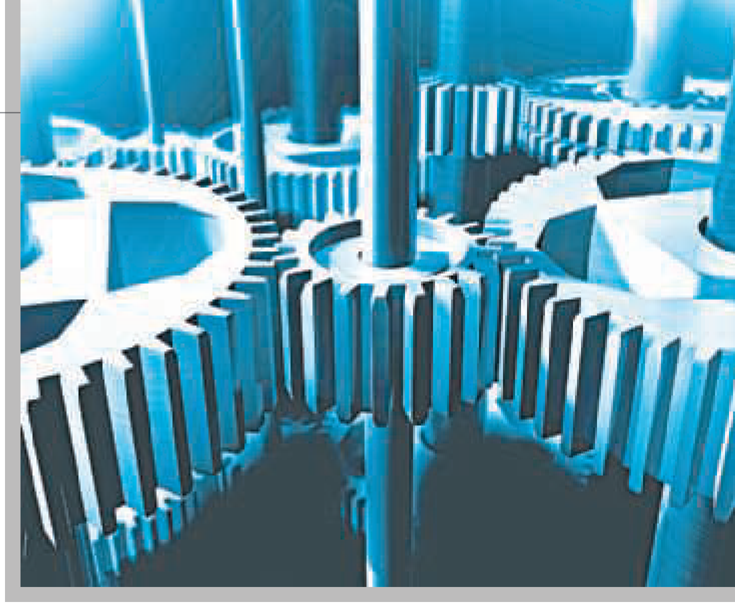
4

Applications of Derivatives



OVERVIEW One of the most important applications of the derivative is its use as a tool for finding the optimal (best) solutions to problems. Optimization problems abound in mathematics, physical science and engineering, business and economics, and biology and medicine. For example, what are the height and diameter of the cylinder of largest volume that can be inscribed in a given sphere? What are the dimensions of the strongest rectangular wooden beam that can be cut from a cylindrical log of given diameter? Based on production costs and sales revenue, how many items should a manufacturer produce to maximize profit? How much does the trachea (windpipe) contract to expel air at maximum speed during a cough? What is the branching angle at which blood vessels minimize the energy loss due to friction as blood flows through the branches?

In this chapter we apply derivatives to find extreme values of functions, to determine and analyze the shapes of graphs, and to solve equations numerically. We also introduce the idea of recovering a function from its derivative. The key to many of these applications is the Mean Value Theorem, which connects the derivative and the average change of a function.

4.1 Extreme Values of Functions on Closed Intervals

This section shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative. Once we can do this, we can solve a variety of optimization problems (see Section 4.5). The domains of the functions we consider are intervals or unions of separate intervals.

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \le f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \ge f(c) \qquad \text{for all } x \text{ in } D.$$

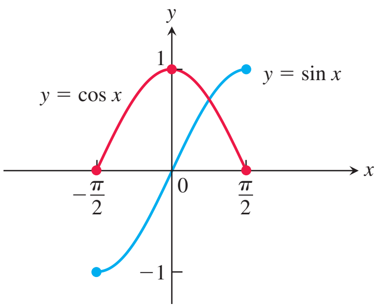


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Maximum and minimum values are called **extreme values** of the function f. Absolute maxima or minima are also referred to as **global** maxima or minima.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.1).

Functions defined by the same equation or formula can have different extrema (maximum or minimum values), depending on the domain. A function might not have a maximum or minimum if the domain is unbounded or fails to contain an endpoint. We see this in the following example.

EXAMPLE 1 The absolute extrema of the following functions on their domains can be seen in Figure 4.2. Each function has the same defining equation, $y = x^2$, but the domains vary.

Function rule	Domain D	Absolute extrema on D
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
(d) $y = x^2$	$(0, 2)$	No absolute extrema

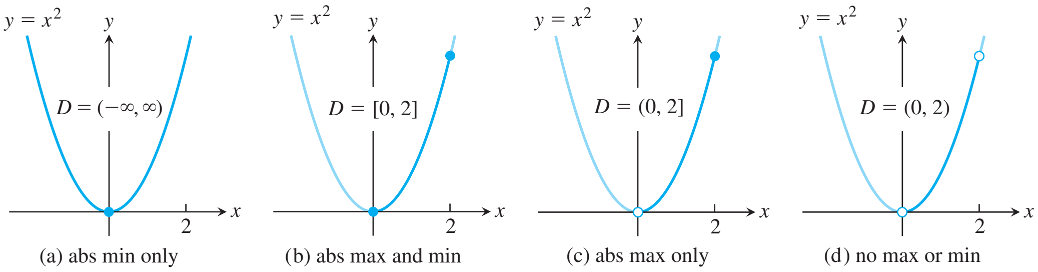


FIGURE 4.2 Graphs for Example 1.

HISTORICAL BIOGRAPHY
Daniel Bernoulli
(1700–1789)
www.goo.gl/JYed90

Some of the functions in Example 1 do not have a maximum or a minimum value. The following theorem asserts that a function which is *continuous* over (or on) a finite *closed* interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval. We look for these extreme values when we graph a function.

THEOREM 1—The Extreme Value Theorem
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.

The proof of the Extreme Value Theorem requires a detailed knowledge of the real number system (see Appendix 7) and we will not give it here. Figure 4.3 illustrates possible locations for the absolute extrema of a continuous function on a closed interval $[a, b]$. As we observed for the function $y = \cos x$, it is possible that an absolute minimum (or absolute maximum) may occur at two or more different points of the interval.

The requirements in Theorem 1 that the interval be closed and finite, and that the function be continuous, are essential. Without them, the conclusion of the theorem need not hold. Example 1 shows that an absolute extreme value may not exist if the interval fails

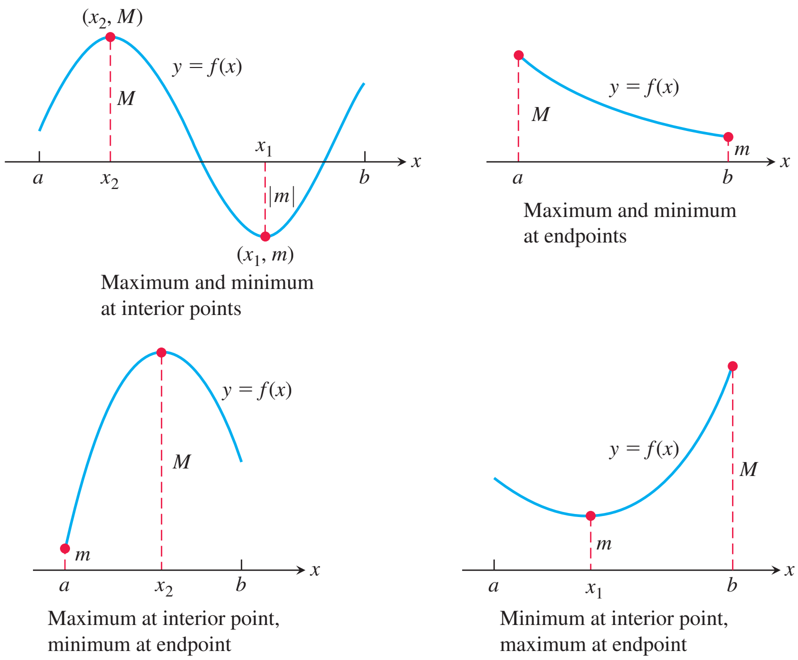


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

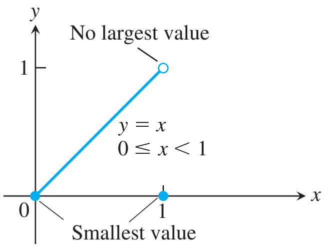


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or a minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of $[0, 1]$ except $x = 1$, yet its graph over $[0, 1]$ does not have a highest point.

to be both closed and finite. The function $y = x$ over $(-\infty, \infty)$ shows that neither extreme value need exist on an infinite interval. Figure 4.4 shows that the continuity requirement cannot be omitted.

Local (Relative) Extreme Values

Figure 4.5 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point *nearby*. The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d. We now define what we mean by local extrema.

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \le f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \ge f(c)$ for all $x \in D$ lying in some open interval containing c.

If the domain of f is the closed interval $[a, b]$, then f has a local maximum at the endpoint $x = a$ if $f(x) \le f(a)$ for all x in some half-open interval $[a, a + \delta)$, $\delta > 0$. Likewise, f has a local maximum at an interior point $x = c$ if $f(x) \le f(c)$ for all x in some open interval $(c - \delta, c + \delta)$, $\delta > 0$, and a local maximum at the endpoint $x = b$ if $f(x) \le f(b)$ for all x in some half-open interval $(b - \delta, b]$, $\delta > 0$. The inequalities are reversed for local minimum values. In Figure 4.5, the function f has local maxima at c and d and local minima at a, e, and b. Local extrema are also called **relative extrema**. Some functions can have infinitely many local extrema, even over a finite interval. One example is the function $f(x) = \sin(1/x)$ on the interval $(0, 1]$. (We graphed this function in Figure 2.40.)

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will*

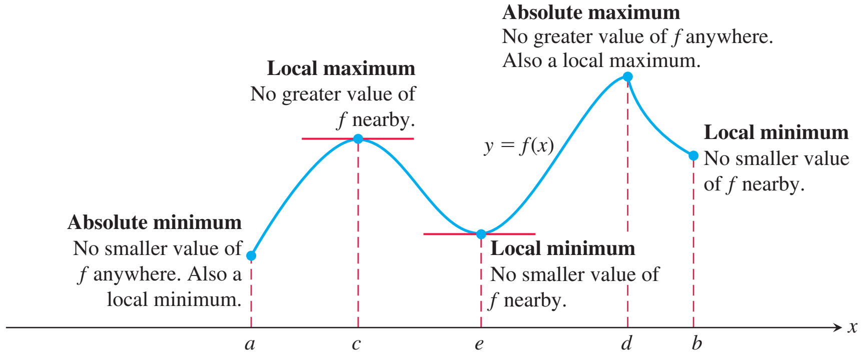


FIGURE 4.5 How to identify types of maxima and minima for a function with domain $a \le x \le b$.

automatically include the absolute maximum if there is one. Similarly, *a list of all local minima will include the absolute minimum if there is one*.

Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

> **THEOREM 2—The First Derivative Theorem for Local Extreme Values**
> If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then
> $$f'(c) = 0.$$

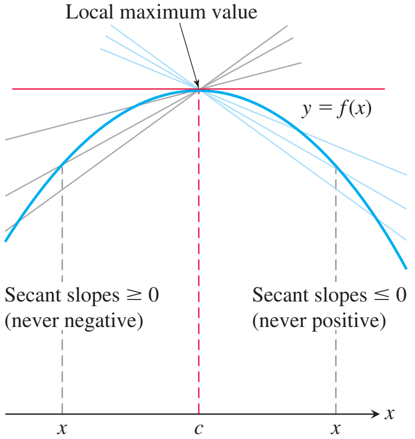


FIGURE 4.6 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

Proof To prove that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ (Figure 4.6) so that $f(x) - f(c) \le 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0. \quad \text{Because } (x - c) > 0 \text{ and } f(x) \le f(c) \tag{1}$$

Similarly,

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0. \quad \text{Because } (x - c) < 0 \text{ and } f(x) \le f(c) \tag{2}$$

Together, Equations (1) and (2) imply $f'(c) = 0$.

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \ge f(c)$, which reverses the inequalities in Equations (1) and (2). ■

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. If we recall that all the domains we consider are intervals or unions of separate intervals, the only places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$, At $x = c$ and $x = e$ in Fig. 4.5
2. interior points where f' is undefined, At $x = d$ in Fig. 4.5
3. endpoints of the domain of f. At $x = a$ and $x = b$ in Fig. 4.5

The following definition helps us to summarize these results.

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

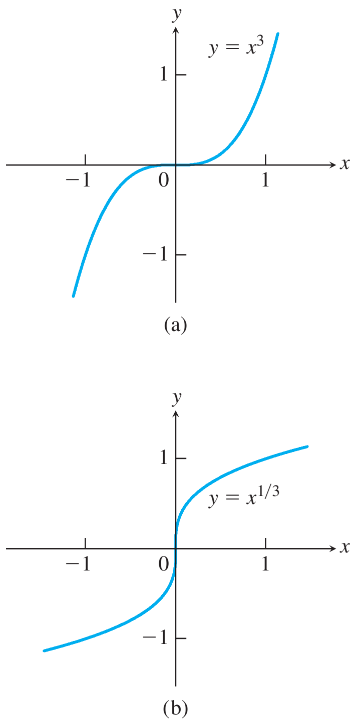


FIGURE 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

Thus the only domain points where a function can assume extreme values are critical points and endpoints. However, be careful not to misinterpret what is being said here. A function may have a critical point at $x = c$ without having a local extreme value there. For instance, both of the functions $y = x^3$ and $y = x^{1/3}$ have critical points at the origin, but neither function has a local extreme value at the origin. Instead, each function has a *point of inflection* there (see Figure 4.7). We define and explore inflection points in Section 4.4.

Most problems that ask for extreme values call for finding the extrema of a continuous function on a closed and finite interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints. Often we can simply list these points and calculate the corresponding function values to find what the largest and smallest values are, and where they are located. However, if the interval is not closed or not finite (such as $a < x < b$ or $a < x < \infty$), we have seen that absolute extrema need not exist. When an absolute maximum or minimum value does exist, it must occur at a critical point or at a right- or left-hand endpoint of the interval.

Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Find all critical points of f on the interval.
2. Evaluate f at all critical points and endpoints.
3. Take the largest and smallest of these values.

EXAMPLE 2 Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point occurs where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = 4$

$f(1) = 1.$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ■

EXAMPLE 3 Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

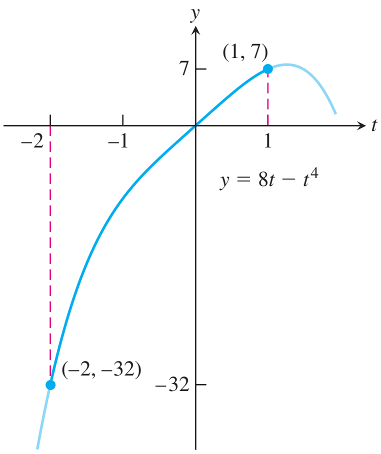


FIGURE 4.8 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 3).

Solution The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$8 - 4t^3 = 0 \qquad \text{or} \qquad t = \sqrt[3]{2} > 1,$$

a point not in the given domain. The function's absolute extrema therefore occur at the endpoints, $g(-2) = -32$ (absolute minimum), and $g(1) = 7$ (absolute maximum). See Figure 4.8. ■

EXAMPLE 4 Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

$$\begin{aligned} &\text{Critical point value:} && f(0) = 0 \\ &\text{Endpoint values:} && f(-2) = (-2)^{2/3} = \sqrt[3]{4} \\ & && f(3) = (3)^{2/3} = \sqrt[3]{9}. \end{aligned}$$

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$ where the graph has a cusp (Figure 4.9). ■

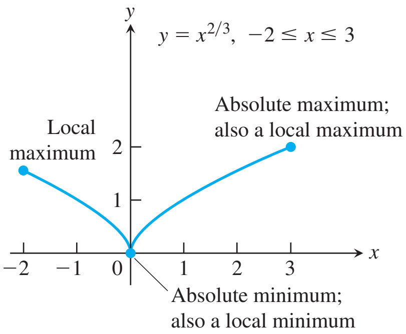


FIGURE 4.9 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

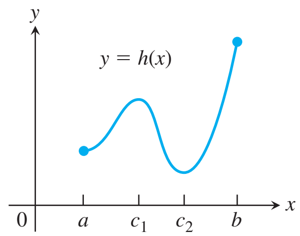
Theorem 1 gives a method to find the absolute maxima and absolute minima of a differentiable function on a finite closed interval. On more general domains, such as $(0, 1)$, $[2, 5)$, $[1, \infty)$, and $(-\infty, \infty)$, absolute maxima and minima may or may not exist. To determine if they exist, and to locate them when they do, we will develop methods to sketch the graph of a differentiable function. With knowledge of the asymptotes of the function, as well as the local maxima and minima, we can deduce the locations of the absolute maxima and minima, if any. For now we can find the absolute maxima and the absolute minima of a function on a finite closed interval by comparing the values of the function at its critical points and at the endpoints of the interval. For a differentiable function, these are the only points where the extrema have the potential to occur.

EXERCISES 4.1

Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Theorem 1.

1.



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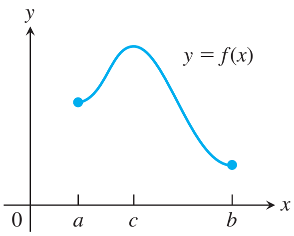

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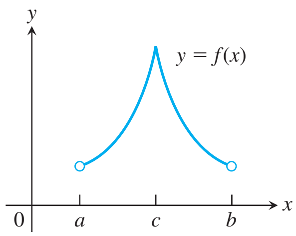

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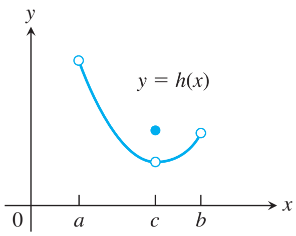

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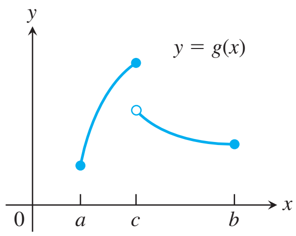

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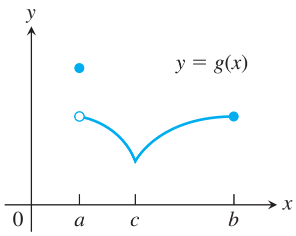
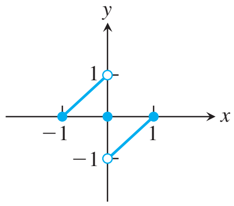


In Exercises 7–10, find the absolute extreme values and where they occur.

7.



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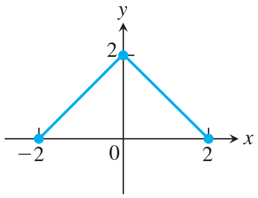

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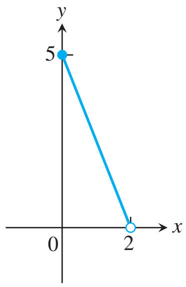

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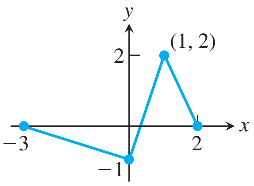


In Exercises 11–14, match the table with a graph.

11.

x	$f'(x)$
a	0
b	0
c	5

12.

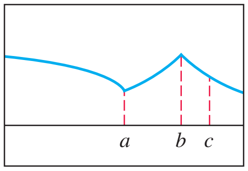
x	$f'(x)$
a	0
b	0
c	-5

13.

x	$f'(x)$
a	does not exist
b	0
c	-2

14.

x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7



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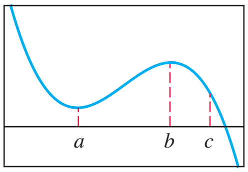

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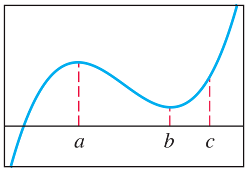
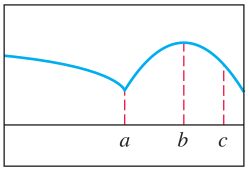


(c)



(d)

In Exercises 15–20, sketch the graph of each function and determine whether the function has any absolute extreme values on its domain. Explain how your answer is consistent with Theorem 1.

15. $f(x) = |x|, \quad -1 < x < 2$

16. $y = \dfrac{6}{x^2 + 2}, \quad -1 < x < 1$

17. $g(x) = \begin{cases} -x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 2 \end{cases}$

18. $h(x) = \begin{cases} \dfrac{1}{x}, & -1 \le x < 0 \\ \sqrt{x}, & 0 \le x \le 4 \end{cases}$

19. $y = 3 \sin x, \quad 0 < x < 2\pi$

20. $f(x) = \begin{cases} x + 1, & -1 \le x < 0 \\ \cos x, & 0 < x \le \dfrac{\pi}{2} \end{cases}$

Absolute Extrema on Finite Closed Intervals

In Exercises 21–36, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

21. $f(x) = \frac{2}{3}x - 5, \quad -2 \le x \le 3$

22. $f(x) = -x - 4, \quad -4 \le x \le 1$

23. $f(x) = x^2 - 1, \quad -1 \le x \le 2$

24. $f(x) = 4 - x^3, \quad -2 \le x \le 1$

25. $F(x) = -\dfrac{1}{x^2}, \quad 0.5 \le x \le 2$

26. $F(x) = -\dfrac{1}{x}, \quad -2 \le x \le -1$

27. $h(x) = \sqrt[3]{x}, \quad -1 \le x \le 8$

28. $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

29. $g(x) = \sqrt{4 - x^2}, \quad -2 \le x \le 1$

30. $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$

31. $f(\theta) = \sin \theta, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{5\pi}{6}$

32. $f(\theta) = \tan \theta, \quad -\dfrac{\pi}{3} \le \theta \le \dfrac{\pi}{4}$

33. $g(x) = \csc x, \quad \dfrac{\pi}{3} \le x \le \dfrac{2\pi}{3}$

34. $g(x) = \sec x, \quad -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{6}$

35. $f(t) = 2 - |t|, \quad -1 \le t \le 3$

36. $f(t) = |t - 5|, \quad 4 \le t \le 7$

In Exercises 37–40, find the function's absolute maximum and minimum values and say where they occur.

37. $f(x) = x^{4/3}, \quad -1 \le x \le 8$

38. $f(x) = x^{5/3}, \quad -1 \le x \le 8$

39. $g(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

40. $h(\theta) = 3\theta^{2/3}, \quad -27 \le \theta \le 8$

Finding Critical Points

In Exercises 41–50, determine all critical points for each function.

41. $y = x^2 - 6x + 7$

42. $f(x) = 6x^2 - x^3$

43. $f(x) = x(4 - x)^3$

44. $g(x) = (x - 1)^2(x - 3)^2$

45. $y = x^2 + \dfrac{2}{x}$

46. $f(x) = \dfrac{x^2}{x - 2}$

47. $y = x^2 - 32\sqrt{x}$

48. $g(x) = \sqrt{2x - x^2}$

49. $y = x^3 + 3x^2 - 24x + 7$

50. $y = x - 3x^{2/3}$

Local Extrema and Critical Points

In Exercises 51–58, find the critical points and domain endpoints for each function. Then find the value of the function at each of these points and identify extreme values (absolute and local).

51. $y = x^{2/3}(x + 2)$

52. $y = x^{2/3}(x^2 - 4)$

53. $y = x\sqrt{4 - x^2}$

54. $y = x^2\sqrt{3 - x}$

55. $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

56. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

57. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

58. $y = \begin{cases} -\dfrac{1}{4}x^2 - \dfrac{1}{2}x + \dfrac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

In Exercises 59 and 60, give reasons for your answers.

59. Let $f(x) = (x - 2)^{2/3}$.

a. Does $f'(2)$ exist?

b. Show that the only local extreme value of f occurs at $x = 2$.

c. Does the result in part (b) contradict the Extreme Value Theorem?

d. Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by a.

60. Let $f(x) = |x^3 - 9x|$.

a. Does $f'(0)$ exist?

b. Does $f'(3)$ exist?

c. Does $f'(-3)$ exist?

d. Determine all extrema of f.

In Exercises 61–62, show that the function has neither an absolute minimum nor an absolute maximum on its natural domain.

61. $y = x^{11} + x^3 + x - 5$

62. $y = 3x + \tan x$

Theory and Examples

63. A minimum with no derivative The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though f is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.

64. Even functions If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of f at $x = -c$? Give reasons for your answer.

65. Odd functions If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of g at $x = -c$? Give reasons for your answer.

66. No critical points or endpoints exist We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

67. The function

$$V(x) = x(10 - 2x)(16 - 2x), \qquad 0 < x < 5,$$

models the volume of a box.

a. Find the extreme values of V.

b. Interpret any values found in part (a) in terms of the volume of the box.

68. Cubic functions Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

a. Show that f can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.

b. How many local extreme values can f have?

69. Maximum height of a vertically moving body The height of a body moving vertically is given by

$$s = -\frac{1}{2}gt^2 + v_0t + s_0, \qquad g > 0,$$

with s in meters and t in seconds. Find the body's maximum height.

70. Peak alternating current Suppose that at any given time t (in seconds) the current i (in amperes) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak current for this circuit (largest magnitude)?

T Graph the functions in Exercises 71–74. Then find the extreme values of the function on the interval and say where they occur.

71. $f(x) = |x - 2| + |x + 3|, \quad -5 \le x \le 5$

72. $g(x) = |x - 1| - |x - 5|, \quad -2 \le x \le 7$

73. $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

74. $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

COMPUTER EXPLORATIONS

In Exercises 75–80, you will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps.

a. Plot the function over the interval to see its general behavior there.

b. Find the interior points where $f' = 0$. (In some exercises, you may have to use the numerical equation solver to approximate a solution.) You may want to plot f' as well.

c. Find the interior points where f' does not exist.

d. Evaluate the function at all points found in parts (b) and (c) and at the endpoints of the interval.

e. Find the function's absolute extreme values on the interval and identify where they occur.

75. $f(x) = x^4 - 8x^2 + 4x + 2, \quad [-20/25, 64/25]$

76. $f(x) = -x^4 + 4x^3 - 4x + 1, \quad [-3/4, 3]$

77. $f(x) = x^{2/3}(3 - x), \quad [-2, 2]$

78. $f(x) = 2 + 2x - 3x^{2/3}, \quad [-1, 10/3]$

79. $f(x) = \sqrt{x} + \cos x, \quad [0, 2\pi]$

80. $f(x) = x^{3/4} - \sin x + \dfrac{1}{2}, \quad [0, 2\pi]$

4.2 The Mean Value Theorem

We know that constant functions have zero derivatives, but could there be a more complicated function whose derivative is always zero? If two functions have identical derivatives over an interval, how are the functions related? We answer these and other questions in this chapter by applying the Mean Value Theorem. First we introduce a special case, known as Rolle's Theorem, which is used to prove the Mean Value Theorem.

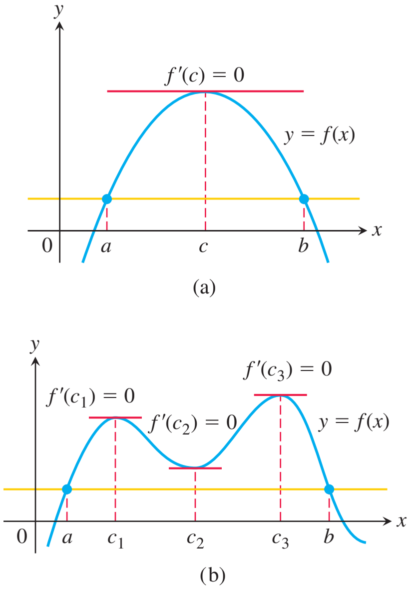


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

HISTORICAL BIOGRAPHY
Michel Rolle
(1652–1719)
www.goo.gl/BfgcNr

Rolle's Theorem

As suggested by its graph, if a differentiable function crosses a horizontal line at two different points, there is at least one point between them where the tangent to the graph is horizontal and the derivative is zero (Figure 4.10). We now state and prove this result.

THEOREM 3—Rolle's Theorem
Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Proof Being continuous, f assumes absolute maximum and minimum values on $[a, b]$ by Theorem 1. These can occur only

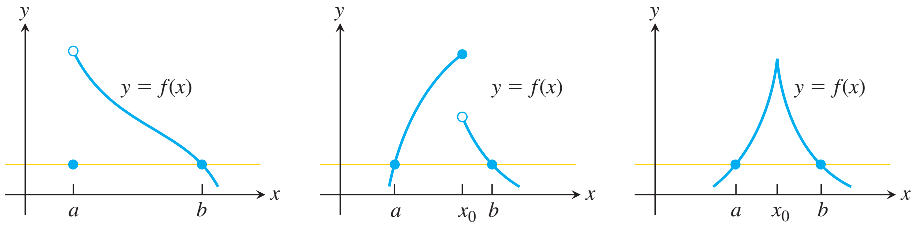
1. at interior points where f' is zero,
2. at interior points where f' does not exist,
3. at endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.

If either the maximum or the minimum occurs at a point c between a and b, then $f'(c) = 0$ by Theorem 2 in Section 4.1, and we have found a point for Rolle's Theorem.

If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b). ■

The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Figure 4.11).



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

FIGURE 4.11 There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

Rolle's Theorem may be combined with the Intermediate Value Theorem to show when there is only one real solution of an equation $f(x) = 0$, as we illustrate in the next example.

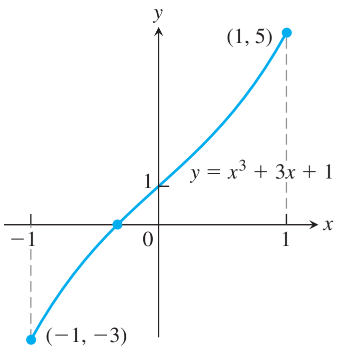


FIGURE 4.12 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0 (Example 1).

EXAMPLE 1 Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Solution We define the continuous function

$$f(x) = x^3 + 3x + 1.$$

Since $f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem tells us that the graph of f crosses the x-axis somewhere in the open interval $(-1, 0)$. (See Figure 4.12.) Now, if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point $x = c$ in between them where f' was zero. However, the derivative

$$f'(x) = 3x^2 + 3$$

is never zero (because it is always positive). Therefore, f has no more than one zero. ■

Our main use of Rolle's Theorem is in proving the Mean Value Theorem.

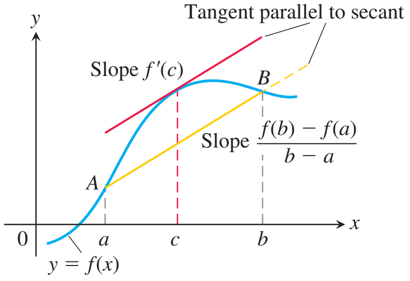


FIGURE 4.13 Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent line parallel to the secant line that joins A and B.

The Mean Value Theorem

The Mean Value Theorem, which was first stated by Joseph-Louis Lagrange, is a slanted version of Rolle's Theorem (Figure 4.13). The Mean Value Theorem guarantees that there is a point where the tangent line is parallel to the secant line that joins A and B.

HISTORICAL BIOGRAPHY
Joseph-Louis Lagrange
(1736–1813)
www.goo.gl/WLub9z

THEOREM 4—The Mean Value Theorem
Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$

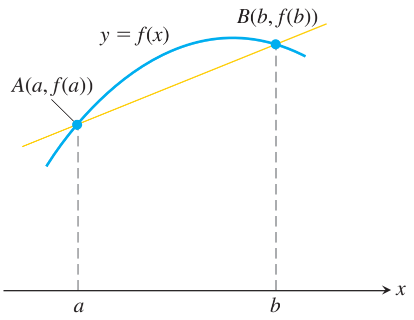


FIGURE 4.14 The graph of f and the secant AB over the interval $[a, b]$.

Proof We picture the graph of f and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$. (See Figure 4.14.) The secant line is the graph of the function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \qquad (2)$$

(point-slope equation). The vertical difference between the graphs of f and g at x is

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned} \qquad (3)$$

Figure 4.15 shows the graphs of f, g, and h together.

The function h satisfies the hypotheses of Rolle's Theorem on $[a, b]$. It is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also, $h(a) = h(b) = 0$ because the graphs of f and g both pass through A and B. Therefore $h'(c) = 0$ at some point $c \in (a, b)$. This is the point we want for Equation (1) in the theorem.

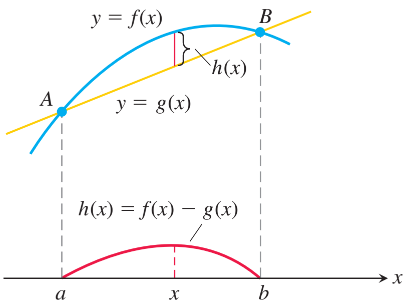


FIGURE 4.15 The secant AB is the graph of the function $g(x)$. The function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x.

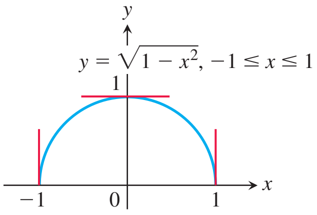


FIGURE 4.16 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.

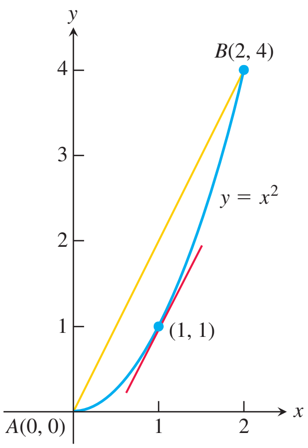


FIGURE 4.17 As we find in Example 2, $c = 1$ is where the tangent is parallel to the secant line.

To verify Equation (1), we differentiate both sides of Equation (3) with respect to x and then set $x = c$:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \qquad \text{Derivative of Eq. (3)}$$

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \text{Evaluated at } x = c$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad h'(c) = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \qquad \text{Rearranged}$$

which is what we set out to prove. ■

The hypotheses of the Mean Value Theorem do not require f to be differentiable at either a or b. One-sided continuity at a and b is enough (Figure 4.16).

EXAMPLE 2 The function $f(x) = x^2$ (Figure 4.17) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists. ■

A Physical Interpretation

We can think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change. Then the Mean Value Theorem says that the instantaneous change at some interior point is equal to the average change over the entire interval.

EXAMPLE 3 If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. The Mean Value Theorem says that at some point during the acceleration the speedometer must read exactly 30 mph (44 ft/sec) (Figure 4.18). ■

Mathematical Consequences

At the beginning of the section, we asked what kind of function has a zero derivative over an interval. The first corollary of the Mean Value Theorem provides the answer that only constant functions have zero derivatives.

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Proof We want to show that f has a constant value on the interval (a, b). We do so by showing that if x_1 and x_2 are any two points in (a, b) with $x_1 < x_2$, then $f(x_1) = f(x_2)$. Now f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$: It is differentiable at every point of $[x_1, x_2]$ and hence continuous at every point as well. Therefore,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

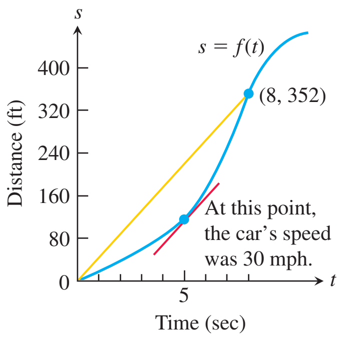


FIGURE 4.18 Distance versus elapsed time for the car in Example 3.

at some point c between x_1 and x_2. Since $f' = 0$ throughout (a, b), this equation implies successively that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0, \qquad f(x_2) - f(x_1) = 0, \qquad \text{and} \qquad f(x_1) = f(x_2). \qquad \blacksquare$$

At the beginning of this section, we also asked about the relationship between two functions that have identical derivatives over an interval. The next corollary tells us that their values on the interval have a constant difference.

COROLLARY 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

Proof At each point $x \in (a, b)$ the derivative of the difference function $h = f - g$ is

$$h'(x) = f'(x) - g'(x) = 0.$$

Thus, $h(x) = C$ on (a, b) by Corollary 1. That is, $f(x) - g(x) = C$ on (a, b), so $f(x) = g(x) + C$. $\blacksquare$

Corollaries 1 and 2 are also true if the open interval (a, b) fails to be finite. That is, they remain true if the interval is (a, ∞), $(-\infty, b)$, or $(-\infty, \infty)$.

Corollary 2 will play an important role when we discuss antiderivatives in Section 4.7. It tells us, for instance, that since the derivative of $f(x) = x^2$ on $(-\infty, \infty)$ is $2x$, any other function with derivative $2x$ on $(-\infty, \infty)$ must have the formula $x^2 + C$ for some value of C (Figure 4.19).

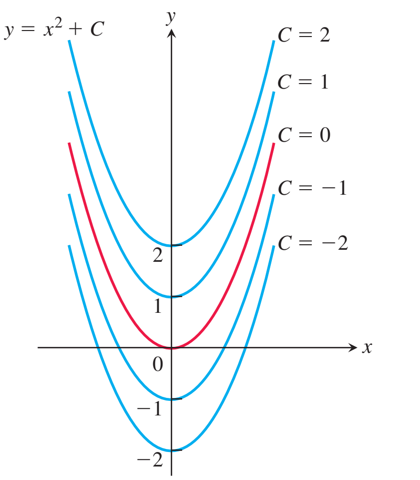


FIGURE 4.19 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for several values of C.

EXAMPLE 4 Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

Solution Since the derivative of $g(x) = -\cos x$ is $g'(x) = \sin x$, we see that f and g have the same derivative. Corollary 2 then says that $f(x) = -\cos x + C$ for some constant C. Since the graph of f passes through the point $(0, 2)$, the value of C is determined from the condition that $f(0) = 2$:

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The function is $f(x) = -\cos x + 3$. $\blacksquare$

Finding Velocity and Position from Acceleration

We can use Corollary 2 to find the velocity and position functions of an object moving along a vertical line. Assume the object or body is falling freely from rest with acceleration 9.8 m/sec^2. We assume the position $s(t)$ of the body is measured positive downward from the rest position (so the vertical coordinate line points *downward*, in the direction of the motion, with the rest position at 0).

We know that the velocity $v(t)$ is some function whose derivative is 9.8. We also know that the derivative of $g(t) = 9.8t$ is 9.8. By Corollary 2,

$$v(t) = 9.8t + C$$

for some constant C. Since the body falls from rest, $v(0) = 0$. Thus

$$9.8(0) + C = 0, \qquad \text{and} \qquad C = 0.$$

The velocity function must be $v(t) = 9.8t$. What about the position function $s(t)$?

We know that $s(t)$ is some function whose derivative is $9.8t$. We also know that the derivative of $f(t) = 4.9t^2$ is $9.8t$. By Corollary 2,

$$s(t) = 4.9t^2 + C$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + C = 0, \qquad \text{and} \qquad C = 0.$$

The position function is $s(t) = 4.9t^2$ until the body hits the ground.

The ability to find functions from their rates of change is one of the very powerful tools of calculus. As we will see, it lies at the heart of the mathematical developments in Chapter 5.

EXERCISES 4.2

Checking the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–6.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$

2. $f(x) = x^{2/3}, \quad [0, 1]$

3. $f(x) = x + \dfrac{1}{x}, \quad \left[\dfrac{1}{2}, 2\right]$

4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$

5. $f(x) = x^3 - x^2, \quad [-1, 2]$

6. $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$

Which of the functions in Exercises 7–12 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

7. $f(x) = x^{2/3}, \quad [-1, 8]$

8. $f(x) = x^{4/5}, \quad [0, 1]$

9. $f(x) = \sqrt{x(1 - x)}, \quad [0, 1]$

10. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$

11. $f(x) = \begin{cases} x^2 - x, & -2 \le x \le -1 \\ 2x^2 - 3x - 3, & -1 < x \le 0 \end{cases}$

12. $f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ 6x - x^2 - 7, & 2 < x \le 3 \end{cases}$

13. The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

14. For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Roots (Zeros)

15. a. Plot the zeros of each polynomial on a line together with the zeros of its first derivative.

i) $y = x^2 - 4$

ii) $y = x^2 + 8x + 15$

iii) $y = x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2$

iv) $y = x^3 - 33x^2 + 216x = x(x - 9)(x - 24)$

b. Use Rolle's Theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ there lies a zero of

$$nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \cdots + a_1.$$

16. Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b). Generalize this result.

17. Show that if $f'' > 0$ throughout an interval $[a, b]$, then f' has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?

18. Show that a cubic polynomial can have at most three real zeros.

Show that the functions in Exercises 19–26 have exactly one zero in the given interval.

19. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$

20. $f(x) = x^3 + \dfrac{4}{x^2} + 7, \quad (-\infty, 0)$

21. $g(t) = \sqrt{t} + \sqrt{1 + t} - 4, \quad (0, \infty)$

22. $g(t) = \dfrac{1}{1 - t} + \sqrt{1 + t} - 3.1, \quad (-1, 1)$

23. $r(\theta) = \theta + \sin^2\left(\dfrac{\theta}{3}\right) - 8, \quad (-\infty, \infty)$

24. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}, \quad (-\infty, \infty)$

25. $r(\theta) = \sec\theta - \dfrac{1}{\theta^3} + 5, \quad (0, \pi/2)$

26. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

Finding Functions from Derivatives

27. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Give reasons for your answer.

28. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.

29. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if

a. $f(0) = 0$ **b.** $f(1) = 0$ **c.** $f(-2) = 3$.

30. What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 31–36, find all possible functions with the given derivative.

31. a. $y' = x$ **b.** $y' = x^2$ **c.** $y' = x^3$

32. a. $y' = 2x$ **b.** $y' = 2x - 1$ **c.** $y' = 3x^2 + 2x - 1$

33. a. $y' = -\dfrac{1}{x^2}$ **b.** $y' = 1 - \dfrac{1}{x^2}$ **c.** $y' = 5 + \dfrac{1}{x^2}$

34. a. $y' = \dfrac{1}{2\sqrt{x}}$ **b.** $y' = \dfrac{1}{\sqrt{x}}$ **c.** $y' = 4x - \dfrac{1}{\sqrt{x}}$

35. a. $y' = \sin 2t$ **b.** $y' = \cos\dfrac{t}{2}$ **c.** $y' = \sin 2t + \cos\dfrac{t}{2}$

36. a. $y' = \sec^2\theta$ **b.** $y' = \sqrt{\theta}$ **c.** $y' = \sqrt{\theta} - \sec^2\theta$

In Exercises 37–40, find the function with the given derivative whose graph passes through the point P.

37. $f'(x) = 2x - 1, \quad P(0, 0)$

38. $g'(x) = \dfrac{1}{x^2} + 2x, \quad P(-1, 1)$

39. $r'(\theta) = 8 - \csc^2\theta, \quad P\left(\dfrac{\pi}{4}, 0\right)$

40. $r'(t) = \sec t \tan t - 1, \quad P(0, 0)$

Finding Position from Velocity or Acceleration

Exercises 41–44 give the velocity $v = ds/dt$ and initial position of an object moving along a coordinate line. Find the object's position at time t.

41. $v = 9.8t + 5, \quad s(0) = 10$ **42.** $v = 32t - 2, \quad s(0.5) = 4$

43. $v = \sin\pi t, \quad s(0) = 0$ **44.** $v = \dfrac{2}{\pi}\cos\dfrac{2t}{\pi}, \quad s(\pi^2) = 1$

Exercises 45–48 give the acceleration $a = d^2s/dt^2$, initial velocity, and initial position of an object moving on a coordinate line. Find the object's position at time t.

45. $a = 32, \quad v(0) = 20, \quad s(0) = 5$

46. $a = 9.8, \quad v(0) = -3, \quad s(0) = 0$

47. $a = -4\sin 2t, \quad v(0) = 2, \quad s(0) = -3$

48. $a = \dfrac{9}{\pi^2}\cos\dfrac{3t}{\pi}, \quad v(0) = 0, \quad s(0) = -1$

Applications

49. Temperature change It took 14 sec for a mercury thermometer to rise from $-19°C$ to $100°C$ when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at the rate of $8.5°C/\text{sec}$.

50. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

51. Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 hours. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea or nautical miles per hour).

52. A marathoner ran the 26.2-mi New York City Marathon in 2.2 hours. Show that at least twice the marathoner was running at exactly 11 mph, assuming the initial and final speeds are zero.

53. Show that at some instant during a 2-hour automobile trip the car's speedometer reading will equal the average speed for the trip.

54. Free fall on the moon On our moon, the acceleration of gravity is $1.6\ \text{m/sec}^2$. If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

Theory and Examples

55. The geometric mean of *a* and *b* The *geometric mean* of two positive numbers a and b is the number $\sqrt{ab}$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval of positive numbers $[a, b]$ is $c = \sqrt{ab}$.

56. The arithmetic mean of *a* and *b* The *arithmetic mean* of two numbers a and b is the number $(a + b)/2$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

T **57.** Graph the function

$$f(x) = \sin x \sin(x + 2) - \sin^2(x + 1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

58. Rolle's Theorem

a. Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1$, and 2.

b. Graph f and its derivative f' together. How is what you see related to Rolle's Theorem?

c. Do $g(x) = \sin x$ and its derivative g' illustrate the same phenomenon as f and f'?

59. Unique solution Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite signs and that $f' \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

60. Parallel tangents Assume that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel or the same line. Illustrate with a sketch.

61. Suppose that $f'(x) \le 1$ for $1 \le x \le 4$. Show that $f(4) - f(1) \le 3$.

62. Suppose that $0 < f'(x) < 1/2$ for all x-values. Show that $f(-1) < f(1) < 2 + f(-1)$.

63. Show that $|\cos x - 1| \le |x|$ for all x-values. (*Hint:* Consider $f(t) = \cos t$ on $[0, x]$.)

64. Show that for any numbers a and b, the sine inequality $|\sin b - \sin a| \leq |b - a|$ is true.

65. If the graphs of two differentiable functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.

66. If $|f(w) - f(x)| \leq |w - x|$ for all values w and x and f is a differentiable function, show that $-1 \leq f'(x) \leq 1$ for all x-values.

67. Assume that f is differentiable on $a \leq x \leq b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b.

68. Let f be a function defined on an interval $[a, b]$. What conditions could you place on f to guarantee that

$$\min f' \leq \frac{f(b) - f(a)}{b - a} \leq \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of f' on $[a, b]$? Give reasons for your answers.

T **69.** Use the inequalities in Exercise 68 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4 \cos x)$ for $0 \leq x \leq 0.1$ and $f(0) = 1$.

T **70.** Use the inequalities in Exercise 68 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \leq x \leq 0.1$ and $f(0) = 2$.

71. Let f be differentiable at every value of x and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.

a. Show that $f(x) \geq 1$ for all x.

b. Must $f'(1) = 0$? Explain.

72. Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point c in (a, b) at which f satisfies the conclusion of the Mean Value Theorem.

4.3 Monotonic Functions and the First Derivative Test

In sketching the graph of a differentiable function, it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This section gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function to identify whether local extreme values are present.

Increasing Functions and Decreasing Functions

As another corollary to the Mean Value Theorem, we show that functions with positive derivatives are increasing functions and functions with negative derivatives are decreasing functions. A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a, b) and $f(x_2) < f(x_1)$ if f' is negative on (a, b). ■

Corollary 3 tells us that $f(x) = \sqrt{x}$ is increasing on the interval $[0, b]$ for any $b > 0$ because $f'(x) = 1/\sqrt{x}$ is positive on $(0, b)$. The derivative does not exist at $x = 0$, but Corollary 3 still applies. The corollary is valid for infinite as well as finite intervals, so $f(x) = \sqrt{x}$ is increasing on $[0, \infty)$.

To find the intervals where a function f is increasing or decreasing, we first find all of the critical points of f. If $a < b$ are two critical points for f, and if the derivative f' is continuous but never zero on the interval (a, b), then by the Intermediate Value Theorem applied to f', the derivative must be everywhere positive on (a, b), or everywhere negative there. One way we can determine the sign of f' on (a, b) is simply by evaluating the derivative at a single point c in (a, b). If $f'(c) > 0$, then $f'(x) > 0$ for all x in (a, b) so f is increasing on $[a, b]$ by Corollary 3; if $f'(c) < 0$, then f is decreasing on $[a, b]$. It doesn't matter which point c we choose in (a, b), since the sign of $f'(c)$ is the same for all choices. Usually we pick c to be a point where it is easy to evaluate $f'(c)$. The next example illustrates how we use this procedure.

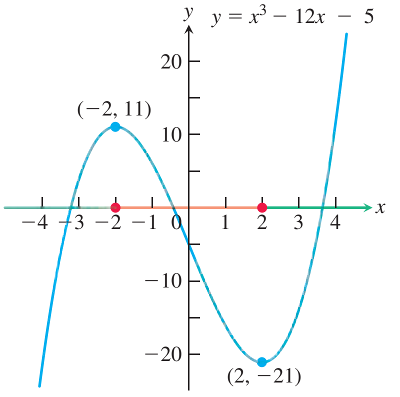


FIGURE 4.20 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

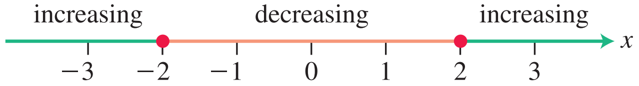
EXAMPLE 1 Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution The function f is everywhere continuous and differentiable. The first derivative

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3(x^2 - 4) \\ &= 3(x + 2)(x - 2) \end{aligned}$$

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f to create nonoverlapping open intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval. We evaluate f' at $x = -3$ in the first interval, $x = 0$ in the second interval and $x = 3$ in the third, since f' is relatively easy to compute at these points. The behavior of f is determined by then applying Corollary 3 to each subinterval. The results are summarized in the following table, and the graph of f is given in Figure 4.20.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing



We used "strict" less-than inequalities to identify the intervals in the summary table for Example 1, since open intervals were specified. Corollary 3 says that we could use $\leq$ inequalities as well. That is, the function f in the example is increasing on $-\infty < x \leq -2$, decreasing on $-2 \leq x \leq 2$, and increasing on $2 \leq x < \infty$. We do not talk about whether a function is increasing or decreasing at a single point.

HISTORICAL BIOGRAPHY
Edmund Halley
(1656–1742)
www.goo.gl/KPGR2w

First Derivative Test for Local Extrema

In Figure 4.21, at the points where f has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. (If the point is an endpoint, there is only one side to consider.) Thus, the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where f has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus, the function is increasing on the left of the maximum value and decreasing on its right. In summary, at a local extreme point, the sign of $f'(x)$ changes.

These observations lead to a test for the presence and nature of local extreme values of differentiable functions.

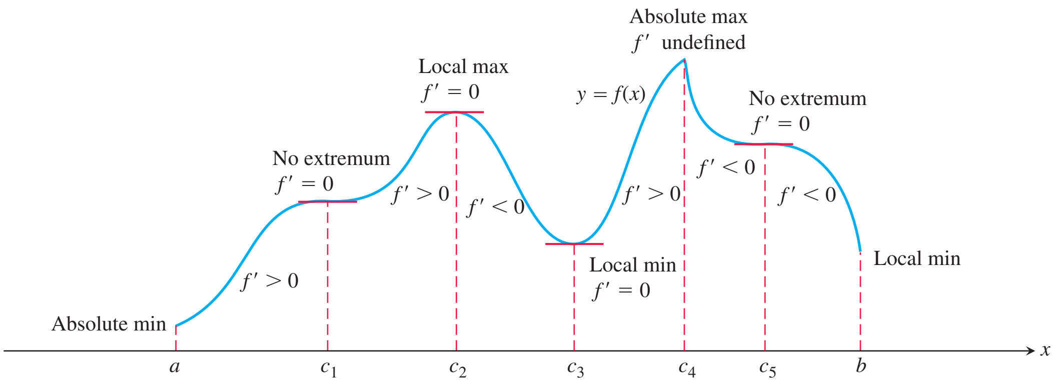


FIGURE 4.21 The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

The test for local extrema at endpoints is similar, but there is only one side to consider in determining whether f is increasing or decreasing, based on the sign of f'.

Proof of the First Derivative Test Part (1). Since the sign of f' changes from negative to positive at c, there are numbers a and b such that $a < c < b$, $f' < 0$ on (a, c), and $f' > 0$ on (c, b). If $x \in (a, c)$, then $f(c) < f(x)$ because $f' < 0$ implies that f is decreasing on $[a, c]$. If $x \in (c, b)$, then $f(c) < f(x)$ because $f' > 0$ implies that f is increasing on $[c, b]$. Therefore, $f(x) \geq f(c)$ for every $x \in (a, b)$. By definition, f has a local minimum at c.

Parts (2) and (3) are proved similarly. ■

EXAMPLE 2 Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} \\ &= \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x - 1)}{3x^{2/3}} \end{aligned}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value.

The critical points partition the x-axis into open intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f between and at the critical points, as summarized in the following table.

Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

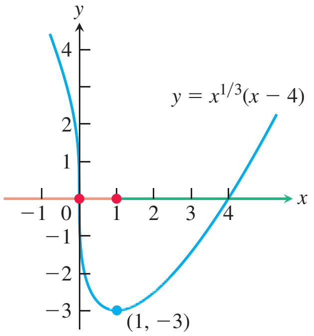


FIGURE 4.22 The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

Corollary 3 to the Mean Value Theorem implies that f decreases on $(-\infty, 0)$, decreases on $(0, 1)$, and increases on $(1, \infty)$. The First Derivative Test for Local Extrema tells us that f does not have an extreme value at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum since f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$. Figure 4.22 shows this value in relation to the function's graph.

Note that $\lim_{x\to 0} f'(x) = -\infty$, so the graph of f has a vertical tangent at the origin. ■

EXAMPLE 3 Within the interval $0 \le x \le 2\pi$, find the critical points of

$$f(x) = \sin^2 x - \sin x - 1.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous over $[0, 2\pi]$ and differentiable over $(0, 2\pi)$, so the critical points occur at the zeros of f' in $(0, 2\pi)$. We find

$$f'(x) = 2 \sin x \cos x - \cos x = (2 \sin x - 1)(\cos x).$$

The first derivative is zero if and only if $\sin x = \frac{1}{2}$ or $\cos x = 0$. So the critical points of f in $(0, 2\pi)$ are $x = \pi/6, x = 5\pi/6, x = \pi/2$, and $x = 3\pi/2$. They partition $[0, 2\pi]$ into open intervals as follows.

Interval	$\left(0, \frac{\pi}{6}\right)$	$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$	$\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
Sign of f'	$-$	$+$	$-$	$+$	$-$
Behavior of f	dec	inc	dec	increasing	decreasing

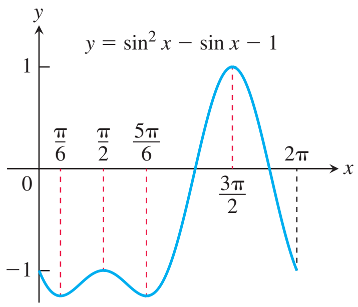


FIGURE 4.23 The graph of the function in Example 3.

The table displays the open intervals on which f is increasing and decreasing. We can deduce from the table that there is a local minimum value of $f(\pi/6) = \frac{1}{4} - \frac{1}{2} - 1 = -\frac{5}{4}$, a local maximum value of $f(\pi/2) = 1 - 1 - 1 = -1$, another local minimum value of $f(5\pi/6) = -\frac{5}{4}$, and another local maximum value of $f(3\pi/2) = 1 - (-1) - 1 = 1$. The endpoint values are $f(0) = f(2\pi) = -1$. The absolute minimum in $[0, 2\pi]$ is $-\frac{5}{4}$ occurring at $x = \pi/6$ and $x = 5\pi/6$; the absolute maximum is 1 occurring at $x = 3\pi/2$. Figure 4.23 shows the graph. ■

EXERCISES 4.3

Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

a. What are the critical points of f?

b. On what open intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x-1)$

2. $f'(x) = (x-1)(x+2)$

3. $f'(x) = (x-1)^2(x+2)$

4. $f'(x) = (x-1)^2(x+2)^2$

5. $f'(x) = (x-1)(x+2)(x-3)$

6. $f'(x) = (x-7)(x+1)(x+5)$

7. $f'(x) = \dfrac{x^2(x-1)}{x+2}, \quad x \neq -2$

8. $f'(x) = \dfrac{(x-2)(x+4)}{(x+1)(x-3)}, \quad x \neq -1, 3$

9. $f'(x) = 1 - \dfrac{4}{x^2}, \quad x \neq 0$

10. $f'(x) = 3 - \dfrac{6}{\sqrt{x}}, \quad x \neq 0$

11. $f'(x) = x^{-1/3}(x+2)$

12. $f'(x) = x^{-1/2}(x-3)$

13. $f'(x) = (\sin x - 1)(2\cos x + 1), \ 0 \le x \le 2\pi$

14. $f'(x) = (\sin x + \cos x)(\sin x - \cos x), \ 0 \le x \le 2\pi$

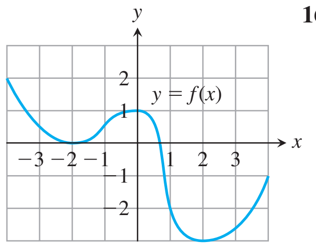
Identifying Extrema

In Exercises 15–40:

a. Find the open intervals on which the function is increasing and decreasing.

b. Identify the function's local and absolute extreme values, if any, saying where they occur.

15.



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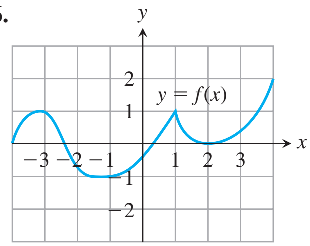

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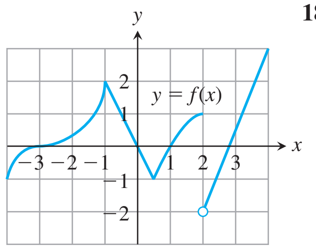

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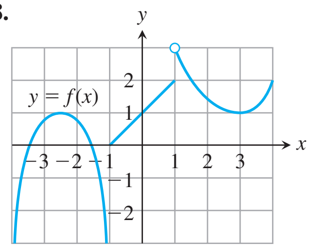


19. $g(t) = -t^2 - 3t + 3$

20. $g(t) = -3t^2 + 9t + 5$

21. $h(x) = -x^3 + 2x^2$

22. $h(x) = 2x^3 - 18x$

23. $f(\theta) = 3\theta^2 - 4\theta^3$

24. $f(\theta) = 6\theta - \theta^3$

25. $f(r) = 3r^3 + 16r$

26. $h(r) = (r+7)^3$

27. $f(x) = x^4 - 8x^2 + 16$

28. $g(x) = x^4 - 4x^3 + 4x^2$

29. $H(t) = \dfrac{3}{2}t^4 - t^6$

30. $K(t) = 15t^3 - t^5$

31. $f(x) = x - 6\sqrt{x-1}$

32. $g(x) = 4\sqrt{x} - x^2 + 3$

33. $g(x) = x\sqrt{8 - x^2}$

34. $g(x) = x^2\sqrt{5 - x}$

35. $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$

36. $f(x) = \dfrac{x^3}{3x^2 + 1}$

37. $f(x) = x^{1/3}(x+8)$

38. $g(x) = x^{2/3}(x+5)$

39. $h(x) = x^{1/3}(x^2 - 4)$

40. $k(x) = x^{2/3}(x^2 - 4)$

In Exercises 41–52:

a. Identify the function's local extreme values in the given domain, and say where they occur.

b. Which of the extreme values, if any, are absolute?

T **c.** Support your findings with a graphing calculator or computer grapher.

41. $f(x) = 2x - x^2, \quad -\infty < x \le 2$

42. $f(x) = (x+1)^2, \quad -\infty < x \le 0$

43. $g(x) = x^2 - 4x + 4, \quad 1 \le x < \infty$

44. $g(x) = -x^2 - 6x - 9, \quad -4 \le x < \infty$

45. $f(t) = 12t - t^3, \quad -3 \le t < \infty$

46. $f(t) = t^3 - 3t^2, \quad -\infty < t \le 3$

47. $h(x) = \dfrac{x^3}{3} - 2x^2 + 4x, \quad 0 \le x < \infty$

48. $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \le 0$

49. $f(x) = \sqrt{25 - x^2}, \quad -5 \le x \le 5$

50. $f(x) = \sqrt{x^2 - 2x - 3}, \quad 3 \le x < \infty$

51. $g(x) = \dfrac{x-2}{x^2 - 1}, \quad 0 \le x < 1$

52. $g(x) = \dfrac{x^2}{4 - x^2}, \quad -2 < x \le 1$

In Exercises 53–60:

a. Find the local extrema of each function on the given interval, and say where they occur.

T **b.** Graph the function and its derivative together. Comment on the behavior of f in relation to the signs and values of f'.

53. $f(x) = \sin 2x, \quad 0 \le x \le \pi$

54. $f(x) = \sin x - \cos x, \quad 0 \le x \le 2\pi$

55. $f(x) = \sqrt{3}\cos x + \sin x, \quad 0 \le x \le 2\pi$

56. $f(x) = -2x + \tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

57. $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}, \quad 0 \le x \le 2\pi$

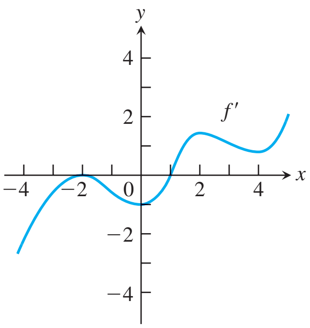
58. $f(x) = -2\cos x - \cos^2 x, \quad -\pi \le x \le \pi$

59. $f(x) = \csc^2 x - 2\cot x, \quad 0 < x < \pi$

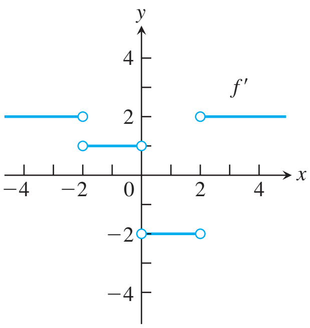
60. $f(x) = \sec^2 x - 2\tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

In Exercises 61 and 62, the graph of f' is given. Assume that f is continuous and determine the x-values corresponding to local minima and local maxima.

61.



62.



Theory and Examples

Show that the functions in Exercises 63 and 64 have local extreme values at the given values of θ, and say which kind of local extreme the function has.

63. $h(\theta) = 3\cos\dfrac{\theta}{2}$, $\quad 0 \le \theta \le 2\pi$, $\quad$ at $\theta = 0$ and $\theta = 2\pi$

64. $h(\theta) = 5\sin\dfrac{\theta}{2}$, $\quad 0 \le \theta \le \pi$, $\quad$ at $\theta = 0$ and $\theta = \pi$

65. Sketch the graph of a differentiable function $y = f(x)$ through the point $(1, 1)$ if $f'(1) = 0$ and

a. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;

b. $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;

c. $f'(x) > 0$ for $x \neq 1$;

d. $f'(x) < 0$ for $x \neq 1$.

66. Sketch the graph of a differentiable function $y = f(x)$ that has

a. a local minimum at $(1, 1)$ and a local maximum at $(3, 3)$;

b. a local maximum at $(1, 1)$ and a local minimum at $(3, 3)$;

c. local maxima at $(1, 1)$ and $(3, 3)$;

d. local minima at $(1, 1)$ and $(3, 3)$.

67. Sketch the graph of a continuous function $y = g(x)$ such that

a. $g(2) = 2$, $0 < g' < 1$ for $x < 2$, $g'(x) \to 1^-$ as $x \to 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;

b. $g(2) = 2$, $g' < 0$ for $x < 2$, $g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$.

68. Sketch the graph of a continuous function $y = h(x)$ such that

a. $h(0) = 0$, $-2 \le h(x) \le 2$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to \infty$ as $x \to 0^+$;

b. $h(0) = 0$, $-2 \le h(x) \le 0$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$.

69. Discuss the extreme-value behavior of the function $f(x) = x\sin(1/x)$, $x \neq 0$. How many critical points does this function have? Where are they located on the x-axis? Does f have an absolute minimum? An absolute maximum? (See Exercise 49 in Section 2.3.)

70. Find the open intervals on which the function $f(x) = ax^2 + bx + c$, $a \neq 0$, is increasing and decreasing. Describe the reasoning behind your answer.

71. Determine the values of constants a and b so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$.

72. Determine the values of constants a, b, c, and d so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(0, 0)$ and a local minimum at the point $(1, -1)$.

4.4 Concavity and Curve Sketching

We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. In this section we see that the second derivative gives us information about how the graph of a differentiable function bends or turns. With this knowledge about the first and second derivatives, coupled with our previous understanding of symmetry and asymptotic behavior studied in Sections 1.1 and 2.6, we can now draw an accurate graph of a function. By organizing all of these ideas into a coherent procedure, we give a method for sketching graphs and revealing visually the key features of functions. Identifying and knowing the locations of these features is of major importance in mathematics and its applications to science and engineering, especially in the graphical analysis and interpretation of data. When the domain of a function is not a finite closed interval, sketching a graph helps to determine whether absolute maxima or absolute minima exist and, if they do exist, where they are located.

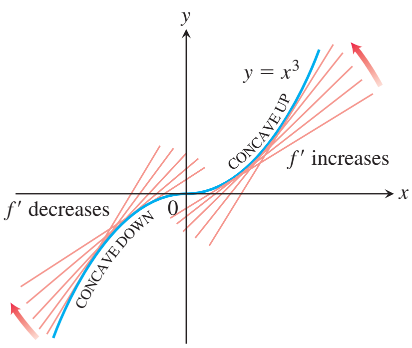


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

Concavity

As you can see in Figure 4.24, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangents. The slopes of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slopes of the tangents are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the *concavity* of the curve.

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

A function whose graph is concave up is also often called **convex**.

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to the first derivative function. We conclude that f' increases if $f'' > 0$ on I, and decreases if $f'' < 0$.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

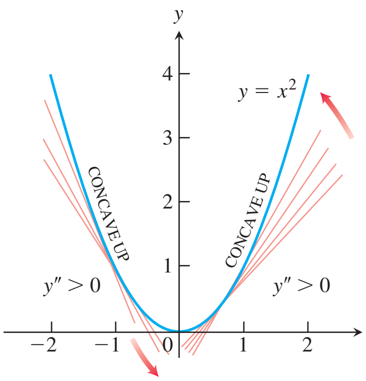


FIGURE 4.25 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

If $y = f(x)$ is twice-differentiable, we will use the notations f'' and y'' interchangeably when denoting the second derivative.

EXAMPLE 1

(a) The curve $y = x^3$ (Figure 4.24) is concave down on $(-\infty, 0)$, where $y'' = 6x < 0$, and concave up on $(0, \infty)$, where $y'' = 6x > 0$.

(b) The curve $y = x^2$ (Figure 4.25) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ■

EXAMPLE 2 Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The first derivative of $y = 3 + \sin x$ is $y' = \cos x$, and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.26). ■

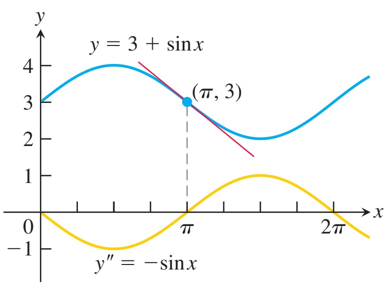


FIGURE 4.26 Using the sign of y'' to determine the concavity of y (Example 2).

Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. Since the first derivative $y' = \cos x$ exists for all x, we see that the curve has a tangent line of slope -1 at the point $(\pi, 3)$. This point is called a *point of inflection* of the curve. Notice from Figure 4.26 that the graph crosses its tangent line at this point and that the second derivative $y'' = -\sin x$ has value 0 when $x = \pi$. In general, we have the following definition.

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

We observed that the second derivative of $f(x) = 3 + \sin x$ is equal to zero at the inflection point $(\pi, 3)$. Generally, if the second derivative exists at a point of inflection $(c, f(c))$, then $f''(c) = 0$. This follows immediately from the Intermediate Value Theorem whenever f'' is continuous over an interval containing $x = c$ because the second derivative changes sign moving across this interval. Even if the continuity assumption is dropped, it is still true that $f''(c) = 0$, provided the second derivative exists (although a more advanced argument is required in this noncontinuous case). Since a tangent line must exist at the point of inflection, either the first derivative $f'(c)$ exists (is finite) or the graph has a vertical tangent at the point. At a vertical tangent neither the first nor second derivative exists. In summary, one of two things can happen at a point of inflection.

> At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

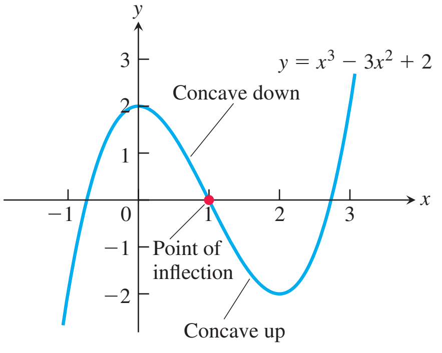


FIGURE 4.27 The concavity of the graph of f changes from concave down to concave up at the inflection point.

EXAMPLE 3 Determine the concavity and find the inflection points of the function

$$f(x) = x^3 - 3x^2 + 2.$$

Solution We start by computing the first and second derivatives.

$$f'(x) = 3x^2 - 6x, \qquad f''(x) = 6x - 6.$$

To determine concavity, we look at the sign of the second derivative $f''(x) = 6x - 6$. The sign is negative when $x < 1$, is 0 at $x = 1$, and is positive when $x > 1$. It follows that the graph of f is concave down on $(-\infty, 1)$, is concave up on $(1, \infty)$, and has an inflection point at the point $(1, 0)$ where the concavity changes.

The graph of f is shown in Figure 4.27. Notice that we did not need to know the shape of this graph ahead of time in order to determine its concavity. ■

The next example illustrates that a function can have a point of inflection where the first derivative exists but the second derivative fails to exist.

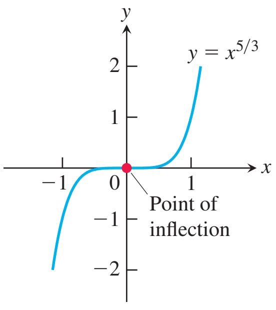


FIGURE 4.28 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$ (Example 4).

EXAMPLE 4 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 4.28. ■

The following example shows that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

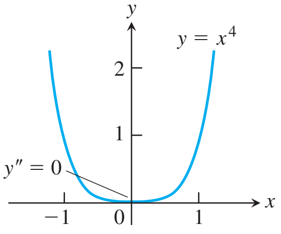


FIGURE 4.29 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 5).

EXAMPLE 5 The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.29). Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign. The curve is concave up everywhere. ■

In the next example a point of inflection occurs at a vertical tangent to the curve where neither the first nor the second derivative exists.

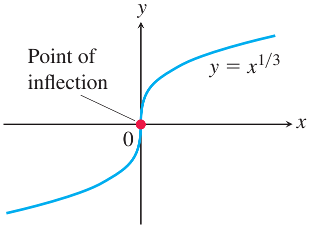


FIGURE 4.30 A point of inflection where y' and y'' fail to exist (Example 6).

EXAMPLE 6 The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 4.30. ■

Caution Example 4 in Section 4.1 (Figure 4.9) shows that the function $f(x) = x^{2/3}$ does not have a second derivative at $x = 0$ and does not have a point of inflection there (there is no change in concavity at $x = 0$). Combined with the behavior of the function in Example 6 above, we see that when the second derivative does not exist at $x = c$, an inflection point may or may not occur there. So we need to be careful about interpreting functional behavior whenever first or second derivatives fail to exist at a point. At such points the graph can have vertical tangents, corners, cusps, or various discontinuities.

To study the motion of an object moving along a line as a function of time, we often are interested in knowing when the object's acceleration, given by the second derivative, is positive or negative. The points of inflection on the graph of the object's position function reveal where the acceleration changes sign.

EXAMPLE 7 A particle is moving along a horizontal coordinate line (positive to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t - 1)(3t - 11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t - 7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left.

Notice that the first derivative ($v = s'$) is zero at the critical points $t = 1$ and $t = 11/3$.

Interval	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	$+$	$-$	$+$
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$, and moving to the left in $(1, 11/3)$. It is momentarily stationary (at rest) at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

Interval	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	$-$	$+$
Graph of s	concave down	concave up

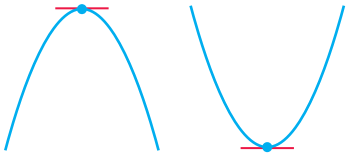
The particle starts out moving to the right while slowing down, and then reverses and begins moving to the left at $t = 1$ under the influence of the leftward acceleration over the time interval $[0, 7/3)$. The acceleration then changes direction at $t = 7/3$ but the particle continues moving leftward, while slowing down under the rightward acceleration. At $t = 11/3$ the particle reverses direction again: moving to the right in the same direction as the acceleration, so it is speeding up.

Second Derivative Test for Local Extrema

Instead of looking for sign changes in f' at critical points, we can sometimes use the following test to determine the presence and nature of local extrema.

THEOREM 5—Second Derivative Test for Local Extrema
Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0$ $\Rightarrow$ local max $f' = 0, f'' > 0$ $\Rightarrow$ local min

Proof Part (1). If $f''(c) < 0$, then $f''(x) < 0$ on some open interval I containing the point c, since f'' is continuous. Therefore, f' is decreasing on I. Since $f'(c) = 0$, the sign of f' changes from positive to negative at c so f has a local maximum at c by the First Derivative Test.

The proof of Part (2) is similar.

For Part (3), consider the three functions $y = x^4$, $y = -x^4$, and $y = x^3$. For each function, the first and second derivatives are zero at $x = 0$. Yet the function $y = x^4$ has a local minimum there, $y = -x^4$ has a local maximum, and $y = x^3$ is increasing in any open interval containing $x = 0$ (having neither a maximum nor a minimum there). Thus the test fails. ■

This test requires us to know f'' *only at c itself* and not in an interval about c. This makes the test easy to apply. That's the good news. The bad news is that the test is inconclusive if $f'' = 0$ or if f'' does not exist at $x = c$. When this happens, use the First Derivative Test for local extreme values.

Together f' and f'' tell us the shape of the function's graph—that is, where the critical points are located and what happens at a critical point, where the function is increasing and where it is decreasing, and how the curve is turning or bending as defined by its concavity. We use this information to sketch a graph of the function that captures its key features.

EXAMPLE 8 Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution The function f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

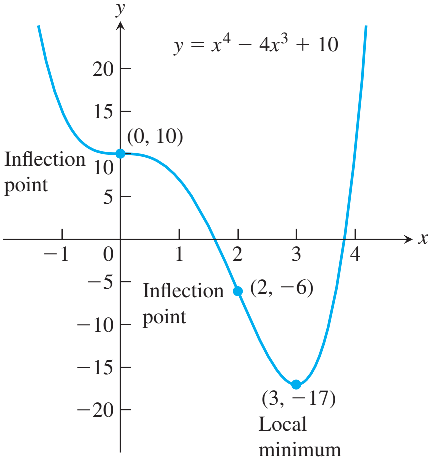


FIGURE 4.31 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 8).

(a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where f is concave up or concave down.

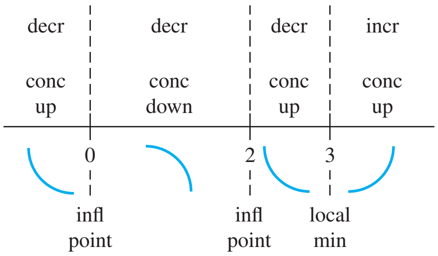
Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	+	−	+
Behavior of f	concave up	concave down	concave up

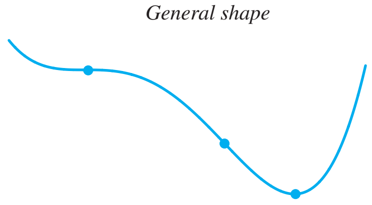
We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

(d) Summarizing the information in the last two tables, we obtain the following.

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

The general shape of the curve is shown in the accompanying figure.





(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.) Figure 4.31 shows the graph of f. ■

The steps in Example 8 give a procedure for graphing the key features of a function. Asymptotes were defined and discussed in Section 2.6. We can find them for rational functions, and the methods in the next section give tools to help find them for more general functions.

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

EXAMPLE 9 Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.1).
2. *Find f' and f''.*

$$f(x) = \frac{(x+1)^2}{1+x^2} \qquad x\text{-intercept at } x=-1,\ y\text{-intercept at } y=1$$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2} \qquad \text{Critical points: } x=-1,\ x=1$$

$$f''(x) = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3} \qquad \text{After some algebra}$$

3. *Behavior at critical points.* The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x=-1$, $f''(-1) = 1 > 0$, yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$, yielding a relative maximum by the Second Derivative test.
4. *Increasing and decreasing.* We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.
5. *Inflection points.* Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}$, 0, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up again on $(\sqrt{3}, \infty)$.
6. *Asymptotes.* Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} \qquad \text{Expanding numerator}$$

$$= \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}. \qquad \text{Dividing by } x^2$$

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \le y \le 2$).
7. The graph of f is sketched in Figure 4.32. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$. ■

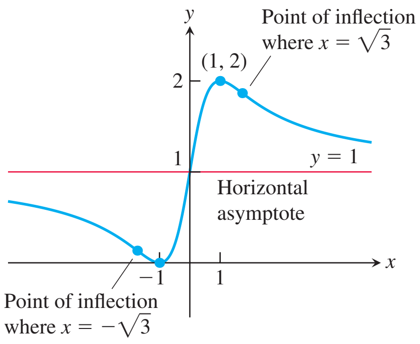


FIGURE 4.32 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 9).

EXAMPLE 10 Sketch the graph of $f(x) = \dfrac{x^2 + 4}{2x}$.

Solution

1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.
2. We calculate the derivatives of the function:

$$
\begin{aligned}
f(x) &= \frac{x^2 + 4}{2x} = \frac{x}{2} + \frac{2}{x} && \text{Function simplified for differentiation} \\
f'(x) &= \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2} && \text{Combine fractions to solve easily } f'(x) = 0. \\
f''(x) &= \frac{4}{x^3} && \text{Exists throughout the entire domain of } f
\end{aligned}
$$

3. The critical points occur only at $x = \pm 2$ where $f'(x) = 0$. Since $f''(-2) < 0$ and $f''(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum at $x = 2$ with $f(2) = 2$.
4. On the interval $(-\infty, -2)$ the derivative f' is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$.
5. There are no points of inflection because $f''(x) < 0$ whenever $x < 0$, $f''(x) > 0$ whenever $x > 0$, and f'' exists everywhere and is never zero throughout the domain of f. The graph is concave down on the interval $(-\infty, 0)$ and concave up on $(0, \infty)$.
6. From the rewritten formula for $f(x)$, we see that

$$
\lim_{x \to 0^+} \left(\frac{x}{2} + \frac{2}{x} \right) = +\infty \quad \text{and} \quad \lim_{x \to 0^-} \left(\frac{x}{2} + \frac{2}{x} \right) = -\infty,
$$

so the y-axis is a vertical asymptote. Also, as $x \to \infty$ or as $x \to -\infty$, the graph of $f(x)$ approaches the line $y = x/2$. Thus $y = x/2$ is an oblique asymptote.

7. The graph of f is sketched in Figure 4.33. ∎

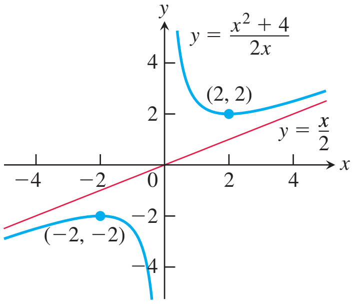


FIGURE 4.33 The graph of $y = \dfrac{x^2 + 4}{2x}$ (Example 10).

EXAMPLE 11 Sketch the graph of $f(x) = \cos x - \dfrac{\sqrt{2}}{2}x$ over $0 \le x \le 2\pi$.

Solution The derivatives of f are

$$
f'(x) = -\sin x - \frac{\sqrt{2}}{2} \quad \text{and} \quad f''(x) = -\cos x.
$$

Both derivatives exist everywhere over the interval $(0, 2\pi)$. Within that open interval, the first derivative is zero when $\sin x = -\sqrt{2}/2$, so the critical points are $x = 5\pi/4$ and $x = 7\pi/4$. Since $f''(5\pi/4) = -\cos(5\pi/4) = \sqrt{2}/2 > 0$, the function has a local minimum value of $f(5\pi/4) \approx -3.48$ (evaluated with a calculator) by the Second Derivative Test. Also, $f''(7\pi/4) = -\cos(7\pi/4) = -\sqrt{2}/2 < 0$, so the function has a local maximum value of $f(7\pi/4) \approx -3.18$.

Examining the second derivative, we find that $f'' = 0$ when $x = \pi/2$ or $x = 3\pi/2$. We conclude that $(\pi/2, f(\pi/2)) \approx (\pi/2, -1.11)$ and $(3\pi/2, f(3\pi/2)) \approx (3\pi/2, -3.33)$ are points of inflection.

Finally, we evaluate f at the endpoints of the interval to find $f(0) = 1$ and $f(2\pi) \approx -3.44$. Therefore, the values $f(0) = 1$ and $f(5\pi/4) \approx -3.48$ are the absolute maximum and absolute minimum values of f over the closed interval $[0, 2\pi]$. The graph of f is sketched in Figure 4.34. ∎

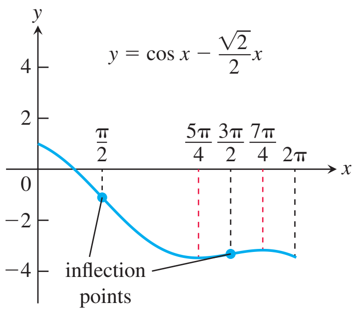


FIGURE 4.34 The graph of the function in Example 11.

Graphical Behavior of Functions from Derivatives

The following figure indicates how the first two derivatives of a functions affect the shape of a graph.

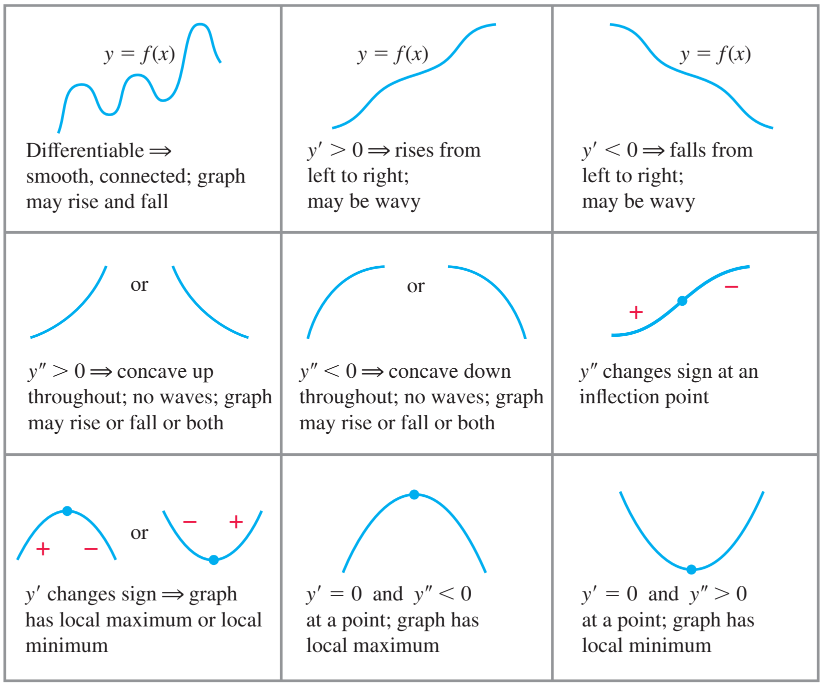


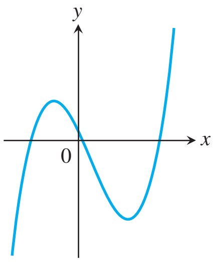
FIGURE 4.35

EXERCISES 4.4

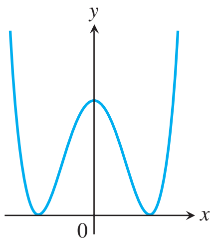
Analyzing Functions from Graphs

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

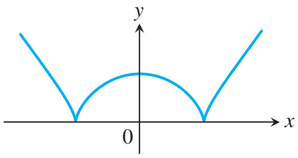
1. $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



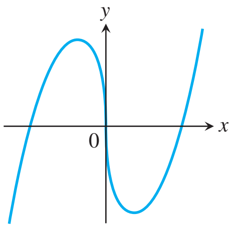
2. $y = \frac{x^4}{4} - 2x^2 + 4$



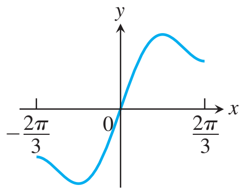
3. $y = \frac{3}{4}(x^2 - 1)^{2/3}$



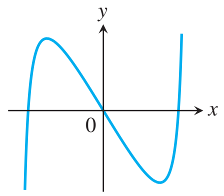
4. $y = \frac{9}{14}x^{1/3}(x^2 - 7)$



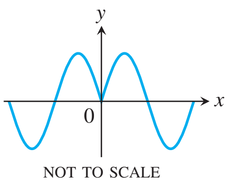
5. $y = x + \sin 2x, \ -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$



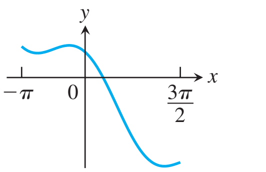
6. $y = \tan x - 4x, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$



7. $y = \sin|x|, \ -2\pi \le x \le 2\pi$



8. $y = 2\cos x - \sqrt{2}x, \ -\pi \le x \le \frac{3\pi}{2}$



Graphing Functions

In Exercises 9–50, identify the coordinates of any local and absolute extreme points and inflection points. Graph the function.

9. $y = x^2 - 4x + 3$

10. $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$

12. $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

14. $y = 1 - 9x - 6x^2 - x^3$

15. $y = (x - 2)^3 + 1$

16. $y = 1 - (x + 1)^3$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

18. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

19. $y = 4x^3 - x^4 = x^3(4 - x)$

20. $y = x^4 + 2x^3 = x^3(x + 2)$

21. $y = x^5 - 5x^4 = x^4(x - 5)$

22. $y = x\left(\dfrac{x}{2} - 5\right)^4$

23. $y = x + \sin x, \quad 0 \le x \le 2\pi$

24. $y = x - \sin x, \quad 0 \le x \le 2\pi$

25. $y = \sqrt{3}x - 2\cos x, \quad 0 \le x \le 2\pi$

26. $y = \dfrac{4}{3}x - \tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

27. $y = \sin x \cos x, \quad 0 \le x \le \pi$

28. $y = \cos x + \sqrt{3}\sin x, \quad 0 \le x \le 2\pi$

29. $y = x^{1/5}$

30. $y = x^{2/5}$

31. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

32. $y = \dfrac{\sqrt{1 - x^2}}{2x + 1}$

33. $y = 2x - 3x^{2/3}$

34. $y = 5x^{2/5} - 2x$

35. $y = x^{2/3}\left(\dfrac{5}{2} - x\right)$

36. $y = x^{2/3}(x - 5)$

37. $y = x\sqrt{8 - x^2}$

38. $y = (2 - x^2)^{3/2}$

39. $y = \sqrt{16 - x^2}$

40. $y = x^2 + \dfrac{2}{x}$

41. $y = \dfrac{x^2 - 3}{x - 2}$

42. $y = \sqrt[3]{x^3 + 1}$

43. $y = \dfrac{8x}{x^2 + 4}$

44. $y = \dfrac{5}{x^4 + 5}$

45. $y = |x^2 - 1|$

46. $y = |x^2 - 2x|$

47. $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

48. $y = \sqrt{|x - 4|}$

49. $y = \dfrac{x}{9 - x^2}$

50. $y = \dfrac{x^2}{1 - x}$

Sketching the General Shape, Knowing y'

Each of Exercises 51–72 gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use Steps 2–4 of the graphing procedure on page 207 to sketch the general shape of the graph of f.

51. $y' = 2 + x - x^2$

52. $y' = x^2 - x - 6$

53. $y' = x(x - 3)^2$

54. $y' = x^2(2 - x)$

55. $y' = x(x^2 - 12)$

56. $y' = (x - 1)^2(2x + 3)$

57. $y' = (8x - 5x^2)(4 - x)^2$

58. $y' = (x^2 - 2x)(x - 5)^2$

59. $y' = \sec^2 x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

60. $y' = \tan x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

61. $y' = \cot\dfrac{\theta}{2}, \quad 0 < \theta < 2\pi$

62. $y' = \csc^2\dfrac{\theta}{2}, \quad 0 < \theta < 2\pi$

63. $y' = \tan^2\theta - 1, \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

64. $y' = 1 - \cot^2\theta, \quad 0 < \theta < \pi$

65. $y' = \cos t, \quad 0 \le t \le 2\pi$

66. $y' = \sin t, \quad 0 \le t \le 2\pi$

67. $y' = (x + 1)^{-2/3}$

68. $y' = (x - 2)^{-1/3}$

69. $y' = x^{-2/3}(x - 1)$

70. $y' = x^{-4/5}(x + 1)$

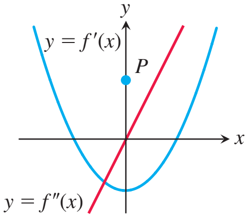
71. $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

72. $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

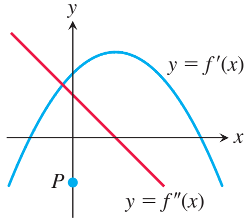
Sketching y from Graphs of y' and y''

Each of Exercises 73–76 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of f, given that the graph passes through the point P.

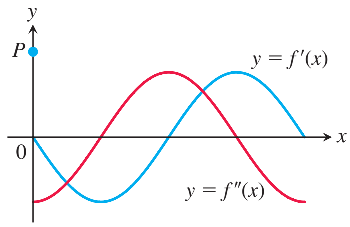
73.



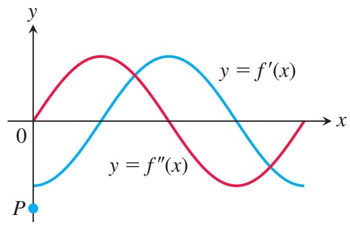
74.



75.



76.



Graphing Rational Functions

Graph the rational functions in Exercises 77–94 using all the steps in the graphing procedure on page 207.

77. $y = \dfrac{2x^2 + x - 1}{x^2 - 1}$

78. $y = \dfrac{x^2 - 49}{x^2 + 5x - 14}$

79. $y = \dfrac{x^4 + 1}{x^2}$

80. $y = \dfrac{x^2 - 4}{2x}$

81. $y = \dfrac{1}{x^2 - 1}$

82. $y = \dfrac{x^2}{x^2 - 1}$

83. $y = -\dfrac{x^2 - 2}{x^2 - 1}$

84. $y = \dfrac{x^2 - 4}{x^2 - 2}$

85. $y = \dfrac{x^2}{x + 1}$

86. $y = -\dfrac{x^2 - 4}{x + 1}$

87. $y = \dfrac{x^2 - x + 1}{x - 1}$

88. $y = -\dfrac{x^2 - x + 1}{x - 1}$

89. $y = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$

90. $y = \dfrac{x^3 + x - 2}{x - x^2}$

91. $y = \dfrac{x}{x^2 - 1}$

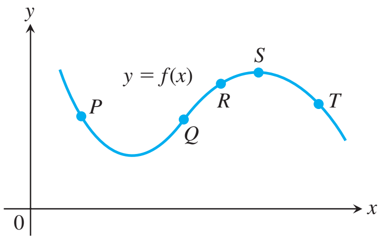
92. $y = \dfrac{x - 1}{x^2(x - 2)}$

93. $y = \dfrac{8}{x^2 + 4}$ (Agnesi's witch)

94. $y = \dfrac{4x}{x^2 + 4}$ (Newton's serpentine)

Theory and Examples

95. The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.



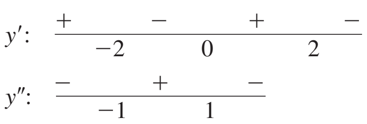
96. Sketch a smooth connected curve $y = f(x)$ with

$$\begin{aligned} &f(-2) = 8, && f'(2) = f'(-2) = 0, \\ &f(0) = 4, && f'(x) < 0 \quad \text{for} \quad |x| < 2, \\ &f(2) = 0, && f''(x) < 0 \quad \text{for} \quad x < 0, \\ &f'(x) > 0 \quad \text{for} \quad |x| > 2, && f''(x) > 0 \quad \text{for} \quad x > 0. \end{aligned}$$

97. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

x	y	Derivatives
$x < 2$		$y' < 0, \quad y'' > 0$
2	1	$y' = 0, \quad y'' > 0$
$2 < x < 4$		$y' > 0, \quad y'' > 0$
4	4	$y' > 0, \quad y'' = 0$
$4 < x < 6$		$y' > 0, \quad y'' < 0$
6	7	$y' = 0, \quad y'' < 0$
$x > 6$		$y' < 0, \quad y'' < 0$

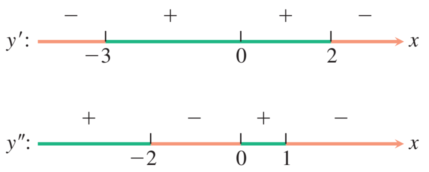
98. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 2)$ and whose first two derivatives have the following sign patterns.



99. Sketch the graph of a twice-differentiable $y = f(x)$ with the following properties. Label coordinates where possible.

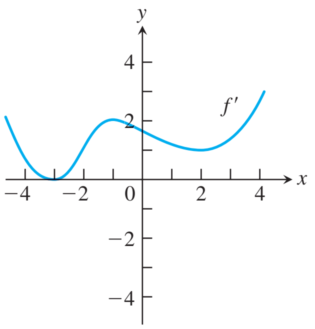
x	y	Derivatives
$x < -2$		$y' > 0, \quad y'' < 0$
-2	-1	$y' = 0, \quad y'' = 0$
$-2 < x < -1$		$y' > 0, \quad y'' > 0$
-1	0	$y' > 0, \quad y'' = 0$
$-1 < x < 0$		$y' > 0, \quad y'' < 0$
0	3	$y' = 0, \quad y'' < 0$
$0 < x < 1$		$y' < 0, \quad y'' < 0$
1	2	$y' < 0, \quad y'' = 0$
$1 < x < 2$		$y' < 0, \quad y'' > 0$
2	0	$y' = 0, \quad y'' > 0$
$x > 2$		$y' > 0, \quad y'' > 0$

100. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-3, -2)$, $(-2, 0)$, $(0, 1)$, $(1, 2)$, and $(2, 3)$ and whose first two derivatives have the following sign patterns.

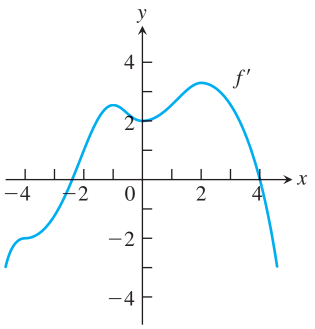


In Exercises 101 and 102, the graph of f' is given. Determine x-values corresponding to inflection points for the graph of f.

101.

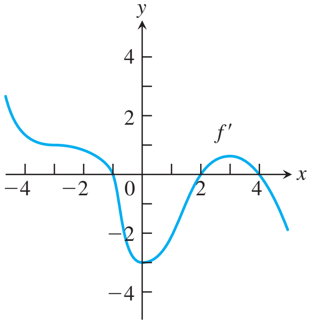


102.

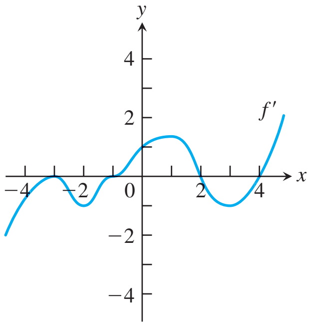


In Exercises 103 and 104, the graph of f' is given. Determine x-values corresponding to local minima, local maxima, and inflection points for the graph of f.

103.

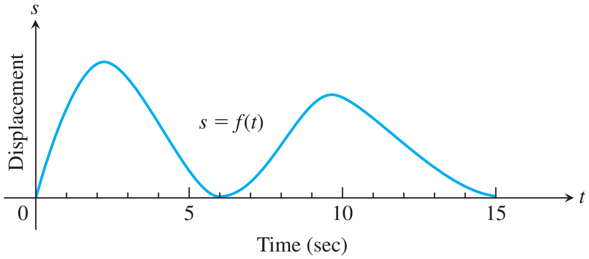


104.

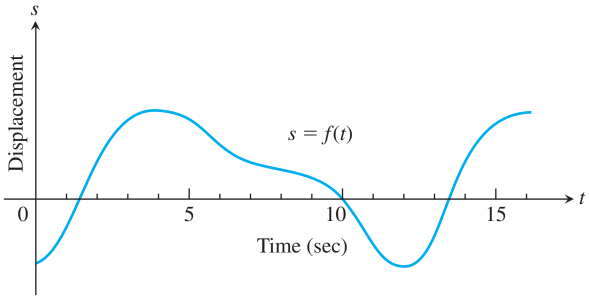


Motion Along a Line The graphs in Exercises 105 and 106 show the position $s = f(t)$ of an object moving up and down on a coordinate line. **(a)** When is the object moving away from the origin? Toward the origin? At approximately what times is the **(b)** velocity equal to zero? **(c)** Acceleration equal to zero? **(d)** When is the acceleration positive? Negative?

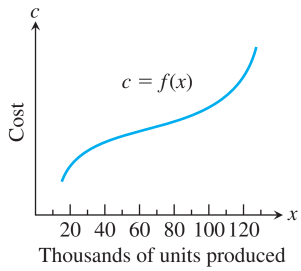
105.



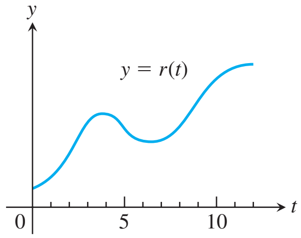
106.



107. Marginal cost The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing x items. At approximately what production level does the marginal cost change from decreasing to increasing?



108. The accompanying graph shows the monthly revenue of the Widget Corporation for the past 12 years. During approximately what time intervals was the marginal revenue increasing? Decreasing?



109. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection? (*Hint:* Draw the sign pattern for y'.)

110. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)(x - 4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

111. For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

112. Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

113. If b, c, and d are constants, for what value of b will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

114. Parabolas

a. Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c, a \neq 0$.

b. When is the parabola concave up? Concave down? Give reasons for your answers.

115. Quadratic curves What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c, a \neq 0$? Give reasons for your answer.

116. Cubic curves What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d, a \neq 0$? Give reasons for your answer.

117. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = (x + 1)(x - 2).$$

For what x-values does the graph of f have an inflection point?

118. Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = x^2(x - 2)^3(x + 3).$$

For what x-values does the graph of f have an inflection point?

119. Find the values of constants a, b, and c so that the graph of $y = ax^3 + bx^2 + cx$ has a local maximum at $x = 3$, local minimum at $x = -1$, and inflection point at $(1, 11)$.

120. Find the values of constants a, b, and c so that the graph of $y = (x^2 + a)/(bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

COMPUTER EXPLORATIONS

In Exercises 121–126, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the x-axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

121. $y = x^5 - 5x^4 - 240$

122. $y = x^3 - 12x^2$

123. $y = \frac{4}{5}x^5 + 16x^2 - 25$

124. $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$

125. Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of f in relation to the signs and values of f' and f''.

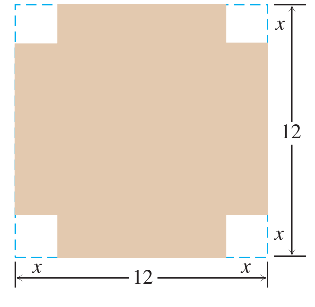
126. Graph $f(x) = x \cos x$ and its second derivative together for $0 \leq x \leq 2\pi$. Comment on the behavior of the graph of f in relation to the signs and values of f''.

4.5 Applied Optimization

What are the dimensions of a rectangle with fixed perimeter having *maximum area*? What are the dimensions for the *least expensive* cylindrical can of a given volume? How many items should be produced for the *most profitable* production run? Each of these questions asks for the best, or optimal, value of a given function. In this section we use derivatives to solve a variety of optimization problems in mathematics, physics, economics, and business.

Solving Applied Optimization Problems

1. *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture.* Label any part that may be important to the problem.
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.



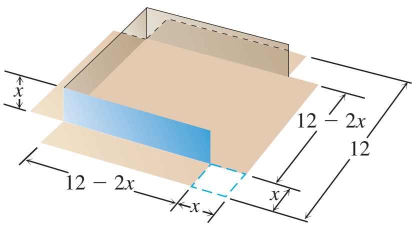


FIGURE 4.36 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

EXAMPLE 1 An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution We start with a picture (Figure 4.36). In the figure, the corner squares are x in. on a side. The volume of the box is a function of this variable:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3. \qquad V = hlw$$

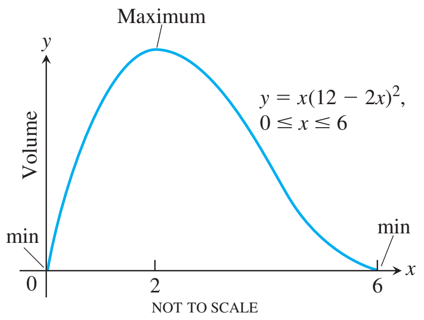


FIGURE 4.37 The volume of the box in Figure 4.36 graphed as a function of x.

Since the sides of the sheet of tin are only 12 in. long, $x \le 6$ and the domain of V is the interval $0 \le x \le 6$.

A graph of V (Figure 4.37) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. To learn more, we examine the first derivative of V with respect to x:

$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x).$$

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are

Critical point value: $V(2) = 128$

Endpoint values: $V(0) = 0, \qquad V(6) = 0.$

The maximum volume is 128 in^3. The cutout squares should be 2 in. on a side. ■

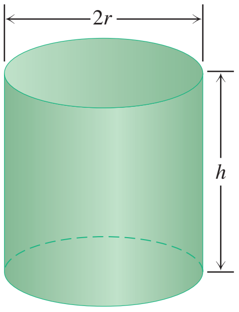


FIGURE 4.38 This one-liter can uses the least material when $h = 2r$ (Example 2).

EXAMPLE 2 You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 4.38). What dimensions will use the least material?

Solution *Volume of can:* If r and h are measured in centimeters, then the volume of the can in cubic centimeters is

$$\pi r^2 h = 1000. \qquad \text{1 liter} = 1000 \text{ cm}^3$$

Surface area of can:
$$A = \underbrace{2\pi r^2}_{\text{circular ends}} + \underbrace{2\pi r h}_{\text{cylindrical wall}}$$

How can we interpret the phrase "least material"? For a first approximation we can ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area as small as possible while satisfying the constraint $\pi r^2 h = 1000 \text{ cm}^3$.

To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for h is easier:

$$h = \frac{1000}{\pi r^2}.$$

Thus,

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) \\ &= 2\pi r^2 + \frac{2000}{r}. \end{aligned}$$

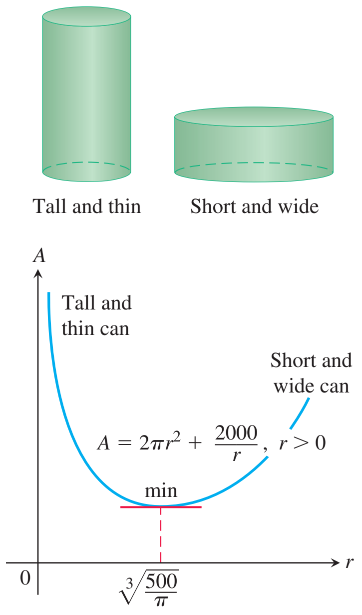


FIGURE 4.39 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

Our goal is to find a value of $r > 0$ that minimizes the value of A. Figure 4.39 suggests that such a value exists.

Notice from the graph that for small r (a tall, thin cylindrical container), the term $2000/r$ dominates (see Section 2.6) and A is large. A very thin cylinder containing 1 liter is so tall that its surface area becomes very large. For large r (a short, wide cylindrical container), the term $2\pi r^2$ dominates and A again is large.

Since A is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$0 = 4\pi r - \frac{2000}{r^2} \qquad \text{Set } dA/dr = 0.$$

$$4\pi r^3 = 2000 \qquad \text{Multiply by } r^2.$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \qquad \text{Solve for } r.$$

What happens at $r = \sqrt[3]{500/\pi}$?

The second derivative

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of A. The graph is therefore everywhere concave up and the value of A at $r = \sqrt[3]{500/\pi}$ is an absolute minimum.

The corresponding value of h (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

The one-liter can that uses the least material has height equal to twice the radius, here with $r \approx 5.42$ cm and $h \approx 10.84$ cm. ■

Examples from Mathematics and Physics

EXAMPLE 3 A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

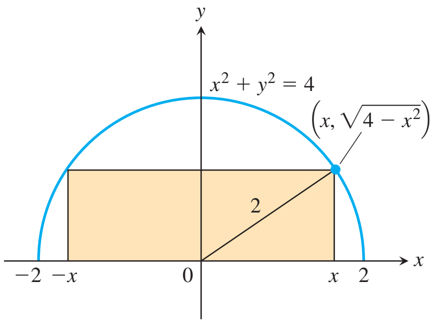


FIGURE 4.40 The rectangle inscribed in the semicircle in Example 3.

Solution Let $\left(x, \sqrt{4 - x^2}\right)$ be the coordinates of the corner of the rectangle obtained by placing the circle and rectangle in the coordinate plane (Figure 4.40). The length, height, and area of the rectangle can then be expressed in terms of the position x of the lower right-hand corner:

$$\text{Length: } 2x, \qquad \text{Height: } \sqrt{4 - x^2}, \qquad \text{Area: } 2x\sqrt{4 - x^2}.$$

Notice that the values of x are to be found in the interval $0 \le x \le 2$, where the selected corner of the rectangle lies.

Our goal is to find the absolute maximum value of the function

$$A(x) = 2x\sqrt{4 - x^2}$$

on the domain $[0, 2]$.

The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

is not defined when $x = 2$ and is equal to zero when

$$\frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} = 0$$

$$-2x^2 + 2(4 - x^2) = 0$$

$$8 - 4x^2 = 0$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}.$$

Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of A's domain and makes the critical-point list. The values of A at the endpoints and at this one critical point are

$$\text{Critical point value:} \quad A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4-2} = 4$$
$$\text{Endpoint values:} \quad A(0) = 0, \qquad A(2) = 0.$$

The area has a maximum value of 4 when the rectangle is $\sqrt{4 - x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long.

HISTORICAL BIOGRAPHY
Willebrord Snell van Royen
(1580–1626)
www.goo.gl/yEeoAi

EXAMPLE 4 The speed of light depends on the medium through which it travels, and is generally slower in denser media.

Fermat's principle in optics states that light travels from one point to another along a path for which the time of travel is a minimum. Describe the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 to a point B in a second medium where its speed is c_2.

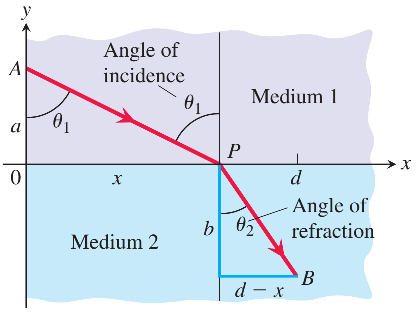


FIGURE 4.41 A light ray refracted (deflected from its path) as it passes from one medium to a denser medium (Example 4).

Solution Since light traveling from A to B follows the quickest route, we look for a path that will minimize the travel time. We assume that A and B lie in the xy-plane and that the line separating the two media is the x-axis (Figure 4.41). We place A at coordinates $(0, a)$ and B at coordinates $(d, -b)$ in the xy-plane.

In a uniform medium, where the speed of light remains constant, "shortest time" means "shortest path," and the ray of light will follow a straight line. Thus the path from A to B will consist of a line segment from A to a boundary point P, followed by another line segment from P to B. Distance traveled equals rate times time, so

$$\text{Time} = \frac{\text{distance}}{\text{rate}}.$$

From Figure 4.41, the time required for light to travel from A to P is

$$t_1 = \frac{AP}{c_1} = \frac{\sqrt{a^2 + x^2}}{c_1}.$$

From P to B, the time is

$$t_2 = \frac{PB}{c_2} = \frac{\sqrt{b^2 + (d-x)^2}}{c_2}.$$

The time from A to B is the sum of these:

$$t = t_1 + t_2 = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d-x)^2}}{c_2}.$$

This equation expresses t as a differentiable function of x whose domain is $[0, d]$. We want to find the absolute minimum value of t on this closed interval. We find the derivative

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d-x}{c_2\sqrt{b^2 + (d-x)^2}}$$

and observe that it is continuous. In terms of the angles θ_1 and θ_2 in Figure 4.41,

$$\frac{dt}{dx} = \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2}.$$

The function t has a negative derivative at $x = 0$ and a positive derivative at $x = d$. Since dt/dx is continuous over the interval $[0, d]$, by the Intermediate Value Theorem for

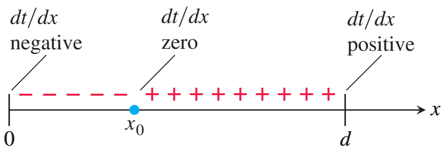


FIGURE 4.42 The sign pattern of dt/dx in Example 4.

continuous functions (Section 2.5), there is a point $x_0 \in [0, d]$ where $dt/dx = 0$ (Figure 4.42). There is only one such point because dt/dx is an increasing function of x (Exercise 60). At this unique point we then have

$$\frac{\sin \theta_1}{c_1} = \frac{\sin \theta_2}{c_2}.$$

This equation is **Snell's Law** or the **Law of Refraction**, and is an important principle in the theory of optics. It describes the path the ray of light follows. ■

Examples from Economics

Suppose that

$$\begin{aligned} r(x) &= \text{the revenue from selling } x \text{ items} \\ c(x) &= \text{the cost of producing the } x \text{ items} \\ p(x) &= r(x) - c(x) = \text{the profit from producing and selling } x \text{ items.} \end{aligned}$$

Although x is usually an integer in many applications, we can learn about the behavior of these functions by defining them for all nonzero real numbers and by assuming they are differentiable functions. Economists use the terms **marginal revenue**, **marginal cost**, and **marginal profit** to name the derivatives $r'(x)$, $c'(x)$, and $p'(x)$ of the revenue, cost, and profit functions. Let's consider the relationship of the profit p to these derivatives.

If $r(x)$ and $c(x)$ are differentiable for x in some interval of production possibilities, and if $p(x) = r(x) - c(x)$ has a maximum value there, it occurs at a critical point of $p(x)$ or at an endpoint of the interval. If it occurs at a critical point, then $p'(x) = r'(x) - c'(x) = 0$ and we see that $r'(x) = c'(x)$. In economic terms, this last equation means that

> At a production level yielding maximum profit, marginal revenue equals marginal cost (Figure 4.43).

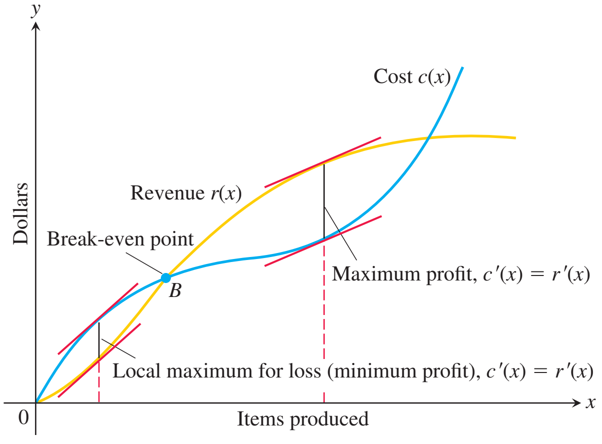


FIGURE 4.43 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of rising labor and material costs and market saturation) and production levels become unprofitable again.

EXAMPLE 5 Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

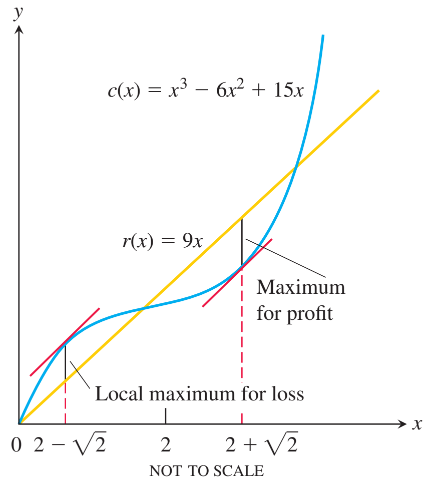


FIGURE 4.44 The cost and revenue curves for Example 5.

Solution Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$.

$$3x^2 - 12x + 15 = 9 \qquad \text{Set } c'(x) = r'(x).$$
$$3x^2 - 12x + 6 = 0$$

The two solutions of the quadratic equation are

$$x_1 = \frac{12 - \sqrt{72}}{6} = 2 - \sqrt{2} \approx 0.586 \qquad \text{and}$$
$$x_2 = \frac{12 + \sqrt{72}}{6} = 2 + \sqrt{2} \approx 3.414.$$

The possible production levels for maximum profit are $x \approx 0.586$ million MP3 players or $x \approx 3.414$ million. The second derivative of $p(x) = r(x) - c(x)$ is $p''(x) = -c''(x)$ since $r''(x)$ is everywhere zero. Thus, $p''(x) = 6(2 - x)$, which is negative at $x = 2 + \sqrt{2}$ and positive at $x = 2 - \sqrt{2}$. By the Second Derivative Test, a maximum profit occurs at about $x = 3.414$ (where revenue exceeds costs) and maximum loss occurs at about $x = 0.586$. The graphs of $r(x)$ and $c(x)$ are shown in Figure 4.44. ■

EXAMPLE 6 A cabinetmaker uses cherry wood to produce 5 desks each day. Each delivery of one container of wood is \$5000, whereas the storage of that material is \$10 per day per unit stored, where a unit is the amount of material needed by her to produce 1 desk. How much material should be ordered each time, and how often should the material be delivered, to minimize her average daily cost in the production cycle between deliveries?

Solution If she asks for a delivery every x days, then she must order $5x$ units to have enough material for that delivery cycle. The *average* amount in storage is approximately one-half of the delivery amount, or $5x/2$. Thus, the cost of delivery and storage for each cycle is approximately

$$\text{Cost per cycle} = \text{delivery costs} + \text{storage costs}$$

$$\text{Cost per cycle} = \underbrace{5000}_{\substack{\text{delivery}\\ \text{cost}}} + \underbrace{\left(\frac{5x}{2}\right)}_{\substack{\text{average}\\ \text{amount stored}}} \cdot \underbrace{x}_{\substack{\text{number of}\\ \text{days stored}}} \cdot \underbrace{10}_{\substack{\text{storage cost}\\ \text{per day}}}$$

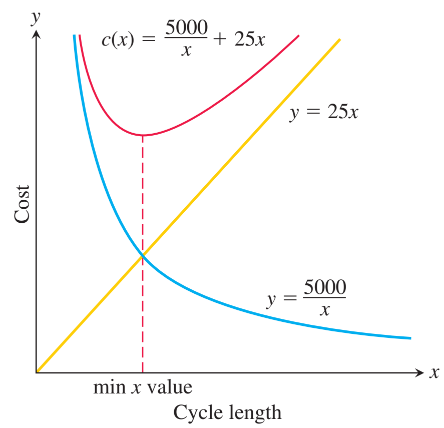


FIGURE 4.45 The average daily cost $c(x)$ is the sum of a hyperbola and a linear function (Example 6).

We compute the *average daily cost* $c(x)$ by dividing the cost per cycle by the number of days x in the cycle (see Figure 4.45).

$$c(x) = \frac{5000}{x} + 25x, \qquad x > 0.$$

As $x \to 0$ and as $x \to \infty$, the average daily cost becomes large. So we expect a minimum to exist, but where? Our goal is to determine the number of days x between deliveries that provides the absolute minimum cost.

We find the critical points by determining where the derivative is equal to zero:

$$c'(x) = -\frac{500}{x^2} + 25 = 0$$
$$x = \pm\sqrt{200} \approx \pm 14.14.$$

Of the two critical points, only $\sqrt{200}$ lies in the domain of $c(x)$. The critical point value of the average daily cost is

$$c(\sqrt{200}) = \frac{5000}{\sqrt{200}} + 25\sqrt{200} = 500\sqrt{2} \approx \$707.11.$$

We note that $c(x)$ is defined over the open interval $(0, \infty)$ with $c''(x) = 10000/x^3 > 0$. Thus, an absolute minimum exists at $x = \sqrt{200} \approx 14.14$ days.

The cabinetmaker should schedule a delivery of $5(14) = 70$ units of wood every 14 days.

EXERCISES 4.5

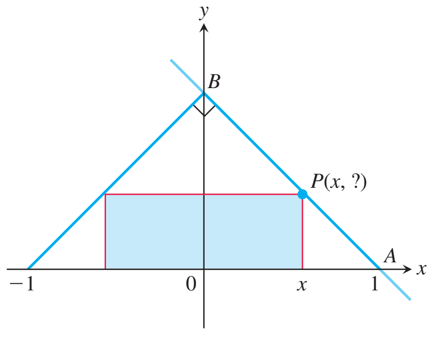
Mathematical Applications

Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph it over the domain that is appropriate to the problem you are solving. The graph will provide insight before you calculate and will furnish a visual context for understanding your answer.

1. **Minimizing perimeter** What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions?

2. Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.

3. The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.
   a. Express the y-coordinate of P in terms of x. (*Hint:* Write an equation for the line AB.)
   b. Express the area of the rectangle in terms of x.
   c. What is the largest area the rectangle can have, and what are its dimensions?



4. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

5. You are planning to make an open rectangular box from an 8-in.-by-15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?

6. You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.

7. **The best fencing plan** A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

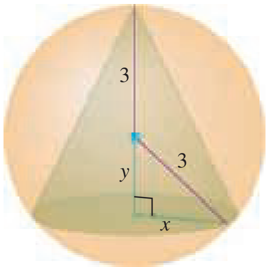
8. **The shortest fence** A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

9. **Designing a tank** Your iron works has contracted to design and build a 500 ft^3, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.
   a. What dimensions do you tell the shop to use?
   b. Briefly describe how you took weight into account.

10. **Catching rainwater** A 1125 ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product xy.
    a. If the total cost is
    $$c = 5(x^2 + 4xy) + 10xy,$$
    what values of x and y will minimize it?
    b. Give a possible scenario for the cost function in part (a).

11. **Designing a poster** You are designing a rectangular poster to contain 50 in^2 of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used?

12. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



13. Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? (*Hint:* $A = (1/2)ab\sin\theta$.)

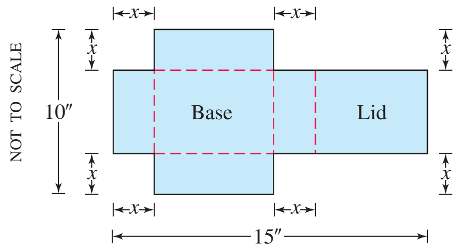
14. **Designing a can** What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 2.

15. **Designing a can** You are designing a 1000 cm^3 right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 2. In Example 2, the ratio of h to r for the most economical can was 2 to 1. What is the ratio now?

T 16. **Designing a box with a lid** A piece of cardboard measures 10 in. by 15 in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.



a. Write a formula $V(x)$ for the volume of the box.

b. Find the domain of V for the problem situation and graph V over this domain.

c. Use a graphical method to find the maximum volume and the value of x that gives it.

d. Confirm your result in part (c) analytically.

T 17. **Designing a suitcase** A 24-in.-by-36-in. sheet of cardboard is folded in half to form a 24-in.-by-18-in. rectangle as shown in the accompanying figure. Then four congruent squares of side length x are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.

a. Write a formula $V(x)$ for the volume of the box.

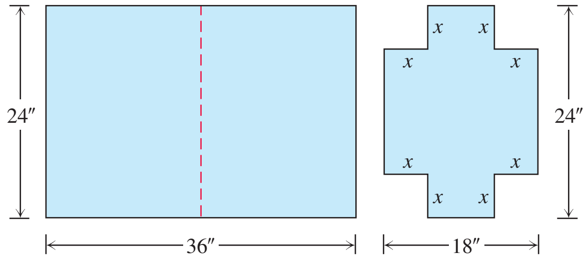
b. Find the domain of V for the problem situation and graph V over this domain.

c. Use a graphical method to find the maximum volume and the value of x that gives it.

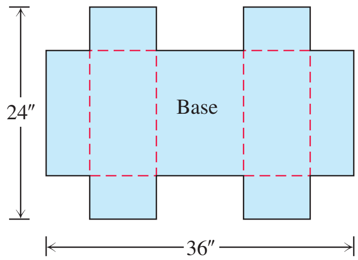
d. Confirm your result in part (c) analytically.

e. Find a value of x that yields a volume of 1120 in^3.

f. Write a paragraph describing the issues that arise in part (b).



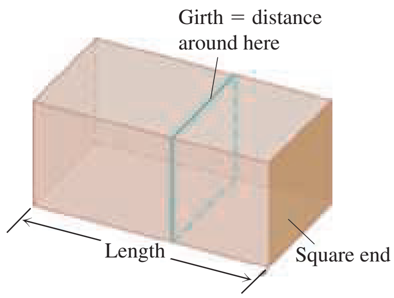
The sheet is then unfolded.



18. A rectangle is to be inscribed under the arch of the curve $y = 4\cos(0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

19. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

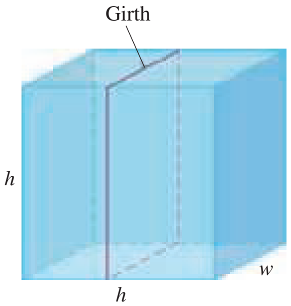
20. a. The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?



T b. Graph the volume of a 108-in. box (length plus girth equals 108 in.) as a function of its length and compare what you see with your answer in part (a).

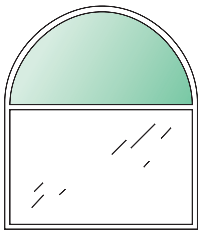
21. (*Continuation of Exercise 20.*)

a. Suppose that instead of having a box with square ends you have a box with square sides so that its dimensions are h by h by w and the girth is $2h + 2w$. What dimensions will give the box its largest volume now?



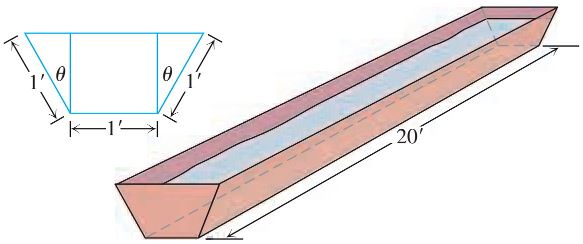
T **b.** Graph the volume as a function of h and compare what you see with your answer in part (a).

22. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



23. A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit of surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and waste in construction.

24. The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?

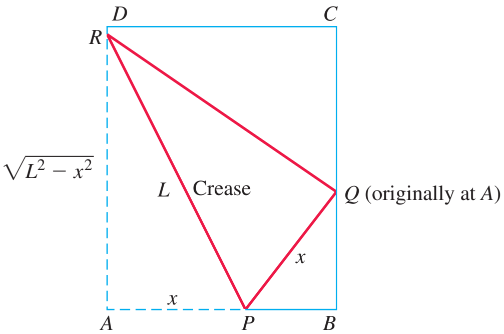


25. Paper folding A rectangular sheet of 8.5-in.-by-11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length L. Try it with paper.

a. Show that $L^2 = 2x^3/(2x - 8.5)$.

b. What value of x minimizes L^2?

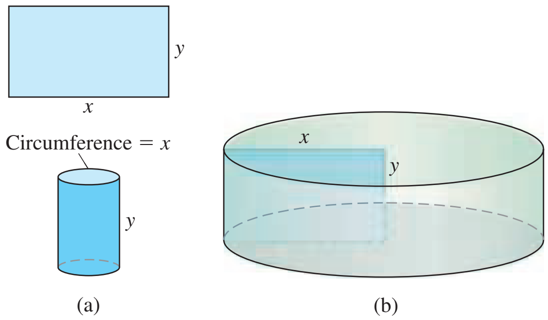
c. What is the minimum value of L?



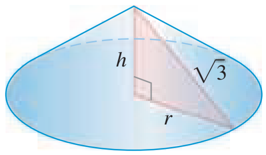
26. Constructing cylinders Compare the answers to the following two construction problems.

a. A rectangular sheet of perimeter 36 cm and dimensions x cm by y cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of x and y give the largest volume?

b. The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder as shown in part (b) of the figure. What values of x and y give the largest volume?



27. Constructing cones A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.



28. Find the point on the line $\frac{x}{a} + \frac{y}{b} = 1$ that is closest to the origin.

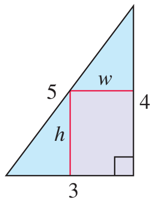
29. Find a positive number for which the sum of it and its reciprocal is the smallest (least) possible.

30. Find a positive number for which the sum of its reciprocal and four times its square is the smallest possible.

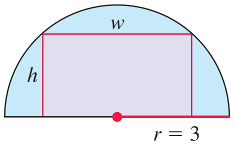
31. A wire b m long is cut into two pieces. One piece is bent into an equilateral triangle and the other is bent into a circle. If the sum of the areas enclosed by each part is a minimum, what is the length of each part?

32. Answer Exercise 31 if one piece is bent into a square and the other into a circle.

33. Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle shown in the accompanying figure.



34. Determine the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 3. (See accompanying figure.)



35. What value of a makes $f(x) = x^2 + (a/x)$ have

a. a local minimum at $x = 2$?

b. a point of inflection at $x = 1$?

36. What values of a and b make $f(x) = x^3 + ax^2 + bx$ have

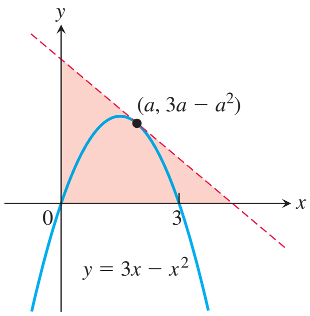
a. a local maximum at $x = -1$ and a local minimum at $x = 3$?

b. a local minimum at $x = 4$ and a point of inflection at $x = 1$?

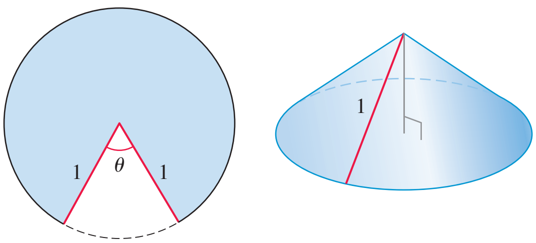
37. A right circular cone is circumscribed in a sphere of radius 1. Determine the height h and radius r of the cone of maximum volume.

38. Find the point on the graph of $y = 20x^3 + 60x - 3x^5 - 5x^4$ with the largest slope.

39. Among all triangles in the first quadrant formed by the x-axis, the y-axis, and tangent lines to the graph of $y = 3x - x^2$, what is the smallest possible area?



40. A cone is formed from a circular piece of material of radius 1 meter by removing a section of angle θ and then joining the two straight edges. Determine the largest possible volume for the cone.



Physical Applications

41. Vertical motion The height above ground of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

with s in feet and t in seconds. Find

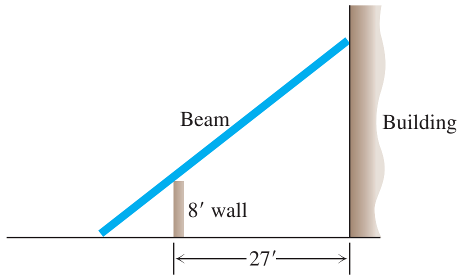
a. the object's velocity when $t = 0$;

b. its maximum height and when it occurs;

c. its velocity when $s = 0$.

42. Quickest route Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

43. Shortest beam The 8-ft wall shown here stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.



44. Motion on a line The positions of two particles on the s-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with s_1 and s_2 in meters and t in seconds.

a. At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?

b. What is the farthest apart that the particles ever get?

c. When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

45. The intensity of illumination at any point from a light source is proportional to the square of the reciprocal of the distance between the point and the light source. Two lights, one having an intensity eight times that of the other, are 6 m apart. How far from the stronger light is the total illumination least?

46. Projectile motion The *range* R of a projectile fired from the origin over horizontal ground is the distance from the origin to the point of impact. If the projectile is fired with an initial velocity v_0 at an angle α with the horizontal, then in Chapter 13 we find that

$$R = \frac{v_0^2}{g}\sin 2\alpha,$$

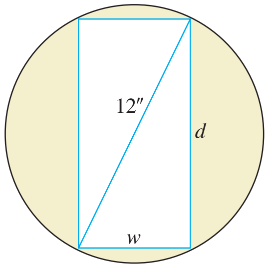
where g is the downward acceleration due to gravity. Find the angle α for which the range R is the largest possible.

T **47. Strength of a beam** The strength S of a rectangular wooden beam is proportional to its width times the square of its depth. (See the accompanying figure.)

a. Find the dimensions of the strongest beam that can be cut from a 12-in.-diameter cylindrical log.

b. Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

d. On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k? Try it.



T **48. Stiffness of a beam** The stiffness S of a rectangular beam is proportional to its width times the cube of its depth.

a. Find the dimensions of the stiffest beam that can be cut from a 12-in.-diameter cylindrical log.

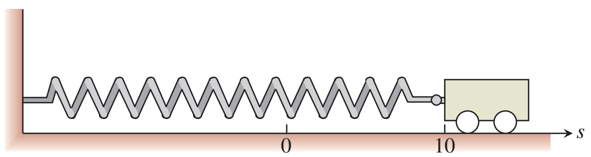
b. Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

c. On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k? Try it.

49. Frictionless cart A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is $s = 10 \cos \pi t$.

a. What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

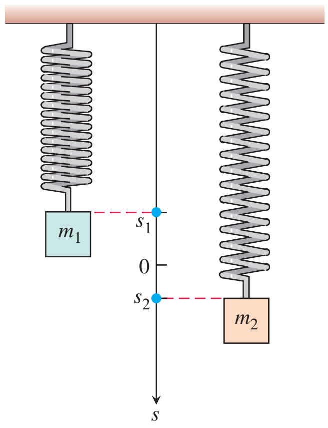
b. Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



50. Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively.

a. At what times in the interval $0 < t$ do the masses pass each other? (*Hint:* $\sin 2t = 2 \sin t \cos t$.)

b. When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2\cos^2 t - 1$.)



51. Distance between two ships At noon, ship A was 12 nautical miles due north of ship B. Ship A was sailing south at 12 knots (nautical miles per hour; a nautical mile is 2000 yd) and continued to do so all day. Ship B was sailing east at 8 knots and continued to do so all day.

a. Start counting time with $t = 0$ at noon and express the distance s between the ships as a function of t.

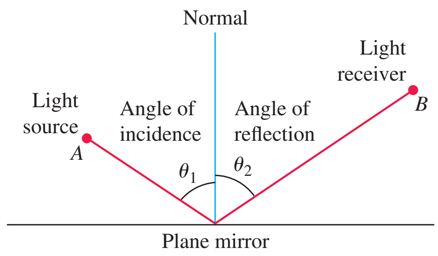
b. How rapidly was the distance between the ships changing at noon? One hour later?

c. The visibility that day was 5 nautical miles. Did the ships ever sight each other?

T **d.** Graph s and ds/dt together as functions of t for $-1 \le t \le 3$, using different colors if possible. Compare the graphs and reconcile what you see with your answers in parts (b) and (c).

e. The graph of ds/dt looks as if it might have a horizontal asymptote in the first quadrant. This in turn suggests that ds/dt approaches a limiting value as $t \to \infty$. What is this value? What is its relation to the ships' individual speeds?

52. Fermat's principle in optics Light from a source A is reflected by a plane mirror to a receiver at point B, as shown in the accompanying figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)



53. Tin pest When metallic tin is kept below 13.2°C, it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious, and indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases, it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, v may be considered to be a function of x alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

$x =$ the amount of product

$a =$ the amount of substance at the beginning

$k =$ a positive constant.

At what value of x does the rate v have a maximum? What is the maximum value of v?

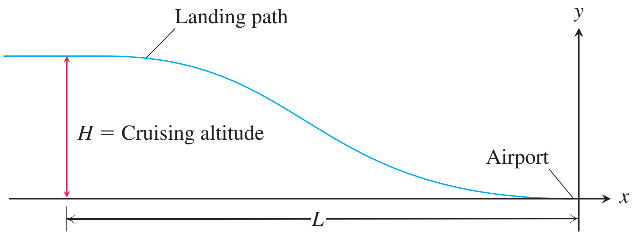
54. Airplane landing path An airplane is flying at altitude H when it begins its descent to an airport runway that is at horizontal ground distance L from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$, where $y(-L) = H$ and $y(0) = 0$.

a. What is dy/dx at $x = 0$?

b. What is dy/dx at $x = -L$?

c. Use the values for dy/dx at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$



Business and Economics

55. It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where a and b are positive constants. What selling price will bring a maximum profit?

56. You operate a tour service that offers the following rates:

$200 per person if 50 people (the minimum number to book the tour) go on the tour.

For each additional person, up to a maximum of 80 people total, the rate per person is reduced by $2.

It costs $6000 (a fixed cost) plus $32 per person to conduct the tour. How many people does it take to maximize your profit?

57. Wilson lot size formula One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

a. Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula*.)

b. Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q. What is the most economical quantity to order now?

58. Production level Prove that the production level (if any) at which average cost is smallest is a level at which the average cost equals marginal cost.

59. Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

60. Production level Suppose that $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost of making x items.

61. You are to construct an open rectangular box with a square base and a volume of 48 ft^3. If material for the bottom costs $\$6/\text{ft}^2$ and material for the sides costs $\$4/\text{ft}^2$, what dimensions will result in the least expensive box? What is the minimum cost?

62. The 800-room Mega Motel chain is filled to capacity when the room charge is $50 per night. For each $10 increase in room charge, 40 fewer rooms are filled each night. What charge per room will result in the maximum revenue per night?

Biology

63. Sensitivity to medicine (*Continuation of Exercise 60, Section 3.3.*) Find the amount of medicine to which the body is most sensitive by finding the value of M that maximizes the derivative dR/dM, where

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$$

and C is a constant.

64. How we cough

a. When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the questions of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity υ can be modeled by the equation

$$\upsilon = c(r_0 - r)r^2 \text{ cm/sec}, \qquad \frac{r_0}{2} \le r \le r_0,$$

where r_0 is the rest radius of the trachea in centimeters and c is a positive constant whose value depends in part on the length of the trachea.

Show that υ is greatest when $r = (2/3)r_0$; that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

T **b.** Take r_0 to be 0.5 and c to be 1 and graph υ over the interval $0 \le r \le 0.5$. Compare what you see with the claim that υ is at a maximum when $r = (2/3)r_0$.

Theory and Examples

65. An inequality for positive integers Show that if a, b, c, and d are positive integers, then

$$\frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd} \ge 16.$$

66. The derivative dt/dx in Example 4

a. Show that

$$f(x) = \frac{x}{\sqrt{a^2 + x^2}}$$

is an increasing function of x.

b. Show that

$$g(x) = \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$$

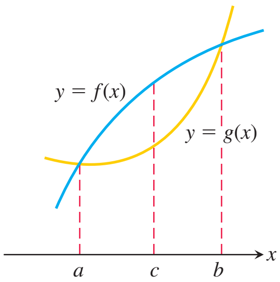
is a decreasing function of x.

c. Show that

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d - x}{c_2\sqrt{b^2 + (d - x)^2}}$$

is an increasing function of x.

67. Let $f(x)$ and $g(x)$ be the differentiable functions graphed here. Point c is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangents to the two curves at c? Give reasons for your answer.



68. You have been asked to determine whether the function $f(x) = 3 + 4\cos x + \cos 2x$ is ever negative.

a. Explain why you need to consider values of x only in the interval $[0, 2\pi]$.

b. Is f ever negative? Explain.

69. a. The function $y = \cot x - \sqrt{2}\csc x$ has an absolute maximum value on the interval $0 < x < \pi$. Find it.

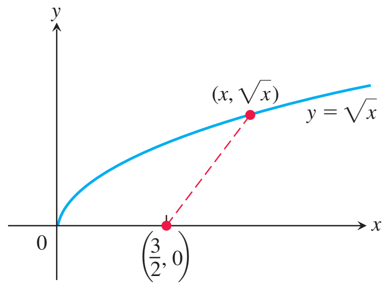
T **b.** Graph the function and compare what you see with your answer in part (a).

70. a. The function $y = \tan x + 3\cot x$ has an absolute minimum value on the interval $0 < x < \pi/2$. Find it.

T **b.** Graph the function and compare what you see with your answer in part (a).

71. a. How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? (*Hint:* If you minimize the *square* of the distance, you can avoid square roots.)

T **b.** Graph the distance function $D(x)$ and $y = \sqrt{x}$ together and reconcile what you see with your answer in part (a).



72. a. How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $(1, \sqrt{3})$?

T **b.** Graph the distance function and $y = \sqrt{16 - x^2}$ together and reconcile what you see with your answer in part (a).

4.6 Newton's Method

For thousands of years, one of the main goals of mathematics has been to find solutions to equations. For linear equations $(ax + b = 0)$, and for quadratic equations $(ax^2 + bx + c = 0)$, we can explicitly solve for a solution. However, for most equations there is no simple formula that gives the solutions.

In this section we study a numerical method called *Newton's method* or the *Newton–Raphson method*, which is a technique to approximate the solutions to an equation $f(x) = 0$. Newton's method estimates the solutions using tangent lines of the graph of $y = f(x)$ near the points where f is zero. A value of x where f is zero is called a *root* of the

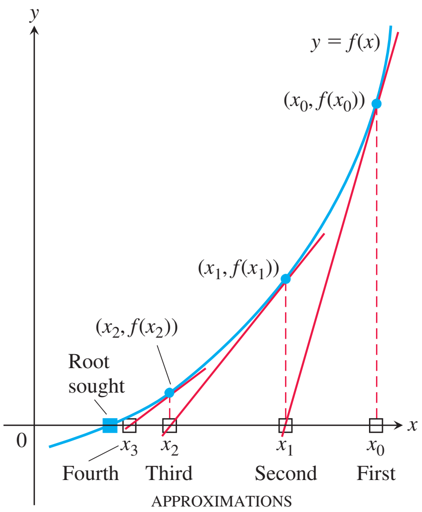


FIGURE 4.46 Newton's method starts with an initial guess x_0 and (under favorable circumstances) improves the guess one step at a time.

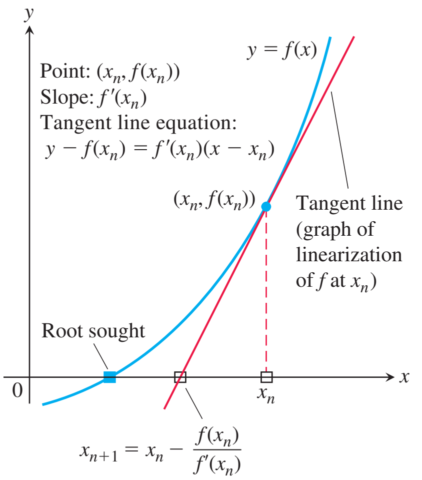


FIGURE 4.47 The geometry of the successive steps of Newton's method. From x_n we go up to the curve and follow the tangent line down to find x_{n+1}.

function f and a *solution* of the equation $f(x) = 0$. Newton's method is both powerful and efficient, and it has numerous applications in engineering and other fields where solutions to complicated equations are needed.

Procedure for Newton's Method

The goal of Newton's method for estimating a solution of an equation $f(x) = 0$ is to produce a sequence of approximations that approach the solution. We pick the first number x_0 of the sequence. Then, under favorable circumstances, the method moves step by step toward a point where the graph of f crosses the x-axis (Figure 4.46). At each step the method approximates a zero of f with a zero of one of its linearizations. Here is how it works.

The initial estimate, x_0, may be found by graphing or just plain guessing. The method then uses the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ to approximate the curve, calling the point x_1 where the tangent meets the x-axis (Figure 4.46). The number x_1 is usually a better approximation to the solution than is x_0. The point x_2 where the tangent to the curve at $(x_1, f(x_1))$ crosses the x-axis is the next approximation in the sequence. We continue, using each approximation to generate the next, until we are close enough to the root to stop.

We can derive a formula for generating the successive approximations in the following way. Given the approximation x_n, the point-slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y = f(x_n) + f'(x_n)(x - x_n).$$

We can find where it crosses the x-axis by setting $y = 0$ (Figure 4.47):

$$0 = f(x_n) + f'(x_n)(x - x_n)$$

$$-\frac{f(x_n)}{f'(x_n)} = x - x_n$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{If } f'(x_n) \neq 0$$

This value of x is the next approximation x_{n+1}. Here is a summary of Newton's method.

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{if } f'(x_n) \neq 0. \tag{1}$$

Applying Newton's Method

Applications of Newton's method generally involve many numerical computations, making them well suited for computers or calculators. Nevertheless, even when the calculations are done by hand (which may be very tedious), they give a powerful way to find solutions of equations.

In our first example, we find decimal approximations to $\sqrt{2}$ by estimating the positive root of the equation $f(x) = x^2 - 2 = 0$.

EXAMPLE 1 Approximate the positive root of the equation

$$f(x) = x^2 - 2 = 0.$$

Solution With $f(x) = x^2 - 2$ and $f'(x) = 2x$, Equation (1) becomes

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n{}^2 - 2}{2x_n} \\ &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n}. \end{aligned}$$

The equation

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

enables us to go from each approximation to the next with just a few keystrokes. With the starting value $x_0 = 1$, we get the results in the first column of the following table. (To five decimal places, or, equivalently, to six digits, $\sqrt{2} = 1.41421$.)

	Error	Number of correct digits
$x_0 = 1$	-0.41421	1
$x_1 = 1.5$	0.08579	1
$x_2 = 1.41667$	0.00246	3
$x_3 = 1.41422$	0.00001	5

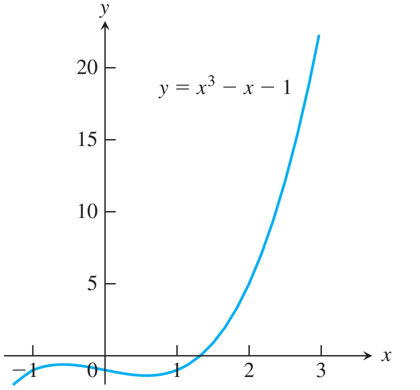


FIGURE 4.48 The graph of $f(x) = x^3 - x - 1$ crosses the x-axis once; this is the root we want to find (Example 2).

Newton's method is the method used by most software applications to calculate roots because it converges so fast (more about this later). If the arithmetic in the table in Example 1 had been carried to 13 decimal places instead of 5, then going one step further would have given $\sqrt{2}$ correctly to more than 10 decimal places.

EXAMPLE 2 Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution The curve crosses the line when $x^3 - x = 1$ or $x^3 - x - 1 = 0$. When does $f(x) = x^3 - x - 1$ equal zero? Since $f(1) = -1$ and $f(2) = 5$, we know by the Intermediate Value Theorem there is a root in the interval (1, 2) (Figure 4.48).

We apply Newton's method to f with the starting value $x_0 = 1$. The results are displayed in Table 4.1 and Figure 4.49.

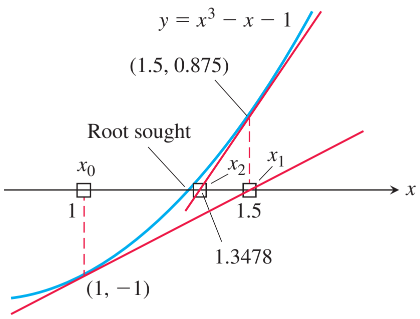


FIGURE 4.49 The first three x-values in Table 4.1 (four decimal places).

TABLE 4.1 **The result of applying Newton's method to $f(x) = x^3 - x - 1$ with $x_0 = 1$**

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	1	-1	2	1.5
1	1.5	0.875	5.75	1.3478 26087
2	1.3478 26087	0.1006 82173	4.4499 05482	1.3252 00399
3	1.3252 00399	0.0020 58362	4.2684 68292	1.3247 18174
4	1.3247 18174	0.0000 00924	4.2646 34722	1.3247 17957
5	1.3247 17957	$-1.8672\text{E-}13$	4.2646 32999	1.3247 17957

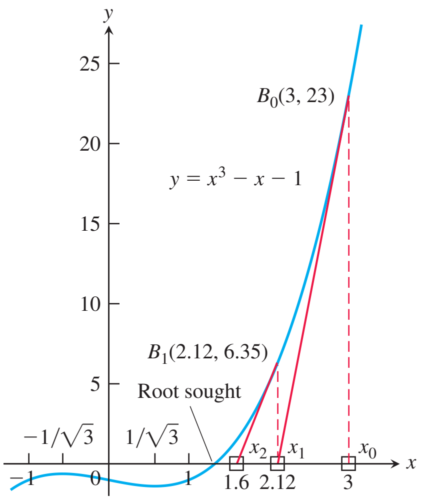


FIGURE 4.50 Any starting value x_0 to the right of $x = 1/\sqrt{3}$ will lead to the root in Example 2.

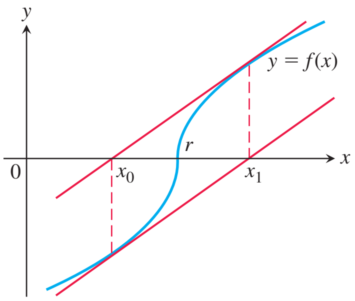


FIGURE 4.51 Newton's method fails to converge. You go from x_0 to x_1 and back to x_0, never getting any closer to r.

At $n = 5$, we come to the result $x_6 = x_5 = 1.3247\ 17957$. When $x_{n+1} = x_n$, Equation (1) shows that $f(x_n) = 0$, up to the accuracy of our computation. We have found a solution of $f(x) = 0$ to nine decimals. ■

In Figure 4.50 we have indicated that the process in Example 2 might have started at the point $B_0(3, 23)$ on the curve, with $x_0 = 3$. Point B_0 is quite far from the x-axis, but the tangent at B_0 crosses the x-axis at about (2.12, 0), so x_1 is still an improvement over x_0. If we use Equation (1) repeatedly as before, with $f(x) = x^3 - x - 1$ and $f'(x) = 3x^2 - 1$, we obtain the nine-place solution $x_7 = x_6 = 1.3247\ 17957$ in seven steps.

Convergence of the Approximations

In Chapter 10 we define precisely the idea of *convergence* for the approximations x_n in Newton's method. Intuitively, we mean that as the number n of approximations increases without bound, the values x_n get arbitrarily close to the desired root r. (This notion is similar to the idea of the limit of a function $g(t)$ as t approaches infinity, as defined in Section 2.6.)

In practice, Newton's method usually gives convergence with impressive speed, but this is not guaranteed. One way to test convergence is to begin by graphing the function to estimate a good starting value for x_0. You can test that you are getting closer to a zero of the function by checking that $|f(x_n)|$ is approaching zero, and you can check that the approximations are converging by evaluating $|x_n - x_{n+1}|$.

Newton's method does not always converge. For instance, if

$$f(x) = \begin{cases} -\sqrt{r - x}, & x < r \\ \sqrt{x - r}, & x \geq r, \end{cases}$$

the graph will be like the one in Figure 4.51. If we begin with $x_0 = r - h$, we get $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.

If Newton's method does converge, it converges to a root. Be careful, however. There are situations in which the method appears to converge but no root is there. Fortunately, such situations are rare.

When Newton's method converges to a root, it may not be the root you have in mind. Figure 4.52 shows two ways this can happen.

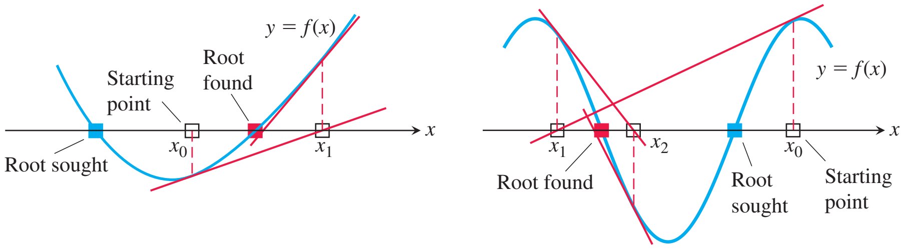


FIGURE 4.52 If you start too far away, Newton's method may miss the root you want.

EXERCISES 4.6

Root Finding

1. Use Newton's method to estimate the solutions of the equation $x^2 + x - 1 = 0$. Start with $x_0 = -1$ for the left-hand solution and with $x_0 = 1$ for the solution on the right. Then, in each case, find x_2.
2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2.
3. Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find x_2.
4. Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2.

5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2.

6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find x_2.

7. Use Newton's method to find an approximate solution of $3 - x = x^3$. Start with $x_0 = 1$ and find x_2.

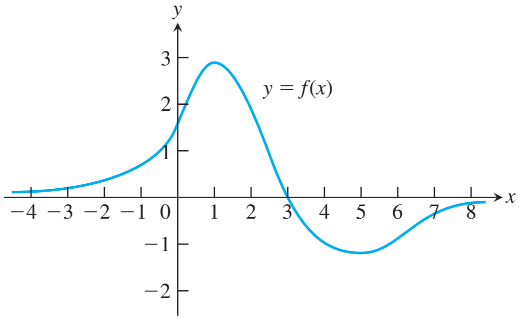
Dependence on Initial Point

8. Using the function shown in the figure, and for each initial estimate x_0, determine graphically what happens to the sequence of Newton's method approximations

a. $x_0 = 0$ **b.** $x_0 = 1$

c. $x_0 = 2$ **d.** $x_0 = 4$

e. $x_0 = 5.5$



9. Guessing a root Suppose that your first guess is lucky, in the sense that x_0 is a root of $f(x) = 0$. Assuming that $f'(x_0)$ is defined and not 0, what happens to x_1 and later approximations?

10. Estimating pi You plan to estimate $\pi/2$ to five decimal places by using Newton's method to solve the equation $\cos x = 0$. Does it matter what your starting value is? Give reasons for your answer.

Theory and Examples

11. Oscillation Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \ge 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_1 = -h$ if $x_0 = h$ and to $x_1 = h$ if $x_0 = -h$. Draw a picture that shows what is going on.

12. Approximations that get worse and worse Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$ and calculate x_1, x_2, x_3, and x_4. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

13. Explain why the following four statements ask for the same information:

i) Find the roots of $f(x) = x^3 - 3x - 1$.

ii) Find the x-coordinates of the intersections of the curve $y = x^3$ with the line $y = 3x + 1$.

iii) Find the x-coordinates of the points where the curve $y = x^3 - 3x$ crosses the horizontal line $y = 1$.

iv) Find the values of x where the derivative of $g(x) = (1/4)x^4 - (3/2)x^2 - x + 5$ equals zero.

14. Locating a planet To calculate a planet's space coordinates, we have to solve equations like $x = 1 + 0.5 \sin x$. Graphing the function $f(x) = x - 1 - 0.5 \sin x$ suggests that the function has a root near $x = 1.5$. Use one application of Newton's method to improve this estimate. That is, start with $x_0 = 1.5$ and find x_1. (The value of the root is 1.49870 to five decimal places.) Remember to use radians.

15. Intersecting curves The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find where.

16. Real solutions of a quartic Use Newton's method to find the two real solutions of the equation $x^4 - 2x^3 - x^2 - 2x + 2 = 0$.

17. a. How many solutions does the equation $\sin 3x = 0.99 - x^2$ have?

b. Use Newton's method to find them.

18. Intersection of curves

a. Does $\cos 3x$ ever equal x? Give reasons for your answer.

b. Use Newton's method to find where.

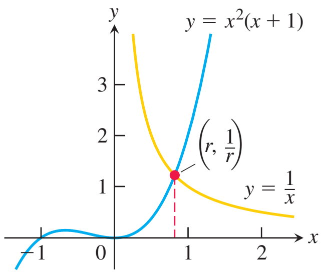
19. Find the four real zeros of the function $f(x) = 2x^4 - 4x^2 + 1$.

20. Estimating pi Estimate π to as many decimal places as your calculator will display by using Newton's method to solve the equation $\tan x = 0$ with $x_0 = 3$.

21. Intersection of curves At what value(s) of x does $\cos x = 2x$?

22. Intersection of curves At what value(s) of x does $\cos x = -x$?

23. The graphs of $y = x^2(x + 1)$ and $y = 1/x$ $(x > 0)$ intersect at one point $x = r$. Use Newton's method to estimate the value of r to four decimal places.

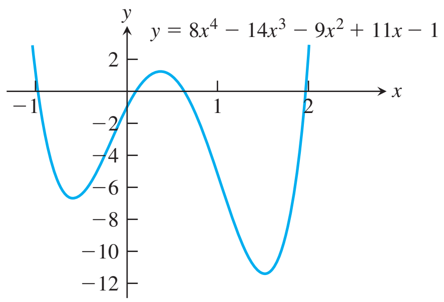


24. The graphs of $y = \sqrt{x}$ and $y = 3 - x^2$ intersect at one point $x = r$. Use Newton's method to estimate the value of r to four decimal places.

25. Use the Intermediate Value Theorem from Section 2.5 to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then find the root to five decimal places.

26. Factoring a quartic Find the approximate values of r_1 through r_4 in the factorization

$$8x^4 - 14x^3 - 9x^2 + 11x - 1 = 8(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$



T **27. Converging to different zeros** Use Newton's method to find the zeros of $f(x) = 4x^4 - 4x^2$ using the given starting values.

a. $x_0 = -2$ and $x_0 = -0.8$, lying in $\left(-\infty, -\sqrt{2}/2\right)$

b. $x_0 = -0.5$ and $x_0 = 0.25$, lying in $\left(-\sqrt{21}/7, \sqrt{21}/7\right)$

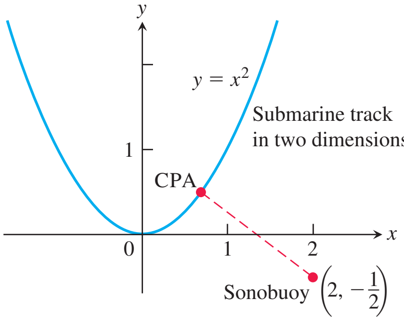
c. $x_0 = 0.8$ and $x_0 = 2$, lying in $\left(\sqrt{2}/2, \infty\right)$

d. $x_0 = -\sqrt{21}/7$ and $x_0 = \sqrt{21}/7$

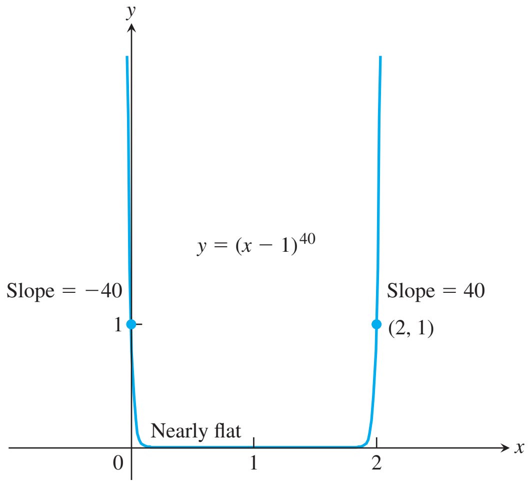
28. The sonobuoy problem In submarine location problems, it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that the submarine travels on the parabolic path $y = x^2$ and that the buoy is located at the point $(2, -1/2)$.

a. Show that the value of x that minimizes the distance between the submarine and the buoy is a solution of the equation $x = 1/(x^2 + 1)$.

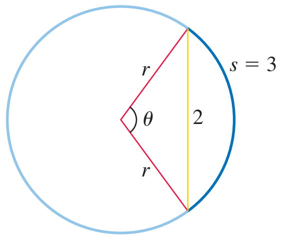
b. Solve the equation $x = 1/(x^2 + 1)$ with Newton's method.



T **29. Curves that are nearly flat at the root** Some curves are so flat that, in practice, Newton's method stops too far from the root to give a useful estimate. Try Newton's method on $f(x) = (x - 1)^{40}$ with a starting value of $x_0 = 2$ to see how close your machine comes to the root $x = 1$. See the accompanying graph.



30. The accompanying figure shows a circle of radius r with a chord of length 2 and an arc s of length 3. Use Newton's method to solve for r and θ (radians) to four decimal places. Assume $0 < \theta < \pi$.



4.7 Antiderivatives

Many problems require that we recover a function from its derivative, or from its rate of change. For instance, the laws of physics tell us the acceleration of an object falling from an initial height, and we can use this to compute its velocity and its height at any time. More generally, starting with a function f, we want to find a function F whose derivative is f. If such a function F exists, it is called an *antiderivative* of f. Antiderivatives are the link connecting the two major elements of calculus: derivatives and definite integrals.

Finding Antiderivatives

> **DEFINITION** A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as F to represent an antiderivative of a function f, G to represent an antiderivative of g, and so forth.

EXAMPLE 1 Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$ **(b)** $g(x) = \cos x$ **(c)** $h(x) = \sec^2 x + \dfrac{1}{2\sqrt{x}}$

Solution We need to think backward here: What function do we know has a derivative equal to the given function?

(a) $F(x) = x^2$ **(b)** $G(x) = \sin x$ **(c)** $H(x) = \tan x + \sqrt{x}$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$, and the derivative of $H(x) = \tan x + \sqrt{x}$ is $\sec^2 x + \left(1/2\sqrt{x}\right)$. ■

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The function $x^2 + 1$ has the same derivative. So does $x^2 + C$ for any constant C. Are there others?

Corollary 2 of the Mean Value Theorem in Section 4.2 gives the answer: Any two antiderivatives of a function differ by a constant. So the functions $x^2 + C$, where C is an **arbitrary constant**, form *all* the antiderivatives of $f(x) = 2x$. More generally, we have the following result.

THEOREM 8 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Thus the most general antiderivative of f on I is a *family* of functions $F(x) + C$ whose graphs are vertical translations of one another. We can select a particular antiderivative from this family by assigning a specific value to C. Here is an example showing how such an assignment might be made.

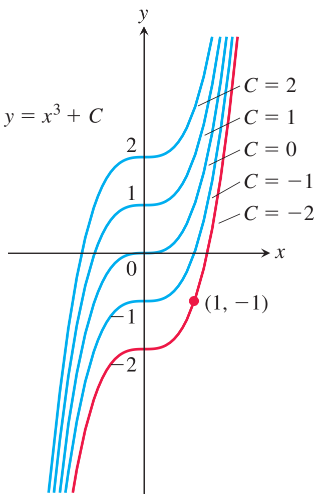


FIGURE 4.53 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

EXAMPLE 2 Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution Since the derivative of x^3 is $3x^2$, the general antiderivative

$$F(x) = x^3 + C$$

gives all the antiderivatives of $f(x)$. The condition $F(1) = -1$ determines a specific value for C. Substituting $x = 1$ into $f(x) = x^3 + C$ gives

$$F(1) = (1)^3 + C = 1 + C.$$

Since $F(1) = -1$, solving $1 + C = -1$ for C gives $C = -2$. So

$$F(x) = x^3 - 2$$

is the antiderivative satisfying $F(1) = -1$. Notice that this assignment for C selects the particular curve from the family of curves $y = x^3 + C$ that passes through the point $(1, -1)$ in the plane (Figure 4.53). ■

By working backward from assorted differentiation rules, we can derive formulas and rules for antiderivatives. In each case there is an arbitrary constant C in the general expression representing all antiderivatives of a given function. Table 4.2 gives antiderivative formulas for a number of important functions.

The rules in Table 4.2 are easily verified by differentiating the general antiderivative formula to obtain the function to its left. For example, the derivative of $(\tan kx)/k + C$ is $\sec^2 kx$, whatever the value of the constants C or $k \neq 0$, and this verifies that Formula 4 gives the general antiderivative of $\sec^2 kx$.

TABLE 4.2 Antiderivative formulas, k a nonzero constant

Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$

EXAMPLE 3 Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$ **(b)** $g(x) = \frac{1}{\sqrt{x}}$ **(c)** $h(x) = \sin 2x$

(d) $i(x) = \cos \frac{x}{2}$

Solution In each case, we can use one of the formulas listed in Table 4.2.

(a) $F(x) = \frac{x^6}{6} + C$ Formula 1 with $n = 5$

(b) $g(x) = x^{-1/2}$, so

$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$ Formula 1 with $n = -1/2$

(c) $H(x) = \frac{-\cos 2x}{2} + C$ Formula 2 with $k = 2$

(d) $I(x) = \frac{\sin(x/2)}{1/2} + C = 2\sin\frac{x}{2} + C$ Formula 3 with $k = 1/2$ ■

Other derivative rules also lead to corresponding antiderivative rules. We can add and subtract antiderivatives and multiply them by constants.

TABLE 4.3 Antiderivative linearity rules

	Function	General antiderivative
1. *Constant Multiple Rule:*	$kf(x)$	$kF(x) + C, \quad k$ a constant
2. *Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

The formulas in Table 4.3 are easily proved by differentiating the antiderivatives and verifying that the result agrees with the original function.

EXAMPLE 4 Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x.$$

Solution We have that $f(x) = 3g(x) + h(x)$ for the functions g and h in Example 3. Since $G(x) = 2\sqrt{x}$ is an antiderivative of $g(x)$ from Example 3b, it follows from the Constant Multiple Rule for antiderivatives that $3G(x) = 3 \cdot 2\sqrt{x} = 6\sqrt{x}$ is an antiderivative of $3g(x) = 3/\sqrt{x}$. Similarly, from Example 3c we know that $H(x) = (-1/2)\cos 2x$ is an antiderivative of $h(x) = \sin 2x$. From the Sum Rule for antiderivatives, we then get that

$$\begin{aligned} F(x) &= 3G(x) + H(x) + C \\ &= 6\sqrt{x} - \frac{1}{2}\cos 2x + C \end{aligned}$$

is the general antiderivative formula for $f(x)$, where C is an arbitrary constant. ■

Initial Value Problems and Differential Equations

Antiderivatives play several important roles in mathematics and its applications. Methods and techniques for finding them are a major part of calculus, and we take up that study in Chapter 8. Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation

$$\frac{dy}{dx} = f(x).$$

This is called a **differential equation**, since it is an equation involving an unknown function y that is being differentiated. To solve it, we need a function $y(x)$ that satisfies the equation. This function is found by taking the antiderivative of $f(x)$. We can fix the arbitrary constant arising in the antidifferentiation process by specifying an initial condition

$$y(x_0) = y_0.$$

This condition means the function $y(x)$ has the value y_0 when $x = x_0$. The combination of a differential equation and an initial condition is called an **initial value problem**. Such problems play important roles in all branches of science.

The most general antiderivative $F(x) + C$ of the function $f(x)$ (such as $x^3 + C$ for the function $3x^2$ in Example 2) gives the **general solution** $y = F(x) + C$ of the differential equation $dy/dx = f(x)$. The general solution gives *all* the solutions of the equation (there are infinitely many, one for each value of C). We **solve** the differential equation by finding its general solution. We then solve the initial value problem by finding the **particular solution** that satisfies the initial condition $y(x_0) = y_0$. In Example 2, the function $y = x^3 - 2$ is the particular solution of the differential equation $dy/dx = 3x^2$ satisfying the initial condition $y(1) = -1$.

Antiderivatives and Motion

We have seen that the derivative of the position function of an object gives its velocity, and the derivative of its velocity function gives its acceleration. If we know an object's acceleration, then by finding an antiderivative we can recover the velocity, and from an antiderivative of the velocity we can recover its position function. This procedure was used as an application of Corollary 2 in Section 4.2. Now that we have a terminology and conceptual framework in terms of antiderivatives, we revisit the problem from the point of view of differential equations.

EXAMPLE 5 A hot-air balloon ascending at the rate of 12 ft/sec is at a height 80 ft above the ground when a package is dropped. How long does it take the package to reach the ground?

Solution Let $v(t)$ denote the velocity of the package at time t, and let $s(t)$ denote its height above the ground. The acceleration of gravity near the surface of the earth is 32 ft/sec^2. Assuming no other forces act on the dropped package, we have

$$\frac{dv}{dt} = -32. \qquad \text{Negative because gravity acts in the direction of decreasing } s$$

This leads to the following initial value problem (Figure 4.54):

Differential equation: $\dfrac{dv}{dt} = -32$

Initial condition: $v(0) = 12.$ Balloon initially rising

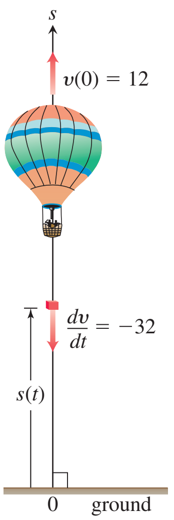


FIGURE 4.54 A package dropped from a rising hot-air balloon (Example 5).

This is our mathematical model for the package's motion. We solve the initial value problem to obtain the velocity of the package.

1. *Solve the differential equation*: The general formula for an antiderivative of -32 is

$$v = -32t + C.$$

Having found the general solution of the differential equation, we use the initial condition to find the particular solution that solves our problem.

2. *Evaluate C*:

$$12 = -32(0) + C \qquad \text{Initial condition } v(0) = 12$$
$$C = 12.$$

The solution of the initial value problem is

$$v = -32t + 12.$$

Since velocity is the derivative of height, and the height of the package is 80 ft at time $t = 0$ when it is dropped, we now have a second initial value problem:

Differential equation: $\dfrac{ds}{dt} = -32t + 12$ Set $v = ds/dt$ in the previous equation.

Initial condition: $s(0) = 80.$

We solve this initial value problem to find the height as a function of t.

1. *Solve the differential equation*: Finding the general antiderivative of $-32t + 12$ gives

$$s = -16t^2 + 12t + C.$$

2. *Evaluate C*:

$$80 = -16(0)^2 + 12(0) + C \qquad \text{Initial condition } s(0) = 80$$
$$C = 80.$$

The package's height above ground at time t is

$$s = -16t^2 + 12t + 80.$$

Use the solution: To find how long it takes the package to reach the ground, we set s equal to 0 and solve for t:

$$-16t^2 + 12t + 80 = 0$$
$$-4t^2 + 3t + 20 = 0$$
$$t = \frac{-3 \pm \sqrt{329}}{-8} \qquad \text{Quadratic formula}$$
$$t \approx -1.89, \qquad t \approx 2.64.$$

The package hits the ground about 2.64 sec after it is dropped from the balloon. (The negative root has no physical meaning.) ■

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function f.

> **DEFINITION** The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by
>
> $$\int f(x)\,dx.$$
>
> The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

After the integral sign in the notation we just defined, the integrand function is always followed by a differential to indicate the variable of integration. We will have more to say about why this is important in Chapter 5. Using this notation, we restate the solutions of Example 1, as follows:

$$\int 2x\,dx = x^2 + C,$$

$$\int \cos x\,dx = \sin x + C,$$

$$\int \left(\sec^2 x + \frac{1}{2\sqrt{x}}\right) dx = \tan x + \sqrt{x} + C$$

This notation is related to the main application of antiderivatives, which will be explored in Chapter 5. Antiderivatives play a key role in computing limits of certain infinite sums, an unexpected and wonderfully useful role that is described in a central result of Chapter 5, the Fundamental Theorem of Calculus.

EXAMPLE 6 Evaluate

$$\int (x^2 - 2x + 5)\,dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\,dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C.}_{\text{arbitrary constant}}$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx \\ &= \int x^2\,dx - 2\int x\,dx + 5\int 1\,dx \\ &= \left(\frac{x^3}{3} + C_1\right) - 2\left(\frac{x^2}{2} + C_2\right) + 5(x + C_3) \\ &= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.\end{aligned}$$

This formula is more complicated than it needs to be. If we combine $C_1, -2C_2$, and $5C_3$ into a single arbitrary constant $C = C_1 - 2C_2 + 5C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and *still* gives all the possible antiderivatives there are. For this reason, we recommend that you go right to the final form even if you elect to integrate term-by-term. Write

$$\int (x^2 - 2x + 5)\,dx = \int x^2\,dx - \int 2x\,dx + \int 5\,dx$$
$$= \frac{x^3}{3} - x^2 + 5x + C.$$

Find the simplest antiderivative you can for each part and add the arbitrary constant of integration at the end.

EXERCISES 4.7

Finding Antiderivatives

In Exercises 1–16, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a. $2x$ b. x^2 c. $x^2 - 2x + 1$
2. a. $6x$ b. x^7 c. $x^7 - 6x + 8$
3. a. $-3x^{-4}$ b. x^{-4} c. $x^{-4} + 2x + 3$
4. a. $2x^{-3}$ b. $\frac{x^{-3}}{2} + x^2$ c. $-x^{-3} + x - 1$
5. a. $\frac{1}{x^2}$ b. $\frac{5}{x^2}$ c. $2 - \frac{5}{x^2}$
6. a. $-\frac{2}{x^3}$ b. $\frac{1}{2x^3}$ c. $x^3 - \frac{1}{x^3}$
7. a. $\frac{3}{2}\sqrt{x}$ b. $\frac{1}{2\sqrt{x}}$ c. $\sqrt{x} + \frac{1}{\sqrt{x}}$
8. a. $\frac{4}{3}\sqrt[3]{x}$ b. $\frac{1}{3\sqrt[3]{x}}$ c. $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$
9. a. $\frac{2}{3}x^{-1/3}$ b. $\frac{1}{3}x^{-2/3}$ c. $-\frac{1}{3}x^{-4/3}$
10. a. $\frac{1}{2}x^{-1/2}$ b. $-\frac{1}{2}x^{-3/2}$ c. $-\frac{3}{2}x^{-5/2}$
11. a. $-\pi \sin \pi x$ b. $3 \sin x$ c. $\sin \pi x - 3 \sin 3x$
12. a. $\pi \cos \pi x$ b. $\frac{\pi}{2} \cos \frac{\pi x}{2}$ c. $\cos \frac{\pi x}{2} + \pi \cos x$
13. a. $\sec^2 x$ b. $\frac{2}{3} \sec^2 \frac{x}{3}$ c. $-\sec^2 \frac{3x}{2}$
14. a. $\csc^2 x$ b. $-\frac{3}{2} \csc^2 \frac{3x}{2}$ c. $1 - 8 \csc^2 2x$
15. a. $\csc x \cot x$ b. $-\csc 5x \cot 5x$ c. $-\pi \csc \frac{\pi x}{2} \cot \frac{\pi x}{2}$
16. a. $\sec x \tan x$ b. $4 \sec 3x \tan 3x$ c. $\sec \frac{\pi x}{2} \tan \frac{\pi x}{2}$

Finding Indefinite Integrals

In Exercises 17–56, find the most general antiderivative or indefinite integral. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

17. $\int (x + 1)\,dx$
18. $\int (5 - 6x)\,dx$
19. $\int \left(3t^2 + \frac{t}{2}\right) dt$
20. $\int \left(\frac{t^2}{2} + 4t^3\right) dt$
21. $\int (2x^3 - 5x + 7)\,dx$
22. $\int (1 - x^2 - 3x^5)\,dx$
23. $\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$
24. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$
25. $\int x^{-1/3}\,dx$
26. $\int x^{-5/4}\,dx$
27. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$
28. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$
29. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$
30. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$
31. $\int 2x(1 - x^{-3})\,dx$
32. $\int x^{-3}(x + 1)\,dx$
33. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\,dt$
34. $\int \frac{4 + \sqrt{t}}{t^3}\,dt$
35. $\int (-2 \cos t)\,dt$
36. $\int (-5 \sin t)\,dt$
37. $\int 7 \sin \frac{\theta}{3}\,d\theta$
38. $\int 3 \cos 5\theta\,d\theta$
39. $\int (-3 \csc^2 x)\,dx$
40. $\int \left(-\frac{\sec^2 x}{3}\right) dx$
41. $\int \frac{\csc \theta \cot \theta}{2}\,d\theta$
42. $\int \frac{2}{5} \sec \theta \tan \theta\,d\theta$
43. $\int (4 \sec x \tan x - 2 \sec^2 x)\,dx$

44. $\displaystyle\int \frac{1}{2}(\csc^2 x - \csc x \cot x)\, dx$

45. $\displaystyle\int (\sin 2x - \csc^2 x)\, dx$

46. $\displaystyle\int (2\cos 2x - 3\sin 3x)\, dx$

47. $\displaystyle\int \frac{1 + \cos 4t}{2}\, dt$

48. $\displaystyle\int \frac{1 - \cos 6t}{2}\, dt$

49. $\displaystyle\int 3x^{\sqrt{3}}\, dx$

50. $\displaystyle\int x^{\sqrt{2}-1}\, dx$

51. $\displaystyle\int (1 + \tan^2 \theta)\, d\theta$

52. $\displaystyle\int (2 + \tan^2 \theta)\, d\theta$

(*Hint:* $1 + \tan^2 \theta = \sec^2 \theta$)

53. $\displaystyle\int \cot^2 x\, dx$

54. $\displaystyle\int (1 - \cot^2 x)\, dx$

(*Hint:* $1 + \cot^2 x = \csc^2 x$)

55. $\displaystyle\int \cos\theta(\tan\theta + \sec\theta)\, d\theta$

56. $\displaystyle\int \frac{\csc\theta}{\csc\theta - \sin\theta}\, d\theta$

Checking Antiderivative Formulas

Verify the formulas in Exercises 57–62 by differentiation.

57. $\displaystyle\int (7x - 2)^3\, dx = \frac{(7x - 2)^4}{28} + C$

58. $\displaystyle\int (3x + 5)^{-2}\, dx = -\frac{(3x + 5)^{-1}}{3} + C$

59. $\displaystyle\int \sec^2(5x - 1)\, dx = \frac{1}{5}\tan(5x - 1) + C$

60. $\displaystyle\int \csc^2\left(\frac{x - 1}{3}\right) dx = -3\cot\left(\frac{x - 1}{3}\right) + C$

61. $\displaystyle\int \frac{1}{(x + 1)^2}\, dx = -\frac{1}{x + 1} + C$

62. $\displaystyle\int \frac{1}{(x + 1)^2}\, dx = \frac{x}{x + 1} + C$

T **63.** Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\displaystyle\int x\sin x\, dx = \frac{x^2}{2}\sin x + C$

b. $\displaystyle\int x\sin x\, dx = -x\cos x + C$

c. $\displaystyle\int x\sin x\, dx = -x\cos x + \sin x + C$

64. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\displaystyle\int \tan\theta\sec^2\theta\, d\theta = \frac{\sec^3\theta}{3} + C$

b. $\displaystyle\int \tan\theta\sec^2\theta\, d\theta = \frac{1}{2}\tan^2\theta + C$

c. $\displaystyle\int \tan\theta\sec^2\theta\, d\theta = \frac{1}{2}\sec^2\theta + C$

65. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\displaystyle\int (2x + 1)^2\, dx = \frac{(2x + 1)^3}{3} + C$

b. $\displaystyle\int 3(2x + 1)^2\, dx = (2x + 1)^3 + C$

c. $\displaystyle\int 6(2x + 1)^2\, dx = (2x + 1)^3 + C$

66. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\displaystyle\int \sqrt{2x + 1}\, dx = \sqrt{x^2 + x + C}$

b. $\displaystyle\int \sqrt{2x + 1}\, dx = \sqrt{x^2 + x} + C$

c. $\displaystyle\int \sqrt{2x + 1}\, dx = \frac{1}{3}\left(\sqrt{2x + 1}\right)^3 + C$

67. Right, or wrong? Give a brief reason why.

$$\int \frac{-15(x + 3)^2}{(x - 2)^4}\, dx = \left(\frac{x + 3}{x - 2}\right)^3 + C$$

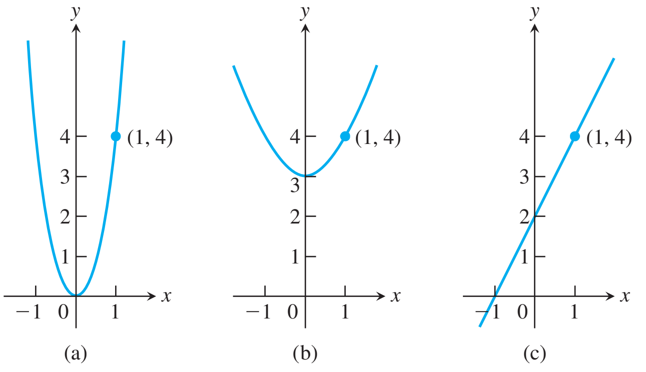
68. Right, or wrong? Give a brief reason why.

$$\int \frac{x\cos(x^2) - \sin(x^2)}{x^2}\, dx = \frac{\sin(x^2)}{x} + C$$

Initial Value Problems

69. Which of the following graphs shows the solution of the initial value problem

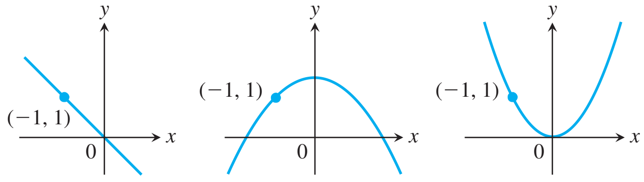
$$\frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1?$$



Give reasons for your answer.

70. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = -x, \quad y = 1 \text{ when } x = -1?$$



(a) (b) (c)

Give reasons for your answer.

Solve the initial value problems in Exercises 71–90.

71. $\dfrac{dy}{dx} = 2x - 7, \quad y(2) = 0$

72. $\dfrac{dy}{dx} = 10 - x, \quad y(0) = -1$

73. $\dfrac{dy}{dx} = \dfrac{1}{x^2} + x, \quad x > 0; \quad y(2) = 1$

74. $\dfrac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$

75. $\dfrac{dy}{dx} = 3x^{-2/3}, \quad y(-1) = -5$

76. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}, \quad y(4) = 0$

77. $\dfrac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

78. $\dfrac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

79. $\dfrac{dr}{d\theta} = -\pi \sin \pi\theta, \quad r(0) = 0$

80. $\dfrac{dr}{d\theta} = \cos \pi\theta, \quad r(0) = 1$

81. $\dfrac{dv}{dt} = \dfrac{1}{2} \sec t \tan t, \quad v(0) = 1$

82. $\dfrac{dv}{dt} = 8t + \csc^2 t, \quad v\left(\dfrac{\pi}{2}\right) = -7$

83. $\dfrac{d^2y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1$

84. $\dfrac{d^2y}{dx^2} = 0; \quad y'(0) = 2, \quad y(0) = 0$

85. $\dfrac{d^2r}{dt^2} = \dfrac{2}{t^3}; \quad \left.\dfrac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$

86. $\dfrac{d^2s}{dt^2} = \dfrac{3t}{8}; \quad \left.\dfrac{ds}{dt}\right|_{t=4} = 3, \quad s(4) = 4$

87. $\dfrac{d^3y}{dx^3} = 6; \quad y''(0) = -8, \quad y'(0) = 0, \quad y(0) = 5$

88. $\dfrac{d^3\theta}{dt^3} = 0; \quad \theta''(0) = -2, \quad \theta'(0) = -\dfrac{1}{2}, \quad \theta(0) = \sqrt{2}$

89. $y^{(4)} = -\sin t + \cos t;$
$y'''(0) = 7, \quad y''(0) = y'(0) = -1, \quad y(0) = 0$

90. $y^{(4)} = -\cos x + 8 \sin 2x;$
$y'''(0) = 0, \quad y''(0) = y'(0) = 1, \quad y(0) = 3$

91. Find the curve $y = f(x)$ in the xy-plane that passes through the point $(9, 4)$ and whose slope at each point is $3\sqrt{x}$.

92. a. Find a curve $y = f(x)$ with the following properties:

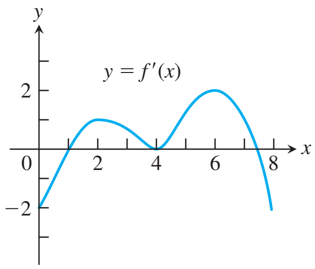
i) $\dfrac{d^2y}{dx^2} = 6x$

ii) Its graph passes through the point $(0, 1)$ and has a horizontal tangent there.

b. How many curves like this are there? How do you know?

In Exercises 93–96, the graph of f' is given. Assume that $f(0) = 1$ and sketch a possible continuous graph of f.

93.



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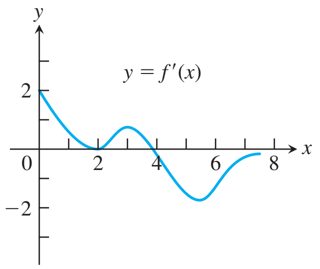

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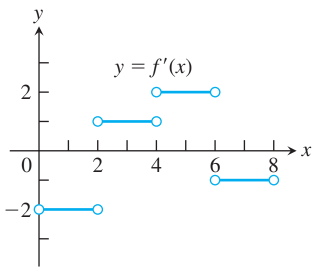

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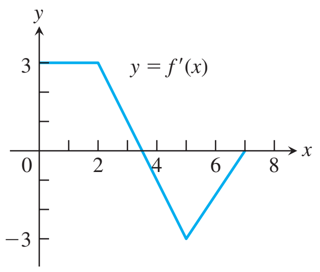
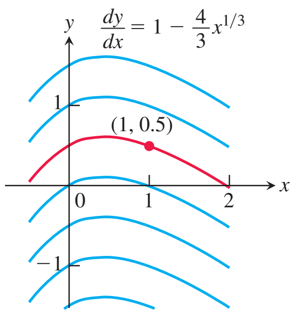


Solution (Integral) Curves

Exercises 97–100 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

97.



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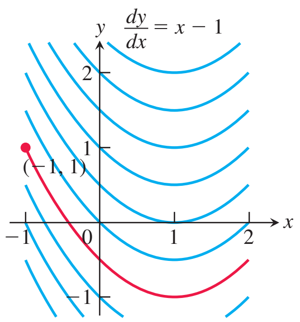

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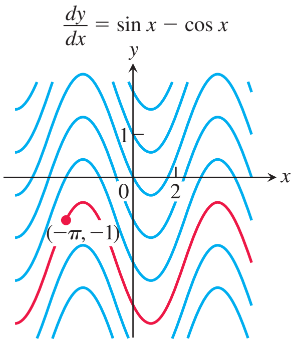
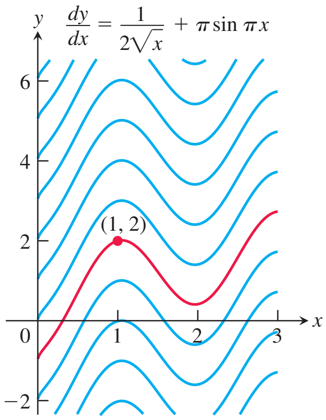


100.



Applications

101. Finding displacement from an antiderivative of velocity

a. Suppose that the velocity of a body moving along the s-axis is

$$\frac{ds}{dt} = v = 9.8t - 3.$$

i) Find the body's displacement over the time interval from $t = 1$ to $t = 3$ given that $s = 5$ when $t = 0$.

ii) Find the body's displacement from $t = 1$ to $t = 3$ given that $s = -2$ when $t = 0$.

iii) Now find the body's displacement from $t = 1$ to $t = 3$ given that $s = s_0$ when $t = 0$.

b. Suppose that the position s of a body moving along a coordinate line is a differentiable function of time t. Is it true that once you know an antiderivative of the velocity function ds/dt you can find the body's displacement from $t = a$ to $t = b$ even if you do not know the body's exact position at either of those times? Give reasons for your answer.

102. Liftoff from Earth A rocket lifts off the surface of Earth with a constant acceleration of $20\ \text{m/sec}^2$. How fast will the rocket be going 1 min later?

103. Stopping a car in time You are driving along a highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 242 ft? To find out, carry out the following steps.

1. Solve the initial value problem

Differential equation: $\frac{d^2s}{dt^2} = -k \qquad (k \text{ constant})$

Initial conditions: $\frac{ds}{dt} = 88$ and $s = 0$ when $t = 0$.

Measuring time and distance from when the brakes are applied

2. Find the value of t that makes $ds/dt = 0$. (The answer will involve k.)

3. Find the value of k that makes $s = 242$ for the value of t you found in Step 2.

104. Stopping a motorcycle The State of Illinois Cycle Rider Safety Program requires motorcycle riders to be able to brake from 30 mph (44 ft/sec) to 0 in 45 ft. What constant deceleration does it take to do that?

105. Motion along a coordinate line A particle moves on a coordinate line with acceleration $a = d^2s/dt^2 = 15\sqrt{t} - (3/\sqrt{t})$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$. Find

a. the velocity $v = ds/dt$ in terms of t.

b. the position s in terms of t.

T **106. The hammer and the feather** When *Apollo 15* astronaut David Scott dropped a hammer and a feather on the moon to demonstrate that in a vacuum all bodies fall with the same (constant) acceleration, he dropped them from about 4 ft above the ground. The television footage of the event shows the hammer and the feather falling more slowly than on Earth, where, in a vacuum, they would have taken only half a second to fall the 4 ft. How long did it take the hammer and feather to fall 4 ft on the moon? To find out, solve the following initial value problem for s as a function of t. Then find the value of t that makes s equal to 0.

Differential equation: $\frac{d^2s}{dt^2} = -5.2\ \text{ft/sec}^2$

Initial conditions: $\frac{ds}{dt} = 0$ and $s = 4$ when $t = 0$

107. Motion with constant acceleration The standard equation for the position s of a body moving with a constant acceleration a along a coordinate line is

$$s = \frac{a}{2}t^2 + v_0 t + s_0, \tag{1}$$

where v_0 and s_0 are the body's velocity and position at time $t = 0$. Derive this equation by solving the initial value problem

Differential equation: $\frac{d^2s}{dt^2} = a$

Initial conditions: $\frac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$.

108. Free fall near the surface of a planet For free fall near the surface of a planet where the acceleration due to gravity has a constant magnitude of g length-units/sec^2, Equation (1) in Exercise 107 takes the form

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \tag{2}$$

where s is the body's height above the surface. The equation has a minus sign because the acceleration acts downward, in the direction of decreasing s. The velocity v_0 is positive if the object is rising at time $t = 0$ and negative if the object is falling.

Instead of using the result of Exercise 107, you can derive Equation (2) directly by solving an appropriate initial value problem. What initial value problem? Solve it to be sure you have the right one, explaining the solution steps as you go along.

109. Suppose that

$$f(x) = \frac{d}{dx}\left(1 - \sqrt{x}\right) \quad \text{and} \quad g(x) = \frac{d}{dx}(x + 2).$$

Find:

a. $\int f(x)\,dx$ **b.** $\int g(x)\,dx$

c. $\int [-f(x)]\,dx$ **d.** $\int [-g(x)]\,dx$

e. $\int [f(x) + g(x)]\,dx$ **f.** $\int_i [f(x) - g(x)]\,dx$

110. Uniqueness of solutions If differentiable functions $y = F(x)$ and $y = g(x)$ both solve the initial value problem

$$\frac{dy}{dx} = f(x), \qquad y(x_0) = y_0,$$

on an interval I, must $F(x) = G(x)$ for every x in I? Give reasons for your answer.

COMPUTER EXPLORATIONS

Use a CAS to solve the initial value problems in Exercises 111–114. Plot the solution curves.

111. $y' = \cos^2 x + \sin x, \quad y(\pi) = 1$

112. $y' = \frac{1}{x} + x, \quad y(1) = -1$

113. $y' = \frac{1}{\sqrt{4 - x^2}}, \quad y(0) = 2$

114. $y'' = \frac{2}{x} + \sqrt{x}, \quad y(1) = 0, \quad y'(1) = 0$

CHAPTER 4 Questions to Guide Your Review

1. What can be said about the extreme values of a function that is continuous on a closed interval?
2. What does it mean for a function to have a local extreme value on its domain? An absolute extreme value? How are local and absolute extreme values related, if at all? Give examples.
3. How do you find the absolute extrema of a continuous function on a closed interval? Give examples.
4. What are the hypotheses and conclusion of Rolle's Theorem? Are the hypotheses really necessary? Explain.
5. What are the hypotheses and conclusion of the Mean Value Theorem? What physical interpretations might the theorem have?
6. State the Mean Value Theorem's three corollaries.
7. How can you sometimes identify a function $f(x)$ by knowing f' and knowing the value of f at a point $x = x_0$? Give an example.
8. What is the First Derivative Test for Local Extreme Values? Give examples of how it is applied.
9. How do you test a twice-differentiable function to determine where its graph is concave up or concave down? Give examples.
10. What is an inflection point? Give an example. What physical significance do inflection points sometimes have?
11. What is the Second Derivative Test for Local Extreme Values? Give examples of how it is applied.
12. What do the derivatives of a function tell you about the shape of its graph?
13. List the steps you would take to graph a polynomial function. Illustrate with an example.
14. What is a cusp? Give examples.
15. List the steps you would take to graph a rational function. Illustrate with an example.
16. Outline a general strategy for solving max-min problems. Give examples.
17. Describe Newton's method for solving equations. Give an example. What is the theory behind the method? What are some of the things to watch out for when you use the method?
18. Can a function have more than one antiderivative? If so, how are the antiderivatives related? Explain.
19. What is an indefinite integral? How do you evaluate one? What general formulas do you know for finding indefinite integrals?
20. How can you sometimes solve a differential equation of the form $dy/dx = f(x)$?
21. What is an initial value problem? How do you solve one? Give an example.
22. If you know the acceleration of a body moving along a coordinate line as a function of time, what more do you need to know to find the body's position function? Give an example.

CHAPTER 4 Practice Exercises

Finding Extreme Values

In Exercises 1–10, find the extreme values (absolute and local) of the function over its natural domain, and where they occur.

1. $y = 2x^2 - 8x + 9$ **2.** $y = x^3 - 2x + 4$

3. $y = x^3 + x^2 - 8x + 5$ **4.** $y = x^3(x - 5)^2$

5. $y = \sqrt{x^2 - 1}$ **6.** $y = x - 4\sqrt{x}$

7. $y = \frac{1}{\sqrt[3]{1 - x^2}}$ **8.** $y = \sqrt{3 + 2x - x^2}$

9. $y = \frac{x}{x^2 + 1}$ **10.** $y = \frac{x + 1}{x^2 + 2x + 2}$

Extreme Values

11. Does $f(x) = x^3 + 2x + \tan x$ have any local maximum or minimum values? Give reasons for your answer.

12. Does $g(x) = \csc x + 2 \cot x$ have any local maximum values? Give reasons for your answer.

13. Does $f(x) = (7 + x)(11 - 3x)^{1/3}$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of f.

14. Find values of a and b such that the function

$$f(x) = \frac{ax + b}{x^2 - 1}$$

has a local extreme value of 1 at $x = 3$. Is this extreme value a local maximum, or a local minimum? Give reasons for your answer.

15. The greatest integer function $f(x) = \lfloor x \rfloor$, defined for all values of x, assumes a local maximum value of 0 at each point of $[0, 1)$. Could any of these local maximum values also be local minimum values of f? Give reasons for your answer.

16. a. Give an example of a differentiable function f whose first derivative is zero at some point c even though f has neither a local maximum nor a local minimum at c.

b. How is this consistent with Theorem 2 in Section 4.1? Give reasons for your answer.

17. The function $y = 1/x$ does not take on either a maximum or a minimum on the interval $0 < x < 1$ even though the function is continuous on this interval. Does this contradict the Extreme Value Theorem for continuous functions? Why?

18. What are the maximum and minimum values of the function $y = |x|$ on the interval $-1 \le x < 1$? Notice that the interval is not closed. Is this consistent with the Extreme Value Theorem for continuous functions? Why?

T **19.** A graph that is large enough to show a function's global behavior may fail to reveal important local features. The graph of $f(x) = (x^8/8) - (x^6/2) - x^5 + 5x^3$ is a case in point.

a. Graph f over the interval $-2.5 \le x \le 2.5$. Where does the graph appear to have local extreme values or points of inflection?

b. Now factor $f'(x)$ and show that f has a local maximum at $x = \sqrt[3]{5} \approx 1.70998$ and local minima at $x = \pm\sqrt{3} \approx \pm 1.73205$.

c. Zoom in on the graph to find a viewing window that shows the presence of the extreme values at $x = \sqrt[3]{5}$ and $x = \sqrt{3}$.

The moral here is that without calculus the existence of two of the three extreme values would probably have gone unnoticed. On any normal graph of the function, the values would lie close enough together to fall within the dimensions of a single pixel on the screen.

(*Source*: *Uses of Technology in the Mathematics Curriculum*, by Benny Evans and Jerry Johnson, Oklahoma State University, published in 1990 under a grant from the National Science Foundation, USE-8950044.)

T **20.** (*Continuation of Exercise 19.*)

a. Graph $f(x) = (x^8/8) - (2/5)x^5 - 5x - (5/x^2) + 11$ over the interval $-2 \le x \le 2$. Where does the graph appear to have local extreme values or points of inflection?

b. Show that f has a local maximum value at $x = \sqrt[7]{5} \approx 1.2585$ and a local minimum value at $x = \sqrt[3]{2} \approx 1.2599$.

c. Zoom in to find a viewing window that shows the presence of the extreme values at $x = \sqrt[7]{5}$ and $x = \sqrt[3]{2}$.

The Mean Value Theorem

21. a. Show that $g(t) = \sin^2 t - 3t$ decreases on every interval in its domain.

b. How many solutions does the equation $\sin^2 t - 3t = 5$ have? Give reasons for your answer.

22. a. Show that $y = \tan\theta$ increases on every open interval in its domain.

b. If the conclusion in part (a) is really correct, how do you explain the fact that $\tan\pi = 0$ is less than $\tan(\pi/4) = 1$?

23. a. Show that the equation $x^4 + 2x^2 - 2 = 0$ has exactly one solution on $[0, 1]$.

T **b.** Find the solution to as many decimal places as you can.

24. a. Show that $f(x) = x/(x + 1)$ increases on every open interval in its domain.

b. Show that $f(x) = x^3 + 2x$ has no local maximum or minimum values.

25. Water in a reservoir As a result of a heavy rain, the volume of water in a reservoir increased by 1400 acre-ft in 24 hours. Show that at some instant during that period the reservoir's volume was increasing at a rate in excess of 225,000 gal/min. (An acre-foot is 43,560 ft^3, the volume that would cover 1 acre to the depth of 1 ft. A cubic foot holds 7.48 gal.)

26. The formula $F(x) = 3x + C$ gives a different function for each value of C. All of these functions, however, have the same derivative with respect to x, namely $F'(x) = 3$. Are these the only differentiable functions whose derivative is 3? Could there be any others? Give reasons for your answers.

27. Show that

$$\frac{d}{dx}\left(\frac{x}{x + 1}\right) = \frac{d}{dx}\left(-\frac{1}{x + 1}\right)$$

even though

$$\frac{x}{x + 1} \ne -\frac{1}{x + 1}.$$

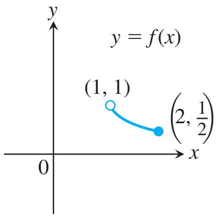
Doesn't this contradict Corollary 2 of the Mean Value Theorem? Give reasons for your answer.

28. Calculate the first derivatives of $f(x) = x^2/(x^2 + 1)$ and $g(x) = -1/(x^2 + 1)$. What can you conclude about the graphs of these functions?

Analyzing Graphs

In Exercises 29 and 30, use the graph to answer the questions.

29. Identify any global extreme values of f and the values of x at which they occur.

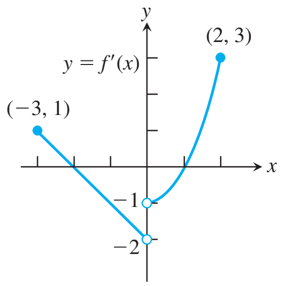


30. Estimate the open intervals on which the function $y = f(x)$ is

a. increasing.

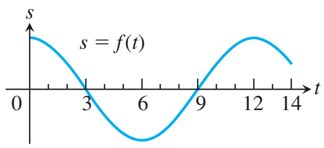
b. decreasing.

c. Use the given graph of f' to indicate where any local extreme values of the function occur, and whether each extreme is a relative maximum or minimum.

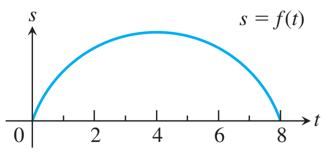


Each of the graphs in Exercises 31 and 32 is the graph of the position function $s = f(t)$ of an object moving on a coordinate line (t represents time). At approximately what times (if any) is each object's **(a)** velocity equal to zero? **(b)** Acceleration equal to zero? During approximately what time intervals does the object move **(c)** forward? **(d)** Backward?

31.



32.



Graphs and Graphing

Graph the curves in Exercises 33–42.

33. $y = x^2 - (x^3/6)$ **34.** $y = x^3 - 3x^2 + 3$

35. $y = -x^3 + 6x^2 - 9x + 3$

36. $y = (1/8)(x^3 + 3x^2 - 9x - 27)$

37. $y = x^3(8 - x)$ **38.** $y = x^2(2x^2 - 9)$

39. $y = x - 3x^{2/3}$ **40.** $y = x^{1/3}(x - 4)$

41. $y = x\sqrt{3 - x}$ **42.** $y = x\sqrt{4 - x^2}$

Each of Exercises 43–48 gives the first derivative of a function $y = f(x)$. **(a)** At what points, if any, does the graph of f have a local maximum, local minimum, or inflection point? **(b)** Sketch the general shape of the graph.

43. $y' = 16 - x^2$

44. $y' = x^2 - x - 6$

45. $y' = 6x(x + 1)(x - 2)$

46. $y' = x^2(6 - 4x)$

47. $y' = x^4 - 2x^2$

48. $y' = 4x^2 - x^4$

In Exercises 49–52, graph each function. Then use the function's first derivative to explain what you see.

49. $y = x^{2/3} + (x - 1)^{1/3}$ **50.** $y = x^{2/3} + (x - 1)^{2/3}$

51. $y = x^{1/3} + (x - 1)^{1/3}$ **52.** $y = x^{2/3} - (x - 1)^{1/3}$

Sketch the graphs of the rational functions in Exercises 53–60.

53. $y = \dfrac{x + 1}{x - 3}$ **54.** $y = \dfrac{2x}{x + 5}$

55. $y = \dfrac{x^2 + 1}{x}$ **56.** $y = \dfrac{x^2 - x + 1}{x}$

57. $y = \dfrac{x^3 + 2}{2x}$ **58.** $y = \dfrac{x^4 - 1}{x^2}$

59. $y = \dfrac{x^2 - 4}{x^2 - 3}$ **60.** $y = \dfrac{x^2}{x^2 - 4}$

Optimization

61. The sum of two nonnegative numbers is 36. Find the numbers if

a. the difference of their square roots is to be as large as possible.

b. the sum of their square roots is to be as large as possible.

62. The sum of two nonnegative numbers is 20. Find the numbers

a. if the product of one number and the square root of the other is to be as large as possible.

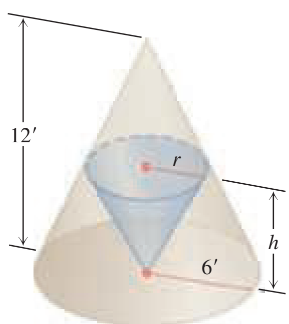
b. if one number plus the square root of the other is to be as large as possible.

63. An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

64. A customer has asked you to design an open-top rectangular stainless steel vat. It is to have a square base and a volume of 32 ft^3, to be welded from quarter-inch plate, and to weigh no more than necessary. What dimensions do you recommend?

65. Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$.

66. The figure here shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of r and h will give the smaller cone the largest possible volume?



67. Manufacturing tires Your company can manufacture x hundred grade A tires and y hundred grade B tires a day, where $0 \le x \le 4$ and

$$y = \frac{40 - 10x}{5 - x}.$$

Your profit on a grade A tire is twice your profit on a grade B tire. What is the most profitable number of each kind to make?

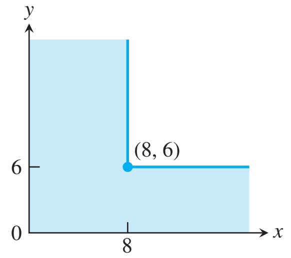
68. Particle motion The positions of two particles on the s-axis are $s_1 = \cos t$ and $s_2 = \cos(t + \pi/4)$.

a. What is the farthest apart the particles ever get?

b. When do the particles collide?

T **69. Open-top box** An open-top rectangular box is constructed from a 10-in.-by-16-in. piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find analytically the dimensions of the box of largest volume and the maximum volume. Support your answers graphically.

70. The ladder problem What is the approximate length (in feet) of the longest ladder you can carry horizontally around the corner of the corridor shown here? Round your answer down to the nearest foot.



Newton's Method

71. Let $f(x) = 3x - x^3$. Show that the equation $f(x) = -4$ has a solution in the interval $[2, 3]$ and use Newton's method to find it.

72. Let $f(x) = x^4 - x^3$. Show that the equation $f(x) = 75$ has a solution in the interval $[3, 4]$ and use Newton's method to find it.

Finding Indefinite Integrals

Find the indefinite integrals (most general antiderivatives) in Exercises 73–88. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

73. $\int (x^3 + 5x - 7)\,dx$

74. $\int \left(8t^3 - \frac{t^2}{2} + t\right) dt$

75. $\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt$

76. $\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4}\right) dt$

77. $\int \frac{dr}{(r + 5)^2}$

78. $\int \frac{6\,dr}{(r - \sqrt{2})^3}$

79. $\int 3\theta\sqrt{\theta^2 + 1}\,d\theta$

80. $\int \frac{\theta}{\sqrt{7 + \theta^2}}\,d\theta$

81. $\int x^3(1 + x^4)^{-1/4}\,dx$

82. $\int (2 - x)^{3/5}\,dx$

83. $\int \sec^2 \frac{s}{10}\,ds$

84. $\int \csc^2 \pi s\,ds$

85. $\int \csc\sqrt{2}\theta \cot\sqrt{2}\theta\,d\theta$

86. $\int \sec\frac{\theta}{3}\tan\frac{\theta}{3}\,d\theta$

87. $\int \sin^2\frac{x}{4}\,dx \quad \left(\textit{Hint: } \sin^2\theta = \frac{1 - \cos 2\theta}{2}\right)$

88. $\int \cos^2\frac{x}{2}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 89–92.

89. $\frac{dy}{dx} = \frac{x^2 + 1}{x^2}, \quad y(1) = -1$

90. $\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^2, \quad y(1) = 1$

91. $\frac{d^2r}{dt^2} = 15\sqrt{t} + \frac{3}{\sqrt{t}}; \quad r'(1) = 8, \quad r(1) = 0$

92. $\frac{d^3r}{dt^3} = -\cos t; \quad r''(0) = r'(0) = 0, \quad r(0) = -1$

CHAPTER 4 Additional and Advanced Exercises

Functions and Derivatives

1. What can you say about a function whose maximum and minimum values on an interval are equal? Give reasons for your answer.

2. Is it true that a discontinuous function cannot have both an absolute maximum and an absolute minimum value on a closed interval? Give reasons for your answer.

3. Can you conclude anything about the extreme values of a continuous function on an open interval? On a half-open interval? Give reasons for your answer.

4. Local extrema Use the sign pattern for the derivative

$$\frac{df}{dx} = 6(x - 1)(x - 2)^2(x - 3)^3(x - 4)^4$$

to identify the points where f has local maximum and minimum values.

5. Local extrema

a. Suppose that the first derivative of $y = f(x)$ is

$$y' = 6(x + 1)(x - 2)^2.$$

At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

b. Suppose that the first derivative of $y = f(x)$ is

$$y' = 6x(x + 1)(x - 2).$$

At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

6. If $f'(x) \le 2$ for all x, what is the most the values of f can increase on $[0, 6]$? Give reasons for your answer.

7. Bounding a function Suppose that f is continuous on $[a, b]$ and that c is an interior point of the interval. Show that if $f'(x) \le 0$ on $[a, c)$ and $f'(x) \ge 0$ on $(c, b]$, then $f(x)$ is never less than $f(c)$ on $[a, b]$.

8. An inequality

a. Show that $-1/2 \le x/(1 + x^2) \le 1/2$ for every value of x.

b. Suppose that f is a function whose derivative is $f'(x) = x/(1 + x^2)$. Use the result in part (a) to show that

$$|f(b) - f(a)| \le \frac{1}{2}|b - a|$$

for any a and b.

9. The derivative of $f(x) = x^2$ is zero at $x = 0$, but f is not a constant function. Doesn't this contradict the corollary of the Mean Value Theorem that says that functions with zero derivatives are constant? Give reasons for your answer.

10. Extrema and inflection points Let $h = fg$ be the product of two differentiable functions of x.

a. If f and g are positive, with local maxima at $x = a$, and if f' and g' change sign at a, does h have a local maximum at a?

b. If the graphs of f and g have inflection points at $x = a$, does the graph of h have an inflection point at a?

In either case, if the answer is yes, give a proof. If the answer is no, give a counterexample.

11. Finding a function Use the following information to find the values of a, b, and c in the formula $f(x) = (x + a)/(bx^2 + cx + 2)$.

a. The values of a, b, and c are either 0 or 1.

b. The graph of f passes through the point $(-1, 0)$.

c. The line $y = 1$ is an asymptote of the graph of f.

12. Horizontal tangent For what value or values of the constant k will the curve $y = x^3 + kx^2 + 3x - 4$ have exactly one horizontal tangent?

Optimization

13. Largest inscribed triangle Points A and B lie at the ends of a diameter of a unit circle and point C lies on the circumference. Is it true that the area of triangle ABC is largest when the triangle is isosceles? How do you know?

14. Proving the second derivative test The Second Derivative Test for Local Maxima and Minima (Section 4.4) says:

a. f has a local maximum value at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$

b. f has a local minimum value at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.

To prove statement (a), let $\varepsilon = (1/2)|f''(c)|$. Then use the fact that

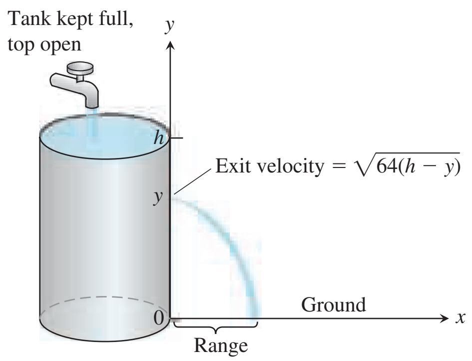
$$f''(c) = \lim_{h \to 0} \frac{f'(c + h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c + h)}{h}$$

to conclude that for some $\delta > 0$,

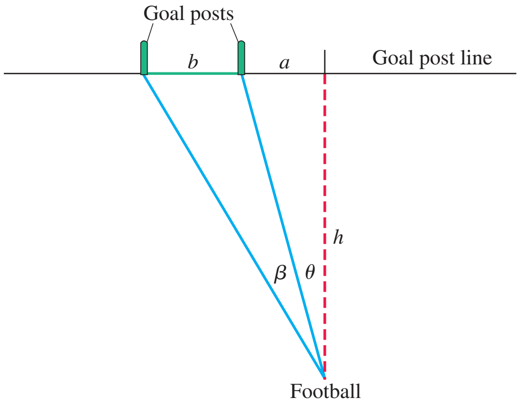
$$0 < |h| < \delta \quad \Rightarrow \quad \frac{f'(c + h)}{h} < f''(c) + \varepsilon < 0.$$

Thus, $f'(c + h)$ is positive for $-\delta < h < 0$ and negative for $0 < h < \delta$. Prove statement (b) in a similar way.

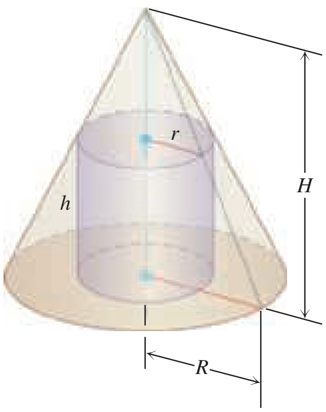
15. Hole in a water tank You want to bore a hole in the side of the tank shown here at a height that will make the stream of water coming out hit the ground as far from the tank as possible. If you drill the hole near the top, where the pressure is low, the water will exit slowly but spend a relatively long time in the air. If you drill the hole near the bottom, the water will exit at a higher velocity but have only a short time to fall. Where is the best place, if any, for the hole? (*Hint:* How long will it take an exiting droplet of water to fall from height y to the ground?)



16. Kicking a field goal An American football player wants to kick a field goal with the ball being on a right hash mark. Assume that the goal posts are b feet apart and that the hash mark line is a distance $a > 0$ feet from the right goal post. (See the accompanying figure.) Find the distance h from the goal post line that gives the kicker his largest angle β. Assume that the football field is flat.

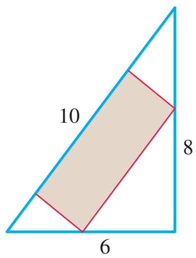


17. A max-min problem with a variable answer Sometimes the solution of a max-min problem depends on the proportions of the shapes involved. As a case in point, suppose that a right circular cylinder of radius r and height h is inscribed in a right circular cone of radius R and height H, as shown here. Find the value of r (in terms of R and H) that maximizes the total surface area of the cylinder (including top and bottom). As you will see, the solution depends on whether $H \le 2R$ or $H > 2R$.



18. Minimizing a parameter Find the smallest value of the positive constant m that will make $mx - 1 + (1/x)$ greater than or equal to zero for all positive values of x.

19. Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle in the accompanying figure.



20. A rectangular box with a square base is inscribed in a right circular cone of height 4 and base radius 3. If the base of the box sits on the base of the cone, what is the largest possible volume of the box?

Theory and Examples

21. Suppose that it costs a company $y = a + bx$ dollars to produce x units per week. It can sell x units per week at a price of $P = c - ex$ dollars per unit. Each of a, b, c, and e represents a positive constant. **(a)** What production level maximizes the profit? **(b)** What is the corresponding price? **(c)** What is the weekly profit at this level of production? **(d)** At what price should each item be sold to maximize profits if the government imposes a tax of t dollars per item sold? Comment on the difference between this price and the price before the tax.

22. Estimating reciprocals without division You can estimate the value of the reciprocal of a number a without ever dividing by a if you apply Newton's method to the function $f(x) = (1/x) - a$. For example, if $a = 3$, the function involved is $f(x) = (1/x) - 3$.

a. Graph $y = (1/x) - 3$. Where does the graph cross the x-axis?

b. Show that the recursion formula in this case is

$$x_{n+1} = x_n(2 - 3x_n),$$

so there is no need for division.

23. To find $x = \sqrt[q]{a}$, we apply Newton's method to $f(x) = x^q - a$. Here we assume that a is a positive real number and q is a positive integer. Show that x_1 is a "weighted average" of x_0 and a/x_0^{q-1}, and find the coefficients m_0, m_1 such that

$$x_1 = m_0 x_0 + m_1\left(\frac{a}{x_0^{\,q-1}}\right), \qquad \begin{array}{l} m_0 > 0, m_1 > 0, \\ m_0 + m_1 = 1. \end{array}$$

What conclusion would you reach if x_0 and $a/x_0^{\,q-1}$ were equal? What would be the value of x_1 in that case?

24. The family of straight lines $y = ax + b$ (a, b arbitrary constants) can be characterized by the relation $y'' = 0$. Find a similar relation satisfied by the family of all circles

$$(x - h)^2 + (y - h)^2 = r^2,$$

where h and r are arbitrary constants. (*Hint:* Eliminate h and r from the set of three equations including the given one and two obtained by successive differentiation.)

25. Assume that the brakes of an automobile produce a constant deceleration of k ft/sec^2. **(a)** Determine what k must be to bring an automobile traveling 60 mi/hr (88 ft/sec) to rest in a distance of 100 ft from the point where the brakes are applied. **(b)** With the same k, how far would a car traveling 30 mi/hr go before being brought to a stop?

26. Let $f(x)$, $g(x)$ be two continuously differentiable functions satisfying the relationships $f'(x) = g(x)$ and $f''(x) = -f(x)$. Let $h(x) = f^2(x) + g^2(x)$. If $h(0) = 5$, find $h(10)$.

27. Can there be a curve satisfying the following conditions? d^2y/dx^2 is everywhere equal to zero and, when $x = 0$, $y = 0$ and $dy/dx = 1$. Give a reason for your answer.

28. Find the equation for the curve in the xy-plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.

29. A particle moves along the x-axis. Its acceleration is $a = -t^2$. At $t = 0$, the particle is at the origin. In the course of its motion, it reaches the point $x = b$, where $b > 0$, but no point beyond b. Determine its velocity at $t = 0$.

30. A particle moves with acceleration $a = \sqrt{t} - \left(1/\sqrt{t}\right)$. Assuming that the velocity $v = 4/3$ and the position $s = -4/15$ when $t = 0$, find

a. the velocity v in terms of t.

b. the position s in terms of t.

31. Given $f(x) = ax^2 + 2bx + c$ with $a > 0$. By considering the minimum, prove that $f(x) \geq 0$ for all real x if and only if $b^2 - ac \leq 0$.

32. Schwarz's inequality

a. In Exercise 31, let

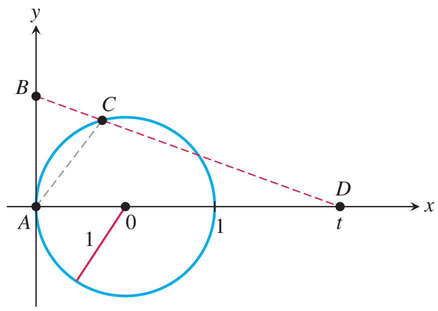
$$f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2,$$

and deduce Schwarz's inequality:

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq \left(a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2\right)\left(b_1{}^2 + b_2{}^2 + \cdots + b_n{}^2\right).$$

b. Show that equality holds in Schwarz's inequality only if there exists a real number x that makes a_ix equal $-b_i$ for every value of i from 1 to n.

33. Consider the unit circle centered at the origin and with a vertical tangent line passing through point A in the accompanying figure. Assume that the lengths of segments AB and AC are equal, and let point D be the intersection of the x-axis with the line passing through points B and C. Find the limit of t as B approaches A.



CHAPTER 4 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyMathLab.

- ***Motion Along a Straight Line: Position → Velocity → Acceleration***
  You will observe the shape of a graph through dramatic animated visualizations of the derivative relations among the position, velocity, and acceleration. Figures in the text can be animated.
- ***Newton's Method: Estimate π to How Many Places?***
  Plot a function, observe a root, pick a starting point near the root, and use Newton's Iteration Procedure to approximate the root to a desired accuracy. The numbers π, e, and $\sqrt{2}$ are approximated.

5

Integrals

OVERVIEW A great achievement of classical geometry was obtaining formulas for the areas and volumes of triangles, spheres, and cones. In this chapter we develop a method, called *integration*, to calculate the areas and volumes of more general shapes. The *definite integral* is the key tool in calculus for defining and calculating areas and volumes. We also use it to compute quantities such as the lengths of curved paths, probabilities, averages, energy consumption, the mass of an object, and the force against a dam's floodgates, to name only a few.

Like the derivative, the definite integral is defined as a limit. The definite integral is a limit of increasingly fine approximations. The idea is to approximate a quantity (such as the area of a curvy region) by dividing it into many small pieces, each of which we can approximate by something simple (such as a rectangle). Summing the contributions of each of the simple pieces gives us an approximation to the original quantity. As we divide the region into more and more pieces, the approximation given by the sum of the pieces will generally improve, converging to the quantity we are measuring. We take a limit as the number of terms increases to infinity, and when the limit exists, the result is a definite integral. We develop this idea in Section 5.3.

We also show that the process of computing these definite integrals is closely connected to finding antiderivatives. This is one of the most important relationships in calculus; it gives us an efficient way to compute definite integrals, providing a simple and powerful method that eliminates the difficulty of directly computing limits of approximations. This connection is captured in the Fundamental Theorem of Calculus.

5.1 Area and Estimating with Finite Sums

The basis for formulating definite integrals is the construction of approximations by finite sums. In this section we consider three examples of this process: finding the area under a graph, the distance traveled by a moving object, and the average value of a function. Although we have yet to define precisely what we mean by the area of a general region in the plane, or the average value of a function over a closed interval, we do have intuitive ideas of what these notions mean. We begin our approach to integration by *approximating* these quantities with simpler finite sums related to these intuitive ideas. We then consider what happens when we take more and more terms in the summation process. In subsequent sections we look at taking the limit of these sums as the number of terms goes to infinity, which leads to a precise definition of the definite integral.

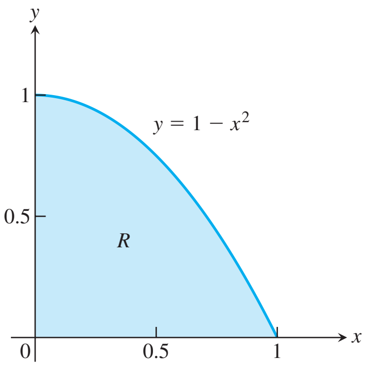


FIGURE 5.1 The area of a region R cannot be found by a simple formula.

Area

Suppose we want to find the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$, and between the vertical lines $x = 0$ and $x = 1$ (see Figure 5.1).

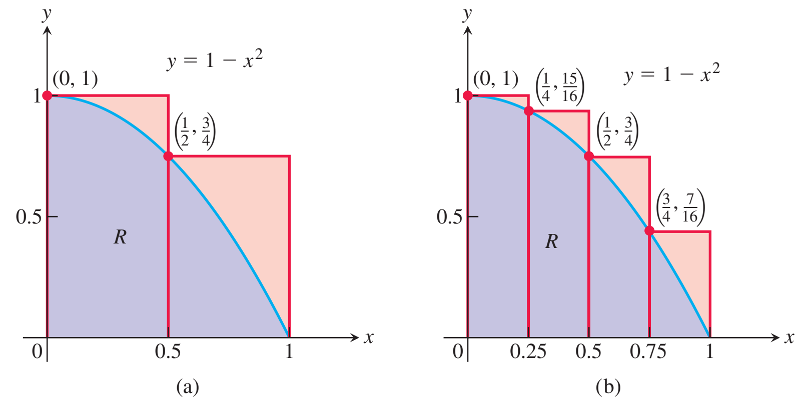


FIGURE 5.2 (a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area by the amount shaded in light red.

Unfortunately, there is no simple geometric formula for calculating the areas of general shapes having curved boundaries like the region R. How, then, can we find the area of R?

While we do not yet have a method for determining the exact area of R, we can approximate it in a simple way. Figure 5.2a shows two rectangles that together contain the region R. Each rectangle has width $1/2$ and they have heights 1 and $3/4$ (left to right). The height of each rectangle is the maximum value of the function f in each subinterval. Because the function f is decreasing, the height is its value at the left endpoint of the subinterval of $[0, 1]$ that forms the base of the rectangle. The total area of the two rectangles approximates the area A of the region R:

$$A \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8} = 0.875.$$

This estimate is larger than the true area A since the two rectangles contain R. We say that 0.875 is an **upper sum** because it is obtained by taking the height of the rectangle corresponding to the maximum (uppermost) value of $f(x)$ over points x lying in the base of each rectangle. In Figure 5.2b, we improve our estimate by using four thinner rectangles, each of width $1/4$, which taken together contain the region R. These four rectangles give the approximation

$$A \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32} = 0.78125,$$

which is still greater than A since the four rectangles contain R.

Suppose instead we use four rectangles contained *inside* the region R to estimate the area, as in Figure 5.3a. Each rectangle has width $1/4$ as before, but the rectangles are shorter and lie entirely beneath the graph of f. The function $f(x) = 1 - x^2$ is decreasing on $[0, 1]$, so the height of each of these rectangles is given by the value of f at the right endpoint of the subinterval forming its base. The fourth rectangle has zero height and therefore contributes no area. Summing these rectangles, whose heights are the minimum value of $f(x)$ over points x in the rectangle's base, gives a **lower sum** approximation to the area:

$$A \approx \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{17}{32} = 0.53125.$$

This estimate is smaller than the area A since the rectangles all lie inside of the region R. The true value of A lies somewhere between these lower and upper sums:

$$0.53125 < A < 0.78125.$$

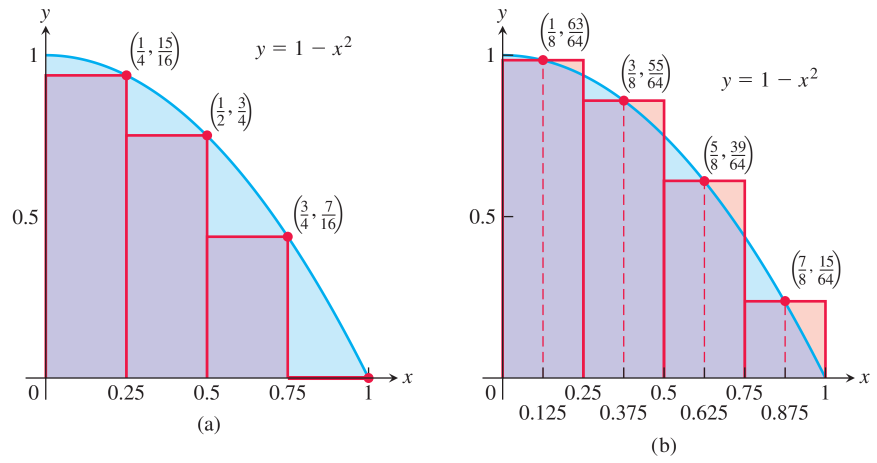


FIGURE 5.3 (a) Rectangles contained in R give an estimate for the area that undershoots the true value by the amount shaded in light blue. (b) The midpoint rule uses rectangles whose height is the value of $y = f(x)$ at the midpoints of their bases. The estimate appears closer to the true value of the area because the light red overshoot areas roughly balance the light blue undershoot areas.

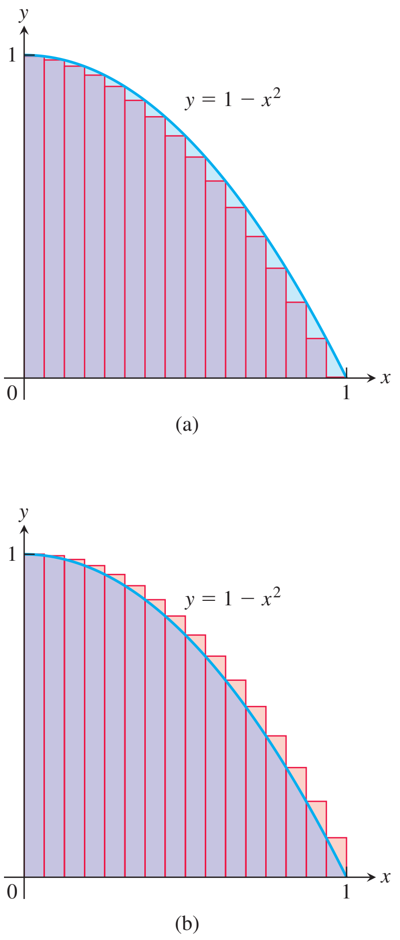


FIGURE 5.4 (a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

Considering both lower and upper sum approximations gives us estimates for the area and a bound on the size of the possible error in these estimates, since the true value of the area lies somewhere between them. Here the error cannot be greater than the difference $0.78125 - 0.53125 = 0.25$.

Yet another estimate can be obtained by using rectangles whose heights are the values of f at the midpoints of the bases of the rectangles (Figure 5.3b). This method of estimation is called the **midpoint rule** for approximating the area. The midpoint rule gives an estimate that is between a lower sum and an upper sum, but it is not quite so clear whether it overestimates or underestimates the true area. With four rectangles of width $1/4$ as before, the midpoint rule estimates the area of R to be

$$A \approx \frac{63}{64} \cdot \frac{1}{4} + \frac{55}{64} \cdot \frac{1}{4} + \frac{39}{64} \cdot \frac{1}{4} + \frac{15}{64} \cdot \frac{1}{4} = \frac{172}{64} \cdot \frac{1}{4} = 0.671875.$$

In each of the sums that we computed, the interval $[a, b]$ over which the function f is defined was subdivided into n subintervals of equal width (or length) $\Delta x = (b - a)/n$, and f was evaluated at a point in each subinterval: c_1 in the first subinterval, c_2 in the second subinterval, and so on. For the upper sum we chose c_k so that $f(c_k)$ was the maximum value of f in the kth subinterval, for the lower sum we chose it so that $f(c_k)$ was the minimum, and for the midpoint rule we chose c_k to be the midpoint of the kth subinterval. In each case the finite sums have the form

$$f(c_1)\,\Delta x + f(c_2)\,\Delta x + f(c_3)\,\Delta x + \cdots + f(c_n)\,\Delta x.$$

By taking more and more rectangles, with each rectangle thinner than before, it appears that these finite sums give better and better approximations to the true area of the region R.

Figure 5.4a shows a lower sum approximation for the area of R using 16 rectangles of equal width. The sum of their areas is 0.634765625, which appears close to the true area, but is still smaller since the rectangles lie inside R.

Figure 5.4b shows an upper sum approximation using 16 rectangles of equal width. The sum of their areas is 0.697265625, which is somewhat larger than the true area because the rectangles taken together contain R. The midpoint rule for 16 rectangles gives a total area approximation of 0.6669921875, but it is not immediately clear whether this estimate is larger or smaller than the true area.

Table 5.1 shows the values of upper and lower sum approximations to the area of R, using up to 1000 rectangles. The values of these approximations appear to be approaching $2/3$. In Section 5.2 we will see how to get an exact value of the area of regions such as R by taking a limit as the base width of each rectangle goes to zero and the number of rectangles goes to infinity. With the techniques developed there, we will be able to show that the area of R is exactly $2/3$.

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint sum	Upper sum
2	0.375	0.6875	0.875
4	0.53125	0.671875	0.78125
16	0.634765625	0.6669921875	0.697265625
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

Distance Traveled

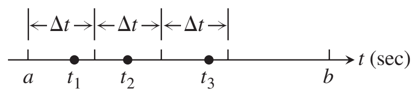
Suppose we know the velocity function $v(t)$ of a car that moves straight down a highway without changing direction, and we want to know how far it traveled between times $t = a$ and $t = b$. The position function $s(t)$ of the car has derivative $v(t)$. If we can find an antiderivative $F(t)$ of $v(t)$ then we can find the car's position function $s(t)$ by setting $s(t) = F(t) + C$. The distance traveled can then be found by calculating the change in position, $s(b) - s(a) = F(b) - F(a)$. However, if the velocity is known only by the readings at various times of a speedometer on the car, then we have no formula from which to obtain an antiderivative for the velocity. So what do we do in this situation?

When we don't know an antiderivative for the velocity $v(t)$, we can approximate the distance traveled by using finite sums in a way similar to the area estimates that we discussed before. We subdivide the interval $[a, b]$ into short time intervals and assume that the velocity on each subinterval is fairly constant. Then we approximate the distance traveled on each time subinterval with the usual distance formula

$$\text{distance} = \text{velocity} \times \text{time}$$

and add the results across $[a, b]$.

Suppose the subdivided interval looks like



with the subintervals all of equal length Δt. Pick a number t_1 in the first interval. If Δt is so small that the velocity barely changes over a short time interval of duration Δt, then the distance traveled in the first time interval is about $v(t_1)\,\Delta t$. If t_2 is a number in the second interval, the distance traveled in the second time interval is about $v(t_2)\,\Delta t$. The sum of the distances traveled over all the time intervals is

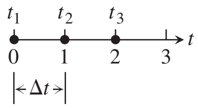
$$D \approx v(t_1)\,\Delta t + v(t_2)\,\Delta t + \cdots + v(t_n)\,\Delta t,$$

where n is the total number of subintervals. This sum is only an approximation to the true distance D, but the approximation increases in accuracy as we take more and more subintervals.

EXAMPLE 1 The velocity function of a projectile fired straight into the air is $f(t) = 160 - 9.8t$ m/sec. Use the summation technique just described to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact value of 435.9 m? (You will learn how to compute the exact value of this and similar quantities in Section 5.4.)

Solution We explore the results for different numbers of subintervals and different choices of evaluation points. Notice that $f(t)$ is decreasing, so choosing left endpoints gives an upper sum estimate; choosing right endpoints gives a lower sum estimate.

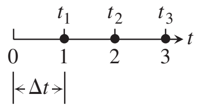
(a) *Three subintervals of length* 1, *with* f *evaluated at left endpoints giving an upper sum*:



With f evaluated at $t = 0$, 1, and 2, we have

$$\begin{aligned} D &\approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + f(t_3)\,\Delta t \\ &= [160 - 9.8(0)](1) + [160 - 9.8(1)](1) + [160 - 9.8(2)](1) \\ &= 450.6. \end{aligned}$$

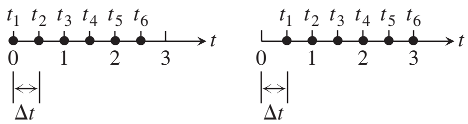
(b) *Three subintervals of length* 1, *with* f *evaluated at right endpoints giving a lower sum*:



With f evaluated at $t = 1$, 2, and 3, we have

$$\begin{aligned} D &\approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + f(t_3)\,\Delta t \\ &= [160 - 9.8(1)](1) + [160 - 9.8(2)](1) + [160 - 9.8(3)](1) \\ &= 421.2. \end{aligned}$$

(c) *With six subintervals of length* $1/2$, *we get*



These estimates give an upper sum using left endpoints: $D \approx 443.25$; and a lower sum using right endpoints: $D \approx 428.55$. These six-interval estimates are somewhat closer than the three-interval estimates. The results improve as the subintervals get shorter.

As we can see in Table 5.2, the left-endpoint upper sums approach the true value 435.9 from above, whereas the right-endpoint lower sums approach it from below. The true value lies between these upper and lower sums. The magnitude of the error in the closest entry is 0.23, a small percentage of the true value.

$$\begin{aligned} \text{Error magnitude} &= |\text{true value} - \text{calculated value}| \\ &= |435.9 - 435.67| = 0.23. \\ \text{Error percentage} &= \frac{0.23}{435.9} \approx 0.05\%. \end{aligned}$$

It would be reasonable to conclude from the table's last entries that the projectile rose about 436 m during its first 3 sec of flight. ■

TABLE 5.2 Travel-distance estimates

Number of subintervals	Length of each subinterval	Upper sum	Lower sum
3	1	450.6	421.2
6	1/2	443.25	428.55
12	1/4	439.58	432.23
24	1/8	437.74	434.06
48	1/16	436.82	434.98
96	1/32	436.36	435.44
192	1/64	436.13	435.67

Displacement Versus Distance Traveled

If an object with position function $s(t)$ moves along a coordinate line without changing direction, we can calculate the total distance it travels from $t = a$ to $t = b$ by summing the distance traveled over small intervals, as in Example 1. If the object reverses direction one or more times during the trip, then we need to use the object's *speed* $|v(t)|$, which is the absolute value of its velocity function, $v(t)$, to find the total distance traveled. Using the velocity itself, as in Example 1, gives instead an estimate to the object's **displacement**, $s(b) - s(a)$, the difference between its initial and final positions. To see the difference, think about what happens when you walk a mile from your home and then walk back. The total distance traveled is two miles, but your displacement is zero, because you end up back where you started.

To see why using the velocity function in the summation process gives an estimate to the displacement, partition the time interval $[a, b]$ into small enough equal subintervals Δt so that the object's velocity does not change very much from time t_{k-1} to t_k. Then $v(t_k)$ gives a good approximation of the velocity throughout the interval. Accordingly, the change in the object's position coordinate, which is its displacement during the time interval, is about

$$v(t_k)\,\Delta t.$$

The change is positive if $v(t_k)$ is positive and negative if $v(t_k)$ is negative.

In either case, the distance traveled by the object during the subinterval is about

$$|v(t_k)|\,\Delta t.$$

The **total distance traveled** over the time interval is approximately the sum

$$|v(t_1)|\Delta t + |v(t_2)|\Delta t + \cdots + |v(t_n)|\,\Delta t.$$

We will revisit these ideas in Section 5.4.

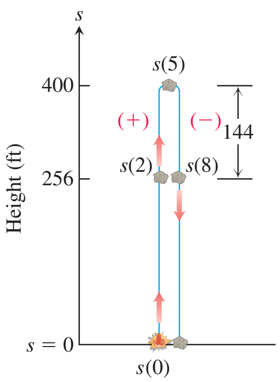


FIGURE 5.5 The rock in Example 2. The height $s = 256$ ft is reached at $t = 2$ and $t = 8$ sec. The rock falls 144 ft from its maximum height when $t = 8$.

EXAMPLE 2 In Example 4 in Section 3.4, we analyzed the motion of a heavy rock blown straight up by a dynamite blast. In that example, we found the velocity of the rock at time t was $v(t) = 160 - 32t$ ft/sec. The rock was 256 ft above the ground 2 sec after the explosion, continued upward to reach a maximum height of 400 ft at 5 sec after the explosion, and then fell back down a distance of 144 ft to reach the height of 256 ft again at $t = 8$ sec after the explosion. (See Figure 5.5.) The total distance traveled in these 8 seconds is $400 + 144 = 544$ ft.

If we follow a procedure like the one presented in Example 1, using the velocity function $v(t)$ in the summation process from $t = 0$ to $t = 8$, we obtain an estimate of the rock's height above the ground at time $t = 8$. Starting at time $t = 0$, the rock traveled upward a total of $256 + 144 = 400$ ft, but then it peaked and traveled downward

TABLE 5.3 Velocity function

t	$v(t)$	t	$v(t)$
0	160	4.5	16
0.5	144	5.0	0
1.0	128	5.5	−16
1.5	112	6.0	−32
2.0	96	6.5	−48
2.5	80	7.0	−64
3.0	64	7.5	−80
3.5	48	8.0	−96
4.0	32		

144 ft, ending at a height of 256 ft at time $t = 8$. The velocity $v(t)$ is positive during the upward travel, but negative while the rock falls back down. When we compute the sum $v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_n)\Delta t$, part of the upward positive distance change is canceled by the negative downward movement, giving in the end an approximation of the displacement from the initial position, equal to a positive change of 256 ft.

On the other hand, if we use the speed $|v(t)|$, which is the absolute value of the velocity function, then distances traveled while moving up and distances traveled while moving down are both counted positively. Both the total upward motion of 400 ft and the downward motion of 144 ft are now counted as positive distances traveled, so the sum $|v(t_1)|\Delta t + |v(t_2)|\Delta t + \cdots + |v(t_n)|\Delta t$ gives us an approximation of 544 ft, the total distance that the rock traveled from time $t = 0$ to time $t = 8$.

As an illustration of our discussion, we subdivide the interval $[0, 8]$ into sixteen subintervals of length $\Delta t = 1/2$ and take the right endpoint of each subinterval as the value of t_k. Table 5.3 shows the values of the velocity function at these endpoints.

Using $v(t)$ in the summation process, we estimate the displacement at $t = 8$:

$$(144 + 128 + 112 + 96 + 80 + 64 + 48 + 32 + 16$$
$$+ 0 - 16 - 32 - 48 - 64 - 80 - 96)\cdot\frac{1}{2} = 192$$
$$\text{Error magnitude} = 256 - 192 = 64$$

Using $|v(t)|$ in the summation process, we estimate the total distance traveled over the time interval $[0, 8]$:

$$(144 + 128 + 112 + 96 + 80 + 64 + 48 + 32 + 16$$
$$+ 0 + 16 + 32 + 48 + 64 + 80 + 96)\cdot\frac{1}{2} = 528$$
$$\text{Error magnitude} = 544 - 528 = 16$$

If we take more and more subintervals of $[0, 8]$ in our calculations, the estimates to the values 256 ft and 544 ft improve, as shown in Table 5.4. ■

TABLE 5.4 Travel estimates for a rock blown straight up during the time interval $[0, 8]$

Number of subintervals	Length of each subinterval	Displacement	Total distance
16	1/2	192.0	528.0
32	1/4	224.0	536.0
64	1/8	240.0	540.0
128	1/16	248.0	542.0
256	1/32	252.0	543.0
512	1/64	254.0	543.5

Average Value of a Nonnegative Continuous Function

The average value of a collection of n numbers $x_1, x_2, \ldots, x_n$ is obtained by adding them together and dividing by n. But what is the average value of a continuous function f on an interval $[a, b]$? Such a function can assume infinitely many values. For example, the temperature at a certain location in a town is a continuous function that goes up and down each day. What does it mean to say that the average temperature in the town over the course of a day is 73 degrees?

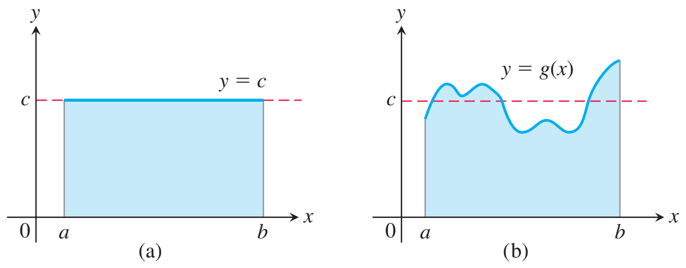


FIGURE 5.6 (a) The average value of $f(x) = c$ on $[a, b]$ is the area of the rectangle divided by $b - a$. (b) The average value of $g(x)$ on $[a, b]$ is the area beneath its graph divided by $b - a$.

When a function is constant, this question is easy to answer. A function with constant value c on an interval $[a, b]$ has average value c. When c is positive, its graph over $[a, b]$ gives a rectangle of height c. The average value of the function can then be interpreted geometrically as the area of this rectangle divided by its width $b - a$ (see Figure 5.6a).

What if we want to find the average value of a nonconstant function, such as the function g in Figure 5.6b? We can think of this graph as a snapshot of the height of some water that is sloshing around in a tank between enclosing walls at $x = a$ and $x = b$. As the water moves, its height over each point changes, but its average height remains the same. To get the average height of the water, we let it settle down until it is level and its height is constant. The resulting height c equals the area under the graph of g divided by $b - a$. We are led to *define* the average value of a nonnegative function on an interval $[a, b]$ to be the area under its graph divided by $b - a$. For this definition to be valid, we need a precise understanding of what is meant by the area under a graph. This will be obtained in Section 5.3, but for now we look at an example.

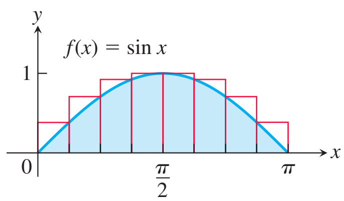


FIGURE 5.7 Approximating the area under $f(x) = \sin x$ between 0 and π to compute the average value of $\sin x$ over $[0, \pi]$, using eight rectangles (Example 3).

EXAMPLE 3 Estimate the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution Looking at the graph of $\sin x$ between 0 and π in Figure 5.7, we can see that its average height is somewhere between 0 and 1. To find the average, we need to calculate the area A under the graph and then divide this area by the length of the interval, $\pi - 0 = \pi$.

We do not have a simple way to determine the area, so we approximate it with finite sums. To get an upper sum approximation, we add the areas of eight rectangles of equal width $\pi/8$ that together contain the region that is beneath the graph of $y = \sin x$ and above the x-axis on $[0, \pi]$. We choose the heights of the rectangles to be the largest value of $\sin x$ on each subinterval. Over a particular subinterval, this largest value may occur at the left endpoint, the right endpoint, or somewhere between them. We evaluate $\sin x$ at this point to get the height of the rectangle for an upper sum. The sum of the rectangular areas then gives an estimate of the total area (Figure 5.7):

$$\begin{aligned} A &\approx \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right)\cdot\frac{\pi}{8} \\ &\approx (.38 + .71 + .92 + 1 + 1 + .92 + .71 + .38)\cdot\frac{\pi}{8} = (6.02)\cdot\frac{\pi}{8} \approx 2.364. \end{aligned}$$

To estimate the average value of $\sin x$ on $[0, \pi]$ we divide the estimated area by the length π of the interval and obtain the approximation $2.364/\pi \approx 0.753$.

Since we used an upper sum to approximate the area, this estimate is greater than the actual average value of $\sin x$ over $[0, \pi]$. If we use more and more rectangles, with each rectangle getting thinner and thinner, we get closer and closer to the exact average value, as

TABLE 5.5 Average value of $\sin x$ on $0 \le x \le \pi$

Number of subintervals	Upper sum estimate
8	0.75342
16	0.69707
32	0.65212
50	0.64657
100	0.64161
1000	0.63712

shown in Table 5.5. Using the techniques covered in Section 5.3, we will later show that the true average value is $2/\pi \approx 0.63662$.

As before, we could just as well have used rectangles lying under the graph of $y = \sin x$ and calculated a lower sum approximation, or we could have used the midpoint rule. In Section 5.3 we will see that in each case, the approximations are close to the true area if all the rectangles are sufficiently thin. ■

Summary

The area under the graph of a positive function, the distance traveled by a moving object that doesn't change direction, and the average value of a nonnegative function f over an interval can all be approximated by finite sums constructed in a certain way. First we subdivide the interval into subintervals, treating f as if it were constant over each subinterval. Then we multiply the width of each subinterval by the value of f at some point within it, and add these products together. If the interval $[a, b]$ is subdivided into n subintervals of equal widths $\Delta x = (b - a)/n$, and if $f(c_k)$ is the value of f at the chosen point c_k in the kth subinterval, this process gives a finite sum of the form

$$f(c_1)\,\Delta x + f(c_2)\,\Delta x + f(c_3)\,\Delta x + \cdots + f(c_n)\,\Delta x.$$

The choices for the c_k could maximize or minimize the value of f in the kth subinterval, or give some value in between. The true value lies somewhere between the approximations given by upper sums and lower sums. In the examples that we looked at, the finite sum approximations improved as we took more subintervals of thinner width.

EXERCISES 5.1

Area

In Exercises 1–4, use finite approximations to estimate the area under the graph of the function using

a. a lower sum with two rectangles of equal width.

b. a lower sum with four rectangles of equal width.

c. an upper sum with two rectangles of equal width.

d. an upper sum with four rectangles of equal width.

1. $f(x) = x^2$ between $x = 0$ and $x = 1$.

2. $f(x) = x^3$ between $x = 0$ and $x = 1$.

3. $f(x) = 1/x$ between $x = 1$ and $x = 5$.

4. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Using rectangles each of whose height is given by the value of the function at the midpoint of the rectangle's base (*the midpoint rule*), estimate the area under the graphs of the following functions, using first two and then four rectangles.

5. $f(x) = x^2$ between $x = 0$ and $x = 1$.

6. $f(x) = x^3$ between $x = 0$ and $x = 1$.

7. $f(x) = 1/x$ between $x = 1$ and $x = 5$.

8. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Distance

9. Distance traveled The accompanying table shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine using 10 subintervals of length 1 with

a. left-endpoint values.

b. right-endpoint values.

Time (sec)	Velocity (cm/sec)	Time (sec)	Velocity (cm/sec)
0	0	6	28
1	30	7	15
2	56	8	5
3	25	9	15
4	38	10	0
5	33		

10. Distance traveled upstream You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with

a. left-endpoint values.

b. right-endpoint values.

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

11. Length of a road You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the accompanying table. Estimate the length of the road using

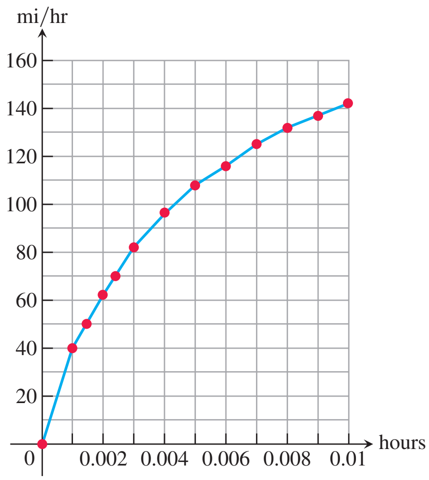
a. left-endpoint values.

b. right-endpoint values.

Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)	Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)
0	0	70	15
10	44	80	22
20	15	90	35
30	35	100	44
40	30	110	30
50	44	120	35
60	35		

12. Distance from velocity data The accompanying table gives data for the velocity of a vintage sports car accelerating from 0 to 142 mi/h in 36 sec (10 thousandths of an hour).

Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		



a. Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

b. Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

13. Free fall with air resistance An object is dropped straight down from a helicopter. The object falls faster and faster but its acceleration (rate of change of its velocity) decreases over time because of air resistance. The acceleration is measured in ft/sec^2 and recorded every second after the drop for 5 sec, as shown:

t	0	1	2	3	4	5
a	32.00	19.41	11.77	7.14	4.33	2.63

a. Find an upper estimate for the speed when $t = 5$.

b. Find a lower estimate for the speed when $t = 5$.

c. Find an upper estimate for the distance fallen when $t = 3$.

14. Distance traveled by a projectile An object is shot straight upward from sea level with an initial velocity of 400 ft/sec.

a. Assuming that gravity is the only force acting on the object, give an upper estimate for its velocity after 5 sec have elapsed. Use $g = 32$ ft/sec^2 for the gravitational acceleration.

b. Find a lower estimate for the height attained after 5 sec.

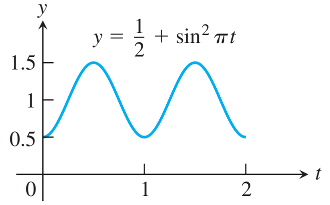
Average Value of a Function

In Exercises 15–18, use a finite sum to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

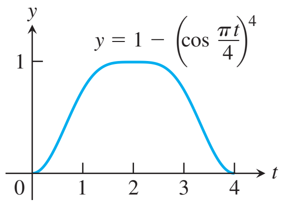
15. $f(x) = x^3$ on $[0, 2]$

16. $f(x) = 1/x$ on $[1, 9]$

17. $f(t) = (1/2) + \sin^2 \pi t$ on $[0, 2]$



18. $f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4$ on $[0, 4]$



Examples of Estimations

19. Water pollution Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

Time (h)	0	1	2	3	4
Leakage (gal/h)	50	70	97	136	190

Time (h)	5	6	7	8
Leakage (gal/h)	265	369	516	720

a. Give an upper and a lower estimate of the total quantity of oil that has escaped after 5 hours.

b. Repeat part (a) for the quantity of oil that has escaped after 8 hours.

c. The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gal of oil, approximately how many more hours will elapse in the worst case before all the oil has spilled? In the best case?

20. Air pollution A power plant generates electricity by burning oil. Pollutants produced as a result of the burning process are removed by scrubbers in the smokestacks. Over time, the scrubbers become less efficient and eventually they must be replaced when the amount of pollution released exceeds government standards. Measurements are taken at the end of each month determining the rate at which pollutants are released into the atmosphere, recorded as follows.

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**
Pollutant release rate (tons/day)	0.20	0.25	0.27	0.34	0.45	0.52

Month	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Pollutant release rate (tons/day)	0.63	0.70	0.81	0.85	0.89	0.95

a. Assuming a 30-day month and that new scrubbers allow only 0.05 ton/day to be released, give an upper estimate of the total tonnage of pollutants released by the end of June. What is a lower estimate?

b. In the best case, approximately when will a total of 125 tons of pollutants have been released into the atmosphere?

21. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n:

a. 4 (square) **b.** 8 (octagon) **c.** 16

d. Compare the areas in parts (a), (b), and (c) with the area of the circle.

22. (*Continuation of Exercise 21.*)

a. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.

b. Compute the limit of the area of the inscribed polygon as $n \to \infty$.

c. Repeat the computations in parts (a) and (b) for a circle of radius r.

COMPUTER EXPLORATIONS

In Exercises 23–26, use a CAS to perform the following steps.

a. Plot the functions over the given interval.

b. Subdivide the interval into $n = 100$, 200, and 1000 subintervals of equal length and evaluate the function at the midpoint of each subinterval.

c. Compute the average value of the function values generated in part (b).

d. Solve the equation $f(x) =$ (average value) for x using the average value calculated in part (c) for the $n = 1000$ partitioning.

23. $f(x) = \sin x$ on $[0, \pi]$ **24.** $f(x) = \sin^2 x$ on $[0, \pi]$

25. $f(x) = x \sin \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$ **26.** $f(x) = x \sin^2 \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

5.2 Sigma Notation and Limits of Finite Sums

While estimating with finite sums in Section 5.1, we encountered sums that had many terms (up to 1000 in Table 5.1, for instance). In this section we introduce a more convenient notation for working with sums that have a large number of terms. After describing this notation and its properties, we consider what happens as the number of terms approaches infinity.

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

Σ is the capital Greek letter Sigma

The Greek letter Σ (capital sigma, corresponding to our letter S), stands for "sum." The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ). Any letter can be used to denote the index, but the letters i, j, k, and n are customary.

The index k ends at $k = n$.

The summation symbol (Greek letter sigma)

$$\sum_{k=1}^{n} a_k$$

a_k is a formula for the kth term.

The index k starts at $k = 1$.

Thus we can write the squares of the numbers 1 through 11 as

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2,$$

and the sum of $f(i)$ for integers i from 1 to 100 as

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i).$$

The starting index does not have to be 1; it can be any integer.

EXAMPLE 1

A sum in sigma notation	**The sum written out, one term for each value of k**	**The value of the sum**
$\sum_{k=1}^{5} k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

■

EXAMPLE 2

Express the sum $1 + 3 + 5 + 7 + 9$ in sigma notation.

Solution The formula generating the terms depends on what we choose the lower limit of summation to be, but the terms generated remain the same. It is often simplest to choose the starting index to be $k = 0$ or $k = 1$, but we can start with any integer.

Starting with $k = 0$: $\quad 1 + 3 + 5 + 7 + 9 = \sum_{k=0}^{4} (2k + 1)$

Starting with $k = 1$: $\quad 1 + 3 + 5 + 7 + 9 = \sum_{k=1}^{5} (2k - 1)$

Starting with $k = 2$: $\quad 1 + 3 + 5 + 7 + 9 = \sum_{k=2}^{6} (2k - 3)$

Starting with $k = -3$: $\quad 1 + 3 + 5 + 7 + 9 = \sum_{k=-3}^{1} (2k + 7)$ ■

When we have a sum such as

$$\sum_{k=1}^{3}(k + k^2)$$

we can rearrange its terms to form two sums:

$$\begin{aligned}\sum_{k=1}^{3}(k + k^2) &= (1 + 1^2) + (2 + 2^2) + (3 + 3^2)\\ &= (1 + 2 + 3) + (1^2 + 2^2 + 3^2) \qquad \text{Regroup terms.}\\ &= \sum_{k=1}^{3} k + \sum_{k=1}^{3} k^2.\end{aligned}$$

This illustrates a general rule for finite sums:

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

This and three other rules are given below. Proofs of these rules can be obtained using mathematical induction (see Appendix 2).

Algebra Rules for Finite Sums

1. *Sum Rule:* $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. *Difference Rule:* $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. *Constant Multiple Rule:* $\displaystyle\sum_{k=1}^{n} ca_k = c\cdot\sum_{k=1}^{n} a_k$ (Any number c)

4. *Constant Value Rule:* $\displaystyle\sum_{k=1}^{n} c = n\cdot c$ (Any number c)

EXAMPLE 3 We demonstrate the use of the algebra rules.

(a) $\displaystyle\sum_{k=1}^{n}(3k - k^2) = 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$ Difference Rule and Constant Multiple Rule

(b) $\displaystyle\sum_{k=1}^{n}(-a_k) = \sum_{k=1}^{n}(-1)\cdot a_k = -1\cdot\sum_{k=1}^{n} a_k = -\sum_{k=1}^{n} a_k$ Constant Multiple Rule

(c) $\displaystyle\sum_{k=1}^{3}(k + 4) = \sum_{k=1}^{3} k + \sum_{k=1}^{3} 4$ Sum Rule

$= (1 + 2 + 3) + (3\cdot 4)$ Constant Value Rule

$= 6 + 12 = 18$

(d) $\displaystyle\sum_{k=1}^{n}\frac{1}{n} = n\cdot\frac{1}{n} = 1$ Constant Value Rule ($1/n$ is constant) ■

HISTORICAL BIOGRAPHY
Carl Friedrich Gauss
(1777–1855)
www.goo.gl/LZMPlA

Over the years people have discovered a variety of formulas for the values of finite sums. The most famous of these are the formula for the sum of the first n integers (Gauss is said to have discovered it at age 8) and the formulas for the sums of the squares and cubes of the first n integers.

EXAMPLE 4 Show that the sum of the first n integers is

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Solution The formula tells us that the sum of the first 4 integers is

$$\frac{(4)(5)}{2} = 10.$$

Addition verifies this prediction:

$$1 + 2 + 3 + 4 = 10.$$

To prove the formula in general, we write out the terms in the sum twice, once forward and once backward.

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \\ n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \end{array}$$

If we add the two terms in the first column we get $1 + n = n + 1$. Similarly, if we add the two terms in the second column we get $2 + (n-1) = n + 1$. The two terms in any column sum to $n + 1$. When we add the n columns together we get n terms, each equal to $n + 1$, for a total of $n(n + 1)$. Since this is twice the desired quantity, the sum of the first n integers is $n(n + 1)/2$. ■

Formulas for the sums of the squares and cubes of the first n integers are proved using mathematical induction (see Appendix 2). We state them here.

The first n squares: $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

The first n cubes: $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Limits of Finite Sums

The finite sum approximations that we considered in Section 5.1 became more accurate as the number of terms increased and the subinterval widths (lengths) narrowed. The next example shows how to calculate a limiting value as the widths of the subintervals go to zero and the number of subintervals grows to infinity.

EXAMPLE 5 Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$ on the x-axis using equal-width rectangles whose widths approach zero and whose number approaches infinity. (See Figure 5.4a.)

Solution We compute a lower sum approximation using n rectangles of equal width $\Delta x = (1 - 0)/n$, and then we see what happens as $n \to \infty$. We start by subdividing $[0, 1]$ into n equal width subintervals

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, \frac{n}{n}\right].$$

Each subinterval has width $1/n$. The function $1 - x^2$ is decreasing on $[0, 1]$, and its smallest value in a subinterval occurs at the subinterval's right endpoint. So a lower sum is constructed with rectangles whose height over the subinterval $[(k-1)/n, k/n]$ is $f(k/n) = 1 - (k/n)^2$, giving the sum

$$f\left(\frac{1}{n}\right)\cdot\frac{1}{n} + f\left(\frac{2}{n}\right)\cdot\frac{1}{n} + \cdots + f\left(\frac{k}{n}\right)\cdot\frac{1}{n} + \cdots + f\left(\frac{n}{n}\right)\cdot\frac{1}{n}.$$

We write this in sigma notation and simplify,

$$\begin{aligned}
\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} &= \sum_{k=1}^{n}\left(1-\left(\frac{k}{n}\right)^2\right)\frac{1}{n} \\
&= \sum_{k=1}^{n}\left(\frac{1}{n}-\frac{k^2}{n^3}\right) \\
&= \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3} && \text{Difference Rule} \\
&= n\cdot\frac{1}{n} - \frac{1}{n^3}\sum_{k=1}^{n}k^2 && \text{Constant Value and Constant Multiple Rules} \\
&= 1-\left(\frac{1}{n^3}\right)\frac{n(n+1)(2n+1)}{6} && \text{Sum of the First } n \text{ Squares} \\
&= 1-\frac{2n^3+3n^2+n}{6n^3}. && \text{Numerator expanded}
\end{aligned}$$

We have obtained an expression for the lower sum that holds for any n. Taking the limit of this expression as $n\to\infty$, we see that the lower sums converge as the number of subintervals increases and the subinterval widths approach zero:

$$\lim_{n\to\infty}\left(1-\frac{2n^3+3n^2+n}{6n^3}\right) = 1-\frac{2}{6} = \frac{2}{3}.$$

The lower sum approximations converge to $2/3$. A similar calculation shows that the upper sum approximations also converge to $2/3$. Any finite sum approximation $\sum_{k=1}^{n} f(c_k)(1/n)$ also converges to the same value, $2/3$. This is because it is possible to show that any finite sum approximation is trapped between the lower and upper sum approximations. For this reason we are led to *define* the area of the region R as this limiting value. In Section 5.3 we study the limits of such finite approximations in a general setting. ■

Riemann Sums

HISTORICAL BIOGRAPHY
Georg Friedrich Bernhard Riemann (1826–1866)
www.goo.gl/hPFV65

The theory of limits of finite approximations was made precise by the German mathematician Bernhard Riemann. We now introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral that will be presented in the next section.

We begin with an arbitrary bounded function f defined on a closed interval $[a, b]$. Like the function pictured in Figure 5.8, f may have negative as well as positive values. We subdivide the interval $[a, b]$ into subintervals, not necessarily of equal widths (or lengths), and form sums in the same way as for the finite approximations in Section 5.1. To do so, we choose $n-1$ points $\{x_1, x_2, x_3, \dots, x_{n-1}\}$ between a and b that are in increasing order, so that

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

To make the notation consistent, we set $x_0 = a$ and $x_n = b$, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The set of all of these points,

$$P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n\},$$

is called a **partition** of $[a, b]$.

The partition P divides $[a, b]$ into the n closed subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n].$$

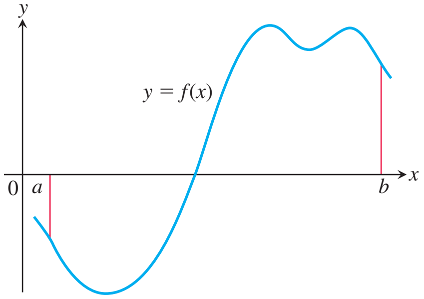
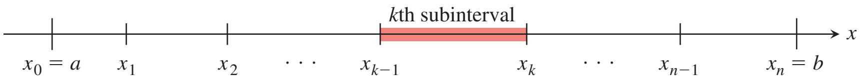


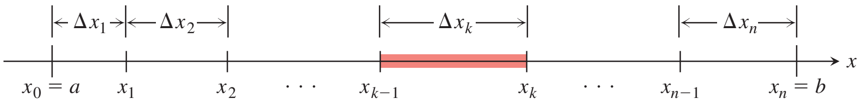
FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

HISTORICAL BIOGRAPHY
Richard Dedekind
(1831–1916)
www.goo.gl/aPN8sH

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the ***k*th subinterval** is $[x_{k-1}, x_k]$ (where k is an integer between 1 and n).



The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1, the width of the second $[x_1, x_2]$ is denoted Δx_2, and the width of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$.



If all n subintervals have equal width, then their common width, which we call Δx, is equal to $(b - a)/n$.

In each subinterval we select some point. The point chosen in the kth subinterval $[x_{k-1}, x_k]$ is called c_k. Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is positive or negative, or on the x-axis if $f(c_k) = 0$ (see Figure 5.9).

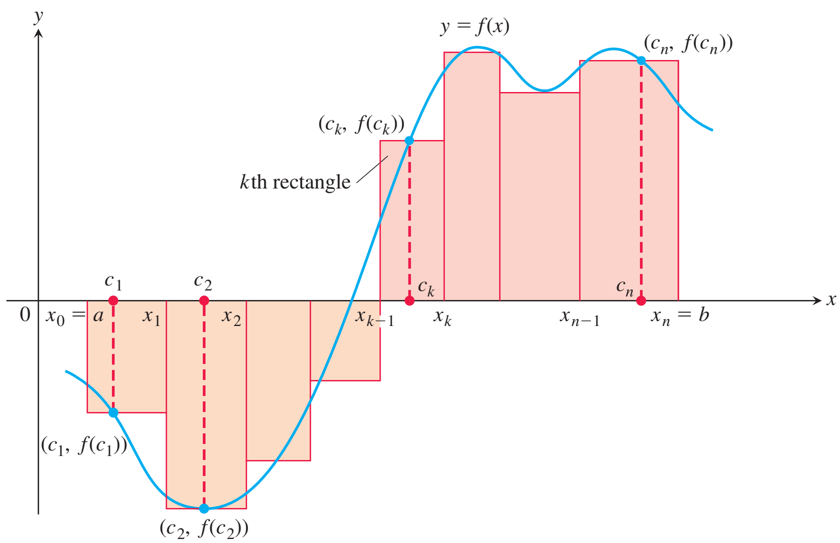


FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis. Figure 5.8 has been repeated and enlarged, the partition of $[a, b]$ and the points c_k have been added, and the corresponding rectangles with heights $f(c_k)$ are shown.

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k. When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.

Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k .$$

The sum S_P is called a **Riemann sum for f on the interval $[a, b]$**. There are many such sums, depending on the partition P we choose, and the choices of the points c_k in the

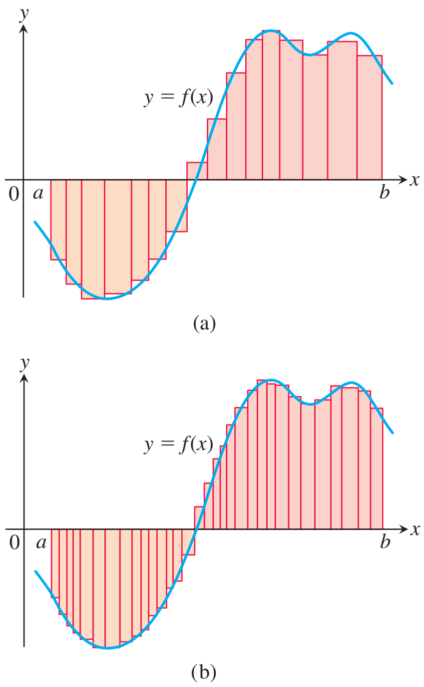


FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.

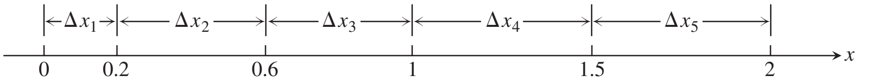
subintervals. For instance, we could choose n subintervals all having equal width $\Delta x = (b - a)/n$ to partition $[a, b]$, and then choose the point c_k to be the right-hand endpoint of each subinterval when forming the Riemann sum (as we did in Example 5). This choice leads to the Riemann sum formula

$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right) \cdot \left(\frac{b-a}{n}\right).$$

Similar formulas can be obtained if instead we choose c_k to be the left-hand endpoint, or the midpoint, of each subinterval.

In the cases in which the subintervals all have equal width $\Delta x = (b - a)/n$, we can make them thinner by simply increasing their number n. When a partition has subintervals of varying widths, we can ensure they are all thin by controlling the width of a widest (longest) subinterval. We define the **norm** of a partition P, written $\|P\|$, to be the largest of all the subinterval widths. If $\|P\|$ is a small number, then all of the subintervals in the partition P have a small width.

EXAMPLE 6 The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$. There are five subintervals of P: $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$:



The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length. ■

Any Riemann sum associated with a partition of a closed interval $[a, b]$ defines rectangles that approximate the region between the graph of a continuous function f and the x-axis. Partitions with norm approaching zero lead to collections of rectangles that approximate this region with increasing accuracy, as suggested by Figure 5.10. We will see in the next section that if the function f is continuous over the closed interval $[a, b]$, then no matter how we choose the partition P and the points c_k in its subintervals, the Riemann sums corresponding to these choices will approach a single limiting value as the subinterval widths (which are controlled by the norm of the partition) approach zero.

EXERCISES 5.2

Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\displaystyle\sum_{k=1}^{2} \frac{6k}{k+1}$ **2.** $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$

3. $\displaystyle\sum_{k=1}^{4} \cos k\pi$ **4.** $\displaystyle\sum_{k=1}^{5} \sin k\pi$

5. $\displaystyle\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$ **6.** $\displaystyle\sum_{k=1}^{4} (-1)^{k} \cos k\pi$

7. Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation?

a. $\displaystyle\sum_{k=1}^{6} 2^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} 2^{k}$ **c.** $\displaystyle\sum_{k=-1}^{4} 2^{k+1}$

8. Which of the following express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation?

a. $\displaystyle\sum_{k=1}^{6} (-2)^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} (-1)^{k} 2^{k}$ **c.** $\displaystyle\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$

9. Which formula is not equivalent to the other two?

a. $\displaystyle\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ **b.** $\displaystyle\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1}$ **c.** $\displaystyle\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2}$

10. Which formula is not equivalent to the other two?

a. $\displaystyle\sum_{k=1}^{4} (k-1)^2$ **b.** $\displaystyle\sum_{k=-1}^{3} (k+1)^2$ **c.** $\displaystyle\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice for the starting index.

11. $1 + 2 + 3 + 4 + 5 + 6$ **12.** $1 + 4 + 9 + 16$

13. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ **14.** $2 + 4 + 6 + 8 + 10$

15. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$

16. $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$

Values of Finite Sums

17. Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of

a. $\sum_{k=1}^{n} 3a_k$ **b.** $\sum_{k=1}^{n} \frac{b_k}{6}$ **c.** $\sum_{k=1}^{n} (a_k + b_k)$

d. $\sum_{k=1}^{n} (a_k - b_k)$ **e.** $\sum_{k=1}^{n} (b_k - 2a_k)$

18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of

a. $\sum_{k=1}^{n} 8a_k$ **b.** $\sum_{k=1}^{n} 250b_k$

c. $\sum_{k=1}^{n} (a_k + 1)$ **d.** $\sum_{k=1}^{n} (b_k - 1)$

Evaluate the sums in Exercises 19–32.

19. a. $\sum_{k=1}^{10} k$ **b.** $\sum_{k=1}^{10} k^2$ **c.** $\sum_{k=1}^{10} k^3$

20. a. $\sum_{k=1}^{13} k$ **b.** $\sum_{k=1}^{13} k^2$ **c.** $\sum_{k=1}^{13} k^3$

21. $\sum_{k=1}^{7} (-2k)$

22. $\sum_{k=1}^{5} \frac{\pi k}{15}$

23. $\sum_{k=1}^{6} (3 - k^2)$

24. $\sum_{k=1}^{6} (k^2 - 5)$

25. $\sum_{k=1}^{5} k(3k + 5)$

26. $\sum_{k=1}^{7} k(2k + 1)$

27. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$

28. $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$

29. a. $\sum_{k=1}^{7} 3$ **b.** $\sum_{k=1}^{500} 7$ **c.** $\sum_{k=3}^{264} 10$

30. a. $\sum_{k=9}^{36} k$ **b.** $\sum_{k=3}^{17} k^2$ **c.** $\sum_{k=18}^{71} k(k - 1)$

31. a. $\sum_{k=1}^{n} 4$ **b.** $\sum_{k=1}^{n} c$ **c.** $\sum_{k=1}^{n} (k - 1)$

32. a. $\sum_{k=1}^{n} \left(\frac{1}{n} + 2n\right)$ **b.** $\sum_{k=1}^{n} \frac{c}{n}$ **c.** $\sum_{k=1}^{n} \frac{k}{n^2}$

33. $\sum_{k=1}^{50} [(k + 1)^2 - k^2]$

34. $\sum_{k=2}^{20} [\sin(k - 1) - \sin k]$

35. $\sum_{k=7}^{30} \left(\sqrt{k - 4} - \sqrt{k - 3}\right)$

36. $\sum_{k=1}^{40} \frac{1}{k(k + 1)}$ $\left(\textit{Hint: } \frac{1}{k(k + 1)} = \frac{1}{k} - \frac{1}{k + 1}\right)$

Riemann Sums

In Exercises 37–42, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^{4} f(c_k)\,\Delta x_k$, given that c_k is the **(a)** left-hand endpoint, **(b)** right-hand endpoint, **(c)** midpoint of the kth subinterval. (Make a separate sketch for each set of rectangles.)

37. $f(x) = x^2 - 1, \quad [0, 2]$

38. $f(x) = -x^2, \quad [0, 1]$

39. $f(x) = \sin x, \quad [-\pi, \pi]$

40. $f(x) = \sin x + 1, \quad [-\pi, \pi]$

41. Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.

42. Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

Limits of Riemann Sums

For the functions in Exercises 43–50, find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into n equal subintervals and using the right-hand endpoint for each c_k. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

43. $f(x) = 1 - x^2$ over the interval $[0, 1]$.

44. $f(x) = 2x$ over the interval $[0, 3]$.

45. $f(x) = x^2 + 1$ over the interval $[0, 3]$.

46. $f(x) = 3x^2$ over the interval $[0, 1]$.

47. $f(x) = x + x^2$ over the interval $[0, 1]$.

48. $f(x) = 3x + 2x^2$ over the interval $[0, 1]$.

49. $f(x) = 2x^3$ over the interval $[0, 1]$.

50. $f(x) = x^2 - x^3$ over the interval $[-1, 0]$.

5.3 The Definite Integral

In this section we consider the limit of general Riemann sums as the norm of the partitions of a closed interval $[a, b]$ approaches zero. This limiting process leads us to the definition of the *definite integral* of a function over a closed interval $[a, b]$.

Definition of the Definite Integral

The definition of the definite integral is based on the fact that for some functions, as the norm of the partitions of $[a, b]$ approaches zero, the values of the corresponding Riemann sums approach a limiting value J. We introduce the symbol ε as a small positive number that specifies how close to J the Riemann sum must be, and the symbol δ as a second small positive number that specifies how small the norm of a partition must be in order for convergence to happen. We now define this limit precisely.

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \varepsilon.$$

The definition involves a limiting process in which the norm of the partition goes to zero.

We have many choices for a partition P with norm going to zero, and many choices of points c_k for each partition. The definite integral exists when we always get the same limit J, no matter what choices are made. When the limit exists we write

$$J = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k,$$

and we say that the *definite integral exists*. The limit of any Riemann sum is always taken as the norm of the partitions approaches zero and the number of subintervals goes to infinity, and furthermore the same limit J must be obtained no matter what choices we make for the points c_k.

Leibniz introduced a notation for the definite integral that captures its construction as a limit of Riemann sums. He envisioned the finite sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ becoming an infinite sum of function values $f(x)$ multiplied by "infinitesimal" subinterval widths dx. The sum symbol $\sum$ is replaced in the limit by the integral symbol $\int$, whose origin is in the letter "S" (for sum). The function values $f(c_k)$ are replaced by a continuous selection of function values $f(x)$. The subinterval widths Δx_k become the differential dx. It is as if we are summing all products of the form $f(x) \cdot dx$ as x goes from a to b. While this notation captures the process of constructing an integral, it is Riemann's definition that gives a precise meaning to the definite integral.

If the definite integral exists, then instead of writing J we write

$$\int_a^b f(x)\,dx.$$

We read this as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x." The component parts in the integral symbol also have names:

Upper limit of integration

The function $f(x)$ is the integrand.

Integral sign

x is the variable of integration.

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

When the definite integral exists, we say that the Riemann sums of f on $[a, b]$ **converge** to the definite integral $J = \int_a^b f(x)\,dx$ and that f is **integrable** over $[a, b]$.

In the cases where the subintervals all have equal width $\Delta x = (b - a)/n$, the Riemann sums have the form

$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right), \qquad \Delta x_k = \Delta x = (b-a)/n \text{ for all } k$$

where c_k is chosen in the kth subinterval. If the definite integral exists, then these Riemann sums converge to the definite integral of f over $[a, b]$, so

$$J = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right). \qquad \text{For equal-width subintervals, } \|P\| \to 0 \text{ is the same as } n \to \infty.$$

If we pick the point c_k to be the right endpoint of the kth subinterval, so that $c_k = a + k\,\Delta x = a + k(b - a)/n$, then the formula for the definite integral becomes

A Formula for the Riemann Sum with Equal-Width Subintervals

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) \tag{1}$$

Equation (1) gives one explicit formula that can be used to compute definite integrals. As long as the definite integral exists, the Riemann sums corresponding to other choices of partitions and locations of points c_k will have the same limit as $n \to \infty$, provided that the norm of the partition approaches zero.

The value of the definite integral of a function over any particular interval depends on the function, not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)\,dt \qquad \text{or} \qquad \int_a^b f(u)\,du \qquad \text{instead of} \qquad \int_a^b f(x)\,dx.$$

No matter how we write the integral, it is still the same number, the limit of the Riemann sums as the norm of the partition approaches zero. Since it does not matter what letter we use, the variable of integration is called a **dummy variable**. In the three integrals given above, the dummy variables are t, u, and x.

Integrable and Nonintegrable Functions

Not every function defined over a closed interval $[a, b]$ is integrable even if the function is bounded. That is, the Riemann sums for some functions might not converge to the same limiting value, or to any value at all. A full development of exactly which functions defined over $[a, b]$ are integrable requires advanced mathematical analysis, but fortunately most functions that commonly occur in applications are integrable. In particular, every *continuous* function over $[a, b]$ is integrable over this interval, and so is every function that has no more than a finite number of jump discontinuities on $[a, b]$. (See Figures 1.9 and 1.10. The latter functions are called *piecewise-continuous functions*, and they are defined in Additional Exercises 11–18 at the end of this chapter.) The following theorem, which is proved in more advanced courses, establishes these results.

THEOREM 1—Integrability of Continuous Functions
If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

The idea behind Theorem 1 for continuous functions is given in Exercises 86 and 87. Briefly, when f is continuous we can choose each c_k so that $f(c_k)$ gives the maximum value of f on the subinterval $[x_{k-1}, x_k]$, resulting in an upper sum. Likewise, we can choose c_k to give the minimum value of f on $[x_{k-1}, x_k]$ to obtain a lower sum. The upper and lower sums can be shown to converge to the same limiting value as the norm of the partition P tends to zero. Moreover, every Riemann sum is trapped between the values of the upper and lower sums, so every Riemann sum converges to the same limit as well. Therefore, the number J in the definition of the definite integral exists, and the continuous function f is integrable over $[a, b]$.

For integrability to fail, a function needs to be sufficiently discontinuous that the region between its graph and the x-axis cannot be approximated well by increasingly thin rectangles. Our first example shows a function that is not integrable over a closed interval.

EXAMPLE 1 The function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

has no Riemann integral over $[0, 1]$. Underlying this is the fact that between any two numbers there is both a rational number and an irrational number. Thus the function jumps up and down too erratically over $[0, 1]$ to allow the region beneath its graph and above the x-axis to be approximated by rectangles, no matter how thin they are. In fact, we will show that upper sum approximations and lower sum approximations converge to different limiting values.

If we choose a partition P of [0, 1], then the lengths of the intervals in the partition sum to 1; that is, $\sum_{k=1}^{n} \Delta x_k = 1$. In each subinterval $[x_{k-1}, x_k]$ there is a rational point, say c_k. Because c_k is rational, $f(c_k) = 1$. Since 1 is the maximum value that f can take anywhere, the upper sum approximation for this choice of c_k's is

$$U = \sum_{k=1}^{n} f(c_k)\, \Delta x_k = \sum_{k=1}^{n} (1)\, \Delta x_k = 1.$$

As the norm of the partition approaches 0, these upper sum approximations converge to 1 (because each approximation is equal to 1).

On the other hand, we could pick the c_k's differently and get a different result. Each subinterval $[x_{k-1}, x_k]$ also contains an irrational point c_k, and for this choice $f(c_k) = 0$. Since 0 is the minimum value that f can take anywhere, this choice of c_k gives us the minimum value of f on the subinterval. The corresponding lower sum approximation is

$$L = \sum_{k=1}^{n} f(c_k)\, \Delta x_k = \sum_{k=1}^{n} (0)\, \Delta x_k = 0.$$

These lower sum approximations converge to 0 as the norm of the partition converges to 0 (because they each equal 0).

Thus making different choices for the points c_k results in different limits for the corresponding Riemann sums. We conclude that the definite integral of f over the interval [0, 1] does not exist, and that f is not integrable over [0, 1]. ■

Theorem 1 says nothing about how to *calculate* definite integrals. A method of calculation will be developed in Section 5.4, through a connection to antiderivatives.

Properties of Definite Integrals

In defining $\int_a^b f(x)\, dx$ as a limit of sums $\sum_{k=1}^{n} f(c_k)\, \Delta x_k$, we moved from left to right across the interval $[a, b]$. What would happen if we instead move right to left, starting with $x_0 = b$ and ending at $x_n = a$? Each Δx_k in the Riemann sum would change its sign, with $x_k - x_{k-1}$ now negative instead of positive. With the same choices of c_k in each subinterval, the sign of any Riemann sum would change, as would the sign of the limit, the integral $\int_b^a f(x)\,dx$. Since we have not previously given a meaning to integrating backward, we are led to define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \qquad \text{a and b interchanged}$$

Although we have only defined the integral over intervals $[a, b]$ with $a < b$, it is convenient to have a definition for the integral over $[a, b]$ when $a = b$, that is, for the integral over an interval of zero width. Since $a = b$ gives $\Delta x = 0$, whenever $f(a)$ exists we define

$$\int_a^a f(x)\,dx = 0. \qquad \text{a is both the lower and the upper limit of integration.}$$

Theorem 2 states some basic properties of integrals, including the two just discussed. These properties, listed in Table 5.6, are very useful for computing integrals. We will refer to them repeatedly to simplify our calculations. Rules 2 through 7 have geometric interpretations, which are shown in Figure 5.11. The graphs in these figures show only positive functions, but the rules apply to general integrable functions, which could take both positive and negative values.

> **THEOREM 2** When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules listed in Table 5.6.

While Rules 1 and 2 are definitions, Rules 3 to 7 of Table 5.6 must be proved. Below we give a proof of Rule 6. Similar proofs can be given to verify the other properties in Table 5.6.

TABLE 5.6 Rules satisfied by definite integrals

1. *Order of Integration:* $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero Width Interval:* $\displaystyle\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists

3. *Constant Multiple:* $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k

4. *Sum and Difference:* $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

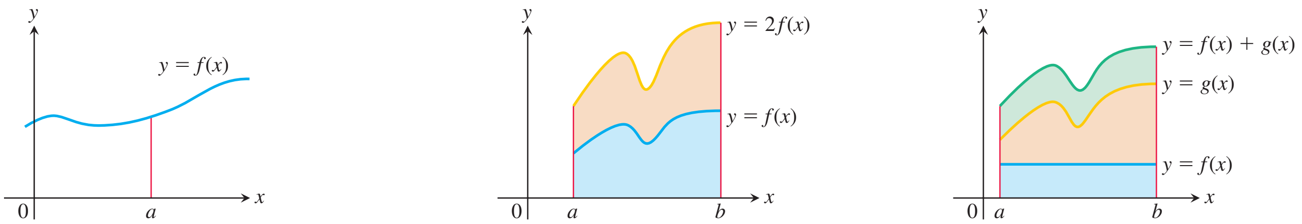
5. *Additivity:* $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$(\min f)\cdot(b - a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b - a).$$

7. *Domination:* If $f(x) \ge g(x)$ on $[a, b]$ then $\displaystyle\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$

If $f(x) \ge 0$ on $[a, b]$ then $\displaystyle\int_a^b f(x)\,dx \ge 0.$ — Special case



(a) *Zero Width Interval:*

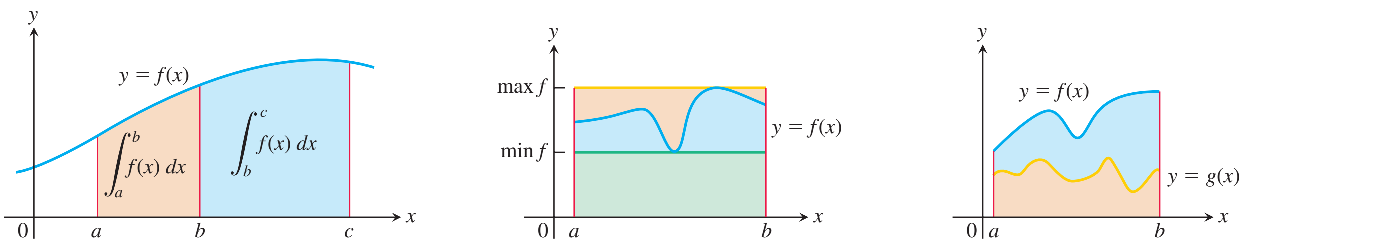
$$\int_a^a f(x)\,dx = 0$$

(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

(e) *Max-Min Inequality:*

$$(\min f)\cdot(b - a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b - a)$$

(f) *Domination:*

If $f(x) \ge g(x)$ on $[a, b]$ then

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

FIGURE 5.11 Geometric interpretations of Rules 2–7 in Table 5.6.

Proof of Rule 6 Rule 6 says that the integral of f over $[a, b]$ is never smaller than the minimum value of f times the length of the interval and never larger than the maximum value of f times the length of the interval. The reason is that for every partition of $[a, b]$ and for every choice of the points c_k,

$$\begin{aligned}
(\min f)\cdot(b - a) &= (\min f)\cdot\sum_{k=1}^{n}\Delta x_k && \sum_{k=1}^{n}\Delta x_k = b - a\\
&= \sum_{k=1}^{n}(\min f)\cdot\Delta x_k && \text{Constant Multiple Rule}\\
&\le \sum_{k=1}^{n} f(c_k)\,\Delta x_k && \min f \le f(c_k)\\
&\le \sum_{k=1}^{n}(\max f)\cdot\Delta x_k && f(c_k) \le \max f\\
&= (\max f)\cdot\sum_{k=1}^{n}\Delta x_k && \text{Constant Multiple Rule}\\
&= (\max f)\cdot(b - a).
\end{aligned}$$

In short, all Riemann sums for f on $[a, b]$ satisfy the inequalities

$$(\min f)\cdot(b - a) \le \sum_{k=1}^{n} f(c_k)\,\Delta x_k \le (\max f)\cdot(b - a).$$

Hence their limit, which is the integral, satisfies the same inequalities. ■

EXAMPLE 2 To illustrate some of the rules, we suppose that

$$\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

Then

1. $\displaystyle\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$ Rule 1

2. $\displaystyle\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$ Rules 3 and 4

$\displaystyle = 2(5) + 3(7) = 31$

3. $\displaystyle\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$ Rule 5 ■

EXAMPLE 3 Show that the value of $\int_0^1 \sqrt{1 + \cos x}\,dx$ is less than or equal to $\sqrt{2}$.

Solution The Max-Min Inequality for definite integrals (Rule 6) says that $(\min f)\cdot(b - a)$ is a *lower bound* for the value of $\int_a^b f(x)\,dx$ and that $(\max f)\cdot(b - a)$ is an *upper bound*. The maximum value of $\sqrt{1 + \cos x}$ on $[0, 1]$ is $\sqrt{1 + 1} = \sqrt{2}$, so

$$\int_{0}^{1} \sqrt{1 + \cos x}\,dx \le \sqrt{2}\cdot(1 - 0) = \sqrt{2}.$$ ■

Area Under the Graph of a Nonnegative Function

We now return to the problem that started this chapter, which is defining what we mean by the *area* of a region having a curved boundary. In Section 5.1 we approximated the area under the graph of a nonnegative continuous function using several types of finite sums of areas of rectangles that approximate the region—upper sums, lower sums, and sums using the midpoints of each subinterval—all of which are Riemann sums constructed in special ways. Theorem 1 guarantees that all of these Riemann sums converge to a single definite integral as the norm of the partitions approaches zero and the number of subintervals goes to infinity. As a result, we can now *define* the area under the graph of a nonnegative integrable function to be the value of that definite integral.

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_{a}^{b} f(x)\,dx.$$

For the first time we have a rigorous definition for the area of a region whose boundary is the graph of a continuous function. We now apply this to a simple example, the area under a straight line, and we verify that our new definition agrees with our previous notion of area.

EXAMPLE 4 Compute $\int_0^b x\,dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.

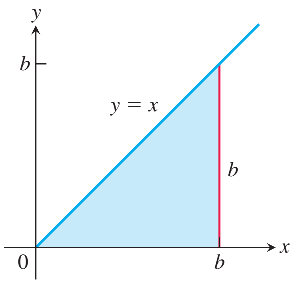


FIGURE 5.12 The region in Example 4 is a triangle.

Solution The region of interest is a triangle (Figure 5.12). We compute the area in two ways.

(a) To compute the definite integral as the limit of Riemann sums, we calculate $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$ for partitions whose norms go to zero. Theorem 1 tells us that it does not matter how we choose the partitions or the points c_k as long as the norms approach zero. All choices give the exact same limit. So we consider the partition P that subdivides the interval $[0, b]$ into n subintervals of equal width $\Delta x = (b - 0)/n = b/n$, and we choose c_k to be the right endpoint in each subinterval. The partition is $P = \left\{0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \cdots, \frac{nb}{n}\right\}$ and $c_k = \frac{kb}{n}$. So

$$
\begin{aligned}
\sum_{k=1}^{n} f(c_k)\,\Delta x &= \sum_{k=1}^{n} \frac{kb}{n}\cdot\frac{b}{n} && f(c_k) = c_k \\
&= \sum_{k=1}^{n} \frac{kb^2}{n^2} \\
&= \frac{b^2}{n^2}\sum_{k=1}^{n} k && \text{Constant Multiple Rule} \\
&= \frac{b^2}{n^2}\cdot\frac{n(n+1)}{2} && \text{Sum of First } n \text{ Integers} \\
&= \frac{b^2}{2}\left(1 + \frac{1}{n}\right).
\end{aligned}
$$

As $n \to \infty$ and $\|P\| \to 0$, this last expression on the right has the limit $b^2/2$. Therefore,

$$\int_0^b x\,dx = \frac{b^2}{2}.$$

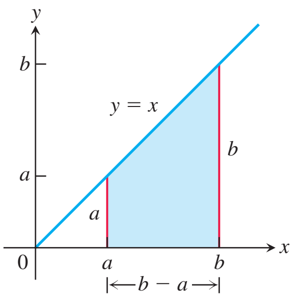
(b) Since the area equals the definite integral for a nonnegative function, we can quickly derive the definite integral by using the formula for the area of a triangle having base length b and height $y = b$. The area is $A = (1/2)\,b\cdot b = b^2/2$. Again we conclude that $\int_0^b x\,dx = b^2/2$. ■

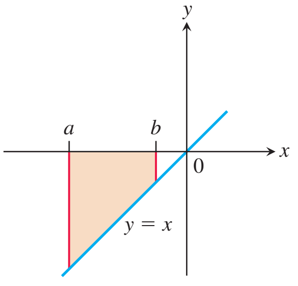
Example 4 can be generalized to integrate $f(x) = x$ over any closed interval $[a, b], 0 < a < b$.

$$
\begin{aligned}
\int_a^b x\,dx &= \int_a^0 x\,dx + \int_0^b x\,dx && \text{Rule 5} \\
&= -\int_0^a x\,dx + \int_0^b x\,dx && \text{Rule 1} \\
&= -\frac{a^2}{2} + \frac{b^2}{2}. && \text{Example 4}
\end{aligned}
$$

In conclusion, we have the following rule for integrating $f(x) = x$:

$$\boxed{\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, \qquad a < b \qquad (2)}$$





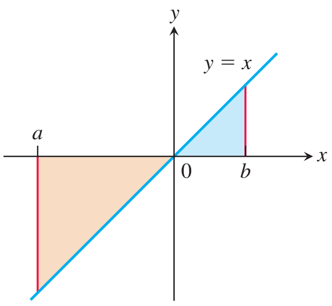


FIGURE 5.13 (a) The area of this trapezoidal region is $A = (b^2 - a^2)/2$. (b) The definite integral in Equation (2) gives the negative of the area of this trapezoidal region. (c) The definite integral in Equation (2) gives the area of the blue triangular region added to the negative of the area of the tan triangular region.

This computation gives the area of the trapezoid in Figure 5.13a. Equation (2) remains valid when a and b are negative, but the interpretation of the definite integral changes. When $a < b < 0$, the definite integral value $(b^2 - a^2)/2$ is a negative number, the negative of the area of a trapezoid dropping down to the line $y = x$ below the x-axis (Figure 5.13b). When $a < 0$ and $b > 0$, Equation (2) is still valid and the definite integral gives the difference between two areas, the area under the graph and above $[0, b]$ minus the area below $[a, 0]$ and over the graph (Figure 5.13c).

The following results can also be established by using a Riemann sum calculation similar to the one that we used in Example 4 (Exercises 63 and 65).

$$\int_a^b c\,dx = c(b - a), \qquad c \text{ any constant} \tag{3}$$

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}, \qquad a < b \tag{4}$$

Average Value of a Continuous Function Revisited

In Section 5.1 we informally introduced the average value of a nonnegative continuous function f over an interval $[a, b]$, leading us to define this average as the area under the graph of $y = f(x)$ divided by $b - a$. In integral notation we write this as

$$\text{Average} = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

This formula gives us a precise definition of the average value of a continuous (or integrable) function, whether it is positive, negative, or both.

Alternatively, we justify this formula through the following reasoning. We start with the idea from arithmetic that the average of n numbers is their sum divided by n. A continuous function f on $[a, b]$ may have infinitely many values, but we can still sample them in an orderly way. We divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$ and evaluate f at a point c_k in each (Figure 5.14). The average of the n sampled values is

$$\begin{aligned}
\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} &= \frac{1}{n}\sum_{k=1}^{n} f(c_k) \\
&= \frac{\Delta x}{b - a}\sum_{k=1}^{n} f(c_k) && \Delta x = \frac{b-a}{n}, \text{ so } \frac{1}{n} = \frac{\Delta x}{b - a} \\
&= \frac{1}{b - a}\sum_{k=1}^{n} f(c_k)\,\Delta x. && \text{Constant Multiple Rule}
\end{aligned}$$

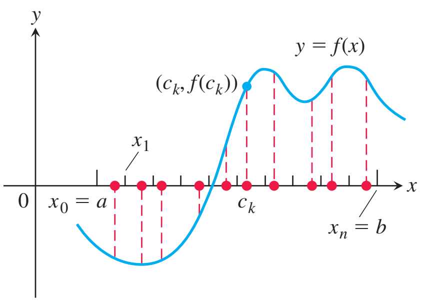


FIGURE 5.14 A sample of values of a function on an interval $[a, b]$.

The average of the samples is obtained by dividing a Riemann sum for f on $[a, b]$ by $(b - a)$. As we increase the number of samples and let the norm of the partition approach zero, the average approaches $(1/(b - a))\int_a^b f(x)\,dx$. Both points of view lead us to the following definition.

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, which is also called its **mean**, is

$$\text{av}(f) = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

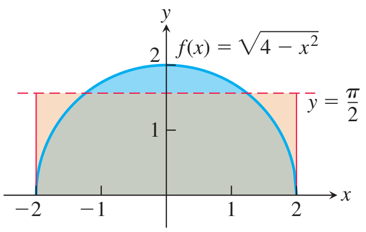


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

EXAMPLE 5 Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

Solution We recognize $f(x) = \sqrt{4 - x^2}$ as the function whose graph is the upper semicircle of radius 2 centered at the origin (Figure 5.15).

Since we know the area inside a circle, we do not need to take the limit of Riemann sums. The area between the semicircle and the x-axis from -2 to 2 can be computed using the geometry formula

$$\text{Area} = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi (2)^2 = 2\pi.$$

Because f is nonnegative, the area is also the value of the integral of f from -2 to 2,

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = 2\pi.$$

Therefore, the average value of f is

$$\operatorname{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^{2} \sqrt{4 - x^2}\, dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

Notice that the average value of f over $[-2, 2]$ is the same as the height of a rectangle over $[-2, 2]$ whose area equals the area of the upper semicircle (see Figure 5.15).

EXERCISES 5.3

Interpreting Limits of Sums as Integrals

Express the limits in Exercises 1–8 as definite integrals.

1. $\lim\limits_{\|P\| \to 0} \sum_{k=1}^{n} c_k^{\,2}\, \Delta x_k$, where P is a partition of $[0, 2]$

2. $\lim\limits_{\|P\| \to 0} \sum_{k=1}^{n} 2c_k^{\,3}\, \Delta x_k$, where P is a partition of $[-1, 0]$

3. $\lim\limits_{\|P\| \to 0} \sum_{k=1}^{n} (c_k^{\,2} - 3c_k)\, \Delta x_k$, where P is a partition of $[-7, 5]$

4. $\lim\limits_{\|P\| \to 0} \sum_{k=1}^{n} \left(\frac{1}{c_k}\right) \Delta x_k$, where P is a partition of $[1, 4]$

5. $\lim\limits_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{1 - c_k}\, \Delta x_k$, where P is a partition of $[2, 3]$

6. $\lim\limits_{\|P\| \to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^{\,2}}\, \Delta x_k$, where P is a partition of $[0, 1]$

7. $\lim\limits_{\|P\| \to 0} \sum_{k=1}^{n} (\sec c_k)\, \Delta x_k$, where P is a partition of $[-\pi/4, 0]$

8. $\lim\limits_{\|P\| \to 0} \sum_{k=1}^{n} (\tan c_k)\, \Delta x_k$, where P is a partition of $[0, \pi/4]$

Using the Definite Integral Rules

9. Suppose that f and g are integrable and that

$$\int_1^2 f(x)\, dx = -4, \quad \int_1^5 f(x)\, dx = 6, \quad \int_1^5 g(x)\, dx = 8.$$

Use the rules in Table 5.6 to find

a. $\int_2^2 g(x)\, dx$ **b.** $\int_5^1 g(x)\, dx$

c. $\int_1^2 3f(x)\, dx$ **d.** $\int_2^5 f(x)\, dx$

e. $\int_1^5 [f(x) - g(x)]\, dx$ **f.** $\int_1^5 [4f(x) - g(x)]\, dx$

10. Suppose that f and h are integrable and that

$$\int_1^9 f(x)\, dx = -1, \quad \int_7^9 f(x)\, dx = 5, \quad \int_7^9 h(x)\, dx = 4.$$

Use the rules in Table 5.6 to find

a. $\int_1^9 -2f(x)\, dx$ **b.** $\int_7^9 [f(x) + h(x)]\, dx$

c. $\int_7^9 [2f(x) - 3h(x)]\, dx$ **d.** $\int_9^1 f(x)\, dx$

e. $\int_1^7 f(x)\, dx$ **f.** $\int_9^7 [h(x) - f(x)]\, dx$

11. Suppose that $\int_1^2 f(x)\, dx = 5$. Find

a. $\int_1^2 f(u)\, du$ **b.** $\int_1^2 \sqrt{3} f(z)\, dz$

c. $\int_2^1 f(t)\, dt$ **d.** $\int_1^2 [-f(x)]\, dx$

12. Suppose that $\int_{-3}^{0} g(t)\,dt = \sqrt{2}$. Find

a. $\displaystyle\int_{0}^{-3} g(t)\,dt$ **b.** $\displaystyle\int_{-3}^{0} g(u)\,du$

c. $\displaystyle\int_{-3}^{0} [-g(x)]\,dx$ **d.** $\displaystyle\int_{-3}^{0} \frac{g(r)}{\sqrt{2}}\,dr$

13. Suppose that f is integrable and that $\int_{0}^{3} f(z)\,dz = 3$ and $\int_{0}^{4} f(z)\,dz = 7$. Find

a. $\displaystyle\int_{3}^{4} f(z)\,dz$ **b.** $\displaystyle\int_{4}^{3} f(t)\,dt$

14. Suppose that h is integrable and that $\int_{-1}^{1} h(r)\,dr = 0$ and $\int_{-1}^{3} h(r)\,dr = 6$. Find

a. $\displaystyle\int_{1}^{3} h(r)\,dr$ **b.** $\displaystyle-\int_{3}^{1} h(u)\,du$

Using Known Areas to Find Integrals

In Exercises 15–22, graph the integrands and use known area formulas to evaluate the integrals.

15. $\displaystyle\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$ **16.** $\displaystyle\int_{1/2}^{3/2} (-2x + 4)\,dx$

17. $\displaystyle\int_{-3}^{3} \sqrt{9 - x^2}\,dx$ **18.** $\displaystyle\int_{-4}^{0} \sqrt{16 - x^2}\,dx$

19. $\displaystyle\int_{-2}^{1} |x|\,dx$ **20.** $\displaystyle\int_{-1}^{1} (1 - |x|)\,dx$

21. $\displaystyle\int_{-1}^{1} (2 - |x|)\,dx$ **22.** $\displaystyle\int_{-1}^{1} \left(1 + \sqrt{1 - x^2}\right) dx$

Use known area formulas to evaluate the integrals in Exercises 23–28.

23. $\displaystyle\int_{0}^{b} \frac{x}{2}\,dx, \quad b > 0$ **24.** $\displaystyle\int_{0}^{b} 4x\,dx, \quad b > 0$

25. $\displaystyle\int_{a}^{b} 2s\,ds, \quad 0 < a < b$ **26.** $\displaystyle\int_{a}^{b} 3t\,dt, \quad 0 < a < b$

27. $f(x) = \sqrt{4 - x^2}$ on **a.** $[-2, 2]$, **b.** $[0, 2]$

28. $f(x) = 3x + \sqrt{1 - x^2}$ on **a.** $[-1, 0]$, **b.** $[-1, 1]$

Evaluating Definite Integrals

Use the results of Equations (2) and (4) to evaluate the integrals in Exercises 29–40.

29. $\displaystyle\int_{1}^{\sqrt{2}} x\,dx$ **30.** $\displaystyle\int_{0.5}^{2.5} x\,dx$ **31.** $\displaystyle\int_{\pi}^{2\pi} \theta\,d\theta$

32. $\displaystyle\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr$ **33.** $\displaystyle\int_{0}^{\sqrt[3]{7}} x^2\,dx$ **34.** $\displaystyle\int_{0}^{0.3} s^2\,ds$

35. $\displaystyle\int_{0}^{1/2} t^2\,dt$ **36.** $\displaystyle\int_{0}^{\pi/2} \theta^2\,d\theta$ **37.** $\displaystyle\int_{a}^{2a} x\,dx$

38. $\displaystyle\int_{a}^{\sqrt{3}} x\,dx$ **39.** $\displaystyle\int_{0}^{\sqrt[3]{b}} x^2\,dx$ **40.** $\displaystyle\int_{0}^{3b} x^2\,dx$

Use the rules in Table 5.6 and Equations (2)–(4) to evaluate the integrals in Exercises 41–50.

41. $\displaystyle\int_{3}^{1} 7\,dx$ **42.** $\displaystyle\int_{0}^{2} 5x\,dx$

43. $\displaystyle\int_{0}^{2} (2t - 3)\,dt$ **44.** $\displaystyle\int_{0}^{\sqrt{2}} \left(t - \sqrt{2}\right) dt$

45. $\displaystyle\int_{2}^{1} \left(1 + \frac{z}{2}\right) dz$ **46.** $\displaystyle\int_{3}^{0} (2z - 3)\,dz$

47. $\displaystyle\int_{1}^{2} 3u^2\,du$ **48.** $\displaystyle\int_{1/2}^{1} 24u^2\,du$

49. $\displaystyle\int_{0}^{2} (3x^2 + x - 5)\,dx$ **50.** $\displaystyle\int_{1}^{0} (3x^2 + x - 5)\,dx$

Finding Area by Definite Integrals

In Exercises 51–54, use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, b]$.

51. $y = 3x^2$ **52.** $y = \pi x^2$

53. $y = 2x$ **54.** $y = \dfrac{x}{2} + 1$

Finding Average Value

In Exercises 55–62, graph the function and find its average value over the given interval.

55. $f(x) = x^2 - 1$ on $\left[0, \sqrt{3}\right]$

56. $f(x) = -\dfrac{x^2}{2}$ on $[0, 3]$

57. $f(x) = -3x^2 - 1$ on $[0, 1]$

58. $f(x) = 3x^2 - 3$ on $[0, 1]$

59. $f(t) = (t - 1)^2$ on $[0, 3]$

60. $f(t) = t^2 - t$ on $[-2, 1]$

61. $g(x) = |x| - 1$ on **a.** $[-1, 1]$, **b.** $[1, 3]$, and **c.** $[-1, 3]$

62. $h(x) = -|x|$ on **a.** $[-1, 0]$, **b.** $[0, 1]$, and **c.** $[-1, 1]$

Definite Integrals as Limits of Sums

Use the method of Example 4a or Equation (1) to evaluate the definite integrals in Exercises 63–70.

63. $\displaystyle\int_{a}^{b} c\,dx$ **64.** $\displaystyle\int_{0}^{2} (2x + 1)\,dx$

65. $\displaystyle\int_{a}^{b} x^2\,dx, \quad a < b$ **66.** $\displaystyle\int_{-1}^{0} (x - x^2)\,dx$

67. $\displaystyle\int_{-1}^{2} (3x^2 - 2x + 1)\,dx$ **68.** $\displaystyle\int_{-1}^{1} x^3\,dx$

69. $\displaystyle\int_{a}^{b} x^3\,dx, \quad a < b$ **70.** $\displaystyle\int_{0}^{1} (3x - x^3)\,dx$

Theory and Examples

71. What values of a and b maximize the value of

$$\int_{a}^{b} (x - x^2)\,dx?$$

(*Hint:* Where is the integrand positive?)

72. What values of a and b minimize the value of

$$\int_a^b (x^4 - 2x^2)\,dx?$$

73. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1+x^2}\,dx.$$

74. (*Continuation of Exercise 73.*) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1+x^2}\,dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1+x^2}\,dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1+x^2}\,dx.$$

75. Show that the value of $\int_0^1 \sin(x^2)\,dx$ cannot possibly be 2.

76. Show that the value of $\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

77. Integrals of nonnegative functions Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \ge 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \ge 0.$$

78. Integrals of nonpositive functions Show that if f is integrable then

$$f(x) \le 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \le 0.$$

79. Use the inequality $\sin x \le x$, which holds for $x \ge 0$, to find an upper bound for the value of $\int_0^1 \sin x\,dx$.

80. The inequality $\sec x \ge 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\,dx$.

81. If $\operatorname{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the constant function $\operatorname{av}(f)$ should have the same integral over $[a, b]$ as f. Does it? That is, does

$$\int_a^b \operatorname{av}(f)\,dx = \int_a^b f(x)\,dx?$$

Give reasons for your answer.

82. It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$.

a. $\operatorname{av}(f+g) = \operatorname{av}(f) + \operatorname{av}(g)$

b. $\operatorname{av}(kf) = k\operatorname{av}(f)$ (any number k)

c. $\operatorname{av}(f) \le \operatorname{av}(g)$ if $f(x) \le g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

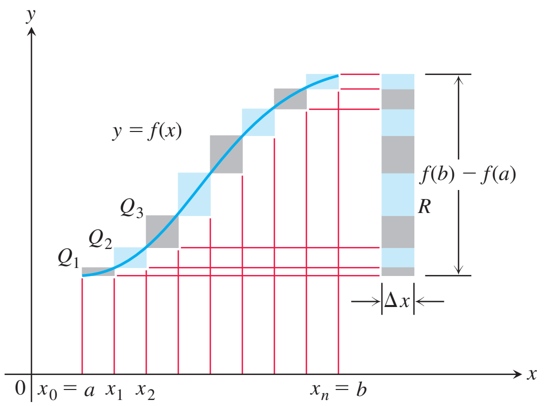
83. Upper and lower sums for increasing functions

a. Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of equal length $\Delta x = (b-a)/n$. Show by referring to the accompanying figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx. (*Hint:* The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \ldots, Q_{n-1}Q_n$ lie approximately along the curve. There is no overlapping when these rectangles are shifted horizontally onto R.)

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \le |f(b) - f(a)|\,\Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of P, and hence that $\lim_{\|P\|\to 0}(U - L) = 0$.



84. Upper and lower sums for decreasing functions (*Continuation of Exercise 83.*)

a. Draw a figure like the one in Exercise 83 for a continuous function $f(x)$ whose values decrease steadily as x moves from left to right across the interval $[a, b]$. Let P be a partition of $[a, b]$ into subintervals of equal length. Find an expression for $U - L$ that is analogous to the one you found for $U - L$ in Exercise 83a.

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of P vary in size. Show that the inequality

$$U - L \le |f(b) - f(a)|\,\Delta x_{\max}$$

of Exercise 83b still holds and hence that $\lim_{\|P\|\to 0}(U - L) = 0$.

85. Use the formula

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \frac{\cos(h/2) - \cos((m + (1/2))h)}{2\sin(h/2)}$$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$ in two steps:

a. Partition the interval $[0, \pi/2]$ into n subintervals of equal length and calculate the corresponding upper sum U; then

b. Find the limit of U as $n \to \infty$ and $\Delta x = (b-a)/n \to 0$.

86. Suppose that f is continuous and nonnegative over $[a, b]$, as in the accompanying figure. By inserting points

$$x_1, x_2, \dots, x_{k-1}, x_k, \dots, x_{n-1}$$

as shown, divide $[a, b]$ into n subintervals of lengths $\Delta x_1 = x_1 - a$, $\Delta x_2 = x_2 - x_1, \dots, \Delta x_n = b - x_{n-1}$, which need not be equal.

a. If $m_k = \min\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **lower sum**

$$L = m_1\,\Delta x_1 + m_2\,\Delta x_2 + \cdots + m_n\,\Delta x_n$$

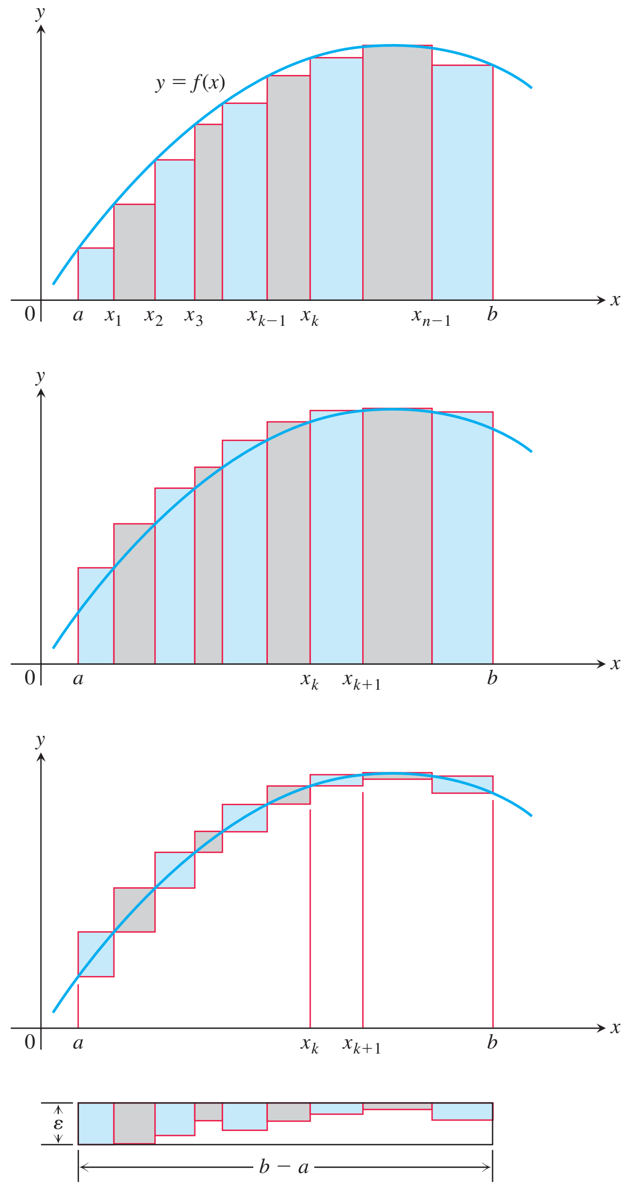
and the shaded regions in the first part of the figure.

b. If $M_k = \max\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **upper sum**

$$U = M_1\,\Delta x_1 + M_2\,\Delta x_2 + \cdots + M_n\,\Delta x_n$$

and the shaded regions in the second part of the figure.

c. Explain the connection between $U - L$ and the shaded regions along the curve in the third part of the figure.



87. We say f is **uniformly continuous** on $[a, b]$ if given any $\varepsilon > 0$, there is a $\delta > 0$ such that if x_1, x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \varepsilon$. It can be shown that a continuous function on $[a, b]$ is uniformly continuous. Use this and the figure for Exercise 86 to show that if f is continuous and $\varepsilon > 0$ is given, it is possible to make $U - L \le \varepsilon \cdot (b - a)$ by making the largest of the Δx_k's sufficiently small.

88. If you average 30 mi/h on a 150-mi trip and then return over the same 150 mi at the rate of 50 mi/h, what is your average speed for the trip? Give reasons for your answer.

COMPUTER EXPLORATIONS

If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals in Exercises 89–94. Use $n = 4$, 10, 20, and 50 subintervals of equal length in each case.

89. $\displaystyle\int_0^1 (1 - x)\,dx = \frac{1}{2}$

90. $\displaystyle\int_0^1 (x^2 + 1)\,dx = \frac{4}{3}$

91. $\displaystyle\int_{-\pi}^{\pi} \cos x\,dx = 0$

92. $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = 1$

93. $\displaystyle\int_{-1}^{1} |x|\,dx = 1$

94. $\displaystyle\int_1^2 \frac{1}{x}\,dx$ (The integral's value is about 0.693.)

In Exercises 95–98, use a CAS to perform the following steps:

a. Plot the functions over the given interval.

b. Partition the interval into $n = 100$, 200, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.

c. Compute the average value of the function values generated in part (b).

d. Solve the equation $f(x) = \text{(average value)}$ for x using the average value calculated in part (c) for the $n = 1000$ partitioning.

95. $f(x) = \sin x$ on $[0, \pi]$

96. $f(x) = \sin^2 x$ on $[0, \pi]$

97. $f(x) = x \sin \dfrac{1}{x}$ on $\left[\dfrac{\pi}{4}, \pi\right]$

98. $f(x) = x \sin^2 \dfrac{1}{x}$ on $\left[\dfrac{\pi}{4}, \pi\right]$

5.4 The Fundamental Theorem of Calculus

HISTORICAL BIOGRAPHY
Sir Isaac Newton (1642–1727)
www.goo.gl/qoKepF

In this section we present the Fundamental Theorem of Calculus, which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals by using an antiderivative of the integrand function rather than by taking limits of Riemann sums as we did in Section 5.3. Leibniz and Newton exploited this relationship and started mathematical developments that fueled the scientific revolution for the next 200 years.

Along the way, we will present an integral version of the Mean Value Theorem, which is another important theorem of integral calculus and is used to prove the Fundamental Theorem. We also find that the net change of a function over an interval is the integral of its rate of change, as suggested by Example 2 in Section 5.1.

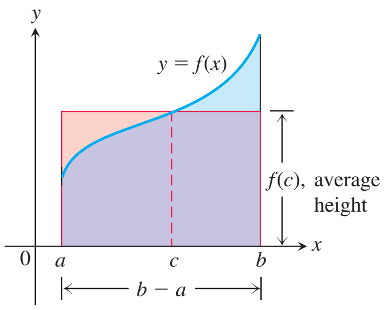


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b-a) = \int_a^b f(x)\,dx.$$

Mean Value Theorem for Definite Integrals

In the previous section we defined the average value of a continuous function over a closed interval $[a, b]$ to be the definite integral $\int_a^b f(x)\,dx$ divided by the length or width $b - a$ of the interval. The Mean Value Theorem for Definite Integrals asserts that this average value is *always* taken on at least once by the function f in the interval.

The graph in Figure 5.16 shows a *positive* continuous function $y = f(x)$ defined over the interval $[a, b]$. Geometrically, the Mean Value Theorem says that there is a number c in $[a, b]$ such that the rectangle with height equal to the average value $f(c)$ of the function and base width $b - a$ has exactly the same area as the region beneath the graph of f from a to b.

THEOREM 3—The Mean Value Theorem for Definite Integrals
If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof If we divide both sides of the Max-Min Inequality (Table 5.6, Rule 6) by $(b - a)$, we obtain

$$\min f \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \max f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions (Section 2.5) says that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $(1/(b-a))\int_a^b f(x)\,dx$ at some point c in $[a, b]$. ■

The continuity of f is important here. It is possible for a discontinuous function to never equal its average value (Figure 5.17).

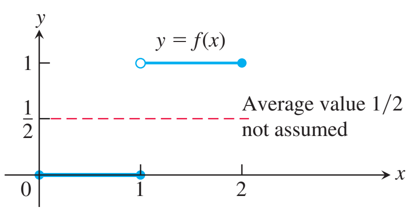


FIGURE 5.17 A discontinuous function need not assume its average value.

EXAMPLE 1 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution The average value of f on $[a, b]$ is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\cdot 0 = 0.$$

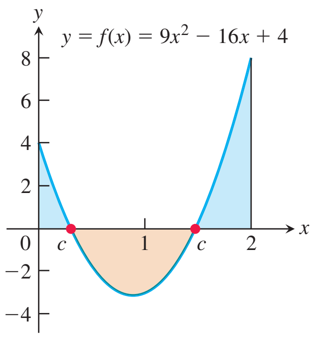


FIGURE 5.18 The function $f(x) = 9x^2 - 16x + 4$ satisfies $\int_0^2 f(x)\, dx = 0$, and there are two values of c in the interval $[0, 2]$ where $f(c) = 0$.

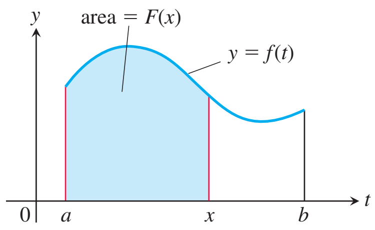


FIGURE 5.19 The function $F(x)$ defined by Equation (1) gives the area under the graph of f from a to x when f is nonnegative and $x > a$.

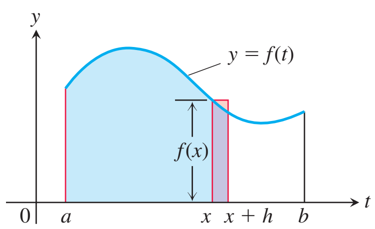


FIGURE 5.20 In Equation (1), $F(x)$ is the area to the left of x. Also, $F(x + h)$ is the area to the left of $x + h$. The difference quotient $[F(x + h) - F(x)]/h$ is then approximately equal to $f(x)$, the height of the rectangle shown here.

By the Mean Value Theorem, f assumes this value at some point $c \in [a, b]$. This is illustrated in Figure 5.18 for the function $f(x) = 9x^2 - 16x + 4$ on the interval $[0, 2]$. ■

Fundamental Theorem, Part 1

It can be very difficult to compute definite integrals by taking the limit of Riemann sums. We now develop a powerful new method for evaluating definite integrals, based on using antiderivatives. This method combines the two strands of calculus. One strand involves the idea of taking the limits of finite sums to obtain a definite integral, and the other strand contains derivatives and antiderivatives. They come together in the Fundamental Theorem of Calculus. We begin by considering how to differentiate a certain type of function that is described as an integral.

If $f(t)$ is an integrable function over a finite interval I, then the integral from any fixed number $a \in I$ to another number $x \in I$ defines a new function F whose value at x is

$$F(x) = \int_a^x f(t)\, dt. \tag{1}$$

For example, if f is nonnegative and x lies to the right of a, then $F(x)$ is the area under the graph from a to x (Figure 5.19). The variable x is the upper limit of integration of an integral, but F is just like any other real-valued function of a real variable. For each value of the input x, there is a single numerical output, in this case the definite integral of f from a to x.

Equation (1) gives a useful way to define new functions (as we will see in Section 7.1), but its key importance is the connection that it makes between integrals and derivatives. If f is a continuous function, then the Fundamental Theorem asserts that F is a differentiable function of x whose derivative is f itself. That is, at each x in the interval $[a, b]$ we have

$$F'(x) = f(x).$$

To gain some insight into why this holds, we look at the geometry behind it.

If $f \geq 0$ on $[a, b]$, then to compute $F'(x)$ from the definition of the derivative we must take the limit as $h \to 0$ of the difference quotient

$$\frac{F(x + h) - F(x)}{h}.$$

If $h > 0$, then $F(x + h)$ is the area under the graph of f from a to $x + h$, while $F(x)$ is the area under the graph of f from a to x. Subtracting the two gives us the area under the graph of f between x and $x + h$ (see Figure 5.20). As shown in Figure 5.20, if h is small, the area under the graph of f from x to $x + h$ is approximated by the area of the rectangle whose height is $f(x)$ and whose base is the interval $[x, x + h]$. That is,

$$F(x + h) - F(x) \approx hf(x).$$

Dividing both sides by h, we see that the value of the difference quotient is very close to the value of $f(x)$:

$$\frac{F(x + h) - F(x)}{h} \approx f(x).$$

This approximation improves as h approaches 0. It is reasonable to expect that $F'(x)$, which is the limit of this difference quotient as $h \to 0$, equals $f(x)$, so that

$$F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = f(x).$$

This equation is true even if the function f is not positive, and it forms the first part of the Fundamental Theorem of Calculus.

THEOREM 4—The Fundamental Theorem of Calculus, Part 1
If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

Before proving Theorem 4, we look at several examples to gain an understanding of what it says. In each of these examples, notice that the independent variable x appears in either the upper or the lower limit of integration (either as part of a formula or by itself). The independent variable on which y depends in these examples is x, while t is merely a dummy variable in the integral.

EXAMPLE 2 Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\,dt$

(b) $y = \int_x^5 3t \sin t\,dt$

(c) $y = \int_1^{x^2} \cos t\,dt$

(d) $y = \int_{1+3x^2}^4 \frac{1}{2 + t}\,dt$

Solution We calculate the derivatives with respect to the independent variable x.

(a) $\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\,dt = x^3 + 1$ Eq. (2) with $f(t) = t^3 + 1$

(b)
$$\begin{aligned}\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t \sin t\,dt &= \frac{d}{dx}\left(-\int_5^x 3t \sin t\,dt\right) && \text{Table 5.6, Rule 1}\\ &= -\frac{d}{dx}\int_5^x 3t \sin t\,dt\\ &= -3x \sin x && \text{Eq. (2) with } f(t) = 3t \sin t\end{aligned}$$

(c) The upper limit of integration is not x but x^2. This makes y a composition of the two functions

$$y = \int_1^u \cos t\,dt \quad \text{and} \quad u = x^2.$$

We must therefore apply the Chain Rule to find dy/dx:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= \left(\frac{d}{du}\int_1^u \cos t\,dt\right)\cdot\frac{du}{dx}\\ &= \cos u \cdot \frac{du}{dx} && \text{Eq. (2) with } f(t) = \cos t\\ &= \cos(x^2)\cdot 2x\\ &= 2x \cos x^2\end{aligned}$$

(d) $$\frac{d}{dx}\int_{1+3x^2}^{4}\frac{1}{2+t}\,dt = \frac{d}{dx}\left(-\int_{4}^{1+3x^2}\frac{1}{2+t}\,dt\right) \qquad \text{Rule 1}$$

$$= -\frac{d}{dx}\int_{4}^{1+3x^2}\frac{1}{2+t}\,dt$$

$$= -\frac{1}{2+(1+3x^2)}\cdot\frac{d}{dx}(1+3x^2) \qquad \text{Eq. (2) and the Chain Rule}$$

$$= -\frac{2x}{1+x^2}$$

Proof of Theorem 4 We prove the Fundamental Theorem, Part 1, by applying the definition of the derivative directly to the function $F(x)$, when x and $x + h$ are in (a, b). This means writing out the difference quotient

$$\frac{F(x+h) - F(x)}{h} \tag{3}$$

and showing that its limit as $h \to 0$ is the number $f(x)$. Doing so, we find that

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt. \qquad \text{Table 5.6, Rule 5}$$

According to the Mean Value Theorem for Definite Integrals, there is some point c between x and $x + h$ where $f(c)$ equals the average value of f on the interval $[x, x + h]$. That is, there is some number c in $[x, x + h]$ such that

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c). \tag{4}$$

As $h \to 0$, $x + h$ approaches x, which forces c to approach x also (because c is trapped between x and $x + h$). Since f is continuous at x, $f(c)$ therefore approaches $f(x)$:

$$\lim_{h\to 0} f(c) = f(x). \tag{5}$$

Hence we have shown that, for any x in (a, b),

$$F'(x) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt$$

$$= \lim_{h\to 0} f(c) \qquad \text{Eq. (4)}$$

$$= f(x), \qquad \text{Eq. (5)}$$

and therefore F is differentiable at x. Since differentiability implies continuity, this also shows that F is continuous on the open interval (a, b). To complete the proof, we just have to show that F is also continuous at $x = a$ and $x = b$. To do this, we make a very similar argument, except that at $x = a$ we need only consider the one-sided limit as $h \to 0^+$, and similarly at $x = b$ we need only consider $h \to 0^-$. This shows that F has a one-sided derivative at $x = a$ and at $x = b$, and therefore Theorem 1 in Section 3.2 implies that F is continuous at those two points. ■

Fundamental Theorem, Part 2 (The Evaluation Theorem)

We now come to the second part of the Fundamental Theorem of Calculus. This part describes how to evaluate definite integrals without having to calculate limits of Riemann sums. Instead we find and evaluate an antiderivative at the upper and lower limits of integration.

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2
If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof Part 1 of the Fundamental Theorem tells us that an antiderivative of f exists, namely

$$G(x) = \int_a^x f(t)\,dt.$$

Thus, if F is *any* antiderivative of f, then $F(x) = G(x) + C$ for some constant C for $a < x < b$ (by Corollary 2 of the Mean Value Theorem for Derivatives, Section 4.2). Since both F and G are continuous on $[a, b]$, we see that the equality $F(x) = G(x) + C$ also holds when $x = a$ and $x = b$ by taking one-sided limits (as $x \to a^+$ and $x \to b^-$).

Evaluating $F(b) - F(a)$, we have

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 \\ &= \int_a^b f(t)\,dt. \end{aligned}$$ ∎

The Evaluation Theorem is important because it says that to calculate the definite integral of f over an interval $[a, b]$ we need do only two things:

1. Find an antiderivative F of f, and
2. Calculate the number $F(b) - F(a)$, which is equal to $\int_a^b f(x)\,dx$.

This process is much easier than using a Riemann sum computation. The power of the theorem follows from the realization that the definite integral, which is defined by a complicated process involving all of the values of the function f over $[a, b]$, can be found by knowing the values of *any* antiderivative F at only the two endpoints a and b. The usual notation for the difference $F(b) - F(a)$ is

$$F(x)\Big]_a^b \quad \text{or} \quad \Big[F(x)\Big]_a^b,$$

depending on whether F has one or more terms.

EXAMPLE 3 We calculate several definite integrals using the Evaluation Theorem, rather than by taking limits of Riemann sums.

(a) $\displaystyle\int_0^\pi \cos x\,dx = \sin x\Big]_0^\pi$ $\qquad \dfrac{d}{dx}\sin x = \cos x$

$$= \sin\pi - \sin 0 = 0 - 0 = 0$$

(b) $$\int_{-\pi/4}^{0} \sec x \tan x \, dx = \sec x \Big]_{-\pi/4}^{0}$$ $\quad \frac{d}{dx}\sec x = \sec x \tan x$

$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

(c) $$\int_{1}^{4} \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_{1}^{4}$$ $\quad \frac{d}{dx}\left(x^{3/2} + \frac{4}{x}\right) = \frac{3}{2}x^{1/2} - \frac{4}{x^2}$

$$= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right]$$

$$= [8 + 1] - [5] = 4$$

Exercise 72 offers another proof of the Evaluation Theorem, bringing together the ideas of Riemann sums, the Mean Value Theorem, and the definition of the definite integral.

The Integral of a Rate

We can interpret Part 2 of the Fundamental Theorem in another way. If F is any antiderivative of f, then $F' = f$. The equation in the theorem can then be rewritten as

$$\int_a^b F'(x)\, dx = F(b) - F(a).$$

Now $F'(x)$ represents the rate of change of the function $F(x)$ with respect to x, so the last equation asserts that the integral of F' is just the *net change* in F as x changes from a to b. Formally, we have the following result.

THEOREM 5—The Net Change Theorem
The net change in a differentiable function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\, dx. \tag{6}$$

EXAMPLE 4 Here are several interpretations of the Net Change Theorem.

(a) If $c(x)$ is the cost of producing x units of a certain commodity, then $c'(x)$ is the marginal cost (Section 3.4). From Theorem 5,

$$\int_{x_1}^{x_2} c'(x)\, dx = c(x_2) - c(x_1),$$

which is the cost of increasing production from x_1 units to x_2 units.

(b) If an object with position function $s(t)$ moves along a coordinate line, its velocity is $v(t) = s'(t)$. Theorem 5 says that

$$\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1),$$

so the integral of velocity is the **displacement** over the time interval $t_1 \le t \le t_2$. On the other hand, the integral of the speed $|v(t)|$ is the **total distance traveled** over the time interval. This is consistent with our discussion in Section 5.1.

If we rearrange Equation (6) as

$$F(b) = F(a) + \int_a^b F'(x)\,dx,$$

we see that the Net Change Theorem also says that the final value of a function $F(x)$ over an interval $[a, b]$ equals its initial value $F(a)$ plus its net change over the interval. So if $\upsilon(t)$ represents the velocity function of an object moving along a coordinate line, this means that the object's final position $s(t_2)$ over a time interval $t_1 \le t \le t_2$ is its initial position $s(t_1)$ plus its net change in position along the line (see Example 4b).

EXAMPLE 5 Consider again our analysis of a heavy rock blown straight up from the ground by a dynamite blast (Example 2, Section 5.1). The velocity of the rock at any time t during its motion was given as $\upsilon(t) = 160 - 32t$ ft/sec.

(a) Find the displacement of the rock during the time period $0 \le t \le 8$.

(b) Find the total distance traveled during this time period.

Solution

(a) From Example 4b, the displacement is the integral

$$\begin{aligned}\int_0^8 \upsilon(t)\,dt &= \int_0^8 (160 - 32t)\,dt = \left[160t - 16t^2\right]_0^8 \\ &= (160)(8) - (16)(64) = 256.\end{aligned}$$

This means that the height of the rock is 256 ft above the ground 8 sec after the explosion, which agrees with our conclusion in Example 2, Section 5.1.

(b) As we noted in Table 5.3, the velocity function $\upsilon(t)$ is positive over the time interval $[0, 5]$ and negative over the interval $[5, 8]$. Therefore, from Example 4b, the total distance traveled is the integral

$$\begin{aligned}\int_0^8 |\upsilon(t)|\,dt &= \int_0^5 |\upsilon(t)|\,dt + \int_5^8 |\upsilon(t)|\,dt \\ &= \int_0^5 (160 - 32t)\,dt - \int_5^8 (160 - 32t)\,dt \qquad |\upsilon(t)| = -(160 - 32t) \text{ over } [5, 8] \\ &= \left[160t - 16t^2\right]_0^5 - \left[160t - 16t^2\right]_5^8 \\ &= \left[(160)(5) - (16)(25)\right] - \left[(160)(8) - (16)(64) - ((160)(5) - (16)(25))\right] \\ &= 400 - (-144) = 544.\end{aligned}$$

Again, this calculation agrees with our conclusion in Example 2, Section 5.1. That is, the total distance of 544 ft traveled by the rock during the time period $0 \le t \le 8$ is (i) the maximum height of 400 ft it reached over the time interval $[0, 5]$ plus (ii) the additional distance of 144 ft the rock fell over the time interval $[5, 8]$. ■

The Relationship Between Integration and Differentiation

The conclusions of the Fundamental Theorem tell us several things. Equation (2) can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

which says that if you first integrate the function f and then differentiate the result, you get the function f back again. Likewise, replacing b by x and x by t in Equation (6) gives

$$\int_a^x F'(t)\,dt = F(x) - F(a),$$

so that if you first differentiate the function F and then integrate the result, you get the function F back (adjusted by an integration constant). In a sense, the processes of integration and differentiation are "inverses" of each other. The Fundamental Theorem also says that every continuous function f has an antiderivative F. It shows the importance of finding antiderivatives in order to evaluate definite integrals easily. Furthermore, it says that the differential equation $dy/dx = f(x)$ has a solution (namely, any of the functions $y = F(x) + C$) when f is a continuous function.

Total Area

Area is always a nonnegative quantity. The Riemann sum approximations contain terms such as $f(c_k)\,\Delta x_k$ that give the area of a rectangle when $f(c_k)$ is positive. When $f(c_k)$ is negative, then the product $f(c_k)\,\Delta x_k$ is the negative of the rectangle's area. When we add up such terms for a negative function, we get the negative of the area between the curve and the x-axis. If we then take the absolute value, we obtain the correct positive area.

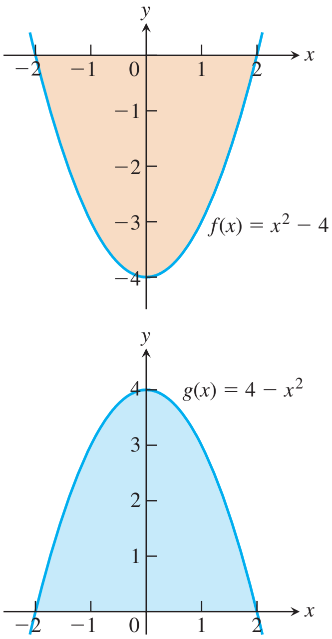


FIGURE 5.21 These graphs enclose the same amount of area with the x-axis, but the definite integrals of the two functions over $[-2, 2]$ differ in sign (Example 6).

EXAMPLE 6 Figure 5.21 shows the graph of $f(x) = x^2 - 4$ and its mirror image $g(x) = 4 - x^2$ reflected across the x-axis. For each function, compute

(a) the definite integral over the interval $[-2, 2]$, and

(b) the area between the graph and the x-axis over $[-2, 2]$.

Solution

(a) $$\int_{-2}^{2} f(x)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) = -\frac{32}{3},$$

and

$$\int_{-2}^{2} g(x)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}.$$

(b) In both cases, the area between the curve and the x-axis over $[-2, 2]$ is $32/3$ square units. Although the definite integral of $f(x)$ is negative, the area is still positive. ■

To compute the area of the region bounded by the graph of a function $y = f(x)$ and the x-axis when the function takes on both positive and negative values, we must be careful to break up the interval $[a, b]$ into subintervals on which the function doesn't change sign. Otherwise we might get cancelation between positive and negative signed areas, leading to an incorrect total. The correct total area is obtained by adding the absolute value of the definite integral over each subinterval where $f(x)$ does not change sign. The term "area" will be taken to mean this *total area*.

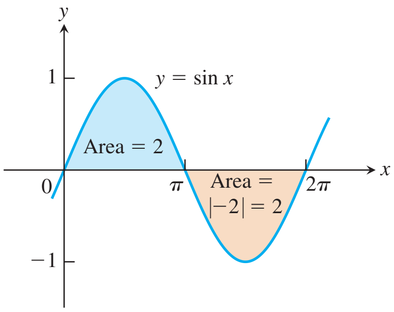


FIGURE 5.22 The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals (Example 7).

EXAMPLE 7 Figure 5.22 shows the graph of the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$,

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.

Solution

(a) The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x\, dx = -\cos x\Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

The definite integral is zero because the portions of the graph above and below the x-axis make canceling contributions.

(b) The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_0^{\pi} \sin x\, dx = -\cos x\Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$

$$\int_{\pi}^{2\pi} \sin x\, dx = -\cos x\Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values,

$$\text{Area} = |2| + |-2| = 4.$$

Summary:
To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

EXAMPLE 8 Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

Solution First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2),$$

the zeros are $x = 0, -1$, and 2 (Figure 5.23). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \ge 0$, and $[0, 2]$, on which $f \le 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals.

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$

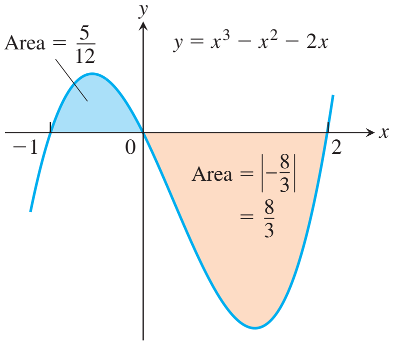


FIGURE 5.23 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

EXERCISES 5.4

Evaluating Integrals

Evaluate the integrals in Exercises 1–28.

1. $\displaystyle\int_0^2 x(x-3)\,dx$

2. $\displaystyle\int_{-1}^1 (x^2-2x+3)\,dx$

3. $\displaystyle\int_{-2}^2 \frac{3}{(x+3)^4}\,dx$

4. $\displaystyle\int_{-1}^1 x^{299}\,dx$

5. $\displaystyle\int_1^4 \left(3x^2-\frac{x^3}{4}\right)dx$

6. $\displaystyle\int_{-2}^3 (x^3-2x+3)\,dx$

7. $\displaystyle\int_0^1 \left(x^2+\sqrt{x}\right)dx$

8. $\displaystyle\int_1^{32} x^{-6/5}\,dx$

9. $\displaystyle\int_0^{\pi/3} 2\sec^2 x\,dx$

10. $\displaystyle\int_0^{\pi} (1+\cos x)\,dx$

11. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\,d\theta$

12. $\displaystyle\int_0^{\pi/3} 4\frac{\sin u}{\cos^2 u}\,du$

13. $\displaystyle\int_{\pi/2}^0 \frac{1+\cos 2t}{2}\,dt$

14. $\displaystyle\int_{-\pi/3}^{\pi/3} \sin^2 t\,dt$

15. $\displaystyle\int_0^{\pi/4} \tan^2 x\,dx$

16. $\displaystyle\int_0^{\pi/6} (\sec x+\tan x)^2\,dx$

17. $\displaystyle\int_0^{\pi/8} \sin 2x\,dx$

18. $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t+\frac{\pi}{t^2}\right)dt$

19. $\displaystyle\int_1^{-1} (r+1)^2\,dr$

20. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\,dt$

21. $\displaystyle\int_{\sqrt{2}}^1 \left(\frac{u^7}{2}-\frac{1}{u^5}\right)du$

22. $\displaystyle\int_{-3}^{-1} \frac{y^5-2y}{y^3}\,dy$

23. $\displaystyle\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^2}\,ds$

24. $\displaystyle\int_1^8 \frac{(x^{1/3}+1)(2-x^{2/3})}{x^{1/3}}\,dx$

25. $\displaystyle\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx$

26. $\displaystyle\int_0^{\pi/3} (\cos x+\sec x)^2\,dx$

27. $\displaystyle\int_{-4}^4 |x|\,dx$

28. $\displaystyle\int_0^{\pi} \frac{1}{2}(\cos x+|\cos x|)\,dx$

In Exercises 29–32, guess an antiderivative for the integrand function. Validate your guess by differentiation and then evaluate the given definite integral. (*Hint:* Keep the Chain Rule in mind when trying to guess an antiderivative. You will learn how to find such antiderivatives in the next section.)

29. $\displaystyle\int_0^{\sqrt{\pi/2}} x\cos x^2\,dx$

30. $\displaystyle\int_1^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

31. $\displaystyle\int_2^5 \frac{x\,dx}{\sqrt{1+x^2}}$

32. $\displaystyle\int_0^{\pi/3} \sin^2 x\cos x\,dx$

Derivatives of Integrals

Find the derivatives in Exercises 33–38.

a. by evaluating the integral and differentiating the result.

b. by differentiating the integral directly.

33. $\displaystyle\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\,dt$

34. $\displaystyle\frac{d}{dx}\int_1^{\sin x} 3t^2\,dt$

35. $\displaystyle\frac{d}{dt}\int_0^{t^4} \sqrt{u}\,du$

36. $\displaystyle\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\,dy$

37. $\displaystyle\frac{d}{dx}\int_0^{x^3} t^{-2/3}\,dt$

38. $\displaystyle\frac{d}{dt}\int_0^{\sqrt{t}} \left(x^4+\frac{3}{\sqrt{1-x^2}}\right)dx$

Find dy/dx in Exercises 39–46.

39. $\displaystyle y=\int_0^x \sqrt{1+t^2}\,dt$

40. $\displaystyle y=\int_1^x \frac{1}{t}\,dt,\quad x>0$

41. $\displaystyle y=\int_{\sqrt{x}}^0 \sin(t^2)\,dt$

42. $\displaystyle y=x\int_2^{x^2} \sin(t^3)\,dt$

43. $\displaystyle y=\int_{-1}^x \frac{t^2}{t^2+4}\,dt-\int_3^x \frac{t^2}{t^2+4}\,dt$

44. $\displaystyle y=\left(\int_0^x (t^3+1)^{10}\,dt\right)^3$

45. $\displaystyle y=\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}},\quad |x|<\frac{\pi}{2}$

46. $\displaystyle y=\int_{\tan x}^0 \frac{dt}{1+t^2}$

Area

In Exercises 47–50, find the total area between the region and the x-axis.

47. $y=-x^2-2x,\quad -3\le x\le 2$

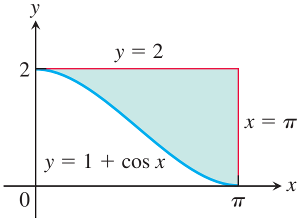
48. $y=3x^2-3,\quad -2\le x\le 2$

49. $y=x^3-3x^2+2x,\quad 0\le x\le 2$

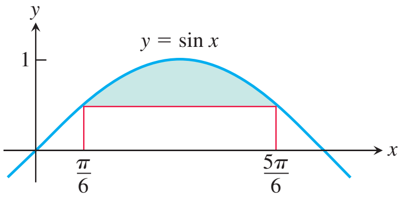
50. $y=x^{1/3}-x,\quad -1\le x\le 8$

Find the areas of the shaded regions in Exercises 51–54.

51.



52.



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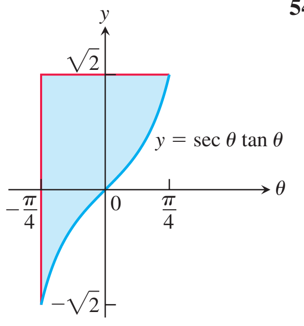
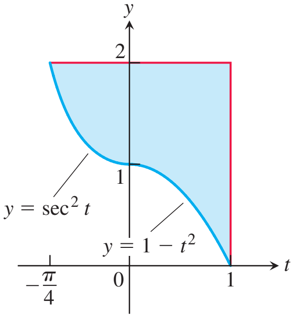


54.



Initial Value Problems

Each of the following functions solves one of the initial value problems in Exercises 55–58. Which function solves which problem? Give brief reasons for your answers.

a. $y = \int_1^x \frac{1}{t}\,dt - 3$ **b.** $y = \int_0^x \sec t\,dt + 4$

c. $y = \int_{-1}^x \sec t\,dt + 4$ **d.** $y = \int_\pi^x \frac{1}{t}\,dt - 3$

55. $\frac{dy}{dx} = \frac{1}{x}, \quad y(\pi) = -3$ **56.** $y' = \sec x, \quad y(-1) = 4$

57. $y' = \sec x, \quad y(0) = 4$ **58.** $y' = \frac{1}{x}, \quad y(1) = -3$

Express the solutions of the initial value problems in Exercises 59 and 60 in terms of integrals.

59. $\frac{dy}{dx} = \sec x, \quad y(2) = 3$ **60.** $\frac{dy}{dx} = \sqrt{1 + x^2}, \quad y(1) = -2$

Theory and Examples

61. Archimedes' area formula for parabolic arches Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - (4h/b^2)x^2$, $-b/2 \le x \le b/2$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis.

62. Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is $2/k$.

63. Cost from marginal cost The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find $c(100) - c(1)$, the cost of printing posters 2–100.

64. Revenue from marginal revenue Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - 2/(x + 1)^2,$$

where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

65. The temperature T(°F) of a room at time t minutes is given by

$$T = 85 - 3\sqrt{25 - t} \quad \text{for} \quad 0 \le t \le 25.$$

a. Find the room's temperature when $t = 0$, $t = 16$, and $t = 25$.

b. Find the room's average temperature for $0 \le t \le 25$.

66. The height H(ft) of a palm tree after growing for t years is given by

$$H = \sqrt{t + 1} + 5t^{1/3} \quad \text{for} \quad 0 \le t \le 8.$$

a. Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b. Find the tree's average height for $0 \le t \le 8$.

67. Suppose that $\int_1^x f(t)\,dt = x^2 - 2x + 1$. Find $f(x)$.

68. Find $f(4)$ if $\int_0^x f(t)\,dt = x \cos \pi x$.

69. Find the linearization of

$$f(x) = 2 - \int_2^{x+1} \frac{9}{1 + t}\,dt$$

at $x = 1$.

70. Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t - 1)\,dt$$

at $x = -1$.

71. Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

a. g is a differentiable function of x.

b. g is a continuous function of x.

c. The graph of g has a horizontal tangent at $x = 1$.

d. g has a local maximum at $x = 1$.

e. g has a local minimum at $x = 1$.

f. The graph of g has an inflection point at $x = 1$.

g. The graph of dg/dx crosses the x-axis at $x = 1$.

72. Another proof of the Evaluation Theorem

a. Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$ be any partition of $[a, b]$, and let F be any antiderivative of f. Show that

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})].$$

b. Apply the Mean Value Theorem to each term to show that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$ for some c_i in the interval (x_{i-1}, x_i). Then show that $F(b) - F(a)$ is a Riemann sum for f on $[a, b]$.

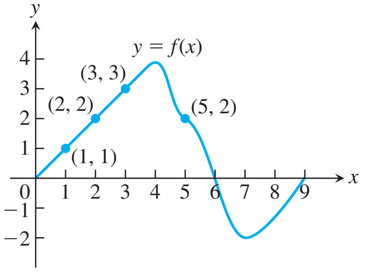
c. From part (b) and the definition of the definite integral, show that

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

73. Suppose that f is the differentiable function shown in the accompanying graph and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



a. What is the particle's velocity at time $t = 5$?

b. Is the acceleration of the particle at time $t = 5$ positive, or negative?

c. What is the particle's position at time $t = 3$?

d. At what time during the first 9 sec does s have its largest value?

e. Approximately when is the acceleration zero?

f. When is the particle moving toward the origin? Away from the origin?

g. On which side of the origin does the particle lie at time $t = 9$?

74. Find $\lim_{x\to\infty} \dfrac{1}{\sqrt{x}} \displaystyle\int_1^x \dfrac{dt}{\sqrt{t}}$.

COMPUTER EXPLORATIONS

In Exercises 75–78, let $F(x) = \int_a^x f(t)\,dt$ for the specified function f and interval $[a, b]$. Use a CAS to perform the following steps and answer the questions posed.

a. Plot the functions f and F together over $[a, b]$.

b. Solve the equation $F'(x) = 0$. What can you see to be true about the graphs of f and F at points where $F'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem coupled with information provided by the first derivative? Explain your answer.

c. Over what intervals (approximately) is the function F increasing and decreasing? What is true about f over those intervals?

d. Calculate the derivative f' and plot it together with F. What can you see to be true about the graph of F at points where $f'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem? Explain your answer.

75. $f(x) = x^3 - 4x^2 + 3x, \quad [0, 4]$

76. $f(x) = 2x^4 - 17x^3 + 46x^2 - 43x + 12, \quad \left[0, \frac{9}{2}\right]$

77. $f(x) = \sin 2x \cos \frac{x}{3}, \quad [0, 2\pi]$

78. $f(x) = x \cos \pi x, \quad [0, 2\pi]$

In Exercises 79–82, let $F(x) = \int_a^{u(x)} f(t)\,dt$ for the specified a, u, and f. Use a CAS to perform the following steps and answer the questions posed.

a. Find the domain of F.

b. Calculate $F'(x)$ and determine its zeros. For what points in its domain is F increasing? Decreasing?

c. Calculate $F''(x)$ and determine its zero. Identify the local extrema and the points of inflection of F.

d. Using the information from parts (a)–(c), draw a rough hand-sketch of $y = F(x)$ over its domain. Then graph $F(x)$ on your CAS to support your sketch.

79. $a = 1, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

80. $a = 0, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

81. $a = 0, \quad u(x) = 1 - x, \quad f(x) = x^2 - 2x - 3$

82. $a = 0, \quad u(x) = 1 - x^2, \quad f(x) = x^2 - 2x - 3$

In Exercises 83 and 84, assume that f is continuous and $u(x)$ is twice-differentiable.

83. Calculate $\dfrac{d}{dx}\displaystyle\int_a^{u(x)} f(t)\,dt$ and check your answer using a CAS.

84. Calculate $\dfrac{d^2}{dx^2}\displaystyle\int_a^{u(x)} f(t)\,dt$ and check your answer using a CAS.

5.5 Indefinite Integrals and the Substitution Method

The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed directly if we can find an antiderivative of the function. In Section 4.8 we defined the **indefinite integral** of the function f with respect to x as the set of *all* antiderivatives of f, symbolized by $\int f(x)\,dx$. Since any two antiderivatives of f differ by a constant, the indefinite integral $\int$ notation means that for any antiderivative F of f,

$$\int f(x)\,dx = F(x) + C,$$

where C is any arbitrary constant. The connection between antiderivatives and the definite integral stated in the Fundamental Theorem now explains this notation:

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(b) + C] - [F(a) + C]$$
$$= [F(x) + C]_a^b = \left[\int f(x)\,dx\right]_a^b.$$

When finding the indefinite integral of a function f, remember that it always includes an arbitrary constant C.

We must distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*. An indefinite integral $\int f(x)\,dx$ is a *function* plus an arbitrary constant C.

So far, we have only been able to find antiderivatives of functions that are clearly recognizable as derivatives. In this section we begin to develop more general techniques for finding antiderivatives of functions we can't easily recognize as derivatives.

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n\frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C. \tag{1}$$

The integral in Equation (1) is equal to the simpler integral

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C,$$

which suggests that the simpler expression du can be substituted for $(du/dx)\,dx$ when computing an integral. Leibniz, one of the founders of calculus, had the insight that indeed this substitution could be done, leading to the *substitution method* for computing integrals. As with differentials, when computing integrals we have

$$du = \frac{du}{dx}\,dx.$$

EXAMPLE 1 Find the integral $\int (x^3 + x)^5(3x^2 + 1)\,dx$.

Solution We set $u = x^3 + x$. Then

$$du = \frac{du}{dx}\,dx = (3x^2 + 1)\,dx,$$

so that by substitution we have

$$\begin{aligned} \int (x^3 + x)^5(3x^2 + 1)\,dx &= \int u^5\,du && \text{Let } u = x^3 + x,\ du = (3x^2+1)\,dx. \\ &= \frac{u^6}{6} + C && \text{Integrate with respect to } u. \\ &= \frac{(x^3 + x)^6}{6} + C && \text{Substitute } x^3 + x \text{ for } u. \end{aligned}$$

■

EXAMPLE 2 Find $\int \sqrt{2x+1}\,dx$.

Solution The integral does not fit the formula

$$\int u^n\,du,$$

with $u = 2x + 1$ and $n = 1/2$, because

$$du = \frac{du}{dx}\,dx = 2\,dx,$$

which is not precisely dx. The constant factor 2 is missing from the integral. However, we can introduce this factor after the integral sign if we compensate for it by introducing a factor of $1/2$ in front of the integral sign. So we write

$$\begin{aligned}
\int \sqrt{2x+1}\,dx &= \frac{1}{2}\int \sqrt{\underbrace{2x+1}_{u}}\cdot\underbrace{2\,dx}_{du} \\
&= \frac{1}{2}\int u^{1/2}\,du && \text{Let } u = 2x+1,\ du = 2\,dx. \\
&= \frac{1}{2}\frac{u^{3/2}}{3/2} + C && \text{Integrate with respect to } u. \\
&= \frac{1}{3}(2x+1)^{3/2} + C && \text{Substitute } 2x+1 \text{ for } u.
\end{aligned}$$

The substitutions in Examples 1 and 2 are instances of the following general rule.

THEOREM 6—The Substitution Rule
If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))\cdot g'(x)\,dx = \int f(u)\,du.$$

Proof By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x))\cdot g'(x)$ whenever F is an antiderivative of f, because

$$\begin{aligned}
\frac{d}{dx}F(g(x)) &= F'(g(x))\cdot g'(x) && \text{Chain Rule} \\
&= f(g(x))\cdot g'(x). && F' = f
\end{aligned}$$

If we make the substitution $u = g(x)$, then

$$\begin{aligned}
\int f(g(x))\,g'(x)\,dx &= \int \frac{d}{dx}F(g(x))\,dx \\
&= F(g(x)) + C && \text{Theorem 8 in Chapter 4} \\
&= F(u) + C && u = g(x) \\
&= \int F'(u)\,du && \text{Theorem 8 in Chapter 4} \\
&= \int f(u)\,du. && F' = f
\end{aligned}$$

■

The use of the variable u in the Substitution Rule is traditional (sometimes it is referred to as u-substitution), but any letter can be used, such as v, t, θ and so forth. The rule provides a method for evaluating an integral of the form $\int f(g(x))g'(x)\,dx$ given that the conditions of Theorem 6 are satisfied. The primary challenge is deciding what expression involving x to substitute for in the integrand. The following examples give helpful ideas.

The Substitution Method to evaluate $\int f(g(x))g'(x)\,dx$

1. Substitute $u = g(x)$ and $du = (du/dx)\,dx = g'(x)\,dx$ to obtain $\int f(u)\,du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

EXAMPLE 3 Find $\displaystyle\int \sec^2(5x + 1)\cdot 5\,dx$

Solution We substitute $u = 5x + 1$ and $du = 5\,dx$. Then,

$$\begin{aligned}\int \sec^2(5x + 1)\cdot 5\,dx &= \int \sec^2 u\,du && \text{Let } u = 5x + 1,\ du = 5\,dx.\\ &= \tan u + C && \frac{d}{du}\tan u = \sec^2 u\\ &= \tan(5x + 1) + C. && \text{Substitute } 5x + 1 \text{ for } u.\end{aligned}$$

■

EXAMPLE 4 Find $\displaystyle\int \cos(7\theta + 3)\,d\theta.$

Solution We let $u = 7\theta + 3$ so that $du = 7\,d\theta$. The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7, using the same procedure as in Example 2. Then,

$$\begin{aligned}\int \cos(7\theta + 3)\,d\theta &= \frac{1}{7}\int \cos(7\theta + 3)\cdot 7\,d\theta && \text{Place factor } 1/7 \text{ in front of integral.}\\ &= \frac{1}{7}\int \cos u\,du && \text{Let } u = 7\theta + 3,\ du = 7\,d\theta.\\ &= \frac{1}{7}\sin u + C && \text{Integrate.}\\ &= \frac{1}{7}\sin(7\theta + 3) + C. && \text{Substitute } 7\theta + 3 \text{ for } u.\end{aligned}$$

There is another approach to this problem. With $u = 7\theta + 3$ and $du = 7\,d\theta$ as before, we solve for $d\theta$ to obtain $d\theta = (1/7)\,du$. Then the integral becomes

$$\begin{aligned}\int \cos(7\theta + 3)\,d\theta &= \int \cos u\cdot\frac{1}{7}\,du && \text{Let } u = 7\theta + 3,\ du = 7\,d\theta, \text{ and } d\theta = (1/7)\,du.\\ &= \frac{1}{7}\sin u + C && \text{Integrate.}\\ &= \frac{1}{7}\sin(7\theta + 3) + C. && \text{Substitute } 7\theta + 3 \text{ for } u.\end{aligned}$$

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 3)$. ■

EXAMPLE 5 Sometimes we observe that a power of x appears in the integrand that is one less than the power of x appearing in the argument of a function we want to integrate.

This observation immediately suggests we try a substitution for the higher power of x. This situation occurs in the following integration.

$$\begin{aligned}\int x^2 \cos x^3\, dx &= \int \cos x^3 \cdot x^2\, dx \\ &= \int \cos u \cdot \frac{1}{3}\, du && \text{Let } u = x^3,\ du = 3x^2\,dx,\ (1/3)\,du = x^2\,dx. \\ &= \frac{1}{3}\int \cos u\, du \\ &= \frac{1}{3}\sin u + C && \text{Integrate with respect to } u. \\ &= \frac{1}{3}\sin x^3 + C && \text{Replace } u \text{ by } x^3. \quad \blacksquare\end{aligned}$$

HISTORICAL BIOGRAPHY
George David Birkhoff
(1884–1944)
www.goo.gl/OYjM2t

It may happen that an extra factor of x appears in the integrand when we try a substitution $u = g(x)$. In that case, it may be possible to solve the equation $u = g(x)$ for x in terms of u. Replacing the extra factor of x with that expression may then result in an integral that we can evaluate. Here is an example of this situation.

EXAMPLE 6 Evaluate $\int x\sqrt{2x + 1}\, dx$.

Solution Our previous experience with the integral in Example 2 suggests the substitution $u = 2x + 1$ with $du = 2\,dx$. Then

$$\sqrt{2x + 1}\, dx = \frac{1}{2}\sqrt{u}\, du.$$

However, in this example the integrand contains an extra factor of x that multiplies the term $\sqrt{2x + 1}$. To adjust for this, we solve the substitution equation $u = 2x + 1$ for x to obtain $x = (u - 1)/2$, and find that

$$x\sqrt{2x + 1}\, dx = \frac{1}{2}(u - 1) \cdot \frac{1}{2}\sqrt{u}\, du.$$

The integration now becomes

$$\begin{aligned}\int x\sqrt{2x + 1}\, dx &= \frac{1}{4}\int (u - 1)\sqrt{u}\, du = \frac{1}{4}\int (u - 1)u^{1/2}\, du && \text{Substitute.} \\ &= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right) du && \text{Multiply terms.} \\ &= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C && \text{Integrate.} \\ &= \frac{1}{10}(2x + 1)^{5/2} - \frac{1}{6}(2x + 1)^{3/2} + C. && \text{Replace } u \text{ by } 2x + 1. \quad \blacksquare\end{aligned}$$

EXAMPLE 7 Sometimes we can use trigonometric identities to transform integrals we do not know how to evaluate into ones we can evaluate using the Substitution Rule.

(a)
$$\begin{aligned}\int \sin^2 x\, dx &= \int \frac{1 - \cos 2x}{2}\, dx && \sin^2 x = \frac{1 - \cos 2x}{2} \\ &= \frac{1}{2}\int (1 - \cos 2x)\, dx \\ &= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C\end{aligned}$$

(b) $\displaystyle\int \cos^2 x\, dx = \int \frac{1 + \cos 2x}{2} dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$ $\qquad \cos^2 x = \dfrac{1 + \cos 2x}{2}$

(c) $$\int (1 - 2\sin^2 x)\sin 2x\, dx = \int (\cos^2 x - \sin^2 x)\sin 2x\, dx$$

$$= \int \cos 2x \sin 2x\, dx \qquad \cos 2x = \cos^2 x - \sin^2 x$$

$$= \int \frac{1}{2}\sin 4x\, dx = \int \frac{1}{8}\sin u\, du \qquad u = 4x,\ du = 4x\, dx$$

$$= -\cos 4x + C.$$

Trying Different Substitutions

The success of the substitution method depends on finding a substitution that changes an integral we cannot evaluate directly into one that we can. Finding the right substitution gets easier with practice and experience. If your first substitution fails, try another substitution, possibly coupled with other algebraic or trigonometric simplifications to the integrand. Several more complicated types of substitutions will be studied in Chapter 8.

EXAMPLE 8 Evaluate $\displaystyle\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}}$.

Solution We will use the substitution method of integration as an exploratory tool: We substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$ or we might even press our luck and take u to be the entire cube root. In this example both substitutions turn out to be successful, but that is not always the case. If one substitution does not help, a different substitution may work instead.

Method 1: Substitute $u = z^2 + 1$.

$$\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2 + 1,\ du = 2z\, dz.$$

$$= \int u^{-1/3}\, du \qquad \text{In the form } \int u^n\, du$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2 + 1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2 + 1.$$

Method 2: Substitute $u = \sqrt[3]{z^2 + 1}$ instead.

$$\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}} = \int \frac{3u^2\, du}{u} \qquad \text{Let } u = \sqrt[3]{z^2 + 1},\ u^3 = z^2 + 1,\ 3u^2\, du = 2z\, dz.$$

$$= 3\int u\, du$$

$$= 3 \cdot \frac{u^2}{2} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}(z^2 + 1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2 + 1)^{1/3}.$$

EXERCISES 5.5

Evaluating Indefinite Integrals

Evaluate the indefinite integrals in Exercises 1–16 by using the given substitutions to reduce the integrals to standard form.

1. $\int 2(2x+4)^5\,dx, \quad u = 2x+4$

2. $\int 7\sqrt{7x-1}\,dx, \quad u = 7x-1$

3. $\int 2x(x^2+5)^{-4}\,dx, \quad u = x^2+5$

4. $\int \frac{4x^3}{(x^4+1)^2}\,dx, \quad u = x^4+1$

5. $\int (3x+2)(3x^2+4x)^4\,dx, \quad u = 3x^2+4x$

6. $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx, \quad u = 1+\sqrt{x}$

7. $\int \sin 3x\,dx, \quad u = 3x$

8. $\int x\sin(2x^2)\,dx, \quad u = 2x^2$

9. $\int \sec 2t\tan 2t\,dt, \quad u = 2t$

10. $\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1-\cos\frac{t}{2}$

11. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1-r^3$

12. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4+4y^2+1$

13. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx, \quad u = x^{3/2}-1$

14. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx, \quad u = -\frac{1}{x}$

15. $\int \csc^2 2\theta\cot 2\theta\,d\theta$

 a. Using $u = \cot 2\theta$ **b.** Using $u = \csc 2\theta$

16. $\int \frac{dx}{\sqrt{5x+8}}$

 a. Using $u = 5x+8$ **b.** Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 17–50.

17. $\int \sqrt{3-2s}\,ds$

18. $\int \frac{1}{\sqrt{5s+4}}\,ds$

19. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

20. $\int 3y\sqrt{7-3y^2}\,dy$

21. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

22. $\int \sqrt{\sin x}\cos^3 x\,dx$

23. $\int \sec^2(3x+2)\,dx$

24. $\int \tan^2 x\sec^2 x\,dx$

25. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

26. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

27. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

28. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

29. $\int x^{1/2}\sin(x^{3/2}+1)\,dx$

30. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

31. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

32. $\int \frac{\sec z\tan z}{\sqrt{\sec z}}\,dz$

33. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$

34. $\int \frac{1}{\sqrt{t}}\cos(\sqrt{t}+3)\,dt$

35. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

36. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

37. $\int \frac{x}{\sqrt{1+x}}\,dx$

38. $\int \sqrt{\frac{x-1}{x^5}}\,dx$

39. $\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

40. $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx$

41. $\int \sqrt{\frac{x^3-3}{x^{11}}}\,dx$

42. $\int \sqrt{\frac{x^4}{x^3-1}}\,dx$

43. $\int x(x-1)^{10}\,dx$

44. $\int x\sqrt{4-x}\,dx$

45. $\int (x+1)^2(1-x)^5\,dx$

46. $\int (x+5)(x-5)^{1/3}\,dx$

47. $\int x^3\sqrt{x^2+1}\,dx$

48. $\int 3x^5\sqrt{x^3+1}\,dx$

49. $\int \frac{x}{(x^2-4)^3}\,dx$

50. $\int \frac{x}{(2x-1)^{2/3}}\,dx$

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 51 and 52.

51. $\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$

 a. $u = \tan x$, followed by $v = u^3$, then by $w = 2+v$

 b. $u = \tan^3 x$, followed by $v = 2+u$

 c. $u = 2+\tan^3 x$

52. $\int \sqrt{1+\sin^2(x-1)}\sin(x-1)\cos(x-1)\,dx$

 a. $u = x-1$, followed by $v = \sin u$, then by $w = 1+v^2$

 b. $u = \sin(x-1)$, followed by $v = 1+u^2$

 c. $u = 1+\sin^2(x-1)$

Evaluate the integrals in Exercises 53 and 54.

53. $\displaystyle\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

54. $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}\,d\theta$

Initial Value Problems

Solve the initial value problems in Exercises 55–60.

55. $\dfrac{ds}{dt} = 12t\,(3t^2-1)^3, \quad s(1) = 3$

56. $\dfrac{dy}{dx} = 4x\,(x^2+8)^{-1/3}, \quad y(0) = 0$

57. $\dfrac{ds}{dt} = 8\sin^2\left(t + \dfrac{\pi}{12}\right), \quad s(0) = 8$

58. $\dfrac{dr}{d\theta} = 3\cos^2\left(\dfrac{\pi}{4} - \theta\right), \quad r(0) = \dfrac{\pi}{8}$

59. $\dfrac{d^2s}{dt^2} = -4\sin\left(2t - \dfrac{\pi}{2}\right), \quad s'(0) = 100, \quad s(0) = 0$

60. $\dfrac{d^2y}{dx^2} = 4\sec^2 2x\tan 2x, \quad y'(0) = 4, \quad y(0) = -1$

61. The velocity of a particle moving back and forth on a line is $v = ds/dt = 6\sin 2t$ m/sec for all t. If $s = 0$ when $t = 0$, find the value of s when $t = \pi/2$ sec.

62. The acceleration of a particle moving back and forth on a line is $a = d^2s/dt^2 = \pi^2\cos\pi t$ m/sec^2 for all t. If $s = 0$ and $v = 8$ m/sec when $t = 0$, find s when $t = 1$ sec.

5.6 Definite Integral Substitutions and the Area Between Curves

There are two methods for evaluating a definite integral by substitution. One method is to find an antiderivative using substitution and then to evaluate the definite integral by applying the Evaluation Theorem. The other method extends the process of substitution directly to *definite* integrals by changing the limits of integration. We will use the new formula that we introduce here to compute the area between two curves.

The Substitution Formula

The following formula shows how the limits of integration change when we apply a substitution to an integral.

THEOREM 7—Substitution in Definite Integrals
If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof Let F denote any antiderivative of f. Then,

$$\begin{aligned}\int_a^b f(g(x))\cdot g'(x)\,dx &= F(g(x))\Big]_{x=a}^{x=b} && \frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)\\ &= F(g(b)) - F(g(a))\\ &= F(u)\Big]_{u=g(a)}^{u=g(b)}\\ &= \int_{g(a)}^{g(b)} f(u)\,du. && \text{Fundamental Theorem, Part 2}\end{aligned}$$

■

To use Theorem 7, make the same u-substitution $u = g(x)$ and $du = g'(x)\,dx$ that you would use to evaluate the corresponding indefinite integral. Then integrate the transformed integral with respect to u from the value $g(a)$ (the value of u at $x = a$) to the value $g(b)$ (the value of u at $x = b$).

EXAMPLE 1 Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

Solution We will show how to evaluate the integral using Theorem 7, and how to evaluate it using the original limits of integration.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 7.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx \qquad \text{Let } u = x^3+1,\ du = 3x^2\,dx. \text{ When } x=-1,\ u=(-1)^3+1=0. \text{ When } x=1,\ u=(1)^3+1=2.$$

$$= \int_0^2 \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2 \qquad \text{Evaluate the new definite integral.}$$

$$= \frac{2}{3}\left[2^{3/2}-0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du \qquad \text{Let } u = x^3+1,\ du = 3x^2\,dx.$$

$$= \frac{2}{3}u^{3/2}+C \qquad \text{Integrate with respect to } u.$$

$$= \frac{2}{3}(x^3+1)^{3/2}+C \qquad \text{Replace } u \text{ by } x^3+1.$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1} \qquad \text{Use the integral just found, with limits of integration for } x.$$

$$= \frac{2}{3}\left[((1)^3+1)^{3/2}-((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2}-0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3} \qquad \blacksquare$$

Which method is better—evaluating the transformed definite integral with transformed limits using Theorem 7, or transforming the integral, integrating, and transforming back to use the original limits of integration? In Example 1, the first method seems easier, but that is not always the case. Generally, it is best to know both methods and to use whichever one seems better at the time.

EXAMPLE 2 We use the method of transforming the limits of integration.

(a) $$\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\,d\theta = \int_1^0 u\cdot(-du) \qquad \text{Let } u=\cot\theta,\ du=-\csc^2\theta\,d\theta,\ -du=\csc^2\theta\,d\theta. \text{ When } \theta=\pi/4,\ u=\cot(\pi/4)=1. \text{ When } \theta=\pi/2,\ u=\cot(\pi/2)=0.$$

$$= -\int_1^0 u\,du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2}-\frac{(1)^2}{2}\right] = \frac{1}{2}$$

(b) $$\int_0^{\pi/2} \frac{2\sin x\cos x}{(1+\sin^2 x)^3}\,dx = \int_1^2 \frac{1}{u^3}\,du$$

Let $u = 1 + \sin^2 x$, $du = 2\sin x \cos x\, dx$.
When $x = 0$, $u = 1$.
When $x = \pi/2$, $u = 2$.

$$= -\frac{1}{2u^2}\Big]_1^2$$

$$= -\frac{1}{8} - \left(-\frac{1}{2}\right) = \frac{3}{8}$$

Definite Integrals of Symmetric Functions

The Substitution Formula in Theorem 7 simplifies the calculation of definite integrals of even and odd functions (Section 1.1) over a symmetric interval $[-a, a]$ (Figure 5.24).

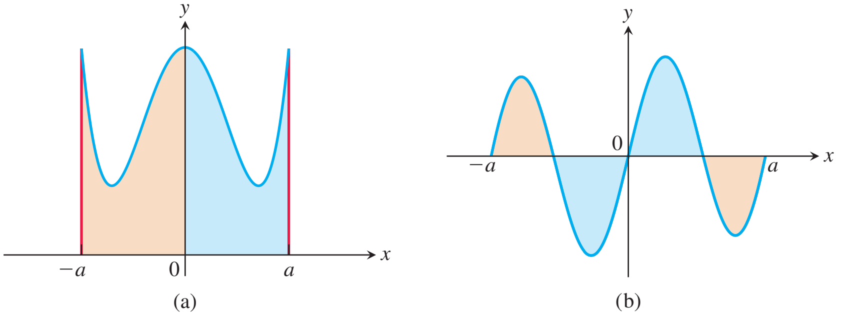


FIGURE 5.24 (a) For f an even function, the integral from $-a$ to a is twice the integral from 0 to a. (b) For f an odd function, the integral from $-a$ to a equals 0.

THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Proof of Part (a)

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

Additivity Rule for Definite Integrals

$$= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$$

Order of Integration Rule

$$= -\int_0^{a} f(-u)(-du) + \int_0^a f(x)\,dx$$

Let $u = -x$, $du = -dx$.
When $x = 0$, $u = 0$.
When $x = -a$, $u = a$.

$$= \int_0^{a} f(-u)\,du + \int_0^a f(x)\,dx$$

$$= \int_0^{a} f(u)\,du + \int_0^a f(x)\,dx$$

f is even, so $f(-u) = f(u)$.

$$= 2\int_0^a f(x)\,dx$$

The proof of part (b) is entirely similar and you are asked to give it in Exercise 86. ■

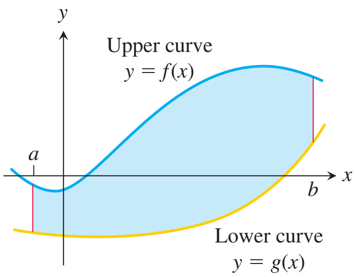


FIGURE 5.25 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

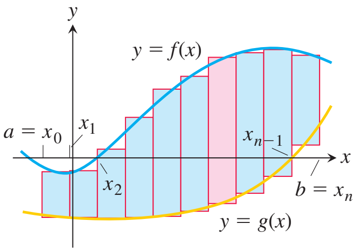


FIGURE 5.26 We approximate the region with rectangles perpendicular to the x-axis.

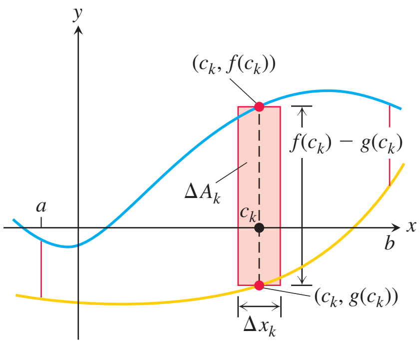


FIGURE 5.27 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

EXAMPLE 3 Evaluate $\displaystyle\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx.$

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$\begin{aligned}\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx &= 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx\\ &= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2\\ &= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.\end{aligned}$$

Areas Between Curves

Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$ (Figure 5.25). The region might accidentally have a shape whose area we could find with geometry, but if f and g are arbitrary continuous functions, we usually have to find the area by computing an integral.

To see what the integral should be, we first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ (Figure 5.26). The area of the kth rectangle (Figure 5.27) is

$$\Delta A_k = \text{height} \times \text{width} = [f(c_k) - g(c_k)]\,\Delta x_k.$$

We then approximate the area of the region by adding the areas of the n rectangles:

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k. \qquad \text{Riemann sum}$$

As $\|P\| \to 0$, the sums on the right approach the limit $\int_a^b [f(x) - g(x)]\,dx$ because f and g are continuous. The area of the region is defined to be the value of this integral. That is,

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k = \int_a^b [f(x) - g(x)]\,dx.$$

DEFINITION If f and g are continuous with $f(x) \ge g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

When applying this definition it is usually helpful to graph the curves. The graph reveals which curve is the upper curve f and which is the lower curve g. It also helps you find the limits of integration if they are not given. You may need to find where the curves intersect to determine the limits of integration, and this may involve solving the equation

$f(x) = g(x)$ for values of x. Then you can integrate the function $f - g$ for the area between the intersections.

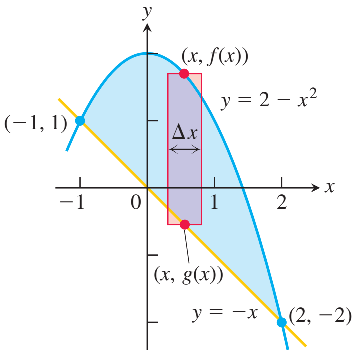


FIGURE 5.28 The region in Example 4 with a typical approximating rectangle from a Riemann sum.

EXAMPLE 4 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 5.28). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$\begin{aligned} 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\ x^2 - x - 2 &= 0 && \text{Rewrite.} \\ (x+1)(x-2) &= 0 && \text{Factor.} \\ x = -1, \qquad x &= 2. && \text{Solve.} \end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

The area between the curves is

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx \\ &= \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} \\ &= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}. \end{aligned}$$

If the formula for a bounding curve changes at one or more points, we subdivide the region into subregions that correspond to the formula changes and apply the formula for the area between curves to each subregion.

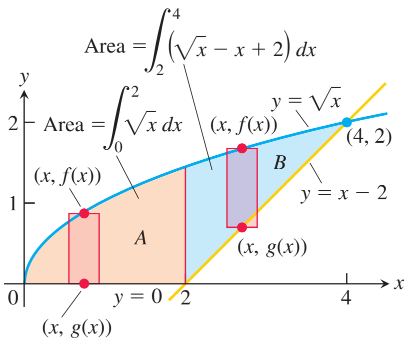


FIGURE 5.29 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 5.

EXAMPLE 5 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution The sketch (Figure 5.29) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (both formulas agree at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure 5.29.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\begin{aligned} \sqrt{x} &= x - 2 && \text{Equate } f(x) \text{ and } g(x). \\ x &= (x-2)^2 = x^2 - 4x + 4 && \text{Square both sides.} \\ x^2 - 5x + 4 &= 0 && \text{Rewrite.} \\ (x-1)(x-4) &= 0 && \text{Factor.} \\ x = 1, \qquad x &= 4. && \text{Solve.} \end{aligned}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

$$\begin{aligned} &\text{For } 0 \le x \le 2: && f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x} \\ &\text{For } 2 \le x \le 4: && f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2 \end{aligned}$$

We add the areas of subregions A and B to find the total area:

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 \left(\sqrt{x} - x + 2\right) dx}_{\text{area of } B}$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

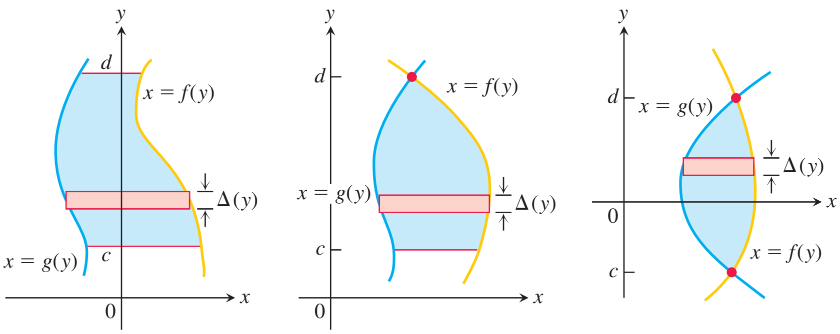
$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these:



use the formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

EXAMPLE 6 Find the area of the region in Example 5 by integrating with respect to y.

Solution We first sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y-values (Figure 5.30). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y:

$$\begin{aligned} y + 2 &= y^2 && \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2. \\ y^2 - y - 2 &= 0 && \text{Rewrite.} \\ (y + 1)(y - 2) &= 0 && \text{Factor.} \\ y = -1, \quad y &= 2 && \text{Solve.} \end{aligned}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

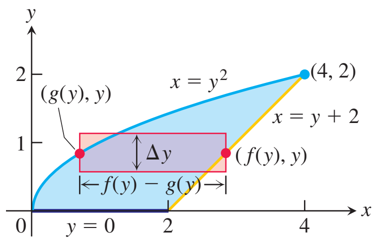


FIGURE 5.30 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 6).

The area of the region is

$$A = \int_c^d [f(y) - g(y)]\,dy = \int_0^2 [y + 2 - y^2]\,dy$$
$$= \int_0^2 [2 + y - y^2]\,dy$$
$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$
$$= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}.$$

This is the result of Example 5, found with less work. ■

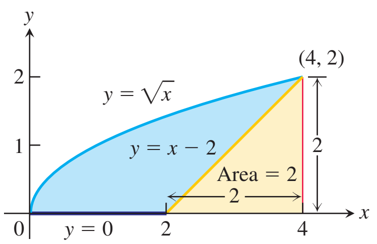


FIGURE 5.31 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

Although it was easier to find the area in Example 5 by integrating with respect to y rather than x (just as we did in Example 6), there is an easier way yet. Looking at Figure 5.31, we see that the area we want is the area between the curve $y = \sqrt{x}$ and the x-axis for $0 \leq x \leq 4$, *minus* the area of an isosceles triangle of base and height equal to 2. So by combining calculus with some geometry, we find

$$\text{Area} = \int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2)$$
$$= \frac{2}{3}x^{3/2}\Big]_0^4 - 2$$
$$= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}.$$

EXAMPLE 7 Find the area of the region bounded below by the line $y = 2 - x$ and above by the curve $y = \sqrt{2x - x^2}$.

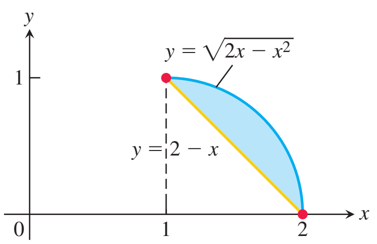


FIGURE 5.32 The region described by the curves in Example 7.

Solution A sketch of the region is displayed in Figure 5.32, and we see that the line and curve intersect at the points (1, 1) and (2, 0). Using vertical rectangles, the area of the region is given by

$$A = \int_1^2 \left(\sqrt{2x - x^2} + x - 2\right) dx.$$

However, we don't know how to find an antiderivative for the term involving the radical, and no simple substitution is apparent.

To use horizontal rectangles, we first need to express each bounding curve as a function of the variable y. The line on the left is easily found to be $x = 2 - y$. For the curve $y = \sqrt{2x - x^2}$ on the right-hand side in Figure 5.32, we have

$$\begin{aligned} y^2 &= 2x - x^2 \\ &= -(x^2 - 2x + 1) + 1 && \text{Complete the square.} \\ &= -(x - 1)^2 + 1. \end{aligned}$$

Solving for x,

$$\begin{aligned} (x - 1)^2 &= 1 - y^2, \\ x &= 1 + \sqrt{1 - y^2}. && x \geq 1,\ 0 \leq y \leq 1 \end{aligned}$$

The area of the region is then given by

$$A = \int_0^1 \left[\left(1 + \sqrt{1 - y^2}\right) - (2 - y)\right] dy = \int_0^1 \left(\sqrt{1 - y^2} + y - 1\right) dy.$$

Again, we don't know yet how to integrate the radical term (although we will see how to do that in Section 8.4). We conclude that neither vertical nor horizontal rectangles lead to an integral we currently can evaluate.

Nevertheless, as we found with Example 6, sometimes a little observation proves to be helpful. If we look again at the algebra for expressing the right-hand side curve $y = \sqrt{2x - x^2}$ as a function of y, we see that $(x - 1)^2 + y^2 = 1$, which is the equation of the unit circle with center shifted to the point $(1, 0)$. From Figure 5.32, we can then see that the area of the region we want is the area of the upper right quarter of the unit circle minus the area of the triangle with vertices $(1, 1)$, $(1, 0)$, and $(2, 0)$. That is, the area is given by

$$A = \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4} \approx 0.285.$$

EXERCISES 5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 7 to evaluate the integrals in Exercises 1–24.

1. a. $\displaystyle\int_0^3 \sqrt{y+1}\,dy$ **b.** $\displaystyle\int_{-1}^0 \sqrt{y+1}\,dy$

2. a. $\displaystyle\int_0^1 r\sqrt{1-r^2}\,dr$ **b.** $\displaystyle\int_{-1}^1 r\sqrt{1-r^2}\,dr$

3. a. $\displaystyle\int_0^{\pi/4} \tan x \sec^2 x\,dx$ **b.** $\displaystyle\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$

4. a. $\displaystyle\int_0^{\pi} 3\cos^2 x \sin x\,dx$ **b.** $\displaystyle\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$

5. a. $\displaystyle\int_0^1 t^3(1+t^4)^3\,dt$ **b.** $\displaystyle\int_{-1}^1 t^3(1+t^4)^3\,dt$

6. a. $\displaystyle\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt$ **b.** $\displaystyle\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\,dt$

7. a. $\displaystyle\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$ **b.** $\displaystyle\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$

8. a. $\displaystyle\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$ **b.** $\displaystyle\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$

9. a. $\displaystyle\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$ **b.** $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

10. a. $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$ **b.** $\displaystyle\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx$

11. a. $\displaystyle\int_0^1 t\sqrt{4+5t}\,dt$ **b.** $\displaystyle\int_1^9 t\sqrt{4+5t}\,dt$

12. a. $\displaystyle\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$

b. $\displaystyle\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\,dt$

13. a. $\displaystyle\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$ **b.** $\displaystyle\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

14. a. $\displaystyle\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

b. $\displaystyle\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

15. $\displaystyle\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$ **16.** $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2}$

17. $\displaystyle\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$ **18.** $\displaystyle\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta$

19. $\displaystyle\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\,dt$

20. $\displaystyle\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\,dt$

21. $\displaystyle\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\,dy$

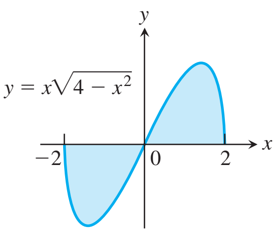
22. $\displaystyle\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$

23. $\displaystyle\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$ **24.** $\displaystyle\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$

Area

Find the total areas of the shaded regions in Exercises 25–40.

25.



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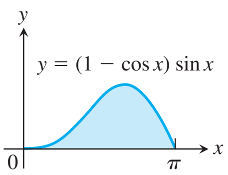

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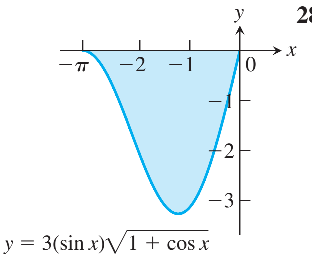

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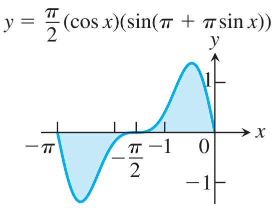

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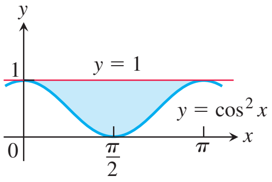

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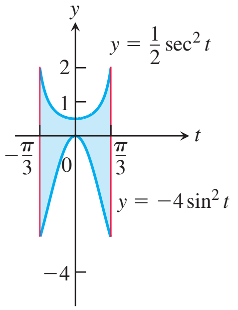

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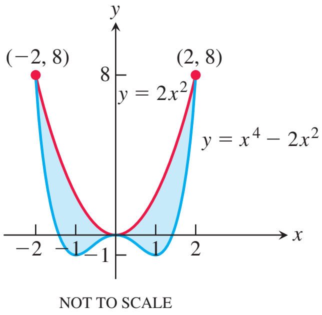

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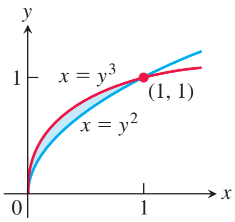

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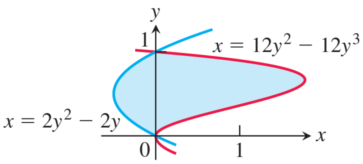

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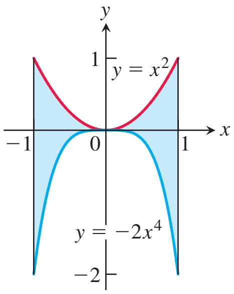

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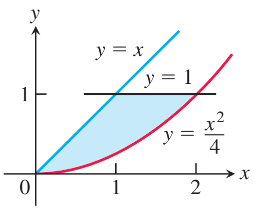

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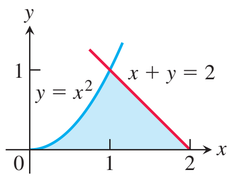

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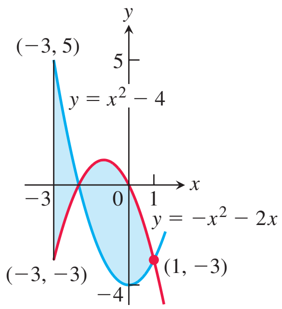

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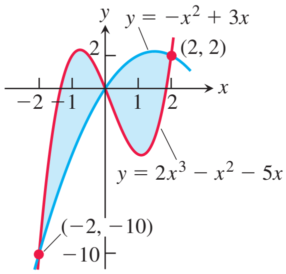

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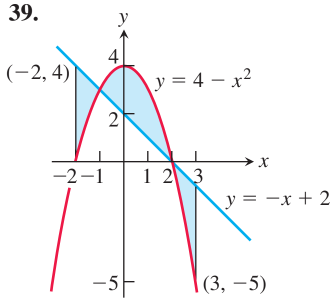

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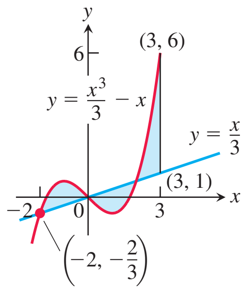


Find the areas of the regions enclosed by the lines and curves in Exercises 41–50.

41. $y = x^2 - 2$ and $y = 2$

42. $y = 2x - x^2$ and $y = -3$

43. $y = x^4$ and $y = 8x$

44. $y = x^2 - 2x$ and $y = x$

45. $y = x^2$ and $y = -x^2 + 4x$

46. $y = 7 - 2x^2$ and $y = x^2 + 4$

47. $y = x^4 - 4x^2 + 4$ and $y = x^2$

48. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

49. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

50. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 51–58.

51. $x = 2y^2$, $x = 0$, and $y = 3$

52. $x = y^2$ and $x = y + 2$

53. $y^2 - 4x = 4$ and $4x - y = 16$

54. $x - y^2 = 0$ and $x + 2y^2 = 3$

55. $x + y^2 = 0$ and $x + 3y^2 = 2$

56. $x - y^{2/3} = 0$ and $x + y^4 = 2$

57. $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

58. $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 59–62.

59. $4x^2 + y = 4$ and $x^4 - y = 1$

60. $x^3 - y = 0$ and $3x^2 - y = 4$

61. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

62. $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 63–70.

63. $y = 2\sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

64. $y = 8\cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

65. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

66. $y = \sin(\pi x/2)$ and $y = x$

67. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

68. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

69. $x = 3\sin y\sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

70. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

Area Between Curves

71. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

72. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

73. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

74. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

75. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

a. Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

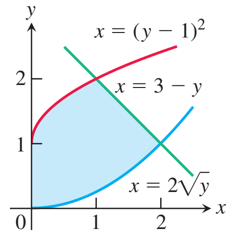
b. Find c by integrating with respect to y. (This puts c in the limits of integration.)

c. Find c by integrating with respect to x. (This puts c into the integrand as well.)

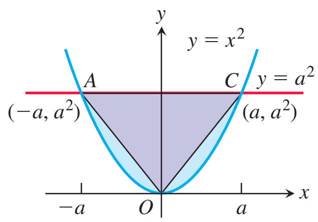
76. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a.** x, **b.** y.

77. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

78. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



79. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

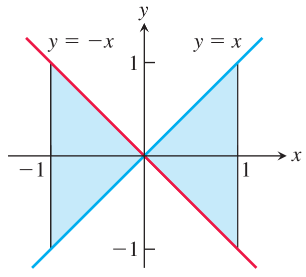


80. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

81. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a. $\displaystyle\int_{-1}^{1} (x - (-x))\,dx = \int_{-1}^{1} 2x\,dx$

b. $\displaystyle\int_{-1}^{1} (-x - (x))\,dx = \int_{-1}^{1} -2x\,dx$



82. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ $(a < b)$ is

$$\int_a^b [f(x) - g(x)]\,dx.$$

Give reasons for your answer.

Theory and Examples

83. Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express

$$\int_1^3 \frac{\sin 2x}{x}\,dx$$

in terms of F.

84. Show that if f is continuous, then

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx.$$

85. Suppose that

$$\int_0^1 f(x)\,dx = 3.$$

Find

$$\int_{-1}^0 f(x)\,dx$$

if **a.** f is odd, **b.** f is even.

86. a. Show that if f is odd on $[-a, a]$, then

$$\int_{-a}^a f(x)\,dx = 0.$$

b. Test the result in part (a) with $f(x) = \sin x$ and $a = \pi/2$.

87. If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x)\,dx}{f(x) + f(a-x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I.

88. By using a substitution, prove that for all positive numbers x and y,

$$\int_x^{xy} \frac{1}{t}\,dt = \int_1^y \frac{1}{t}\,dt.$$

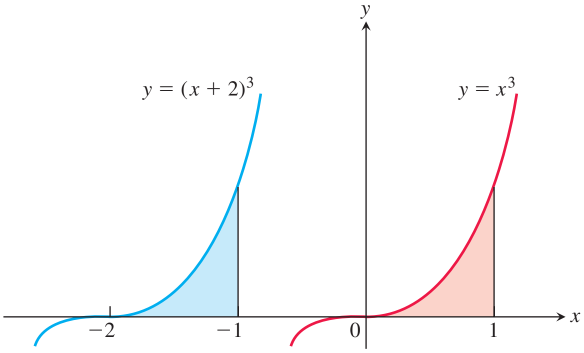
The Shift Property for Definite Integrals A basic property of definite integrals is their invariance under translation, as expressed by the equation

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx. \qquad (1)$$

The equation holds whenever f is integrable and defined for the necessary values of x. For example in the accompanying figure, show that

$$\int_{-2}^{-1} (x+2)^3\,dx = \int_0^1 x^3\,dx$$

because the areas of the shaded regions are congruent.



89. Use a substitution to verify Equation (1).

90. For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x+c)$ over $[a-c, b-c]$ to convince yourself that Equation (1) is reasonable.

a. $f(x) = x^2, \quad a = 0, \quad b = 1, \quad c = 1$

b. $f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad c = \pi/2$

c. $f(x) = \sqrt{x-4}, \quad a = 4, \quad b = 8, \quad c = 5$

COMPUTER EXPLORATIONS

In Exercises 91–94, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

a. Plot the curves together to see what they look like and how many points of intersection they have.

b. Use the numerical equation solver in your CAS to find all the points of intersection.

c. Integrate $|f(x) - g(x)|$ over consecutive pairs of intersection values.

d. Sum together the integrals found in part (c).

91. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}, \quad g(x) = x - 1$

92. $f(x) = \frac{x^4}{2} - 3x^3 + 10, \quad g(x) = 8 - 12x$

93. $f(x) = x + \sin(2x), \quad g(x) = x^3$

94. $f(x) = x^2 \cos x, \quad g(x) = x^3 - x$

CHAPTER 5 Questions to Guide Your Review

1. How can you sometimes estimate quantities like distance traveled, area, and average value with finite sums? Why might you want to do so?
2. What is sigma notation? What advantage does it offer? Give examples.
3. What is a Riemann sum? Why might you want to consider such a sum?
4. What is the norm of a partition of a closed interval?
5. What is the definite integral of a function f over a closed interval $[a, b]$? When can you be sure it exists?

6. What is the relation between definite integrals and area? Describe some other interpretations of definite integrals.

7. What is the average value of an integrable function over a closed interval? Must the function assume its average value? Explain.

8. Describe the rules for working with definite integrals (Table 5.6). Give examples.

9. What is the Fundamental Theorem of Calculus? Why is it so important? Illustrate each part of the theorem with an example.

10. What is the Net Change Theorem? What does it say about the integral of velocity? The integral of marginal cost?

11. Discuss how the processes of integration and differentiation can be considered as "inverses" of each other.

12. How does the Fundamental Theorem provide a solution to the initial value problem $dy/dx = f(x)$, $y(x_0) = y_0$, when f is continuous?

13. How is integration by substitution related to the Chain Rule?

14. How can you sometimes evaluate indefinite integrals by substitution? Give examples.

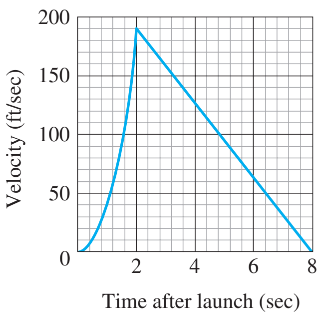
15. How does the method of substitution work for definite integrals? Give examples.

16. How do you define and calculate the area of the region between the graphs of two continuous functions? Give an example.

CHAPTER 5 Practice Exercises

Finite Sums and Estimates

1. The accompanying figure shows the graph of the velocity (ft/sec) of a model rocket for the first 8 sec after launch. The rocket accelerated straight up for the first 2 sec and then coasted to reach its maximum height at $t = 8$ sec.

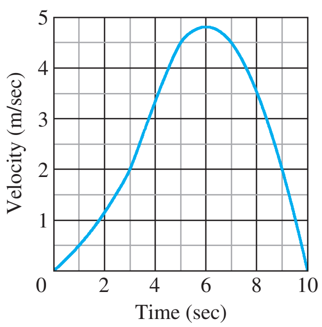


a. Assuming that the rocket was launched from ground level, about how high did it go? (This is the rocket in Section 3.4, Exercise 17, but you do not need to do Exercise 17 to do the exercise here.)

b. Sketch a graph of the rocket's height above ground as a function of time for $0 \le t \le 8$.

2. a. The accompanying figure shows the velocity (m/sec) of a body moving along the s-axis during the time interval from $t = 0$ to $t = 10$ sec. About how far did the body travel during those 10 sec?

b. Sketch a graph of s as a function of t for $0 \le t \le 10$, assuming $s(0) = 0$.



3. Suppose that $\sum_{k=1}^{10} a_k = -2$ and $\sum_{k=1}^{10} b_k = 25$. Find the value of

a. $\displaystyle\sum_{k=1}^{10} \frac{a_k}{4}$ b. $\displaystyle\sum_{k=1}^{10} (b_k - 3a_k)$

c. $\displaystyle\sum_{k=1}^{10} (a_k + b_k - 1)$ d. $\displaystyle\sum_{k=1}^{10} \left(\frac{5}{2} - b_k\right)$

4. Suppose that $\sum_{k=1}^{20} a_k = 0$ and $\sum_{k=1}^{20} b_k = 7$. Find the values of

a. $\displaystyle\sum_{k=1}^{20} 3a_k$ b. $\displaystyle\sum_{k=1}^{20} (a_k + b_k)$

c. $\displaystyle\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7}\right)$ d. $\displaystyle\sum_{k=1}^{20} (a_k - 2)$

Definite Integrals

In Exercises 5–8, express each limit as a definite integral. Then evaluate the integral to find the value of the limit. In each case, P is a partition of the given interval and the numbers c_k are chosen from the subintervals of P.

5. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} (2c_k - 1)^{-1/2}\,\Delta x_k$, where P is a partition of $[1, 5]$

6. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k(c_k^2 - 1)^{1/3}\,\Delta x_k$, where P is a partition of $[1, 3]$

7. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(\cos\left(\frac{c_k}{2}\right)\right)\Delta x_k$, where P is a partition of $[-\pi, 0]$

8. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sin c_k)(\cos c_k)\,\Delta x_k$, where P is a partition of $[0, \pi/2]$

9. If $\int_{-2}^{2} 3f(x)\,dx = 12$, $\int_{-2}^{5} f(x)\,dx = 6$, and $\int_{-2}^{5} g(x)\,dx = 2$, find the values of the following.

a. $\displaystyle\int_{-2}^{2} f(x)\,dx$ b. $\displaystyle\int_{2}^{5} f(x)\,dx$

c. $\displaystyle\int_{5}^{-2} g(x)\,dx$ d. $\displaystyle\int_{-2}^{5} (-\pi g(x))\,dx$

e. $\displaystyle\int_{-2}^{5} \left(\frac{f(x) + g(x)}{5}\right)dx$

10. If $\int_0^2 f(x)\,dx = \pi$, $\int_0^2 7g(x)\,dx = 7$, and $\int_0^1 g(x)\,dx = 2$, find the values of the following.

a. $\displaystyle\int_0^2 g(x)\,dx$ **b.** $\displaystyle\int_1^2 g(x)\,dx$

c. $\displaystyle\int_2^0 f(x)\,dx$ **d.** $\displaystyle\int_0^2 \sqrt{2}\,f(x)\,dx$

e. $\displaystyle\int_0^2 (g(x) - 3f(x))\,dx$

Area

In Exercises 11–14, find the total area of the region between the graph of f and the x-axis.

11. $f(x) = x^2 - 4x + 3, \quad 0 \le x \le 3$

12. $f(x) = 1 - (x^2/4), \quad -2 \le x \le 3$

13. $f(x) = 5 - 5x^{2/3}, \quad -1 \le x \le 8$

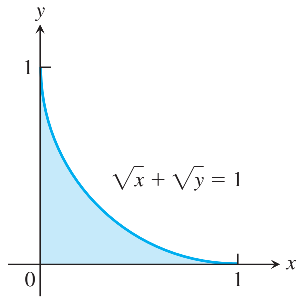
14. $f(x) = 1 - \sqrt{x}, \quad 0 \le x \le 4$

Find the areas of the regions enclosed by the curves and lines in Exercises 15–26.

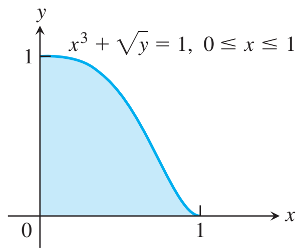
15. $y = x, \quad y = 1/x^2, \quad x = 2$

16. $y = x, \quad y = 1/\sqrt{x}, \quad x = 2$

17. $\sqrt{x} + \sqrt{y} = 1, \quad x = 0, \quad y = 0$



18. $x^3 + \sqrt{y} = 1, \quad x = 0, \quad y = 0, \quad \text{for} \quad 0 \le x \le 1$



19. $x = 2y^2, \quad x = 0, \quad y = 3$ **20.** $x = 4 - y^2, \quad x = 0$

21. $y^2 = 4x, \quad y = 4x - 2$

22. $y^2 = 4x + 4, \quad y = 4x - 16$

23. $y = \sin x, \quad y = x, \quad 0 \le x \le \pi/4$

24. $y = |\sin x|, \quad y = 1, \quad -\pi/2 \le x \le \pi/2$

25. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \le x \le \pi$

26. $y = 8\cos x, \quad y = \sec^2 x, \quad -\pi/3 \le x \le \pi/3$

27. Find the area of the "triangular" region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$.

28. Find the area of the "triangular" region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.

29. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of f and the x-axis.

30. Find the area of the region cut from the first quadrant by the curve $x^{1/2} + y^{1/2} = a^{1/2}$.

31. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and the lines $x = y$ and $y = -1$.

32. Find the total area of the region between the curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le 3\pi/2$.

Initial Value Problems

33. Show that $y = x^2 + \displaystyle\int_1^x \frac{1}{t}\,dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}; \quad y'(1) = 3, \quad y(1) = 1.$$

34. Show that $y = \int_0^x \left(1 + 2\sqrt{\sec t}\right) dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = \sqrt{\sec x}\,\tan x; \quad y'(0) = 3, \quad y(0) = 0.$$

Express the solutions of the initial value problems in Exercises 35 and 36 in terms of integrals.

35. $\dfrac{dy}{dx} = \dfrac{\sin x}{x}, \quad y(5) = -3$

36. $\dfrac{dy}{dx} = \sqrt{2 - \sin^2 x}, \quad y(-1) = 2$

Evaluating Indefinite Integrals

Evaluate the integrals in Exercises 37–46.

37. $\displaystyle\int 2(\cos x)^{-1/2} \sin x\,dx$ **38.** $\displaystyle\int (\tan x)^{-3/2} \sec^2 x\,dx$

39. $\displaystyle\int (2\theta + 1 + 2\cos(2\theta + 1))\,d\theta$

40. $\displaystyle\int \left(\frac{1}{\sqrt{2\theta - \pi}} + 2\sec^2(2\theta - \pi)\right) d\theta$

41. $\displaystyle\int \left(t - \frac{2}{t}\right)\left(t + \frac{2}{t}\right) dt$ **42.** $\displaystyle\int \frac{(t+1)^2 - 1}{t^4}\,dt$

43. $\displaystyle\int \sqrt{t}\,\sin(2t^{3/2})\,dt$ **44.** $\displaystyle\int (\sec\theta\tan\theta)\sqrt{1 + \sec\theta}\,d\theta$

45. $\displaystyle\int \frac{\sin 2\theta - \cos 2\theta}{(\sin 2\theta + \cos 2\theta)^3}\,d\theta$ **46.** $\displaystyle\int \cos\theta \cdot \sin(\sin\theta)\,d\theta$

Evaluating Definite Integrals

Evaluate the integrals in Exercises 47–68.

47. $\displaystyle\int_{-1}^1 (3x^2 - 4x + 7)\,dx$ **48.** $\displaystyle\int_0^1 (8s^3 - 12s^2 + 5)\,ds$

49. $\displaystyle\int_1^2 \frac{4}{v^2}\,dv$ **50.** $\displaystyle\int_1^{27} x^{-4/3}\,dx$

51. $\displaystyle\int_1^4 \frac{dt}{t\sqrt{t}}$ **52.** $\displaystyle\int_1^4 \frac{\left(1 + \sqrt{u}\right)^{1/2}}{\sqrt{u}}\,du$

53. $\displaystyle\int_0^1 \frac{36\,dx}{(2x+1)^3}$ **54.** $\displaystyle\int_0^1 \frac{dr}{\sqrt[3]{(7 - 5r)^2}}$

55. $\int_{1/8}^{1} x^{-1/3}(1 - x^{2/3})^{3/2}\,dx$

56. $\int_{0}^{1/2} x^3(1 + 9x^4)^{-3/2}\,dx$

57. $\int_{0}^{\pi} \sin^2 5r\,dr$

58. $\int_{0}^{\pi/4} \cos^2\left(4t - \frac{\pi}{4}\right)dt$

59. $\int_{0}^{\pi/3} \sec^2\theta\,d\theta$

60. $\int_{\pi/4}^{3\pi/4} \csc^2 x\,dx$

61. $\int_{\pi}^{3\pi} \cot^2\frac{x}{6}\,dx$

62. $\int_{0}^{\pi} \tan^2\frac{\theta}{3}\,d\theta$

63. $\int_{-\pi/3}^{0} \sec x \tan x\,dx$

64. $\int_{\pi/4}^{3\pi/4} \csc z \cot z\,dz$

65. $\int_{0}^{\pi/2} 5(\sin x)^{3/2}\cos x\,dx$

66. $\int_{-\pi/2}^{\pi/2} 15\sin^4 3x\cos 3x\,dx$

67. $\int_{0}^{\pi/2} \frac{3\sin x\cos x}{\sqrt{1 + 3\sin^2 x}}dx$

68. $\int_{0}^{\pi/4} \frac{\sec^2 x}{(1 + 7\tan x)^{2/3}}dx$

Average Values

69. Find the average value of $f(x) = mx + b$
   - **a.** over $[-1, 1]$
   - **b.** over $[-k, k]$

70. Find the average value of
   - **a.** $y = \sqrt{3x}$ over $[0, 3]$
   - **b.** $y = \sqrt{ax}$ over $[0, a]$

71. Let f be a function that is differentiable on $[a, b]$. In Chapter 2 we defined the average rate of change of f over $[a, b]$ to be

$$\frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change of f at x to be $f'(x)$. In this chapter we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have

$$\frac{f(b) - f(a)}{b - a} = \text{average value of } f' \text{ on } [a, b].$$

Is this the case? Give reasons for your answer.

72. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval? Give reasons for your answer.

T 73. Compute the average value of the temperature function

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25$$

for a 365-day year. (See Exercise 86, Section 3.6.) This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F, which is slightly higher than the average value of $f(x)$.

T 74. **Specific heat of a gas** Specific heat C_v is the amount of heat required to raise the temperature of one mole (gram molecule) of a gas with constant volume by 1°C. The specific heat of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Find the average value of C_v for $20° \le T \le 675°\text{C}$ and the temperature at which it is attained.

Differentiating Integrals

In Exercises 75–78, find dy/dx.

75. $y = \int_{2}^{x} \sqrt{2 + \cos^3 t}\,dt$

76. $y = \int_{2}^{7x^2} \sqrt{2 + \cos^3 t}\,dt$

77. $y = \int_{x}^{1} \frac{6}{3 + t^4}dt$

78. $y = \int_{\sec x}^{2} \frac{1}{t^2 + 1}dt$

Theory and Examples

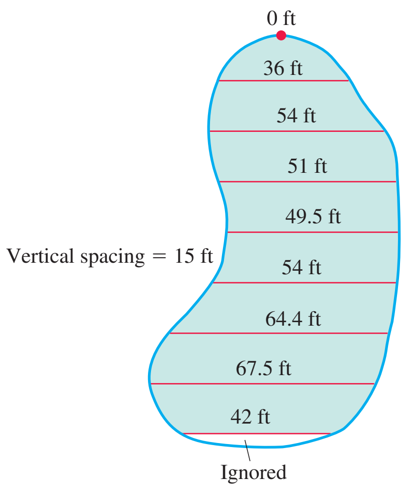
79. Is it true that every function $y = f(x)$ that is differentiable on $[a, b]$ is itself the derivative of some function on $[a, b]$? Give reasons for your answer.

80. Suppose that $F(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4}\,dx$ in terms of F and give a reason for your answer.

81. Find dy/dx if $y = \int_x^1 \sqrt{1 + t^2}\,dt$. Explain the main steps in your calculation.

82. Find dy/dx if $y = \int_{\cos x}^{0} (1/(1 - t^2))\,dt$. Explain the main steps in your calculation.

83. **A new parking lot** To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$10,000. The cost to clear the land will be \$0.10 a square foot, and the lot will cost \$2.00 a square foot to pave. Can the job be done for \$10,000? Use a lower sum estimate to see. (Answers may vary slightly, depending on the estimate used.)



84. Skydivers A and B are in a helicopter hovering at 6400 ft. Skydiver A jumps and descends for 4 sec before opening her parachute. The helicopter then climbs to 7000 ft and hovers there. Forty-five seconds after A leaves the aircraft, B jumps and descends for 13 sec before opening his parachute. Both skydivers descend at 16 ft/sec with parachutes open. Assume that the skydivers fall freely (no effective air resistance) before their parachutes open.
   - **a.** At what altitude does A's parachute open?
   - **b.** At what altitude does B's parachute open?
   - **c.** Which skydiver lands first?

CHAPTER 5 Additional and Advanced Exercises

Theory and Examples

1. a. If $\int_0^1 7f(x)\,dx = 7$, does $\int_0^1 f(x)\,dx = 1$?

b. If $\int_0^1 f(x)\,dx = 4$ and $f(x) \geq 0$, does

$$\int_0^1 \sqrt{f(x)}\,dx = \sqrt{4} = 2?$$

Give reasons for your answers.

2. Suppose $\int_{-2}^{2} f(x)\,dx = 4$, $\int_{2}^{5} f(x)\,dx = 3$, $\int_{-2}^{5} g(x)\,dx = 2$.

Which, if any, of the following statements are true?

a. $\int_5^2 f(x)\,dx = -3$

b. $\int_{-2}^{5} (f(x) + g(x)) = 9$

c. $f(x) \leq g(x)$ on the interval $-2 \leq x \leq 5$

3. Initial value problem Show that

$$y = \frac{1}{a}\int_0^x f(t) \sin a(x - t)\,dt$$

solves the initial value problem

$$\frac{d^2y}{dx^2} + a^2 y = f(x), \qquad \frac{dy}{dx} = 0 \text{ and } y = 0 \text{ when } x = 0.$$

(*Hint:* $\sin(ax - at) = \sin ax \cos at - \cos ax \sin at$.)

4. Proportionality Suppose that x and y are related by the equation

$$x = \int_0^y \frac{1}{\sqrt{1 + 4t^2}}\,dt.$$

Show that d^2y/dx^2 is proportional to y and find the constant of proportionality.

5. Find $f(4)$ if

a. $\int_0^{x^2} f(t)\,dt = x \cos \pi x$ **b.** $\int_0^{f(x)} t^2\,dt = x \cos \pi x.$

6. Find $f(\pi/2)$ from the following information.

i) f is positive and continuous.

ii) The area under the curve $y = f(x)$ from $x = 0$ to $x = a$ is

$$\frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a.$$

7. The area of the region in the xy-plane enclosed by the x-axis, the curve $y = f(x)$, $f(x) \geq 0$, and the lines $x = 1$ and $x = b$ is equal to $\sqrt{b^2 + 1} - \sqrt{2}$ for all $b > 1$. Find $f(x)$.

8. Prove that

$$\int_0^x \left(\int_0^u f(t)\,dt\right) du = \int_0^x f(u)(x - u)\,du.$$

(*Hint:* Express the integral on the right-hand side as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to x.)

9. Finding a curve Find the equation for the curve in the xy-plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.

10. Shoveling dirt You sling a shovelful of dirt up from the bottom of a hole with an initial velocity of 32 ft/sec. The dirt must rise 17 ft above the release point to clear the edge of the hole. Is that enough speed to get the dirt out, or had you better duck?

Piecewise Continuous Functions

Although we are mainly interested in continuous functions, many functions in applications are piecewise continuous. A function $f(x)$ is **piecewise continuous on a closed interval *I*** if f has only finitely many discontinuities in I, the limits

$$\lim_{x \to c^-} f(x) \quad \text{and} \quad \lim_{x \to c^+} f(x)$$

exist and are finite at every interior point of I, and the appropriate one-sided limits exist and are finite at the endpoints of I. All piecewise continuous functions are integrable. The points of discontinuity subdivide I into open and half-open subintervals on which f is continuous, and the limit criteria above guarantee that f has a continuous extension to the closure of each subinterval. To integrate a piecewise continuous function, we integrate the individual extensions and add the results. The integral of

$$f(x) = \begin{cases} 1 - x, & -1 \leq x < 0 \\ x^2, & 0 \leq x < 2 \\ -1, & 2 \leq x \leq 3 \end{cases}$$

(Figure 5.33) over $[-1, 3]$ is

$$\begin{aligned} \int_{-1}^{3} f(x)\,dx &= \int_{-1}^{0} (1 - x)\,dx + \int_0^2 x^2\,dx + \int_2^3 (-1)\,dx \\ &= \left[x - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_0^2 + \left[-x\right]_2^3 \\ &= \frac{3}{2} + \frac{8}{3} - 1 = \frac{19}{6}. \end{aligned}$$

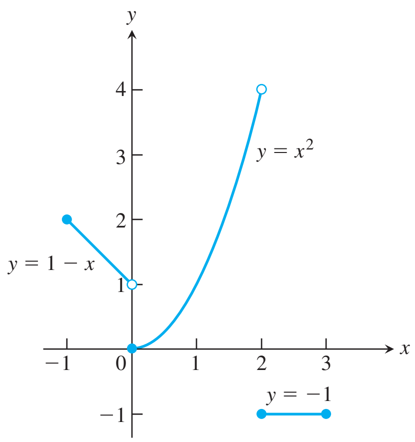


FIGURE 5.33 Piecewise continuous functions like this are integrated piece by piece.

The Fundamental Theorem applies to piecewise continuous functions with the restriction that $(d/dx)\int_a^x f(t)\,dt$ is expected to equal $f(x)$ only at values of x at which f is continuous. There is a similar restriction on Leibniz's Rule (see Exercises 27–30).

Graph the functions in Exercises 11–16 and integrate them over their domains.

11. $f(x) = \begin{cases} x^{2/3}, & -8 \le x < 0 \\ -4, & 0 \le x \le 3 \end{cases}$

12. $f(x) = \begin{cases} \sqrt{-x}, & -4 \le x < 0 \\ x^2 - 4, & 0 \le x \le 3 \end{cases}$

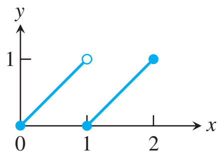
13. $g(t) = \begin{cases} t, & 0 \le t < 1 \\ \sin \pi t, & 1 \le t \le 2 \end{cases}$

14. $h(z) = \begin{cases} \sqrt{1 - z}, & 0 \le z < 1 \\ (7z - 6)^{-1/3}, & 1 \le z \le 2 \end{cases}$

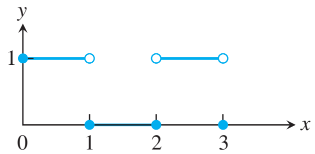
15. $f(x) = \begin{cases} 1, & -2 \le x < -1 \\ 1 - x^2, & -1 \le x < 1 \\ 2, & 1 \le x \le 2 \end{cases}$

16. $h(r) = \begin{cases} r, & -1 \le r < 0 \\ 1 - r^2, & 0 \le r < 1 \\ 1, & 1 \le r \le 2 \end{cases}$

17. Find the average value of the function graphed in the accompanying figure.



18. Find the average value of the function graphed in the accompanying figure.



Approximating Finite Sums with Integrals

In many applications of calculus, integrals are used to approximate finite sums—the reverse of the usual procedure of using finite sums to approximate integrals.

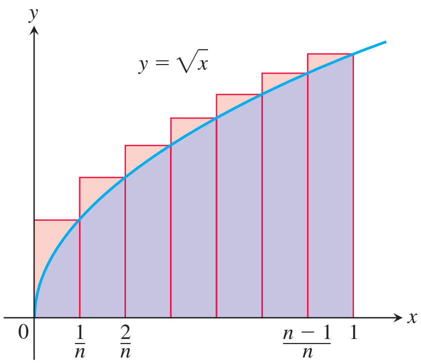
For example, let's estimate the sum of the square roots of the first n positive integers, $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$. The integral

$$\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\bigg]_0^1 = \frac{2}{3}$$

is the limit of the upper sums

$$S_n = \sqrt{\frac{1}{n}}\cdot\frac{1}{n} + \sqrt{\frac{2}{n}}\cdot\frac{1}{n} + \cdots + \sqrt{\frac{n}{n}}\cdot\frac{1}{n}$$

$$= \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}}.$$



Therefore, when n is large, S_n will be close to $2/3$ and we will have

$$\text{Root sum} = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} = S_n \cdot n^{3/2} \approx \frac{2}{3}n^{3/2}.$$

The following table shows how good the approximation can be.

n	Root sum	$(2/3)n^{3/2}$	Relative error
10	22.468	21.082	$1.386/22.468 \approx 6\%$
50	239.04	235.70	1.4%
100	671.46	666.67	0.7%
1000	21,097	21,082	0.07%

19. Evaluate

$$\lim_{n\to\infty} \frac{1^5 + 2^5 + 3^5 + \cdots + n^5}{n^6}$$

by showing that the limit is

$$\int_0^1 x^5\,dx$$

and evaluating the integral.

20. See Exercise 19. Evaluate

$$\lim_{n\to\infty} \frac{1}{n^4}(1^3 + 2^3 + 3^3 + \cdots + n^3).$$

21. Let $f(x)$ be a continuous function. Express

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right]$$

as a definite integral.

22. Use the result of Exercise 21 to evaluate

a. $\displaystyle\lim_{n\to\infty} \frac{1}{n^2}(2 + 4 + 6 + \cdots + 2n),$

b. $\displaystyle\lim_{n\to\infty} \frac{1}{n^{16}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15}),$

c. $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right).$

What can be said about the following limits?

d. $\lim_{n\to\infty} \frac{1}{n^{17}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15})$

e. $\lim_{n\to\infty} \frac{1}{n^{15}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15})$

23. a. Show that the area A_n of an n-sided regular polygon in a circle of radius r is

$$A_n = \frac{nr^2}{2}\sin\frac{2\pi}{n}.$$

b. Find the limit of A_n as $n \to \infty$. Is this answer consistent with what you know about the area of a circle?

24. Let

$$S_n = \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{(n-1)^2}{n^3}.$$

To calculate $\lim_{n\to\infty} S_n$, show that

$$S_n = \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right]$$

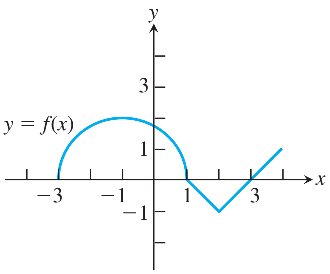
and interpret S_n as an approximating sum of the integral

$$\int_0^1 x^2\,dx.$$

(*Hint:* Partition $[0, 1]$ into n intervals of equal length and write out the approximating sum for inscribed rectangles.)

Defining Functions Using the Fundamental Theorem

25. A function defined by an integral The graph of a function f consists of a semicircle and two line segments as shown. Let $g(x) = \int_1^x f(t)\,dt$.



a. Find $g(1)$. **b.** Find $g(3)$. **c.** Find $g(-1)$.

d. Find all values of x on the open interval $(-3, 4)$ at which g has a relative maximum.

e. Write an equation for the line tangent to the graph of g at $x = -1$.

f. Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-3, 4)$.

g. Find the range of g.

26. A differential equation Show that both of the following conditions are satisfied by $y = \sin x + \int_x^{\pi} \cos 2t\,dt + 1$:

i) $y'' = -\sin x + 2\sin 2x$

ii) $y = 1$ and $y' = -2$ when $x = \pi$.

Leibniz's Rule In applications, we sometimes encounter functions defined by integrals that have variable upper limits of integration and variable lower limits of integration at the same time. We can find the derivative of such an integral by a formula called **Leibniz's Rule**.

Leibniz's Rule

If f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

To prove the rule, let F be an antiderivative of f on $[a, b]$. Then

$$\int_{u(x)}^{v(x)} f(t)\,dt = F(v(x)) - F(u(x)).$$

Differentiating both sides of this equation with respect to x gives the equation we want:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = \frac{d}{dx}\left[F(v(x)) - F(u(x))\right]$$

$$= F'(v(x))\frac{dv}{dx} - F'(u(x))\frac{du}{dx} \qquad \text{Chain Rule}$$

$$= f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

Use Leibniz's Rule to find the derivatives of the functions in Exercises 27–29.

27. $f(x) = \int_{1/x}^{x} \frac{1}{t}\,dt$

28. $f(x) = \int_{\cos x}^{\sin x} \frac{1}{1-t^2}\,dt$

29. $g(y) = \int_{\sqrt{y}}^{2\sqrt{y}} \sin t^2\,dt$

30. Use Leibniz's Rule to find the value of x that maximizes the value of the integral

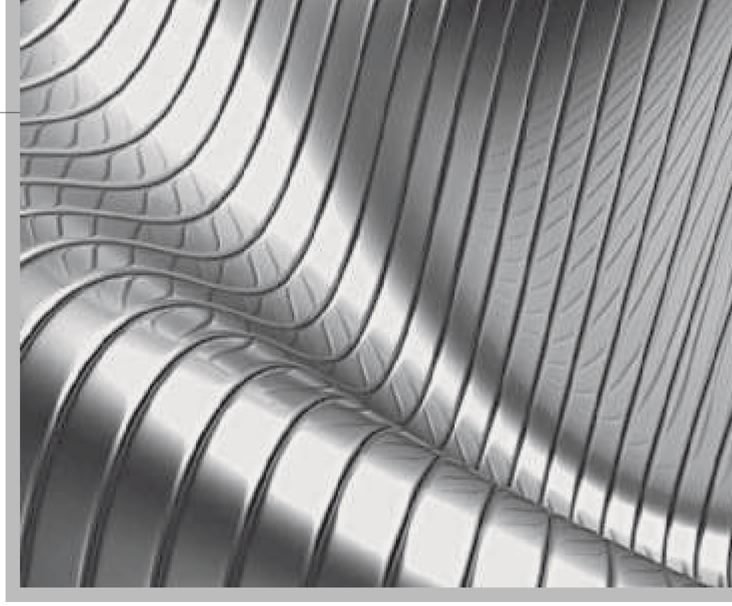
$$\int_x^{x+3} t(5-t)\,dt.$$

CHAPTER 5 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyMathLab.

- ***Using Riemann Sums to Estimate Areas, Volumes, and Lengths of Curves***
  Visualize and approximate areas and volumes in Part I.
- ***Riemann Sums, Definite Integrals, and the Fundamental Theorem of Calculus***
  Parts I, II, and III develop Riemann sums and definite integrals. Part IV continues the development of the Riemann sum and definite integral using the Fundamental Theorem to solve problems previously investigated.
- ***Rain Catchers, Elevators, and Rockets***
  Part I illustrates that the area under a curve is the same as the area of an appropriate rectangle for examples taken from the chapter. You will compute the amount of water accumulating in basins of different shapes as the basin is filled and drained.
- ***Motion Along a Straight Line, Part II***
  You will observe the shape of a graph through dramatic animated visualizations of the derivative relations among position, velocity, and acceleration. Figures in the text can be animated using this software.
- ***Bending of Beams***
  Study bent shapes of beams, determine their maximum deflections, concavity, and inflection points, and interpret the results in terms of a beams's compression and tension.



6

Applications of Definite Integrals

OVERVIEW In Chapter 5 we saw that a continuous function over a closed interval has a definite integral, which is the limit of Riemann sum approximations for the function. We found a way to evaluate definite integrals using the Fundamental Theorem of Calculus. We saw that the area under a curve and the area between two curves could be defined and computed as definite integrals. In this chapter we will see some of the many additional applications of definite integrals. We will use the definite integral to define and find volumes, lengths of plane curves, and areas of surfaces of revolution. We will see how integrals are used to solve physical problems involving the work done by a force, and how they give the location of an object's center of mass. The integral arises in these and other applications in which we can approximate a desired quantity by Riemann sums. The limit of those Riemann sums, which is the quantity we seek, is given by a definite integral.

6.1 Volumes Using Cross-Sections

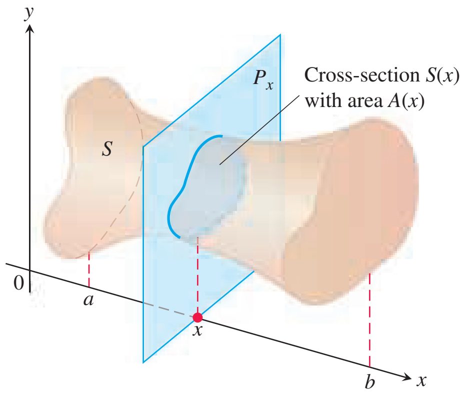


FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

In this section we define volumes of solids by using the areas of their cross-sections. A **cross-section** of a solid S is the planar region formed by intersecting S with a plane (Figure 6.1). We present three different methods for obtaining the cross-sections appropriate to finding the volume of a particular solid: the method of slicing, the disk method, and the washer method.

Suppose that we want to find the volume of a solid S like the one pictured in Figure 6.1. At each point x in the interval $[a, b]$ we form a cross-section $S(x)$ by intersecting S with a plane perpendicular to the x-axis through the point x, which gives a planar region whose area is $A(x)$. We will show that if A is a continuous function of x, then the volume of the solid S is the definite integral of $A(x)$. This method of computing volumes is known as the **method of slicing**.

Before showing how this method works, we need to extend the definition of a cylinder from the usual cylinders of classical geometry (which have circular, square, or other regular bases) to cylindrical solids that have more general bases. As shown in Figure 6.2, if the

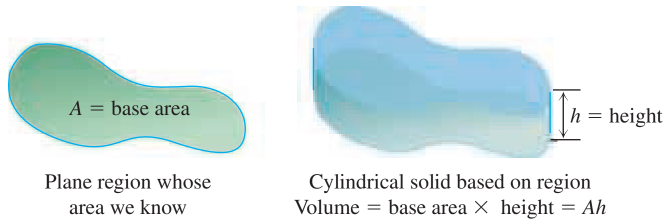


FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.

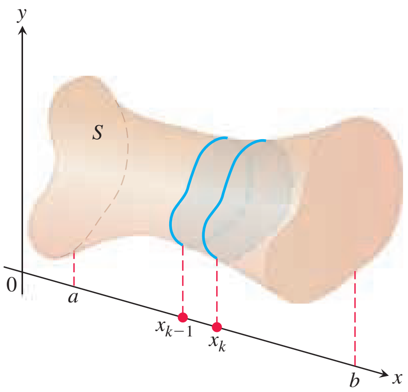


FIGURE 6.3 A typical thin slab in the solid S.

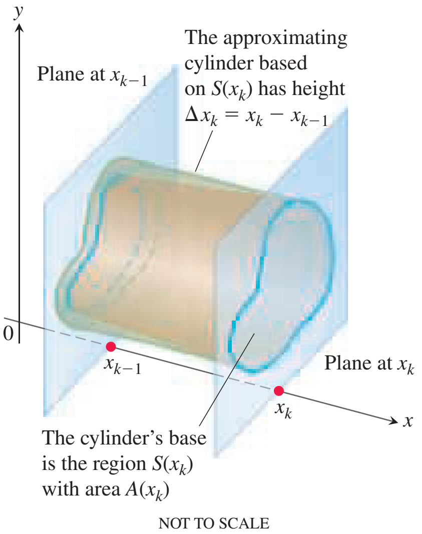


FIGURE 6.4 The solid thin slab in Figure 6.3 is shown enlarged here. It is approximated by the cylindrical solid with base $S(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

cylindrical solid has a base whose area is A and its height is h, then the volume of the cylindrical solid is

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

In the method of slicing, the base will be the cross-section of S that has area $A(x)$, and the height will correspond to the width Δx_k of subintervals formed by partitioning the interval $[a, b]$ into finitely many subintervals $[x_{k-1}, x_k]$.

Slicing by Parallel Planes

We partition $[a, b]$ into subintervals of width (length) Δx_k and slice the solid, as we would a loaf of bread, by planes perpendicular to the x-axis at the partition points $a = x_0 < x_1 < \cdots < x_n = b$. These planes slice S into thin "slabs" (like thin slices of a loaf of bread). A typical slab is shown in Figure 6.3. We approximate the slab between the plane at x_{k-1} and the plane at x_k by a cylindrical solid with base area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.4). The volume V_k of this cylindrical solid is $A(x_k) \cdot \Delta x_k$, which is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{th slab} \approx V_k = A(x_k)\,\Delta x_k.$$

The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\,\Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. The approximation given by this Riemann sum converges to the definite integral of $A(x)$ as $n \to \infty$:

$$\lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k = \int_a^b A(x)\,dx.$$

Therefore, we define this definite integral to be the volume of the solid S.

DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

This definition applies whenever $A(x)$ is integrable, and in particular when $A(x)$ is continuous. To apply this definition to calculate the volume of a solid using cross-sections perpendicular to the x-axis, take the following steps:

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ to find the volume.

EXAMPLE 1 A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

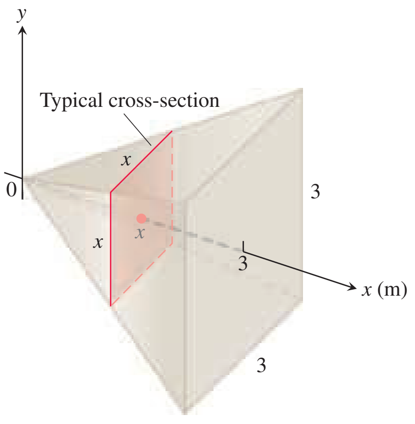


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

Solution

1. *A sketch*. We draw the pyramid with its altitude along the x-axis and its vertex at the origin and include a typical cross-section (Figure 6.5). Note that by positioning the pyramid in this way, we have vertical cross-sections that are squares, whose areas are easy to calculate.
2. *A formula for* $A(x)$. The cross-section at x is a square x meters on a side, so its area is
$$A(x) = x^2.$$
3. *The limits of integration*. The squares lie on the planes from $x = 0$ to $x = 3$.
4. *Integrate to find the volume*:
$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\bigg]_0^3 = 9\text{ m}^3. \qquad \blacksquare$$

EXAMPLE 2 A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

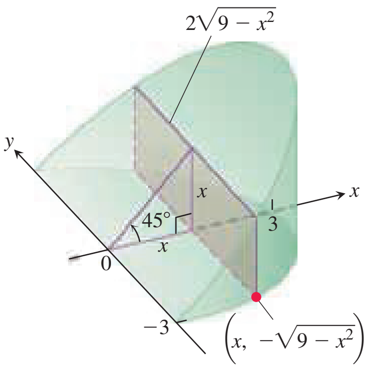


FIGURE 6.6 The wedge of Example 2, sliced perpendicular to the x-axis. The cross-sections are rectangles.

Solution We draw the wedge and sketch a typical cross-section perpendicular to the x-axis (Figure 6.6). The base of the wedge in the figure is the semicircle with $x \ge 0$ that is cut from the circle $x^2 + y^2 = 9$ by the 45° plane when it intersects the y-axis. For any x in the interval $[0, 3]$, the y-values in this semicircular base vary from $y = -\sqrt{9 - x^2}$ to $y = \sqrt{9 - x^2}$. When we slice through the wedge by a plane perpendicular to the x-axis, we obtain a cross-section at x which is a rectangle of height x whose width extends across the semicircular base. The area of this cross-section is

$$\begin{aligned} A(x) &= (\text{height})(\text{width}) = (x)\left(2\sqrt{9 - x^2}\right) \\ &= 2x\sqrt{9 - x^2}. \end{aligned}$$

The rectangles run from $x = 0$ to $x = 3$, so we have

$$\begin{aligned} V &= \int_a^b A(x)\,dx = \int_0^3 2x\sqrt{9 - x^2}\,dx \\ &= -\frac{2}{3}(9 - x^2)^{3/2}\bigg]_0^3 \\ &= 0 + \frac{2}{3}(9)^{3/2} \\ &= 18. \qquad \blacksquare \end{aligned}$$

Let $u = 9 - x^2$, $du = -2x\,dx$, integrate, and substitute back.

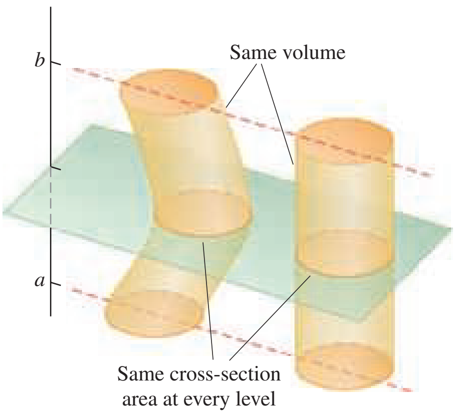


FIGURE 6.7 *Cavalieri's principle:* These solids have the same volume (imagine each solid as a stack of coins).

HISTORICAL BIOGRAPHY
Bonaventura Cavalieri
(1598–1647)
www.goo.gl/kPgDdQ

EXAMPLE 3 Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume (Figure 6.7). This follows immediately from the definition of volume, because the cross-sectional area function $A(x)$ and the interval $[a, b]$ are the same for both solids. $\blacksquare$

Solids of Revolution: The Disk Method

The solid generated by rotating (or revolving) a planar region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we first observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, where $R(x)$ is the distance from the axis of revolution to the planar region's boundary. The area is then

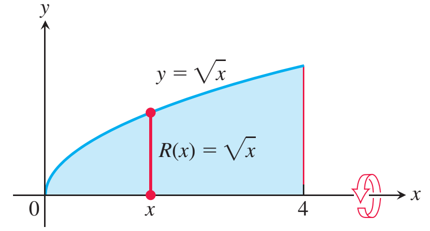
$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

Therefore, the definition of volume gives us the following formula.

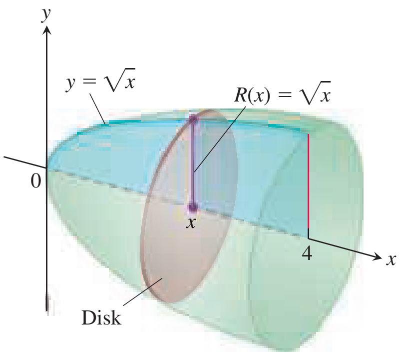
Volume by Disks for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\,[R(x)]^2\,dx.$$

This method for calculating the volume of a solid of revolution is often called the **disk method** because a cross-section is a circular disk of radius $R(x)$.



(a)



(b)

FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

EXAMPLE 4 The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.8). The volume is

$$\begin{aligned} V &= \int_a^b \pi\,[R(x)]^2\,dx \\ &= \int_0^4 \pi\,[\sqrt{x}]^2\,dx \qquad \text{Radius } R(x) = \sqrt{x} \text{ for rotation around } x\text{-axis.} \\ &= \pi \int_0^4 x\,dx = \pi \frac{x^2}{2}\bigg]_0^4 = \pi \frac{(4)^2}{2} = 8\pi. \end{aligned}$$

EXAMPLE 5 The circle

$$x^2 + y^2 = a^2$$

is rotated about the x-axis to generate a sphere. Find its volume.

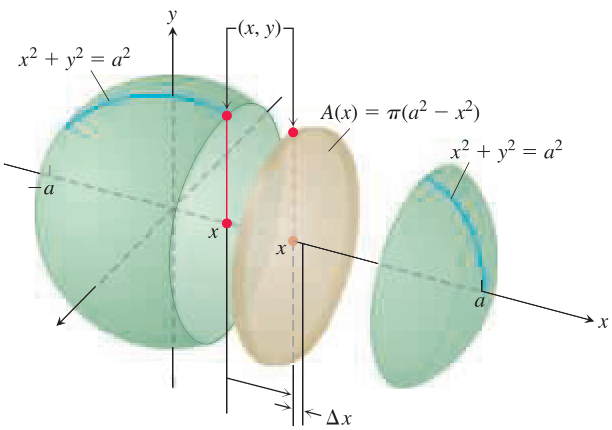


FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x-axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5).

Solution We imagine the sphere cut into thin slices by planes perpendicular to the x-axis (Figure 6.9). The cross-sectional area at a typical point x between $-a$ and a is

$$A(x) = \pi y^2 = \pi(a^2 - x^2). \qquad R(x) = \sqrt{a^2 - x^2} \text{ for rotation around } x\text{-axis.}$$

Therefore, the volume is

$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

The axis of revolution in the next example is not the x-axis, but the rule for calculating the volume is the same: Integrate $\pi(\text{radius})^2$ between appropriate limits.

EXAMPLE 6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.10). The volume is

$$\begin{aligned} V &= \int_1^4 \pi\left[R(x)\right]^2 dx \\ &= \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx && \text{Radius } R(x) = \sqrt{x} - 1 \text{ for rotation around } y = 1. \\ &= \pi\int_1^4 \left[x - 2\sqrt{x} + 1\right] dx && \text{Expand integrand.} \\ &= \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}. && \text{Integrate.} \end{aligned}$$

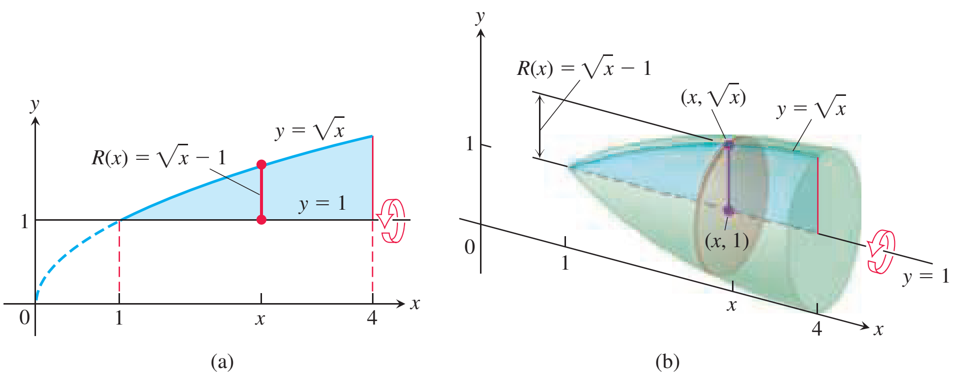


FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6.

To find the volume of a solid generated by revolving a region between the y-axis and a curve $x = R(y)$, $c \le y \le d$, about the y-axis, we use the same method with x replaced by y. In this case, the area of the circular cross-section is

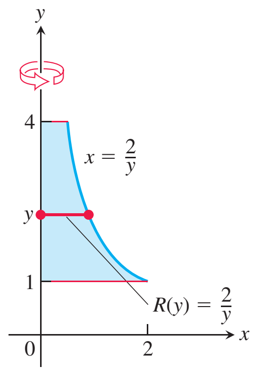
$$A(y) = \pi\left[\text{radius}\right]^2 = \pi\left[R(y)\right]^2,$$

and the definition of volume gives us the following formula.

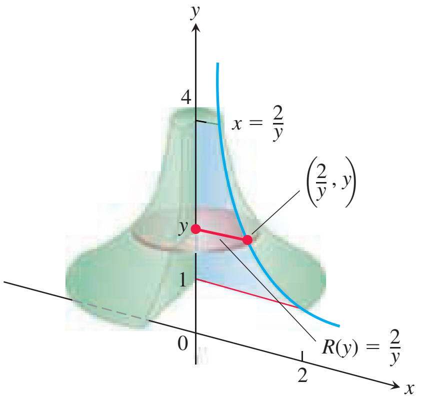
Volume by Disks for Rotation About the y-Axis

$$V = \int_c^d A(y)\,dy = \int_c^d \pi\left[R(y)\right]^2 dy.$$

EXAMPLE 7 Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.



(a)



(b)

FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.11). The volume is

$$V = \int_1^4 \pi[R(y)]^2\,dy$$

$$= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy \qquad \text{Radius } R(y) = \frac{2}{y} \text{ for rotation around } y\text{-axis}$$

$$= \pi\int_1^4 \frac{4}{y^2}\,dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right] = 3\pi.$$

EXAMPLE 8 Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.12). Note that the cross-sections are perpendicular to the line $x = 3$ and have y-coordinates from $y = -\sqrt{2}$ to $y = \sqrt{2}$. The volume is

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2\,dy \qquad y = \pm\sqrt{2} \text{ when } x = 3$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2\,dy \qquad \text{Radius } R(y) = 3 - (y^2 + 1) \text{ for rotation around axis } x = 3.$$

$$= \pi\int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4]\,dy \qquad \text{Expand integrand.}$$

$$= \pi\left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} \qquad \text{Integrate.}$$

$$= \frac{64\pi\sqrt{2}}{15}.$$

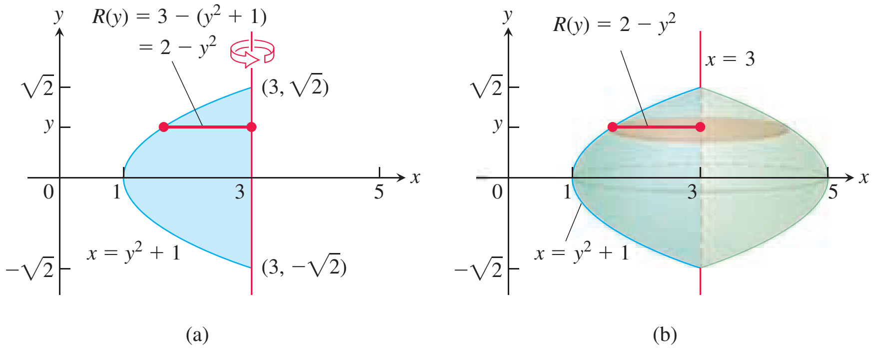


FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8.

Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, then the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are *washers* (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

Outer radius: $R(x)$

Inner radius: $r(x)$

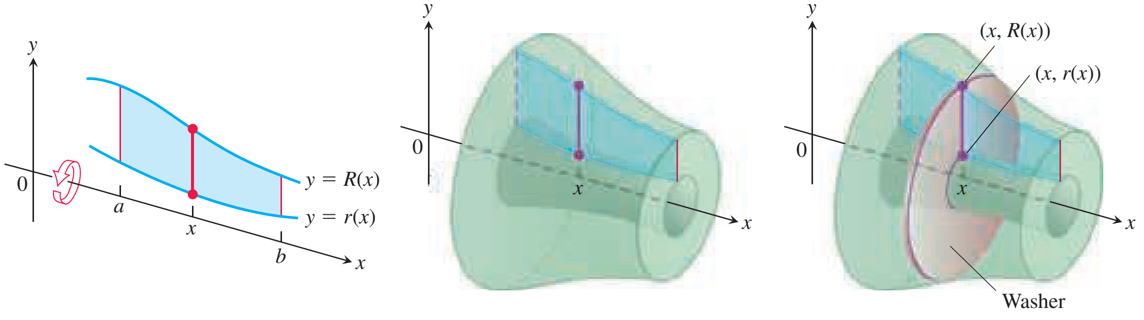
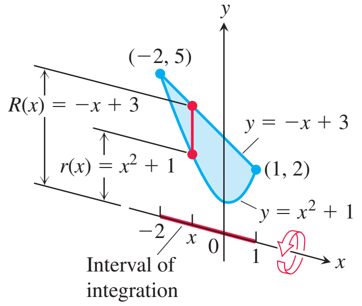


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

The washer's area is the area of a circle of radius $R(x)$ minus the area of a circle of radius $r(x)$:

$$A(x) = \pi\,[R(x)]^2 - \pi\,[r(x)]^2 = \pi\left([R(x)]^2 - [r(x)]^2\right).$$

Consequently, the definition of volume in this case gives us the following formula.

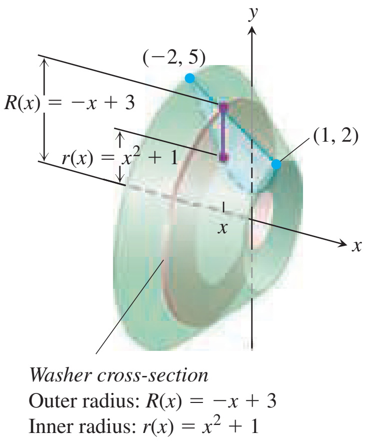


(a)

Volume by Washers for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx.$$

This method for calculating the volume of a solid of revolution is called the **washer method** because a thin slab of the solid resembles a circular washer with outer radius $R(x)$ and inner radius $r(x)$.



(b)

FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x-axis, the line segment generates a washer.

EXAMPLE 9 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution We use the same four steps for calculating the volume of a solid that were discussed earlier in this section.

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Figure 6.14a).
2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region.

   These radii are the distances of the ends of the line segment from the axis of revolution (see Figure 6.14).

$$\text{Outer radius:}\quad R(x) = -x + 3$$
$$\text{Inner radius:}\quad r(x) = x^2 + 1$$

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line in Figure 6.14a.

$$\begin{aligned} x^2 + 1 &= -x + 3 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \\ x = -2, \quad x &= 1 \qquad \text{Limits of integration} \end{aligned}$$

4. Evaluate the volume integral.

$$V = \int_a^b \pi\left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right) dx \qquad \text{Rotation around } x\text{-axis}$$

$$= \int_{-2}^{1} \pi\left((-x+3)^2 - (x^2+1)^2\right) dx \qquad \text{Values from Steps 2 and 3}$$

$$= \pi\int_{-2}^{1} (8 - 6x - x^2 - x^4)\, dx \qquad \text{Simplify algebraically.}$$

$$= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5} \qquad \text{Integrate.}$$

To find the volume of a solid formed by revolving a region about the y-axis, we use the same procedure as in Example 9, but integrate with respect to y instead of x. In this situation the line segment sweeping out a typical washer is perpendicular to the y-axis (the axis of revolution), and the outer and inner radii of the washer are functions of y.

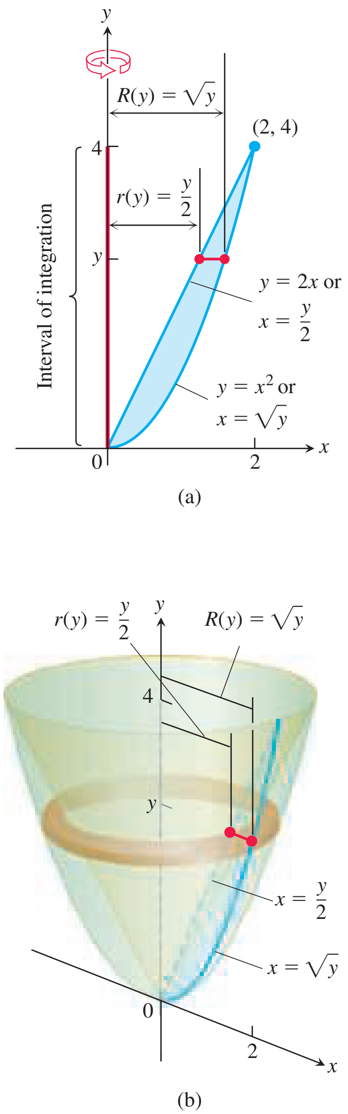


FIGURE 6.15 (a) The region being rotated about the y-axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

EXAMPLE 10 The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y-axis). See Figure 6.15a.

The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figure 6.15).

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$V = \int_c^d \pi\left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy \qquad \text{Rotation around } y\text{-axis}$$

$$= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right) dy \qquad \text{Substitute for radii and limits of integration.}$$

$$= \pi\int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi$$

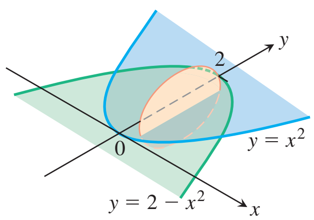
EXERCISES 6.1

Volumes by Slicing

Find the volumes of the solids in Exercises 1–10.

1. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

2. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

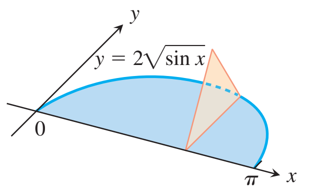


3. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

5. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross-sections perpendicular to the x-axis are

a. equilateral triangles with bases running from the x-axis to the curve as shown in the accompanying figure.



b. squares with bases running from the x-axis to the curve.

6. The solid lies between planes perpendicular to the x-axis at $x = -\pi/3$ and $x = \pi/3$. The cross-sections perpendicular to the x-axis are

a. circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.

b. squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$.

7. The base of a solid is the region bounded by the graphs of $y = 3x$, $y = 6$, and $x = 0$. The cross-sections perpendicular to the x-axis are

a. rectangles of height 10.

b. rectangles of perimeter 20.

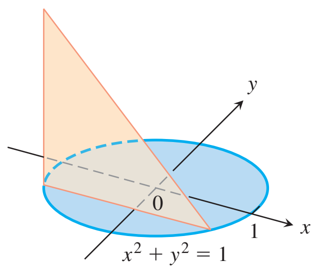
8. The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = x/2$. The cross-sections perpendicular to the x-axis are

a. isosceles triangles of height 6.

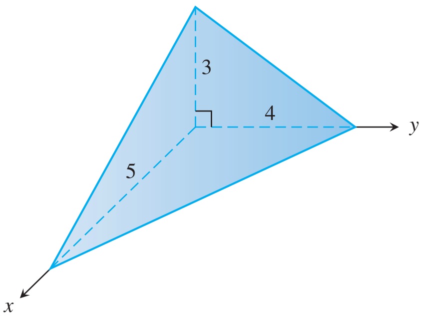
b. semicircles with diameters running across the base of the solid.

9. The solid lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross-sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$.

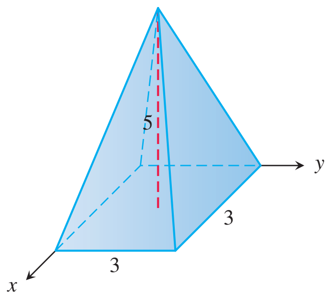
10. The base of the solid is the disk $x^2 + y^2 \le 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.



11. Find the volume of the given right tetrahedron. (*Hint:* Consider slices perpendicular to one of the labeled edges.)



12. Find the volume of the given pyramid, which has a square base of area 9 and height 5.

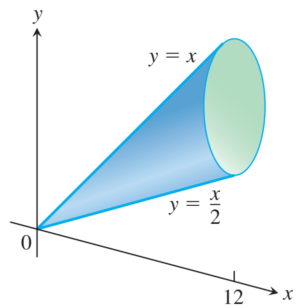


13. A twisted solid A square of side length s lies in a plane perpendicular to a line L. One vertex of the square lies on L. As this square moves a distance h along L, the square turns one revolution about L to generate a corkscrew-like column with square cross-sections.

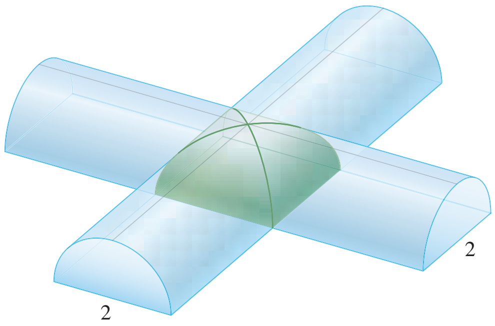
a. Find the volume of the column.

b. What will the volume be if the square turns twice instead of once? Give reasons for your answer.

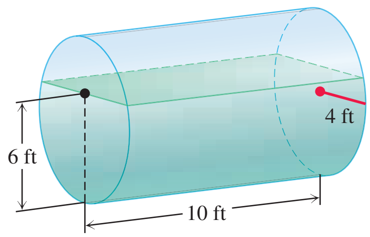
14. Cavalieri's principle A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 12$. The cross-sections by planes perpendicular to the x-axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$ as shown in the accompanying figure. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



15. Intersection of two half-cylinders Two half-cylinders of diameter 2 meet at a right angle in the accompanying figure. Find the volume of the solid region common to both half-cylinders. (*Hint*: Consider slices parallel to the base of the solid.)



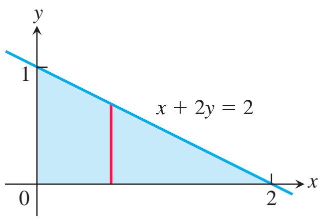
16. Gasoline in a tank A gasoline tank is in the shape of a right circular cylinder (lying on its side) of length 10 ft and radius 4 ft. Set up an integral that represents the volume of the gas in the tank if it is filled to a depth of 6 ft. You will learn how to compute this integral in Chapter 8 (or you may use geometry to find its value).



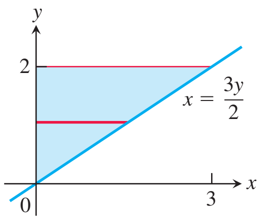
Volumes by the Disk Method

In Exercises 17–20, find the volume of the solid generated by revolving the shaded region about the given axis.

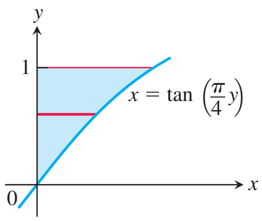
17. About the x-axis



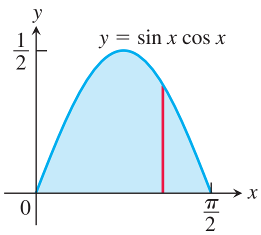
18. About the y-axis



19. About the y-axis



20. About the x-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 21–26 about the x-axis.

21. $y = x^2, \quad y = 0, \quad x = 2$

22. $y = x^3, \quad y = 0, \quad x = 2$

23. $y = \sqrt{9 - x^2}, \quad y = 0$

24. $y = x - x^2, \quad y = 0$

25. $y = \sqrt{\cos x}, \quad 0 \le x \le \pi/2, \quad y = 0, \quad x = 0$

26. $y = \sec x, \quad y = 0, \quad x = -\pi/4, \quad x = \pi/4$

In Exercises 27 and 28, find the volume of the solid generated by revolving the region about the given line.

27. The region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$

28. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x$, $0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 29–34 about the y-axis.

29. The region enclosed by $x = \sqrt{5}y^2, \quad x = 0, \quad y = -1, \quad y = 1$

30. The region enclosed by $x = y^{3/2}, \quad x = 0, \quad y = 2$

31. The region enclosed by $x = \sqrt{2 \sin 2y}, \quad 0 \le y \le \pi/2, \quad x = 0$

32. The region enclosed by $x = \sqrt{\cos(\pi y/4)}, \quad -2 \le y \le 0, \quad x = 0$

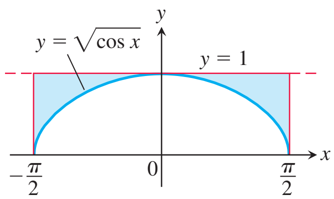
33. $x = 2/\sqrt{y + 1}, \quad x = 0, \quad y = 0, \quad y = 3$

34. $x = \sqrt{2}y/(y^2 + 1), \quad x = 0, \quad y = 1$

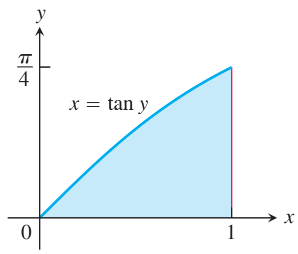
Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 35 and 36 about the indicated axes.

35. The x-axis



36. The y-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 37–42 about the x-axis.

37. $y = x, \quad y = 1, \quad x = 0$

38. $y = 2\sqrt{x}, \quad y = 2, \quad x = 0$

39. $y = x^2 + 1, \quad y = x + 3$

40. $y = 4 - x^2, \quad y = 2 - x$

41. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$

42. $y = \sec x, \quad y = \tan x, \quad x = 0, \quad x = 1$

In Exercises 43–46, find the volume of the solid generated by revolving each region about the y-axis.

43. The region enclosed by the triangle with vertices (1, 0), (2, 1), and (1, 1)

44. The region enclosed by the triangle with vertices (0, 1), (1, 0), and (1, 1)

45. The region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis, and on the right by the line $x = 2$

46. The region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$

In Exercises 47 and 48, find the volume of the solid generated by revolving each region about the given axis.

47. The region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis, and on the right by the line $x = 1$, about the line $x = -1$

48. The region in the second quadrant bounded above by the curve $y = -x^3$, below by the x-axis, and on the left by the line $x = -1$, about the line $x = -2$

Volumes of Solids of Revolution

49. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

a. the x-axis. **b.** the y-axis.

c. the line $y = 2$. **d.** the line $x = 4$.

50. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

a. the line $x = 1$. **b.** the line $x = 2$.

51. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

a. the line $y = 1$. **b.** the line $y = 2$.

c. the line $y = -1$.

52. By integration, find the volume of the solid generated by revolving the triangular region with vertices (0, 0), (b, 0), (0, h) about

a. the x-axis. **b.** the y-axis.

Theory and Applications

53. The volume of a torus The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b$ $(b > a)$ to generate a solid shaped like a doughnut and called a *torus*. Find its volume. (*Hint:* $\int_{-a}^{a} \sqrt{a^2 - y^2}\, dy = \pi a^2/2$, since it is the area of a semicircle of radius a.)

54. Volume of a bowl A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y-axis.

a. Find the volume of the bowl.

b. Related rates If we fill the bowl with water at a constant rate of 3 cubic units per second, how fast will the water level in the bowl be rising when the water is 4 units deep?

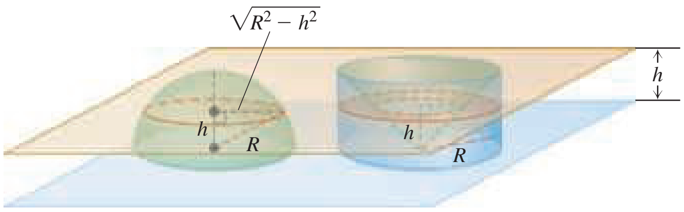
55. Volume of a bowl

a. A hemispherical bowl of radius a contains water to a depth h. Find the volume of water in the bowl.

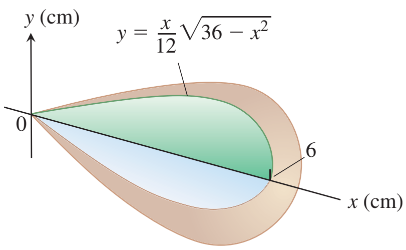
b. Related rates Water runs into a sunken concrete hemispherical bowl of radius 5 m at the rate of 0.2 m^3/sec. How fast is the water level in the bowl rising when the water is 4 m deep?

56. Explain how you could estimate the volume of a solid of revolution by measuring the shadow cast on a table parallel to its axis of revolution by a light shining directly above it.

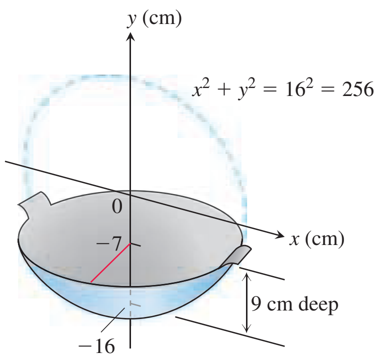
57. Volume of a hemisphere Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross-sections with the cross-sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed, as suggested by the accompanying figure.



58. Designing a plumb bob Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here. Find the plumb bob's volume. If you specify a brass that weighs 8.5 g/cm^3, how much will the plumb bob weigh (to the nearest gram)?



59. Designing a wok You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that holds about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? (1 L = 1000 cm^3)

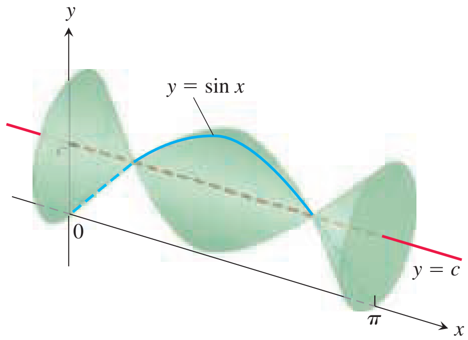


60. Max-min The arch $y = \sin x$, $0 \le x \le \pi$, is revolved about the line $y = c$, $0 \le c \le 1$, to generate the solid in the accompanying figure.

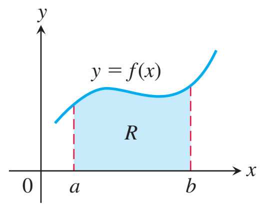
a. Find the value of c that minimizes the volume of the solid. What is the minimum volume?

b. What value of c in $[0, 1]$ maximizes the volume of the solid?

T **c.** Graph the solid's volume as a function of c, first for $0 \le c \le 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from $[0, 1]$? Does this make sense physically? Give reasons for your answers.



61. Consider the region R bounded by the graphs of $y = f(x) > 0$, $x = a > 0$, $x = b > a$, and $y = 0$ (see accompanying figure). If the volume of the solid formed by revolving R about the x-axis is 4π, and the volume of the solid formed by revolving R about the line $y = -1$ is 8π, find the area of R.



62. Consider the region R given in Exercise 61. If the volume of the solid formed by revolving R around the x-axis is 6π, and the volume of the solid formed by revolving R around the line $y = -2$ is 10π, find the area of R.

6.2 Volumes Using Cylindrical Shells

In Section 6.1 we defined the volume of a solid to be the definite integral $V = \int_a^b A(x)\,dx$, where $A(x)$ is an integrable cross-sectional area of the solid from $x = a$ to $x = b$. The area $A(x)$ was obtained by slicing through the solid with a plane perpendicular to the x-axis. However, this method of slicing is sometimes awkward to apply, as we will illustrate in our first example. To overcome this difficulty, we use the same integral definition for volume, but obtain the area by slicing through the solid in a different way.

Slicing with Cylinders

Suppose we slice through the solid using circular cylinders of increasing radii, like cookie cutters. We slice straight down through the solid so that the axis of each cylinder is parallel to the y-axis. The vertical axis of each cylinder is always the same line, but the radii of the cylinders increase with each slice. In this way the solid is sliced up into thin cylindrical shells of constant thickness that grow outward from their common axis, like circular tree rings. Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx. This slab interpretation allows us to apply the same integral definition for volume as before. The following example provides some insight.

EXAMPLE 1 The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid (see Figure 6.16). Find the volume of the solid.

Solution Using the washer method from Section 6.1 would be awkward here because we would need to express the x-values of the left and right sides of the parabola in Figure 6.16a in terms of y. (These x-values are the inner and outer radii for a typical washer, requiring us to solve $y = 3x - x^2$ for x, which leads to a complicated formula for x.) Instead of rotating a horizontal strip of thickness Δy, we rotate a *vertical strip* of thickness Δx. This rotation produces a *cylindrical shell* of height y_k above a point x_k within the base of the vertical strip and of thickness Δx. An example of a cylindrical shell is shown as the orange-shaded region in Figure 6.17. We can think of the cylindrical shell shown in the figure as approximating a slice of the solid obtained by cutting straight down through it, parallel to the axis of revolution, all the way around close to the inside hole. We then cut

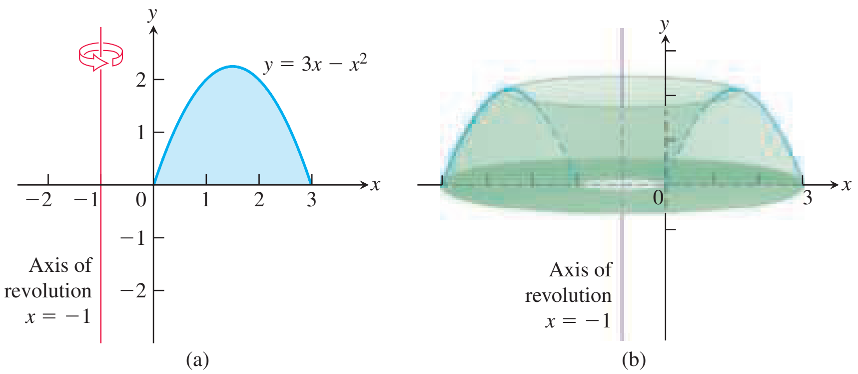


FIGURE 6.16 (a) The graph of the region in Example 1, before revolution. (b) The solid formed when the region in part (a) is revolved about the axis of revolution $x = -1$.

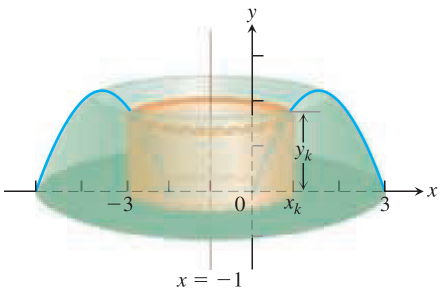


FIGURE 6.17 A cylindrical shell of height y_k obtained by rotating a vertical strip of thickness Δx_k about the line $x = -1$. The outer radius of the cylinder occurs at x_k, where the height of the parabola is $y_k = 3x_k - x_k^2$ (Example 1).

another cylindrical slice around the enlarged hole, then another, and so on, obtaining n cylinders. The radii of the cylinders gradually increase, and the heights of the cylinders follow the contour of the parabola: shorter to taller, then back to shorter (Figure 6.16a). The sum of the volumes of the shells is a Riemann sum that approximates the volume of the entire solid.

Each shell sits over a subinterval $[x_{k-1}, x_k]$ in the x-axis. The thickness of the shell is $\Delta x_k = x_k - x_{k-1}$. Because the parabola is rotated around the line $x = -1$, the outer radius of the shell is $1 + x_k$. The height of the shell is the height of the parabola at some point in the interval $[x_{k-1}, x_k]$, or approximately $y_k = f(x_k) = 3x_k - x_k^2$. If we unroll this cylinder and flatten it out, it becomes (approximately) a rectangular slab with thickness Δx_k (see Figure 6.18). The height of the rectangular slab is approximately $y_k = 3x_k - x_k^2$, and its length is the circumference of the shell, which is approximately $2\pi \cdot \text{radius} = 2\pi(1 + x_k)$. Hence the volume of the shell is approximately the volume of the rectangular slab, which is

$$\begin{aligned}\Delta V_k &= \text{circumference} \times \text{height} \times \text{thickness} \\ &= 2\pi(1 + x_k) \cdot \left(3x_k - x_k^2\right) \cdot \Delta x_k.\end{aligned}$$

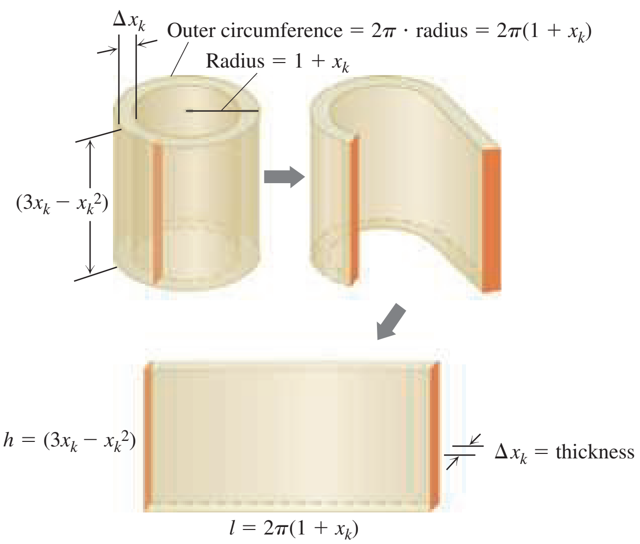


FIGURE 6.18 Cutting and unrolling a cylindrical shell gives a nearly rectangular solid (Example 1).

Summing together the volumes ΔV_k of the individual cylindrical shells over the interval $[0, 3]$ gives the Riemann sum

$$\sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi(x_k + 1)\left(3x_k - x_k{}^2\right)\Delta x_k.$$

Taking the limit as the thickness $\Delta x_k \to 0$ and $n \to \infty$ gives the volume integral

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi(x_k + 1)\left(3x_k - x_k^2\right) \Delta x_k \\ &= \int_0^3 2\pi(x + 1)(3x - x^2)\, dx \\ &= \int_0^3 2\pi(3x^2 + 3x - x^3 - x^2)\, dx \\ &= 2\pi\int_0^3 (2x^2 + 3x - x^3)\, dx \\ &= 2\pi\left[\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{4}x^4\right]_0^3 = \frac{45\pi}{2}. \end{aligned}$$

We now generalize this procedure to more general solids.

The Shell Method

Suppose that the region bounded by the graph of a nonnegative continuous function $y = f(x)$ and the x-axis over the finite closed interval $[a, b]$ lies to the right of the vertical line $x = L$ (see Figure 6.19a). We assume $a \geq L$, so the vertical line may touch the region but cannot pass through it. We generate a solid S by rotating this region about the vertical line L.

Let P be a partition of the interval $[a, b]$ by the points $a = x_0 < x_1 < \cdots < x_n = b$. As usual, we choose a point c_k in each subinterval $[x_{k-1}, x_k]$. In Example 1 we chose c_k to be the endpoint x_k, but now it will be more convenient to let c_k be the midpoint of the subinterval $[x_{k-1}, x_k]$. We approximate the region in Figure 6.19a with rectangles based on this partition of $[a, b]$. A typical approximating rectangle has height $f(c_k)$ and width

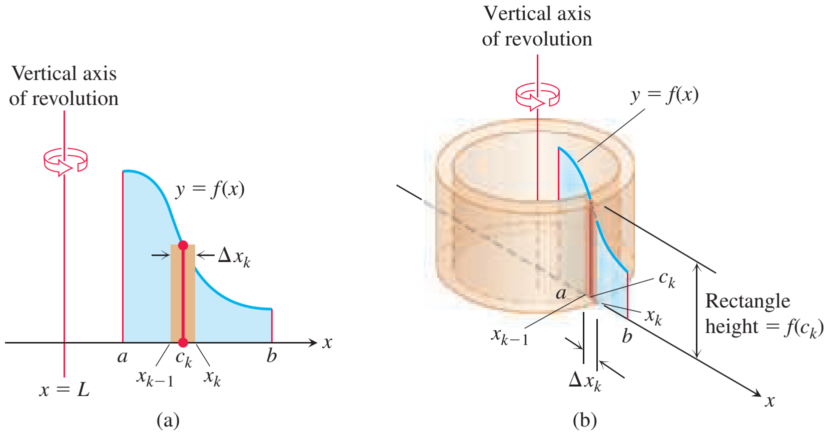


FIGURE 6.19 When the region shown in (a) is revolved about the vertical line $x = L$, a solid is produced which can be sliced into cylindrical shells. A typical shell is shown in (b).

The volume of a cylindrical shell of height h with inner radius r and outer radius R is

$$\pi R^2 h - \pi r^2 h = 2\pi\left(\frac{R + r}{2}\right)(h)(R - r).$$

$\Delta x_k = x_k - x_{k-1}$. If this rectangle is rotated about the vertical line $x = L$, then a shell is swept out, as in Figure 6.19b. A formula from geometry tells us that the volume of the shell swept out by the rectangle is

$$\begin{aligned}\Delta V_k &= 2\pi \times \text{average shell radius} \times \text{shell height} \times \text{thickness}\\ &= 2\pi \cdot (c_k - L) \cdot f(c_k) \cdot \Delta x_k. \qquad R = x_k - L \text{ and } r = x_{k-1} - L\end{aligned}$$

We approximate the volume of the solid S by summing the volumes of the shells swept out by the n rectangles:

$$V \approx \sum_{k=1}^{n} \Delta V_k.$$

The limit of this Riemann sum as each $\Delta x_k \to 0$ and $n \to \infty$ gives the volume of the solid as a definite integral:

$$\begin{aligned}V = \lim_{n\to\infty} \sum_{k-1}^{n} \Delta V_k &= \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx\\ &= \int_a^b 2\pi(x - L)f(x)\, dx.\end{aligned}$$

We refer to the variable of integration, here x, as the **thickness variable**. To emphasize the *process* of the shell method, we state the general formula in terms of the shell radius and shell height. This will allow for rotations about a horizontal line L as well.

Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \ge 0, L \le a \le x \le b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi\binom{\text{shell}}{\text{radius}}\binom{\text{shell}}{\text{height}}\, dx.$$

EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.20a). Label the segment's height (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.20b, but you need not do that.)

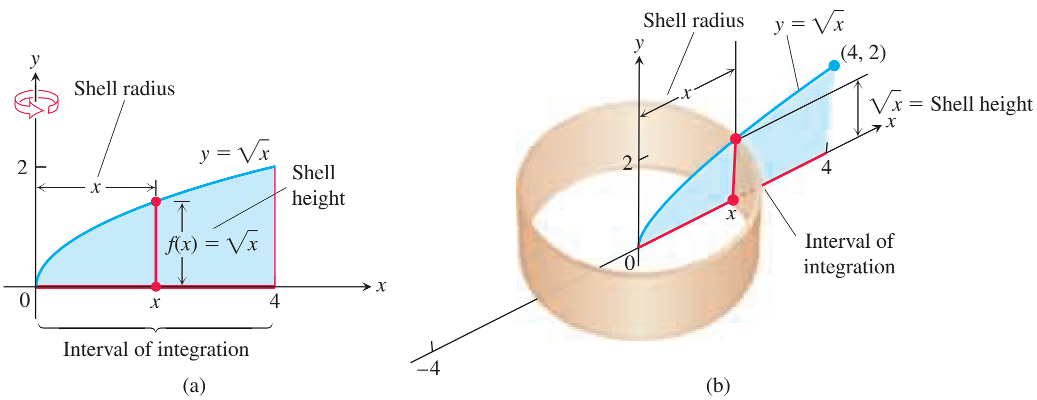


FIGURE 6.20 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx.

The shell thickness variable is x, so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is

$$\begin{aligned} V &= \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx \\ &= \int_0^4 2\pi(x)\left(\sqrt{x}\right) dx \\ &= 2\pi\int_0^4 x^{3/2}\, dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}. \end{aligned}$$

So far, we have used vertical axes of revolution. For horizontal axes, we replace the x's with y's.

EXAMPLE 3 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid by the shell method.

Solution This is the solid whose volume was found by the disk method in Example 4 of Section 6.1. Now we find its volume by the shell method. First, sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.21a). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.21b, but you need not do that.)

In this case, the shell thickness variable is y, so the limits of integration for the shell formula method are $a = 0$ and $b = 2$ (along the y-axis in Figure 6.21). The volume of the solid is

$$\begin{aligned} V &= \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy \\ &= \int_0^2 2\pi(y)(4 - y^2)\, dy \\ &= 2\pi\int_0^2 (4y - y^3)\, dy \\ &= 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi. \end{aligned}$$

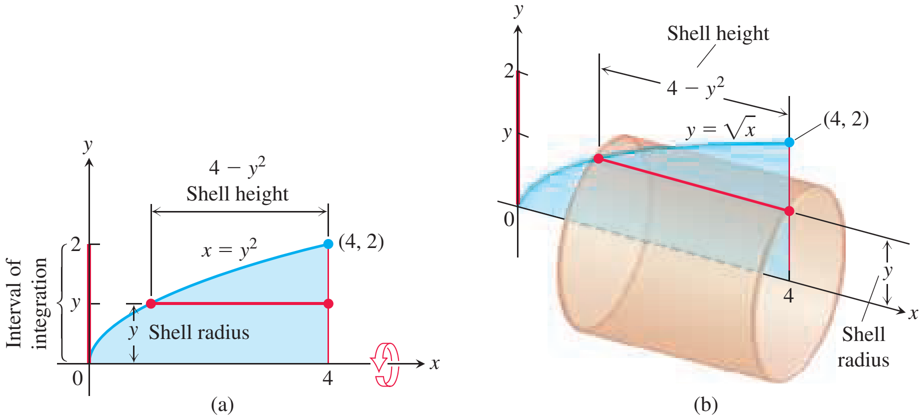


FIGURE 6.21 (a) The region, shell dimensions, and interval of integration in Example 3. (b) The shell swept out by the horizontal segment in part (a) with a width Δy.

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find the limits of integration* for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

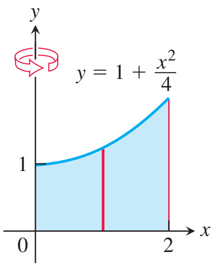
The shell method gives the same answer as the washer method when both are used to calculate the volume of a region. We do not prove that result here, but it is illustrated in Exercises 37 and 38. Both volume formulas are actually special cases of a general volume formula we will look at when studying double and triple integrals in Chapter 15. That general formula also allows for computing volumes of solids other than those swept out by regions of revolution.

EXERCISES 6.2

Revolution About the Axes

In Exercises 1–6, use the shell method to find the volumes of the solids generated by revolving the shaded region about the indicated axis.

1.



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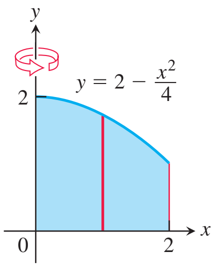
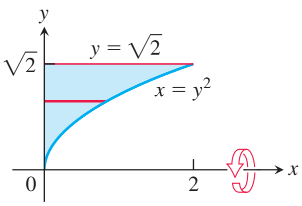
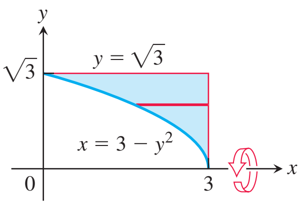


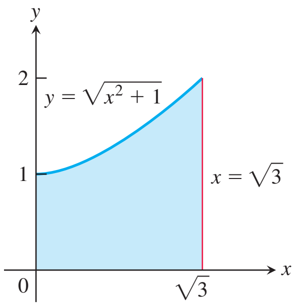
3.



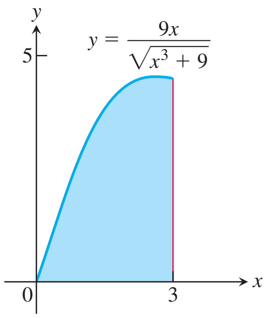
4.



5. The y-axis



6. The y-axis



Revolution About the y-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–12 about the y-axis.

7. $y = x, \quad y = -x/2, \quad x = 2$

8. $y = 2x, \quad y = x/2, \quad x = 1$

9. $y = x^2, \quad y = 2 - x, \quad x = 0, \quad$ for $x \geq 0$

10. $y = 2 - x^2, \quad y = x^2, \quad x = 0$

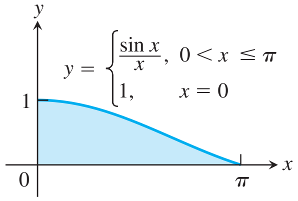
11. $y = 2x - 1, \quad y = \sqrt{x}, \quad x = 0$

12. $y = 3/\left(2\sqrt{x}\right), \quad y = 0, \quad x = 1, \quad x = 4$

13. Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \leq \pi \\ 1, & x = 0 \end{cases}$

a. Show that $x f(x) = \sin x$, $0 \leq x \leq \pi$.

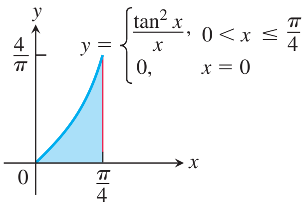
b. Find the volume of the solid generated by revolving the shaded region about the y-axis in the accompanying figure.



14. Let $g(x) = \begin{cases} (\tan x)^2/x, & 0 < x \leq \pi/4 \\ 0, & x = 0 \end{cases}$

a. Show that $x\, g(x) = (\tan x)^2$, $0 \leq x \leq \pi/4$.

b. Find the volume of the solid generated by revolving the shaded region about the y-axis in the accompanying figure.



Revolution About the x-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x-axis.

15. $x = \sqrt{y}, \quad x = -y, \quad y = 2$

16. $x = y^2, \quad x = -y, \quad y = 2, \quad y \geq 0$

17. $x = 2y - y^2, \quad x = 0$

18. $x = 2y - y^2, \quad x = y$

19. $y = |x|, \quad y = 1$

20. $y = x, \quad y = 2x, \quad y = 2$

21. $y = \sqrt{x}, \quad y = 0, \quad y = x - 2$

22. $y = \sqrt{x}, \quad y = 0, \quad y = 2 - x$

Revolution About Horizontal and Vertical Lines

In Exercises 23–26, use the shell method to find the volumes of the solids generated by revolving the regions bounded by the given curves about the given lines.

23. $y = 3x, \quad y = 0, \quad x = 2$

a. The y-axis **b.** The line $x = 4$

c. The line $x = -1$ **d.** The x-axis

e. The line $y = 7$ **f.** The line $y = -2$

24. $y = x^3, \quad y = 8, \quad x = 0$

a. The y-axis **b.** The line $x = 3$

c. The line $x = -2$ **d.** The x-axis

e. The line $y = 8$ **f.** The line $y = -1$

25. $y = x + 2, \quad y = x^2$

a. The line $x = 2$ **b.** The line $x = -1$

c. The x-axis **d.** The line $y = 4$

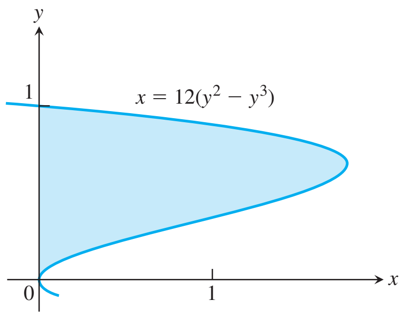
26. $y = x^4, \quad y = 4 - 3x^2$

a. The line $x = 1$ **b.** The x-axis

In Exercises 27 and 28, use the shell method to find the volumes of the solids generated by revolving the shaded regions about the indicated axes.

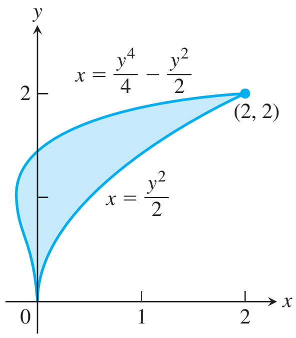
27. a. The x-axis **b.** The line $y = 1$

c. The line $y = 8/5$ **d.** The line $y = -2/5$



28. a. The x-axis **b.** The line $y = 2$

c. The line $y = 5$ **d.** The line $y = -5/8$



Choosing the Washer Method or Shell Method

For some regions, both the washer and shell methods work well for the solid generated by revolving the region about the coordinate axes, but this is not always the case. When a region is revolved about the y-axis, for example, and washers are used, we must integrate with respect to y. It may not be possible, however, to express the integrand in terms of y. In such a case, the shell method allows us to integrate with respect to x instead. Exercises 29 and 30 provide some insight.

29. Compute the volume of the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about each coordinate axis using

a. the shell method. **b.** the washer method.

30. Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4$, $y = x$, and $x = 0$ about

a. the x-axis using the washer method.

b. the y-axis using the shell method.

c. the line $x = 4$ using the shell method.

d. the line $y = 8$ using the washer method.

In Exercises 31–36, find the volumes of the solids generated by revolving the regions about the given axes. If you think it would be better to use washers in any given instance, feel free to do so.

31. The triangle with vertices (1, 1), (1, 2), and (2, 2) about

a. the x-axis **b.** the y-axis

c. the line $x = 10/3$ **d.** the line $y = 1$

32. The region bounded by $y = \sqrt{x}$, $y = 2$, $x = 0$ about

a. the x-axis **b.** the y-axis

c. the line $x = 4$ **d.** the line $y = 2$

33. The region in the first quadrant bounded by the curve $x = y - y^3$ and the y-axis about

a. the x-axis **b.** the line $y = 1$

34. The region in the first quadrant bounded by $x = y - y^3$, $x = 1$, and $y = 1$ about

a. the x-axis **b.** the y-axis

c. the line $x = 1$ **d.** the line $y = 1$

35. The region bounded by $y = \sqrt{x}$ and $y = x^2/8$ about

a. the x-axis **b.** the y-axis

36. The region bounded by $y = 2x - x^2$ and $y = x$ about

a. the y-axis **b.** the line $x = 1$

37. The region in the first quadrant that is bounded above by the curve $y = 1/x^{1/4}$, on the left by the line $x = 1/16$, and below by the line $y = 1$ is revolved about the x-axis to generate a solid. Find the volume of the solid by

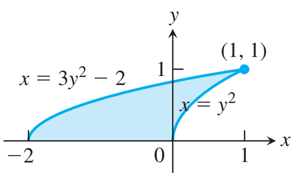
a. the washer method. **b.** the shell method.

38. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $x = 1/4$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by

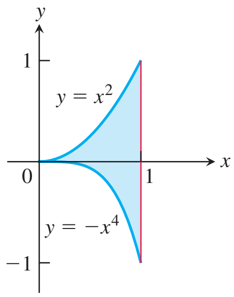
a. the washer method. **b.** the shell method.

Theory and Examples

39. The region shown here is to be revolved about the x-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Explain.



40. The region shown here is to be revolved about the y-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Give reasons for your answers.



41. A bead is formed from a sphere of radius 5 by drilling through a diameter of the sphere with a drill bit of radius 3.

a. Find the volume of the bead.

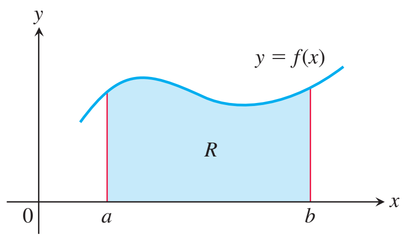
b. Find the volume of the removed portion of the sphere.

42. A Bundt cake, well known for having a ringed shape, is formed by revolving around the y-axis the region bounded by the graph of $y = \sin(x^2 - 1)$ and the x-axis over the interval $1 \le x \le \sqrt{1 + \pi}$. Find the volume of the cake.

43. Derive the formula for the volume of a right circular cone of height h and radius r using an appropriate solid of revolution.

44. Derive the equation for the volume of a sphere of radius r using the shell method.

45. Consider the region R bounded by the graphs of $y = f(x) > 0$, $x = a > 0$, and $x = b > a$. If the volume of the solid formed by revolving R about the y-axis is 2π, and the volume formed by revolving R about the line $x = -2$ is 10π, find the area of R.



46. Consider the region R given in Exercise 45. If the area of region R is 1, and the volume of the solid formed by revolving R about the line $x = -3$ is 10π, find the volume of the solid formed by revolving R about the y-axis.

6.3 Arc Length

We know what is meant by the length of a straight-line segment, but without calculus, we have no precise definition of the length of a general winding curve. If the curve is the graph of a continuous function defined over an interval, then we can find the length of the curve using a procedure similar to that we used for defining the area between the curve and the x-axis. We divide the curve into many pieces, and we approximate each piece by a straight-line segment. The sum of the lengths of these segments is an approximation to the total curve length that we seek. The total length of the curve is the limiting value of these approximations as the number of segments goes to infinity.

Length of a Curve $y = f(x)$

Suppose the curve whose length we want to find is the graph of the function $y = f(x)$ from $x = a$ to $x = b$. In order to derive an integral formula for the length of the curve, we assume that f has a continuous derivative at every point of $[a, b]$. Such a function is called **smooth**, and its graph is a **smooth curve** because it does not have any breaks, corners, or cusps.

We partition the interval $[a, b]$ into n subintervals with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. If $y_k = f(x_k)$, then the corresponding point $P_k(x_k, y_k)$ lies on the curve. Next we connect successive points P_{k-1} and P_k with straight-line segments that, taken together, form a polygonal path whose length approximates the length of the curve (Figure 6.22). If we set $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$, then a representative line segment in the path has length (see Figure 6.23)

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2},$$

so the length of the curve is approximated by the sum

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \tag{1}$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer. In order to evaluate this limit, we use the Mean Value Theorem, which tells us that there is a point c_k, with $x_{k-1} < c_k < x_k$, such that

$$\Delta y_k = f'(c_k)\, \Delta x_k.$$

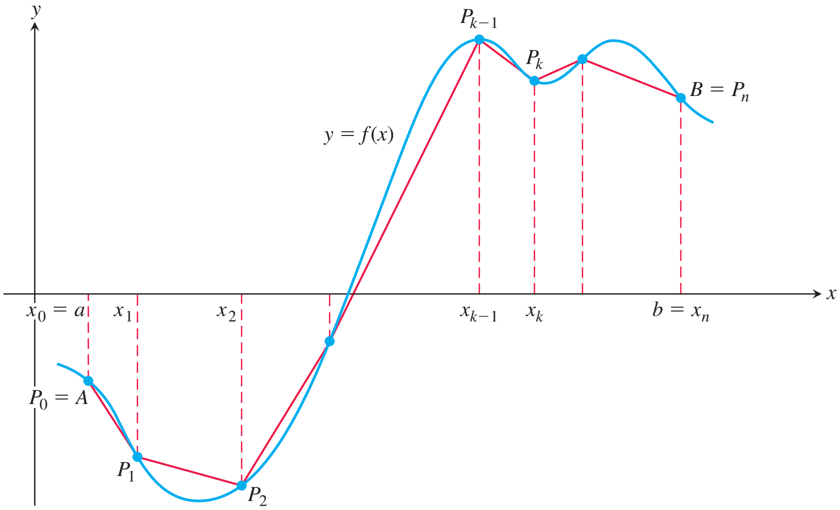


FIGURE 6.22 The length of the polygonal path $P_0P_1P_2 \cdots P_n$ approximates the length of the curve $y = f(x)$ from point A to point B.

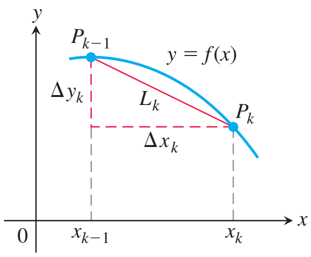


FIGURE 6.23 The arc $P_{k-1}P_k$ of the curve $y = f(x)$ is approximated by the straight-line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

Substituting this for Δy_k, the sums in Equation (1) take the form

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2} = \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\, \Delta x_k. \tag{2}$$

This is a Riemann sum whose limit we can evaluate. Because $\sqrt{1 + [f'(x)]^2}$ is continuous on $[a, b]$, the limit of the Riemann sum on the right-hand side of Equation (2) exists and has the value

$$\lim_{n\to\infty} \sum_{k=1}^{n} L_k = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\, \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We define the length of the curve to be this integral.

> **DEFINITION** If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral
>
> $$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{3}$$

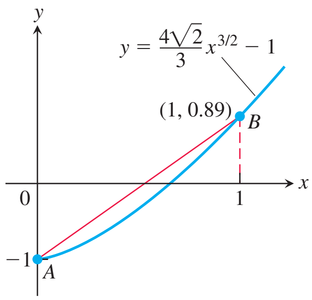


FIGURE 6.24 The length of the curve is slightly larger than the length of the line segment joining points A and B (Example 1).

EXAMPLE 1 Find the length of the curve shown in Figure 6.24, which is the graph of the function

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution We use Equation (3) with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \qquad \text{If } x = 1, \text{ then } y \approx 0.89$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.$$

The length of the curve over $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx \qquad \text{Eq. (3) with } a = 0,\ b = 1.$$

$$= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6} \approx 2.17. \qquad \text{Let } u = 1 + 8x, \text{ integrate, and replace } u \text{ by } 1 + 8x.$$

Notice that the length of the curve is slightly larger than the length of the straight-line segment joining the points $A = (0, -1)$ and $B = \left(1, 4\sqrt{2}/3 - 1\right)$ on the curve (see Figure 6.24):

$$2.17 > \sqrt{1^2 + (1.89)^2} \approx 2.14. \qquad \text{Decimal approximations}$$

■

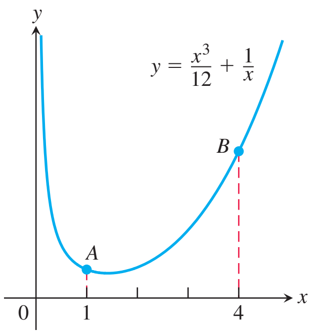


FIGURE 6.25 The curve in Example 2, where $A = (1, 13/12)$ and $B = (4, 67/12)$.

EXAMPLE 2 Find the length of the graph of

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \qquad 1 \le x \le 4.$$

Solution A graph of the function is shown in Figure 6.25. To use Equation (3), we find

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

so

$$1+[f'(x)]^2 = 1+\left(\frac{x^2}{4}-\frac{1}{x^2}\right)^2 = 1+\left(\frac{x^4}{16}-\frac{1}{2}+\frac{1}{x^4}\right)$$
$$= \frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4} = \left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2.$$

The length of the graph over $[1, 4]$ is

$$L = \int_1^4 \sqrt{1+[f'(x)]^2}\,dx = \int_1^4 \left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx$$
$$= \left[\frac{x^3}{12}-\frac{1}{x}\right]_1^4 = \left(\frac{64}{12}-\frac{1}{4}\right)-\left(\frac{1}{12}-1\right) = \frac{72}{12} = 6.$$

Dealing with Discontinuities in dy/dx

Even if the derivative dy/dx does not exist at some point on a curve, it is possible that dx/dy could exist. This can happen, for example, when a curve has a vertical tangent. In this case, we may be able to find the curve's length by expressing x as a function of y and applying the following analogue of Equation (3):

Formula for the Length of $x = g(y), c \le y \le d$

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d \sqrt{1+[g'(y)]^2}\,dy. \tag{4}$$

EXAMPLE 3 Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Equation (3).

We therefore rewrite the equation to express x in terms of y:

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2} \qquad \text{Raise both sides to the power } 3/2.$$

$$x = 2y^{3/2}. \qquad \text{Solve for } x.$$

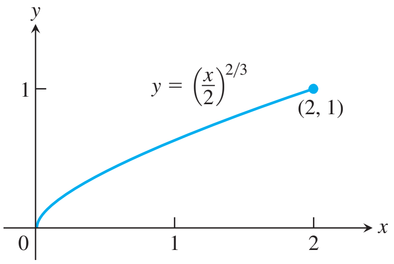


FIGURE 6.26 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 3).

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (see Figure 6.26).

The derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

is continuous on $[0, 1]$. We may therefore use Equation (4) to find the curve's length:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy \qquad \text{Eq. (4) with } c = 0,\ d = 1.$$

$$= \frac{1}{9} \cdot \frac{2}{3}(1 + 9y)^{3/2} \Big]_0^1 \qquad \text{Let } u = 1 + 9y,\ du/9 = dy, \text{ integrate, and substitute back.}$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \approx 2.27.$$

The Differential Formula for Arc Length

If $y = f(x)$ and if f' is continuous on $[a, b]$, then by the Fundamental Theorem of Calculus we can define a new function

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt. \tag{5}$$

From Equation (3) and Figure 6.22, we see that this function $s(x)$ is continuous and measures the length along the curve $y = f(x)$ from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$ for each $x \in [a, b]$. The function s is called the **arc length function** for $y = f(x)$. From the Fundamental Theorem, the function s is differentiable on (a, b) and

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Then the differential of arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{6}$$

A useful way to remember Equation (6) is to write

$$ds = \sqrt{dx^2 + dy^2}, \tag{7}$$

which can be integrated between appropriate limits to give the total length of a curve. From this point of view, all the arc length formulas are simply different expressions for the equation $L = \int ds$. Figure 6.27a gives the exact interpretation of ds corresponding to Equation (7). Figure 6.27b is not strictly accurate, but it can be thought of as a simplified approximation of Figure 6.27a. That is, $ds \approx \Delta s$.

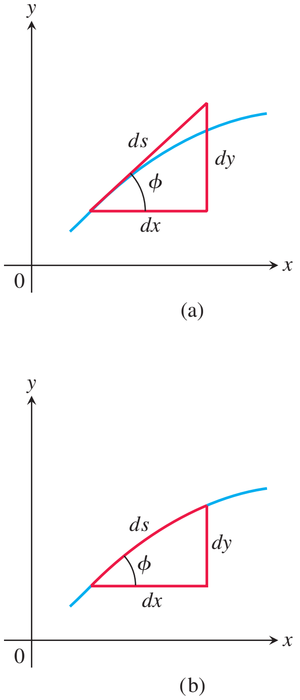


FIGURE 6.27 Diagrams for remembering the equation $ds = \sqrt{dx^2 + dy^2}$.

EXAMPLE 4 Find the arc length function for the curve in Example 2, taking $A = (1, 13/12)$ as the starting point (see Figure 6.25).

Solution In the solution to Example 2, we found that

$$1 + [f'(x)]^2 = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2.$$

Therefore the arc length function is given by

$$s(x) = \int_1^x \sqrt{1 + [f'(t)]^2}\, dt = \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2}\right) dt$$

$$= \left[\frac{t^3}{12} - \frac{1}{t}\right]_1^x = \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}.$$

To compute the arc length along the curve from $A = (1, 13/12)$ to $B = (4, 67/12)$, for instance, we simply calculate

$$s(4) = \frac{4^3}{12} - \frac{1}{4} + \frac{11}{12} = 6.$$

This is the same result we obtained in Example 2.

EXERCISES 6.3

Finding Lengths of Curves

Find the lengths of the curves in Exercises 1–12. If you have graphing software, you may want to graph these curves to see what they look like.

1. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$
2. $y = x^{3/2}$ from $x = 0$ to $x = 4$
3. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
4. $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$
5. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
6. $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$
7. $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5, \quad 1 \le x \le 8$
8. $y = (x^3/3) + x^2 + x + 1/(4x + 4), \quad 0 \le x \le 2$
9. $y = \dfrac{x^3}{3} + \dfrac{1}{4x}, \quad 1 \le x \le 3$
10. $y = \dfrac{x^5}{5} + \dfrac{1}{12x^3}, \quad \dfrac{1}{2} \le x \le 1$
11. $x = \displaystyle\int_0^y \sqrt{\sec^4 t - 1}\, dt, \quad -\pi/4 \le y \le \pi/4$
12. $y = \displaystyle\int_{-2}^x \sqrt{3t^4 - 1}\, dt, \quad -2 \le x \le -1$

T Finding Integrals for Lengths of Curves

In Exercises 13–20, do the following.

a. Set up an integral for the length of the curve.

b. Graph the curve to see what it looks like.

c. Use your grapher's or computer's integral evaluator to find the curve's length numerically.

13. $y = x^2, \quad -1 \le x \le 2$
14. $y = \tan x, \quad -\pi/3 \le x \le 0$
15. $x = \sin y, \quad 0 \le y \le \pi$
16. $x = \sqrt{1 - y^2}, \quad -1/2 \le y \le 1/2$
17. $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$
18. $y = \sin x - x \cos x, \quad 0 \le x \le \pi$
19. $y = \displaystyle\int_0^x \tan t\, dt, \quad 0 \le x \le \pi/6$
20. $x = \displaystyle\int_0^y \sqrt{\sec^2 t - 1}\, dt, \quad -\pi/3 \le y \le \pi/4$

Theory and Examples

21. **a.** Find a curve with a positive derivative through the point (1, 1) whose length integral (Equation 3) is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx.$$

b. How many such curves are there? Give reasons for your answer.

22. **a.** Find a curve with a positive derivative through the point (0, 1) whose length integral (Equation 4) is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy.$$

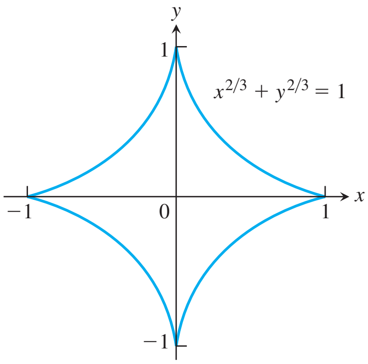
b. How many such curves are there? Give reasons for your answer.

23. Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt$$

from $x = 0$ to $x = \pi/4$.

24. **The length of an astroid** The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (see the accompanying figure). Find the length of this particular astroid by finding the length of half the first-quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \le x \le 1$, and multiplying by 8.



25. **Length of a line segment** Use the arc length formula (Equation 3) to find the length of the line segment $y = 3 - 2x$, $0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

26. **Circumference of a circle** Set up an integral to find the circumference of a circle of radius r centered at the origin. You will learn how to evaluate the integral in Section 8.3.

27. If $9x^2 = y(y-3)^2$, show that

$$ds^2 = \frac{(y+1)^2}{4y}\,dy^2.$$

28. If $4x^2 - y^2 = 64$, show that

$$ds^2 = \frac{4}{y^2}\left(5x^2 - 16\right)dx^2.$$

29. Is there a smooth (continuously differentiable) curve $y = f(x)$ whose length over the interval $0 \le x \le a$ is always $\sqrt{2}a$? Give reasons for your answer.

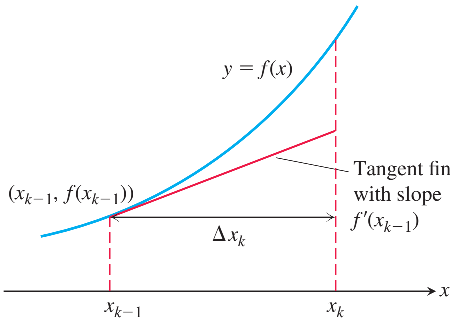
30. Using tangent fins to derive the length formula for curves Assume that f is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$, construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$, as shown in the accompanying figure.

a. Show that the length of the kth tangent fin over the interval $[x_{k-1}, x_k]$ equals $\sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\,\Delta x_k)^2}$.

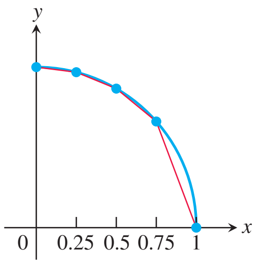
b. Show that

$$\lim_{n\to\infty}\sum_{k=1}^{n}(\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2}\,dx,$$

which is the length L of the curve $y = f(x)$ from a to b.



31. Approximate the arc length of one-quarter of the unit circle (which is $\pi/2$) by computing the length of the polygonal approximation with $n = 4$ segments (see accompanying figure).



32. Distance between two points Assume that the two points (x_1, y_1) and (x_2, y_2) lie on the graph of the straight line $y = mx + b$. Use the arc length formula (Equation 3) to find the distance between the two points.

33. Find the arc length function for the graph of $f(x) = 2x^{3/2}$ using $(0, 0)$ as the starting point. What is the length of the curve from $(0, 0)$ to $(1, 2)$?

34. Find the arc length function for the curve in Exercise 8, using $(0, 1/4)$ as the starting point. What is the length of the curve from $(0, 1/4)$ to $(1, 59/24)$?

COMPUTER EXPLORATIONS

In Exercises 35–40, use a CAS to perform the following steps for the given graph of the function over the closed interval.

a. Plot the curve together with the polygonal path approximations for $n = 2, 4, 8$ partition points over the interval. (See Figure 6.22.)

b. Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.

c. Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ with the actual length given by the integral. How does the actual length compare with the approximations as n increases? Explain your answer.

35. $f(x) = \sqrt{1 - x^2}, \quad -1 \le x \le 1$

36. $f(x) = x^{1/3} + x^{2/3}, \quad 0 \le x \le 2$

37. $f(x) = \sin(\pi x^2), \quad 0 \le x \le \sqrt{2}$

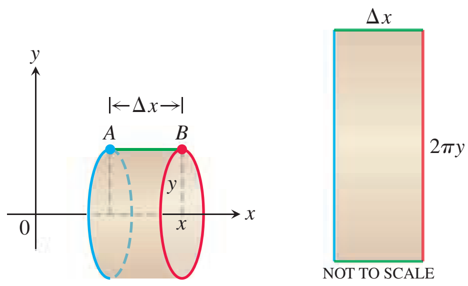
38. $f(x) = x^2 \cos x, \quad 0 \le x \le \pi$

39. $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad -\dfrac{1}{2} \le x \le 1$

40. $f(x) = x^3 - x^2, \quad -1 \le x \le 1$

6.4 Areas of Surfaces of Revolution

When you jump rope, the rope sweeps out a surface in the space around you similar to what is called a *surface of revolution*. The surface surrounds a volume of revolution, and many applications require that we know the area of the surface rather than the volume it encloses. In this section we define areas of surfaces of revolution. More general surfaces are treated in Chapter 16.



(a) (b)

FIGURE 6.28 (a) A cylindrical surface generated by rotating the horizontal line segment AB of length Δx about the x-axis has area $2\pi y \Delta x$. (b) The cut and rolled-out cylindrical surface as a rectangle.

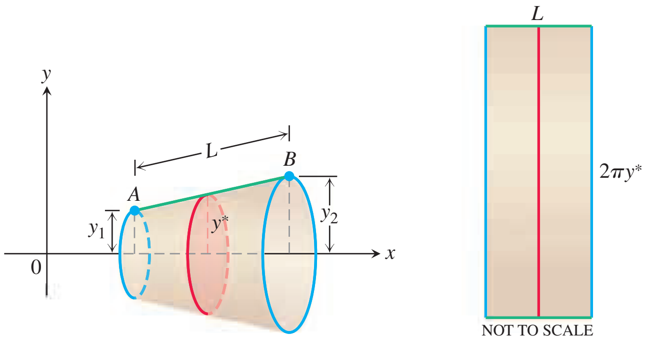
Defining Surface Area

If you revolve a region in the plane that is bounded by the graph of a function over an interval, it sweeps out a solid of revolution, as we saw earlier in the chapter. However, if you revolve only the bounding curve itself, it does not sweep out any interior volume but rather a surface that surrounds the solid and forms part of its boundary. Just as we were interested in defining and finding the length of a curve in the last section, we are now interested in defining and finding the area of a surface generated by revolving a curve about an axis.

Before considering general curves, we begin by rotating horizontal and slanted line segments about the x-axis. If we rotate the horizontal line segment AB having length Δx about the x-axis (Figure 6.28a), we generate a cylinder with surface area $2\pi y \Delta x$. This area is the same as that of a rectangle with side lengths Δx and $2\pi y$ (Figure 6.28b). The length $2\pi y$ is the circumference of the circle of radius y generated by rotating the point (x, y) on the line AB about the x-axis.

Suppose the line segment AB has length L and is slanted rather than horizontal. Now when AB is rotated about the x-axis, it generates a frustum of a cone (Figure 6.29a). From classical geometry, the surface area of this frustum is $2\pi y^* L$, where $y^* = (y_1 + y_2)/2$ is the average height of the slanted segment AB above the x-axis. This surface area is the same as that of a rectangle with side lengths L and $2\pi y^*$ (Figure 6.29b).

Let's build on these geometric principles to define the area of a surface swept out by revolving more general curves about the x-axis. Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative continuous function $y = f(x), a \le x \le b$, about the x-axis. We partition the closed interval $[a, b]$ in the usual way and use the points in the partition to subdivide the graph into short arcs. Figure 6.30 shows a typical arc PQ and the band it sweeps out as part of the graph of f.



(a) (b)

FIGURE 6.29 (a) The frustum of a cone generated by rotating the slanted line segment AB of length L about the x-axis has area $2\pi y^* L$. (b) The area of the rectangle for $y^* = \dfrac{y_1 + y_2}{2}$, the average height of AB above the x-axis.

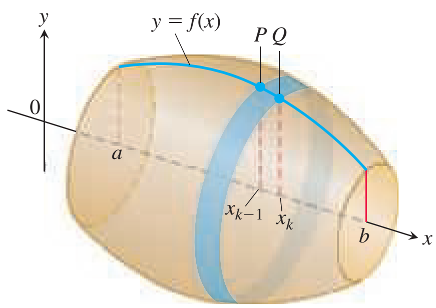


FIGURE 6.30 The surface generated by revolving the graph of a nonnegative function $y = f(x), a \le x \le b$, about the x-axis. The surface is a union of bands like the one swept out by the arc PQ.

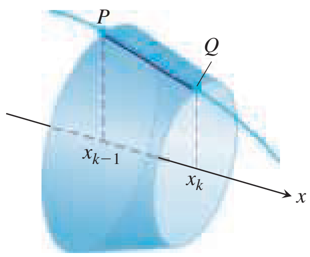


FIGURE 6.31 The line segment joining P and Q sweeps out a frustum of a cone.

As the arc PQ revolves about the x-axis, the line segment joining P and Q sweeps out a frustum of a cone whose axis lies along the x-axis (Figure 6.31). The surface area of this frustum approximates the surface area of the band swept out by the arc PQ. The surface area of the frustum of the cone shown in Figure 6.31 is $2\pi y^* L$, where y^* is the average height of the line segment joining P and Q, and L is its length (just as before). Since $f \ge 0$, from Figure 6.32 we see that the average height of the line segment is $y^* = (f(x_{k-1}) + f(x_k))/2$, and the slant length is $L = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. Therefore,

$$\begin{aligned}\text{Frustum surface area} &= 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.\end{aligned}$$

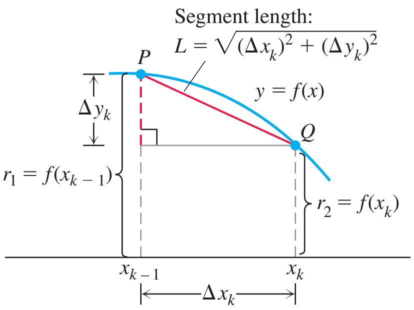


FIGURE 6.32 Dimensions associated with the arc and line segment PQ.

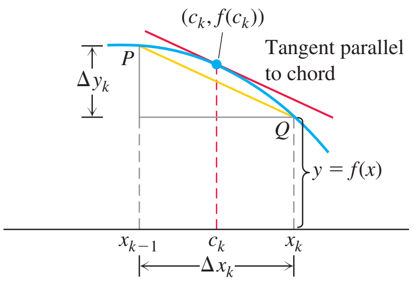


FIGURE 6.33 If f is smooth, the Mean Value Theorem guarantees the existence of a point c_k where the tangent is parallel to segment PQ.

The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc PQ, is approximated by the frustum area sum

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \tag{1}$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer. To find the limit, we first need to find an appropriate substitution for Δy_k. If the function f is differentiable, then by the Mean Value Theorem, there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ (Figure 6.33). At this point,

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k},$$

$$\Delta y_k = f'(c_k)\,\Delta x_k.$$

With this substitution for Δy_k, the sums in Equation (1) take the form

$$\begin{aligned}&\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (f'(c_k)\,\Delta x_k)^2}\\ &= \sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{1 + (f'(c_k))^2}\,\Delta x_k.\end{aligned} \tag{2}$$

These sums are not the Riemann sums of any function because the points x_{k-1}, x_k, and c_k are not the same. However, the points x_{k-1}, x_k, and c_k are very close to each other, and so we expect (and it can be proved) that as the norm of the partition of $[a, b]$ goes to zero, the sums in Equation (2) converge to the integral

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

We therefore define this integral to be the area of the surface swept out by the graph of f from a to b.

DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx. \tag{3}$$

Note that the square root in Equation (3) is similar to the one that appears in the formula for the arc length differential of the generating curve in Equation (6) of Section 6.3.

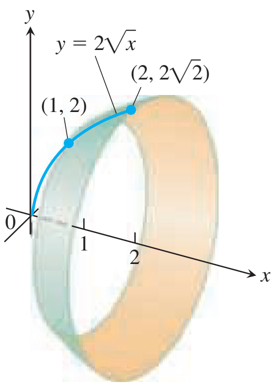


FIGURE 6.34 In Example 1 we calculate the area of this surface.

EXAMPLE 1 Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis (Figure 6.34).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{Eq. (3)}$$

with

$$a = 1, \qquad b = 2, \qquad y = 2\sqrt{x}, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}}.$$

First, we perform some algebraic manipulation on the radical in the integrand to transform it into an expression that is easier to integrate.

$$\begin{aligned}\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}\\ &= \sqrt{1+\frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}\end{aligned}$$

With these substitutions, we have

$$\begin{aligned}S &= \int_1^2 2\pi \cdot 2\sqrt{x}\,\frac{\sqrt{x+1}}{\sqrt{x}}\,dx = 4\pi\int_1^2 \sqrt{x+1}\,dx\\ &= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3}-2\sqrt{2}\right).\end{aligned}$$

Revolution About the y-Axis

For revolution about the y-axis, we interchange x and y in Equation (3).

Surface Area for Revolution About the y-Axis
If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy. \qquad (4)$$

EXAMPLE 2 The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y-axis to generate the cone in Figure 6.35. Find its lateral surface area (which excludes the base area).

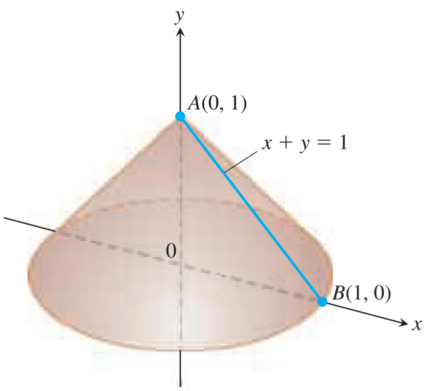


FIGURE 6.35 Revolving line segment AB about the y-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Equation (4) gives the same result, we take

$$c = 0, \qquad d = 1, \qquad x = 1 - y, \qquad \frac{dx}{dy} = -1,$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+(-1)^2} = \sqrt{2}$$

and calculate

$$\begin{aligned}S &= \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 2\pi(1-y)\sqrt{2}\,dy\\ &= 2\pi\sqrt{2}\left[y-\frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1-\frac{1}{2}\right)\\ &= \pi\sqrt{2}.\end{aligned}$$

The results agree, as they should.

EXERCISES 6.4

Finding Integrals for Surface Area

In Exercises 1–8:

a. Set up an integral for the area of the surface generated by revolving the given curve about the indicated axis.

T **b.** Graph the curve to see what it looks like. If you can, graph the surface too.

T **c.** Use your utility's integral evaluator to find the surface's area numerically.

1. $y = \tan x, \quad 0 \le x \le \pi/4; \quad x$-axis

2. $y = x^2, \quad 0 \le x \le 2; \quad x$-axis

3. $xy = 1, \quad 1 \le y \le 2; \quad y$-axis

4. $x = \sin y, \quad 0 \le y \le \pi; \quad y$-axis

5. $x^{1/2} + y^{1/2} = 3$ from $(4, 1)$ to $(1, 4)$; x-axis

6. $y + 2\sqrt{y} = x, \quad 1 \le y \le 2; \quad y$-axis

7. $x = \displaystyle\int_0^y \tan t\, dt, \quad 0 \le y \le \pi/3; \quad y$-axis

8. $y = \displaystyle\int_1^x \sqrt{t^2 - 1}\, dt, \quad 1 \le x \le \sqrt{5}; \quad x$-axis

Finding Surface Area

9. Find the lateral (side) surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$, about the x-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

10. Find the lateral surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$, about the y-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

11. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the x-axis. Check your result with the geometry formula

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

12. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the y-axis. Check your result with the geometry formula

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

Find the areas of the surfaces generated by revolving the curves in Exercises 13–23 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

13. $y = x^3/9, \quad 0 \le x \le 2; \quad x$-axis

14. $y = \sqrt{x}, \quad 3/4 \le x \le 15/4; \quad x$-axis

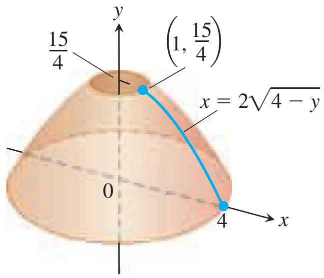
15. $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x$-axis

16. $y = \sqrt{x + 1}, \quad 1 \le x \le 5; \quad x$-axis

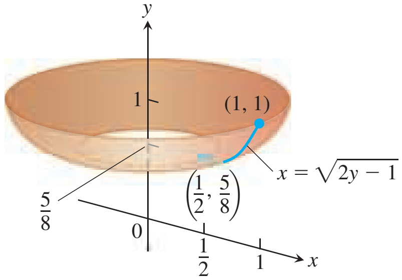
17. $x = y^3/3, \quad 0 \le y \le 1; \quad y$-axis

18. $x = (1/3)y^{3/2} - y^{1/2}, \quad 1 \le y \le 3; \quad y$-axis

19. $x = 2\sqrt{4 - y}, \quad 0 \le y \le 15/4; \quad y$-axis



20. $x = \sqrt{2y - 1}, \quad 5/8 \le y \le 1; \quad y$-axis



21. $y = (1/2)(x^2 + 1), \quad 0 \le x \le 1; \quad y$-axis

22. $y = (1/3)(x^2 + 2)^{3/2}, \quad 0 \le x \le \sqrt{2}; \quad y$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dx, and evaluate the integral $S = \int 2\pi x\, ds$ with appropriate limits.)

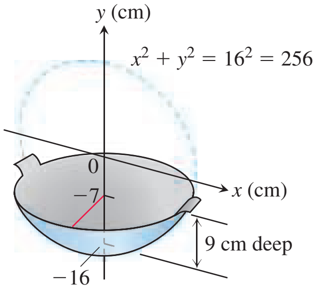
23. $x = (y^4/4) + 1/(8y^2), \quad 1 \le y \le 2; \quad x$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dy, and evaluate the integral $S = \int 2\pi y\, ds$ with appropriate limits.)

24. Write an integral for the area of the surface generated by revolving the curve $y = \cos x, -\pi/2 \le x \le \pi/2$, about the x-axis. In Section 8.4 we will see how to evaluate such integrals.

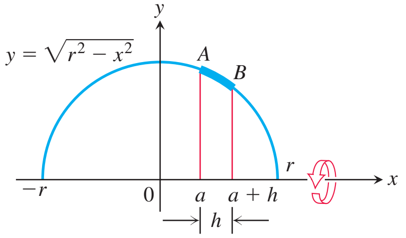
25. Testing the new definition Show that the surface area of a sphere of radius a is still $4\pi a^2$ by using Equation (3) to find the area of the surface generated by revolving the curve $y = \sqrt{a^2 - x^2}$, $-a \le x \le a$, about the x-axis.

26. Testing the new definition The lateral (side) surface area of a cone of height h and base radius r should be $\pi r\sqrt{r^2 + h^2}$, the semiperimeter of the base times the slant height. Show that this is still the case by finding the area of the surface generated by revolving the line segment $y = (r/h)x, 0 \le x \le h$, about the x-axis.

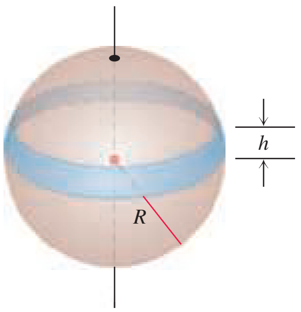
T **27. Enameling woks** Your company decided to put out a deluxe version of a wok you designed. The plan is to coat it inside with white enamel and outside with blue enamel. Each enamel will be sprayed on 0.5 mm thick before baking. (See accompanying figure.) Your manufacturing department wants to know how much enamel to have on hand for a production run of 5000 woks. What do you tell them? (Neglect waste and unused material and give your answer in liters. Remember that $1\text{ cm}^3 = 1$ mL, so $1\text{ L} = 1000\text{ cm}^3$.)



28. Slicing bread Did you know that if you cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the x-axis to generate a sphere. Let AB be an arc of the semicircle that lies above an interval of length h on the x-axis. Show that the area swept out by AB does not depend on the location of the interval. (It does depend on the length of the interval.)



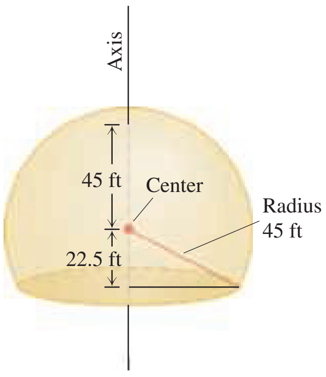
29. The shaded band shown here is cut from a sphere of radius R by parallel planes h units apart. Show that the surface area of the band is $2\pi Rh$.



30. Here is a schematic drawing of the 90-ft dome used by the U.S. National Weather Service to house radar in Bozeman, Montana.

a. How much outside surface is there to paint (not counting the bottom)?

T **b.** Express the answer to the nearest square foot.

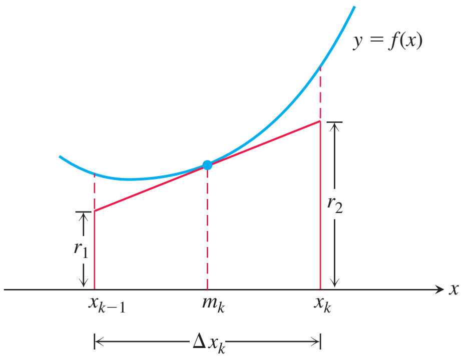


31. An alternative derivation of the surface area formula Assume f is smooth on $[a, b]$ and partition $[a, b]$ in the usual way. In the kth subinterval $[x_{k-1}, x_k]$, construct the tangent line to the curve at the midpoint $m_k = (x_{k-1} + x_k)/2$, as in the accompanying figure.

a. Show that

$$r_1 = f(m_k) - f'(m_k)\frac{\Delta x_k}{2} \quad \text{and} \quad r_2 = f(m_k) + f'(m_k)\frac{\Delta x_k}{2}.$$

b. Show that the length L_k of the tangent line segment in the kth subinterval is $L_k = \sqrt{(\Delta x_k)^2 + (f'(m_k)\,\Delta x_k)^2}$.



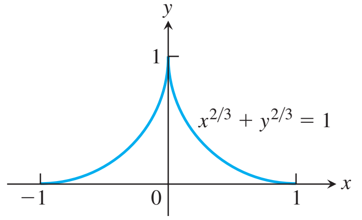
c. Show that the lateral surface area of the frustum of the cone swept out by the tangent line segment as it revolves about the x-axis is $2\pi f(m_k)\sqrt{1 + (f'(m_k))^2}\,\Delta x_k$.

d. Show that the area of the surface generated by revolving $y = f(x)$ about the x-axis over $[a, b]$ is

$$\lim_{n\to\infty}\sum_{k=1}^{n}\begin{pmatrix}\text{lateral surface area}\\ \text{of } k\text{th frustum}\end{pmatrix} = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

32. The surface of an astroid Find the area of the surface generated by revolving about the x-axis the portion of the astroid $x^{2/3} + y^{2/3} = 1$ shown in the accompanying figure.

(*Hint:* Revolve the first-quadrant portion $y = (1 - x^{2/3})^{3/2}$, $0 \le x \le 1$, about the x-axis and double your result.)



6.5 Work and Fluid Forces

In everyday life, *work* means an activity that requires muscular or mental effort. In science, the term refers specifically to a force acting on an object and the object's subsequent displacement. This section shows how to calculate work. The applications run from compressing railroad car springs and emptying subterranean tanks to forcing subatomic particles to collide and lifting satellites into orbit.

Work Done by a Constant Force

When an object moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F in the direction of motion, we define the **work** W done by the force on the object with the formula

$$W = Fd \qquad \text{(Constant-force formula for work).} \tag{1}$$

From Equation (1) we see that the unit of work in any system is the unit of force multiplied by the unit of distance. In SI units (SI stands for *Système International*, or International System), the unit of force is a newton, the unit of distance is a meter, and the unit of work is a newton-meter ($\text{N} \cdot \text{m}$). This combination appears so often it has a special name, the **joule**. Taking gravitational acceleration at sea level to be 9.8 m/sec^2, to lift one kilogram one meter requires work of 9.8 joules. This is seen by multiplying the force of 9.8 newtons exerted on one kilogram by the one-meter distance moved. In the British system, the unit of work is the foot-pound, a unit sometimes used in applications. It requires one foot-pound of work to lift a one pound weight a distance of one foot.

Joules
The joule, abbreviated J, is named after the English physicist James Prescott Joule (1818–1889). The defining equation is

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}).$$

In symbols, $1 \text{ J} = 1 \text{ N} \cdot \text{m}$.

EXAMPLE 1 Suppose you jack up the side of a 2000-lb car 1.25 ft to change a tire. The jack applies a constant vertical force of about 1000 lb in lifting the side of the car (but because of the mechanical advantage of the jack, the force you apply to the jack itself is only about 30 lb). The total work performed by the jack on the car is $1000 \times 1.25 = 1250$ ft-lb. In SI units, the jack has applied a force of 4448 N through a distance of 0.381 m to do $4448 \times 0.381 \approx 1695$ J of work. ■

Work Done by a Variable Force Along a Line

If the force you apply varies along the way, as it will if you are stretching or compressing a spring, the formula $W = Fd$ has to be replaced by an integral formula that takes the variation in F into account.

Suppose that the force performing the work acts on an object moving along a straight line, which we take to be the x-axis. We assume that the magnitude of the force is a continuous function F of the object's position x. We want to find the work done over the interval from $x = a$ to $x = b$. We partition $[a, b]$ in the usual way and choose an arbitrary point c_k in each subinterval $[x_{k-1}, x_k]$. If the subinterval is short enough, the continuous function F will not vary much from x_{k-1} to x_k. The amount of work done across the interval will be about $F(c_k)$ times the distance Δx_k, the same as it would be if F were constant and we could apply Equation (1). The total work done from a to b is therefore approximated by the Riemann sum

$$\text{Work} \approx \sum_{k=1}^{n} F(c_k)\, \Delta x_k.$$

We expect the approximation to improve as the norm of the partition goes to zero, so we define the work done by the force from a to b to be the integral of F from a to b:

$$\lim_{n\to\infty} \sum_{k=1}^{n} F(c_k)\, \Delta x_k = \int_a^b F(x)\, dx.$$

DEFINITION The **work** done by a variable force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx. \tag{2}$$

The units of the integral are joules if F is in newtons and x is in meters, and foot-pounds if F is in pounds and x is in feet. So the work done by a force of $F(x) = 1/x^2$ newtons in moving an object along the x-axis from $x = 1$ m to $x = 10$ m is

$$W = \int_1^{10} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg]_1^{10} = -\frac{1}{10} + 1 = 0.9 \text{ J}.$$

Hooke's Law for Springs: $F = kx$

One calculation for work arises in finding the work required to stretch or compress a spring. **Hooke's Law** says that the force required to hold a stretched or compressed spring x units from its natural (unstressed) length is proportional to x. In symbols,

$$F = kx. \tag{3}$$

The constant k, measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or **spring constant**) of the spring. Hooke's Law, Equation (3), gives good results as long as the force doesn't distort the metal in the spring. We assume that the forces in this section are too small to do that.

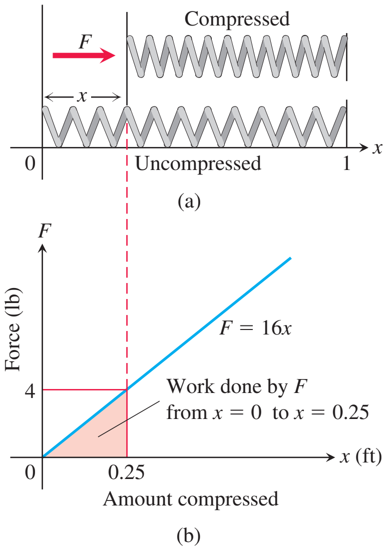


FIGURE 6.36 The force F needed to hold a spring under compression increases linearly as the spring is compressed (Example 2).

EXAMPLE 2 Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is $k = 16$ lb/ft.

Solution We picture the uncompressed spring laid out along the x-axis with its movable end at the origin and its fixed end at $x = 1$ ft (Figure 6.36). This enables us to describe the force required to compress the spring from 0 to x with the formula $F = 16x$. To compress the spring from 0 to 0.25 ft, the force must increase from

$$F(0) = 16 \cdot 0 = 0 \text{ lb} \quad \text{to} \quad F(0.25) = 16 \cdot 0.25 = 4 \text{ lb}.$$

The work done by F over this interval is

$$W = \int_0^{0.25} 16x\,dx = 8x^2\bigg]_0^{0.25} = 0.5 \text{ ft-lb}. \qquad \text{Eq. (2) with } a = 0,\ b = 0.25,\ F(x) = 16x \qquad \blacksquare$$

EXAMPLE 3 A spring has a natural length of 1 m. A force of 24 N holds the spring stretched to a total length of 1.8 m.

(a) Find the force constant k.

(b) How much work will it take to stretch the spring 2 m beyond its natural length?

(c) How far will a 45-N force stretch the spring?

Solution

(a) *The force constant.* We find the force constant from Equation (3). A force of 24 N maintains the spring at a position where it is stretched 0.8 m from its natural length, so

$$24 = k(0.8) \qquad \text{Eq. (3) with } F = 24,\ x = 0.8$$
$$k = 24/0.8 = 30 \text{ N/m}.$$

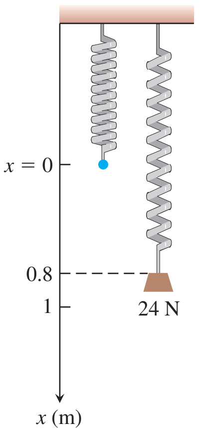


FIGURE 6.37 A 24-N weight stretches this spring 0.8 m beyond its unstressed length (Example 3).

(b) *The work to stretch the spring* 2 m. We imagine the unstressed spring hanging along the x-axis with its free end at $x = 0$ (Figure 6.37). The force required to stretch the spring x m beyond its natural length is the force required to hold the free end of the spring x units from the origin. Hooke's Law with $k = 30$ says that this force is

$$F(x) = 30x.$$

The work done by F on the spring from $x = 0$ m to $x = 2$ m is

$$W = \int_0^2 30x\,dx = 15x^2\Big]_0^2 = 60 \text{ J}.$$

(c) *How far will a* 45-N *force stretch the spring?* We substitute $F = 45$ in the equation $F = 30x$ to find

$$45 = 30x, \qquad \text{or} \qquad x = 1.5 \text{ m}.$$

A 45-N force will keep the spring stretched 1.5 m beyond its natural length. ■

Lifting Objects and Pumping Liquids from Containers

The work integral is useful for calculating the work done in lifting objects whose weights vary with their elevation.

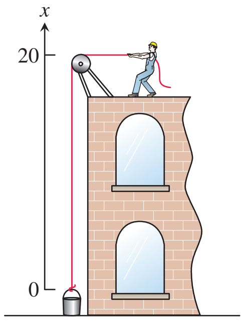


FIGURE 6.38 Lifting the bucket in Example 4.

EXAMPLE 4 A 5-kg bucket is lifted from the ground into the air by pulling in 20 m of rope at a constant speed (Figure 6.38). The rope weighs 0.08 kg/m. How much work was spent lifting the bucket and rope?

Solution The weight of the bucket is obtained by multiplying the mass (5 kg) and the acceleration due to gravity, approximately 9.8 m/s^2. So the bucket's weight is $(5)(9.8) = 49$ N, and the work done lifting it alone is weight $\times$ distance $= (49)(20) = 980$ J.

The weight of the rope varies with the bucket's elevation, because less of it is freely hanging as the bucket is raised. When the bucket is x m off the ground, the remaining portion of the rope still being lifted weighs $(0.08)(20 - x)(9.8)$ N. So the work in lifting the rope is

$$\text{Work on rope} = \int_0^{20} (0.08)(20 - x)(9.8)\,dx = \int_0^{20} (15.68 - 0.784x)\,dx$$

$$= \Big[15.68x - 0.392x^2\Big]_0^{20} = 313.6 - 156.8 = 156.8 \text{ J}.$$

The total work for the bucket and rope combined is

$$980 + 156.8 = 1136.8 \text{ J}.$$ ■

How much work does it take to pump all or part of the liquid from a container? Engineers often need to know the answer in order to design or choose the right pump, or to compute the cost to transport water or some other liquid from one place to another. To find out how much work is required to pump the liquid, we imagine lifting the liquid out one thin horizontal slab at a time and applying the equation $W = Fd$ to each slab. We then evaluate the integral that this leads to as the slabs become thinner and more numerous.

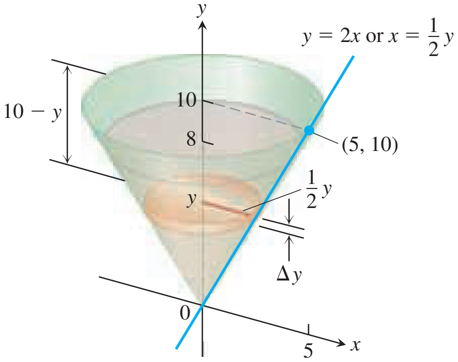


FIGURE 6.39 The olive oil and tank in Example 5.

EXAMPLE 5 The conical tank in Figure 6.39 is filled to within 2 ft of the top with olive oil weighing 57 lb/ft^3. How much work does it take to pump the oil to the rim of the tank?

Solution We imagine the oil divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval $[0, 8]$.

The typical slab between the planes at y and $y + \Delta y$ has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{1}{2}y\right)^2 \Delta y = \frac{\pi}{4}y^2 \Delta y \text{ ft}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight,

$$F(y) = 57\,\Delta V = \frac{57\pi}{4}y^2 \Delta y \text{ lb.}$$ Weight = (weight per unit volume) × volume

The distance through which $F(y)$ must act to lift this slab to the level of the rim of the cone is about $(10 - y)$ ft, so the work done lifting the slab is about

$$\Delta W = \frac{57\pi}{4}(10 - y)y^2 \Delta y \text{ ft-lb.}$$

Assuming there are n slabs associated with the partition of $[0, 8]$, and that $y = y_k$ denotes the plane associated with the kth slab of thickness Δy_k, we can approximate the work done lifting all of the slabs with the Riemann sum

$$W \approx \sum_{k=1}^{n} \frac{57\pi}{4}(10 - y_k)y_k{}^2 \Delta y_k \text{ ft-lb.}$$

The work of pumping the oil to the rim is the limit of these sums as the norm of the partition goes to zero and the number of slabs tends to infinity:

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{57\pi}{4}(10 - y_k)y_k^2 \Delta y_k = \int_0^8 \frac{57\pi}{4}(10 - y)y^2\, dy \\ &= \frac{57\pi}{4}\int_0^8 (10y^2 - y^3)\, dy \\ &= \frac{57\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 \approx 30{,}561 \text{ ft-lb.} \end{aligned}$$

Fluid Pressure and Forces

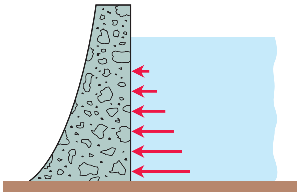


FIGURE 6.40 To withstand the increasing pressure, dams are built thicker as they go down.

Dams are built thicker at the bottom than at the top (Figure 6.40) because the pressure against them increases with depth. The pressure at any point on a dam depends only on how far below the surface the point is and not on how much the surface of the dam happens to be tilted at that point. The pressure, in pounds per square foot at a point h feet below the surface, is always $62.4h$. The number 62.4 is the weight-density of freshwater in pounds per cubic foot. The pressure h feet below the surface of any fluid is the fluid's *weight-density* times h.

Weight-density

A fluid's weight-density w is its weight per unit volume. Typical values (lb/ft^3) are listed below.

Gasoline	42
Mercury	849
Milk	64.5
Molasses	100
Olive oil	57
Seawater	64
Freshwater	62.4

The Pressure-Depth Equation

In a fluid that is standing still, the pressure p at depth h is the fluid's weight-density w times h:

$$p = wh. \tag{4}$$

In a container of fluid with a flat horizontal base, the total force exerted by the fluid against the base can be calculated by multiplying the area of the base by the pressure at the base. We can do this because total force equals force per unit area (pressure) times area. (See Figure 6.41.) If F, p, and A are the total force, pressure, and area, then

$$\begin{aligned} F &= \text{total force} = \text{force per unit area} \times \text{area} \\ &= \text{pressure} \times \text{area} = pA \\ &= whA. \end{aligned}$$ $p = wh$ from Eq. (4)

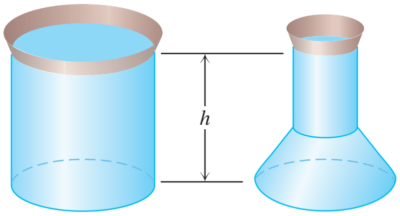


FIGURE 6.41 These containers are filled with water to the same depth and have the same base area. The total force is therefore the same on the bottom of each container. The containers' shapes do not matter here.

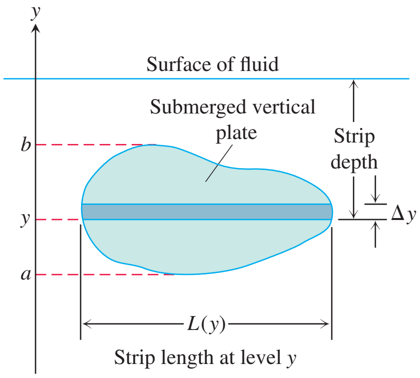


FIGURE 6.42 The force exerted by a fluid against one side of a thin, flat horizontal strip is about $\Delta F = \text{pressure} \times \text{area} = w \times (\text{strip depth}) \times L(y)\,\Delta y$.

Fluid Force on a Constant-Depth Surface

$$F = pA = whA \tag{5}$$

For example, the weight-density of freshwater is 62.4 lb/ft^3, so the fluid force at the bottom of a 10 ft $\times$ 20 ft rectangular swimming pool 3 ft deep is

$$\begin{aligned} F = whA &= (62.4\ \text{lb}/\text{ft}^3)(3\ \text{ft})(10 \cdot 20\ \text{ft}^2) \\ &= 37{,}440\ \text{lb}. \end{aligned}$$

For a flat plate submerged *horizontally*, like the bottom of the swimming pool just discussed, the downward force acting on its upper face due to liquid pressure is given by Equation (5). If the plate is submerged *vertically*, however, then the pressure against it will be different at different depths and Equation (5) no longer is usable in that form (because h varies).

Suppose we want to know the force exerted by a fluid against one side of a vertical plate submerged in a fluid of weight-density w. To find it, we model the plate as a region extending from $y = a$ to $y = b$ in the xy-plane (Figure 6.42). We partition $[a, b]$ in the usual way and imagine the region to be cut into thin horizontal strips by planes perpendicular to the y-axis at the partition points. The typical strip from y to $y + \Delta y$ is Δy units wide by $L(y)$ units long. We assume $L(y)$ to be a continuous function of y.

The pressure varies across the strip from top to bottom. If the strip is narrow enough, however, the pressure will remain close to its bottom-edge value of $w \times$ (strip depth). The force exerted by the fluid against one side of the strip will be about

$$\begin{aligned} \Delta F &= (\text{pressure along bottom edge}) \times (\text{area}) \\ &= w \cdot (\text{strip depth}) \cdot L(y)\,\Delta y. \end{aligned}$$

Assume there are n strips associated with the partition of $a \le y \le b$ and that y_k is the bottom edge of the kth strip having length $L(y_k)$ and width Δy_k. The force against the entire plate is approximated by summing the forces against each strip, giving the Riemann sum

$$F \approx \sum_{k=1}^{n} w \cdot (\text{strip depth})_k \cdot L(y_k)\,\Delta y_k. \tag{6}$$

The sum in Equation (6) is a Riemann sum for a continuous function on $[a, b]$, and we expect the approximations to improve as the norm of the partition goes to zero. The force against the plate is the limit of these sums:

$$\lim_{n\to\infty} \sum_{k=1}^{n} w \cdot (\text{strip depth})_k \cdot L(y_k)\,\Delta y_k = \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\,dy.$$

The Integral for Fluid Force Against a Vertical Flat Plate

Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y. Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\,dy. \tag{7}$$

EXAMPLE 6 A flat isosceles right-triangular plate with base 6 ft and height 3 ft is submerged vertically, base up, 2 ft below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.

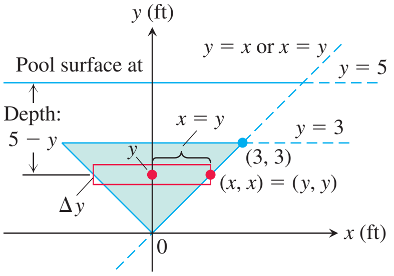


FIGURE 6.43 To find the force on one side of the submerged plate in Example 6, we can use a coordinate system like the one here.

Solution We establish a coordinate system to work in by placing the origin at the plate's bottom vertex and running the y-axis upward along the plate's axis of symmetry (Figure 6.43). The surface of the pool lies along the line $y = 5$ and the plate's top edge along the line $y = 3$. The plate's right-hand edge lies along the line $y = x$, with the upper-right vertex at (3, 3). The length of a thin strip at level y is

$$L(y) = 2x = 2y.$$

The depth of the strip beneath the surface is $(5 - y)$. The force exerted by the water against one side of the plate is therefore

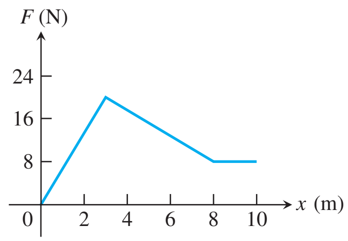
$$\begin{aligned} F &= \int_a^b w \cdot \begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix} \cdot L(y)\, dy \qquad \text{Eq. (7)} \\ &= \int_0^3 62.4(5 - y)2y\, dy \\ &= 124.8 \int_0^3 (5y - y^2)\, dy \\ &= 124.8 \left[\frac{5}{2}y^2 - \frac{y^3}{3} \right]_0^3 = 1684.8 \text{ lb.} \end{aligned}$$

EXERCISES 6.5

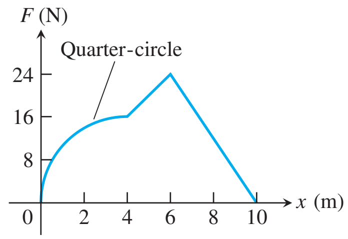
Springs

The graphs of force functions (in newtons) are given in Exercises 1 and 2. How much work is done by each force in moving an object 10 m?

1.



2.



3. **Spring constant** It took 1800 J of work to stretch a spring from its natural length of 2 m to a length of 5 m. Find the spring's force constant.

4. **Stretching a spring** A spring has a natural length of 10 in. An 800-lb force stretches the spring to 14 in.
   a. Find the force constant.
   b. How much work is done in stretching the spring from 10 in. to 12 in.?
   c. How far beyond its natural length will a 1600-lb force stretch the spring?

5. **Stretching a rubber band** A force of 2 N will stretch a rubber band 2 cm (0.02 m). Assuming that Hooke's Law applies, how far will a 4-N force stretch the rubber band? How much work does it take to stretch the rubber band this far?

6. **Stretching a spring** If a force of 90 N stretches a spring 1 m beyond its natural length, how much work does it take to stretch the spring 5 m beyond its natural length?

7. **Subway car springs** It takes a force of 21,714 lb to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 8 in. to its fully compressed height of 5 in.
   a. What is the assembly's force constant?
   b. How much work does it take to compress the assembly the first half inch? the second half inch? Answer to the nearest in.-lb.

8. **Bathroom scale** A bathroom scale is compressed 1/16 in. when a 150-lb person stands on it. Assuming that the scale behaves like a spring that obeys Hooke's Law, how much does someone who compresses the scale 1/8 in. weigh? How much work is done compressing the scale 1/8 in.?

Work Done by a Variable Force

9. **Lifting a rope** A mountain climber is about to haul up a 50-m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m?

10. **Leaky sandbag** A bag of sand originally weighing 144 lb was lifted at a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted to 18 ft. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

11. Lifting an elevator cable An electric elevator with a motor at the top has a multistrand cable weighing 4.5 lb/ft. When the car is at the first floor, 180 ft of cable are paid out, and effectively 0 ft are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

12. Force of attraction When a particle of mass m is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is k/x^2. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time it reaches $x = a$, $0 < a < b$.

13. Leaky bucket Assume the bucket in Example 4 is leaking. It starts with 5 liters of water (5 kg) and leaks at a constant rate. It finishes draining just as it reaches the top. How much work was spent lifting the water alone? (*Hint:* Do not include the rope and bucket, and find the proportion of water left at elevation x m.)

14. (*Continuation of Exercise 13.*) The workers in Example 4 and Exercise 13 changed to a larger bucket that held 10 liters (10 kg) of water, but the new bucket had an even larger leak so that it, too, was empty by the time it reached the top. Assuming that the water leaked out at a steady rate, how much work was done lifting the water alone? (Do not include the rope and bucket.)

Pumping Liquids from Containers

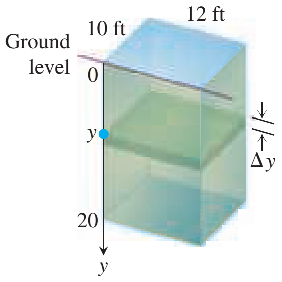
15. Pumping water The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs 62.4 lb/ft^3.

a. How much work does it take to empty the tank by pumping the water back to ground level once the tank is full?

b. If the water is pumped to ground level with a (5/11)-horsepower (hp) motor (work output 250 ft-lb/sec), how long will it take to empty the full tank (to the nearest minute)?

c. Show that the pump in part (b) will lower the water level 10 ft (halfway) during the first 25 min of pumping.

d. The weight of water What are the answers to parts (a) and (b) in a location where water weighs 62.26 lb/ft^3? 62.59 lb/ft^3?



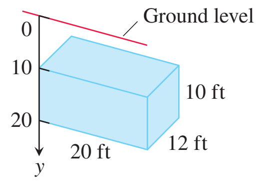
16. Emptying a cistern The rectangular cistern (storage tank for rainwater) shown has its top 10 ft below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level.

a. How much work will it take to empty the cistern?

b. How long will it take a 1/2-hp pump, rated at 275 ft-lb/sec, to pump the tank dry?

c. How long will it take the pump in part (b) to empty the tank halfway? (It will be less than half the time required to empty the tank completely.)

d. The weight of water What are the answers to parts (a) through (c) in a location where water weighs 62.26 lb/ft^3? 62.59 lb/ft^3?



17. Pumping oil How much work would it take to pump oil from the tank in Example 5 to the level of the top of the tank if the tank were completely full?

18. Pumping a half-full tank Suppose that, instead of being full, the tank in Example 5 is only half full. How much work does it take to pump the remaining oil to a level 4 ft above the top of the tank?

19. Emptying a tank A vertical right-circular cylindrical tank measures 30 ft high and 20 ft in diameter. It is full of kerosene weighing 51.2 lb/ft^3. How much work does it take to pump the kerosene to the level of the top of the tank?

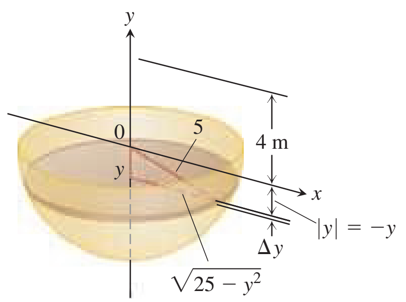
20. a. Pumping milk Suppose that the conical container in Example 5 contains milk (weighing 64.5 lb/ft^3) instead of olive oil. How much work will it take to pump the contents to the rim?

b. Pumping oil How much work will it take to pump the oil in Example 5 to a level 3 ft above the cone's rim?

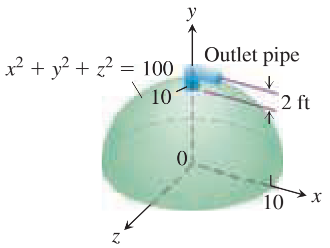
21. The graph of $y = x^2$ on $0 \le x \le 2$ is revolved about the y-axis to form a tank that is then filled with salt water from the Dead Sea (weighing approximately 73 lb/ft^3). How much work does it take to pump all of the water to the top of the tank?

22. A right-circular cylindrical tank of height 10 ft and radius 5 ft is lying horizontally and is full of diesel fuel weighing 53 lb/ft^3. How much work is required to pump all of the fuel to a point 15 ft above the top of the tank?

23. Emptying a water reservoir We model pumping from spherical containers the way we do from other containers, with the axis of integration along the vertical axis of the sphere. Use the figure here to find how much work it takes to empty a full hemispherical water reservoir of radius 5 m by pumping the water to a height of 4 m above the top of the reservoir. Water weighs 9800 N/m^3.



24. You are in charge of the evacuation and repair of the storage tank shown here. The tank is a hemisphere of radius 10 ft and is full of benzene weighing 56 lb/ft^3. A firm you contacted says it can empty the tank for 1/2¢ per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 ft above the top of the tank. If you have $5000 budgeted for the job, can you afford to hire the firm?



Work and Kinetic Energy

25. Kinetic energy If a variable force of magnitude $F(x)$ moves an object of mass m along the x-axis from x_1 to x_2, the object's velocity v can be written as dx/dt (where t represents time). Use Newton's second law of motion $F = m(dv/dt)$ and the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

to show that the net work done by the force in moving the object from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x)\,dx = \frac{1}{2}mv_2{}^2 - \frac{1}{2}mv_1{}^2,$$

where v_1 and v_2 are the object's velocities at x_1 and x_2. In physics, the expression $(1/2)mv^2$ is called the *kinetic energy* of an object of mass m moving with velocity v. Therefore, *the work done by the force equals the change in the object's kinetic energy*, and we can find the work by calculating this change.

In Exercises 26–30, use the result of Exercise 25.

26. Tennis A 2-oz tennis ball was served at 160 ft/sec (about 109 mph). How much work was done on the ball to make it go this fast? (To find the ball's mass from its weight, express the weight in pounds and divide by 32 ft/sec^2, the acceleration of gravity.)

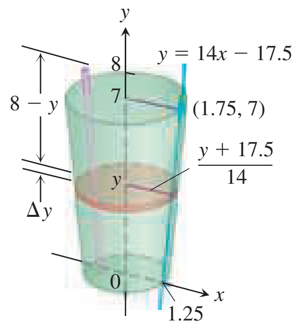
27. Baseball How many foot-pounds of work does it take to throw a baseball 90 mph? A baseball weighs 5 oz, or 0.3125 lb.

28. Golf A 1.6-oz golf ball is driven off the tee at a speed of 280 ft/sec (about 191 mph). How many foot-pounds of work are done on the ball getting it into the air?

29. Tennis At the 2012 Busan Open Challenger Tennis Tournament is Busan, South Korea, the Australian Samuel Groth hit a serve measured at 263 kph (163.4 mph). How much work was required by Groth to serve a 0.056699-kg (2-oz) tennis ball at that speed?

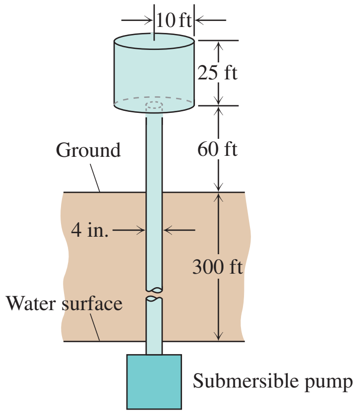
30. Softball How much work has to be performed on a 6.5-oz softball to pitch it 132 ft/sec (90 mph)?

31. Drinking a milkshake The truncated conical container shown here is full of strawberry milkshake that weighs 4/9 oz/in^3. As you can see, the container is 7 in. deep, 2.5 in. across at the base, and 3.5 in. across at the top (a standard size at Brigham's in Boston). The straw sticks up an inch above the top. About how much work does it take to suck up the milkshake through the straw (neglecting friction)? Answer in inch-ounces.



Dimensions in inches

32. Water tower Your town has decided to drill a well to increase its water supply. As the town engineer, you have determined that a water tower will be necessary to provide the pressure needed for distribution, and you have designed the system shown here. The water is to be pumped from a 300-ft well through a vertical 4-in. pipe into the base of a cylindrical tank 20 ft in diameter and 25 ft high. The base of the tank will be 60 ft above ground. The pump is a 3-hp pump, rated at 1650 ft · lb/sec. To the nearest hour, how long will it take to fill the tank the first time? (Include the time it takes to fill the pipe.) Assume that water weighs 62.4 lb/ft^3.



NOT TO SCALE

33. Putting a satellite in orbit The strength of Earth's gravitational field varies with the distance r from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}.$$

Here, $M = 5.975 \times 10^{24}$ kg is Earth's mass, $G = 6.6720 \times 10^{-11}$ N · m^2 kg^{-2} is the universal gravitational constant, and r is measured in meters. The work it takes to lift a 1000-kg satellite from Earth's surface to a circular orbit 35,780 km above Earth's center is therefore given by the integral

$$\text{Work} = \int_{6{,}370{,}000}^{35{,}780{,}000} \frac{1000MG}{r^2}\,dr \text{ joules}.$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spent lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

34. Forcing electrons together Two electrons r meters apart repel each other with a force of

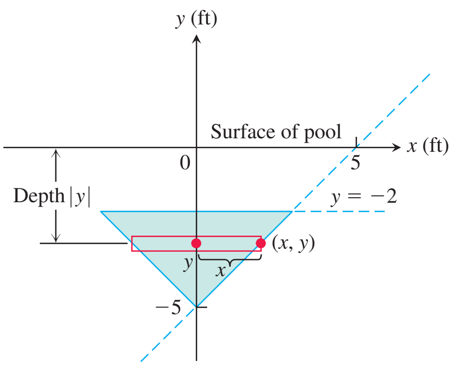
$$F = \frac{23 \times 10^{-29}}{r^2} \text{ newtons.}$$

a. Suppose one electron is held fixed at the point (1, 0) on the x-axis (units in meters). How much work does it take to move a second electron along the x-axis from the point $(-1, 0)$ to the origin?

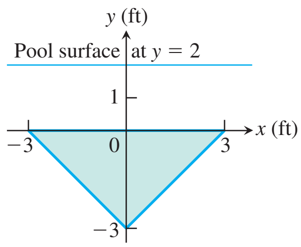
b. Suppose an electron is held fixed at each of the points $(-1, 0)$ and (1, 0). How much work does it take to move a third electron along the x-axis from (5, 0) to (3, 0)?

Finding Fluid Forces

35. Triangular plate Calculate the fluid force on one side of the plate in Example 6 using the coordinate system shown here.



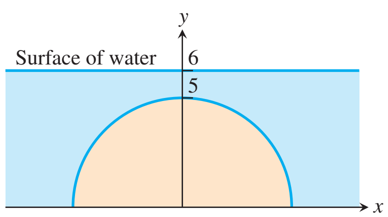
36. Triangular plate Calculate the fluid force on one side of the plate in Example 6 using the coordinate system shown here.



37. Rectangular plate In a pool filled with water to a depth of 10 ft, calculate the fluid force on one side of a 3 ft by 4 ft rectangular plate if the plate rests vertically at the bottom of the pool

a. on its 4-ft edge. **b.** on its 3-ft edge.

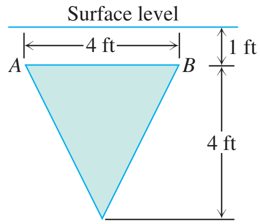
38. Semicircular plate Calculate the fluid force on one side of a semicircular plate of radius 5 ft that rests vertically on its diameter at the bottom of a pool filled with water to a depth of 6 ft.



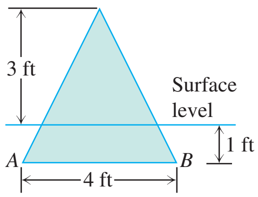
39. Triangular plate The isosceles triangular plate shown here is submerged vertically 1 ft below the surface of a freshwater lake.

a. Find the fluid force against one face of the plate.

b. What would be the fluid force on one side of the plate if the water were seawater instead of freshwater?



40. Rotated triangular plate The plate in Exercise 39 is revolved 180° about line AB so that part of the plate sticks out of the lake, as shown here. What force does the water exert on one face of the plate now?



41. New England Aquarium The viewing portion of the rectangular glass window in a typical fish tank at the New England Aquarium in Boston is 63 in. wide and runs from 0.5 in. below the water's surface to 33.5 in. below the surface. Find the fluid force against this portion of the window. The weight-density of seawater is 64 lb/ft^3. (In case you were wondering, the glass is 3/4 in. thick and the tank walls extend 4 in. above the water to keep the fish from jumping out.)

42. Semicircular plate A semicircular plate 2 ft in diameter sticks straight down into freshwater with the diameter along the surface. Find the force exerted by the water on one side of the plate.

43. Tilted plate Calculate the fluid force on one side of a 5 ft by 5 ft square plate if the plate is at the bottom of a pool filled with water to a depth of 8 ft and

a. lying flat on its 5 ft by 5 ft face.

b. resting vertically on a 5-ft edge.

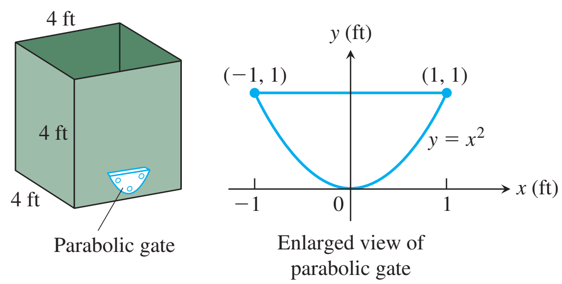
c. resting on a 5-ft edge and tilted at 45° to the bottom of the pool.

44. Tilted plate Calculate the fluid force on one side of a right-triangular plate with edges 3 ft, 4 ft, and 5 ft if the plate sits at the bottom of a pool filled with water to a depth of 6 ft on its 3-ft edge and tilted at 60° to the bottom of the pool.

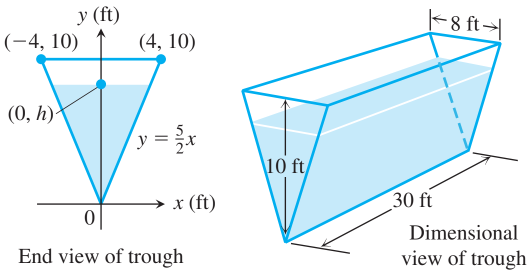
45. The cubical metal tank shown here has a parabolic gate held in place by bolts and designed to withstand a fluid force of 160 lb without rupturing. The liquid you plan to store has a weight-density of 50 lb/ft^3.

a. What is the fluid force on the gate when the liquid is 2 ft deep?

b. What is the maximum height to which the container can be filled without exceeding the gate's design limitation?



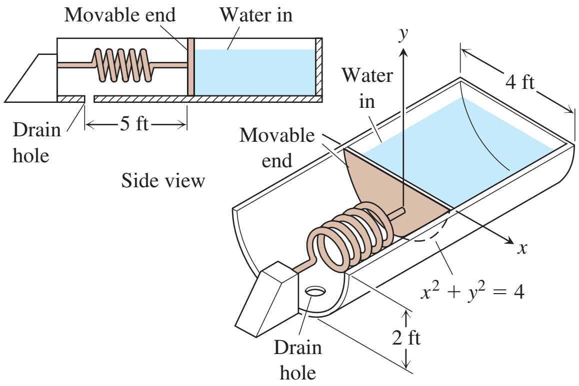
46. The end plates of the trough shown here were designed to withstand a fluid force of 6667 lb. How many cubic feet of water can the tank hold without exceeding this limitation? Round down to the nearest cubic foot. What is the value of h?



47. A vertical rectangular plate a units long by b units wide is submerged in a fluid of weight-density w with its long edges parallel to the fluid's surface. Find the average value of the pressure along the vertical dimension of the plate. Explain your answer.

48. (*Continuation of Exercise 47.*) Show that the force exerted by the fluid on one side of the plate is the average value of the pressure (found in Exercise 47) times the area of the plate.

49. Water pours into the tank shown here at the rate of 4 ft^3/min. The tank's cross-sections are 4-ft-diameter semicircles. One end of the tank is movable, but moving it to increase the volume compresses a spring. The spring constant is $k = 100$ lb/ft. If the end of the tank moves 5 ft against the spring, the water will drain out of a safety hole in the bottom at the rate of 5 ft^3/min. Will the movable end reach the hole before the tank overflows?



50. Watering trough The vertical ends of a watering trough are squares 3 ft on a side.

a. Find the fluid force against the ends when the trough is full.

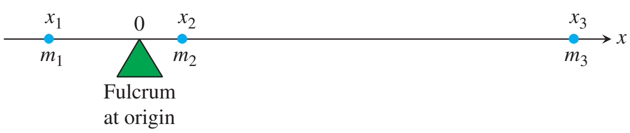
b. How many inches do you have to lower the water level in the trough to reduce the fluid force by 25%?

6.6 Moments and Centers of Mass

Many structures and mechanical systems behave as if their masses were concentrated at a single point, called the *center of mass* (Figure 6.44). It is important to know how to locate this point, and doing so is basically a mathematical enterprise. Here we consider masses distributed along a line or region in the plane. Masses distributed across a region or curve in three-dimensional space are treated in Chapters 15 and 16.

Masses Along a Line

We develop our mathematical model in stages. The first stage is to imagine masses m_1, m_2, and m_3 on a rigid x-axis supported by a fulcrum at the origin.



The resulting system might balance, or it might not, depending on how large the masses are and how they are arranged along the x-axis.

Each mass m_k exerts a downward force $m_k g$ (the weight of m_k) equal to the magnitude of the mass times the acceleration due to gravity. Note that gravitational acceleration is downward, hence negative. Each of these forces has a tendency to turn the x-axis about the origin, the way a child turns a seesaw. This turning effect, called a **torque**, is measured by multiplying the force $m_k g$ by the signed distance x_k from the point of application to the origin. By convention, a positive torque induces a counterclockwise turn. Masses to the left of the origin exert positive (counterclockwise) torque. Masses to the right of the origin exert negative (clockwise) torque.

The sum of the torques measures the tendency of a system to rotate about the origin. This sum is called the **system torque**.

$$\text{System torque} = m_1 g x_1 + m_2 g x_2 + m_3 g x_3 \tag{1}$$

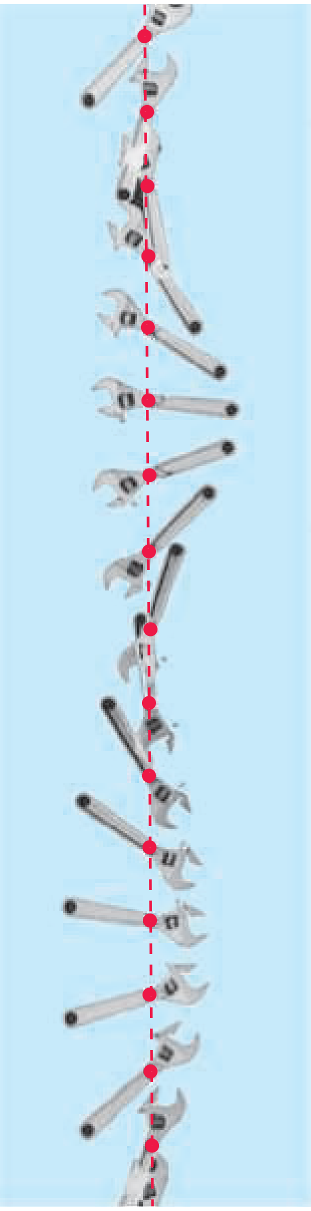


FIGURE 6.44 A wrench gliding on ice turning about its center of mass as the center glides in a vertical line. (*Source: PSSC Physics*, 2nd ed., Reprinted by permission of Education Development Center, Inc.)

The system will balance if and only if its torque is zero.

If we factor out the g in Equation (1), we see that the system torque is

$$\underbrace{g}_{\text{a feature of the environment}} \cdot \underbrace{(m_1x_1 + m_2x_2 + m_3x_3)}_{\text{a feature of the system}}.$$

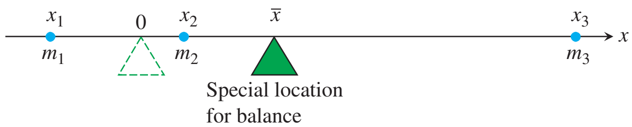
Thus, the torque is the product of the gravitational acceleration g, which is a feature of the environment in which the system happens to reside, and the number $(m_1x_1 + m_2x_2 + m_3x_3)$, which is a feature of the system itself.

The number $(m_1x_1 + m_2x_2 + m_3x_3)$ is called the **moment of the system about the origin**. It is the sum of the **moments** m_1x_1, m_2x_2, m_3x_3 of the individual masses.

$$M_0 = \text{Moment of system about origin} = \sum m_kx_k$$

(We shift to sigma notation here to allow for sums with more terms.)

We usually want to know where to place the fulcrum to make the system balance; that is, we want to know at what point $\bar{x}$ to place the fulcrum to make the torques add to zero.



The torque of each mass about the fulcrum in this special location is

$$\begin{aligned}\text{Torque of } m_k \text{ about } \bar{x} &= \begin{pmatrix}\text{signed distance}\\ \text{of } m_k \text{ from } \bar{x}\end{pmatrix}\begin{pmatrix}\text{downward}\\ \text{force}\end{pmatrix}\\ &= (x_k - \bar{x})m_kg.\end{aligned}$$

When we write the equation that says that the sum of these torques is zero, we get an equation we can solve for $\bar{x}$:

$$\sum (x_k - \bar{x})m_kg = 0 \qquad \text{Sum of the torques equals zero.}$$

$$\bar{x} = \frac{\sum m_kx_k}{\sum m_k}. \qquad \text{Solved for } \bar{x}$$

This last equation tells us to find $\bar{x}$ by dividing the system's moment about the origin by the system's total mass:

$$\bar{x} = \frac{\sum m_kx_k}{\sum m_k} = \frac{\text{system moment about origin}}{\text{system mass}}. \tag{2}$$

The point $\bar{x}$ is called the system's **center of mass**.

Thin Wires

Instead of a discrete set of masses arranged in a line, suppose that we have a straight wire or rod located on interval $[a, b]$ on the x-axis. Suppose further that this wire is not homogeneous, but rather the density varies continuously from point to point. If a short segment of a rod containing the point x with length Δx has mass Δm, then the density at x is given by

$$\delta(x) = \lim_{\Delta x \to 0} \Delta m/\Delta x.$$

We often write this formula in one of the alternative forms $\delta = dm/dx$ and $dm = \delta\, dx$.

Partition the interval $[a, b]$ into finitely many subintervals $[x_{k-1}, x_k]$. If we take n subintervals and replace the portion of a wire along a subinterval of length Δx_k containing x_k by a point mass located at x_k with mass $\Delta m_k = \delta(x_k)\,\Delta x_k$, then we obtain a collection of point masses that have approximately the same total mass and same moment as the wire.

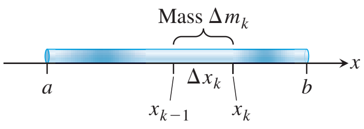


FIGURE 6.45 A rod of varying density can be modeled by a finite number of point masses of mass $\Delta m_k = \delta(x_k)\,\Delta x_k$ located at points x_k along the rod.

The mass M of the wire and the moment M_0 are approximated by the Riemann sums

$$M \approx \sum_{k=1}^{n} \Delta m_k = \sum_{k=1}^{n} \delta(x_k)\,\Delta x_k, \qquad M_0 \approx \sum_{k=1}^{n} x_k\,\Delta m_k = \sum_{k=1}^{n} x_k\,\delta(x_k)\,\Delta x_k.$$

By taking a limit of these Riemann sums as the length of the intervals in the partition approaches zero, we get integral formulas for the mass and the moment of the wire about the origin. The mass M, moment about the origin M_0, and center of mass $\bar{x}$ are

$$M = \int_a^b \delta(x)\,dx, \qquad M_0 = \int_a^b x\,\delta(x)\,dx, \qquad \bar{x} = \frac{M_0}{M} = \frac{\displaystyle\int_a^b x\,\delta(x)\,dx}{\displaystyle\int_a^b \delta(x)\,dx}.$$

EXAMPLE 1 Find the mass M and the center of mass $\bar{x}$ of a rod lying on the x-axis over the interval $[1, 2]$ whose density is given by $\delta(x) = 2 + 3x^2$.

Solution The mass of the rod is obtained by integrating the density,

$$M = \int_1^2 (2 + 3x^2)\,dx = \Big[2x + x^3\Big]_1^2 = (4 + 8) - (2 + 1) = 9,$$

and the center of mass is

$$\bar{x} = \frac{M_0}{M} = \frac{\displaystyle\int_1^2 x(2 + 3x^2)\,dx}{9} = \frac{\left[x^2 + \dfrac{3x^4}{4}\right]_1^2}{9} = \frac{19}{12}.$$

Masses Distributed over a Plane Region

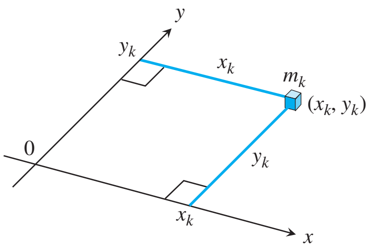


FIGURE 6.46 Each mass m_k has a moment about each axis.

Suppose that we have a finite collection of masses located in the plane, with mass m_k at the point (x_k, y_k) (see Figure 6.46). The mass of the system is

$$\text{System mass:} \qquad M = \sum m_k.$$

Each mass m_k has a moment about each axis. Its moment about the x-axis is $m_k y_k$, and its moment about the y-axis is $m_k x_k$. The moments of the entire system about the two axes are

$$\text{Moment about } x\text{-axis:} \qquad M_x = \sum m_k y_k,$$
$$\text{Moment about } y\text{-axis:} \qquad M_y = \sum m_k x_k.$$

The x-coordinate of the system's center of mass is defined to be

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}. \tag{3}$$

With this choice of $\bar{x}$, as in the one-dimensional case, the system balances about the line $x = \bar{x}$ (Figure 6.47).

The y-coordinate of the system's center of mass is defined to be

$$\bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}. \tag{4}$$

With this choice of $\bar{y}$, the system balances about the line $y = \bar{y}$ as well. The torques exerted by the masses about the line $y = \bar{y}$ cancel out. Thus, as far as balance is concerned, the system behaves as if all its mass were at the single point $(\bar{x}, \bar{y})$. We call this point the system's **center of mass**.

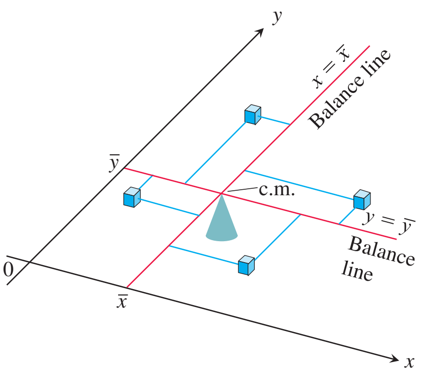


FIGURE 6.47 A two-dimensional array of masses balances on its center of mass.

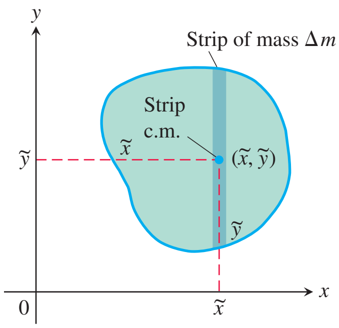


FIGURE 6.48 A plate cut into thin strips parallel to the y-axis. The moment exerted by a typical strip about each axis is the moment its mass Δm would exert if concentrated at the strip's center of mass $(\tilde{x}, \tilde{y})$.

Thin, Flat Plates

In many applications, we need to find the center of mass of a thin, flat plate: a disk of aluminum, say, or a triangular sheet of steel. In such cases, we assume the distribution of mass to be continuous, and the formulas we use to calculate $\bar{x}$ and $\bar{y}$ contain integrals instead of finite sums. The integrals arise in the following way.

Imagine that the plate occupying a region in the xy-plane is cut into thin strips parallel to one of the axes (in Figure 6.48, the y-axis). The center of mass of a typical strip is $(\tilde{x}, \tilde{y})$. We treat the strip's mass Δm as if it were concentrated at $(\tilde{x}, \tilde{y})$. The moment of the strip about the y-axis is then $\tilde{x}\,\Delta m$. The moment of the strip about the x-axis is $\tilde{y}\,\Delta m$. Equations (3) and (4) then become

$$\bar{x} = \frac{M_y}{M} = \frac{\sum \tilde{x}\,\Delta m}{\sum \Delta m}, \qquad \bar{y} = \frac{M_x}{M} = \frac{\sum \tilde{y}\,\Delta m}{\sum \Delta m}.$$

These sums are Riemann sums for integrals, and they approach these integrals in the limit as the strips become narrower and narrower. We write these integrals symbolically as

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} \quad \text{and} \quad \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm}.$$

Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the *xy*-Plane

$$\begin{aligned}
&\text{Moment about the } x\text{-axis:} && M_x = \int \tilde{y}\,dm \\
&\text{Moment about the } y\text{-axis:} && M_y = \int \tilde{x}\,dm \\
&\text{Mass:} && M = \int dm \qquad (5)\\
&\text{Center of mass:} && \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}
\end{aligned}$$

Density of a plate
A material's density is its mass per unit area. For wires, rods, and narrow strips, the density is given by mass per unit length.

The differential dm in these integrals is the mass of the strip. For this section, we assume the density δ of the plate is a constant or a continuous function of x. Then $dm = \delta\,dA$, which is the mass per unit area δ times the area dA of the strip.

To evaluate the integrals in Equations (5), we picture the plate in the coordinate plane and sketch a strip of mass parallel to one of the coordinate axes. We then express the strip's mass dm and the coordinates $(\tilde{x}, \tilde{y})$ of the strip's center of mass in terms of x or y. Finally, we integrate $\tilde{y}\,dm$, $\tilde{x}\,dm$, and dm between limits of integration determined by the plate's location in the plane.

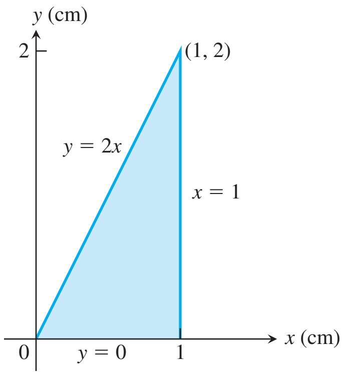


FIGURE 6.49 The plate in Example 2.

EXAMPLE 2 The triangular plate shown in Figure 6.49 has a constant density of $\delta = 3\ \mathrm{g/cm^2}$. Find

(a) the plate's moment M_y about the y-axis.

(b) the plate's mass M.

(c) the x-coordinate of the plate's center of mass (c.m.).

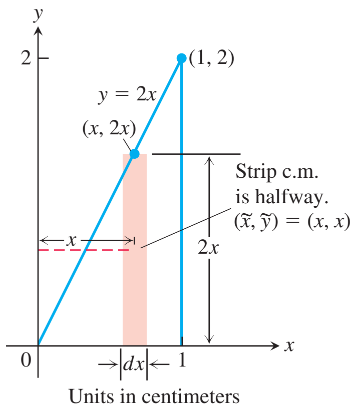


FIGURE 6.50 Modeling the plate in Example 2 with vertical strips.

Solution **Method 1: Vertical Strips** (Figure 6.50)

(a) The moment M_y: The typical vertical strip has the following relevant data.

$$\begin{aligned}
\text{center of mass (c.m.):}\quad & (\tilde{x}, \tilde{y}) = (x, x)\\
\text{length:}\quad & 2x\\
\text{width:}\quad & dx\\
\text{area:}\quad & dA = 2x\,dx\\
\text{mass:}\quad & dm = \delta\,dA = 3 \cdot 2x\,dx = 6x\,dx\\
\text{distance of c.m. from } y\text{-axis:}\quad & \tilde{x} = x
\end{aligned}$$

The moment of the strip about the y-axis is

$$\tilde{x}\,dm = x \cdot 6x\,dx = 6x^2\,dx.$$

The moment of the plate about the y-axis is therefore

$$M_y = \int \tilde{x}\,dm = \int_0^1 6x^2\,dx = 2x^3\Big]_0^1 = 2\text{ g}\cdot\text{cm}.$$

(b) The plate's mass:

$$M = \int dm = \int_0^1 6x\,dx = 3x^2\Big]_0^1 = 3\text{ g}.$$

(c) The x-coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2\text{ g}\cdot\text{cm}}{3\text{ g}} = \frac{2}{3}\text{ cm}.$$

By a similar computation, we could find M_x and $\bar{y} = M_x/M$.

Method 2: Horizontal Strips (Figure 6.51)

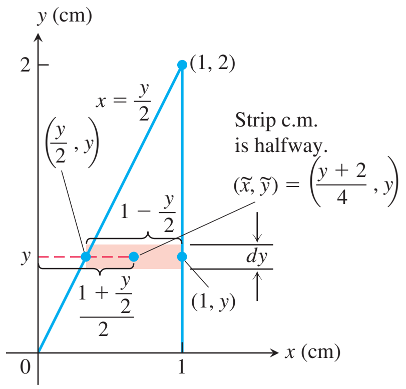


FIGURE 6.51 Modeling the plate in Example 2 with horizontal strips.

(a) The moment M_y: The y-coordinate of the center of mass of a typical horizontal strip is y (see the figure), so

$$\tilde{y} = y.$$

The x-coordinate is the x-coordinate of the point halfway across the triangle. This makes it the average of $y/2$ (the strip's left-hand x-value) and 1 (the strip's right-hand x-value):

$$\tilde{x} = \frac{(y/2) + 1}{2} = \frac{y}{4} + \frac{1}{2} = \frac{y+2}{4}.$$

We also have

$$\begin{aligned}
\text{length:}\quad & 1 - \frac{y}{2} = \frac{2-y}{2}\\
\text{width:}\quad & dy\\
\text{area:}\quad & dA = \frac{2-y}{2}\,dy\\
\text{mass:}\quad & dm = \delta\,dA = 3 \cdot \frac{2-y}{2}\,dy\\
\text{distance of c.m. to } y\text{-axis:}\quad & \tilde{x} = \frac{y+2}{4}.
\end{aligned}$$

The moment of the strip about the y-axis is

$$\tilde{x}\,dm = \frac{y+2}{4} \cdot 3 \cdot \frac{2-y}{2}\,dy = \frac{3}{8}(4 - y^2)\,dy.$$

The moment of the plate about the y-axis is

$$M_y = \int \tilde{x}\, dm = \int_0^2 \frac{3}{8}(4 - y^2)\, dy = \frac{3}{8}\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{3}{8}\left(\frac{16}{3}\right) = 2 \text{ g} \cdot \text{cm}.$$

(b) The plate's mass:

$$M = \int dm = \int_0^2 \frac{3}{2}(2 - y)\, dy = \frac{3}{2}\left[2y - \frac{y^2}{2}\right]_0^2 = \frac{3}{2}(4 - 2) = 3 \text{ g}.$$

(c) The x-coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2 \text{ g} \cdot \text{cm}}{3 \text{ g}} = \frac{2}{3} \text{ cm}.$$

By a similar computation, we could find M_x and $\bar{y}$. ■

If the distribution of mass in a thin, flat plate has an axis of symmetry, the center of mass will lie on this axis. If there are two axes of symmetry, the center of mass will lie at their intersection. These facts often help to simplify our work.

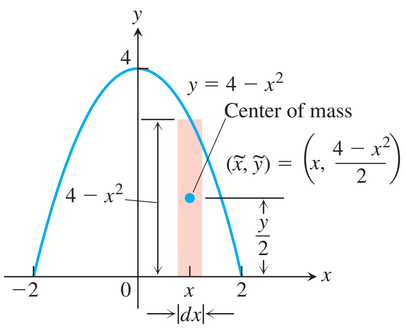


FIGURE 6.52 Modeling the plate in Example 3 with vertical strips.

EXAMPLE 3 Find the center of mass of a thin plate covering the region bounded above by the parabola $y = 4 - x^2$ and below by the x-axis (Figure 6.52). Assume the density of the plate at the point (x, y) is $\delta = 2x^2$, which is twice the square of the distance from the point to the y-axis.

Solution The mass distribution is symmetric about the y-axis, so $\bar{x} = 0$. We model the distribution of mass with vertical strips, since the density is given as a function of the variable x. The typical vertical strip (see Figure 6.52) has the following relevant data.

center of mass (c.m.):	$(\tilde{x}, \tilde{y}) = \left(x, \frac{4 - x^2}{2}\right)$
length:	$4 - x^2$
width:	dx
area:	$dA = (4 - x^2)\, dx$
mass:	$dm = \delta\, dA = \delta(4 - x^2)\, dx$
distance from c.m. to x-axis:	$\tilde{y} = \frac{4 - x^2}{2}$

The moment of the strip about the x-axis is

$$\tilde{y}\, dm = \frac{4 - x^2}{2} \cdot \delta(4 - x^2)\, dx = \frac{\delta}{2}(4 - x^2)^2\, dx.$$

The moment of the plate about the x-axis is

$$M_x = \int \tilde{y}\, dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\, dx = \int_{-2}^{2} x^2(4 - x^2)^2\, dx$$
$$= \int_{-2}^{2} (16x^2 - 8x^4 + x^6)\, dx = \frac{2048}{105}.$$

The mass of the plate is

$$M = \int dm = \int_{-2}^{2} \delta(4 - x^2)\, dx = \int_{-2}^{2} 2x^2(4 - x^2)\, dx$$
$$= \int_{-2}^{2} (8x^2 - 2x^4)\, dx = \frac{256}{15}.$$

Therefore,

$$\bar{y} = \frac{M_x}{M} = \frac{2048}{105} \cdot \frac{15}{256} = \frac{8}{7}.$$

The plate's center of mass is

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{7}\right).$$

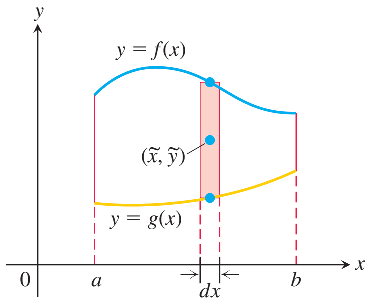


FIGURE 6.53 Modeling the plate bounded by two curves with vertical strips. The strip c.m. is halfway, so $\tilde{y} = \frac{1}{2}[f(x) + g(x)]$.

Plates Bounded by Two Curves

Suppose a plate covers a region that lies between two curves $y = g(x)$ and $y = f(x)$, where $f(x) \geq g(x)$ and $a \leq x \leq b$. The typical vertical strip (see Figure 6.53) has

$$\begin{aligned}
\text{center of mass (c.m.):} \quad & (\tilde{x}, \tilde{y}) = \left(x, \tfrac{1}{2}[f(x) + g(x)]\right) \\
\text{length:} \quad & f(x) - g(x) \\
\text{width:} \quad & dx \\
\text{area:} \quad & dA = [f(x) - g(x)]\,dx \\
\text{mass:} \quad & dm = \delta\,dA = \delta[f(x) - g(x)]\,dx.
\end{aligned}$$

The moment of the plate about the y-axis is

$$M_y = \int x\,dm = \int_a^b x\delta[f(x) - g(x)]\,dx,$$

and the moment about the x-axis is

$$\begin{aligned}
M_x &= \int y\,dm = \int_a^b \frac{1}{2}[f(x) + g(x)] \cdot \delta[f(x) - g(x)]\,dx \\
&= \int_a^b \frac{\delta}{2}[f^2(x) - g^2(x)]\,dx.
\end{aligned}$$

These moments give us the following formulas.

$$\bar{x} = \frac{1}{M}\int_a^b \delta x[f(x) - g(x)]\,dx \tag{6}$$

$$\bar{y} = \frac{1}{M}\int_a^b \frac{\delta}{2}[f^2(x) - g^2(x)]\,dx \tag{7}$$

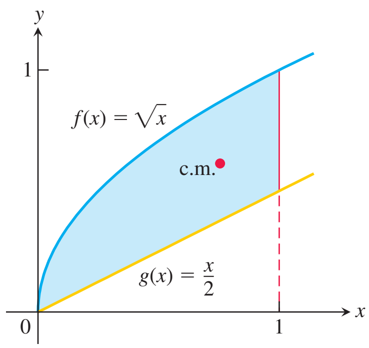


FIGURE 6.54 The region in Example 4.

EXAMPLE 4 Find the center of mass for the thin plate bounded by the curves $g(x) = x/2$ and $f(x) = \sqrt{x}$, $0 \leq x \leq 1$ (Figure 6.54), using Equations (6) and (7) with the density function $\delta(x) = x^2$.

Solution We first compute the mass of the plate, using $dm = \delta[f(x) - g(x)]\,dx$:

$$M = \int_0^1 x^2\left(\sqrt{x} - \frac{x}{2}\right)dx = \int_0^1 \left(x^{5/2} - \frac{x^3}{2}\right)dx = \left[\frac{2}{7}x^{7/2} - \frac{1}{8}x^4\right]_0^1 = \frac{9}{56}.$$

Then from Equations (6) and (7) we get

$$\begin{aligned}
\bar{x} &= \frac{56}{9}\int_0^1 x^2 \cdot x\left(\sqrt{x} - \frac{x}{2}\right) dx \\
&= \frac{56}{9}\int_0^1 \left(x^{7/2} - \frac{x^4}{2}\right) dx \\
&= \frac{56}{9}\left[\frac{2}{9}x^{9/2} - \frac{1}{10}x^5\right]_0^1 = \frac{308}{405},
\end{aligned}$$

and

$$\begin{aligned}
\bar{y} &= \frac{56}{9}\int_0^1 \frac{x^2}{2}\left(x - \frac{x^2}{4}\right) dx \\
&= \frac{28}{9}\int_0^1 \left(x^3 - \frac{x^4}{4}\right) dx \\
&= \frac{28}{9}\left[\frac{1}{4}x^4 - \frac{1}{20}x^5\right]_0^1 = \frac{252}{405}.
\end{aligned}$$

The center of mass is shown in Figure 6.54.

Centroids

The center of mass in Example 4 is not located at the geometric center of the region. This is due to the region's nonuniform density. When the density function is constant, it cancels out of the numerator and denominator of the formulas for $\bar{x}$ and $\bar{y}$. Thus, when the density is constant, the location of the center of mass is a feature of the geometry of the object and not of the material from which it is made. In such cases, engineers may call the center of mass the **centroid** of the shape, as in "Find the centroid of a triangle or a solid cone." To do so, just set δ equal to 1 and proceed to find $\bar{x}$ and $\bar{y}$ as before, by dividing moments by masses.

EXAMPLE 5 Find the center of mass (centroid) of a thin wire of constant density δ shaped like a semicircle of radius a.

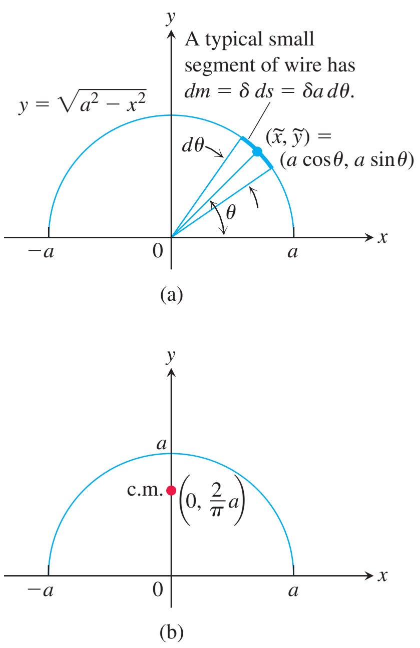


FIGURE 6.55 The semicircular wire in Example 5. (a) The dimensions and variables used in finding the center of mass. (b) The center of mass does not lie on the wire.

Solution We model the wire with the semicircle $y = \sqrt{a^2 - x^2}$ (Figure 6.55). The distribution of mass is symmetric about the y-axis, so $\bar{x} = 0$. To find $\bar{y}$, we imagine the wire divided into short subarc segments. If $(\tilde{x}, \tilde{y})$ is the center of mass of a subarc and θ is the angle between the x-axis and the radial line joining the origin to $(\tilde{x}, \tilde{y})$, then $\tilde{y} = a\sin\theta$ is a function of the angle θ measured in radians (see Figure 6.55a). The length ds of the subarc containing $(\tilde{x}, \tilde{y})$ subtends an angle of $d\theta$ radians, so $ds = a\, d\theta$. Thus a typical subarc segment has these relevant data for calculating $\bar{y}$:

$$\begin{aligned}
\text{length:}\quad & ds = a\, d\theta \\
\text{mass:}\quad & dm = \delta\, ds = \delta a\, d\theta \qquad \text{Mass per unit length times length} \\
\text{distance of c.m. to } x\text{-axis:}\quad & \tilde{y} = a\sin\theta.
\end{aligned}$$

Hence,

$$\bar{y} = \frac{\int \tilde{y}\, dm}{\int dm} = \frac{\int_0^\pi a\sin\theta \cdot \delta a\, d\theta}{\int_0^\pi \delta a\, d\theta} = \frac{\delta a^2\left[-\cos\theta\right]_0^\pi}{\delta a\pi} = \frac{2}{\pi}a.$$

The center of mass lies on the axis of symmetry at the point $(0, 2a/\pi)$, about two-thirds of the way up from the origin (Figure 6.55b). Notice how δ cancels in the equation for $\bar{y}$, so we could have set $\delta = 1$ everywhere and obtained the same value for $\bar{y}$.

In Example 5 we found the center of mass of a thin wire lying along the graph of a differentiable function in the xy-plane. In Chapter 16 we will learn how to find the center of mass of a wire lying along a more general smooth curve in the plane or in space.

Fluid Forces and Centroids

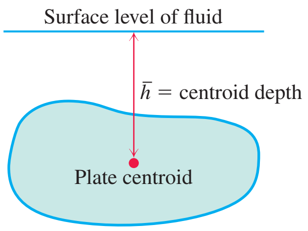


FIGURE 6.56 The force against one side of the plate is $w \cdot \bar{h} \cdot$ plate area.

If we know the location of the centroid of a submerged flat vertical plate (Figure 6.56), we can take a shortcut to find the force against one side of the plate. From Equation (7) in Section 6.5, and the definition of the moment about the x-axis, we have

$$\begin{aligned} F &= \int_a^b w \times (\text{strip depth}) \times L(y)\, dy \\ &= w\int_a^b (\text{strip depth}) \times L(y)\, dy \\ &= w \times (\text{moment about surface level line of region occupied by plate}) \\ &= w \times (\text{depth of plate's centroid}) \times (\text{area of plate}). \end{aligned}$$

Fluid Forces and Centroids

The force of a fluid of weight-density w against one side of a submerged flat vertical plate is the product of w, the distance $\bar{h}$ from the plate's centroid to the fluid surface, and the plate's area:

$$F = w\bar{h}A. \tag{8}$$

EXAMPLE 6 A flat isosceles triangular plate with base 6 ft and height 3 ft is submerged vertically, base up with its vertex at the origin, so that the base is 2 ft below the surface of a swimming pool. (This is Example 6, Section 6.5.) Use Equation (8) to find the force exerted by the water against one side of the plate.

Solution The centroid of the triangle (Figure 6.43) lies on the y-axis, one-third of the way from the base to the vertex, so $\bar{h} = 3$ (where $y = 2$), since the pool's surface is $y = 5$. The triangle's area is

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(6)(3) = 9.$$

Hence,

$$F = w\bar{h}A = (62.4)(3)(9) = 1684.8 \text{ lb.}$$

■

The Theorems of Pappus

In the fourth century, an Alexandrian Greek named Pappus discovered two formulas that relate centroids to surfaces and solids of revolution. The formulas provide shortcuts to a number of otherwise lengthy calculations.

THEOREM 1—Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A. \tag{9}$$

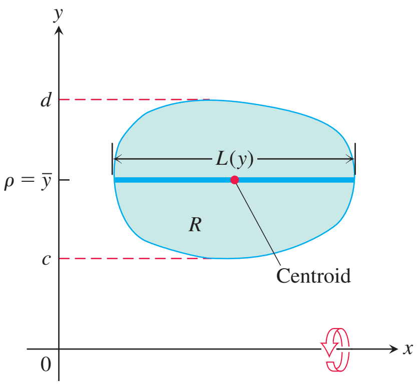


FIGURE 6.57 The region R is to be revolved (once) about the x-axis to generate a solid. A 1700-year-old theorem says that the solid's volume can be calculated by multiplying the region's area by the distance traveled by its centroid during the revolution.

Proof We draw the axis of revolution as the x-axis with the region R in the first quadrant (Figure 6.57). We let $L(y)$ denote the length of the cross-section of R perpendicular to the y-axis at y. We assume $L(y)$ to be continuous.

By the method of cylindrical shells, the volume of the solid generated by revolving the region about the x-axis is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\,dy = 2\pi \int_c^d y\,L(y)\,dy. \tag{10}$$

The y-coordinate of R's centroid is

$$\bar{y} = \frac{\int_c^d \tilde{y}\,dA}{A} = \frac{\int_c^d y\,L(y)\,dy}{A}, \qquad \tilde{y} = y,\ dA = L(y)\,dy$$

so that

$$\int_c^d y\,L(y)\,dy = A\bar{y}.$$

Substituting $A\bar{y}$ for the last integral in Equation (10) gives $V = 2\pi\bar{y}A$. With ρ equal to $\bar{y}$, we have $V = 2\pi\rho A$. ■

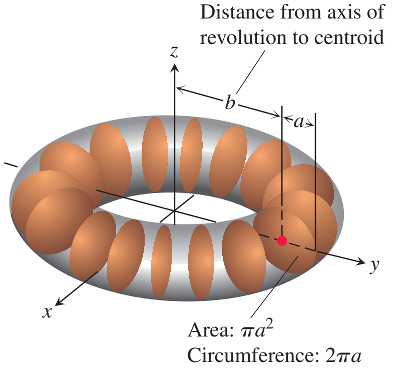


FIGURE 6.58 With Pappus's first theorem, we can find the volume of a torus without having to integrate (Example 7).

EXAMPLE 7 Find the volume of the torus (doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \ge a$ from its center (Figure 6.58).

Solution We apply Pappus's Theorem for volumes. The centroid of a disk is located at its center, the area is $A = \pi a^2$, and $\rho = b$ is the distance from the centroid to the axis of revolution (see Figure 6.58). Substituting these values into Equation (9), we find the volume of the torus to be

$$V = 2\pi(b)(\pi a^2) = 2\pi^2 ba^2.$$ ■

The next example shows how we can use Equation (9) in Pappus's Theorem to find one of the coordinates of the centroid of a plane region of known area A when we also know the volume V of the solid generated by revolving the region about the other coordinate axis. That is, if $\bar{y}$ is the coordinate we want to find, we revolve the region around the x-axis so that $\bar{y} = \rho$ is the distance from the centroid to the axis of revolution. The idea is that the rotation generates a solid of revolution whose volume V is an already known quantity. Then we can solve Equation (9) for ρ, which is the value of the centroid's coordinate $\bar{y}$.

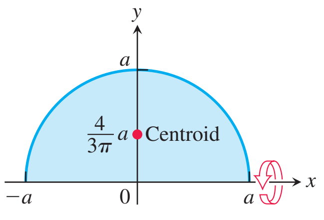


FIGURE 6.59 With Pappus's first theorem, we can locate the centroid of a semicircular region without having to integrate (Example 8).

EXAMPLE 8 Locate the centroid of a semicircular region of radius a.

Solution We consider the region between the semicircle $y = \sqrt{a^2 - x^2}$ (Figure 6.59) and the x-axis and imagine revolving the region about the x-axis to generate a solid sphere. By symmetry, the x-coordinate of the centroid is $\bar{x} = 0$. With $\bar{y} = \rho$ in Equation (9), we have

$$\bar{y} = \frac{V}{2\pi A} = \frac{(4/3)\pi a^3}{2\pi(1/2)\pi a^2} = \frac{4}{3\pi}a.$$ ■

THEOREM 2—Pappus's Theorem for Surface Areas
If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length L of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \qquad (11)$$

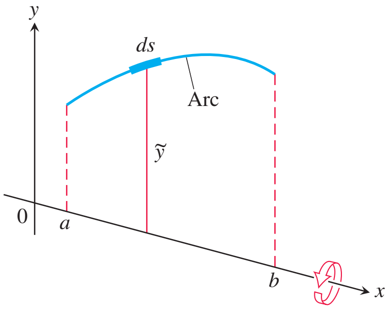


FIGURE 6.60 Figure for proving Pappus's Theorem for surface area. The arc length differential ds is given by Equation (6) in Section 6.3.

The proof we give assumes that we can model the axis of revolution as the x-axis and the arc as the graph of a continuously differentiable function of x.

Proof We draw the axis of revolution as the x-axis with the arc extending from $x = a$ to $x = b$ in the first quadrant (Figure 6.60). The area of the surface generated by the arc is

$$S = \int_{x=a}^{x=b} 2\pi y\, ds = 2\pi \int_{x=a}^{x=b} y\, ds. \qquad (12)$$

The y-coordinate of the arc's centroid is

$$\bar{y} = \frac{\int_{x=a}^{x=b} \widetilde{y}\, ds}{\int_{x=a}^{x=b} ds} = \frac{\int_{x=a}^{x=b} y\, ds}{L}.$$

$L = \int ds$ is the arc's length and $\widetilde{y} = y$.

Hence

$$\int_{x=a}^{x=b} y\, ds = \bar{y}L.$$

Substituting $\bar{y}L$ for the last integral in Equation (12) gives $S = 2\pi\bar{y}L$. With ρ equal to $\bar{y}$, we have $S = 2\pi\rho L$. ■

EXAMPLE 9 Use Pappus's area theorem to find the surface area of the torus in Example 7.

Solution From Figure 6.58, the surface of the torus is generated by revolving a circle of radius a about the z-axis, and $b \geq a$ is the distance from the centroid to the axis of revolution. The arc length of the smooth curve generating this surface of revolution is the circumference of the circle, so $L = 2\pi a$. Substituting these values into Equation (11), we find the surface area of the torus to be

$$S = 2\pi(b)(2\pi a) = 4\pi^2 ba.$$ ■

EXERCISES 6.6

Mass of a wire

In Exercises 1–6, find the mass M and center of mass $\bar{x}$ of the linear wire covering the given interval and having the given density $\delta(x)$.

1. $1 \leq x \leq 4, \quad \delta(x) = \sqrt{x}$
2. $-3 \leq x \leq 3, \quad \delta(x) = 1 + 3x^2$
3. $0 \leq x \leq 3, \quad \delta(x) = \dfrac{1}{x+1}$
4. $1 \leq x \leq 2, \quad \delta(x) = \dfrac{8}{x^3}$
5. $\delta(x) = \begin{cases} 4, & 0 \leq x \leq 2 \\ 5, & 2 < x \leq 3 \end{cases}$
6. $\delta(x) = \begin{cases} 2 - x, & 0 \leq x < 1 \\ x, & 1 \leq x \leq 2 \end{cases}$

Thin Plates with Constant Density

In Exercises 18, find the center of mass of a thin plate of constant density δ covering the given region.

7. The region bounded by the parabola $y = x^2$ and the line $y = 4$

8. The region bounded by the parabola $y = 25 - x^2$ and the x-axis

9. The region bounded by the parabola $y = x - x^2$ and the line $y = -x$

10. The region enclosed by the parabolas $y = x^2 - 3$ and $y = -2x^2$

11. The region bounded by the y-axis and the curve $x = y - y^3$, $0 \le y \le 1$

12. The region bounded by the parabola $x = y^2 - y$ and the line $y = x$

13. The region bounded by the x-axis and the curve $y = \cos x$, $-\pi/2 \le x \le \pi/2$

14. The region between the curve $y = \sec^2 x$, $-\pi/4 \le x \le \pi/4$ and the x-axis

15. a. The region cut from the first quadrant by the circle $x^2 + y^2 = 9$

b. The region bounded by the x-axis and the semicircle $y = \sqrt{9 - x^2}$

Compare your answer in part (b) with the answer in part (a).

16. The region bounded by the parabolas $y = 2x^2 - 4x$ and $y = 2x - x^2$

17. The region between the curve $y = 1/\sqrt{x}$ and the x-axis from $x = 1$ to $x = 16$

18. The region bounded above by the curve $y = 1/x^3$, below by the curve $y = -1/x^3$, and on the left and right by the lines $x = 1$ and $x = a > 1$. Also, find $\lim_{a\to\infty} \bar{x}$.

19. Consider a region bounded by the graphs of $y = x^4$ and $y = x^5$. Show that the center of mass lies outside the region.

20. Consider a thin plate of constant density δ lies in the region bounded by the graphs of $y = \sqrt{x}$ and $x = 2y$. Find the plate's

a. moment about the x-axis.

b. moment about the y-axis.

c. moment about the line $x = 5$.

d. moment about the line $x = -1$.

e. moment about the line $y = 2$.

f. moment about the line $y = -3$.

g. mass.

h. center of mass.

Thin Plates with Varying Density

21. Find the center of mass of a thin plate covering the region between the x-axis and the curve $y = 2/x^2$, $1 \le x \le 2$, if the plate's density at the point (x, y) is $\delta(x) = x^2$.

22. Find the center of mass of a thin plate covering the region bounded below by the parabola $y = x^2$ and above by the line $y = x$ if the plate's density at the point (x, y) is $\delta(x) = 12x$.

23. The region bounded by the curves $y = \pm 4/\sqrt{x}$ and the lines $x = 1$ and $x = 4$ is revolved about the y-axis to generate a solid.

a. Find the volume of the solid.

b. Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = 1/x$.

c. Sketch the plate and show the center of mass in your sketch.

24. The region between the curve $y = 2/x$ and the x-axis from $x = 1$ to $x = 4$ is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

b. Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = \sqrt{x}$.

c. Sketch the plate and show the center of mass in your sketch.

Centroids of Triangles

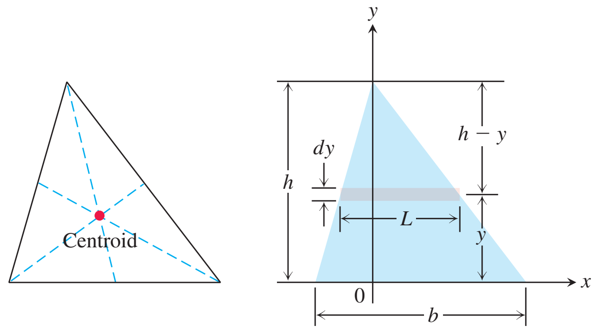
25. The centroid of a triangle lies at the intersection of the triangle's medians You may recall that the point inside a triangle that lies one-third of the way from each side toward the opposite vertex is the point where the triangle's three medians intersect. Show that the centroid lies at the intersection of the medians by showing that it too lies one-third of the way from each side toward the opposite vertex. To do so, take the following steps.

i) Stand one side of the triangle on the x-axis as in part (b) of the accompanying figure. Express dm in terms of L and dy.

ii) Use similar triangles to show that $L = (b/h)(h - y)$. Substitute this expression for L in your formula for dm.

iii) Show that $\bar{y} = h/3$.

iv) Extend the argument to the other sides.



(a) (b)

Use the result in Exercise 25 to find the centroids of the triangles whose vertices appear in Exercises 26–30. Assume $a, b > 0$.

26. $(-1, 0), (1, 0), (0, 3)$

27. $(0, 0), (1, 0), (0, 1)$

28. $(0, 0), (a, 0), (0, a)$

29. $(0, 0), (a, 0), (0, b)$

30. $(0, 0), (a, 0), (a/2, b)$

Thin Wires

31. Constant density Find the moment about the x-axis of a wire of constant density that lies along the curve $y = \sqrt{x}$ from $x = 0$ to $x = 2$.

32. Constant density Find the moment about the x-axis of a wire of constant density that lies along the curve $y = x^3$ from $x = 0$ to $x = 1$.

33. Variable density Suppose that the density of the wire in Example 5 is $\delta = k \sin\theta$ (k constant). Find the center of mass.

34. Variable density Suppose that the density of the wire in Example 5 is $\delta = 1 + k|\cos\theta|$ (k constant). Find the center of mass.

Plates Bounded by Two Curves

In Exercises 35–38, find the centroid of the thin plate bounded by the graphs of the given functions. Use Equations (6) and (7) with $\delta = 1$ and $M =$ area of the region covered by the plate.

35. $g(x) = x^2$ and $f(x) = x + 6$

36. $g(x) = x^2(x + 1)$, $f(x) = 2$, and $x = 0$

37. $g(x) = x^2(x - 1)$ and $f(x) = x^2$

38. $g(x) = 0$, $f(x) = 2 + \sin x$, $x = 0$, and $x = 2\pi$

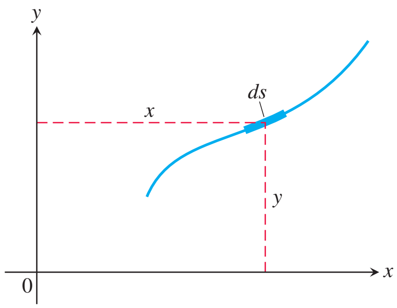
(*Hint:* $\int x \sin x\, dx = \sin x - x \cos x + C.$)

Theory and Examples

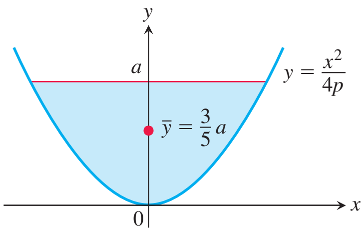
Verify the statements and formulas in Exercises 39 and 40.

39. The coordinates of the centroid of a differentiable plane curve are

$$\bar{x} = \frac{\int x\, ds}{\text{length}}, \qquad \bar{y} = \frac{\int y\, ds}{\text{length}}.$$



40. Whatever the value of $p > 0$ in the equation $y = x^2/(4p)$, the y-coordinate of the centroid of the parabolic segment shown here is $\bar{y} = (3/5)a$.



The Theorems of Pappus

41. The square region with vertices (0, 2), (2, 0), (4, 2), and (2, 4) is revolved about the x-axis to generate a solid. Find the volume and surface area of the solid.

42. Use a theorem of Pappus to find the volume generated by revolving about the line $x = 5$ the triangular region bounded by the coordinate axes and the line $2x + y = 6$ (see Exercise 25).

43. Find the volume of the torus generated by revolving the circle $(x - 2)^2 + y^2 = 1$ about the y-axis.

44. Use the theorems of Pappus to find the lateral surface area and the volume of a right-circular cone.

45. Use Pappus's Theorem for surface area and the fact that the surface area of a sphere of radius a is $4\pi a^2$ to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.

46. As found in Exercise 45, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface swept out by revolving the semicircle about the line $y = a$.

47. The area of the region R enclosed by the semiellipse $y = (b/a)\sqrt{a^2 - x^2}$ and the x-axis is $(1/2)\pi ab$, and the volume of the ellipsoid generated by revolving R about the x-axis is $(4/3)\pi ab^2$. Find the centroid of R. Notice that the location is independent of a.

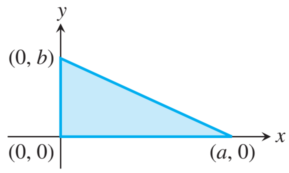
48. As found in Example 8, the centroid of the region enclosed by the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 4a/3\pi)$. Find the volume of the solid generated by revolving this region about the line $y = -a$.

49. The region of Exercise 48 is revolved about the line $y = x - a$ to generate a solid. Find the volume of the solid.

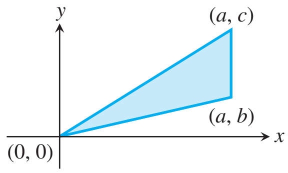
50. As found in Exercise 45, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface generated by revolving the semicircle about the line $y = x - a$.

In Exercises 51 and 52, use a theorem of Pappus to find the centroid of the given triangle. Use the fact that the volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.

51.



52.



CHAPTER 6 Questions to Guide Your Review

1. How do you define and calculate the volumes of solids by the method of slicing? Give an example.
2. How are the disk and washer methods for calculating volumes derived from the method of slicing? Give examples of volume calculations by these methods.
3. Describe the method of cylindrical shells. Give an example.
4. How do you find the length of the graph of a smooth function over a closed interval? Give an example. What about functions that do not have continuous first derivatives?
5. How do you define and calculate the area of the surface swept out by revolving the graph of a smooth function $y = f(x)$, $a \le x \le b$, about the x-axis? Give an example.

6. How do you define and calculate the work done by a variable force directed along a portion of the x-axis? How do you calculate the work it takes to pump a liquid from a tank? Give examples.

7. How do you calculate the force exerted by a liquid against a portion of a flat vertical wall? Give an example.

8. What is a center of mass? a centroid?

9. How do you locate the center of mass of a thin flat plate of material? Give an example.

10. How do you locate the center of mass of a thin plate bounded by two curves $y = f(x)$ and $y = g(x)$ over $a \le x \le b$?

CHAPTER 6 Practice Exercises

Volumes

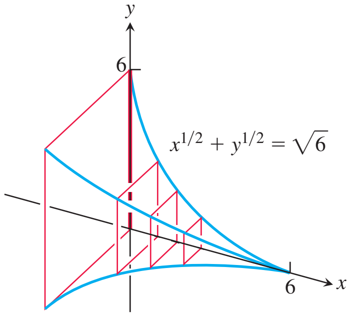
Find the volumes of the solids in Exercises 1–18.

1. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$.

2. The base of the solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross-sections of the solid perpendicular to the x-axis are equilateral triangles whose bases stretch from the line to the curve.

3. The solid lies between planes perpendicular to the x-axis at $x = \pi/4$ and $x = 5\pi/4$. The cross-sections between these planes are circular disks whose diameters run from the curve $y = 2\cos x$ to the curve $y = 2\sin x$.

4. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 6$. The cross-sections between these planes are squares whose bases run from the x-axis up to the curve $x^{1/2} + y^{1/2} = \sqrt{6}$.



5. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections of the solid perpendicular to the x-axis between these planes are circular disks whose diameters run from the curve $x^2 = 4y$ to the curve $y^2 = 4x$.

6. The base of the solid is the region bounded by the parabola $y^2 = 4x$ and the line $x = 1$ in the xy-plane. Each cross-section perpendicular to the x-axis is an equilateral triangle with one edge in the plane. (The triangles all lie on the same side of the plane.)

7. Find the volume of the solid generated by revolving the region bounded by the x-axis, the curve $y = 3x^4$, and the lines $x = 1$ and $x = -1$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 1$; **(d)** the line $y = 3$.

8. Find the volume of the solid generated by revolving the "triangular" region bounded by the curve $y = 4/x^3$ and the lines $x = 1$ and $y = 1/2$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 2$; **(d)** the line $y = 4$.

9. Find the volume of the solid generated by revolving the region bounded on the left by the parabola $x = y^2 + 1$ and on the right by the line $x = 5$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 5$.

10. Find the volume of the solid generated by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 4$; **(d)** the line $y = 4$.

11. Find the volume of the solid generated by revolving the "triangular" region bounded by the x-axis, the line $x = \pi/3$, and the curve $y = \tan x$ in the first quadrant about the x-axis.

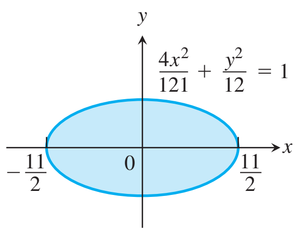
12. Find the volume of the solid generated by revolving the region bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$, and $y = 2$ about the line $y = 2$.

13. Find the volume of the solid generated by revolving the region between the x-axis and the curve $y = x^2 - 2x$ about **(a)** the x-axis; **(b)** the line $y = -1$; **(c)** the line $x = 2$; **(d)** the line $y = 2$.

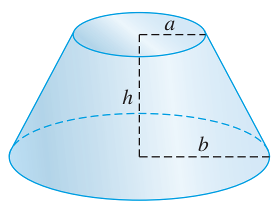
14. Find the volume of the solid generated by revolving about the x-axis the region bounded by $y = 2\tan x$, $y = 0$, $x = -\pi/4$, and $x = \pi/4$. (The region lies in the first and third quadrants and resembles a skewed bowtie.)

15. Volume of a solid sphere hole A round hole of radius $\sqrt{3}$ ft is bored through the center of a solid sphere of a radius 2 ft. Find the volume of material removed from the sphere.

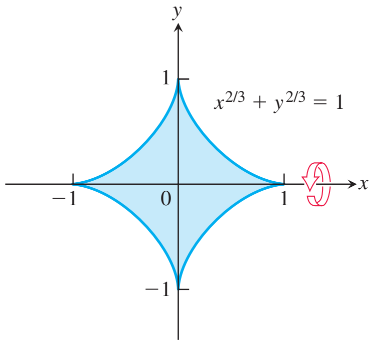
16. Volume of a football The profile of a football resembles the ellipse shown here. Find the football's volume to the nearest cubic inch.



17. Find the volume of the given circular frustum of height h and radii a and b.



18. The graph of $x^{2/3} + y^{2/3} = 1$ is called an astroid and is given below. Find the volume of the solid formed by revolving the region enclosed by the astroid about the x-axis.



Lengths of Curves

Find the lengths of the curves in Exercises 19–22.

19. $y = x^{1/2} - (1/3)x^{3/2}, \quad 1 \le x \le 4$

20. $x = y^{2/3}, \quad 1 \le y \le 8$

21. $y = (5/12)x^{6/5} - (5/8)x^{4/5}, \quad 1 \le x \le 32$

22. $x = (y^3/12) + (1/y), \quad 1 \le y \le 2$

Areas of Surfaces of Revolution

In Exercises 23–26, find the areas of the surfaces generated by revolving the curves about the given axes.

23. $y = \sqrt{2x + 1}, \quad 0 \le x \le 3; \quad x$-axis

24. $y = x^3/3, \quad 0 \le x \le 1; \quad x$-axis

25. $x = \sqrt{4y - y^2}, \quad 1 \le y \le 2; \quad y$-axis

26. $x = \sqrt{y}, \quad 2 \le y \le 6; \quad y$-axis

Work

27. **Lifting equipment** A rock climber is about to haul up 100 N (about 22.5 lb) of equipment that has been hanging beneath her on 40 m of rope that weighs 0.8 newton per meter. How much work will it take? (*Hint:* Solve for the rope and equipment separately, then add.)

28. **Leaky tank truck** You drove an 800-gal tank truck of water from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-ft elevation change in 50 min. Assuming that the water leaked out at a steady rate, how much work was spent in carrying water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8 lb/U.S. gal.

29. **Earth's attraction** The force of attraction on an object below Earth's surface is directly proportional to its distance from Earth's center. Find the work done in moving a weight of w lb located a mi below Earth's surface up to the surface itself. Assume Earth's radius is a constant r mi.

30. **Garage door spring** A force of 200 N will stretch a garage door spring 0.8 m beyond its unstressed length. How far will a 300-N force stretch the spring? How much work does it take to stretch the spring this far from its unstressed length?

31. **Pumping a reservoir** A reservoir shaped like a right-circular cone, point down, 20 ft across the top and 8 ft deep, is full of water. How much work does it take to pump the water to a level 6 ft above the top?

32. **Pumping a reservoir** (*Continuation of Exercise 31.*) The reservoir is filled to a depth of 5 ft, and the water is to be pumped to the same level as the top. How much work does it take?

33. **Pumping a conical tank** A right-circular conical tank, point down, with top radius 5 ft and height 10 ft is filled with a liquid whose weight-density is 60 lb/ft^3. How much work does it take to pump the liquid to a point 2 ft above the tank? If the pump is driven by a motor rated at 275 ft-lb/sec (1/2 hp), how long will it take to empty the tank?

34. **Pumping a cylindrical tank** A storage tank is a right-circular cylinder 20 ft long and 8 ft in diameter with its axis horizontal. If the tank is half full of olive oil weighing 57 lb/ft^3, find the work done in emptying it through a pipe that runs from the bottom of the tank to an outlet that is 6 ft above the top of the tank.

35. Assume that a spring does not follow Hooke's Law. Instead, the force required to stretch the spring x ft from its natural length is $F(x) = 10x^{3/2}$ lb. How much work does it take to
   a. stretch the spring 4 ft from its natural length?
   b. stretch the spring from an initial 1 ft past its natural length to 5 ft past its natural length?

36. Assume that a spring does not follow Hooke's Law. Instead, the force required to stretch the spring x m from its natural length is $F(x) = k\sqrt{5 + x^2}$ N.
   a. If a 3-N force stretches the spring 2 m, find the value of k.
   b. How much work is required to stretch the spring 1 m from its natural length?

Centers of Mass and Centroids

37. Find the centroid of a thin, flat plate covering the region enclosed by the parabolas $y = 2x^2$ and $y = 3 - x^2$.

38. Find the centroid of a thin, flat plate covering the region enclosed by the x-axis, the lines $x = 2$ and $x = -2$, and the parabola $y = x^2$.

39. Find the centroid of a thin, flat plate covering the "triangular" region in the first quadrant bounded by the y-axis, the parabola $y = x^2/4$, and the line $y = 4$.

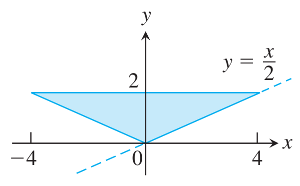
40. Find the centroid of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$.

41. Find the center of mass of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$ if the density function is $\delta(y) = 1 + y$. (Use horizontal strips.)

42. **a.** Find the center of mass of a thin plate of constant density covering the region between the curve $y = 3/x^{3/2}$ and the x-axis from $x = 1$ to $x = 9$.
   b. Find the plate's center of mass if, instead of being constant, the density is $\delta(x) = x$. (Use vertical strips.)

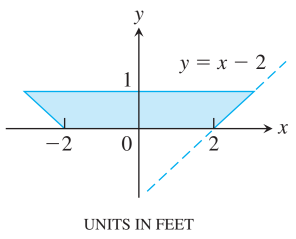
Fluid Force

43. **Trough of water** The vertical triangular plate shown here is the end plate of a trough full of water ($w = 62.4$). What is the fluid force against the plate?



UNITS IN FEET

44. Trough of maple syrup The vertical trapezoidal plate shown here is the end plate of a trough full of maple syrup weighing 75 lb/ft^3. What is the force exerted by the syrup against the end plate of the trough when the syrup is 10 in. deep?



45. Force on a parabolic gate A flat vertical gate in the face of a dam is shaped like the parabolic region between the curve $y = 4x^2$ and the line $y = 4$, with measurements in feet. The top of the gate lies 5 ft below the surface of the water. Find the force exerted by the water against the gate ($w = 62.4$).

T **46.** You plan to store mercury ($w = 849$ lb/ft^3) in a vertical rectangular tank with a 1 ft square base side whose interior side wall can withstand a total fluid force of 40,000 lb. About how many cubic feet of mercury can you store in the tank at any one time?

CHAPTER 6 Additional and Advanced Exercises

Volume and Length

1. A solid is generated by revolving about the x-axis the region bounded by the graph of the positive continuous function $y = f(x)$, the x-axis, the fixed line $x = a$, and the variable line $x = b$, $b > a$. Its volume, for all b, is $b^2 - ab$. Find $f(x)$.

2. A solid is generated by revolving about the x-axis the region bounded by the graph of the positive continuous function $y = f(x)$, the x-axis, and the lines $x = 0$ and $x = a$. Its volume, for all $a > 0$, is $a^2 + a$. Find $f(x)$.

3. Suppose that the increasing function $f(x)$ is smooth for $x \geq 0$ and that $f(0) = a$. Let $s(x)$ denote the length of the graph of f from $(0, a)$ to $(x, f(x))$, $x > 0$. Find $f(x)$ if $s(x) = Cx$ for some constant C. What are the allowable values for C?

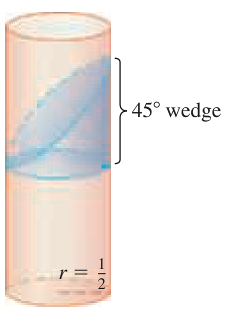
4. a. Show that for $0 < \alpha \leq \pi/2$,

$$\int_0^{\alpha} \sqrt{1 + \cos^2 \theta}\, d\theta > \sqrt{\alpha^2 + \sin^2 \alpha}.$$

b. Generalize the result in part (a).

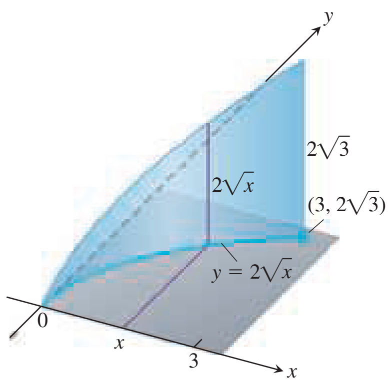
5. Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x$ and $y = x^2$ about the line $y = x$.

6. Consider a right-circular cylinder of diameter 1. Form a wedge by making one slice parallel to the base of the cylinder completely through the cylinder, and another slice at an angle of 45° to the first slice and intersecting the first slice at the opposite edge of the cylinder (see accompanying diagram). Find the volume of the wedge.

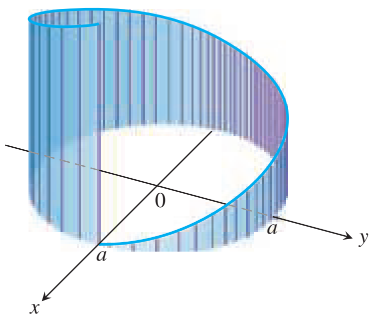


Surface Area

7. At points on the curve $y = 2\sqrt{x}$, line segments of length $h = y$ are drawn perpendicular to the xy-plane. (See accompanying figure.) Find the area of the surface formed by these perpendiculars from $(0, 0)$ to $\left(3, 2\sqrt{3}\right)$.



8. At points on a circle of radius a, line segments are drawn perpendicular to the plane of the circle, the perpendicular at each point P being of length ks, where s is the length of the arc of the circle measured counterclockwise from $(a, 0)$ to P and k is a positive constant, as shown here. Find the area of the surface formed by the perpendiculars along the arc beginning at $(a, 0)$ and extending once around the circle.



Work

9. A particle of mass m starts from rest at time $t = 0$ and is moved along the x-axis with constant acceleration a from $x = 0$ to $x = h$ against a variable force of magnitude $F(t) = t^2$. Find the work done.

10. Work and kinetic energy Suppose a 1.6-oz golf ball is placed on a vertical spring with force constant $k = 2\text{ lb/in}$. The spring is compressed 6 in. and released. About how high does the ball go (measured from the spring's rest position)?

Centers of Mass

11. Find the centroid of the region bounded below by the x-axis and above by the curve $y = 1 - x^n$, n an even positive integer. What is the limiting position of the centroid as $n \to \infty$?

12. If you haul a telephone pole on a two-wheeled carriage behind a truck, you want the wheels to be 3 ft or so behind the pole's center of mass to provide an adequate "tongue" weight. The 40-ft wooden telephone poles used by Verizon have a 27-in. circumference at the top and a 43.5-in. circumference at the base. About how far from the top is the center of mass?

13. Suppose that a thin metal plate of area A and constant density δ occupies a region R in the xy-plane, and let M_y be the plate's moment about the y-axis. Show that the plate's moment about the line $x = b$ is

a. $M_y - b\delta A$ if the plate lies to the right of the line, and

b. $b\delta A - M_y$ if the plate lies to the left of the line.

14. Find the center of mass of a thin plate covering the region bounded by the curve $y^2 = 4ax$ and the line $x = a$, $a =$ positive constant, if the density at (x, y) is directly proportional to **(a)** x, **(b)** $|y|$.

15. a. Find the centroid of the region in the first quadrant bounded by two concentric circles and the coordinate axes, if the circles have radii a and b, $0 < a < b$, and their centers are at the origin.

b. Find the limits of the coordinates of the centroid as a approaches b and discuss the meaning of the result.

16. A triangular corner is cut from a square 1 ft on a side. The area of the triangle removed is 36 in^2. If the centroid of the remaining region is 7 in. from one side of the original square, how far is it from the remaining sides?

Fluid Force

17. A triangular plate ABC is submerged in water with its plane vertical. The side AB, 4 ft long, is 6 ft below the surface of the water, while the vertex C is 2 ft below the surface. Find the force exerted by the water on one side of the plate.

18. A vertical rectangular plate is submerged in a fluid with its top edge parallel to the fluid's surface. Show that the force exerted by the fluid on one side of the plate equals the average value of the pressure up and down the plate times the area of the plate.

CHAPTER 6 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyMathLab.

- ***Using Riemann Sums to Estimate Areas, Volumes, and Lengths of Curves***
  Visualize and approximate areas and volumes in **Part I** and **Part II**: Volumes of Revolution; and **Part III**: Lengths of Curves.
- ***Modeling a Bungee Cord Jump***
  Collect data (or use data previously collected) to build and refine a model for the force exerted by a jumper's bungee cord. Use the work-energy theorem to compute the distance fallen for a given jumper and a given length of bungee cord.