

# ASSIGNMENTS OF COMPLEX ANALYSIS

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### 1. FIRST ASSIGNMENT

1. Graph the following four sets:

(A)  $\{z \in \mathbb{C} \mid |z - 3 - 2i| \leq 1\}$

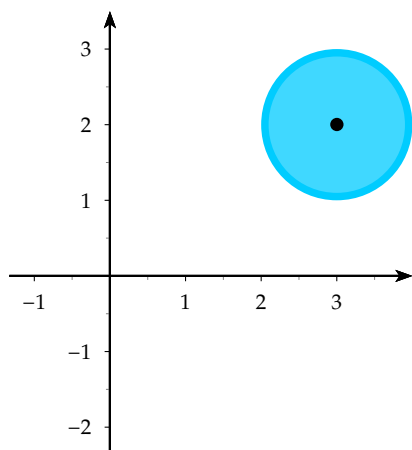
(B)  $\{z \in \mathbb{C} \mid \Im(z) = 2\}$

(C)  $\left\{z \in \mathbb{C} \setminus \{0\} \mid 0 < \arg z < \frac{\pi}{6}\right\}$

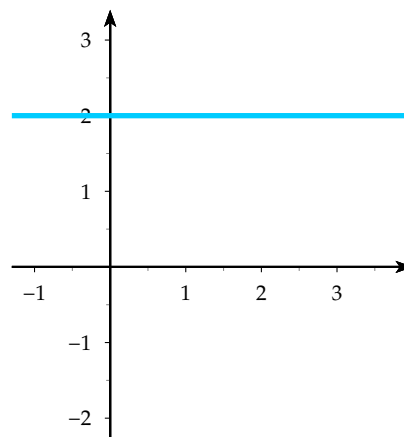
(D)  $\{z \in \mathbb{C} \mid |z - 1| < |z|\}$

#### Solution

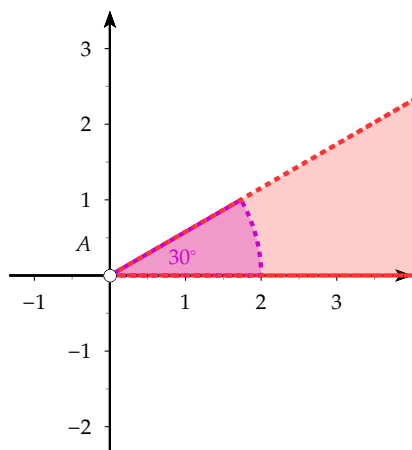
In the following figures the  $x$ -axis is the **real** axis and the  $y$ -axis the **imaginary** one.



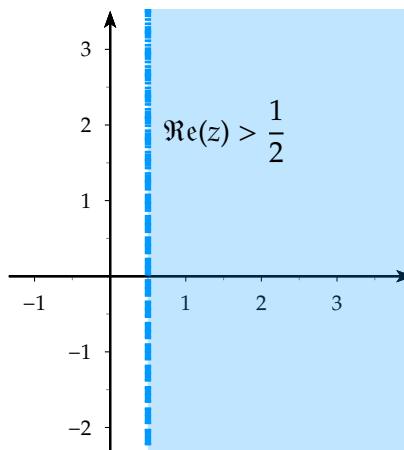
(A)  $|z - 3 - 2i| \leq 1$  is the **unit** ( $r=1$ ) disc (circle+interior) with center  $(3, 2)$ .



(B) With  $\Im(z) = 2$ , it is clear that the locus of these  $z$  is the line  $y=2$ .



(C) It is the colored (light red) region included between the lines  $y = \tan\left(\frac{\pi}{6}\right)x$  and  $y=0$  without the point  $(0,0)$  and of course **without** the points of the lines mentioned above.



(D) Raise both sides to the power of two, it is the blue sketched region below without the points of the vertical line  $x = \frac{1}{2}$ .

2. Find and plot the 6th roots of unity. For this problem you need to submit both a graph showing your answer as well as the calculation that led you to this graph.

### Solution

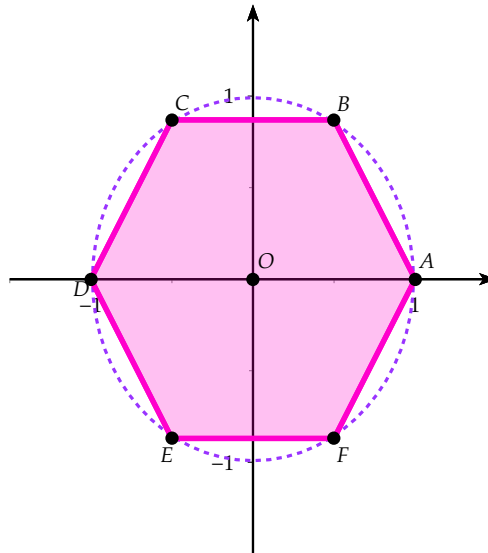
$$z^6 = 1 \Rightarrow |z|^6 e^{6i\theta} = 1^6 e^{0 \cdot i} \Rightarrow \begin{cases} |z| = 1 \\ 6\theta = 2k\pi \Rightarrow \theta = \frac{k\pi}{3}, k = 0, 1, 2, 3, 4, 5 \end{cases}, \text{ so the points we seek are}$$

$$\left\{ (1, 0), \left( \cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right) \right), \left( \cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right) \right), (\cos(\pi), \sin(\pi)), \left( \cos\left(\frac{4\pi}{3}\right), \sin\left(\frac{4\pi}{3}\right) \right), \left( \cos\left(\frac{5\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right) \right) \right\}$$

or

$$\left\{ (1, 0), \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right), (-1, 0), \left( -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right), \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \right\}$$

An easy way to construct the figure is to take  $z_1 = 1$ , draw the point (1,0) and then from that point create a **regular** hexagon of side 1 centered at the origin of the axis, which of course is also inscribed in the circle of radius 1 and center (0,0). The six solutions are represented by the points A, B, C, D, E, F which are the vertices of this regular hexagon, as seen in the following figure.



## 2. SECOND ASSIGNMENT

1. Find the image of the set  $U = \left\{ z \in \mathbb{C} \mid -\frac{\pi}{2} \leq \Re(z) \leq \frac{\pi}{2} \right\}$  under the function  $f(z) = \sin(z)$ . To do so please answer the following questions:

- What is the image of the line segment  $L_1 = \left( \frac{\pi}{2}, \frac{\pi}{2} \right)$  (in the real axis) under  $f$ ?
- What is the image of the imaginary axis  $L_2 = \{iy \mid y \in \mathbb{R}\}$  under  $f$ ?
- What is the image of the vertical line  $L_3 = \left\{ -\frac{\pi}{2} + iy \mid y \in \mathbb{R} \right\}$  under  $f$ ?
- What is the image of the vertical line  $L_4 = \left\{ \frac{\pi}{2} + iy \mid y \in \mathbb{R} \right\}$  under  $f$ ?
- Given your above observations, what do you guess the image of the set  $U$  is under  $f$ ?

### Solution

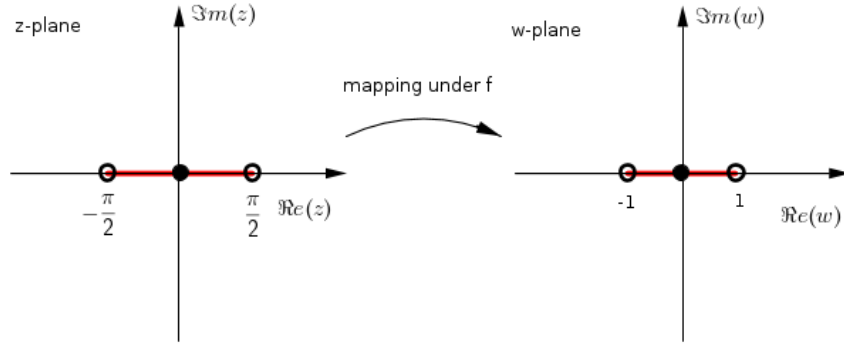
Before we begin answering the questions we need to do some calculations.

$$f(z) = \sin(z) = \sin(x + yi) = \sin(x) \cos(yi) + \sin(iy) \cos(x) = \sin(x) \cosh(y) + i \cdot \cos(x) \sinh(y)$$

so  $u(x, y) = \sin(x) \cosh(y)$  and  $v(x, y) = \cos(x) \sinh(y)$ .

- $L_1$  is the line segment  $\Re \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$  in the  $z$ -plane, so for  $x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$  and  $y = 0$  we get  $w(x, 0) = \langle u(x, 0), v(x, 0) \rangle = \langle \sin(x), 0 \rangle$

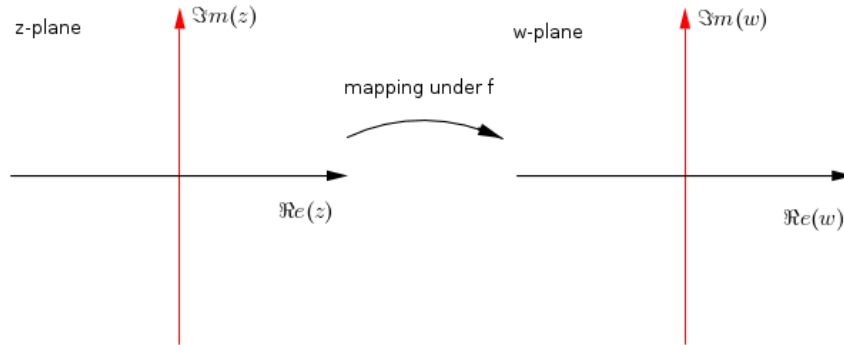
and for  $x = \pm \frac{\pi}{2}$ ,  $w\left(\pm \frac{\pi}{2}, 0\right) = \pm 1, f$  which means that under  $L_1$  is mapped into the line segment  $\Re(-1, 1)$  in the  $w$ -plane.



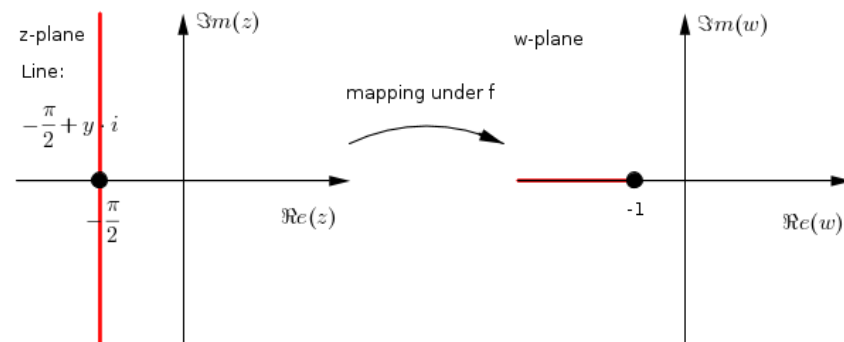
b. Following the same steps as in answer (a) we get,

$$w(0, y) = \langle u(0, y), v(0, y) \rangle = \langle 0, \sinh(y) \rangle$$

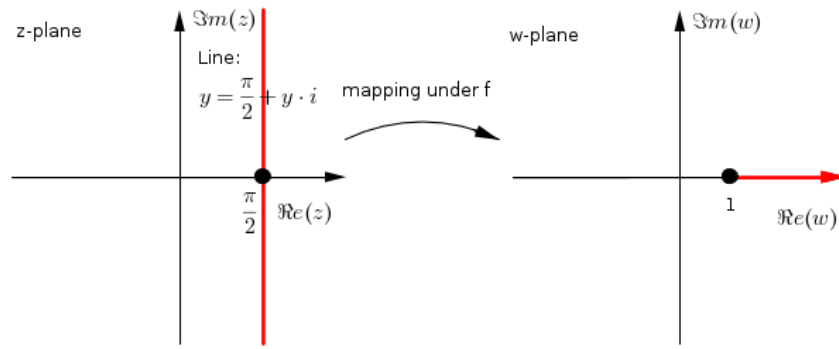
and since the range of  $\sinh$  is  $(-\infty, +\infty)$  for  $y \in \mathbb{R}$ , we deduce that  $f$  maps the  $yi$  line or the  $\Im(z)$  axis into the *imaginary* axis  $(\Im)(w)$  of the  $w$ -plane.



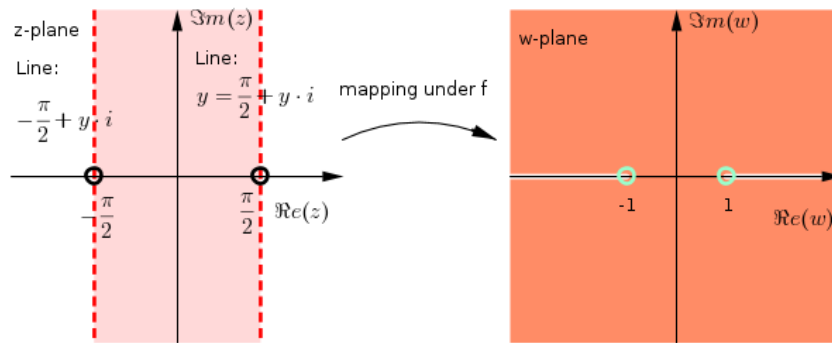
c. For  $x = -\frac{\pi}{2}$  we get  $w\left(-\frac{\pi}{2}, y\right) = \langle -\cosh(y), 0 \rangle$  and since the range of  $\cosh$  is  $[1, +\infty)$  for  $y \in \mathbb{R}$ , we deduce that  $f$  maps the line  $y = -\frac{\pi}{2} + yi$  of the  $z$ -plane into the line  $\Re(-\infty, -1]$  of the  $w$ -plane, we can imagine that an invisible hand bends the two infinite branches of the line  $y = -\frac{\pi}{2} + yi$  regarding as the center of the infinite vertical line the point  $\left(-\frac{\pi}{2}, 0\right)$  leftwards, until both branched become one with the *real axis* and then the same force moves the new line's edge horizontally to the point  $(-1, 0)$ . We will see the same process in the next question but in the opposite direction.



d. As in (c),  $x = \frac{\pi}{2}$  and so  $w\left(\frac{\pi}{2}, y\right) = \langle \cosh(y), 0 \rangle$ , so  $f$  maps the line  $y = \frac{\pi}{2} + yi$  of the  $z$ -plane into the line  $\Re[1, +\infty]$  of the  $w$ -plane.



e. It is clear that  $U = \left\{ z \in \mathbb{C} \mid -\frac{\pi}{2} < \Re(z) < \frac{\pi}{2} \right\}$  is mapped into the whole  $w$ -plane slit along the rays  $\Re(-\infty, 1]$  and  $\Re[1, +\infty)$  (white lines in the next figure excluding also from the  $w$ -plane the points  $(-1,0)$  and  $(1,0)$ ).



2. Let  $u(x, y) = x^2 - y^2 - y$ . Find a real-valued function  $v(x, y)$  such that  $v(0, 0) = 1$  and together,  $u$  and  $v$  satisfy the *Cauchy-Riemann* equations in the entire complex plane.

To do so, please follow these steps:

- Find the partial derivatives  $u_x(x, y)$  and  $u_y(x, y)$ .
- Using these partial derivatives and the Cauchy-Riemann equations, give equations for the partial derivatives  $v_x(x, y)$  and  $v_y(x, y)$ .
- Find functions  $v(x, y)$  that satisfy the equation for the partial derivative with respect to  $y$ .
- Now find a function  $v(x, y)$  that satisfies both equations for the partial derivatives at the same time.
- Finally, check whether the function you found in the previous step satisfies  $v(0, 0) = 1$ . If not, modify the function so that it does.

### Solution

$$u(x, y) = x^2 - y^2 - y$$

$$a. u_x(x, y) = 2x \text{ and } u_y(x, y) = -2y - 1.$$

$$b. \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} 2x = v_y \\ -2y - 1 = -v_x \end{cases} \Rightarrow \begin{cases} v_y = 2x \\ v_x = 2y + 1 \end{cases} \Rightarrow \begin{cases} v_x = 2y + 1 \\ v_y = 2x \end{cases}$$

c. From the previous answer we came up with the equation and by integrating with respect to  $x$  and treating  $y$  as a *constant* we get,

$$v_x(x, y) = 2y + 1 \Rightarrow \int v_x(x, y) dx = \int (2y + 1) dx \Rightarrow v(x, y) = 2xy + x + ay + c \quad (1)$$

for  $a, c \in \mathbb{R}$ .

d. Following the same process as in (c), but now by integrating with respect to  $y$  and treating  $x$  as a *constant* we get

$$v_y(x, y) = 2x \Rightarrow \int v_y(x, y) dy = \int 2x dy \Rightarrow v(x, y) = 2xy + bx + d \quad (2)$$

for  $b, d \in \mathbb{R}$ .

e. Equating relations (1) and (2) from answers c and d respectively we get

$$\begin{aligned} 2xy + x + ay + c &= 2xy + bx + d \\ (1 - b)x + ay + c - d &= 0 \end{aligned}$$

so  $a = 0, b = 1$  and  $c = d$  and  $\boxed{v(x, y) = 2xy + x + c}$  (3).

f. From (3)  $v(0, 0) = c \Leftrightarrow c = 1$  for the function  $v$  is  $v(x, y) = 2xy + x + 1$ .

### 3. THIRD ASSIGNMENT

1. Sketch the image under the map  $f(z) = \text{Log}(z)$  of the open half annulus  $A = \left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} < |z| < e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$ .

Recall:  $\text{Log}(z)$  denotes the *principal branch* of logarithm, that is,  $\text{Log}(z) = \ln |z| + i\text{Arg}(z)$ , where  $-\pi < \text{Arg}(z) \leq \pi$  is the principal argument of  $z$ .

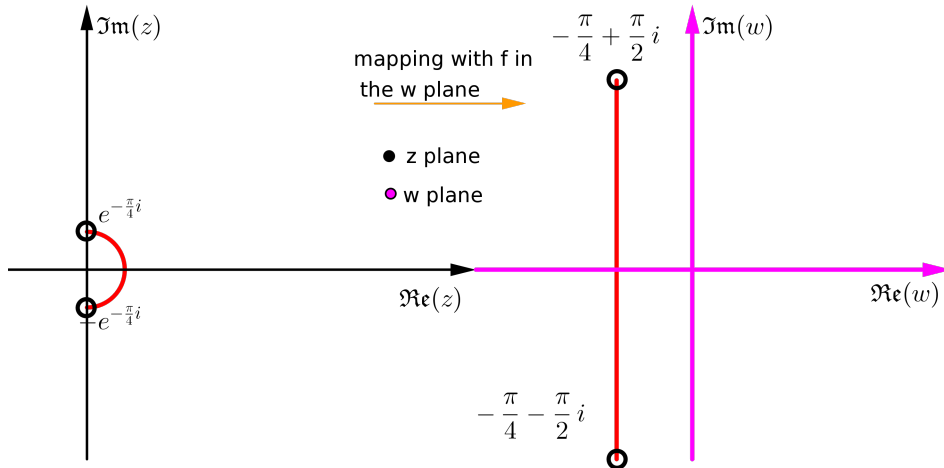
The image you create (either by hand or using a computer graphing program) should contain two graphs: On one set of coordinate axes, sketch the half annulus  $A$ , on a second set of axes sketch its image under  $f$ . Please highlight the following parts of your graph:

1. The set  $\left\{ z \in \mathbb{C} \mid |z| = e^{-\frac{\pi}{4}}, \Re(z) > 0 \right\}$  (this is a part of the boundary of  $A$ ) as well as the image of this boundary portion under  $f$  in your second graph.
2. The set  $\left\{ z \in \mathbb{C} \mid |z| = e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$  (this is another part of the boundary of  $A$ ) as well as the image of this boundary portion under  $f$  in your second graph (use a different color for these sets than you used in the first part if possible).
3. The set  $\left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} \leq \Im(z) \leq e^{\frac{\pi}{4}}, \Re(z) = 0 \right\}$  (this is yet another part of the boundary of  $A$ ) as well as the image of this boundary portion under  $f$  in your second graph (use a third color if possible).
4. The set  $\left\{ z \in \mathbb{C} \mid -e^{\frac{\pi}{4}} \leq \Im(z) \leq -e^{-\frac{\pi}{4}}, \Re(z) = 0 \right\}$  (this is the fourth part of the boundary of  $A$ ) as well as the image of this boundary portion under  $f$  in your second graph (use a fourth color if possible).
5. The set  $A$  (on your first graph) and its image  $f(A)$  (on your second graph).

### Solution

First of all before we begin with the step by step solution we should always have in our mind that  $\text{Log}(z) = \ln |z| + i\text{Arg}(z)$ , for  $-\pi < \text{Arg}(z) \leq \pi$ .

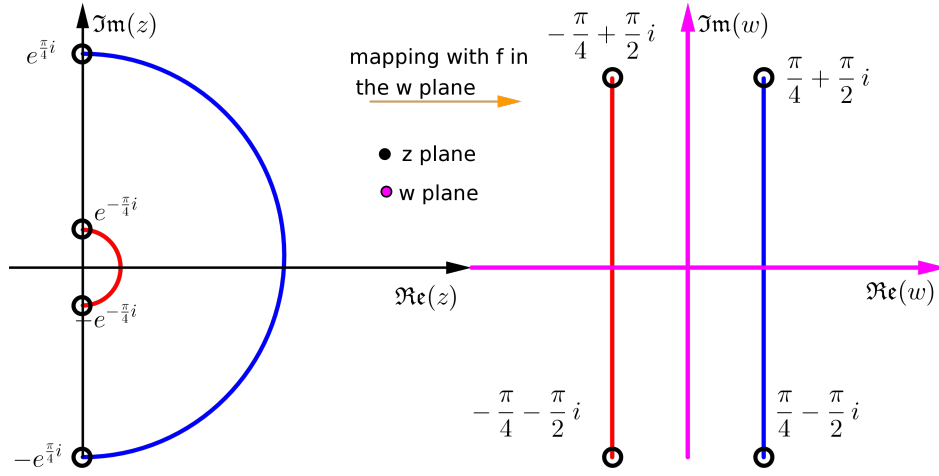
1. We are given the semicircle  $C_1 = \left\{ z \in \mathbb{C} \mid |z| = e^{-\frac{\pi}{4}}, \Re(z) > 0 \right\}$  and we will find its image under  $f(z) = \text{Log}(z)$ . So for every  $z \in C_1$  we have  $\text{Arg}(z) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$  and  $\text{Log}(z) = \ln(e^{-\frac{\pi}{4}}) + i\text{Arg}(z) = -\frac{\pi}{4} + i\text{Arg}(z)$  so  $f$  maps  $C_1$  into the vertical line  $z = -\frac{\pi}{4} + i\text{Arg}(z)$  of the  $w$ -plane as seen in the next figure (red semicircle without the ending points).



2. We are given the semicircle  $C_2 = \left\{ z \in \mathbb{C} \mid |z| = e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$  and following the same steps as in (1) we get

$$\text{Log}(z) = \ln \left( e^{\frac{\pi}{4}} \right) + i\text{Arg}(z) = \frac{\pi}{4} + i\text{Arg}(z)$$

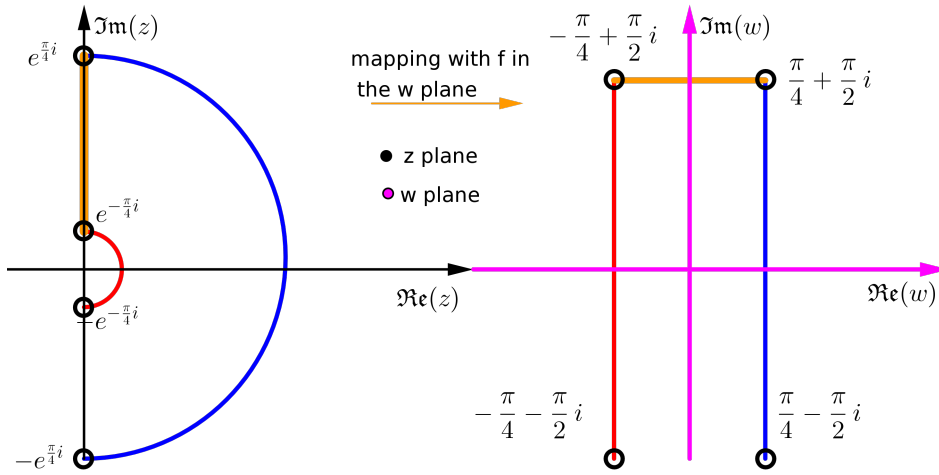
with  $\text{Arg}(z) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$ . So the blue semicircle  $C_2$  without its ending points is mapped into the vertical blue line  $z = \frac{\pi}{4} + i\text{Arg}(z)$  of the  $w$ -plane without its ending points as seen in the next figure.



3. The set  $C_3 = \left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} \leq \Im(z) \leq e^{\frac{\pi}{4}}, \Re(z) = 0 \right\}$  is the vertical line segment which lies in the imaginary axis of the  $z$ -plane with ending points (included)  $z = e^{-\frac{\pi}{4}}i$  and  $z = e^{\frac{\pi}{4}}i$ . In this case  $\text{Arg}(z) = \frac{\pi}{2}$  and  $\Re(z) = 0$ , so

$$\text{Log}(z) = \ln |y \cdot i| + \frac{\pi}{2}i = \ln(y) + i\frac{\pi}{2}$$

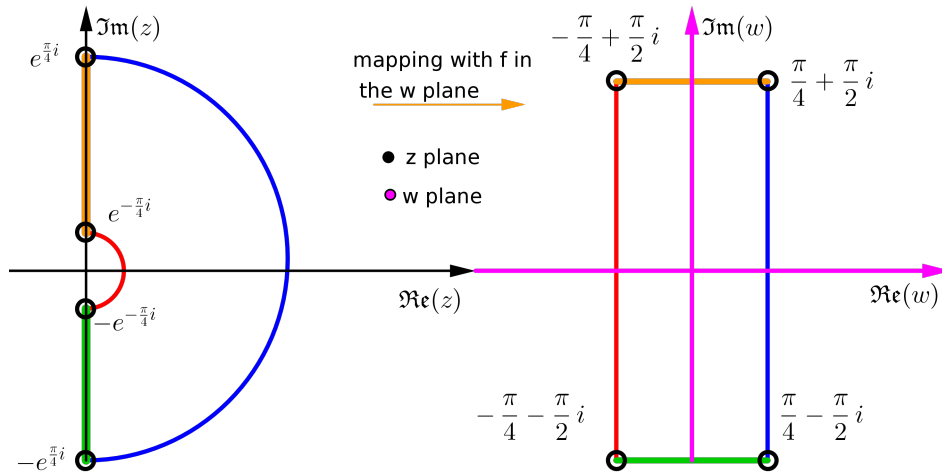
which means that the closed line segment  $C_3$  is mapped under  $f$  into the closed horizontal line segment  $t + i\frac{\pi}{2}$  with  $t \in \left[-\frac{\pi}{4}, \frac{\pi}{2}\right]$ . In the next figure we see the orange closed vertical line segment  $C_3$  mapped into the orange closed horizontal line segment of the  $w$ -plane.



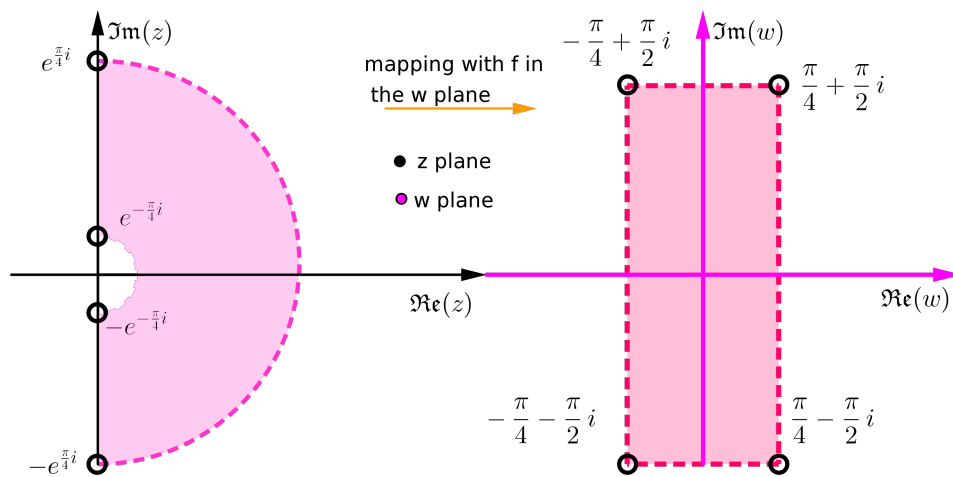
4. The set  $C_4 = \left\{ z \in \mathbb{C} \mid -e^{\frac{\pi}{4}} \leq \Im(z) \leq -e^{-\frac{\pi}{4}}, \Re(z) = 0 \right\}$  is the vertical line segment which lies in the imaginary axis of the  $z$ -plane with ending points (included)  $z = -e^{\frac{\pi}{4}}i$  and  $z = -e^{-\frac{\pi}{4}}i$ . In this case  $\text{Arg}(z) = -\frac{\pi}{2}$  and  $\Re(z) = 0$ , so

$$\text{Log}(z) = \ln |y \cdot i| - \frac{\pi}{2}i = \ln(y) - i\frac{\pi}{2}$$

which means that the closed line segment  $C_4$  is mapped under  $f$  into the closed horizontal line segment  $t - i\frac{\pi}{2}$  with  $t \in \left[-\frac{\pi}{4}, \frac{\pi}{2}\right]$ . In the next figure we see the green closed vertical line segment  $C_4$  mapped into the green closed horizontal line segment of the  $w$ -plane.



5.  $A = \left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} < |z| < e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$  under  $f$  is shown in the next figure.

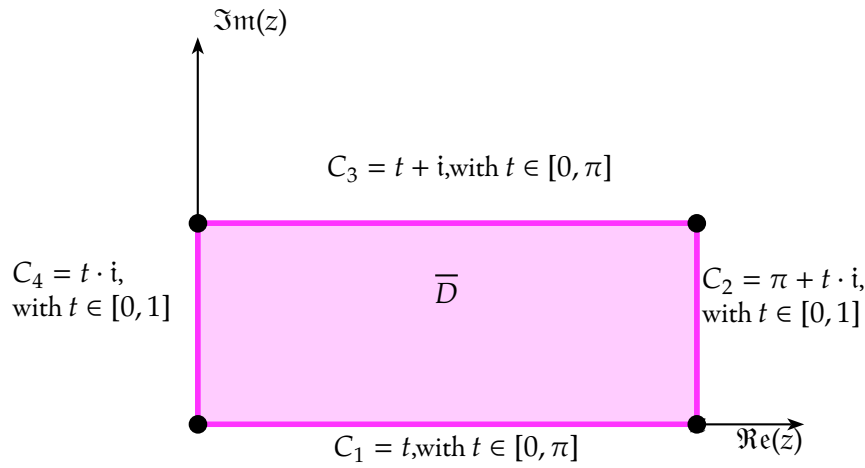


----- | boundary not included  
O | point not included

2. Let  $f(z) = \sin(z)$  and consider the domain  $D = \{z \in \mathbb{C} \mid 0 < \Re(z) < \pi, 0 < \Im(z) < 1\}$  (an open rectangle). Find the maximum of  $|f(z)|$  on  $\overline{D} = D \cup \partial D$  as well as the  $z$ -value(s) at which  $|f|$  attains this maximum value.

### Solution

We want to find the maximum of  $|f|$  over the rectangle  $\overline{D} = D \cup \partial D$ . From the maximum modulus principle the  $\max_{z \in \overline{D}} |f(z)|$  is attained on the boundary  $\partial D$ . In next figure it is clear that we must seek for the maximum of  $|f|$  on the lines  $C_1, C_2, C_3, C_4$ .



Now will examine what happens on each line separately.

$$C_1 : \max_{z \in C_1} \sin(z) = \max_{t \in [0, \pi]} |\sin(t)| = \sin\left(\frac{\pi}{2}\right) = 1, \text{ attained at } z = \frac{\pi}{2}.$$

$$C_2 : \max_{z \in C_2} \sin(z) = \max_{t \in [0, 1]} \sin(\pi + ti) = \max_{t \in [0, 1]} -i \sinh(t) = \max_{t \in [0, 1]} \sinh(t) = \sinh(1), \text{ attained at } z = \pi + i.$$

$$C_3 : \sin(x + yi) = \sin(x) \cosh(y) + i \cos(x) \sinh(y), \text{ so}$$

$$\begin{aligned} \max_{z \in C_3} \sin(z) &= \max_{t \in [0, \pi]} \sin(t) \cosh(1) + i \cos(t) \sinh(1) \\ &= \max_{t \in [0, \pi]} \sqrt{(\sin(t) \cosh(1))^2 + (\cos(t) \sinh(1))^2} \\ &= \cosh(1) \end{aligned}$$

$$, \text{ attained at } z = \frac{\pi}{2} + i.$$

$$C_4 : \max_{z \in C_4} |\sin(z)| = \max_{t \in [0, 1]} |\sin(t \cdot i)| = \max_{t \in [0, 1]} |i \cdot \sinh(t)| = \sinh(1), \text{ attained at } z = i.$$

So summing up the above we get  $\max_{z \in \bar{D}} |f(z)| = \left| f\left(\frac{\pi}{2} + i\right) \right| = \cosh(1)$ , which means that the **maximum** modulus of  $f$  is  $\cosh(1)$

$$\text{attained at } z = \frac{\pi}{2} + i.$$

$$\dagger \text{ Recall that } \sinh(z) = \frac{e^z - e^{-z}}{2} \text{ and } \cosh(z) = \frac{e^z + e^{-z}}{2}.$$