Question 1

$$f = \begin{cases} 3x^2, & x < 1 \\ ax + b, & x \geq 1 \end{cases}$$

If lagrange mean value theorem applies, then the function's continuous and differentiable.

For continuity:

$3x^2 = ax + b$ at $x = 1$

$\Rightarrow 3 = a + b$

For differentiability:

$6x = a$ at $x = 1$

$\Rightarrow \underline{a = 6}$

$\therefore \underline{b = -3}$

$$\therefore f = \begin{cases} 3x^2, & x < 1 \\ 6x - 3, & x \geq 1 \end{cases}$$

Finding c

$4f'(c) = f(2) - f(-2)$

$f(2) = 6(2) - 3 = 9$

$f(-2) = 3(-2)^2 = 12$

$\therefore f'(c) = \underline{-\frac{3}{4}}$

Since slope < 1 in the $2^{nd}$ quadrant

$\therefore$ c belongs to the $2^{nd}$ quadrant

$\therefore$ c is on the line $f = 3x^2$

$\therefore$ To find c:

$$f'(c) = -\frac{3}{4} = 6x$$

$$\Rightarrow x = -\frac{3}{24}$$

$\therefore$ The point $c = -\frac{3}{24}$ has a slope of $-\frac{3}{4}$

Question 2

$$f(t) = \frac{100}{t^2 + 12}$$

Speed $= f'(t)$

$$f'(t) = \frac{-100(2t)}{(t^2+12)^2}$$

$$= \frac{-200t}{(t^2+12)^2}$$

Extrema at $f''(t) = 0$

$$f''(t) = \frac{-200(t^2+12)^2 + 200t(2t)(2)(t^2+12)}{(t^2+12)^4}$$

$$= \frac{800t^2(t^2+12) - 200(t^2+12)^2}{(t^2+12)^4}$$

$$= \frac{800t^2 - 200(t^2+12)}{(t^2+12)^3}$$

$$= \frac{600t^2 - 2400}{(t^2+12)^3}$$

$$= \frac{600(t^2-4)}{(t^2+12)^3}$$

Finding $f''(t) = 0$

$600(t^2-4) = 0$

$\Rightarrow t^2 - 4 = 0$

$\Rightarrow t = \pm 2$

$$f'(-2) = \frac{400}{(4+12)^2} = \frac{400}{256}$$

$$\therefore f'(-2) = 1.56$$

$$f'(2) = \frac{-400}{256} = -1.56$$

$\therefore$ The maximum speed is 1.56 ft/s.

Question 3

$y = \sqrt{x}$

$y = 0$

$x = 9$

About $x = 9$

Radius $= 9 - y^2$

Limits $= [0, 3]$

$$\text{volume} = \pi \int_0^3 (9 - y^2)^2 \, dy$$

$$= \pi \int_0^3 (81 - 18y^2 + y^4) \, dy$$

$$= \pi \left[ 81y - 6y^3 + \frac{y^5}{5} \right]_0^3$$

$$= \pi \left( 243 - 162 + \frac{243}{5} \right)$$

$$= \frac{648\pi}{5} \text{ units}^3$$

Question 4

$u = 2x - 3y + 4z$

$v = 4x^2 + 9y^2 + 16z^2$

$w = 3xy + 6yz - 4xz$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} 2 & -3 & 4 \\ 8x & 18y & 32z \\ 3y-4z & 3x+6z & 6y-4x \end{vmatrix}$$

$$= 2\left[108y^2 - 72xy - 96xz - 192z^2\right]$$

$$+3\left[48xy - 32x^2 - 96yz + 128z^2\right]$$

$$+4\left[24x^2 + 48xz - 54y^2 + 72yz\right]$$

$$= 216y^2 - 144xy - 192xz - 384z^2 + 144xy - 96x^2 - 288yz$$

$$+ 384z^2 + 96x^2 + 192xz - 216y^2 + 288yz$$

$$= \underline{0}$$

$\therefore$ u, v, w are related

The relation is:

$\underline{u^2 = v + 4w}$

Question 5

$$f = \frac{1}{1-x-y}$$

$a, b = 0, 0$

$$f_x = f_y = \frac{1}{(1-x-y)^2}$$

$$f_{xx} = f_{xy} = f_{yy} = \frac{2}{(1-x-y)^3}$$

$$f_{xxx} = f_{xxy} = f_{xyy} = f_{yyy} = \frac{6}{(1-x-y)^4}$$

At (0,0)

$f_x = f_y = 1$

$f_{xx} = f_{xy} = f_{yy} = 2$

$f_{xxx} = f_{xxy} = f_{xyy} = f_{yyy} = 6$

Quadratic Approximation:

$$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0)}{2} + \frac{y^2 f_{yy}(0,0)}{2}$$

$$\Rightarrow f(x,y) = 1 + x + y + x^2 + 2xy + y^2$$

Cubic Approximation:

$$f(x,y) = 1 + x + y + x^2 + 2xy + y^2 + x^3 + 3x^2y + 3xy^2 + y^3$$

Question 6

$V = xyz$

$x + y + z = 129$

(or) $g = x + y + z - 129$

$$\frac{\partial V}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial V}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$\frac{\partial V}{\partial z} = \lambda \frac{\partial g}{\partial z}$$

$\Rightarrow yz = \lambda$

$\Rightarrow xz = \lambda$

$\Rightarrow xy = \lambda$

$\therefore x = y = z$

Substituiting in g:

$3x = 129$

$\Rightarrow x = 43$

$\therefore x = y = z = 43$

$V_{max} = 43^3 = 79507$ units$^3$

Question 7

$$\int_0^1 \tan^{-1}(1-x)\,dx$$

$$= \int_0^1 \int_0^{\tan^{-1}(1-x)} dy\,dx$$

changing variables:

Min $x = 0$

Max $x = 1 - \tan y$

Min $y = 0$

Max $y = \pi/4$

$$= \int_0^{\pi/4} \int_0^{1-\tan y} dx\,dy$$

$$= \int_0^{\pi/4} (1 - \tan y)\,dy$$

$$= \Big[ y - \log(\sec y) \Big]_0^{\pi/4}$$

$$= \frac{\pi}{4} - \log\sqrt{2}$$

$$= \frac{\pi}{4} - \frac{\log 2}{2}$$

Question 8

$f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$

$= \log(x^2+ab) - \log x - \log(a+b)$

The function is continuous on $[a,b]$ since $a>0, b>a$

Checking differentiability on $(a,b)$:

$\frac{2x}{x^2+ab} - \frac{1}{x} = f'(x)$

$\Rightarrow f'(x) = \frac{x^2-ab}{x(x^2+ab)}$

$\therefore$ Only not differentiable if $x^2+ab=0$

$\Rightarrow x = \sqrt{-ab}$

Since $a>0, b>a$, $f(x)$ is always differentiable.

$f(a) = \log\left(\frac{a^2+ab}{a^2+ab}\right) = \log 1 = 0$

$f(b) = \log\left(\frac{b^2+ab}{b^2+ab}\right) = 0$

$\therefore f(a) = f(b)$

$\therefore$ With the above three conditions, Rolle's Theorem applies here.

$f'(c) = 0$

$\Rightarrow c^2 - ab = 0$

$\Rightarrow c = \sqrt{ab}$, ($c$ can't be $-\sqrt{ab}$ as $c \in [a,b]$)