ASSIGNMENTS OF COMPLEX ANALYSIS
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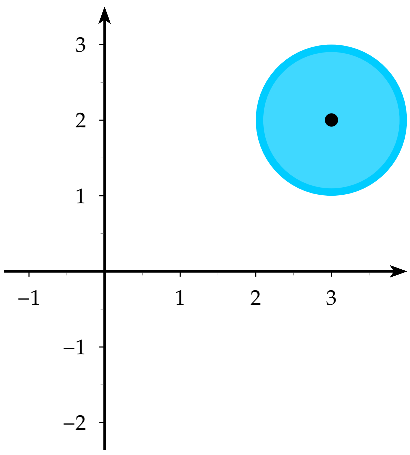
1. First Assignment

1. Graph the following four sets:

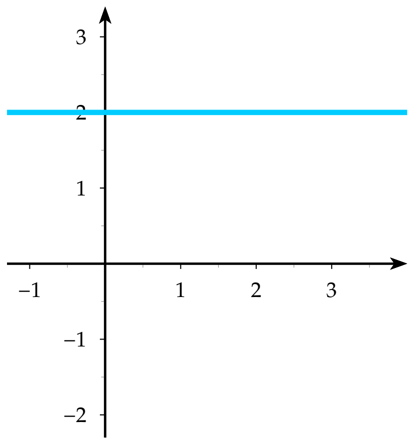
A. $\{z \in \mathbb{C} \mid |z-3-2i| \leqslant 1\}$
B. $\{z \in \mathbb{C} \mid \mathfrak{Im}(z) = 2\}$
C. $\left\{z \in \mathbb{C} \setminus \{0\} \;\middle|\; 0 < \arg(z) < \dfrac{\pi}{6}\right\}$
D. $\{z \in \mathbb{C} \mid |z-1| < |z|\}$

Solution

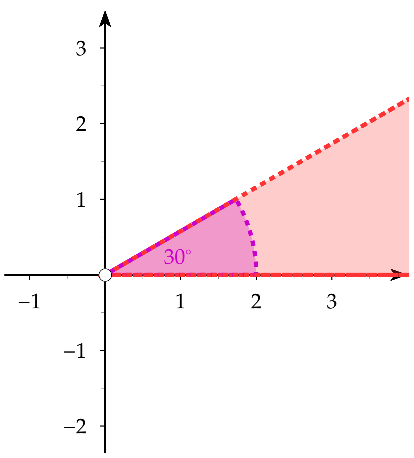
In the following figures the *x-axis* is the **real** axis and the *y-axis* the **imaginary** one.



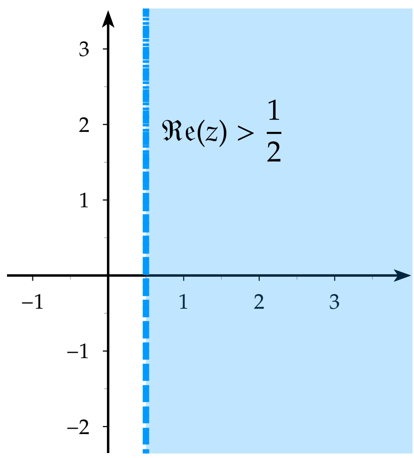
(A) $|z-3-2i| \leqslant 1$ is the **unit** (r=1) disc (circle+interior) with center (3,2).



(B) With $\mathfrak{Im}(z) = 2$, it is clear that the locus of these z is the line y=2.



(C) It is the colored (light red) region included between the lines $y = \tan\left(\frac{\pi}{6}\right)$ and y=0 without the point (0,0) and of course **without** the points of the lines mentioned above.



(D) Raise both sides to the power of two, it is the blue sketched region below without the points of the vertical line $x = \frac{1}{2}$.

2. Find and plot the 6th roots of unity. For this problem you need to submit both a graph showing your answer as well as the calculation that led you to this graph.

Solution

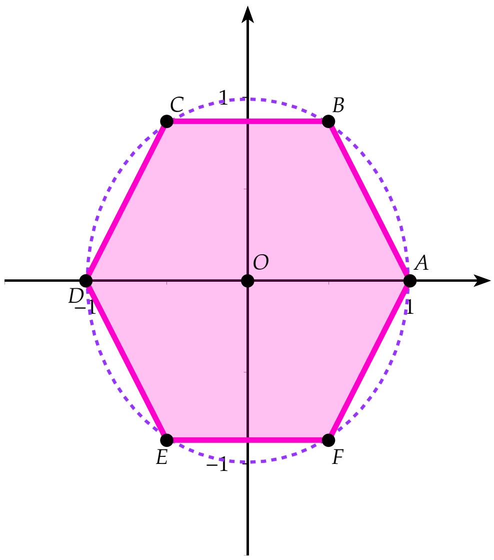
$$z^6 = 1 \Rightarrow |z|^6 e^{6i\theta} = 1^6 e^{0\cdot i} \Rightarrow \begin{cases} |z| = 1 \\ 6\theta = 2k\pi \Rightarrow \theta = \dfrac{k\pi}{3} \quad ,k = 0,1,2,3,4,5 \end{cases}$$, so the points we seek are

$$\left\{(1,0),\left(\cos\left(\frac{\pi}{3}\right),\sin\left(\frac{\pi}{3}\right)\right),\left(\cos\left(\frac{2\pi}{3}\right),\sin\left(\frac{2\pi}{3}\right)\right),(\cos(\pi),\sin(\pi)),\left(\cos\left(\frac{4\pi}{3}\right),\sin\left(\frac{4\pi}{3}\right)\right),\left(\cos\left(\frac{5\pi}{3}\right),\sin\left(\frac{5\pi}{3}\right)\right)\right\}$$

or

$$\left\{(1,0),\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),(-1,0),\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right),\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)\right\}$$

An easy way to construct the figure is to take $z_1 = 1$, draw the point (1,0) and then from that point create a **regular** hexagon of side 1 centered at the origin of the axis, which of course is also inscribed in the circle of radius 1 and center (0,0). The six solutions are represented by the points A, B, C, D, E, F which are the vertices of this regular hexagon, as seen in the following figure.



2. Second Assignment

1. Find the image of the set $U = \left\{z \in \mathbb{C} \;\middle|\; -\frac{\pi}{2} \leqslant \Re\mathrm{e}(z) \leqslant \frac{\pi}{2}\right\}$ under the function $f(z) = \sin(z)$. To do so please answer the following questions:

a. What is the image of the line segment $L_1 = \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (in the real axis) under f?
b. What is the image of the imaginary axis $L_2 = \left\{iy \;\middle|\; y \in \mathbb{R}\right\}$ under f?
c. What is the image of the vertical line $L_3 = \left\{-\frac{\pi}{2} + iy \;\middle|\; y \in \mathbb{R}\right\}$ under f?
d. What is the image of the vertical line $L_4 = \left\{\frac{\pi}{2} + iy \;\middle|\; y \in \mathbb{R}\right\}$ under f?
e. Given your above observations, what do you guess the image of the set U is under f?

Solution

Before we begin answering the questions we need to do some calculations.

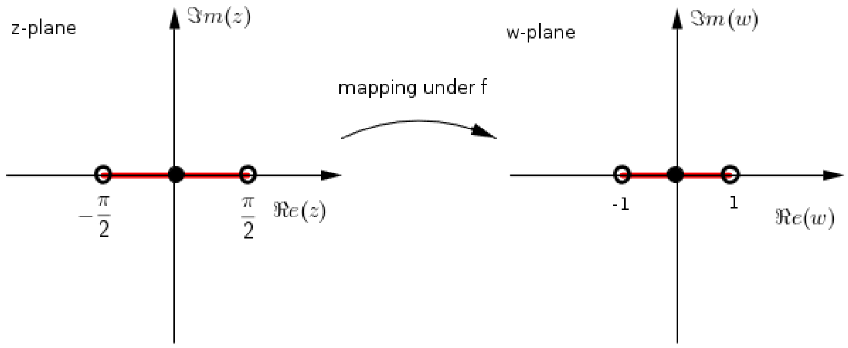
$$f(z) = \sin(z) = \sin(x + yi) = \sin(x)\cos(yi) + \sin(iy)\cos(x) = \sin(x)\cosh(y) + i\cdot\cos(x)\sinh(y)$$

so $u(x,y) = \sin(x)\cosh(y)$ and $v(x,y) = \cos(x)\sinh(y)$.

a. L_1 is the line segment $\Re\mathrm{e}\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ in the *z-plane*, so for $x \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $y = 0$ we get

$$w(x,0) = \langle u(x,0), v(x,0)\rangle = \langle \sin(x), 0\rangle$$

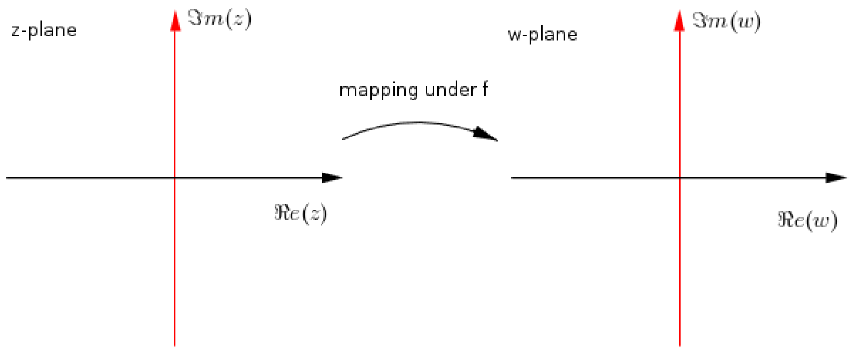
and for $x = \pm\dfrac{\pi}{2}$, $w\left(\pm\dfrac{\pi}{2}, 0\right) = \pm 1$, which means that under f, L_1 is mapped into the line segment $\Re\mathfrak{e}(-1,1)$ in the *w-plane*.



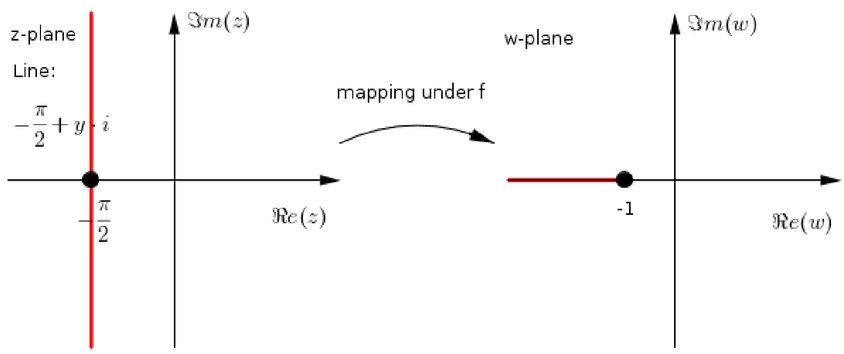
b. Following the same steps as in answer (a) we get,

$$w(0, y) = \langle u(0, y), v(0, y)\rangle = \langle 0, \sinh(y)\rangle$$

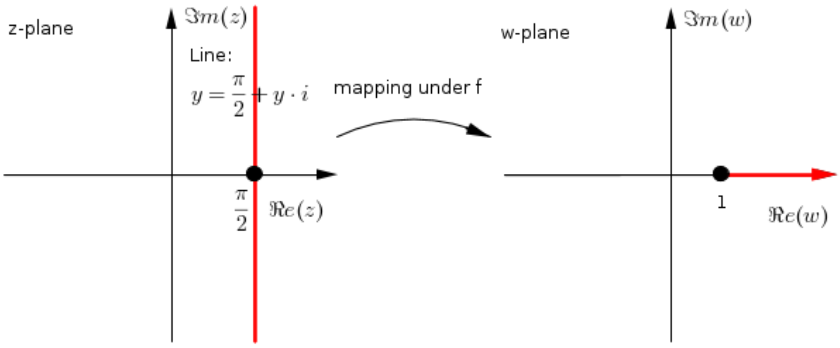
and since the range of sinh is $(-\infty, +\infty)$ for $y \in \mathbb{R}$, we deduce that f maps the $y\mathrm{i}$ line or the $\Im\mathfrak{m}(z)$ axis into the *imaginary* axis $\Im\mathfrak{m}(w)$ of the *w-plane*.



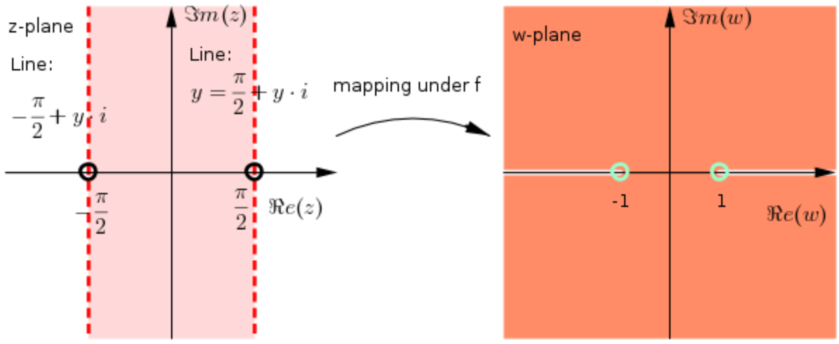
c. For $x = -\dfrac{\pi}{2}$ we get $w\left(-\dfrac{\pi}{2}, y\right) = \langle -\cosh(y), 0\rangle$ and since the range of cosh is $[1, +\infty)$ for $y \in \mathbb{R}$, we deduce that f maps the line $y = -\dfrac{\pi}{2} + y\mathrm{i}$ of the *z-plane* into the line $\Re\mathfrak{e}(-\infty, -1]$ of the *w-plane*, we can imagine that an invisible hand bends the two infinite branches of the line $y = -\dfrac{\pi}{2} + y\mathrm{i}$ regarding as the center of the infinite vertical line the point $\left(-\dfrac{\pi}{2}, 0\right)$ leftwards, until both branched become one with the *real axis* and then the same force moves the new line's edge horizontally to the point (-1,0). We will see the same process in the next question but in the opposite direction.



d. As in (c), $x = \frac{\pi}{2}$ and so $w\left(\frac{\pi}{2}, y\right) = \langle \cosh(y), 0 \rangle$, so f maps the line $y = \frac{\pi}{2} + yi$ of the *z-plane* into the line $\Re\mathrm{e}[1, +\infty]$ of the *w-plane*.



e. It is clear that $U = \left\{ z \in \mathbb{C} \;\middle|\; -\frac{\pi}{2} < \Re\mathrm{e}(z) < \frac{\pi}{2} \right\}$ is mapped into the whole *w-plane* slit along the rays $\Re\mathrm{e}(-\infty, 1]$ and $\Re\mathrm{e}[1, +\infty)$ (white lines in the next figure excluding also from the *w-plane* the points (-1,0) and (1,0)).



2. Let $u(x, y) = x^2 - y^2 - y$. Find a real-valued function $v(x, y)$ such that $v(0, 0) = 1$ and together, u and v satisfy the *Cauchy-Riemann* equations in the entire complex plane.

To do so, please follow these steps:

a. Find the partial derivatives $u_x(x, y)$ and $u_y(x, y)$.
b. Using these partial derivatives and the Cauchy-Riemann equations, give equations for the partial derivatives $v_x(x, y)$ and $v_y(x, y)$.
c. Find functions $v(x, y)$ that satisfy the equation for the partial derivative with respect to x.
d. Find functions $v(x, y)$ that satisfy the equation for the partial derivative with respect to y.
e. Now find a function $v(x, y)$ that satisfies both equations for the partial derivatives at the same time.
f. Finally, check whether the function you found in the previous step satisfies $v(0, 0) = 1$. If not, modify the function so that it does.

Solution

$u(x, y) = x^2 - y^2 - y$

a. $u_x(x, y) = 2x$ and $u_y(x, y) = -2y - 1$.

b. $\begin{cases} u_x &= v_y \\ u_y &= -v_x \end{cases} \Rightarrow \begin{cases} 2x &= v_y \\ -2y - 1 &= -v_x \end{cases} \Rightarrow \begin{cases} v_y &= 2x \\ v_x &= 2y + 1 \end{cases} \Rightarrow \begin{cases} v_x &= 2y + 1 \\ v_y &= 2x \end{cases}$

c. From the previous answer we came up with the equation and by integrating with respect to x and treating y as a ***constant*** we get

$$v_x(x, y) = 2y + 1 \Rightarrow \int v_x(x, y)\, \mathrm{d}x = \int 2y + 1 \, \mathrm{d}x \Rightarrow v(x, y) = 2xy + x + ay + c \tag{1}$$

for $a, c \in \mathbb{R}$.

d. Following the same process as in (c), but now by integrating with respect to y and treating x as a ***constant*** we get

$$v_y(x, y) = 2x \Rightarrow \int v_y(x, y)\, \mathrm{d}y = \int 2x \, \mathrm{d}y \Rightarrow v(x, y) = 2xy + bx + d \tag{2}$$

for $b, d \in \mathbb{R}$.
e. Equating relations (1) and (2) from answers c and d respectively we get

$$\begin{aligned} 2xy + x + ay + c &= 2xy + bx + d \\ (1-b)x + ay + c - d &= 0 \end{aligned}$$

so $a = 0, b = 1$ and $c = d$ and $\boxed{v(x,y) = 2xy + x + c}$ (3).
f. From (3) $v(0,0) = c \Leftrightarrow c = 1$ for the function v is $v(x,y) = 2xy + x + 1$.

3. Third Assignment

1. Sketch the image under the map $f(z) = \mathrm{Log}(z)$ of the open half annulus $A = \left\{ z \in \mathbb{C} \;\middle|\; e^{-\frac{\pi}{4}} < |z| < e^{\frac{\pi}{4}}, \Re\mathfrak{e}(z) > 0 \right\}$.
Recall: $\mathrm{Log}(z)$ denotes the *principal branch* of logarithm, that is, $\mathrm{Log}(z) = \log|z| + i\mathrm{Arg}(z)$, where $-\pi < \mathrm{Arg}(z) \leqslant \pi$ is the principal argument of z.
The image you create (either by hand or using a computer graphing program) should contain two graphs: On one set of coordinate axes, sketch the half annulus A, on a second set of axes sketch its image under f. Please highlight the following parts of your graph:
a. The set $\left\{ z \in \mathbb{C} \;\middle|\; |z| = e^{-\frac{\pi}{4}}, \Re\mathfrak{e}(z) > 0 \right\}$ (this is a part of the boundary of A) as well as the image of this boundary portion under f in your second graph.
b. The set $\left\{ z \in \mathbb{C} \;\middle|\; |z| = e^{\frac{\pi}{4}}, \Re\mathfrak{e}(z) > 0 \right\}$ (this is another part of the boundary of A) as well as the image of this boundary portion under f in your second graph (use a different color for these sets than you used in the first part if possible).
c. The set $\left\{ z \in \mathbb{C} \;\middle|\; e^{-\frac{\pi}{4}} \leqslant \Im\mathfrak{m}(z) \leqslant e^{\frac{\pi}{4}}, \Re\mathfrak{e}(z) = 0 \right\}$ (this is yet another part of the boundary of A) as well as the image of this boundary portion under f in your second graph (use a third color if possible).
d. The set $\left\{ z \in \mathbb{C} \;\middle|\; -e^{\frac{\pi}{4}} \leqslant \Im\mathfrak{m}(z) \leqslant -e^{-\frac{\pi}{4}}, \Re\mathfrak{e}(z) = 0 \right\}$ (this is the fourth part of the boundary of A) as well as the image of this boundary portion under f in your second graph (use a fourth color if possible).
e. The set A (on your first graph) and its image $f(A)$ (on your second graph).

Solution

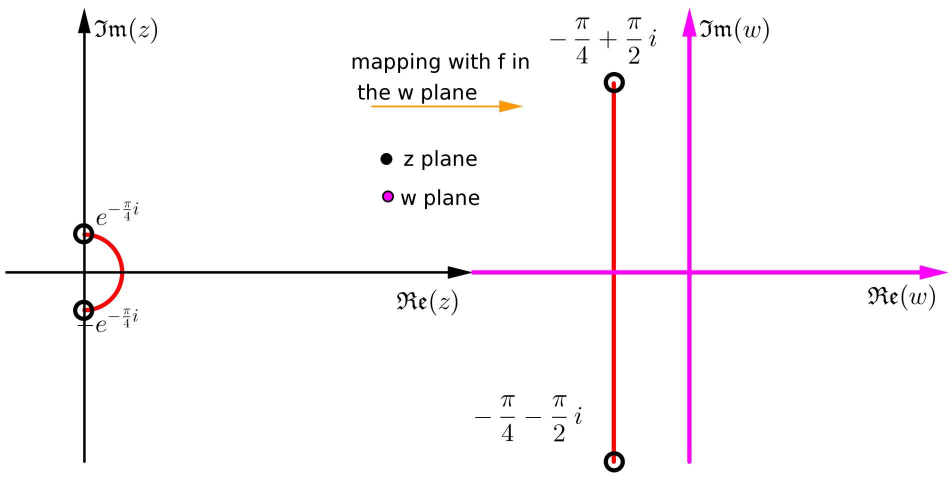
First of all before we begin with the step by step solution we should always have in our mind that:

$$\mathrm{Log}(z) = \log|z| + i\mathrm{Arg}(z), \text{ for } -\pi < \mathrm{Arg}(z) \leqslant \pi$$

a. We are given the semicircle $C_1 = \left\{ z \in \mathbb{C} \;\middle|\; |z| = e^{-\frac{\pi}{4}}, \Re\mathfrak{e}(z) > 0 \right\}$ and we will find its image under $f(z) = \mathrm{Log}(z)$. So for every $z \in C_1$ we have $\mathrm{Arg}(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and

$$\mathrm{Log}(z) = \log(e^{-\frac{\pi}{4}}) + i\mathrm{Arg}(z) = -\frac{\pi}{4} + i\mathrm{Arg}(z)$$

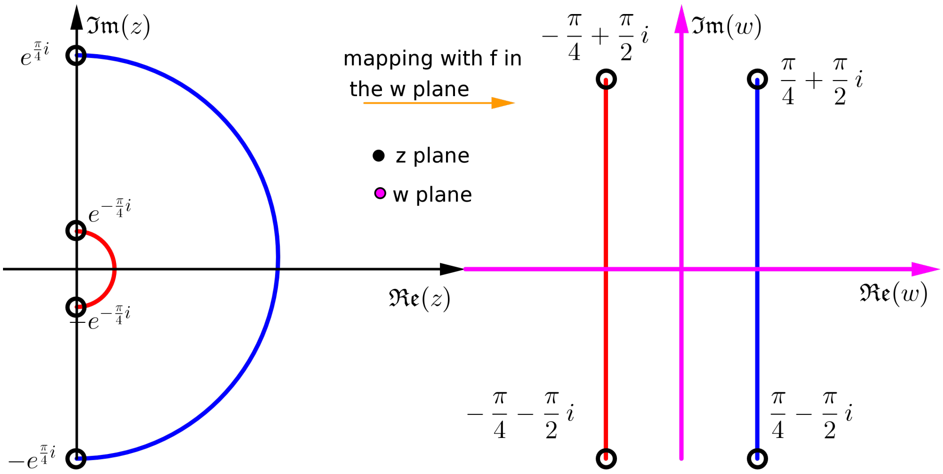
so f maps C_1 into the vertical line $z = -\frac{\pi}{4} + i\mathrm{Arg}(z)$ of the *w-plane* as seen in the next figure (red semicircle without the ending points) with $\mathrm{Arg}(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



b. We are given the semicircle $C_2 = \left\{ z \in \mathbb{C} \;\middle|\; |z| = e^{\frac{\pi}{4}}, \Re\mathfrak{e}(z) > 0 \right\}$ and following the same steps as in (1) we get

$$\mathrm{Log}(z) = \log\left(e^{\frac{\pi}{4}}\right) + i\mathrm{Arg}(z) = \frac{\pi}{4} + i\mathrm{Arg}(z)$$

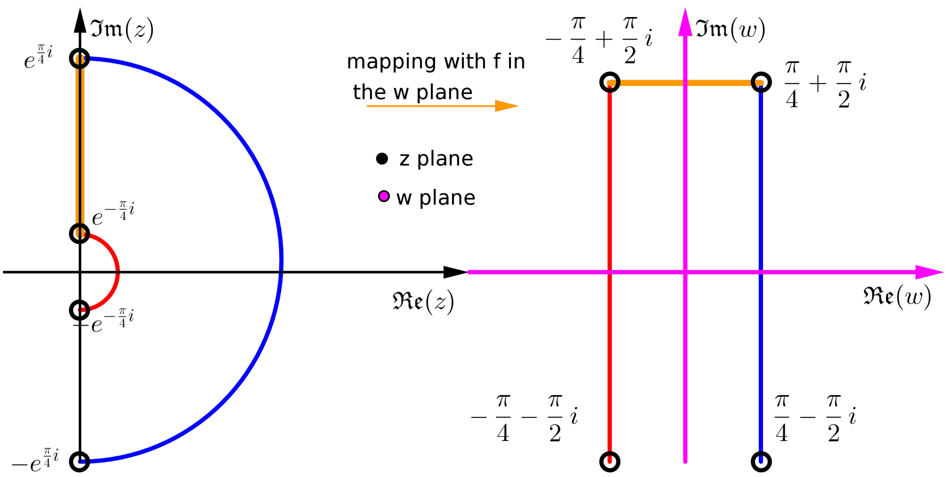
with $\mathrm{Arg}(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So the blue semicircle C_2 without its ending points is mapped into the vertical blue line $z = \frac{\pi}{4} + \mathrm{i}\mathrm{Arg}(z)$ of the *w-plane* without its ending points as seen in the next figure.



c. The set $C_3 = \left\{z \in \mathbb{C} \;\middle|\; e^{-\frac{\pi}{4}} \leqslant \Im\mathfrak{m}(z) \leqslant e^{\frac{\pi}{4}}, \Re\mathfrak{e}(z) = 0\right\}$ is the vertical line segment which lies in the imaginary axis of the *z-plane* with ending points (included) $z = e^{-\frac{\pi}{4}}\mathrm{i}$ and $z = e^{\frac{\pi}{4}}\mathrm{i}$. In this case $\mathrm{Arg}(z) = \frac{\pi}{2}$ and $\Re\mathfrak{e}(z) = 0$, so

$$\mathrm{Log}(z) = \log|y \cdot \mathrm{i}| + \mathrm{i}\mathrm{Arg}(z) = \log|y| + \mathrm{i}\frac{\pi}{2}$$

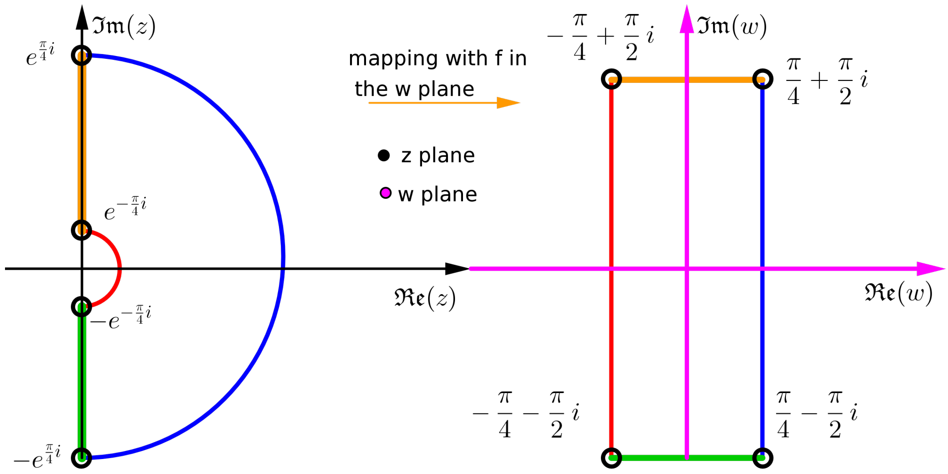
which means that that the **closed** line segment C_3 is mapped under f into the **closed** horizontal line line segment $t + \mathrm{i}\frac{\pi}{2}$ with $t \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. In the next figure we see the orange closed vertical line segment C_3 mapped into the orange closed horizontal line segment of the *w-plane*.



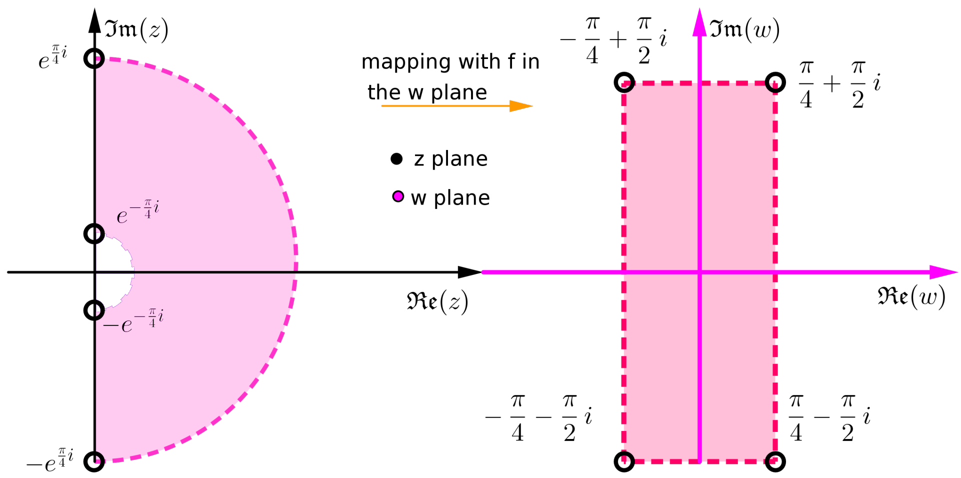
d. The set $C_4 = \left\{z \in \mathbb{C} \;\middle|\; -e^{\frac{\pi}{4}} \leqslant \Im\mathfrak{m}(z) \leqslant -e^{-\frac{\pi}{4}}, \Re\mathfrak{e}(z) = 0\right\}$ is the vertical line segment which lies in the imaginary axis of the *z-plane* with ending points (included) $z = -e^{-\frac{\pi}{4}}\mathrm{i}$ and $z = -e^{\frac{\pi}{4}}\mathrm{i}$. In this case $\mathrm{Arg}(z) = -\frac{\pi}{2}$ and $\Re\mathfrak{e}(z) = 0$, so

$$\mathrm{Log}(z) = \log|y \cdot \mathrm{i}| - \mathrm{i}\mathrm{Arg}(z) = \log|y| - \mathrm{i}\frac{\pi}{2}$$

which means that that the **closed** line segment C_4 is mapped under f into the **closed** horizontal line line segment $t - \mathrm{i}\frac{\pi}{2}$ with $t \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. In the next figure we see the green closed vertical line segment C_4 mapped into the green closed horizontal line segment of the *w-plane*.



e. $A = \left\{z \in \mathbb{C} \,\middle|\, e^{-\frac{\pi}{4}} < |z| < e^{\frac{\pi}{4}i}, \mathfrak{Re}(z) > 0\right\}$ under f is shown in the next figure.



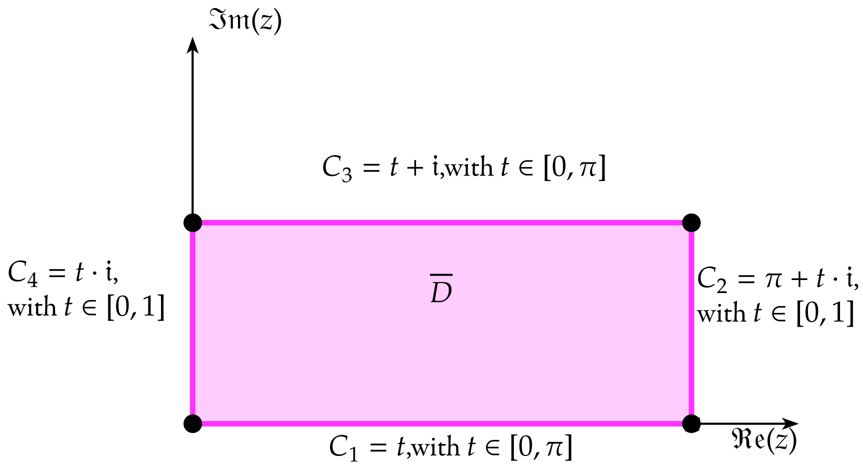
Region Mapping under f.

O	point not included
- - -	boundary not included

2. Let $f(z) = \sin(z)$ and consider the domain $D = \{z \in \mathbb{C} \mid 0 < \mathfrak{Re}(z) < \pi, 0 < \mathfrak{Im}(z) < 1\}$ (an open rectangle). Find the maximum of $|f(z)|$ on $\overline{D} = D \bigcup \partial D$ as well as the z-value(s) at which $|f|$ attains this maximum value.

Solution

We want to find the maximum of $|f|$ over the rectangle $\overline{D} = D \bigcup \partial D$. From the **maximum modulus principle** the $\max_{z \in \overline{D}} |f(z)|$ is attained on the boundary ∂D. In next figure it is clear that we must seek for the maximum of $|f|$ on the lines C_1, C_2, C_3, C_4.



Now will examine what happens on each line separately.
$C_1 : \max_{z \in C_1} |\sin(z)| = \max_{t \in [0,\pi]} |\sin(t)| = \sin\left(\frac{\pi}{2}\right) = 1$, attained at $z = \frac{\pi}{2}$.
$C_2 : \max_{z \in C_2} |\sin(z)| = \max_{t \in [0,1]} |\sin(\pi + t\mathrm{i})| = \max_{t \in [0,1]} |-\mathrm{i} \sinh(t)| = \max_{t \in [0,1]} |\sinh(t)| = \sinh(1)$, attained at $z = \pi + \mathrm{i}$.
C_3 : Recall that $\sin(z) = \sin(x + y\mathrm{i}) = \sin(x)\cosh(y) + \mathrm{i}\cos(x)\sinh(y)$.

$$
\begin{aligned}
\max_{z \in C_3} |\sin(z)| &= \max_{t \in [0,\pi]} |\sin(t)\cosh(1) + \mathrm{i}\cos(t)\sinh(1)| \\
&= \max_{t \in [0,\pi]} \sqrt{\sin(t)^2\cosh(1)^2 + \cos(t)^2\sinh(1)^2} \\
&= \sqrt{\cosh(1)^2} \quad \text{attained at } t = \frac{\pi}{2} \\
&= \cosh(1) \ \ddagger
\end{aligned}
$$

so the maximum value of the modulus of f on C_3 is $\cosh(1)$ and it is attained at $z = \frac{\pi}{2} + \mathrm{i}$.
$C_4 : \max_{z \in C_4} |\sin(z)| = \max_{t \in [0,1]} |\sin(t \cdot \mathrm{i})| = \max_{t \in [0,1]} |\mathrm{i} \cdot \sinh(t)| = \sinh(1)$, attained at $z = \mathrm{i}$.

So summing up the above we conclude that the **maximum** modulus of f is $\cosh(1)$ attained at $z = \frac{\pi}{2} + \mathrm{i}$.

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‡ Recall that $\sinh(z) = \frac{e^z - e^{-z}}{2}$ and $\cosh(z) = \frac{e^z + e^{-z}}{2}$.