Question 1

$$(P \to (Q \wedge R)) \wedge (\neg P \to (\neg Q \wedge \neg R))$$

$$= (\neg P \vee (Q \wedge R)) \wedge (P \vee (\neg Q \wedge \neg R))$$

$$= ((\neg P \vee Q) \wedge (\neg P \vee R)) \wedge ((P \vee \neg Q) \wedge (P \vee \neg R))$$

$$= (\neg P \vee Q) \wedge (\neg P \vee R) \wedge (P \vee \neg Q) \wedge (P \vee \neg R)$$

$$= (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge$$
$$(\neg P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge$$
$$(P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R)$$

$$S = (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge$$
$$(P \vee \neg Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee \neg R)$$

$\uparrow$

PCNF $= \prod M(1, 2, 3, 4, 5, 6)$

$$\neg S = \prod M(0, 7)$$
$$= (P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$$

$$\Rightarrow S = \neg((P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$$

$$= (\neg P \vee \neg Q \vee \neg R) \vee (P \vee Q \vee R)$$

$\uparrow$

PDNF

Question 2

$R \rightarrow \neg Q$

$R \vee S$

$S \rightarrow \neg Q$

$P \rightarrow Q \Rightarrow \neg P$

Proof by contrapositive:
Need to show:

$\neg(\neg P) \Rightarrow \neg(P \rightarrow Q)$

$= P \Rightarrow \neg(\neg P \vee Q)$

$= P \Rightarrow P \wedge \neg Q$

(1)	$R \vee S$	Rule P
(2)	$R \rightarrow \neg Q$	Rule P
(3)	$S \rightarrow \neg Q$	Rule P
(4)	$\neg Q \vee S$	Rule T (Modus ponens 1,2 1&3)
(5)	$\neg Q \vee \neg Q$	Rule T (Modus ponens 3,4)
(6)	$\neg Q$	Rule T (5)
(7)	P	Rule P
(8)	$P \wedge \neg Q$	Rule T (6,7)

Question 3

$$(\exists x)(F(x) \wedge S(x)) \to (y)(M(y) \to W(y))$$

$$\exists y (M(y) \wedge \neg W(y))$$

conclusion : $x(F(x) \to \neg S(x))$

(1) $\exists x (F(x) \wedge S(x)) \to (y)(M(y) \to W(y))$ Rule P

(2) $x(F(x) \wedge S(x)) \to y(M(y) \to W(y))$ Existential generalisation (1)

(3) $\neg(y(M(y) \to W(y)) \to \neg(x(F(x) \wedge S(x))$ Contrapositive (2) Rule T

(4) $y(M(y) \wedge \neg W(y)) \to x(\neg F(x) \vee \neg S(x))$ Rule T (3)

(5) $y(M(y) \wedge \neg W(y)) \to x(\cancel{\neg} F(x) \to \neg S(x))$ Rule T (4)

(6) $\exists y (M(y) \wedge \neg W(y))$ Rule P

(7) $y(M(y) \wedge \neg W(y))$ Existential generalisation (6)

11) $x(F(x) \to \neg S(x))$ Rule T (Modus ponens 5, 7)

Question 4.a.

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{R}$$

Closure property:

let $\begin{bmatrix} w & x \\ y & z \end{bmatrix} \in S$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$= \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$ $\in S$, since scalar multiplication and addition b/w $\mathbb{R} = \mathbb{R}$

Identity property:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\therefore$ For any a, b, c, d, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

always gives the same result

$\therefore$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity element of S

Inverse property:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \cancel{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$$

$\text{Det} = ad - bc$

$$\text{inverse} = \frac{1}{\text{Det}} \cdot \text{Adjunct} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\therefore$ Det can equal 0

$\therefore$ Inverse property not satisfied

$\therefore$ S is not a group under the inverse property

Question 4. b

$E = \{2x \mid x \geq 0\}$

$\langle E, + \rangle$

Closure:

Let $a, b \in E$

$a = 2m$, $b = 2n$, $m, n \in Q^+$

$a + b$

$= 2m + 2n$

$= 2(m + n)$

$m + n \in Q^+$

$\therefore 2(m+n) \in E$

$= a + b \in E$

Associative:

Let $a, b, c \in E$

$(a+b)+c$

$= a + b + c$

$= a + (b+c)$

$\therefore (a+b)+c = a+(b+c)$

Identity:

The additive identity is 0

$\therefore 0 \in E$

$\therefore$ Identity satisfied

$\therefore \langle E, + \rangle$ is a semigroup and monoid

$\langle E, * \rangle$ $\quad * =$ multiplication

Closure

Let $a, b \in E$

$a = 2m, \ b = 2n \quad m, n \in Q^+$

$a \times b$

$= 2m \times 2n$

$= 4mn$

$= 2 \times 2mn$

$\therefore 2 \times 2mn \in E$

$\Rightarrow a \times b \in E$

Associative:

Let $a, b \in E$

$(a \times b) \times c$

$= a \times b \times c$

$= a \times (b \times c)$

Identity:

Let $a, b \in E$

Identity of an element a under standard multiplication $= 1$

$a \times 1 = a$

But $1 \notin E$

$\therefore E$ is a semi-group but not a monoid

Question 5.a

with Q^+

$\forall x \, (y > x \mid x, y \in Q^+)$

without Q^+

$\forall x \, (y > x \mid y, x \in \mathbb{R})$

Question 5.b.

$P \rightarrow (\neg P \rightarrow Q)$

$\Rightarrow P \rightarrow (P \vee Q)$

$\Rightarrow P \rightarrow \neg(\neg P \wedge \neg Q)$

$\Rightarrow P \rightarrow (\neg P \uparrow \neg Q)$

$\Rightarrow \neg P \vee (\neg P \uparrow \neg Q)$

$\Rightarrow \neg(P \wedge \neg(\neg P \uparrow \neg Q))$

$\Rightarrow P \uparrow \neg(\neg P \uparrow \neg Q)$

NAND

$P \rightarrow (\neg P \rightarrow Q)$

$\Rightarrow P \rightarrow (P \vee Q)$

$\Rightarrow P \rightarrow \neg(P \downarrow Q)$

$\Rightarrow \neg P \vee \neg(P \downarrow Q)$

$\Rightarrow \neg\neg(\neg P \vee \neg(P \downarrow Q))$

$\Rightarrow \neg(\neg P \downarrow \neg(P \downarrow Q))$

NOR