

ASSIGNMENTS OF COMPLEX ANALYSIS

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1. FIRST ASSIGNMENT

1. Graph the following four sets:

(A) $\{z \in \mathbb{C} \mid |z - 3 - 2i| \leq 1\}$

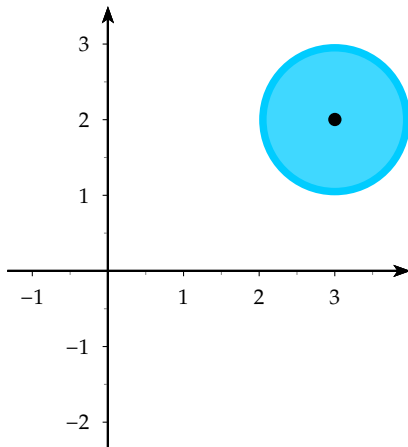
(B) $\{z \in \mathbb{C} \mid \Im(z) = 2\}$

(C) $\left\{z \in \mathbb{C} \setminus \{0\} \mid 0 < \arg z < \frac{\pi}{6}\right\}$

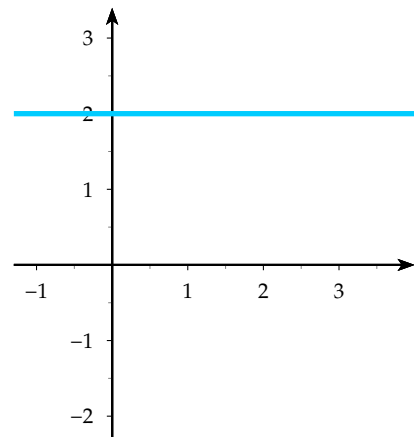
(D) $\{z \in \mathbb{C} \mid |z - 1| < |z|\}$

Solution

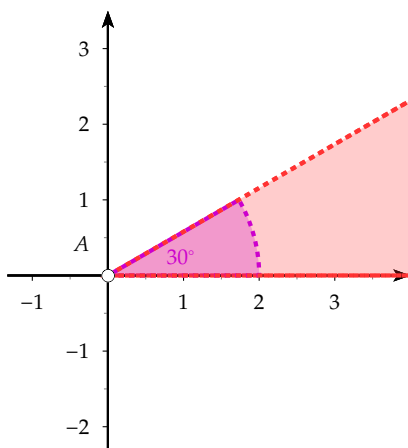
In the following figures the x -axis is the **real** axis and the y -axis the **imaginary** one.



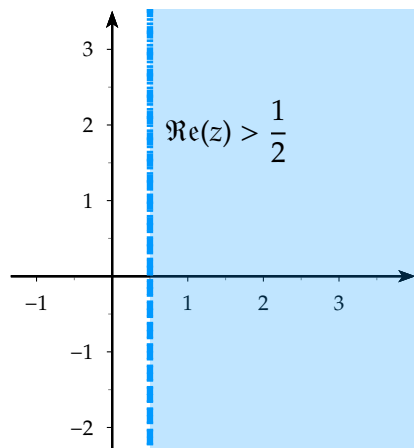
(A) $|z - 3 - 2i| \leq 1$ is the **unit** ($r=1$) disc (circle+interior) with center $(3, 2)$.



(B) With $\Im(z) = 2$, it is clear that the locus of these z is the line $y=2$.



(C) It is the colored (light red) region included between the lines $y = \tan\left(\frac{\pi}{6}\right)x$ and $y=0$ without the point $(0,0)$ and of course **without** the points of the lines mentioned above.



(D) Raise both sides to the power of two, it is the blue sketched region below without the points of the vertical line $x = \frac{1}{2}$.

2. Find and plot the 6th roots of unity. For this problem you need to submit both a graph showing your answer as well as the calculation that led you to this graph.

Solution

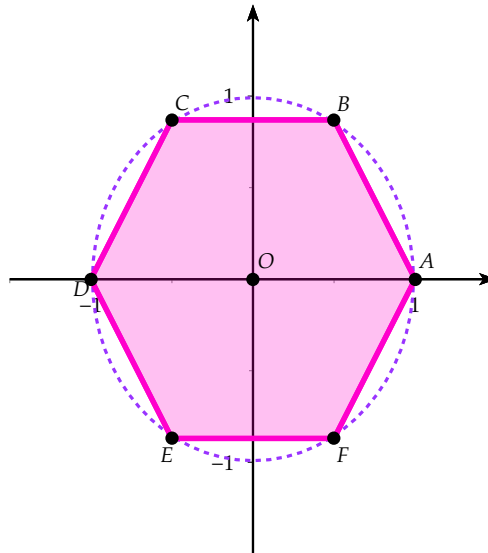
$$z^6 = 1 \Rightarrow |z|^6 e^{6i\theta} = 1^6 e^{0 \cdot i} \Rightarrow \begin{cases} |z| = 1 \\ 6\theta = 2k\pi \Rightarrow \theta = \frac{k\pi}{3}, k = 0, 1, 2, 3, 4, 5 \end{cases}, \text{ so the points we seek are}$$

$$\left\{ (1, 0), \left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right) \right), \left(\cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right) \right), (\cos(\pi), \sin(\pi)), \left(\cos\left(\frac{4\pi}{3}\right), \sin\left(\frac{4\pi}{3}\right) \right), \left(\cos\left(\frac{5\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right) \right) \right\}$$

or

$$\left\{ (1, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2} \right), (-1, 0), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2} \right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \right\}$$

An easy way to construct the figure is to take $z_1 = 1$, draw the point (1,0) and then from that point create a **regular** hexagon of side 1 centered at the origin of the axis, which of course is also inscribed in the circle of radius 1 and center (0,0). The six solutions are represented by the points A, B, C, D, E, F which are the vertices of this regular hexagon, as seen in the following figure.



2. SECOND ASSIGNMENT

1. Find the image of the set $U = \left\{ z \in \mathbb{C} \mid -\frac{\pi}{2} \leq \Re(z) \leq \frac{\pi}{2} \right\}$ under the function $f(z) = \sin(z)$. To do so please answer the following questions:

- What is the image of the line segment $L_1 = \left[\frac{\pi}{2}, \frac{\pi}{2} \right]$ (in the real axis) under f ?
- What is the image of the imaginary axis $L_2 = \{iy \mid y \in \mathbb{R}\}$ under f ?
- What is the image of the vertical line $L_3 = \left\{ -\frac{\pi}{2} + iy \mid y \in \mathbb{R} \right\}$ under f ?
- What is the image of the vertical line $L_4 = \left\{ \frac{\pi}{2} + iy \mid y \in \mathbb{R} \right\}$ under f ?
- Given your above observations, what do you guess the image of the set U is under f ?

Solution

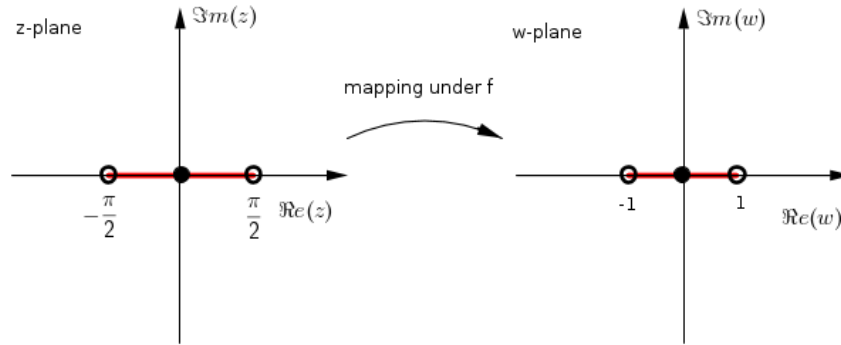
Before we begin answering the questions we need to do some calculations.

$$f(z) = \sin(z) = \sin(x + yi) = \sin(x) \cos(yi) + \sin(iy) \cos(x) = \sin(x) \cosh(y) + i \cdot \cos(x) \sinh(y)$$

so $u(x, y) = \sin(x) \cosh(y)$ and $v(x, y) = \cos(x) \sinh(y)$.

- L_1 is the line segment $\Re \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$ in the z -plane, so for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$ and $y = 0$ we get $w(x, 0) = \langle u(x, 0), v(x, 0) \rangle = \langle \sin(x), 0 \rangle$

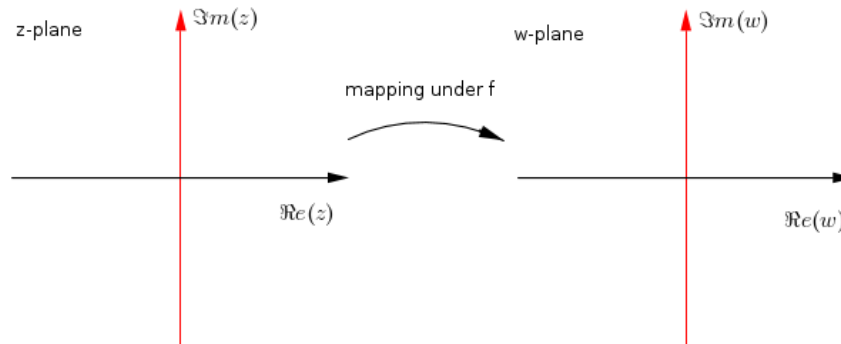
and for $x = \pm \frac{\pi}{2}$, $w\left(\pm \frac{\pi}{2}, 0\right) = \pm 1, f$ which means that under L_1 is mapped into the line segment $\Re(-1, 1)$ in the w -plane.



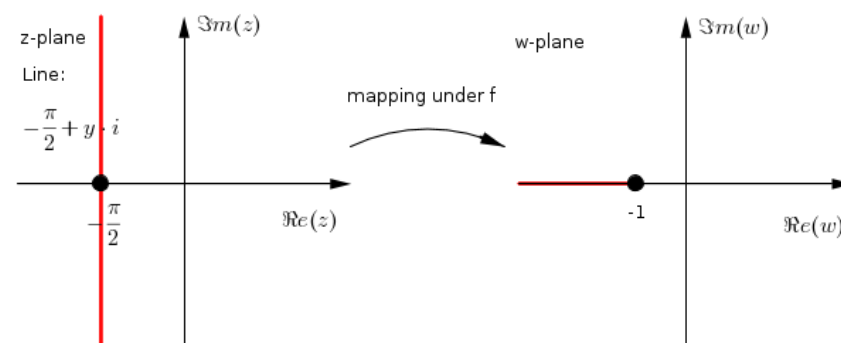
b. Following the same steps as in answer (a) we get,

$$w(0, y) = \langle u(0, y), v(0, y) \rangle = \langle 0, \sinh(y) \rangle$$

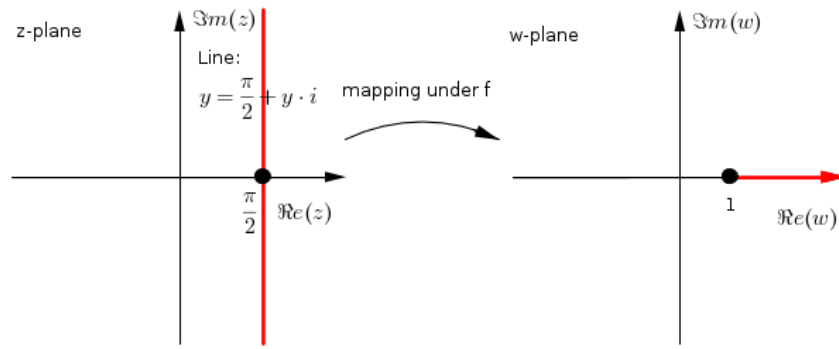
and since the range of \sinh is $(-\infty, +\infty)$ for $y \in \mathbb{R}$, we deduce that f maps the yi line or the $\Im(z)$ axis into the *imaginary* axis $(\Im)(w)$ of the w -plane.



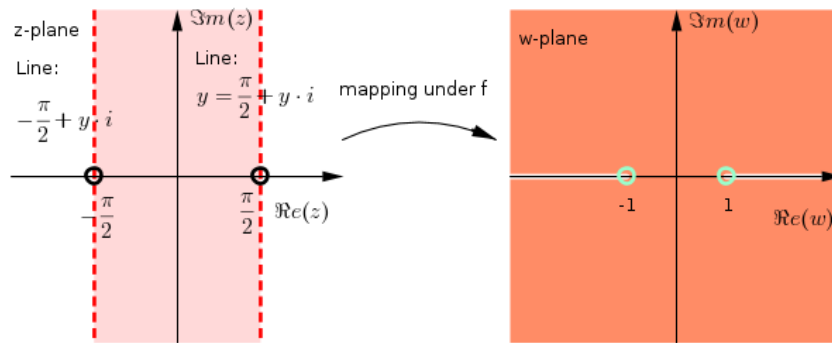
c. For $x = -\frac{\pi}{2}$ we get $w\left(-\frac{\pi}{2}, y\right) = \langle -\cosh(y), 0 \rangle$ and since the range of \cosh is $[1, +\infty)$ for $y \in \mathbb{R}$, we deduce that f maps the line $y = -\frac{\pi}{2} + yi$ of the z -plane into the line $\Re(-\infty, -1]$ of the w -plane, we can imagine that an invisible hand bends the two infinite branches of the line $y = -\frac{\pi}{2} + yi$ regarding as the center of the infinite vertical line the point $\left(-\frac{\pi}{2}, 0\right)$ leftwards, until both branched become one with the *real axis* and then the same force moves the new line's edge horizontally to the point $(-1, 0)$. We will see the same process in the next question but in the opposite direction.



d. As in (c), $x = \frac{\pi}{2}$ and so $w\left(\frac{\pi}{2}, y\right) = \langle \cosh(y), 0 \rangle$, so f maps the line $y = \frac{\pi}{2} + yi$ of the z -plane into the line $\Re[1, +\infty]$ of the w -plane.



e. It is clear that $U = \left\{ z \in \mathbb{C} \mid -\frac{\pi}{2} < \Re(z) < \frac{\pi}{2} \right\}$ is mapped into the whole w -plane slit along the rays $\Re(-\infty, 1]$ and $\Re[1, +\infty)$ (white lines in the next figure excluding also from the w -plane the points $(-1,0)$ and $(1,0)$).



2. Let $u(x, y) = x^2 - y^2 - y$. Find a real-valued function $v(x, y)$ such that $v(0, 0) = 1$ and together, u and v satisfy the *Cauchy-Riemann* equations in the entire complex plane.

To do so, please follow these steps:

- Find the partial derivatives $u_x(x, y)$ and $u_y(x, y)$.
- Using these partial derivatives and the Cauchy-Riemann equations, give equations for the partial derivatives $v_x(x, y)$ and $v_y(x, y)$.
- Find functions $v(x, y)$ that satisfy the equation for the partial derivative with respect to y .
- Now find a function $v(x, y)$ that satisfies both equations for the partial derivatives at the same time.
- Finally, check whether the function you found in the previous step satisfies $v(0, 0) = 1$. If not, modify the function so that it does.

Solution

$$u(x, y) = x^2 - y^2 - y$$

$$a. u_x(x, y) = 2x \text{ and } u_y(x, y) = -2y - 1.$$

$$b. \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} 2x = v_y \\ -2y - 1 = -v_x \end{cases} \Rightarrow \begin{cases} v_y = 2x \\ v_x = 2y + 1 \end{cases} \Rightarrow \begin{cases} v_x = 2y + 1 \\ v_y = 2x \end{cases}$$

c. From the previous answer we came up with the equation and by integrating with respect to x and treating y as a *constant* we get,

$$v_x(x, y) = 2y + 1 \Rightarrow \int v_x(x, y) dx = \int 2y + 1 dx \Rightarrow v(x, y) = 2xy + x + ay + c \quad (1)$$

for $a, c \in \mathbb{R}$.

d. Following the same process as in (c), but now by integrating with respect to y and treating x as a *constant* we get

$$v_y(x, y) = 2x \Rightarrow \int v_y(x, y) dy = \int 2x dy \Rightarrow v(x, y) = 2xy + bx + d \quad (2)$$

for $b, d \in \mathbb{R}$.

e. Equating relations (1) and (2) from answers c and d respectively we get

$$\begin{aligned} 2xy + x + ay + c &= 2xy + bx + d \\ (1 - b)x + ay + c - d &= 0 \end{aligned}$$

so $a = 0, b = 1$ and $c = d$ and $\boxed{v(x, y) = 2xy + x + c}$ (3).

f. From (3) $v(0, 0) = c \Leftrightarrow c = 1$ for the function v is $v(x, y) = 2xy + x + 1$.

3. THIRD ASSIGNMENT

1. Sketch the image under the map $f(z) = \text{Log}(z)$ of the open half annulus $A = \left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} < |z| < e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$.

Recall: $\text{Log}(z)$ denotes the *principal branch* of logarithm, that is, $\text{Log}(z) = \ln |z| + i\text{Arg}(z)$, where $-\pi < \text{Arg}(z) \leq \pi$ is the principal argument of z .

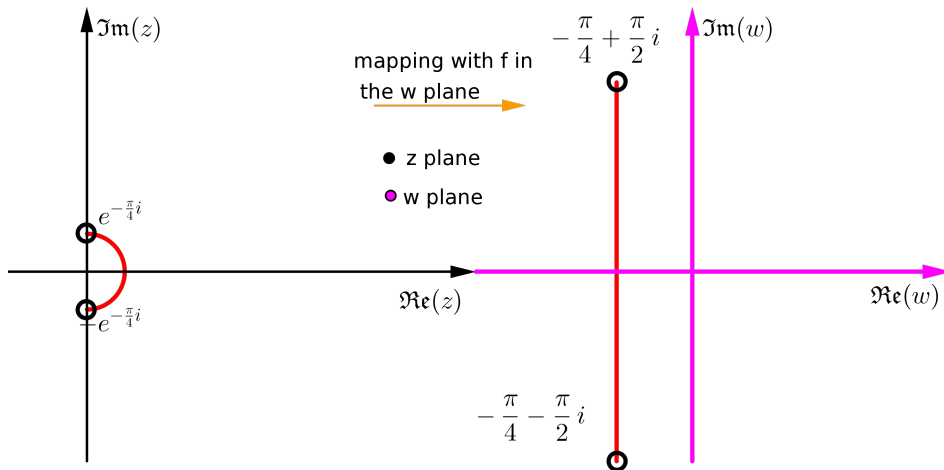
The image you create (either by hand or using a computer graphing program) should contain two graphs: On one set of coordinate axes, sketch the half annulus A , on a second set of axes sketch its image under f . Please highlight the following parts of your graph:

1. The set $\left\{ z \in \mathbb{C} \mid |z| = e^{-\frac{\pi}{4}}, \Re(z) > 0 \right\}$ (this is a part of the boundary of A) as well as the image of this boundary portion under f in your second graph.
2. The set $\left\{ z \in \mathbb{C} \mid |z| = e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$ (this is another part of the boundary of A) as well as the image of this boundary portion under f in your second graph (use a different color for these sets than you used in the first part if possible).
3. The set $\left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} \leq \Im(z) \leq e^{\frac{\pi}{4}}, \Re(z) = 0 \right\}$ (this is yet another part of the boundary of A) as well as the image of this boundary portion under f in your second graph (use a third color if possible).
4. The set $\left\{ z \in \mathbb{C} \mid -e^{\frac{\pi}{4}} \leq \Im(z) \leq -e^{-\frac{\pi}{4}}, \Re(z) = 0 \right\}$ (this is the fourth part of the boundary of A) as well as the image of this boundary portion under f in your second graph (use a fourth color if possible).
5. The set A (on your first graph) and its image $f(A)$ (on your second graph).

Solution

First of all before we begin with the step by step solution we should always have in our mind that $\text{Log}(z) = \ln |z| + i\text{Arg}(z)$, for $-\pi < \text{Arg}(z) \leq \pi$.

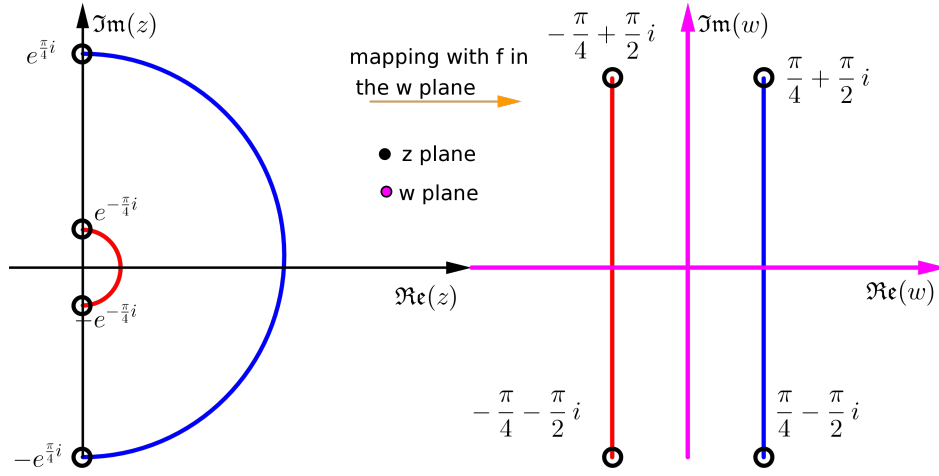
1. We are given the semicircle $C_1 = \left\{ z \in \mathbb{C} \mid |z| = e^{-\frac{\pi}{4}}, \Re(z) > 0 \right\}$ and we will find its image under $f(z) = \text{Log}(z)$. So for every $z \in C_1$ we have $\text{Arg}(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\text{Log}(z) = \ln(e) - \frac{\pi}{2} + i\text{Arg}(z)$ so f maps C_1 into the vertical line $z = -\frac{\pi}{4} + i\text{Arg}(z)$ of the w -plane as seen in the next figure (red semicircle without the ending points).



2. We are given the semicircle $C_2 = \left\{ z \in \mathbb{C} \mid |z| = e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$ and following the same steps as in (1) we get

$$\text{Log}(z) = \ln\left(e^{\frac{\pi}{4}}\right) + i\text{Arg}(z) = \frac{\pi}{4} + i\text{Arg}(z)$$

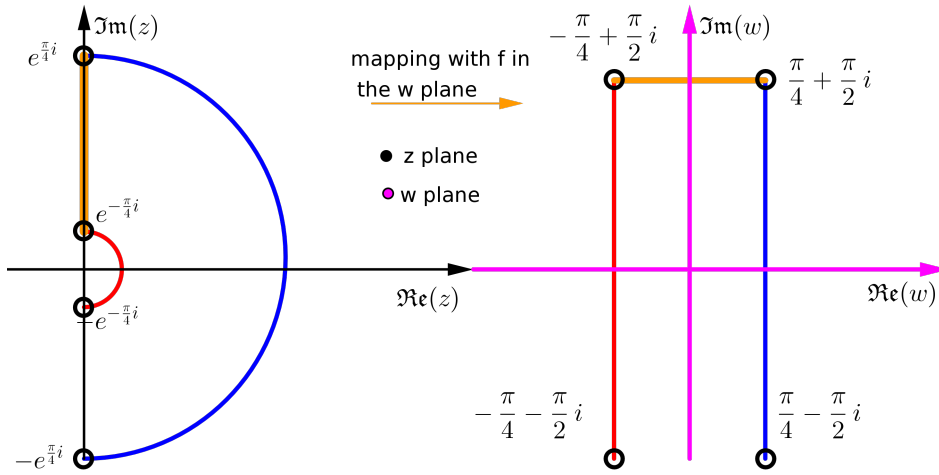
with $\text{Arg}(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So the blue semicircle C_2 without its ending points is mapped into the vertical blue line $z = \frac{\pi}{4} + i\text{Arg}(z)$ of the w -plane without its ending points as seen in the next figure.



3. The set $C_3 = \left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} \leq \Im(z) \leq e^{\frac{\pi}{4}}, \Re(z) = 0 \right\}$ is the vertical line segment which lies in the imaginary axis of the z -plane with ending points (included) $z = e^{-\frac{\pi}{4}}i$ and $z = e^{\frac{\pi}{4}}i$. In this case $\text{Arg}(z) = \frac{\pi}{2}$ and $\Re(z) = 0$, so

$$\text{Log}(z) = \ln |y \cdot i| + \frac{\pi}{2}i = \ln(y) + i\frac{\pi}{2}$$

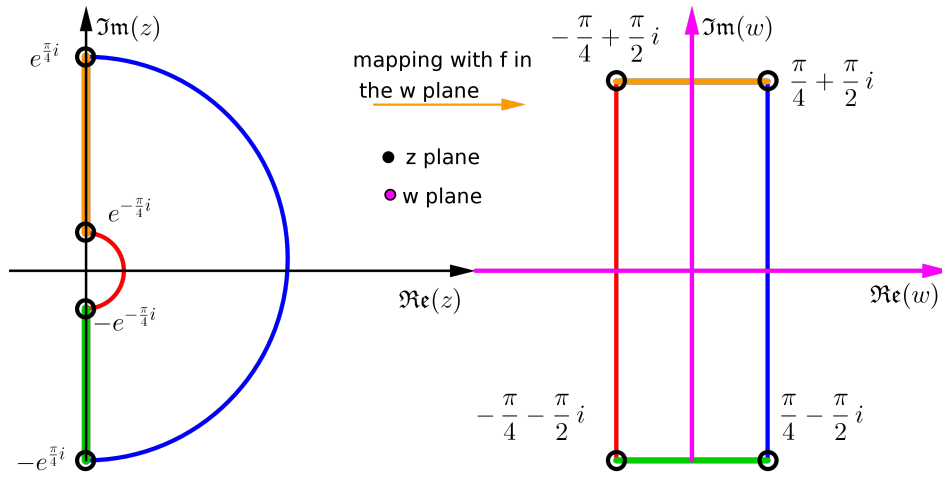
which means that the closed line segment C_3 is mapped under f into the closed horizontal line segment $t + i\frac{\pi}{2}$ with $t \in \left[-\frac{\pi}{4}, \frac{\pi}{2}\right]$. In the next figure we see the orange closed vertical line segment C_3 mapped into the orange closed horizontal line segment of the w -plane.



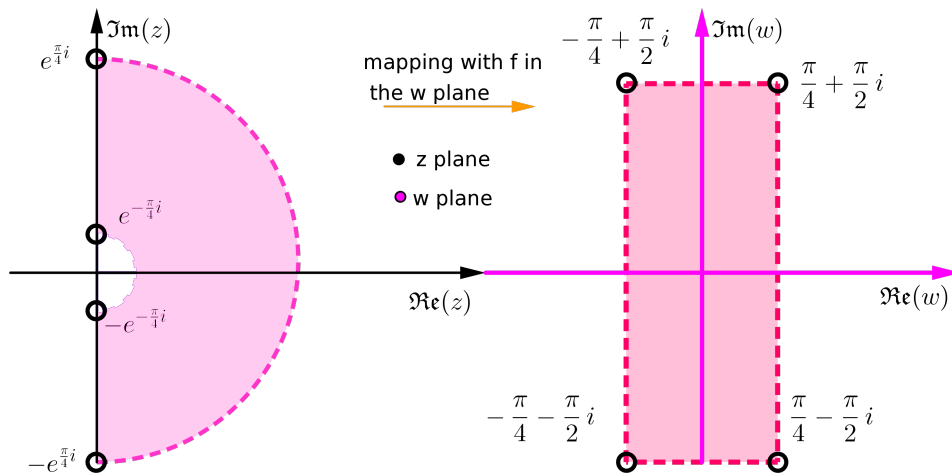
4. The set $C_4 = \left\{ z \in \mathbb{C} \mid -e^{\frac{\pi}{4}} \leq \Im(z) \leq -e^{-\frac{\pi}{4}}, \Re(z) = 0 \right\}$ is the vertical line segment which lies in the imaginary axis of the z -plane with ending points (included) $z = -e^{\frac{\pi}{4}}i$ and $z = -e^{-\frac{\pi}{4}}i$. In this case $\text{Arg}(z) = -\frac{\pi}{2}$ and $\Re(z) = 0$, so

$$\text{Log}(z) = \ln |y \cdot i| - \frac{\pi}{2}i = \ln(y) - i\frac{\pi}{2}$$

which means that the closed line segment C_4 is mapped under f into the closed horizontal line segment $t - i\frac{\pi}{2}$ with $t \in \left[-\frac{\pi}{4}, \frac{\pi}{2}\right]$. In the next figure we see the green closed vertical line segment C_4 mapped into the green closed horizontal line segment of the w -plane.



5. $A = \left\{ z \in \mathbb{C} \mid e^{-\frac{\pi}{4}} < |z| < e^{\frac{\pi}{4}}, \Re(z) > 0 \right\}$ under f is shown in the next figure.

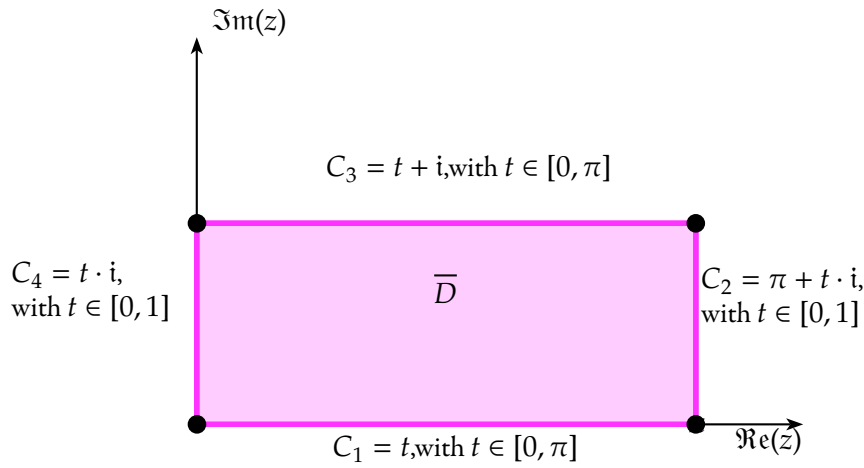


----- | boundary not included
O | point not included

2. Let $f(z) = \sin(z)$ and consider the domain $D = \{z \in \mathbb{C} \mid 0 < \Re(z) < \pi, 0 < \Im(z) < 1\}$ (an open rectangle). Find the maximum of $|f(z)|$ on $\overline{D} = D \cup \partial D$ as well as the z -value(s) at which $|f|$ attains this maximum value.

Solution

We want to find the maximum of $|f|$ over the rectangle $\overline{D} = D \cup \partial D$. From the maximum modulus principle the $\max_{z \in \overline{D}} |f(z)|$ is attained on the boundary ∂D . In next figure it is clear that we must seek for the maximum of $|f|$ on the lines C_1, C_2, C_3, C_4 .



Now will examine what happens on each line separately.

$$C_1 : \max_{z \in C_1} \sin(z) = \max_{t \in [0, \pi]} |\sin(t)| = \sin\left(\frac{\pi}{2}\right) = 1, \text{ attained at } z = \frac{\pi}{2}.$$

$$C_2 : \max_{z \in C_2} \sin(z) = \max_{t \in [0, 1]} \sin(\pi + ti) = \max_{t \in [0, 1]} -i \sinh(t) = \max_{t \in [0, 1]} \sinh(t) = \sinh(1), \text{ attained at } z = \pi + i.$$

$$C_3 : \sin(x + yi) = \sin(x) \cosh(y) + i \cos(x) \sinh(y), \text{ so}$$

$$\begin{aligned} \max_{z \in C_3} \sin(z) &= \max_{t \in [0, \pi]} \sin(t) \cosh(1) + i \cos(t) \sinh(1) \\ &= \max_{t \in [0, \pi]} \sqrt{(\sin(t) \cosh(1))^2 + (\cos(t) \sinh(1))^2} \\ &= \cosh(1) \end{aligned}$$

$$\text{, attained at } z = \frac{\pi}{2} + i.$$

$$C_4 : \max_{z \in C_4} |\sin(z)| = \max_{t \in [0, 1]} |\sin(t \cdot i)| = \max_{t \in [0, 1]} |i \cdot \sinh(t)| = \sinh(1), \text{ attained at } z = i.$$

So summing up the above we get $\max_{z \in \overline{D}} |f(z)| = \left| f\left(\frac{\pi}{2} + i\right) \right| = \cosh(1)$, which means that the **maximum** modulus of f is $\cosh(1)$

$$\text{attained at } z = \frac{\pi}{2} + i.$$

$$\dagger \text{ Recall that } \sinh(z) = \frac{e^z - e^{-z}}{2} \text{ and } \cosh(z) = \frac{e^z + e^{-z}}{2}.$$