Portfolio Selection

Lecture Three

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Outline

- Bit of statistics review.
- Optimal portfolio choice with 1 risky and 1 risk-free asset
- Optimal portfolio choice with 2 risky assets
- Optimal portfolio choice with 2 risky and 1 risk-free asset
- Try to remember what is *possible* versus what is *preferable*.

Allocation

- Risky Assets vs. Risk-Free

"The fundamental decision of investing is the allocation of your assets: How much should you own in stock? How much should you own in bonds? How much should you own in cash reserves? That decision accounts for an astonishing 94% of the difference in total returns achieved by institutional investors. There is no reason to believe that the same relationship does not hold for individual investors." - John C. Bogle

Probabilities

- Think about different (possibly infinite) states of the world $\{s_i\}$.
- For this class (and much of finance) states are defined by returns.
- How to think about *risk*?

Expected Value

- Probability-weighted average of possible state-dependent outcomes

- Suppose that a return R_i on asset i is equal to $R_i(s)$ in state s. And that states' probabilities of occurrence are given by $p(s)$, for $s = 1, ..., S$.

$$E[R_i] = \sum_{s=1}^{S} R_i(s)p(s)$$

Variance and Standard Deviation

- Measures fluctuation of a variable around its mean.

$$
\begin{aligned}
V[R_i] &= \sigma_i^2 = E\left[(R_i(s) - E[R_i])^2\right] \\
&= \sum_{s=1}^{S} p(s)[R_i(s) - E(R_i)]^2
\end{aligned}
$$

- The standard deviation is the square root of the variance.
- Volatility is another word for ‘standard deviation’.

Covariance

- The covariance between two random variables is expressed as:

$$
\begin{aligned}
Cov(R_i, R_j) &= E\left[(R_i - E[R_i])(R_j - E[R_j])\right] \\
&= \sum_{s=1}^{S} p(s)[(R_i(s) - E[R_i])(R_j(s) - E[R_j])
\end{aligned}
$$

- Positive covariance means assets tend to be unusually high at the same time.

- Negative covariance means one asset tends to be unusually low when the other is unusually high and vice-versa .

Correlation

- Very similar to correlation, but normalized by standard deviations

$$Cov(R_i, R_j) = \rho_{ij} = \frac{Cov(R_i, R_j)}{\sigma_i \sigma_j}$$

- The correlation is always between -1 and +1. Always has the same sign as covariance.

Covariance

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\end{aligned}
$$

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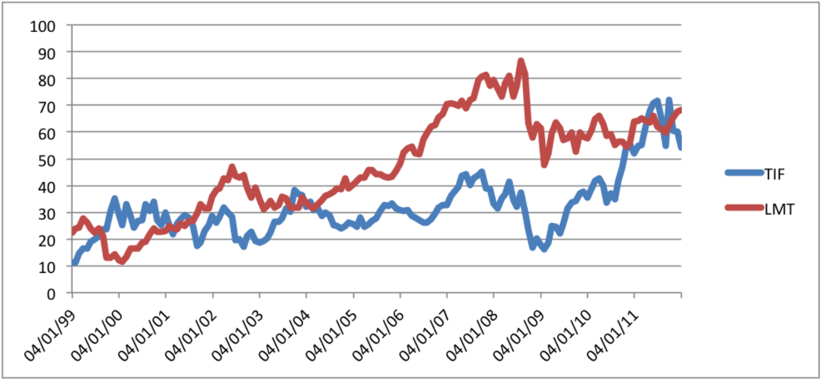
- Negative covariance means assets tend to be unusually low at the same time.

Historical Price Comparison



Summary Statistics (Monthly)

	TIF	LMT	S&P
Arith. Mean	1.70%	0.97%	0.19%
Variance	0.0145	0.0059	0.0021
σ	12.04%	7.70%	4.56%
$\rho(R_i, R_{TIF})$	1		
$\rho(R_i, R_{LMT})$	0.18	1	
$\rho(R_i, R_{SPY})$	0.73	0.23	1

Portfolio

- N assets (with returns $R_1, ..., R_N$)
- Weights $w_1, ..., w_N$ are defined as:

$$w_i = \frac{\text{total \$ value of stock } i\text{'s position}}{\text{total\$ value of all holdings}}$$

- $w_1 + ... + w_N = 1.$
- Negative weights are possible (what do they mean?).

Sample Portfolio

- What if I held 50% in LMT and 50% in TIF?

Sample Portfolio

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- Expected Return is

$$E[R_p] = E[0.5R_L + 0.5R_T] = 0.5E[R_L] + 0.5E[R_T] = 1.335\%$$

Sample Portfolio

- What if I held 50% in LMT and 50% in TIF?
- Expected Return is

$$E[R_p] = E[0.5R_L + 0.5R_T] = 0.5E[R_L] + 0.5E[R_T] = 1.335\%$$

- Volatility is:

$$\begin{aligned} V[R_p] &= V[0.5R_L + 0.5R_T] \\ &= 0.25V[R_L] + 0.25V[R_T] + 0.5\sigma_L\sigma_T\rho_{LT} \\ &= 0.00593 \end{aligned}$$

$\sigma_p = 7.7\%$

One Risky One Riskless

- Suppose I invest w in a risky asset, and $1 - w$ in a risk-free asset?
- Risk-free asset is denoted R_f.

$$\begin{aligned} E[R_f] &= R_f \\ V[R_f] &= 0 \\ Cov[R_f, R_i] &= 0 \end{aligned}$$

- Assume that R_f is constant over time.

One Risky One Riskless

- Expected Return is:

$$\begin{aligned} E[R_p] &= E[wR_i + (1-w)R_f] \\ &= wE[R_i] + (1-w)R_f \\ &= R_f + w\,\overbrace{E[\,\underbrace{R_i - R_f}_{\text{excess return}}\,]}^{\text{risk premium}} \end{aligned}$$

- Variance is:

$$\begin{aligned} V[R_p] &= V[wR_i + (1-w)R_f] \\ &= w^2\sigma_i^2 + (1-w)^2\sigma_f^2 + 2w(1-w)\sigma_i\sigma_f\rho_{if} = w^2\sigma_i^2 \end{aligned}$$

- Standard deviation can be expressed as $\sigma_p = |w|\sigma_i$.

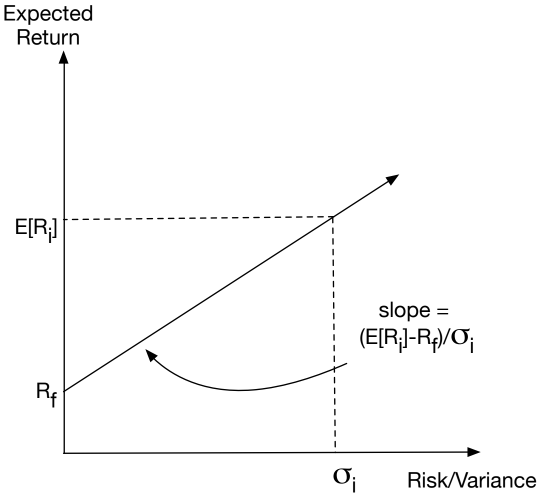
Example 1 - Investment

- Rewriting from the last slide:

$$\begin{aligned} E[R_p] &= R_f + wE[R_i - R_f] \\ &= R_f + \frac{E[R_i - R_f]}{\sigma_i}\sigma_p \\ &= R_f + \text{Sharpe Ratio}\sigma_p \end{aligned}$$

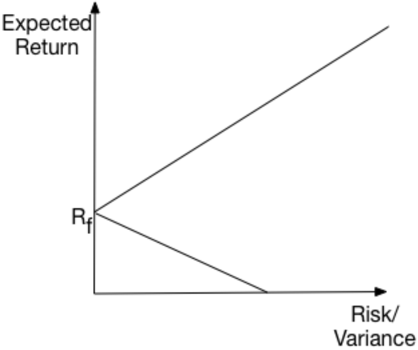
- **The Capital Allocation Line**

Capital Allocation Line



Which point would you choose?

Capital Allocation Line



Which point would you choose?

Capital Allocation Line

- Suppose the risky asset is the US stock market:

$$E[R_{US}] = 12.13\%, \sigma_{US} = 15.98\%$$

- Risk-free is the US T-bill, $R_f = 5\%$.
- Then the Sharpe Ratio is $\frac{0.1213-0.05}{0.1598} = 0.446$
- SR is the return premium per unit of risk.

Capital Allocation Line

- What if I want a 20% return?

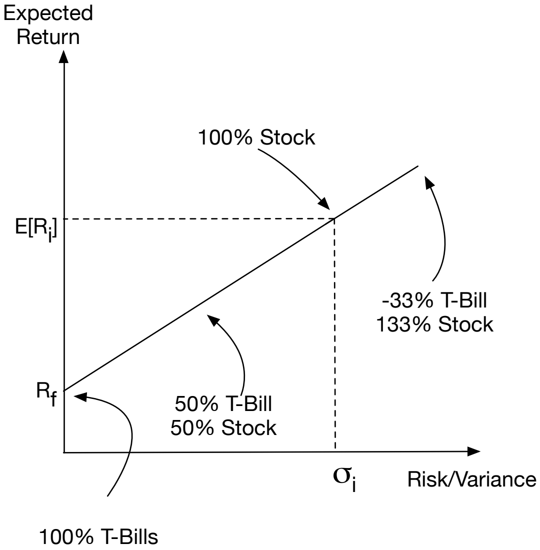
Capital Allocation Line

- What if I want a 20% return?

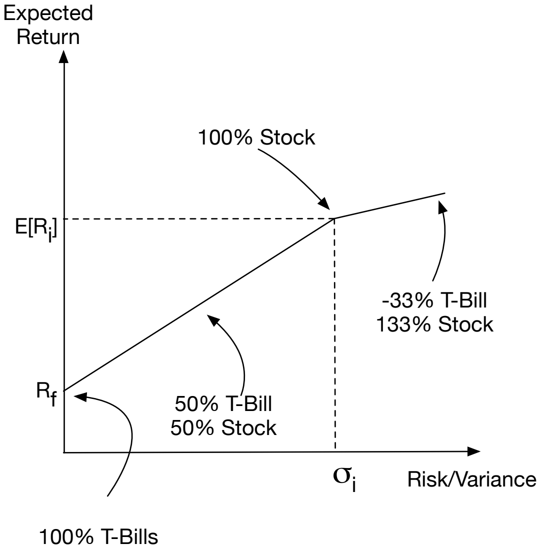
$$E[R_p] = 0.2 = w \cdot 0.1213 + (1 - w) \cdot 0.05$$

- $w = 2.1$, so $1 - w < 0$.
- What about the real world?

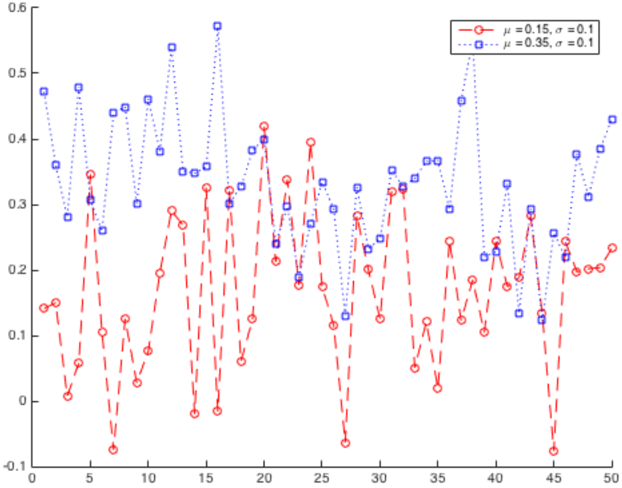
Borrowing


Expected
Return
100% Stock
E[R_i]
-33% T-Bill
133% Stock
R_f
50% T-Bill
50% Stock
σ_i
Risk/Variance
100% T-Bills

Borrowing with Costs



Preferences



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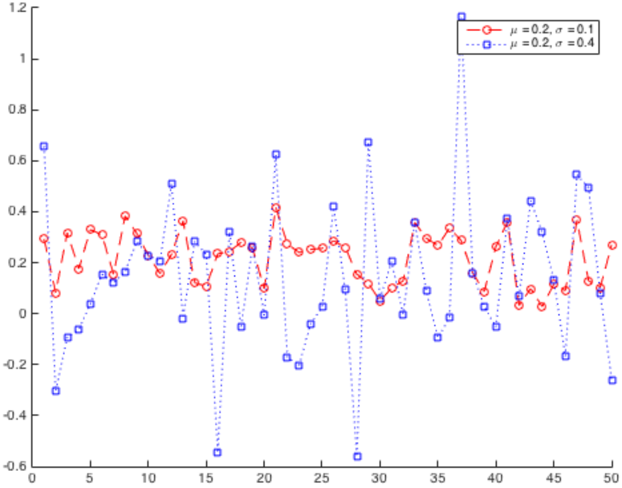
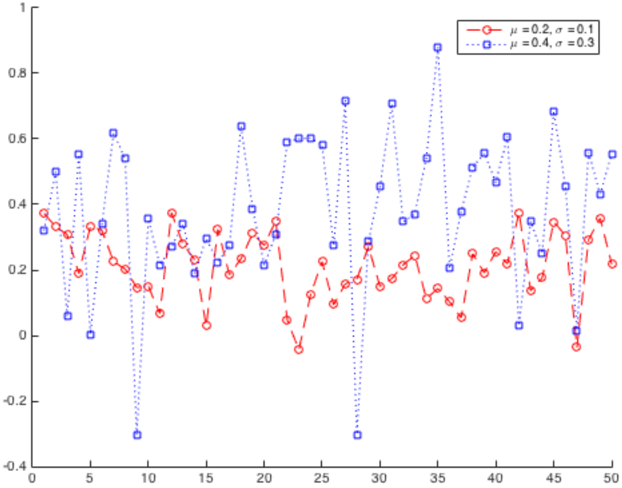


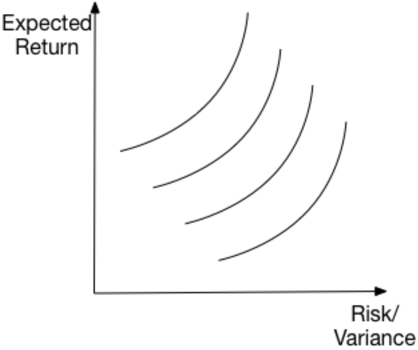
Preferences



Preferences

- Investors prefer more to less.
- Investors do not like risk.
- Want to maximize $E[R_p]$.
- Want to minimize σ_p^2.

Indifference Curves



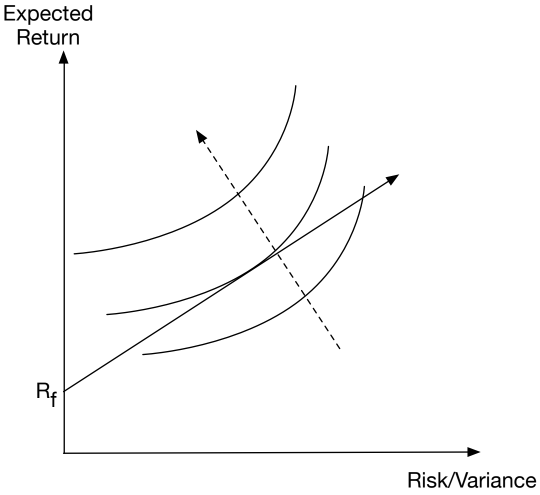
Indifference Curves

- Expresses sets of portfolios that would make investors equally happy.
- They must slope upwards
- They must be convex
- They cannot intersect each other
- Indifference curves increase in value from right to left.

Indifference Curves

- Indifference curves represent what investors want.
- Capital Allocation Line represents what investors *can* get.

Indifference Curves



Utility Function

- General rules for utility functions $U(R_p)$:
  - If $A > B$, then $U(A) > U(B)$.
  - $U'(R) > 0$.
  - $U''(R) < 0$.
- Usually we will use mean-variance utility:

$$E\left[U(R_p)\right] = E(R_p) - 0.5A \cdot V(R_p)$$

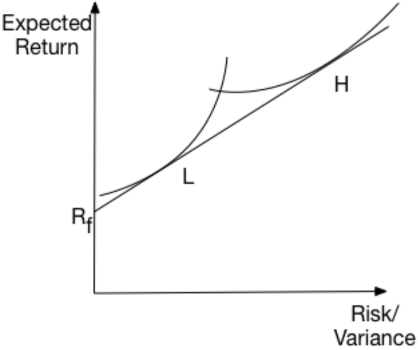
- A measures risk aversion. What happens if $A = 0$? $A < 0$?

Portfolio Choice

Asset	Expected Return	Standard Deviation
Low Risk Stock L	7%	5%
High Risk Stock H	13%	20%

Investor	Investment L	Investment H
$A = 2$	$U = 0.0675$	$U = 0.0638$
$A = 5$	$U = 0.0900$	$U = 0.0300$

Portfolio Choice



Example 2

- What is the risk and return of an optimal portfolio with two risky assets:
- Asset A: $E[A] = 13.6\%$, $V[A] = 15.4\%$.
- Asset B: $E[B] = 15\%$, $V[B] = 23\%$.
- $\rho_{AB} = 27\%$, $w_A = 60\%$, $w_B = 1 - w_A = 40\%$.
- $E[R_p] = 0.6 \cdot 0.136 + 0.4 \cdot 0.15 = 0.142$
- $V[R_p] = 0.6^2 0.154 + 0.4^2 0.23 + 2 \cdot 0.6 \cdot 0.4 \cdot \sqrt{0.154 \cdot 0.23} \cdot 0.27 = 12\%$
- Less risky than either asset.

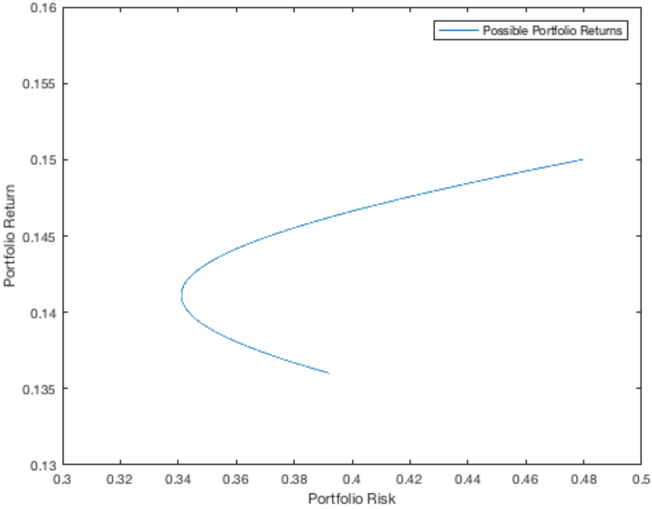
Example 2

- If we generalize to weights w and $1 - w$, we get:

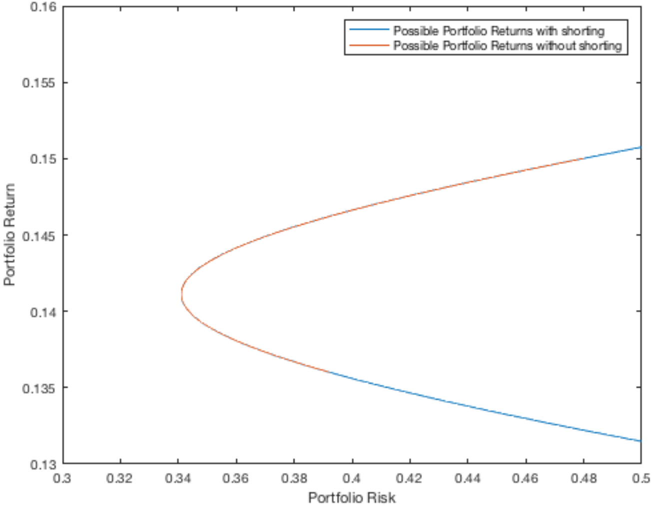
$$V[R_p] = w^2 0.154^2 + (1-w)^2 0.23^2 + 2w(1-w)0.154 \cdot 0.23 \cdot 0.27$$

- Quadratic in w.

Example 2: Possible Portfolios



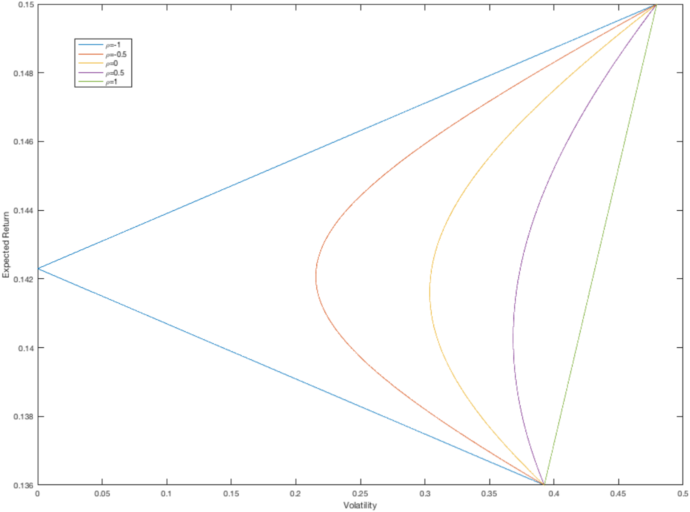
Example 2: Possible Portfolios



Portfolio Terms

- Investment Opportunity Set
- Efficient Portfolio
- Efficient Frontier
- Minimum Variance Portfolio

Different Correlations



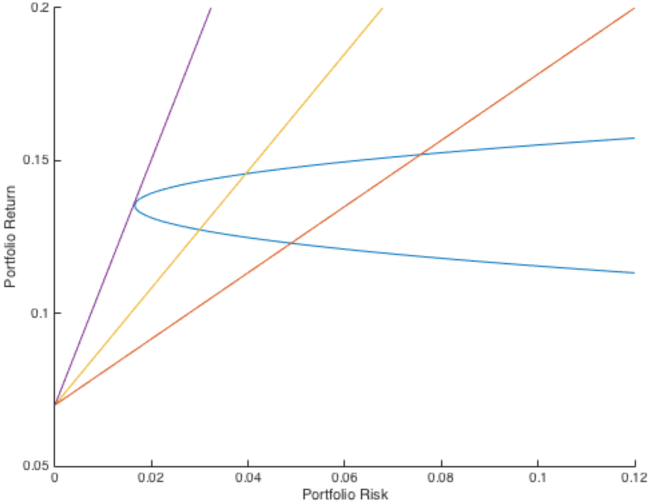
Perfect Positive Correlation

- When $\rho_{AB} = 1$, $\sigma = w_A \sigma_A + (1 - w_A)\sigma_B$
- The set is a straight line.
- When $\rho_{AB} = -1$, $\sigma = |w_A \sigma_A - w_B \sigma_B|$
- The set is two straight lines to the y-axis.

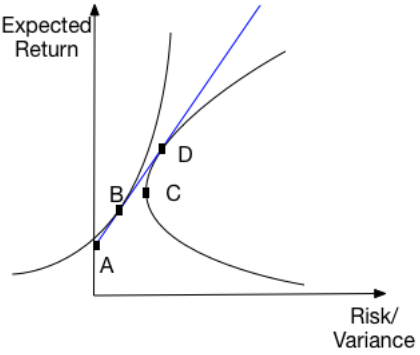
Example 3

- What is the risk and return of an optimal portfolio with two risky assets and a risk free asset:
  - Asset A: $E[A] = 13\%,\ V[A] = 15\%$.
  - Asset B: $E[B] = 15\%,\ V[B] = 23\%$.
  - $E[R_p] = w_A \cdot 0.13 + w_B \cdot 0.15$
  - $V[R_p] = w_A^2 0.15 + w_B^2 0.23 + 2 \cdot w_A \cdot w_B \cdot \sqrt{0.15 \cdot 0.23} \cdot \rho_{AB}$

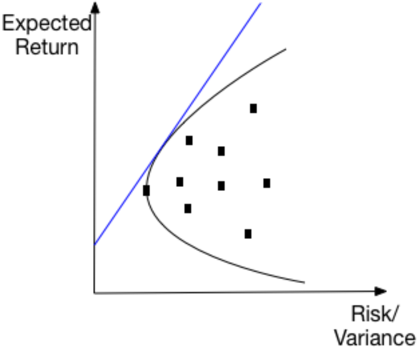
Example 3: Diagram



Example 3: Diagram



Example 3: Many Risky Assets



Special case: Equal-Weighted Portfolios

- Assume $w_i = \frac{1}{N}$ for all i, and that $Cov(R_i, R_j) = 0$ for all i, j.
- Variance becomes:

$$\begin{aligned} \sigma_p^2 &= \frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^2 \\ &= \frac{1}{N} E[\sigma_i^2] \end{aligned}$$

- Risk decreases with the number of assets.

General case: Equal-Weighted Portfolios

- Assume $w_i = \frac{1}{N}$ for all i, but general covariance structure
- Variance becomes:

$$
\begin{aligned}
\sigma_p^2 &= \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2 + \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j>i}^{N} Cov(R_i, R_j) \\
&= \frac{1}{N}E[\sigma_i^2] + \frac{N-1}{N}E[Cov]
\end{aligned}
$$

- Risk decreases with the number of assets.

Special case: Equal-Weighted Portfolios

- Assume that $\sigma_i = 60\%$ for all i.
- Assume that $\rho_{ij} = 0.3$ for all i, j.
- Assume an equally weighted portfolio of N assets.

$$
\begin{aligned}
Cov(R_i, R_j) &= \rho_{ij}\sigma_i\sigma_j = 0.108 \\
V[R_i] &= 0.36 \\
V[R_p] &= \frac{1}{N}0.36 + \frac{N-1}{N}0.108 \\
&\rightarrow 0.108
\end{aligned}
$$

- Standard deviation declines with the number of assets.

Risk Classification

- Diversifiable risk vs. Undiversifiable risk.
- Systemic risk vs. Idiosyncratic risk.
- What do investors need to be compensated for?