Lecture : 5

Linear Algebra (Review)

" You can't add apples and oranges "

two dimensional vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $v_1 \rightarrow$ first component

column vector $v_2 \rightarrow$ second component

Vector Addition $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ & $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, $\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$

Scalar Multiplication $2\vec{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}$ and $-\vec{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$

linear combination $= c\vec{v} + d\vec{w} = \vec{R}$; c & d are constant

By changing the values of c & d linear combination of two non-parallel vectors $\vec{v}$ & $\vec{w}$ can fill entire two dimentional space.

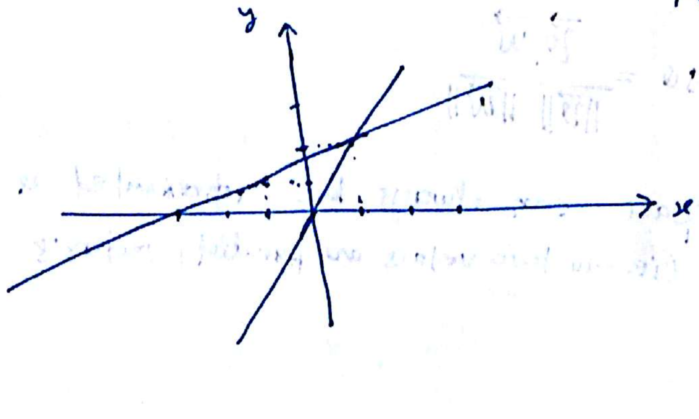
In other words any vector in the plane formed by $\vec{v}$ & $\vec{w}$ can be represented as the linear combination of $\vec{v}$ & $\vec{w}$.

Example: $2x - y = 0$

$-x + 2y = 3$ $\Rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$A x = b$

Row Picture

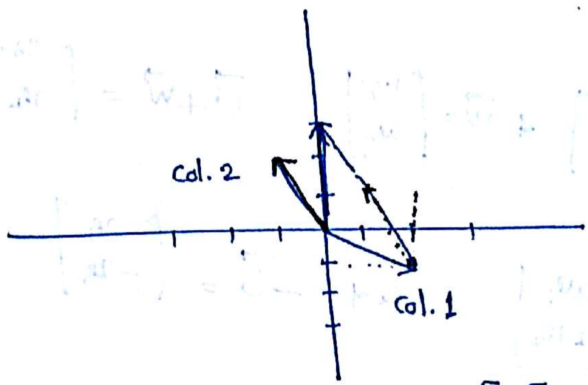


$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

column Picture

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

linear combination of column vectors



$$1. \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Inner product (Dot product) of vector $\vec{v}$ & $\vec{w}$

$$\vec{v} . \vec{w} = v_1 w_1 + v_2 w_2$$

$$\text{if } \vec{v} . \vec{w} = 0 \Rightarrow \vec{v} \perp \vec{w}$$

length $\|v\|$ of vector $\vec{v}$

$$\text{length} = \text{norm}(\vec{v})$$
$$= \sqrt{\vec{v} . \vec{v}}$$

Angle between

$$\cos\theta = \frac{\vec{v} . \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}$$

NOTE

A vector in n dimensional space can always be represented as linear combination of n non-parallel (ie. no two vectors are parallel) vectors

Matrix multiplication with a vector. $Ax = b$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Ax is a comb. of columns of A.

* $\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$
$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \cancel{[a \; b \; c]} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \times \text{column}(1) + b \times \text{column}(2) + c \times \text{column}(3)$$

* $$[a \quad b \quad c] \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = a \times \text{row}(1) + b \times \text{row}(2) + c \times \text{row}(3)$$

for example <u>to subtract 3 × row(1) from row(2)</u>

$$\begin{array}{r} \text{to keep 1st row unchanged} \rightarrow \\ \text{to subtract } 3 \times \text{row}(1) \text{ from row}(2) \rightarrow \\ \text{to keep 3rd row unchanged} \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

<u>Permutation Matrix (P)</u>

<u>Exchange rows</u>

$$\overset{P}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

0×1^{st} row $+ 1 \times$ second row $= 2^{nd}$ row

1×1^{st} row $+ 0 \times 2^{nd}$ row $= 1^{st}$ row

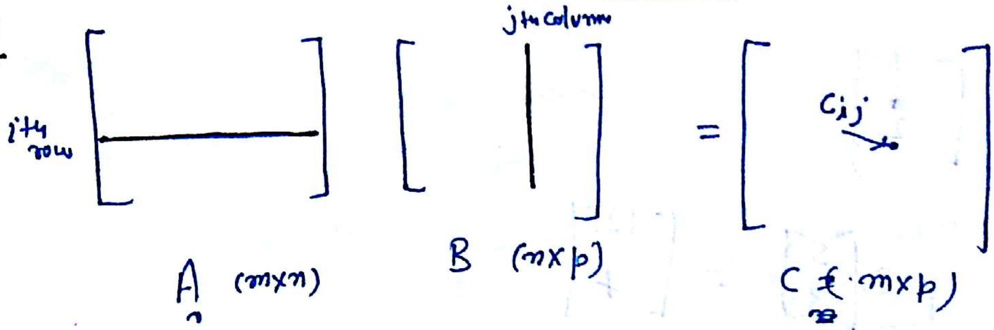
<u>Column Exchange</u>

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{P}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

0×1^{st} col. $+ 1 \times 2^{nd}$ col. $= 2^{nd}$ column | 1×1^{st} col. $+ 0 \times 2^{nd}$ column $= 1^{st}$ column.

Multiplication of two Matrices A & B

Method-I



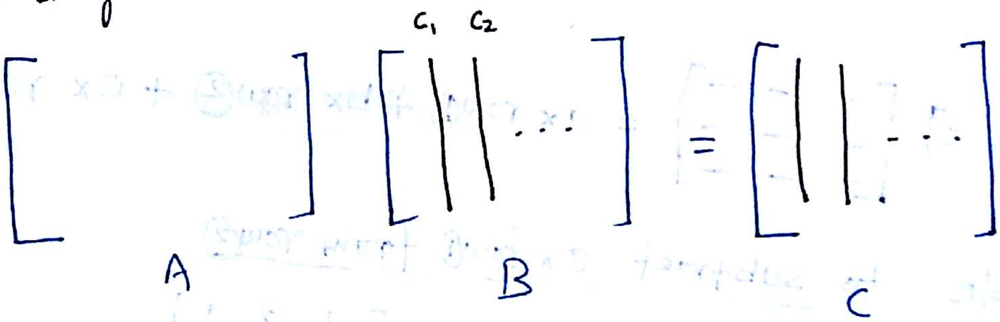
General element of matrix C

$$c_{ij} = (\text{i}^{th} \text{ row vector of A}) \cdot (\text{j}^{th} \text{ column vector of B})$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots$$

$$= \sum_{k=1}^{n} a_{ik}\, b_{kj}.$$

Method-II (using column vector)



Matrix A × 1^{st} column of B = 1^{st} column of Matrix C

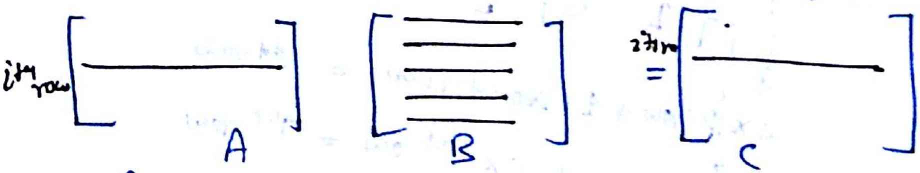
Matrix A × 2^{nd} col. of B = 2^{nd} col. of C

... and so on

in general Matrix A × j^{th} col. vector of B = j^{th} col. vector C

i.e column of C are linear combination of column of A.

Method-III (using row vector)



Rows of C are linear combination of rows of B

(i^{th} row vector of A) × Matrix B = i^{th} row of B.

Method-IV

Outer product

(column vector of A) x (row vector of B) = ?

$(m \times 1)$ $(1 \times p)$

Example

$$\underset{A}{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & 6 \end{bmatrix}} = \underset{C}{\begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}}$$

AB = Sum of (cols of A) x (rows of B)

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}$$

Block Multiplication

$$\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]} \underset{B}{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]} = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right]$$

$C_1 = A_1B_1 + A_2B_3$

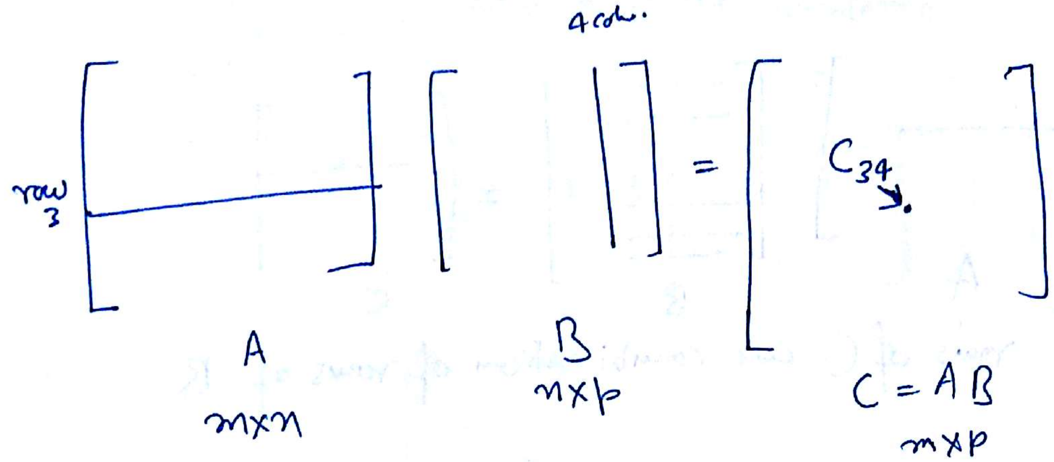
$C_2 = A_1B_2 + A_2B_4$

~~and so on~~

$C_3 = A_3B_1 + A_4B_3$

$C_4 = A_3B_2 + A_4B_4$

Inverces



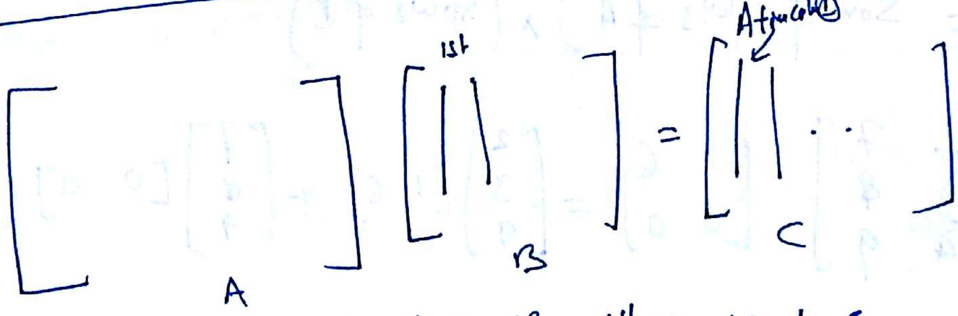
$$C_{34} = (\text{row3 of } A) \cdot (4 \text{ colum of } B)$$

dot product

$$C_{ij} = (i^{th} \text{ row vector of } A) \cdot (j^{th} \text{ column vector of } B)$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots$$

$$= \sum_{k=1}^{n} a_{ik}b_{kj}$$



Matrix × 1^{st} column of B = 1^{st} colum of Result C

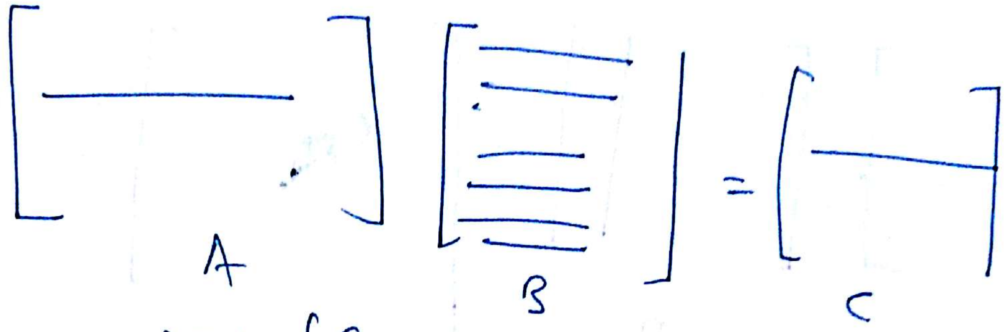
similar Matrix × 2^{nd} colu B = 2^{nd} colum of Ans. C

and so on

Matrix × j^{th} column of B = j^{th} colum of Ans C

i.e. column of C are linear combination of column of A

III way



rows of C are combination of rows of B

IV th way

Column of A $\times$ row of B

$m \times 1$ $\qquad 1 \times p$

Exam

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

AB = Sum of (Cols of A) $\times$ (rows of B)

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

V th way.

Block Multiplication

$A_1B_1 + A_2A_3$

$$\underbrace{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]}_{A} \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right]$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

Matrix x Column = column

Now

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = \begin{matrix} 1 \times row1 \\ + 2 \times row2 \\ + 7 \times row3 \end{matrix}$$

1x3 3x3

row x Matrix = row

step 1: Subtract 3x row1 from row2

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

3rd row not change

as 1st row is not change

E_{21}
↓
Elemination Matrix to obtain 2,1

Step: Subtract 2 x row2 from row3

$$E_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

upper triangular Matrix

$$E_{32}(E_{21}A) = U$$

$$(E_{32}E_{21})A = U$$

Permutation Matrix → Exchange rows or column Exchange.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

P

$$\cancel{\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

P

Solving Eqn by elimination

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y + z = 2$$

$$Ax = b.$$

$$A = \begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \longrightarrow \begin{matrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{matrix} \rightarrow \begin{matrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{matrix}$$

1st pivot

second pivot

$R_2 - 3R_1$

$R_3 - 2R_2$

↑ 3rd pivot

$$U = \begin{matrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{matrix}$$

U → upper triangular Matrix three pivots 1, 2, & 5

→ Pivot is never zero ; if zero comes switch rows to make it non-zero. If its not possible ⇒ no solution

Augmented matrix

$$\begin{matrix} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \\ & A & & b \end{matrix} \rightarrow \begin{matrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{matrix} \rightarrow \begin{matrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & 10 \\ & & U & c \end{matrix}$$

Back substitution

$$x + 2y + z = 2$$
$$2y - 2z = 6$$
$$5z = -10 \Rightarrow z = -2$$
$$y = 1$$
$$x = 2$$

$$E[A\ I] = [I\ \ ?]$$

elimination matrices (product)

ie $EA = I$ tells us $B = A^{-1}$

→ $EI = A^{-1}I = A^{-1}$

Inverse of product AB

$$(AB)(B^{-1}A^{-1}) = I$$

$$B^{-1}A^{-1}AB = I$$

$$AA^{-1} = I$$

$$(A^{-1})^T A^T = I$$

$(A^{-1})^T$ → Inverse of A^T

i.e. inverse of AT is Transpose of A^{-1}

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$E_{21}A = U$

$E_{21}^{-1}E_{21}A = E_{21}^{-1}U$

$A = E_{21}^{-1}U$

$= LU$

$$A = L\ \ U$$

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$= \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}$$

D ↑ diagonal

Inverse (Square Matrices)

$A^{-1}A = I = AA^{-1}$ (for square matrices not for rectangular matrices)

If A^{-1} exists, $\therefore$ A is invertible or non singular

Singular case : No inverse

Example: $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

A will not have inverse as there exists a non zero vector X such that $Ax = 0$

here $x = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $Ax = 0$

Let A^{-1} exist ; $\Rightarrow A^{-1}Ax = 0$

$\Rightarrow x = 0$, but $x \neq 0$

ie A^{-1} does not exist.

ie. column vector are pointing in same direction

Invertible Matrix

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

A x column j of A^{-1} = column j of I

Gauss - Jordan (Solve 2 eqns at once

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

augmented Matrix

$$\rightarrow \left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

A I

eliminate to

$$\rightarrow \left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

use elimination upward

$$\rightarrow \left[\begin{array}{cc:cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

Example

$$E_{32}E_{31}E_{21}A = U \quad \left(\begin{array}{c}\text{needed} \\ \text{no row} \\ \text{exchange case}\end{array}\right)$$

$$A = \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}U$$
$$= L\,U$$

let E_{31} not needed

$$\begin{array}{ccc} E_{32} & E_{21} & E \\ \begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix} & \begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix} & = \begin{bmatrix}1&0&0\\-2&1&0\\10&-5&1\end{bmatrix}\end{array}$$

$$\begin{array}{ccc} E_{21}^{-1} & E_{32}^{-1} & L \\ \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix} & \begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix} & = \begin{bmatrix}1&0&0\\2&1&0\\0&5&1\end{bmatrix}\end{array}$$

$$E\,A = U$$

$$\underline{A = L\,U}$$

If no ~~no~~ row exchanges

take multipliers (for elemination) directly go to L

Row exchanges

Permutations 3x3

~~At~~ Permutation matrix with no row exchange is identity matrix. $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$3! = 3\times2\times1$

Exchang row 2 & row 1

$$P_{12} = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, \quad P_{13} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

$$P_{23} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}, \quad \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}, \quad \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

6 possible row exchanges

1 2 3
3 2 1
1 3 2
2 1 3
3 1 2
2 3 1

group of 6 matrix, multiple of these result in same group

Inverse also in same group

ie. $\boxed{P^{-1} = P^T}$

for 4x4
$4! = 4\times3\times2\times1 = 24$

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Permutations P: execute row exchanges

when row exchange becomes necessary to have non zero pivot

$A = LU$ (not applicable)

becomes $PA = LU$ // for any invertible matrix A

Permutations

$P =$ identity matrix with reordered rows. ($n!$)

$P^{-1} = P^T$ or $P^T P = I$

Transpose $(A^T)_{ij} = A_{ji}$

Symmetric matrices $A_{ij} = A_{ji}$ ie $A^T = A$

To get a Symmetric matrix for a Matrix R

$R^T R$ is always Symmetric

$(R^T R)^T = R^T \cdot (R^T)^T = R^T R$

or $(R R^T)^T = (R^T)^T \cdot R^T = R R^T$

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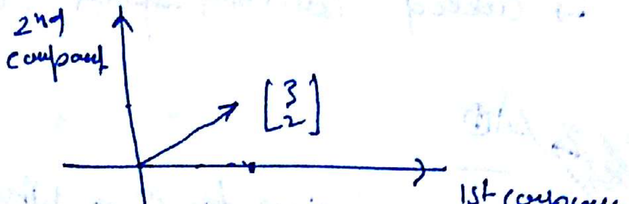
Vector Spaces & Subspaces

Examples: R^2 = all 2-dim. real vectors e.g. $\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}, \ldots$

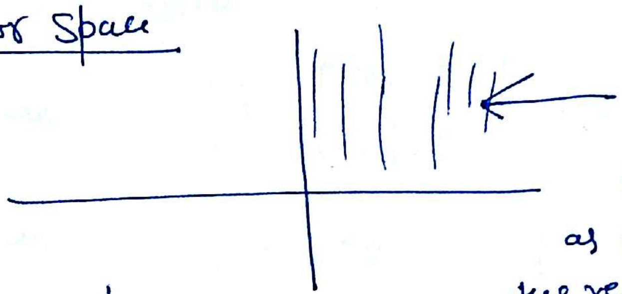


R^2 is all such vectors 2D space

Similarly R^3 = all 3-dim. real vectors with real components.

R^n = all column vector in n ~~com~~ real component in n-dim.

Not a vector Space



if we consider only quadrant space

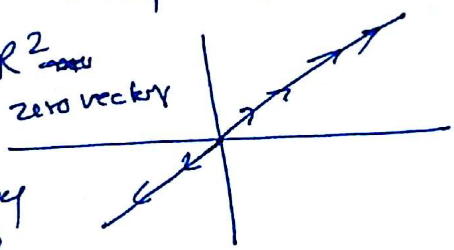
it is not a vector space as if we multiply some scalar the resultant vector may be out of this space.

Examples

a vector space inside R^2

i.e. subspace of R^2

A line in R^2 going through zero vector



i.e. every subspace must have $\vec{0}$.

Subspace of R^2

1. all of R^2
2. any line through $\begin{bmatrix}0\\0\end{bmatrix}$ (infinite length)
3. zero vector only

Subs. of R^3

1. all of R^3
2. a plane passing thr. $\begin{bmatrix}0\\0\\0\end{bmatrix}$
3. a line "
4. zero vector only

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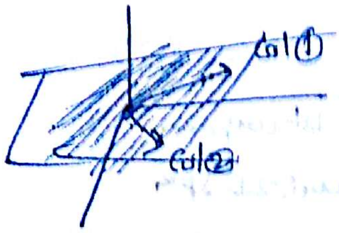
$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

Columns are in R^3

all their linear combination form a subspace

↳ called column space $C(A)$



subspace for is a plane form col(1) & col(2)

Spanning a space

Vectors: $v_1, \dots, v_\ell$, span a space

means, the space consists of all lin. combination of these vectors

Basis for a space is a sequence of vectors

(* Not unique) $v_1, v_2, \dots, v_d$ with two properties:

1. They are independent
2. They span the space

i.e. sufficient no. of indep. vector not less, not more

Example: Space is $\mathbb{R}^3$ || for space $\mathbb{R}^n$, n vectors give basis if the $n \times n$ matrix with these cols is invertible.

One basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ (not the only.)

other basis Example $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$

> Although basis for given space not unique, but every basis for the space has the same number of vectors, which is dimension of the space. Also, $\dim(C(A)) = \text{rank}(A)$

Example: let the space is $C(A)$, $N(A)$ (for $Ax = 0$)

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

rank $= 2$, dim of $C(A) = 2$

$N(A)$: $\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$

$\dim N(A)$ = no. of free variable $= n - r$ ($n \to$ no. of colm, $r \to$ rank)

* Rank(A) = No. of Pivot Columns = Dimension of column space $C(A)$

Basis: 1st two column,
or col. 1 & 3
col 2 & 4
or combination like $\begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}6\\4\\6\end{bmatrix}$ etc.

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linear Independence

Let A is $m \times n$ matrix $m < n$.

Then there are nonzero solution to $Ax=0$
(more unknown than equations)

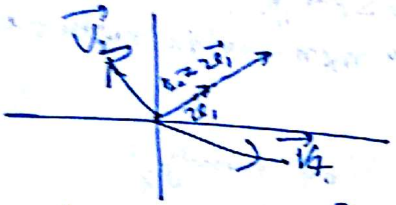
Reason: There will be free variables

Independence

Vector $\vec{x}_1, \vec{x}_2, \dots \vec{x}_n$ are linearly independent if.

iff No combination gives zero vector (except the zero comb.)

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n \neq 0 \quad \text{(except all } c_i = 0\text{)}$$



$$A = \begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $\vec{V}_1 = \vec{0}$
$\vec{V}_2 =$ any V_4
$5V_4 + 0\vec{V}_2 = 0$
$\Rightarrow$ depen

$\vec{v}_1$ & $\vec{V}_3$ are ~~de~~ independent
But $\vec{v}_1, \vec{V}_3$ and any $\vec{V}_4$
is dependent.

<u>Repeat</u> A: when $V_1, \dots, V_n$ are column of A

They are independent if null space of A is only {zero vector} | rank $= n$

They are dependent if $Ac = 0$ for some non-zero c in null space
↳ | rank $< n$

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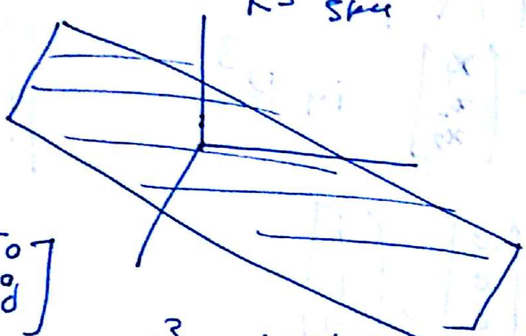
Vector space

Requirement

$\vec{V}+\vec{W}$ and $c\vec{V}$ are in vector space

all lin. combi; $c\vec{V}+d\vec{w}$ are in the space

Subspace : A vector space in side a vector space



Example

any plane through $\begin{bmatrix}0\\0\\0\end{bmatrix}$

is subspace in R^3 vector space

or a line passing through $\begin{bmatrix}0\\0\\0\end{bmatrix}$

Column space of $A_{4\times3}$ is subspace of R^4

$C(A)$

$$A = \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \end{array}$$

$C(A)$: all linear combinat of columns

since. $C_3 = c_1+c_2$ ie column. vector ③ is not independent (only two independent col. vector to

Hence. col. space is 2-D ~~part~~ plane constituted by linear combination of c_1 & c_2.

Solution of $\vec{A}x=\vec{b}$ e.g. $\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1\\x_2\\x_3\end{bmatrix}}_{x} = \underbrace{\begin{bmatrix}2\\3\\4\\5\end{bmatrix}}_{b}$

exits only if $\vec{b}$ is a linear combination of column vector of A ; for given examp $\vec{x}=\begin{bmatrix}0\\0\\1\end{bmatrix}$

i.e. The solution to exist $\vec{b}$ must lies in col. space of A - $C(A)$

Null space of A :

Null space of A is consituted by a bunch of vectors $\vec{x}$ such that

$$A\vec{x} = 0$$

ie space formed solution vectors of above eqn.

Example :

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

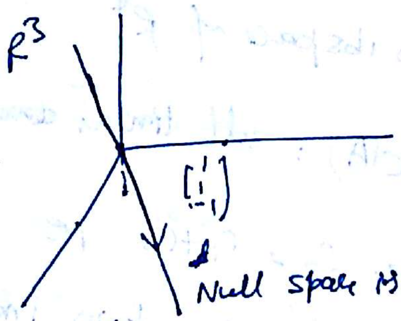
in gen matrix A of size m×n
column space is in R^m
& null space in R^n

Now vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in R^3

soln

$N(A)$ contain $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

gin $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ all ie. $c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ are solution.



Null space is a line in this example

Now how we can say it is a space.

i:e it satisfy to condition
check that solutions to $Ax = 0$ always give a

subspace → if $\vec{v}$ & $\vec{w}$ are solution then $a\vec{v} + b\vec{w}$ is also a solution

ie (1) $A\vec{v} = 0$ and $A\vec{w} = 0$

then (2) $A(a\vec{v} + b\vec{w}) = 0$

$$\Rightarrow a\underbrace{A\vec{v}}_{=0} + b\underbrace{A\vec{w}}_{=0} = 0$$

(3) Also $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is part of solutions. hence it is a subspace.

Four fundamental subspaces   A is m×n matrix

Column space $C(A)$ in $\mathbb{R}^m$

Null Space $N(A)$ in $\mathbb{R}^n$

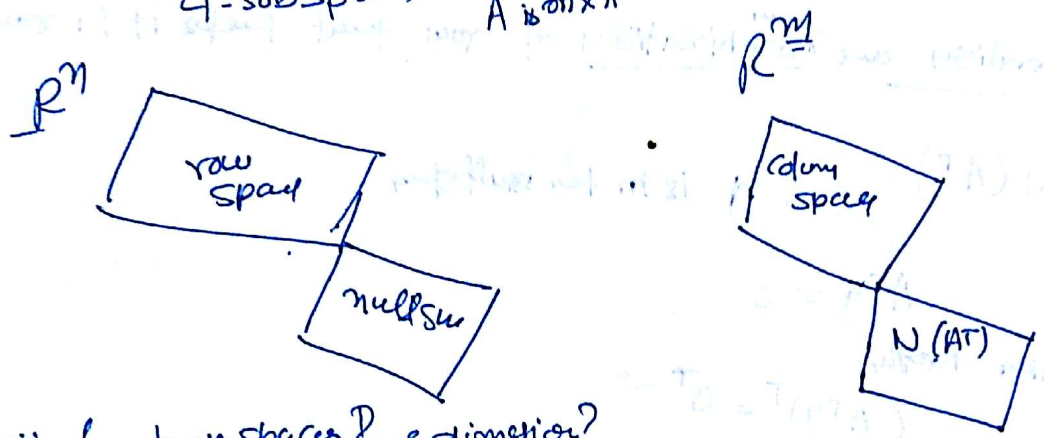
Row space = all linear combs of rows

or = all combs. of the column of A^T

$= C(A^T)$ in $\mathbb{R}^n$

Null space of A^T = $N(A^T)$ = left Null space of A in $\mathbb{R}^m$

4-subspaces   A is m×n



Basis of these spaces? & dimension?

$\dim C(A) = \text{rank}(A)$

one Basis is pivot column (in original matrix before reduction)

Also $\dim(C(A^T)) = \text{rank}(A)$

$\dim N(A) = n - r$ = no. of free variable

or rank(A)

one Basis of null is free column

$\dim N(A^T) = m - r$

no. of rows / or no. of column in A^T

Exam

$$A = \begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix} \to \begin{bmatrix}1&2&3&1\\0&-1&-1&0\\0&0&0&0\end{bmatrix} \to \begin{bmatrix}1&2&3&1\\0&1&1&0\\0&0&0&0\end{bmatrix}$$

~~back~~ upward elimination $\to \begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix} = R$

Remember $C(R) \neq C(A)$

But same row space $\xrightarrow{\text{d.r.t}}$ as we perform Row operations

Basis for row space is first 'r' rows of R (rank)

Row operations are linear combination of row that keeps it in row space.

4th space : $N(A^T)$, let y is in the null space

$$A^T y = 0$$

take transpose.

$$(A^T y)^T = 0^T$$

$$y^T A = [0]$$

$y^T \to$ row vector $\quad [y^T][A] = [0]$ thats why called left Null space

Basis Gauss-Jordan

$$\text{rref}\,[A_{m\times n}\ I_{m\times m}] \longrightarrow [R_{m\times n}\ E_{m\times m}]$$

$$\Rightarrow E[A_{m\times n}\ I_{m\times m}] \to [R_{m\times n}\ E_{m\times m}]$$

where $E_{m\times m}$ represents all the step (operation) performed to obtain Reduced form ie rref.

Remember if A is invertible square matrix R is Identity matrix I (described in chap. 2), then E was A^{-1}

$$AI = \left[\begin{array}{cccc|ccc}1&2&3&1&1&0&0\\1&1&2&1&0&1&0\\1&2&3&1&0&0&1\end{array}\right] \to \left[\begin{array}{cccc|ccc}1&2&3&1&1&0&0\\0&-1&-1&0&-1&1&0\\0&0&0&0&-1&0&1\end{array}\right] \to \left[\begin{array}{cccc|ccc}1&2&3&1&1&0&0\\0&1&1&0&1&-1&0\\0&0&0&0&-1&0&1\end{array}\right]$$

upward elimination $\longrightarrow$

$$\left[\begin{array}{cccc|ccc}1&0&1&1&-1&+2&0\\0&1&1&0&1&-1&0\\0&0&0&0&-1&0&1\end{array}\right]$$

with the left block labelled R and the right block labelled E.

Dimension of $N(A^T) = m - r = 3-2=1.$

check

$$EA = \begin{bmatrix}-1&2&0\\1&-1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix} = \begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix}$$

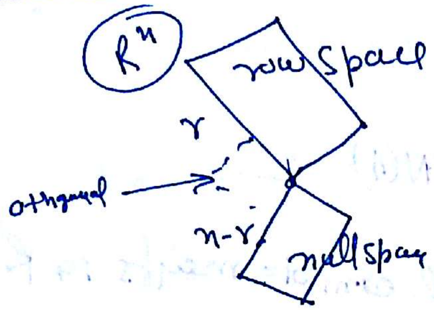
R ✓

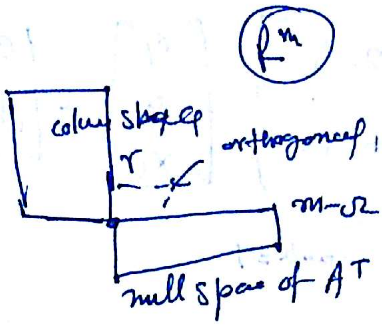
also $y^T A = [0] \to$ combination that give zero row

then $y^T = [-1, 0\ 1]$ $\quad y = \begin{bmatrix}-1\\0\\1\end{bmatrix}$

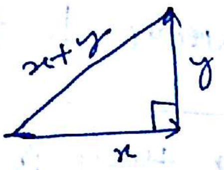
Orthogonality

$A_{m \times n}$





Orthogonal vectors



$\vec{x}^T\vec{y} = 0$

Pythogaras

$$\|\vec{x}\|^2 + \|\vec{y}\|^2 = \|\vec{x}+\vec{y}\|^2$$

$$\vec{x}^T\vec{x} + \vec{y}^T\vec{y} \neq (\bar{x}+\bar{y})^T(\bar{x}+\bar{y})$$

as $x^Ty = y^Tx$

$$= \cancel{x^Tx} + \cancel{y^Ty} + x^Ty + y^Tx$$

$$0 = 2x^Ty$$

$$\Rightarrow x^Ty = 0 \quad \text{or } y^Tx = 0$$

Note $\vec{0}$ is orthogonal to every vector

Subspace S is orthogonal to subspace T

means: every vector in S is orthogonal to every vector in T

Row space is orthogonal to nullspace ($\vec{x}$)

Why?

for null space $Ax = 0$

$$\begin{bmatrix} \text{row 1 to } A \\ \text{row 2 to } A \\ \cdots \\ \text{row } m \text{ of } A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

ie row vector are orthogonal to $\vec{x}$ vectors (which constitute null space).

But rowspace contain all linear combination of row vectors, which also give zero.

similarly: col. spaces & null space of A^T are orthogonal.

Example

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$n = 3$, rank $= 1$ dim $N(A) = 2$

~~Null~~ space and row space are orthogonal complements in R^n

Null space contains all vector $\perp$ to row space.

Solve $Ax = b$, where there is no solution $A_{m \times n}$, $A^T_{n \times m}$

$\underbrace{A^T A}_{n \times n}$ with A^T: $n \times m$, A: $m \times n$ → Square matrix → Symmetric $((A^TA)^T = A^TA)$

$$Ax = b$$

$$A^TAx = A^Tb$$

$m = 2$: $n = 2$

Example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$ ~~can be solved~~

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

→ invertible

$N(A^TA) = N(A)$

rank of A^TA = rank of A

A^TA is invertible exactly if A has independent columns

R = reduced row echelon form (zeros above & below pivots)

echelon form (staircase)

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

by upward elimination

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2/2$

$$\rightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

by Matlab $R = rref(A)$ reduced row echelon form of A

Notice the pivot rows & pivot column are indent[?] $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. get back to equation

$$x_1 + 2x_2 \quad -2x_4 = 0$$
$$x_3 + 2x_4 = 0$$

in the pivot cols $\underbrace{\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}}_{I}$ free cols $\underbrace{\boxed{\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}}}_{F}$

compare with two special solution already obtained

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \leftarrow r \text{ pivot rows}$$

↑ r pivot columns, ↑ n-r free columns

$Rx = 0$

Null space Matrix (N)

(columns = special soln's

$RN = 0$, $N = \begin{bmatrix} -F \\ I \end{bmatrix}$

$Rx = 0$

$$[\,I\ F\,]\begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

$Ax=0$

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

when we performe elimination solution for $Ax=0$ is not changing ie Null space remain same. Although colum vector (& row vector also) changes so is column space also changes.

$$\rightarrow A=\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}$$ by $R_2\to R_2-2R_1$, $R_3\to R_3-3R_1$

$$\rightarrow A=\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}=U$$

we have only two pivot!

Rank of A

$r=$ No. of pivots
$=2$ (for above above exa)

No of free variables for $m\times n$ Matrix
$=n-r$
$=4-2=2$

we two pivot column C_1 & C_3, while C_2 & C_4 are free columns.

These variable x_2 & x_4 (corresponding to free column) can be assigned any valy.

let $x_2=1$, $x_4=0$

whe $x_1+2x_2+2x_3+2x_4=0$
$2x_3+4x_4=0$
$\Rightarrow x_3=0$ & $x_1=-2$

$$\Rightarrow x=\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$$

$Ax=0$ can be w $x=c_1\begin{bmatrix}-2\\1\\:\end{bmatrix}$

or any multiple of $\vec{x}$ will also constitu a solutn

Now let $x_2=0$, $x_4=1$, $\Rightarrow x_3=-2$, $x_1=2$ $x=\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$ or any multiple $x=c_2\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$

whole Null space is linear combination of $\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$ & $\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$

ie $x=c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$

R = reduced row echelon form (zeros above & below pivots)

echelon form (staircase)
$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

by upward elimination

$$\to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2/2$

$$\to \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

by Matlab $R = \text{rref}(A)$ reduced row echelon form of A

Notice the pivot rows & pivot column are identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. get back to equation

$$x_1 + 2x_2 \quad -2x_4 = 0$$
$$x_3 + 2x_4 = 0$$

in the pivot cols

$$\underbrace{\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}}_{I}$$

free cols

$$\underbrace{\boxed{\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}}}_{F}$$

compare with two special solution already obtain

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \leftarrow r \text{ pivot rows}$$

↑ r pivot columns, ↑ n-r free columns

$Rx = 0$

$$[I \; F]\begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

$Rx = 0$

Null space Matrix (N)

(columns = special soln's

$RN = 0$, $N = \begin{bmatrix} -F \\ I \end{bmatrix}$

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Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

row exchange

$$\rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivot for ... free col^y

rank $r = 2$

$(3-2)=1$

for $Ax = 0$

at free variable $x_3 = 1$

$x_1 + 2x_2 + 3x_3 = 0$

$2x_2 + 3x_3 = 0$

$$x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

then whole null space

$$x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = c\begin{bmatrix} -F \\ I \end{bmatrix}$$

N = null space matrix

i.e.

rref

$$\rightarrow R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

---

complete solution of $Ax = b$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Augmented matrix $[A \; b]$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

Pivot columns

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix} \Rightarrow 0 = b_3 - b_2 - b_1$$

condition to have solution $Ax = b$

let $b_1 = 1$

Solvability condition on b

$Ax = b$ solvable when b is in $C(A)$ i.e column space of A

If a comb. of rows of A gives zero row, the same combination of the entries of b must give 0

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To find complete solution to $Ax=b$

step ① a particular soln

x particular : set all free variable to zero
or x_p — Solve $Ax=b$ for pivot variable

i.e. $x_2 = 0$ & $x_4 = 0$

$x_1 + 2x_3 = 1$ $x_1 = -2$

$2x_3 = 3$, $\Rightarrow x_3 = 3/2$

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

step 2

X nullspace : or x_n

$A x_p = b$

$A x_n = 0$

$A(x_p + x_n) = b$

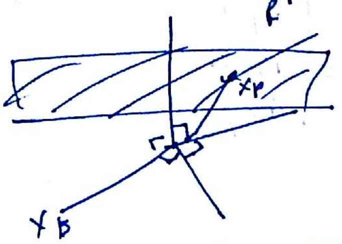
complete soln
$x = x_p + x_n$

If we have one soln, we can be obtain another solution by adding any vector from null space ($Ax=0$)

for above example
$$x_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

2-D subspace through $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

two dimensional plane passing through $\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$ not passing through $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$



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m by n matrix A of rank 'r' (no. of pivot)

we know $r \le m$, $r \le n$

Full column rank ie $r = n$ (all column have pivot ie. no free variable)

$N(A)$ = only zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Soln to $Ax=b$, $x = x_p$ (unique soln if it exists at all)
ie ~~either~~ zero or 1 solution.

for example
$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$$

$$rref(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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full row rank ($r = m$) every row have pivot

can solve $Ax = b$ for b ~~(transpose)~~ exists

left with $n - m = n - r$ free variable

Example

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$rref(A) = \begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix} \quad F$$

Full rank

case-IV. $r = m = n \Rightarrow$ Invertible

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$N(A) =$ zero vector only

$R = [I]$

1 solution for $Ax = b$

Summary $r = n < m$

$$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

0 or 1 solution $Ax = b$

$r = m < n$

$R = [I \; F]$

Eigenvalues - Eigenvectors

like in calculus input x to a function f, and output $f(x)$

in linear algebra input vector X acts on Matrix A results in output Ax

generally vector Ax have different direction than x,

A family of input vector X, for a matrix A, giving Ax in the same direction as X.

Ax parallel to x called Eigenvectors

$$Ax = \lambda x$$

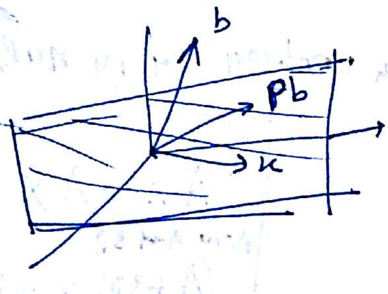
$x \rightarrow$ eigen vector, $\lambda \rightarrow$ eigen value.

If A is singular, $\lambda = 0$ ie an eigen value.

---

eigen value & vectors for Projection matrix

Example ①



Any x in the plane : $Px = X$, $\lambda = 1$

any vector $\perp$ to the plane : $Px = 0$, $\lambda = 0$

Example ②

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ permutation matrix, exchange component of vector

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda = 1$

also $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; $Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -x = \lambda, = -1$,

---

facts : ✓ sum of λ's = sum of diagonal elem

$= a_{11} + a_{22} + a_{33} + \dots + a_{nn}$

✓ for $n \times n$ matrix, there will be n eigen values,

=

How to solve $Ax = \lambda x$

Rewrite: $(A - \lambda I)x = 0$

for above equation to be true, for Non zero X

$\underline{A - \lambda I}$ must be singular matrix

$\det(A - \lambda I) = 0$ find λ first

Example

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)^2 - 1 = 0$

$\Rightarrow \lambda^2 - 6\lambda + 8 = 0$

$\lambda^2 - (\text{trace})\lambda + \det = 0$

trace $= d_1 + d_2$

$(\lambda - 4)(\lambda - 2) = 0$

$\lambda_1 = 4$

$\lambda_2 = 2$

case $\lambda = 4$

$A - \lambda I$

$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix} = 0 \quad (A - \lambda I)x = 0$

Eigen vectors are in null space

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2$. $(A - \cancel{2I})x = 0$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix} = 0$

$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

If $Ax = \lambda x$

Now $A + 3I$

$(A + 3I)x = \lambda x + 3x$

$= (\lambda + 3)x$

If $Ax = \lambda x$

& B has eigenvalue α,

$Bx = \alpha x$ ✗

~~$(A+B)x = (\lambda + \alpha)x$~~

Not true

as B has different eigen vector say y

$By = \alpha y$

$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \lambda_1 = 1, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda_2 = -1 \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad \lambda_1 = 4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = 2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Notice that $A_2 = A_1 + 3I$ but same eigen vectors

$\lambda'_1 = \lambda_1 + 3$

Example Rotation Matrix

Say rotate every vector by 90° $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\text{trace} = 0 + 0 = \lambda_1 + \lambda_2$$

$$\det = 1 = \lambda_1 \lambda_2$$

$$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \begin{matrix} \lambda_1 = i \\ \lambda_2 = -i \end{matrix}$$

Eigen values are not real one, but are complex

If Matrix is symmetric then eigen values will be real

Example

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \quad \det.(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda) = 0$$

$$\lambda_1 = 3, \; \lambda_2 = 3$$

Solve $(A - \lambda I)x = 0$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad X_2 =$ No 2nd independent eigen vector

Diagonalization

Suppose we have n linearly independent eigen vectors of A put them in columns of S
so that S^{-1} exist.

$$Ax_1 = \lambda_1 x_1$$
$$\vdots \quad \vdots$$
etc

$$AS = A\underbrace{\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}}_{S}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & & \lambda_n x_n \\ | & | & & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{\substack{\text{diagonal matrix} \\ \text{(Eigen value Matrix)}}} = S\Lambda$$

$$AS = S\Lambda$$

$$S^{-1}AS = S^{-1}S\Lambda = \Lambda$$

Also $\quad ASS^{-1} = S\Lambda S^{-1}$

$$A = S\Lambda S^{-1}$$

Application $\quad$ If $\quad Ax = \lambda x$

multiply by A

$$A^2x = \lambda Ax = \lambda^2 x$$ i.e. A^2 has e.values λ^2 & same e. vectors

---

Also

$$A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

Also tells the same fact e.vects S is same & e.vals Λ^2

its true $\quad A^k = S\Lambda^k S^{-1}$

why real eigen value?

let it be complex

$$Ax = \lambda x \xRightarrow{\text{also}} \bar{A}\bar{x} = \bar{\lambda}\bar{x}$$ conjugate.

since A is real $\bar{A} = A$ $\quad A\bar{x} = \bar{\lambda}\bar{x}$

take Trans of eqn

$$\bar{x}^T A^T = \bar{x}^T \bar{\lambda}$$

Now $A^T = A$, $\quad \bar{x}^T A = \bar{x}^T \bar{\lambda}$

$$\bar{x}^T A x = \bar{x}^T \bar{\lambda} x \quad (1)$$

Now $\cancel{A\bar{x} = \bar{\lambda}\bar{x}}$

$\cancel{\bar{x}^T A x = \bar{\lambda}\bar{x}^T\bar{x}}$

for $Ax = \lambda x$

$$\bar{x}^T A x = \lambda \bar{x}^T x \quad (2)$$

from (1) & (2), $\quad \lambda \bar{x}^T x = \bar{\lambda}\bar{x}^T x$

$$\Rightarrow \underline{\lambda = \bar{\lambda}} \quad \underline{\text{real}}$$

Let $\underline{x_1}$ & $\underline{x_2}$ be two eig. vector with eig. value λ_1 & λ_2 for an Symmetric matrix A.

To show $\underline{x_1} \cdot \underline{x_2} = 0$

$$\lambda_1 \underline{x}_1^T \underline{x}_2 = (\lambda_1 \underline{x}_1^T)\underline{x}_2$$
$$= (A\underline{x}_1)^T \underline{x}_2$$
$$= \underline{x}_1^T A^T \underline{x}_2$$

as (Now $A^T = A$) $\Rightarrow$ $$= \underline{x}_1^T (A\underline{x}_2)$$
$$= \underline{x}_1^T \lambda_2 \underline{x}_2$$
$$= \lambda_2 \underline{x}_1^T \underline{x}_2$$

here $(\lambda_1 - \lambda_2)\underline{x}_1^T \underline{x}_2 = 0$, but $\lambda_1 \neq \lambda_2 \neq 0$

$$\Rightarrow \underline{x}_1^T \underline{x}_2 = 0$$

ie. eigen vector for symmetric matrix are orthogonal.

Theoreym

$A^K \to 0$ as $K \to \infty$

if all $|\lambda_i| < 1$

A is sure to have n independent vectors (and be diagonalizable)

If all the λ's are different (i.e. No repeated λ's)

Repeated eigenvalues may or may not have n indep. evectors.

Example

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0$$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = 2$$

$$(A-\lambda I)x = 0, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \end{bmatrix} = 0 \qquad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ only one

---

Symmetric Matrix: $A^T = A$

① The eigenvalues are REAL

② The Eigen vectors are PERPENDICULAR (orthonormal eigenvectors if unit magnitude) can be chosen

Usual case $A = S\Lambda S^{-1}$

for symmetric case $A = Q\Lambda Q^{-1}$
$= Q\Lambda Q^T$

Q = column of orthonormal

$Q^{-1} = Q^T$

Optimization

Example

Machine Type	Product				Total time available per week
	P_1	P_2	P_3	P_4	
M1	2.7 (time)	3	4.6	3	3000
M2	2	7	2	5.1	9500
M3	2.4	4	6.1	3	6300
Unit Profit	7.5	4.6	9.2	5.25	

if unit cost

x_j be the number of units produced for product j (per week)

Maximize $\quad z = 7.5x_1 + 4.6x_2 + 9.2x_3 + 5.25x_4$

minimize

Subject to
$$2.7x_1 + 3x_2 + 4.6x_3 + 3x_4 \le 3000$$
$$2x_1 + 7x_2 + 2x_3 + 5.1x_4 \le 9500$$
$$2.4x_1 + 4x_2 + 6.1x_3 + 3x_4 \le 6300$$

i.e. to find How much quantity of each product should be produce to maximize profit.

→ George Dantzig – 1940 first optimization linear programming

Since then many optimization technique has been developed & widely used in operational research, Artificial Intelligence optimization & control & in Industry

* Linear Problems → obj. fun & constraints are linear fn
* Nonlinear Problem → " " non linear
* Discrete "
* Continuous "

Mathematical background

Matrix Algebra

$m \times n$ matrix : an arrangement of mn objects

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$m \neq n$ Rectangular

a_{ij} : → elements

$a_{11}, a_{22}, \ldots a_{mn}$ → diagonal elements if we have three independ col. vec ~~a b~~ a b c & want A, B, C are orthogonal vectors

$$q_1 = \frac{A}{\|A\|}, \quad q_2 = \frac{B}{\|B\|}, \quad q_3 = \frac{C}{\|C\|}$$

$$A = a$$

$$B = b - \frac{A^Tb}{A^TA} A$$

Now $C \perp A$ & $C \perp B$

$$C = c - \frac{A^Tc}{A^TA} A - \frac{B^Tc}{B^TB} B$$

Example $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$A = a$, $B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}_{0}$

$$Q = \begin{array}{c} \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \\ \begin{array}{cc} q_1 & q_2 \end{array} \end{array}$$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}} \rightarrow \underset{Q}{\begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}}$$

A & Q both have same column space

$$\underset{A}{\begin{bmatrix} a_1 & a_2 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} q_1 & q_2 \end{bmatrix}} \begin{bmatrix} q_1^Ta_1 & \\ q_1^Ta_2 & \end{bmatrix}$$

R → upper triangular matrix

$q_1^Ta_2 = 0$

check in the book

~~Rectangular Example~~

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

Q has orthonormal columns

to project onto its column space

projection Matrix = $P = Q(Q^TQ)^{-1}Q^T$ | Rem. $(Q^TQ) = I$

$= QQ^T$

$= \{ I \text{ if } Q \text{ is Square} \}$

(if Q is square matrix it covers whole space, and projected vector is vector itself)

for b's

Rem: $A^TA\hat{x} = A^Tb$ — ith component of $\hat{x}$

Now A is Q

$\underbrace{Q^TQ}_{I}\hat{x} = Q^Tb \Rightarrow \hat{x} = Q^Tb$ i.e. $\boxed{\hat{x}_i = q_i^T . b}$

projection on its basis vector is $q_i^T . b$

we have independent vector Matrix A and want to convert in orthonormal Q.

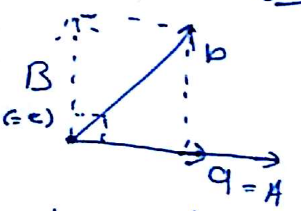
goal Orthogonal vector A, B from a & b

$\downarrow$

orthonormal vec $q_1 = \frac{A}{\|A\|}$ $q_2 = \frac{B}{\|B\|}$

Gram-Schmidt

let two vector a, b independent



let the first vector direction such ~~a = A~~ $A = a$

$B = b - p$ $\quad p = \text{projection} = \frac{A^Tb}{A^TA}A$ $\quad$ A → vec not matrix

$= b - \frac{A^Tb}{A^TA}A$

check Now if $A \perp B$ $\quad A^TB = A^T\left(b - \frac{A^Tb}{A^TA}A\right) = 0$

$= A^Tb - \frac{A^Tb}{A^TA} \cdot A^TA = 0$

If A has independent columns

Then A^TA is invertible

Suppose $A^TA x = 0$, then x must be 0 (if A^TA is invertible as Null space have only zero vector)

~~Proof~~ taking x^T both side

$$x^T A^T A x = 0$$

$$\Rightarrow (Ax)^T Ax = 0$$

i.e. $\|Ax\|^2 = 0 \Rightarrow Ax = 0$

if A has independent columns and $Ax = 0$

then $x = 0$ (must).

Colums are definitely independent if they are perpendicular

Example: column are perpendicular unit vector (orthonormal)

Orthogonal basis $q_1, \dots, q_n$

Orthogonal Matrix Q : ~~Square~~

Orthonormal Vectors

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

$$Q^TQ = \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix}$$

$$Q^TQ = I$$

if Q is square then $Q^TQ = I$

tells us $Q^T = Q^{-1}$

Examples ① permutation mat $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$QQ^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = I$$

② $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

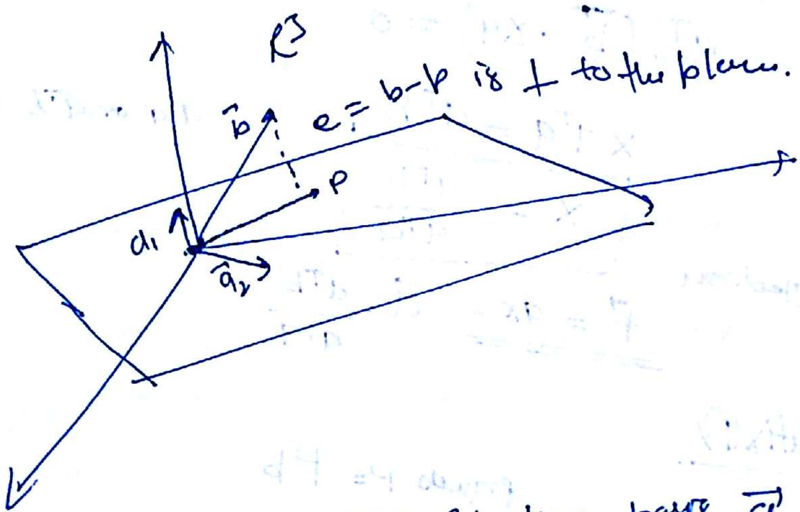
why project?

Because $Ax = b$ may have no solution (more equations than unknowns $m > n$)

Ax - always remain in column space of A, where $\vec{b}$ may not be, which result in no solution

So, we can try for best possible solution by projecting b in column space of A and solve

$$Ax = p \quad \text{instead} \quad \left\{ p \rightarrow \text{proj } b \text{ onto } C(A) \right.$$



two describe plane $C(A)$ let two basis $\vec{a}_1$ & $\vec{a}_2$ (need not to be perpendicular, but independent)

$$\text{plane of } a_1 \,\&\, a_2 = \text{col space of } A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$$

$$\text{Projection} = \hat{x}_1 a_1 + \hat{x}_2 a_2$$ p is (linear comb. of a_1 & a_2)

$$p = A\hat{x} \quad \text{find } \hat{x}$$

key: $b - A\hat{x}$ is $\perp$ to the plane. (ie. $\perp$ to a_1 & a_2 equally)

$$\left.\begin{aligned} a_1^T(b - A\hat{x}) &= 0 \\ a_2^T(b - A\hat{x}) &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

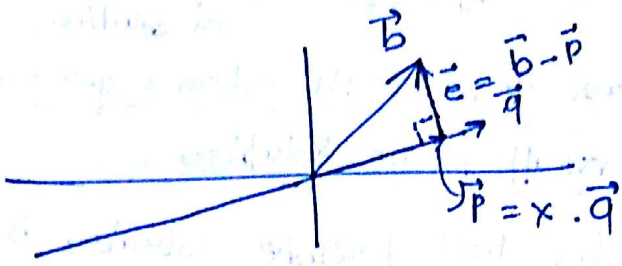
or
$$A^T(b - A\hat{x}) = 0$$
$$A^TA\hat{x} = A^Tb$$

$A^Te = 0$ i.e. e is in $N(A^T)$

Also $\Rightarrow$ $e \perp C(A)$ column of A

Projection Matrix

Projections: $C(\vec{P})$



$\vec{a}$ is perpendicular to $\vec{e} = \vec{b} - \vec{p}$

$\Rightarrow \quad \vec{a}^T . (\vec{b} - x\vec{a}) = 0$

for simplicity $\quad a^T . (\vec{b} - xa) = 0$

$x a^T a = a^T b$ $\quad \because a^T a$ or $a^T b$ are scaller num

$$\boxed{x = \frac{a^T b}{a^T a}}$$

projection $\quad \vec{p} = ax = \vec{a} . \frac{a^T b}{a^T a}$

Projection Matrix (P)

projects $p = Pb$

matrix $\quad \boxed{P = \frac{a a^T}{a^T a}}$

Columnspace $\quad C(P) =$ line Through $\vec{a}$

rank $(P) = 1$

also $\quad \boxed{P^T = P} \Rightarrow$ symetric Matrix

- If you take projection twice or More then it remains Same

$\boxed{P^2 = P}$ or $P^k = P$ $\quad k$ is any natural no.

A^TA is not scalar as earlier a^Ta, A^TA $n \times n$ matrix

$$\hat{x} = (A^TA)^{-1}A^Tb$$

projection $p = A\hat{x}$

$$= A(A^TA)^{-1}A^Tb$$

Projection Matrix $\boxed{P = A(A^TA)^{-1}A^T}$.

Remember: Since A is not square invertible matrix we can't do $AA^{-1}(A^T)^{-1}A^T$,
i.e. $(AB)^{-1} = B^{-1}A^{-1}$ can not used here.

Properties

- $P^T = P$ → symmetric
- $P^2 = P$ → (multiple projection do not alter results)

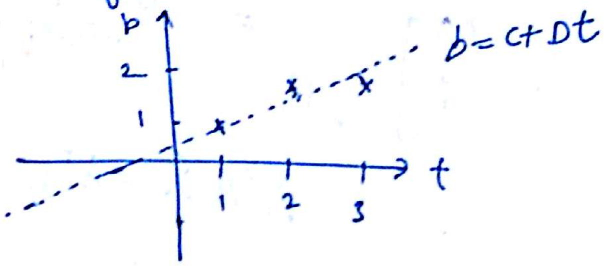
$$\left(P^2 = A(A^TA)^{-1}\underbrace{A^TA(A^TA)^{-1}}_{I}A^T = A(A^TA)^{-1}A^T \right.$$

Example (Application)

Least Squares

Fitting by a line

Given data point to be fitted by best line



$(1,1)$, $(2,2)$, $(3,2)$

pt 1, $C + D = 1$

$C + 2D = 2$

$C + 3D = 2$

we can't solve
but can get best approx

we can't solve

$$Ax = b$$

but

$$A^TAx = A^Tb \quad \text{can be solved}$$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

Projection matrix $P = A(A^TA)^{-1}A^T$

Extreme cases: if b in col space $C(A)$; $Pb = b$

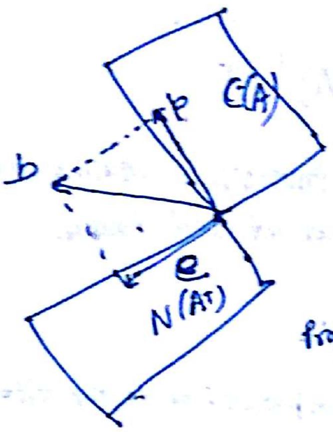
if $b \perp$ to $C(A)$; $Pb = 0$
(i.e. in the $N(A^T)$)

If b is in column space (line comb. of col. of A i.e. Ax) stat

projection $p = Pb = A\underbrace{(A^TA)^{-1}A^TA}_{I}x = Ax$

if b is in $N(A^T)$ then $A^Tb = 0$

$p = Pb = A(A^TA)^{-1}\underbrace{A^Tb}_{0} = 0$



$p + e = b$

$p = Pb$

projection on $\perp$ space ($N(A^T)$)

$e = \underline{(I - P)}\,b$

↳ projection Matrix for $N(A^T)$

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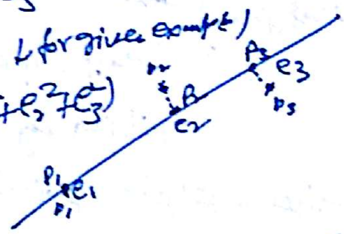
going back to line fitting problem

to get best fitted line, we minimize errors

↳ for given example)



Minimize: $\|Ax - b\|^2 = \|e\|^2 \; (= e_1^2 + e_2^2 + e_3^2)$

$= (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$

find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, P

$A^TA\hat{x} = A^Tb$

best line $\underline{\frac{2}{3} + \frac{1}{2}t}$ $\Rightarrow p = A\hat{x}$

$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}\begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

$\hat{D} = 1/2$

$\hat{C} = 2/3$

$\begin{bmatrix} 3 & 6 \\ 0 & 2 \end{bmatrix}$