QUESTION 1

$$y'' - 2y' + 3y = x^3 + \cos x$$

$$AE \Rightarrow m^2 - 2m + 3 = 0$$

$$\Rightarrow m = \frac{2 \pm \sqrt{4-12}}{2}$$

$$= \frac{2 \pm i2\sqrt{2}}{2}$$

$$= 1 \pm i\sqrt{2}$$

$$\therefore CF = e^{x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$$

$$PI = y = Ax^3 + Bx^2 + Cx + D + E\cos x + F\sin x$$

$$y' = 3Ax^2 + 2Bx + C - E\sin x + F\cos x$$

$$y'' = 6Ax + 2B - E\cos x - F\sin x$$

$$\therefore 3A = 1 \Rightarrow A = 1/3$$

$$3B + (-6A) = 0 \Rightarrow B = 2/3$$

$$3C - 4B + 6A = 0 \Rightarrow C = 2/9$$

$$3D - 2C + 2B = 0 \Rightarrow D = -8/27$$

$$3E - 2F - E = 1 \Rightarrow 2E - 2F = 1$$

$$3F + 2E - F = 0 \Rightarrow 2E + 2F = 0$$

$$E = 1/4$$

$$\therefore F = -1/4$$

$$\therefore GS = e^{x}(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x) + \frac{x^3}{3} + \frac{2x^2}{3} + \frac{2x}{9} - \frac{8}{27} + \frac{\cos x}{4} - \frac{\sin x}{4}$$

Question 2

$$y'' + 9y = \sec 3x$$

$$AE \Rightarrow m^2 + 9 = 0$$

$$\Rightarrow m = \pm 3i$$

$$\therefore CF = c_1 \cos 3x + c_2 \sin 3x$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x$$

$$= 3$$

$$u_1 = -\int \frac{\sin 3x . \sec 3x}{3}$$

$$= -\frac{1}{3}\int \tan 3x$$

$$= \frac{1}{3}\log \cos x$$

$$u_2 = \int \frac{\cos 3x . \sec 3x}{3}$$

$$= \frac{x}{3}$$

$$\therefore GS = \cos 3x\left(c_1 + \frac{1}{3}\log \cos x\right) + \sin 3x\left(c_2 + \frac{x}{3}\right)$$

Question 3

$(z-y)p + (x-z)q = y-x$

$P = z-y$

$Q = x-z$

$R = y-x$

let $l = m = n = 1$

$lP + mQ + nR = 0$

$\therefore x + y + z = a$

Let $l = z+y, m = x+z, n = y+x$

$lP + mQ + nR = 0$

$\therefore xz + xy + xy + zy + zy + zx = b$

$\Rightarrow 2xz + 2xy + 2yz = b$

$\Rightarrow xz + yx + zy = b$

$\therefore f(x+y+z, xz + yx + zy) = 0$

Question 4

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0$$

Let $u = X \cdot Y$

$$\therefore \frac{\partial u}{\partial x} = Y \frac{dX}{dx}$$

$$\frac{\partial u}{\partial y} = X \frac{dY}{dy}$$

$$x^2 \cdot Y \frac{dX}{dx} + y^2 X \frac{dY}{dy} = 0$$

$$\Rightarrow x^2 \cdot Y \frac{dX}{dx} = -y^2 X \frac{dY}{dy} = a$$

$$\Rightarrow \frac{x^2}{X} \frac{dX}{dx} = -\frac{y^2}{Y} \frac{dY}{dy} = a$$

$$\frac{x^2}{X} = a \frac{dx}{dX}$$

$$\Rightarrow \frac{dX}{X} = \frac{a}{x^2} dx$$

$$\Rightarrow \log X + c = -\frac{a}{x}$$

$$\Rightarrow X = e^{-a/x + c}$$

$$\frac{dY}{Y} = -\frac{a}{y^2} dy$$

$$\Rightarrow \ln Y = \frac{a}{y} + c$$

$$\Rightarrow Y = e^{a/y + c}$$

$$\therefore u = ce^{-a/x + a/y}$$

Question 5

$$f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $l = \pi/\omega$

$$a_0 = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin\omega t \, dt + \int_{\pi/\omega}^{2\pi/\omega} 0 \, dt$$

$$= \frac{\omega}{\pi}\left[-\frac{\cos\omega t}{\omega}\right]_0^{\pi/\omega}$$

$$= \frac{\omega}{\pi}\left(\frac{1}{\omega}\right) - \frac{\omega}{\pi}\left(\frac{-1}{\omega}\right)$$

$$= \frac{2}{\pi}$$

$$a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin\omega t \cdot \cos n\omega t \, dt$$

$$= \frac{\omega}{2\pi}\int_0^{\pi/\omega} \sin(\omega + n\omega)t + \sin(\omega - n\omega)t \, dt$$

$$= \frac{\omega}{2\pi}\left[-\frac{\cos(\omega+n\omega)t}{\omega+n\omega} - \frac{\cos(\omega-n\omega)t}{\omega-n\omega}\right]_0^{\pi/\omega}$$

$$= \frac{\omega}{2\pi}\left(-\frac{(-1)^{n+1}}{\omega+n\omega} - \frac{(-1)^{n-1}}{\omega-n\omega} + \frac{1}{\omega+n\omega} + \frac{1}{-\omega+n\omega}\right)$$

$$a_n = \frac{1}{2\pi}\left(\frac{(-1)^n}{1+n} + \frac{(-1)^n}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right)$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^n+1}{1+n} + \frac{(-1)^n+1}{n-1}\right]$$

$$b_n = \frac{\omega}{2\pi}\int_0^{\pi/\omega} \sin\omega t \cdot \sin n\omega t \, dt$$

$$= \frac{\omega}{2\pi}\int_0^{\pi/\omega} \cos(n\omega+\omega)t - \cos(n\omega-\omega)t \, dt$$

$$= \frac{\omega}{2\pi}\left[\cancel{\cos}\ \frac{\sin(n\omega+\omega)t}{(n+1)\omega} - \frac{\sin(n\omega-\omega)t}{(n-1)\omega}\right]_0^{\pi/\omega}$$

$$\neq \cancel{\frac{\omega}{2\pi}}$$

$$= \frac{1}{2\pi}(0-0-0+0)$$

$$= 0$$

$$\therefore \sin\omega t = \frac{1}{\pi} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n+1}{1+n} + \frac{(-1)^n+1}{n-1}\right]\cos n\omega t$$

$$= \frac{1}{\pi} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{((-1)^n+1)2}{n^2-1}\cos n\omega t$$

Question 6

$$f(x) = \begin{cases} -\frac{\pi x}{2}, & -\pi < x < 0 \\ \frac{\pi x}{2}, & 0 < x < \pi \end{cases}$$

$l = \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi x}{2}\,dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi x}{2}\,dx$$

$$= -\left[\frac{x^2}{4}\right]_{-\pi}^{0} + \left[\frac{x^2}{4}\right]_{0}^{\pi}$$

$$= +\frac{\pi^2}{4} + \frac{\pi^2}{4} = \frac{\pi^2}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi x}{2}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi x}{2}\cos nx\,dx$$

$$= -\frac{1}{2}\left[\frac{x}{n}\sin nx + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{2}\left[\frac{x}{n}\sin nx + \frac{\cos nx}{n^2}\right]_{0}^{\pi}$$

$$= -\frac{1}{2}\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right) + \frac{1}{2}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)$$

$$= \frac{(-1)^n - 1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi x}{2}\, \overset{\sin}{\cancel{\cos}}\, nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi x}{2} \sin nx \, dx$$

$$= -\frac{1}{2}\left[-\frac{x}{n}\cos nx + \frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \frac{1}{2}\left[-\frac{x}{n}\cos nx + \frac{\sin nx}{n^2}\right]_{0}^{\pi}$$

$$= +\frac{1}{2}\left(\frac{\pi}{n}(-1)^n\right) + \frac{1}{2}\left(-\frac{\pi}{n}(-1)^n\right)$$

$$= 0$$

$$\therefore\ f(x) = \frac{\pi^2}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2}\right)\cos nx$$