An introduction

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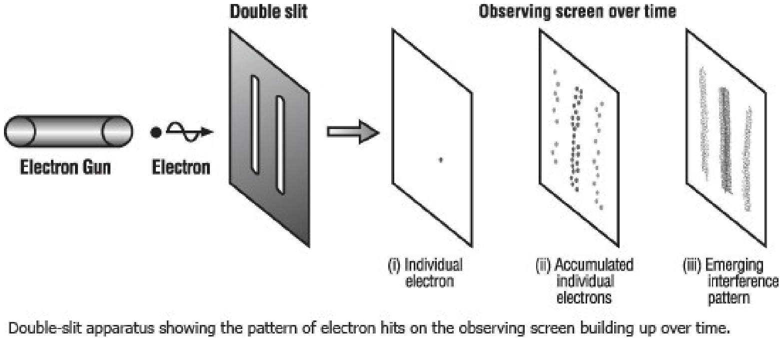
Young's Double Slit Experiment

Particles behave like waves



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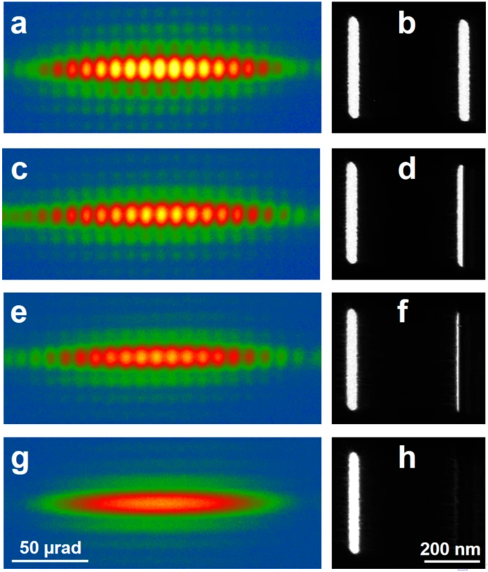
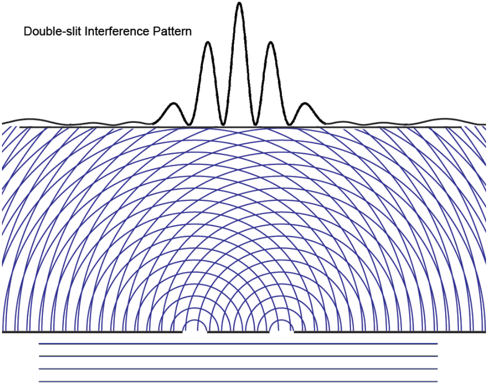
Quantum world if fascinating

Particles behave like waves

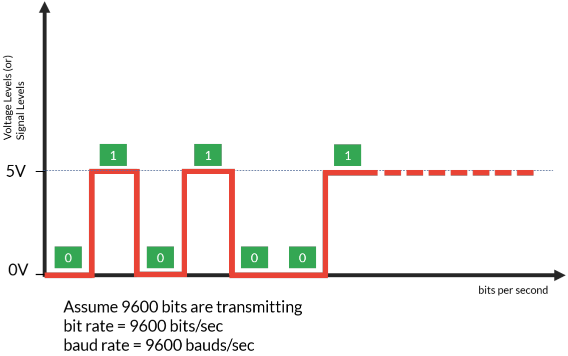


The state of a particle after passing through either one of the slits can be described as a *wave* function (probability distribution) namely $\Psi = (\alpha_0\psi_0 + \alpha_1\psi_1)$ with $\{\alpha_0, \alpha_1\} \in \mathbb{C}$



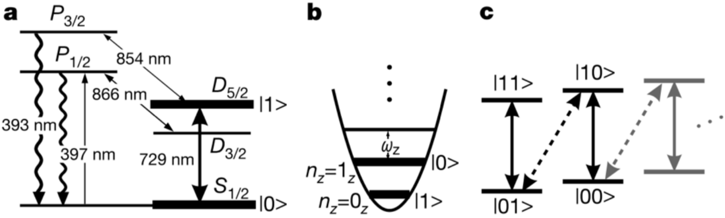
Basic Unit of information: Bits

Traditional computation works with 0 and 1 as basic units of information. A physical realization of this is voltage from 0V to 5V



Basic Unit of information: Qubits

Quantum computation works with $|0\rangle$ and $|1\rangle$ as basic units of information. A physical realization of this would be a spin $1/2$ particle.



Computational basis states

Qubits can be in different states *other* than $|0\rangle$ or $|1\rangle$. It is possible to form *linear combinations* of states, called superpositions:

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

The numbers α_0 and α_1 are complex numbers and $|\alpha_0|^2 + |\alpha_1|^2 = 1$.

Where $|0\rangle$ and $|1\rangle$ are vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{C}^2$.

A superposition state is a linear combination $\psi = \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Quantum NOT gate

Classical computer circuits consists of wires and logic gates. E.g. the NOT gate which has a truth table $0 \rightarrow 1$ and $1 \rightarrow 0$.

The analogous quantum operation would take

$$\alpha_0 |0\rangle + \alpha_1 |1\rangle$$

to

$$\alpha_0 |1\rangle + \alpha_1 |0\rangle$$

Since a quantum state can be represented as a vector, we are looking for a matrix such that:

$$\mathrm{X} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_0 \end{bmatrix}$$

Quantum Gates

Quantum gates are represented by matrices applied on our vectors (qubits). Are all matrices valid Quantum Gates?...no.

Recall that $|\alpha_0|^2 + |\alpha_1|^2 = 1$ for a quantum state

$$\alpha_0 |0\rangle + \alpha_1 |1\rangle$$

. This must also hold for

$$\alpha_0^{'} |0\rangle + \alpha_1^{'} |1\rangle$$

after the gate has acted. It turns out that the appropriate condition on the matrix representing the gate is tha the matrix U be unitary. That is (with $U^\dagger$ is the adjoint of U)

$$U^\dagger U = I$$

Quantum Gates

In classical computers we only have one non-trivial gate for one bit (NOT gate). In the case of Quantum computers we have many!

For example the NOT gate:

$$\mathrm{X} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The *Haddamard* gate

$$\mathrm{H} \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Quantum Gates

Some single qubit matrices are used quite frequently and are very useful.

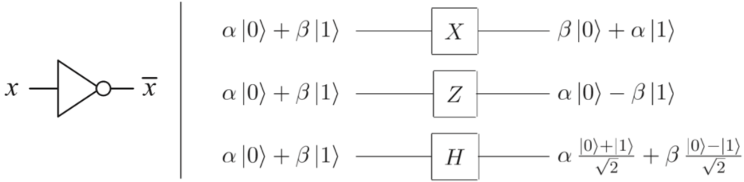


Figure: Single bit (left) and qubit (right) logic gates.

Two qubit representation

If V and W are vector spaces with bases $\{v_1 ... v_n\}, \{w_1 ... w_n\}$, the *tensor product* $V \times W$ of V and W is a nm-dimensional vector space which is spanned by elements of the form $v \otimes w$.
These elements behave bilinearly, meaning:

$$\alpha(v \otimes w) = \alpha v \otimes w = v \otimes \alpha w$$

$$u \otimes v + w \otimes v = (u + w) \otimes v$$

$$u \otimes v + v \otimes w = u \otimes (v + w)$$

Using Dirack "ket" notation, we can write a two qubuit basis space as:

$$\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$$

Problem to be solved by Deutsch's algorithm

Given a black box U_f implementing some unknown function f : $\{0,1\} \rightarrow \{0,1\}$, determine wheter f is "constant" or "balanced".

Here "constant" means f always outputs the same bit, i.e. $f(0) = f(1)$, and "balanced" means f outputs different bits on different inputs, i.e. $f(0) \neq f(1)$.

There is an easy solution, simply evaluate f on inputs 0 and 1, i.e. compute $f(0)$ and $f(1)$, and then check if $f(0) = f(1)$. This is the "classical" solution which requires two "queries" or calls to U_f. Can we solve it in just one query?

Classically, the answer turns out to be no. But quantumly this can be achieved in a single query

A bit about black boxes

First we need to understand how to model a function in the quantum world. It turns out postulate 2 of quantum mechanics says, all quantum operations must be unitary and hence reversible.

This is a bit non-trivial since in general given the output of a function $f(x)$, it is not always possible to invert f to obtain x. We have to compute $f(x)$ in such a way as to guarantee that the computation can be undone. This is achieved in the following setup:

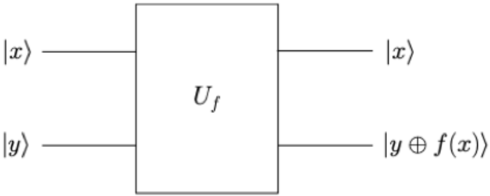


Figure: U_f is a unitary operator mapping $|x\rangle |y\rangle \rightarrow |x\rangle |x \oplus y\rangle$ for any $x, y \in \{|x\rangle, |y\rangle\}$

A naive idea

Since we are leveraging Quantum Mechanics, what happens if we query U_f with input state $|x\rangle$ replaced with $\alpha|0\rangle + \beta|1\rangle$ and output state $|y\rangle$ with $|0\rangle$?

Intuitively, we have set the input register to both inputs 0 and 1, so we expect U_f to return a superspoition of both possible outputs, $f(0)$ and $f(1)$. Inded, by linearity of U_f, the output of the circuit will beamer

$$\begin{aligned}|\psi\rangle &= U_f(\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle = \alpha U_f|0\rangle|0\rangle + \beta U_f|1\rangle|0\rangle \\ &= \alpha|0\rangle|f(0)\rangle + \beta|1\rangle|f(1)\rangle\end{aligned}$$

Great! We seem to have obtained both outputs of f with just a single query! Unfortunately, upon measurement, we would only be able to obtain information about one of the two terms in superposition.

We failed obtaining $f(0)$ and $f(1)$ simultaneously. Luckily our goal is not that, but evaluating $f(0) \oplus f(1)$ will help us deterimine if our function is constant or balanced. The solution is Deutsch's algorithm

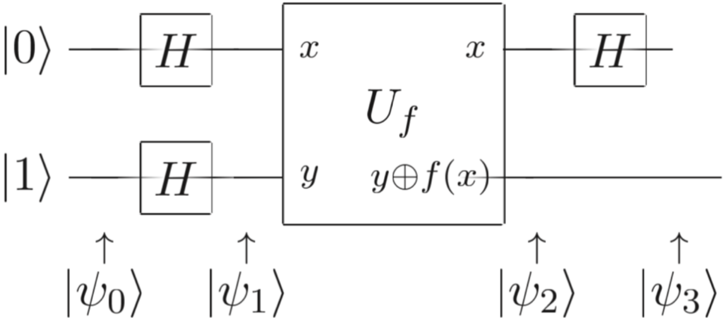


Figure: Quantum circuit implementing Deutsch's algorithm

(Quantum) walk through

Not obvious at all why it works (generally true about all quantum algorithms). Let's first look at the start of the circuit ($|\psi_0\rangle$), then after applying the Haddamards ($|\psi_1\rangle$), then
after applying U_f ($|\psi_2\rangle$) and after applying the last Hadamard ($|\psi_3\rangle$)

$$|\psi_0\rangle = |0\rangle\,|1\rangle\,,$$
$$|\psi_1\rangle = |+\rangle\,|-\rangle = \frac{1}{2}(|0\rangle\,|0\rangle - |0\rangle\,|1\rangle + |1\rangle\,|0\rangle - |1\rangle\,|1\rangle)$$

After applying the operator U_f, we have

$$|\psi_2\rangle = \frac{1}{2}(|0\rangle\,|f(0)\rangle - |0\rangle\,|1+f(0)\rangle + |1\rangle\,|f(1)\rangle - |1\rangle\,|1+f(1)\rangle)$$

Constant f

By definition, if f is constant, then $f(0) = f(1)$ So we can simplify $|\psi_2\rangle$ as follows

$$\begin{aligned}
|\psi_2\rangle &= \frac{1}{2}(|0\rangle|f(0)\rangle - |0\rangle|1 \oplus f(0)\rangle + |1\rangle|f(0)\rangle - |1\rangle|1 \oplus f(0)\rangle) \\
&= \frac{1}{2}((|0\rangle + |1\rangle) \otimes |f(0)\rangle - (|0\rangle + |1\rangle) \otimes |1 \oplus f(0)\rangle) \\
&= \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|f(0)\rangle - |1 \oplus f(0)\rangle) \\
&= \frac{1}{\sqrt{2}}|+\rangle \otimes (|f(0)\rangle - |1 \oplus f(0)\rangle)
\end{aligned} \tag{1}$$

Applying the last Haddamard on the top qubit we obtain:

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes (|f(0)\rangle - |1 \oplus f(0)\rangle)$$

Balanced f

By definition, if f is constant, then $f(0) \neq f(1)$. Since f is a binary function, this means $f(0) \oplus 1 = f(1)$ and $f(1) \oplus 1 = f(0)$ we can then simplify:

$$\begin{aligned} |\psi_2\rangle &= \frac{1}{2}(|0\rangle |f(0)\rangle - |0\rangle |f(1)\rangle + |1\rangle |f(1)\rangle - |1\rangle |f(0)\rangle) \\ &= \frac{1}{2}((|0\rangle - |1\rangle) \otimes |f(0)\rangle - (|0\rangle - |1\rangle) \otimes |f(1)\rangle) \\ &= \frac{1}{2}(|0\rangle - |1\rangle) \otimes (|f(0)\rangle - |f(1)\rangle) \\ &= \frac{1}{\sqrt{2}} |-\rangle \otimes (|f(0)\rangle - |f(1)\rangle) \end{aligned} \tag{2}$$

Applying the last Haddamard on the top qubit we obtain:

$$|\psi_3\rangle = \frac{1}{\sqrt{2}} |1\rangle \otimes (|f(0)\rangle - |f(1)\rangle)$$

Final thoughts

Even though this quantum algorithm reduces "only" in half the amount of queries we have to make to our function. It turns out that there are other algorithms which provide an exponential advantage which use some of the ideas presented here.

The fasinating world of quantum is finally begining to yield some of the advantages that I was not expecting to come to life during my lifetime.

Incredibly we can now leverage real quantum computers and the future of Quantum Computing seems more realistic than 4 years ago (https://quantum-computing.ibm.com/)

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