values $(a,b,c,d) \in \mathbb{R}^4$ such that $a-b-c+d \neq 0$, or equivalently $(\alpha,\beta) \in \mathbb{R}^2$ such that $\alpha \neq -\beta$. We can use this last equilibrium to further characterise the admissable set of Definition 1.1. def:restricted_admissable_set

Definition 1.2. Let Γ be the admissable set in Definition 1.1 and let $\Gamma' = \{\gamma \in \Gamma \ : \ x_* \notin \mathbb{R}\}$ then we define a *restricted* admissable set

$$\Gamma_* = \Gamma \setminus \Gamma'\,.$$

Essentially with the definition of the restricted admissable set Γ_* we attempt to characterise a smaller portion of the initial admissable set Γ by exploiting the parametrised equilibrium x_*.

Specifically, if the parameter vector $\gamma \in \Gamma$ does not allow for x_* to exist (i.e. $\gamma \in \Gamma'$) then any trajectory starting in Ω will trivially stay in Ω asymptotically in forward time. This is true because at least one between $x_1 = 0$ and $x_2 = 1$ is globally stable in Ω (with the exception of the other equilibrium) $\forall\gamma \in \Gamma'$. We omit the trivial proof of this statement and instead redirect to Figure 1a for a visual depiction.

Conversely if $\gamma \in \Gamma$ is such that x_* does exist (i.e. $\gamma \in \Gamma \setminus \Gamma'$) then we ask that it allows for x_* to be in Ω. This last condition is necessary to ensure that such equilibrium is not spurious in the context of the game-theoretic setting of (1) i.e. we cannot have a fraction of a population to be less than 0 nor more than 1. This last condition is met by imposing the following inequalities on (4)

$$0 \leq \frac{d-b}{a-b-c+d} = \frac{\beta}{\beta+\alpha} \leq 1\,,$$

which yield

$$\Gamma_* = \{(\alpha,\beta) \in \mathbb{R}^2 \ : \ \operatorname{sign}(\alpha) = \operatorname{sign}(\beta)\}\,.$$ {eq:admissable_set_explicit}

With (5) we now summarise the informations regarding the stability of the autonomous system with the following statement. lemma:stability

Lemma 1.1. *Let $\Gamma = \Gamma' \cup \Gamma_*$ be the admissable set in Definition 1.2 then the following hold true*

1. ***coordination game**: if $\gamma \in \Gamma_*$ s.t. $\alpha,\ \beta > 0$ then x_1, x_2 are stable and x_* is unstable;*
2. ***dominant strategy**: if $\gamma \in \Gamma'$ s.t. $\alpha < 0$ and $\beta > 0$ then x_1 is stable and x_2 is unstable;*
3. ***dominant strategy**: if $\gamma \in \Gamma'$ s.t. $\alpha > 0$ and $\beta < 0$ then x_1 is unstable and x_2 is stable;*
4. ***anti-coordination**: if $\gamma \in \Gamma_*$ s.t. $\alpha,\ \beta < 0$ then x_1, x_2 are unstable and x_* is stable.*

Proof. This proof is trivial and essentially a direct calculation argument of the linearisation of (3) around its equilibria [3, Appendix A, p. 15]. $\square$

1.1.2 Bifurcation structure

subsubsec:bifurcations

Following Lemma 1.1 we can easily identify local codim−1 bifurcations of the replicator system as per the following result.

Theorem 1.1

Let the conditions of Lemma 1.1 be satisfied, then the subsets $\alpha = 0$ and $\beta = 0$ of the admissable set Γ are locii of transcritical bifurcations for (3).

Proof. Given a scalar (i.e. 1−dimensional) dynamical system $f(x;\mu)$ in state variable $x \in \mathbb{R}$ and (fixed) parameter value $\mu \in \mathbb{R}$ the conditions for a transcritical bifurcation of an equilibrium x_0 at parameter value μ_0 are the following

1. $(D_x f)(x = x_0; \mu = \mu_0) = 0;$
2. $(D_\mu f)(x = x_0; \mu = \mu_0) = 0;$
3. $(D_{xx} f)(x = x_0; \mu = \mu_0) \neq 0;$
4. $\Big((D_{\mu x} f) - (D_{xx} f)(D_{\mu\mu} f)\Big)(x = x_0; \mu = \mu_0) > 0;$

The linearised dynamics of (3) reads

$$(D_x f)(x) = -3(\alpha+\beta)x^2 + 2(\alpha+2\beta)x - \beta\,,$$

which evaluated at the 3 equilibria yields

$$\begin{aligned}
(D_x f)(x = x_1) &= -\,\beta\,,\\
(D_x f)(x = x_2) &= -\,\alpha\,,\\
(D_x f)(x = x_*) &= -\,\alpha(3\alpha^2 + \alpha(3\beta+4) + 2\beta + 1)\,.
\end{aligned}$$

It is straightforward from the above to realise that x_1 approaches loss of hyperbolicity from a state of stable attraction as $\beta \nearrow 0$ and, similarly, so does x_2 as $\alpha \nearrow 0$. Furthermore, from (4) it is also straightforward to see that

$$x_* \to x_1\,,\ \beta \to 0 \quad \text{and} \quad x_* \to x_2\,,\ \alpha \to 0\,.$$

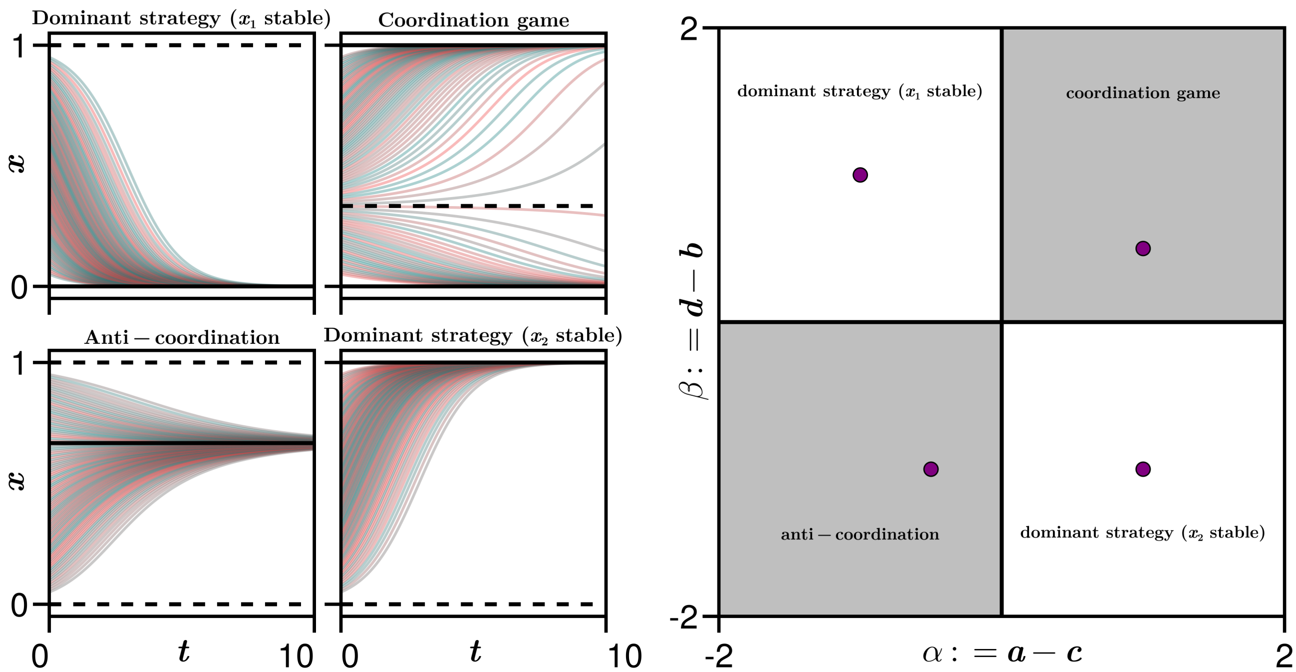
To prove that $\alpha = 0$ and $\beta = 0$ are transcritical bifurcations for x_* we essentially need to verify that the conditions 1. − 4. listed above hold for $x = x_*$ and $\{\alpha, \beta\} \ni \mu = 0$. This can be done by direct calculation for $\alpha = 0$ and $\beta = 0$ individually by fixing one of the two parameters and letting μ being the bifurcating one. We omit such calculations which, alebit trivial, are long and tedious to detail here and remark that this is a standard procedure in classical bifurcation theory (see e.g. [1]). obs:separatrices

Observation 1.1. From Definition 1.2 it follows trivially that Γ' and Γ_* are disjoint except for a subset of measure 1 in $\mathbb{R}^2$ that correspond to the union of the locii of transcritical bifurcations $\alpha = 0$ and $\beta = 0$.

The meaning of 1.1 is that the locii of transcritical bifurcations in $\mathbb{R}^2$ act as separatrices in the bifurcation set of (3) and as a result separate the interior of Γ' from the interior of the restricted admissable set Γ_*.

Bifurcation set

par:bif_set



(a) fig:autonomous_solutions

(b) fig:autonomous_bifset

Figure 1: Properties of the autonomous system (1) in qualitatively different regions of the admissable set Γ as per Lemma 1.1. (a) Asymptotic stability in Ω of different initial conditions for qualitatively different values of the parameter vector $\gamma := (\alpha, \beta) \in \Gamma$. Solid black lines indicate globally stable equilibria; dashed black lines indicate globally unstable equilibria. (b) Bifurcation set in $\mathbb{R}^2$ with separatrices (solid black lines) indicating the transcritical bifurcations outlined in Theorem 1.1. Purple dots in each region indicate the position in the admissable set ($\Gamma_* \subset \Gamma$ is shaded in gray) for which the solutions on the left a fig:autonomous

2 Non-autonomous augmentation

sec:nonautonomous_augmentation

$$\begin{cases} \dot{x} = x(1-x)(\lambda x - \beta(1-x))\,, \\ \dot{\lambda} = \frac{1}{2}(\operatorname{sech}^2(\varepsilon t + \delta) + 1)\,. \end{cases} \qquad \{eq:nonautonomous\_shifted\}$$

To analyse and prove rate-induced tipping in (6) we will refer to the rigorous framework introduced in [2].

References

[1] Glendinning, P. *Stability, instability and chaos: an introduction to the theory of nonlinear differential equations*. Cambridge University Press, 1994.

[2] Ashwin, P., Perryman, C., and Wieczorek, S. “Parameter shifts for nonautonomous systems in low dimension: bifurcation- and rate-induced tipping”. *Nonlinearity*, 30 (2017). DOI.

[3] Zino, L., Ye, M., Calafiore, G. C., and Rizzo, A. “Equilibrium selection in replicator equations using adaptive-gain control”. *Ieee transactions on automatic control* (2025). DOI.