Rate-induced tipping in a game-theoretic system

1 Autonomous replicator equation

sec:aut_replicator

We consider the following 1−dimensional dynamical system

$$\dot{x} = f(x) = x(1-x)(r_1(x) - r_2(x))\,,$$

{eq:replicator_implicit}

where $r_1(x) = ax + b(1-x)$, $r_2(x) = cx + d(1-x)$ and $a, b, c, d \in \mathbb{R}$. Rearranging the terms of the last factor in (1) yields

$$\begin{aligned}\dot{x} =& x(1-x)(ax + b(1-x) - cx - d(1-x)) = \\ =& x(1-x)((a-c)x - (d-b)(1-x))\,.\end{aligned}$$

{eq:replicator_full}

This system's parameter space is therefore $\mathbb{R}^4$. We can define $\mathbb{R} \ni \alpha := a - c$ and $\mathbb{R} \ni \beta := d - b$ so that we can write (2) in a compactified form

$$\dot{x} = x(1-x)(\alpha x - \beta(1-x))\,.$$

{eq:replicator_reduced}

In the following we will use (2) or (3) depending on the conditions we want to derive.

1.1 Game-theoretic setting

subsec:aut_setting

The replicator equation, in its abstract form (1), models a population game where players adopt one of two strategies to maximise payoff [3]. In particular, the dynamic variable x models a fraction of a population of players adopting said strategy (say 1). It follows that a fraction of $1 - x$ of the population will adopt the other strategy (say 2). The 4−dimensional parameter $(a, b, c, d) \in \mathbb{R}^4$ models the payoff of adopting strategy 1 and is encoded as entries of a 2×2 matrix A. As such we restrict the parameter space to reflect such property.

def:admissable_set

Definition 1.1 (Admissable set)**.** Let (1) be our dynamical system, $\Omega = [0, 1] \subset \mathbb{R}$ be a target subset of its phase space, $x_0 := x(0) \in \Omega$ be an initial condition for (1) and $\phi_t(x_0)$ be the (forward) flow of the initial condition x_0 under (1). Then we denote

$$\Gamma := \{(a, b, c, d) \in \mathbb{R}^4 \ :\ \phi_t(x_0) \in \Omega \quad \forall x_0 \in \Omega,\, t > 0\}\,,$$

as the subset of admissable values in the parameter space of (1).

We will further characterise the admissable set Γ by looking at equilibria of the replicator equation.

1.1.1 Equilibria and stability

subsubsec:equilibria

The system has 3 equilibria: 2 *fixed* ones (meaning that they do not depend on the parameter vector) at $x = 0 =: x_1$ and $x = 1 =: x_2$ and 1 parametrised equilibrium

$$x_* = \frac{d-b}{a-b-c+d} = \frac{\beta}{\beta+\alpha}\,,$$

{eq:unstable_eq}

which we expressed both in terms of the full (2) and reduced (3) formulations of the system (1). Notice that x_* only exists in $\mathbb{R}$ for parameter values $(a, b, c, d) \in \mathbb{R}^4$ such that $a - b - c + d \neq 0$, or equivalently $(\alpha, \beta) \in \mathbb{R}^2$ such that $\alpha \neq -\beta$. We can use this last equilibrium to further characterise the admissable set of Definition 1.1.

def:restricted_admissable_set

Definition 1.2. Let Γ be the admissable set in Definition 1.1 and let $\Gamma' = \{\gamma \in \Gamma \ : \ x_* \notin \mathbb{R}\}$ then we define a *restricted* admissable set

$$\Gamma_* = \Gamma \setminus \Gamma'\,.$$

Essentially with the definition of the restricted admissable set Γ_* we attempt to characterise a smaller portion of the initial admissable set Γ by exploiting the parametrised equilibrium x_*. Specifically, if the parameter vector $\gamma \in \Gamma$ does not allow for x_* to exist (i.e. $\gamma \in \Gamma'$) then any trajectory starting in Ω will trivially stay in Ω asymptotically in forward time. This is true because at least one between $x_1 = 0$ and $x_2 = 1$ is almost globally stable in Ω $\forall \gamma \in \Gamma'$ (i.e. $\phi_t(x_0) \to x_1$ for all $x_0 \in \Omega$ with the exception of $x_0 = x_2$ and viceversa). We omit the trivial proof of this statement and instead redirect to Figure 1a for a visual depiction. Conversely if $\gamma \in \Gamma$ is such that x_* does exist (i.e. $\gamma \in \Gamma \setminus \Gamma'$) then we ask that it allows for x_* to be in Ω. This last condition is necessary to ensure that such equilibrium is not spurious in the context of the game-theoretic setting of (1) i.e. we cannot have a fraction of a population to be less than 0 nor more than 1. This last condition is met by imposing the following inequalities on (4)

$$0 \leq \frac{d-b}{a-b-c+d} = \frac{\beta}{\beta+\alpha} \leq 1\,,$$

which yield

$$\Gamma_* = \{(\alpha, \beta) \in \mathbb{R}^2 \ : \ \mathrm{sign}(\alpha) = \mathrm{sign}(\beta)\}\,.$$

{eq:admissable_set_explicit}

With (5) we now summarise the informations regarding the stability of the autonomous system with the following statement.

lemma:stability

Lemma 1.1. *Let $\Gamma = \Gamma' \cup \Gamma_*$ be the admissable set in Definition 1.2 then the following hold true*

1. ***coordination game**: if $\gamma \in \Gamma_*$ s.t. $\alpha,\ \beta > 0$ then x_1, x_2 are stable and x_* is unstable;*
2. ***dominant strategy**: if $\gamma \in \Gamma'$ s.t. $\alpha < 0$ and $\beta > 0$ then x_1 is stable and x_2 is unstable;*
3. ***dominant strategy**: if $\gamma \in \Gamma'$ s.t. $\alpha > 0$ and $\beta < 0$ then x_1 is unstable and x_2 is stable;*
4. ***anti-coordination**: if $\gamma \in \Gamma_*$ s.t. $\alpha,\ \beta < 0$ then x_1, x_2 are unstable and x_* is stable.*

Proof. This proof is trivial and essentially a direct calculation argument of the linearisation of (3) around its equilibria [3, Appendix A, p. 15]. □

1.1.2 Bifurcation structure

subsubsec:bifurcations

Following Lemma 1.1 we can easily identify local codim−1 bifurcations of the replicator system as per the following result.

Theorem 1.1

Let the conditions of Lemma 1.1 be satisfied, then the subsets $\alpha = 0$ and $\beta = 0$ of the admissable set Γ are locii of transcritical bifurcations for (3).

Proof. Given a scalar (i.e. 1−dimensional) dynamical system $f(x;\mu)$ in state variable $x \in \mathbb{R}$ and (fixed) parameter value $\mu \in \mathbb{R}$ the conditions for a transcritical bifurcation of an equilibrium x_0 at parameter value μ_0 are the following

1. $(D_x f)(x = x_0; \mu = \mu_0) = 0$;
2. $(D_\mu f)(x = x_0; \mu = \mu_0) = 0$;
3. $(D_{xx} f)(x = x_0; \mu = \mu_0) \neq 0$;
4. $\Big((D_{\mu x} f) - (D_{xx} f)(D_{\mu\mu} f)\Big)(x = x_0; \mu = \mu_0) > 0$;

The linearised dynamics of (3) reads

$$(D_x f)(x) = -3(\alpha+\beta)x^2 + 2(\alpha+2\beta)x - \beta\,,$$

which evaluated at the 3 equilibria yields

$$\begin{aligned}(D_x f)(x = x_1) &= -\,\beta\,,\\ (D_x f)(x = x_2) &= -\,\alpha\,,\\ (D_x f)(x = x_*) &= -\,\alpha(3\alpha^2 + \alpha(3\beta+4) + 2\beta + 1)\,.\end{aligned}$$

It is straightforward from the above to realise that x_1 approaches loss of hyperbolicity from a state of stable attraction as $\beta \nearrow 0$ and, similarly, so does x_2 as $\alpha \nearrow 0$. Furthermore, from (4) it is also straightforward to see that

$$x_* \to x_1\,,\ \beta \to 0 \quad \text{and} \quad x_* \to x_2\,,\ \alpha \to 0\,.$$

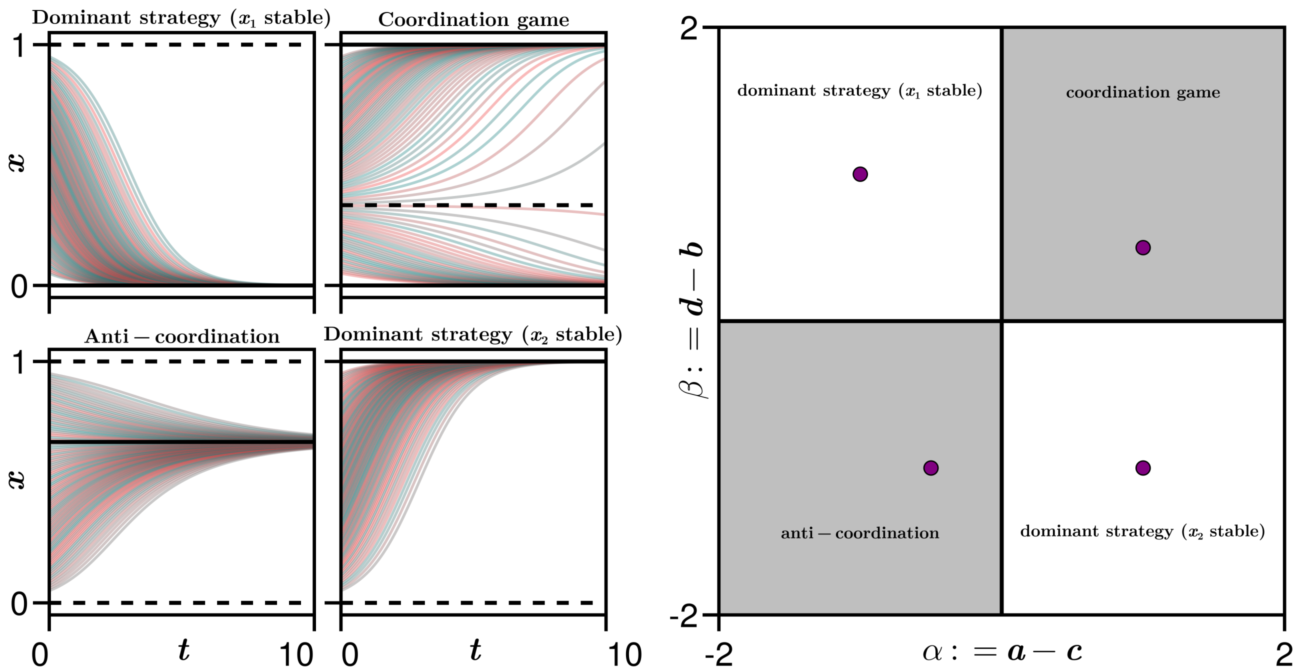
To prove that $\alpha = 0$ and $\beta = 0$ are transcritical bifurcations for x_* we essentially need to verify that the conditions 1. − 4. listed above hold for $x = x_*$ and $\{\alpha, \beta\} \ni \mu = 0$. This can be done by direct calculation for $\alpha = 0$ and $\beta = 0$ individually by fixing one of the two parameters and letting μ being the bifurcating one. We omit such calculations which, albeit trivial, are long and tedious to detail here and remark that this is a standard procedure in classical bifurcation theory (see e.g. [1]). obs:separatrices

Observation 1.1. From Definition 1.2 it follows trivially that Γ' and Γ_* are disjoint except for a subset of measure 1 in $\mathbb{R}^2$ that correspond to the union of the locii of transcritical bifurcations $\alpha = 0$ and $\beta = 0$.

The meaning of 1.1 is that the locii of transcritical bifurcations in $\mathbb{R}^2$ act as separatrices in the bifurcation set of (3) and as a result separate the interior of Γ' from the interior of the restricted admissable set Γ_*.

Bifurcation set

par:bif_set



(a) fig:autonomous_solutions

(b) fig:autonomous_bifset

Figure 1: Properties of the autonomous system (1) in qualitatively different regions of the admissable set Γ as per Lemma 1.1. (a) Asymptotic stability in Ω of different initial conditions for qualitatively different values of the parameter vector $\gamma := (\alpha, \beta) \in \Gamma$. Solid black lines indicate globally stable equilibria; dashed black lines indicate globally unstable equilibria. (b) Bifurcation set in $\mathbb{R}^2$ with separatrices (solid black lines) indicating the transcritical bifurcations outlined in Theorem 1.1. Purple dots in each region indicate the position in the admissable set ($\Gamma_* \subset \Gamma$ is shaded in gray) for which the solutions on the left a fig:autonomous

2 Non-autonomous augmentation

sec:nonautonomous_augmentation

In order ro reproduce rate-induce tipping (R-tipping) for a non-autonomous formulation of (3) we need the phase space to contain at least two stable equilibria. This restrict ourselves to the subset $\tilde{\Gamma}_* := \mathbb{R}^+ \times \mathbb{R}^+ \subset \Gamma_*$ of the restricted admissable set Γ_* which corresponds to the regime of a coordination game as per Lemma 1.1 and as depicted in Figure 1. def:r_tipping

Definition 2.1 (Irreversible R-tipping)**.** Let $\mathbb{B}(x_j, t)$ be the basin of attraction of the stable equilibrium x_j, $j \in \mathbb{N}$, at time $t \in \mathbb{R}$, $T > 0$ denoting a time instant and $x_0 := x(-T) \in \mathring{\mathbb{B}}(x_j, -T)$ being an initial condition that a time $-T$ lies within the interior of the basin of x_j. Then we say that the forward solution $x_0(t) := \phi_t(x_0)$ has undergone irreversible R-tipping if $x_0(t) \to x_k$, $t \to +\infty$, with $k \neq j$.

With the above we specify the condition for which R-tipping can be observed for a non-autonomous augmentation of the replicator system in a regime of coordination game. obs:r_tipping_replicator

Observation 2.1. In the case of (3) then $j \in \{1, 2\}$, $\forall (\alpha, \beta) \in \tilde{\Gamma}_*$. The basins $\mathbb{B}(x_{1/2}, t)$ are therefore uniquely identified, at each time instant $t \in \mathbb{R}$, by the location of x_* in Ω. More precisely

$$\mathbb{B}(x_1, t) = [x_1 = 0, x_3)\,,$$
$$\mathbb{B}(x_2, t) = (x_3, x_2 = 1]\,.$$

Furthermore $\Omega = \mathbb{B}(x_1, t) \cup x_* \cup \mathbb{B}(x_2, t)$, $\forall t \in \mathbb{R}$.

We now have the tools to formulate the non-autonomous replicator equation in a way that it guarantees the necessary conditions for irreversible R-tipping.

2.1 Parameter shift

subsec:shift

With Definition 2.1 we know that we must model the non-autonomous part of the system in such a way that an initial condition x_0, which at time $-T$ starts in the basin of x_1 (or x_2), when propagated forward in time will have x_2 (or x_1) as its $\omega-$limit set. In other words we want the trajectory $x_0(t)$ to somehow cross the boundary of the basin of attraction at some finite time t, which eventually is the cause of the R-tipping. With Observation 2.1 we know that such boundary for the replicator equation is given by x_* which is unstable for all (α, β) in $\tilde{\Gamma}_*$. Therefore one way to reproduce R-tipping is to make either (or both) the parameters in $\tilde{\Gamma}_*$ to change over time as this will *move* x_* in Ω and consequentally also change the boundary of the two basins of attraction.

2.1.1 Smooth monotonic ramp

subsubsec:ramp

To analyse and prove rate-induced tipping in (10) we will refer to the rigorous framework introduced in [2]. In it a non-autonomous dynamical system that presents R-tipping requires a C^2-smooth and bounded parameter shift $\Lambda(t)$ which is not necessarily monotonic. The parameter shift $\Lambda(t)$ shall connect two parameter values λ_-, λ_+ in the time asymptotic regime, i.e.

$$\text{range}(\Lambda(t)) = (\lambda_-, \lambda_+) \quad \text{and} \quad \lim_{t\to\pm\infty} \Lambda(t) = \lambda_\pm\,. \qquad \text{\{eq:generic\_shift\}}$$

Furthermore we ask for our parameter shift to be quasi-stationary at almost every time with the exception of a small subset (t_a, t_b). In other words we ask

$$\dot{\Lambda}(t) \approx 0\,,\ \forall t \in \mathbb{R} \setminus (t_a, t_b)\,. \qquad \text{\{eq:generic\_shift\_derivative\}}$$

Conditions (6)-(7) are necessary to ensure that the system has a stable attractor in both the past and future limits and that such attractor does not drift too fast once the transient regime is ended or before it started. Furthermore, even if the parameter shift does not need to be monotonic (as per [2]), we will impose this further constraint for simplicity.

2.1.2 Modelling the shift

subsubsec:modelling

We will model the parameter shift $\Lambda(t)$ in such a way that the unstable equilibrium x_* connects x_1 and x_2 in the past and forward limit respectively. In this way we condition the basin of attraction x_2 to shrink from almost the entirity of the phase space Ω to almost only containing x_2 itself as time runs forward. To do that we further reduce the dimensionality of our parameter space by letting $\alpha = 1 - \beta$. This condition entails that $x_* = \beta$ which significantly simplifies our modelling procedure. In fact now we can impose $\lambda(t) := \beta$ and choose an appropriate parameter shift $\Lambda(t)$ that satifies (6)-(7). This is done easily since $x_* \to x_{1/2}$, $t \to \pm\infty$ immediately translates in $\lambda_- = x_1 = 0$ and $\lambda_+ = x_2 = 1$. We can now pick any sigmoid function for $\Lambda(t)$ that connects 0 to 1 in $t \to \pm\infty$. One choice is the hyperbolic tangent shift

$$\Lambda(t) = \frac{1}{2}(\tanh(\varepsilon t + \delta) + 1)\,, \qquad \text{\{eq:tanh\_shift\}}$$

which we parametrised in its ramp rate by $\varepsilon > 0$ and its centering (in time) by $\delta \in \mathbb{R}$. Notice that by varying ε we change the Lipschitz constant L of $\Lambda(t)$ (essentially making the shift to ramp faster in the transient regime for larger values of ε) whereas by varying δ we change the subset (t_a, t_b) in (7) when the transient regime occurs. Values $\varepsilon \to +\infty$ and $\delta \to 0$ make the selected shift (8) to converge to the sign function.

Remark. *Any C^1-smooth, bounded function f is Lipschitz with constant L that coincides with* $\sup\{|f'(x_j)|\}$, *where $x_j \in$ dom f are stationary points of f'.*

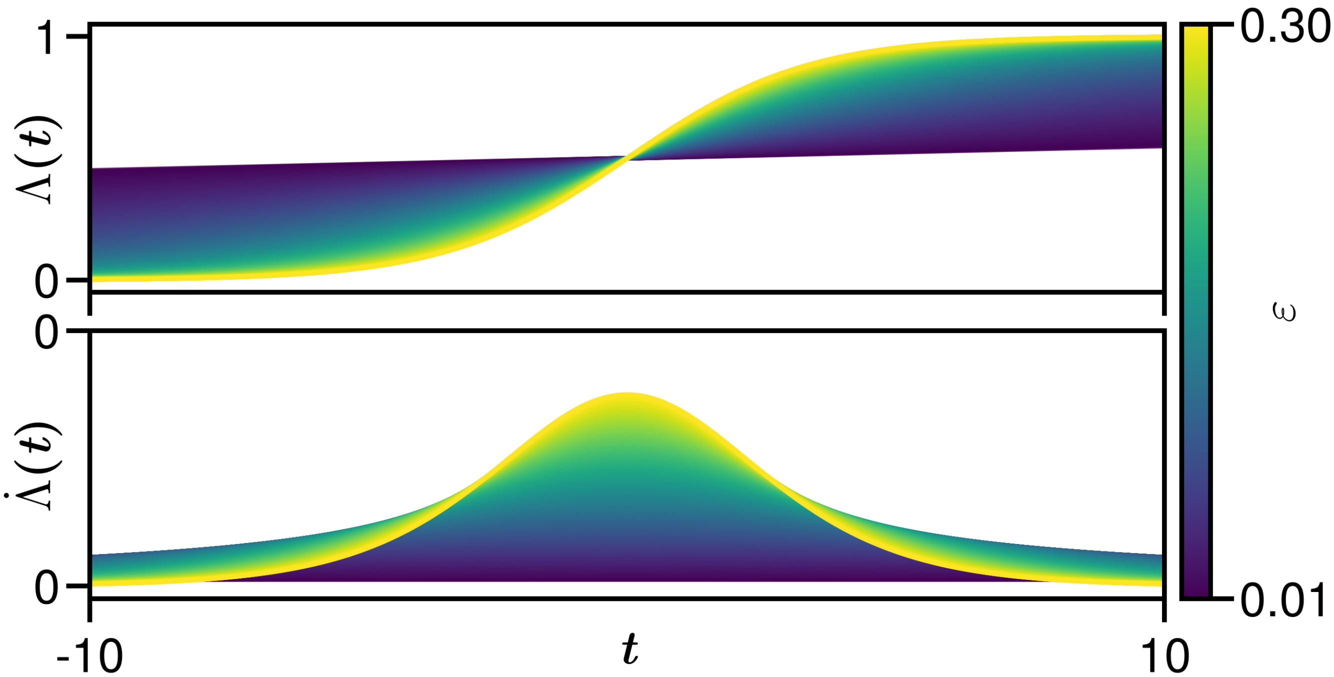


Figure 2: Example of a sigmoid parameter shift $\Lambda(t) =$ (top) and its time derivative $\dot{\Lambda}(t) =$ (bottom). Different magnitudes of the shift's ramp $\varepsilon > 0$ are color coded as per the colorbar on the left. fig:sigmoid_example

With the above we can formulate the non-autonomous augmented replicator system

$$\begin{cases} \dot{x}(t) = x(1-x)(x - \lambda(t))\,, \\ \dot{\lambda}(t) = \frac{\varepsilon}{2}\,\mathrm{sech}^2(\varepsilon t + \delta)\,, \end{cases} \qquad \text{\{eq:nonautonomous\_shifted\_implicit\}}$$

where $\dot{\lambda}(t) = \frac{d}{dt}\Lambda(t) = \dot{\Lambda}(t)$.

We would like to rewrite (9) so that it can be interpreted as a 2−dimensional autonomous system. To do that we drop the time dependence on λ and $\dot\lambda$ and notice that

$$\tanh(\varepsilon t+\delta)=2\lambda-1 \quad \text{and} \quad \dot\lambda=\frac{\varepsilon}{2}\operatorname{sech}^2(\varepsilon t+\delta)=\frac{\varepsilon}{2}\Big(1-\tanh^2(\varepsilon t+\delta)\Big)=\frac{\varepsilon}{2}\left(1-(2\lambda-1)^2\right),$$

which, after trivial calculations, allows us to rewrite the non-autonomous system as

$$\begin{cases}\dot x = x(1-x)(x-\lambda(t))\,,\\ \dot\lambda = 2\,\varepsilon\lambda(1-\lambda)\,.\end{cases}$$

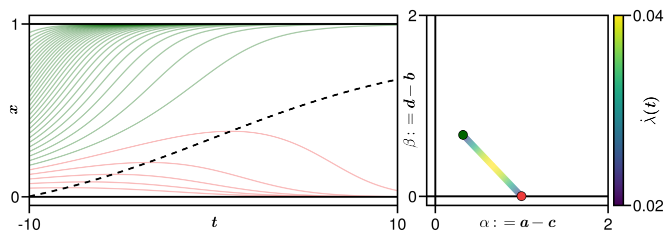
{eq:nonautonomous_shifted}

2.2 Numerical results

subsec:results

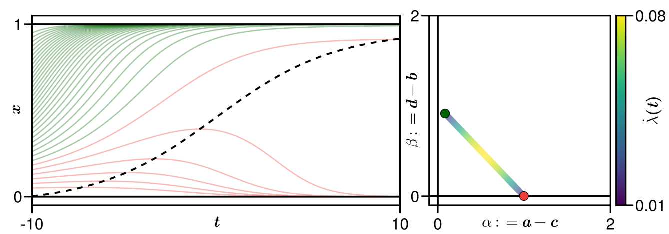
In the following we will solve the non-autonomous system (10) for different values of the shift ramp $\varepsilon>0$ and infere qualitatevely how it affects the properties of the a set of solutions.

We truncate the time domain to be finite, i.e. we choose $T>0$ so that $\{-T,T\}$ represents the time horizons for the past and future limit systems.



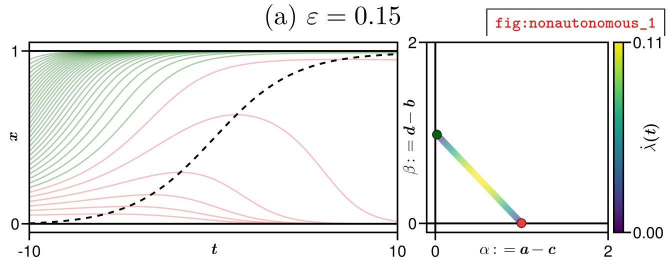
(a) $\varepsilon=0.15$

fig:nonautonomous_1



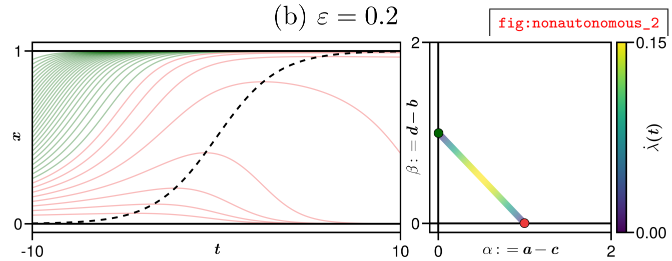
(b) $\varepsilon=0.2$

fig:nonautonomous_2



(c) $\varepsilon=0.25$

fig:nonautonomous_3



(d) $\varepsilon=0.3$

fig:nonautonomous_4

Figure 3: Solutions of the non-autonomous replicator system with a hyperbolic tangent shift of the parameter formulated in (10). Different shift rates ε are depicted with increasing values sorted left to right and top to bottom (the values for ε are reported in the caption of each subplot). The time truncation is set to $T=10$ and we set the $\delta=0$ so that the shift's time derivative is maximised at time $t=0$. For each subplot the left (rectangular) box displays the timeseries of solutions of a set of initial conditions in $\mathbb{B}(x_2,-T)$: green lines indicate those solutions that have not undergone R-tipping at time $T=10$, red lines those that did, black solid lines indicate the stable equilibria x_1, x_2 and the dashed black line indicate the unstable equilibria x_*. Conversely the right (square) box indicates the trajectory of the shifted parameter $\lambda(t)=\beta=1-\alpha$ in $\tilde\Gamma_*$ from its past limit value (red dot) to its future limit value (green dot). The color along the curve indicates the magnitude of its rate of change at different timesteps (i.e. the value of $\dot\Lambda(t)$) as encoded on the colorbar on the left.

fig:nonautonomous

References

[1] Glendinning, P. *Stability, instability and chaos: an introduction to the theory of nonlinear differential equations*. Cambridge University Press, 1994.

[2] Ashwin, P., Perryman, C., and Wieczorek, S. "Parameter shifts for nonautonomous systems in low dimension: bifurcation- and rate-induced tipping". *Nonlinearity*, 30 (2017). DOI.

[3] Zino, L., Ye, M., Calafiore, G. C., and Rizzo, A. "Equilibrium selection in replicator equations using adaptive-gain control". *Ieee transactions on automatic control* (2025). DOI.