

ASSIGNMENTS OF COMPLEX ANALYSIS
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1. FIRST ASSIGNMENT

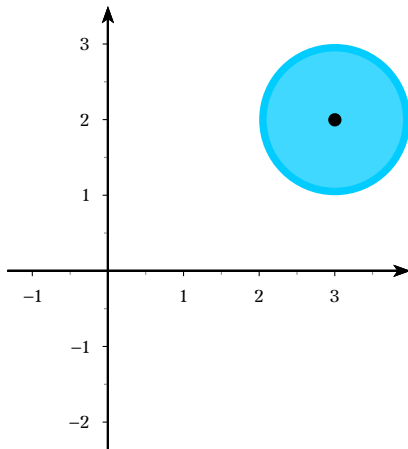
1. Graph the following four sets:

- (A) $\{z \in \mathbb{C} \mid |z - 3 - 2i| \leq 1\}$
- (B) $\{z \in \mathbb{C} \mid \Im(z) = 2\}$
- (C) $\left\{z \in \mathbb{C} \setminus \{0\} \mid 0 < \arg z < \frac{\pi}{6}\right\}$
- (D) $\{z \in \mathbb{C} \mid |z - 1| < |z|\}$

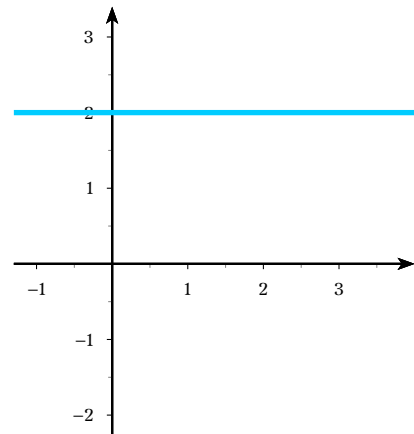
Solution

In the following figures the x -axis is the real axis and the y -axis the imaginary one.

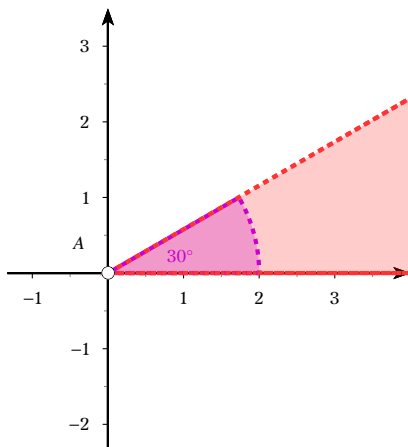
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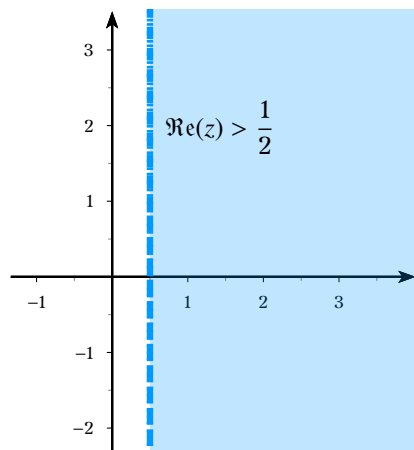
(A) $|z - 3 - 2i| \leq 1$ is the unit ($r=1$) disc (circle+interior) with center $(3, 2)$.



(B) With $\Im(z) = 2$, it is clear that the locus of these z is the line $y=2$.



(C) It is the colored (light red) region included between the lines $y = \tan\left(\frac{\pi}{6}\right)x$ and $y=0$ without the point $(0,0)$ and of course without the points of the lines mentioned above.



(D) Raise both sides to the power of two, it is the blue sketched region below without the points of the vertical line $x = \frac{1}{2}$.

2. Find and plot the 6th roots of unity. For this problem you need to submit both a graph showing your answer as well as the calculation that led you to this graph.

Solution

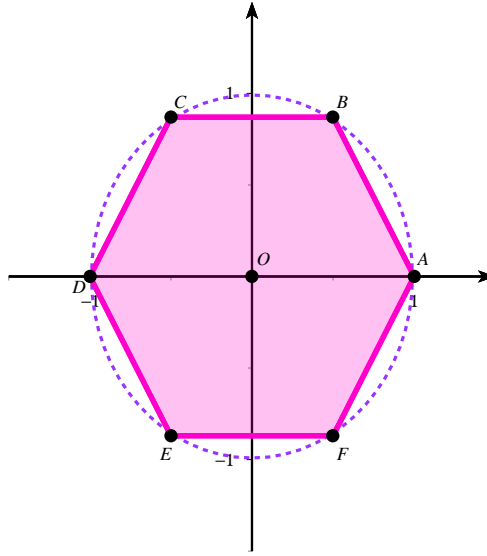
$$z^6 = 1 \Rightarrow |z|^6 e^{6i\theta} = 1^6 e^{0 \cdot i} \Rightarrow \begin{cases} |z| = 1 \\ 6\theta = 2k\pi \Rightarrow \theta = \frac{k\pi}{3}, k = 0, 1, 2, 3, 4, 5 \end{cases}, \text{ so the points we seek are}$$

$$\left\{ (1, 0), \left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right) \right), \left(\cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right) \right), (\cos(\pi), \sin(\pi)), \left(\cos\left(\frac{4\pi}{3}\right), \sin\left(\frac{4\pi}{3}\right) \right), \left(\cos\left(\frac{5\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right) \right) \right\}$$

or

$$\left\{ (1, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2} \right), (-1, 0), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2} \right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \right\}$$

An easy way to construct the figure is to take $z_1 = 1$, draw the point (1,0) and then from that point create a regular hexagon of side 1 centered at the origin of the axis, which of course is also inscribed in the circle of radius 1 and center (0,0). The six solutions are represented by the red points A, B, C, D, E, F which are the vertices of this regular hexagon, as seen in the following figure.



2. SECOND ASSIGNMENT

1. Find the image of the set $U = \left\{ z \in \mathbb{C} \mid -\frac{\pi}{2} \leq \Re(z) \leq \frac{\pi}{2} \right\}$ under the function $f(z) = \sin(z)$. To do so please answer the following questions:
 - a. What is the image of the line segment $L_1 = \left(\frac{\pi}{2}, \frac{\pi}{2} \right)$ (in the real axis) under f ?
 - b. What is the image of the imaginary axis $L_2 = \{iy \mid y \in \mathbb{R}\}$ under f ?
 - c. What is the image of the vertical line $L_3 = \left\{ -\frac{\pi}{2} + iy \mid y \in \mathbb{R} \right\}$ under f ?
 - d. What is the image of the vertical line $L_4 = \left\{ \frac{\pi}{2} + iy \mid y \in \mathbb{R} \right\}$ under f ?
 - e. Given your above observations, what do you guess the image of the set U is under f ?

Solution

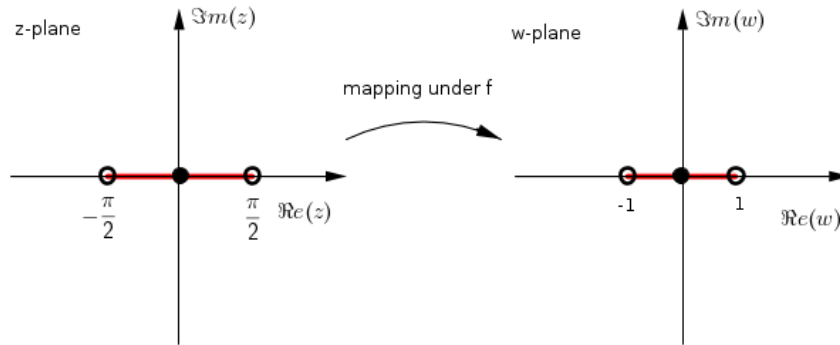
Before we begin answering the questions we need to do some calculations.

$$f(z) = \sin(z) = \sin(x + yi) = \sin(x) \cos(yi) + \sin(iy) \cos(x) = \sin(x) \cosh(y) + i \cdot \cos(x) \sinh(y)$$

so $u(x, y) = \sin(x) \cosh(y)$ and $v(x, y) = \cos(x) \sinh(y)$.

- a. L_1 is the line segment $\Re\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ in the z -plane, so for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $y = 0$ we get $w(x, 0) = \langle u(x, 0), v(x, 0) \rangle = \langle \sin(x), 0 \rangle$ and for

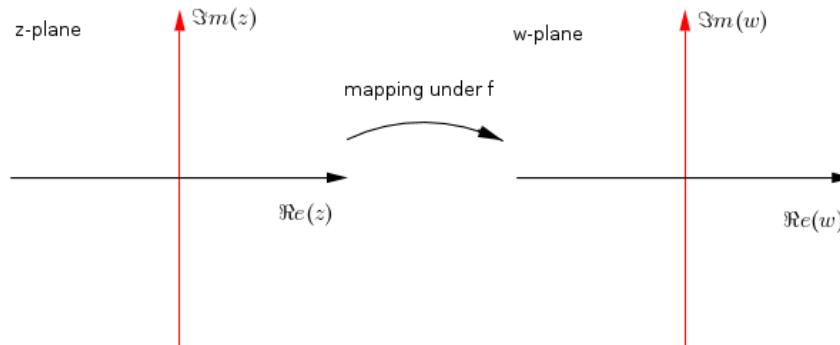
$x = \pm \frac{\pi}{2}$, $w\left(\pm \frac{\pi}{2}, 0\right) = \pm 1$, which means that under f , L_1 is mapped into the line segment $\Re(-1, 1)$ in the w -plane.



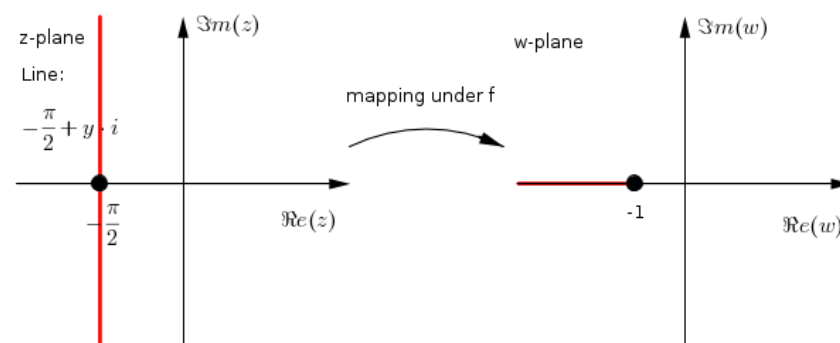
b. Following the same steps as in answer (a) we get,

$$w(0, y) = \langle u(0, y), v(0, y) \rangle = \langle 0, \sinh(y) \rangle$$

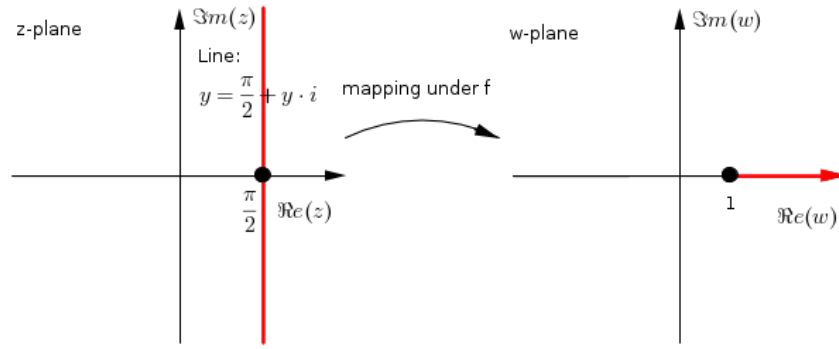
and since the range of \sinh is $(-\infty, +\infty)$ for $y \in \mathbb{R}$, we deduce that f maps the y i line or the $\Im(z)$ axis into the imaginary axis ($\Im(w)$) of the w -plane.



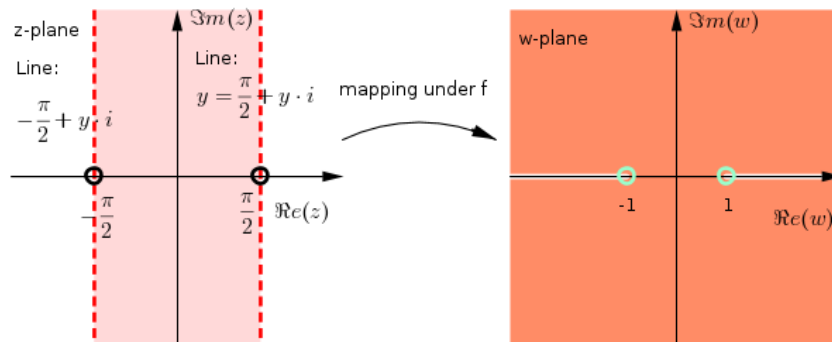
c. Again for $x = -\frac{\pi}{2}$ we get $w\left(-\frac{\pi}{2}, y\right) = \langle -\cosh(y), 0 \rangle$ and since the range of \cosh is $[1, +\infty)$ for $y \in \mathbb{R}$, we deduce that f maps the line $y = -\frac{\pi}{2} + yi$ of the z -plane into the line $\Re(-\infty, -1]$ of the w -plane, we can imagine that an invisible hand bends the two infinite branches of the line $y = -\frac{\pi}{2} + yi$ regarding as the center of the infinite vertical line the point $\left(-\frac{\pi}{2}, 0\right)$ leftwards, until both branched become one with the real axis and then the same force moves the new line's edge horizontally to the point $(-1, 0)$. We will see the same process in the next question but in the opposite direction.



d. As in (c), $x = \frac{\pi}{2}$ and so $w\left(\frac{\pi}{2}, y\right) = \langle \cosh(y), 0 \rangle$, so f maps the line $y = \frac{\pi}{2} + yi$ of the z -plane into the line $\Re[1, +\infty]$ of the w -plane.



e. It is clear that $U = \left\{ z \in \mathbb{C} \mid -\frac{\pi}{2} < \Re(z) < \frac{\pi}{2} \right\}$ is mapped into the whole w -plane slit along the rays $\Re(-\infty, 1]$ and $\Re[1, +\infty)$ (white lines in the next figure excluding also from the w -plane the points $(-1, 0)$ and $(1, 0)$).



2. Let $u(x, y) = x^2 - y^2 - y$. Find a real-valued function $v(x, y)$ such that $v(0, 0) = 1$ and together, u and v satisfy the *Cauchy-Riemann* equations in the entire complex plane.

To do so, please follow these steps:

- Find the partial derivatives $u_x(x, y)$ and $u_y(x, y)$.
- Using these partial derivatives and the Cauchy-Riemann equations, give equations for the partial derivatives $v_x(x, y)$ and $v_y(x, y)$.
- Find functions $v(x, y)$ that satisfy the equation for the partial derivative with respect to y .
- Now find a function $v(x, y)$ that satisfies both equations for the partial derivatives at the same time.
- Finally, check whether the function you found in the previous step satisfies $v(0, 0) = 1$. If not, modify the function so that it does.

Solution

$$u(x, y) = x^2 - y^2 - y$$

$$\text{a. } u_x(x, y) = 2x \text{ and } u_y(x, y) = -2y - 1.$$

$$\text{b. } \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} 2x = v_y \\ -2y - 1 = -v_x \end{cases} \Rightarrow \begin{cases} v_y = 2x \\ v_x = 2y + 1 \end{cases} \Rightarrow \begin{cases} v_x = 2y + 1 \\ v_y = 2x \end{cases}$$

c. From the previous answer we came up with the equation and by integrating with respect to x and treating y as a *constant* we get,

$$v_x(x, y) = 2y + 1 \Rightarrow \int v_x(x, y) dx = \int (2y + 1) dx \Rightarrow v(x, y) = 2xy + x + ay + c \quad (1)$$

for $a, c \in \mathbb{R}$.

d. Following the same process as in (c), but now by integrating with respect to y and treating x as a *constant* we get

$$v_y(x, y) = 2x \Rightarrow \int v_y(x, y) dy = \int 2x dy \Rightarrow v(x, y) = 2xy + bx + d \quad (2)$$

for $b, d \in \mathbb{R}$.

e. Equating relations (1) and (2) from answers c and d respectively we get

$$\begin{aligned} 2xy + x + ay + c &= 2xy + bx + d \\ (1 - b)x + ay + c - d &= 0 \end{aligned}$$

so $a=0, b=1$ and $c=d$ and $\boxed{v(x, y) = 2xy + x + c} \quad (3).$

f. From (3) $v(0, 0) = c \Leftrightarrow c = 1$ for the function v is $v(x, y) = 2xy + x + 1$.

3. THIRD ASSIGNMENT

1. Sketch the image under the map $f(z) = \text{Log}(z)$ of the open half annulus $A = \left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} < |z| < e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$.

Recall: $\text{Log}(z)$ denotes the principal branch of logarithm, that is, $\text{Log}(z) = \ln|z| + i\text{Arg}(z)$, where $-\pi < \text{Arg}(z) \leq \pi$ is the principal argument of z .

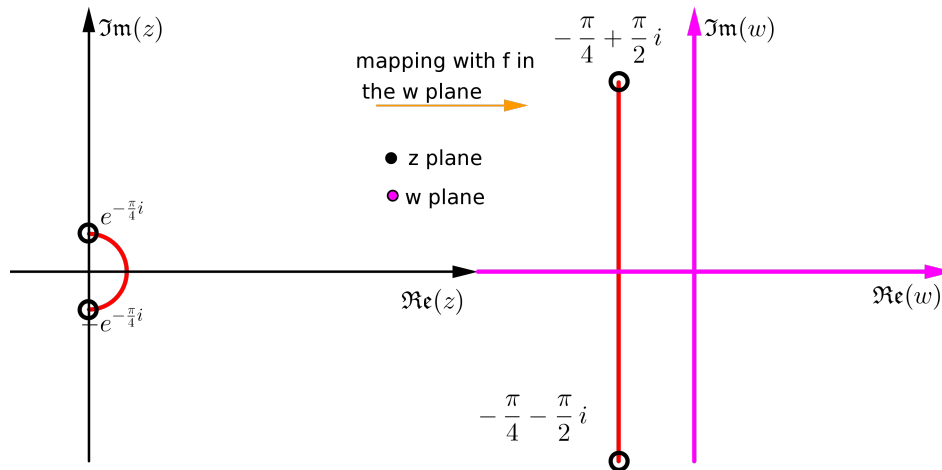
The image you create (either by hand or using a computer graphing program) should contain two graphs: On one set of coordinate axes, sketch the half annulus A , on a second set of axes sketch its image under f . Please highlight the following parts of your graph:

1. The set $\left\{ z \in \mathbb{C} \mid |z| = e^{-\frac{\pi}{4}}, \Re(z) > 0 \right\}$ (this is a part of the boundary of A) as well as the image of this boundary portion under f in your second graph.
2. The set $\left\{ z \in \mathbb{C} \mid |z| = e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$ (this is another part of the boundary of A) as well as the image of this boundary portion under f in your second graph (use a different color for these sets than you used in the first part if possible).
3. The set $\left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} \leq \Im(z) \leq e^{\frac{\pi}{4}}, \Re(z) = 0 \right\}$ (this is yet another part of the boundary of A) as well as the image of this boundary portion under f in your second graph (use a third color if possible).
4. The set $\left\{ z \in \mathbb{C} \mid -e^{\frac{\pi}{4}} \leq \Im(z) \leq -e^{-\frac{\pi}{4}}, \Re(z) = 0 \right\}$ (this is the fourth part of the boundary of A) as well as the image of this boundary portion under f in your second graph (use a fourth color if possible).
5. The set A (on your first graph) and its image $f(A)$ (on your second graph).

Solution

First of all before we begin with the step by step solution we should always have in our mind that $\text{Log}(z) = \ln|z| + i\text{Arg}(z)$, for $-\pi < \text{Arg}(z) \leq \pi$.

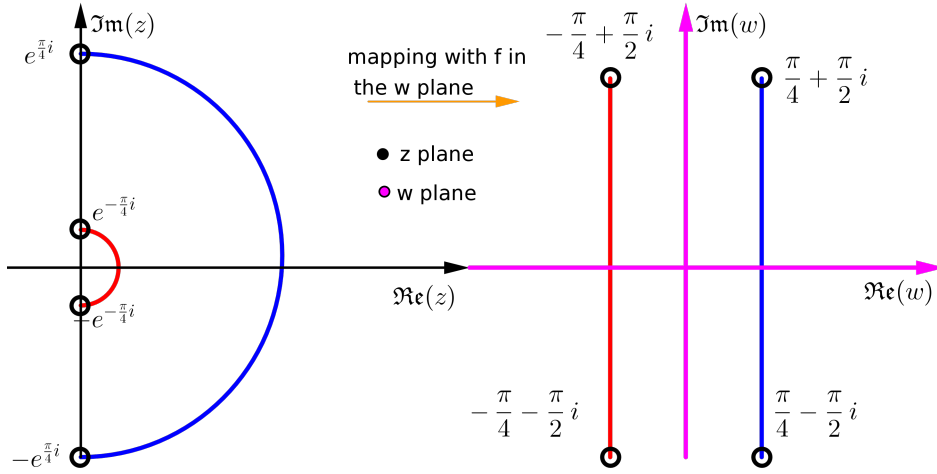
1. We are given the semicircle $C_1 = \left\{ z \in \mathbb{C} \mid |z| = e^{-\frac{\pi}{4}}, \Re(z) > 0 \right\}$ and we will find its image under $f(z) = \text{Log}(z)$. So for every $z \in C_1$ we have $\text{Arg}(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\text{Log}(z) = \ln(e^{-\frac{\pi}{4}}) + i\text{Arg}(z) = -\frac{\pi}{4} + i\text{Arg}(z)$ so f maps C_1 into the vertical line $z = -\frac{\pi}{4} + i\text{Arg}(z)$ of the w -plane as seen in the next figure (red semicircle without the ending points).



2. We are given the semicircle $C_2 = \left\{ z \in \mathbb{C} \mid |z| = e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$ and following the same steps as in (1) we get

$$\text{Log}(z) = \ln\left(e^{\frac{\pi}{4}}\right) + i\text{Arg}(z) = \frac{\pi}{4} + i\text{Arg}(z)$$

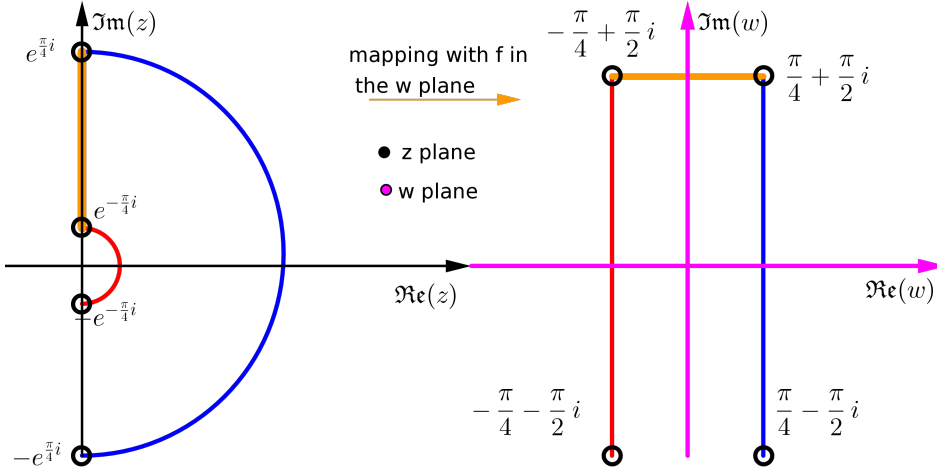
with $\text{Arg}(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So the blue semicircle C_2 without its ending points is mapped into the vertical blue line $z = \frac{\pi}{4} + i\text{Arg}(z)$ of the w -plane without its ending points as seen in the next figure.



3. The set $C_3 = \left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} \leq \Im(z) \leq e^{\frac{\pi}{4}}, \Re(z) = 0 \right\}$ is the vertical line segment which lies in the imaginary axis of the z -plane with ending points (included) $z = e^{-\frac{\pi}{4}}i$ and $z = e^{\frac{\pi}{4}}i$. In this case $\text{Arg}(z) = \frac{\pi}{2}$ and $\Re(z) = 0$, so

$$\text{Log}(z) = \ln|y \cdot i| + \frac{\pi}{2}i = \ln(y) + i\frac{\pi}{2}$$

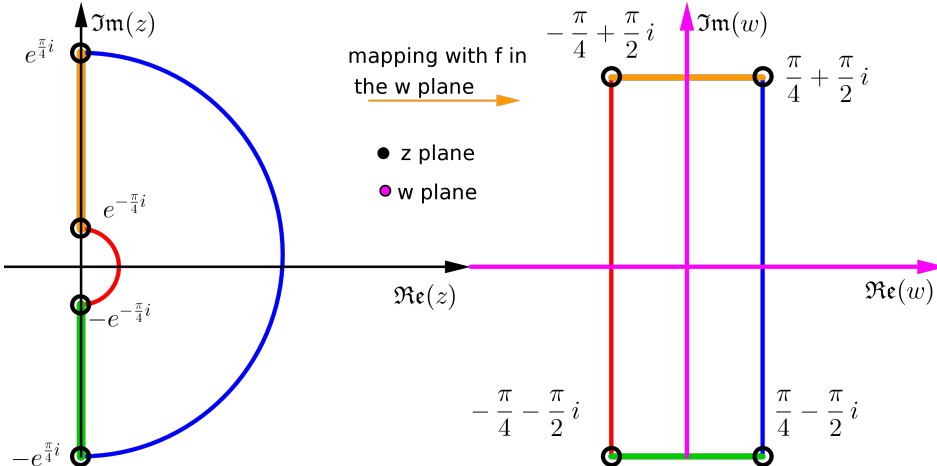
which means that the closed line segment C_3 is mapped under f into the closed horizontal line segment $t + i\frac{\pi}{2}$ with $t \in \left[-\frac{\pi}{4}, \frac{\pi}{2}\right]$. In the next figure we see the orange closed vertical line segment C_3 mapped into the orange closed horizontal line segment of the w -plane.



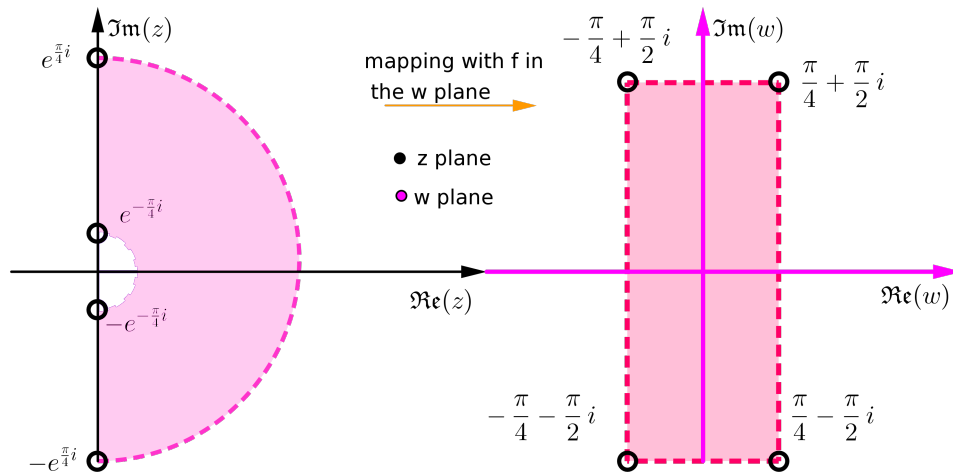
4. The set $C_4 = \left\{ z \in \mathbb{C} \mid -e^{\frac{\pi}{4}} \leq \Im(z) \leq -e^{-\frac{\pi}{4}}, \Re(z) = 0 \right\}$ is the vertical line segment which lies in the imaginary axis of the z -plane with ending points (included) $z = -e^{-\frac{\pi}{4}}i$ and $z = -e^{\frac{\pi}{4}}i$. In this case $\text{Arg}(z) = -\frac{\pi}{2}$ and $\Re(z) = 0$, so

$$\text{Log}(z) = \ln|y \cdot i| + \frac{\pi}{2}i = \ln(y) + i\frac{\pi}{2}$$

which means that the closed line segment C_4 is mapped under f into the closed horizontal line segment $ti\frac{\pi}{2}$ with $t \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. In the next figure we see the green closed vertical line segment C_4 mapped into the green closed horizontal line segment of the w -plane.



5. $A = \left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} < |z| < e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$ under f is shown in the next figure.

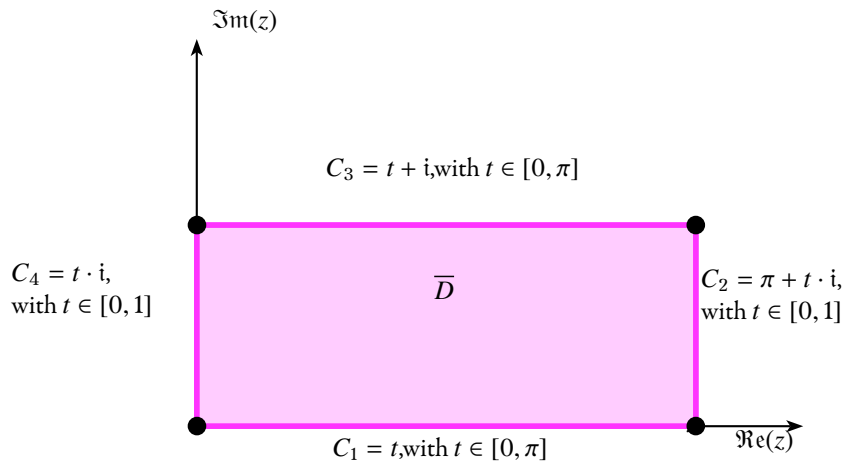


- - - - - | boundary not included
 O | point not included

2. Let $f(z) = \sin(z)$ and consider the domain $D = \{z \in \mathbb{C} \mid 0 < \Re(z) < \pi, 0 < \Im(z) < 1\}$ (an open rectangle). Find the maximum of $|f(z)|$ on $\overline{D} = D \cup \partial D$ as well as the z -value(s) at which $|f|$ attains this maximum value.

Solution

We want to find the maximum of $|f|$ over the rectangle $\overline{D} = D \cup \partial D$. From the maximum modulus principle the $\max_{z \in \overline{D}} |f(z)|$ is attained on the boundary ∂D . In next figure it is clear that we must seek for the maximum of $|f|$ on the lines C_1, C_2, C_3, C_4 .



Now will examine what happens on each line separately.

$$C_1 : \max_{z \in C_1} \sin(z) = \max_{t \in [0, \pi]} |\sin(t)| = \sin\left(\frac{\pi}{2}\right) = 1, \text{ attained at } z = \frac{\pi}{2}.$$

$$C_2 : \max_{z \in C_2} \sin(z) = \max_{t \in [0, 1]} \sin(\pi + ti) = \max_{t \in [0, 1]} -i \sinh(t) = \max_{t \in [0, 1]} \sinh(t) = \sinh(1), \text{ attained at } z = \pi + i.$$

$$C_3 : \sin(x + yi) = \sin(x) \cosh(y) + i \cos(x) \sinh(y), \text{ so}$$

$$\begin{aligned}
 \max_{z \in C_3} \sin(z) &= \max_{t \in [0, \pi]} \sin(t) \cosh(1) + i \cos(t) \sinh(1) \\
 &= \max_{t \in [0, \pi]} \sqrt{(\sin(t) \cosh(1))^2 + (\cos(t) \sinh(1))^2} \\
 &= \cosh(1)
 \end{aligned}$$

, attained at $z = \frac{\pi}{2} + i$.

$$C_4 : \max_{z \in C_4} |\sin(z)| = \max_{t \in [0,1]} |\sin(t \cdot i)| = \max_{t \in [0,1]} |i \cdot \sinh(t)| = \sinh(1), \text{ attained at } z = i.$$

So summing up the above we get $\max_{z \in \overline{D}} |f(z)| = \left| f\left(\frac{\pi}{2} + i\right) \right| = \cosh(1)$, which means that the maximum modulus of f is $\cosh(1)$ attained at $z = \frac{\pi}{2} + i$.

$$\dagger \text{ Recall that } \sinh(z) = \frac{e^z - e^{-z}}{2} \text{ and } \cosh(z) = \frac{e^z + e^{-z}}{2}.$$