1.

i. $3u + 2v = y^2 - x^2 + 16xy$

$$3u_x + 2v_x = -2x + 16y \quad - (1)$$

$$3u_y + 2v_y = 2y + 16x \quad - (2)$$

$u_x = v_y$; ECcas

$u_y = -v_x$

(1) $3u_x + -2u_y = -2x + 16y \quad - (3)$

(2) $3u_y + 2u_x = 2y + 16x \quad - (4)$

$3 \times (3) + 2 \times (4)$

$13u_x = 52y + 26x$

$\Rightarrow \underline{u_x = 4y + 2x} \quad - (5)$

(1) $3v_y + (+2)v_x = -2x + 16y \quad - (6)$

(2) $-3v_x + 2v_y = 2y + 16x \quad - (7)$

$2 \times (6) - 3 \times (7)$

$+4v_x + 9v_x = -52x + 26y$

$\Rightarrow \underline{v_x = -4x + 2y}$

using 5 and 8

$$f_x = u_x + i v_x$$

$$\Rightarrow f_x(z,0) = u_x(z,0) + i\, v_x(z,0)$$

$$\Rightarrow f(z) = \int (4y + 2x + i(-4x + 2y))\, dz$$

let $x = z,\ y = 0$

$$f(z) = \int (2z - 4iz)\, dz$$

$$\Rightarrow f(z) = z^2 - \frac{4iz^2}{2} + c$$

$$\Rightarrow f(z) = (1 - 2i) z^2 + c$$

ii. $\varphi = e^{-x}(2xy\cos y - (x^2 - y^2)\sin y)$

$$\frac{\partial \varphi}{\partial x} = -e^{-x}(2xy\cos y - (x^2-y^2)\sin y) + e^{-x}(2y\cos y - 2x\sin y)$$

$$\frac{\partial^2 \varphi}{\partial x^2} = +e^{-x}(2xy\cos y - (x^2-y^2)\sin y) + -e^{-x}(2y\cos y - 2x\sin y) + e^{x}(-2\sin y) - e^{-x}(2y\cos y \mp 2x\sin y)$$

$$\frac{\partial \varphi}{\partial y} = e^{-x}(2x\cos y \mp 2xy\sin y - (-2y)\sin y - (x^2-y^2)\cos y)$$

$$\frac{\partial^2 \varphi}{\partial y^2} = e^{-x}(-2x\sin y - 2xy\cos y - 2x\sin y + 2y\cos y + 2\sin y + (x^2 + -y^2)\sin y + 2y\cos y)$$

$$\frac{\partial^2 \varphi}{\partial x^2} = -\frac{\partial^2 \varphi}{\partial y^2}$$

$\therefore$ The function is harmonic and can represent a path of electrical flow

Using Milne Thomson Method:

$f = \varphi_x + i\,\phi_x$

$\Rightarrow f = \varphi_x + i(-\varphi_y)$

~~$\Rightarrow f(z) = \varphi_x(z)$~~

$\Rightarrow \cancel{f(z,0)}\ f(z) = \int \varphi_x(z,0) - i\varphi_y(z,0)\,dz$

$\Rightarrow f(z) = \int +e^{-z}(0) - ie^{-z}(2z - z^2)\,dz$

$= -i\int e^{-z}(2z - z^2)\,dz$

$= -i\left(-e^{-z}(2z - z^2) - (e^{-z})(2 - 2z) + (-e^{-z})(-2)\right)$

$= -i(-e^{-z}(2z - z^2 + 2 - 2z - 2)$

$= ie^{-z}(-z^2)$

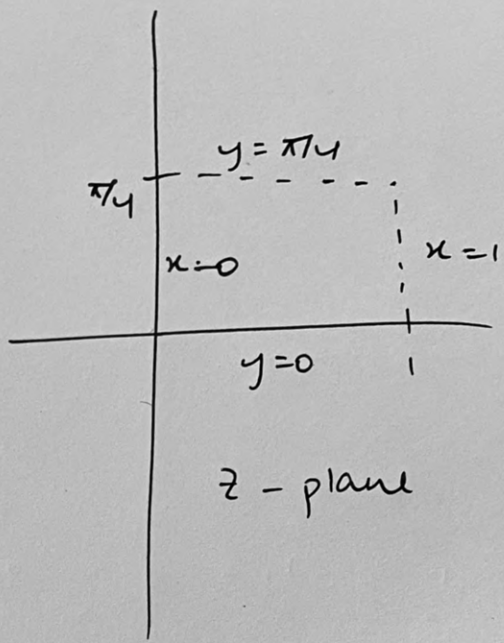
$= -iz^2e^{-z}$

2.

i. $w = e^{-z}$

$R: 0 \le x \le 1$

$0 \le y \le \pi/4$



$y = 0 ; 0 \le x \le 1$

$w = e^{-(x + 0i)}$

$\Rightarrow w = e^{-x} \; ; \; u = e^{-x}$

$x = 1 ; 0 \le y \le \pi/4$

$w = e^{-(1 + iy)}$

$\Rightarrow w = \frac{1}{e}(\cos y \mp i \sin y)$

$u = \frac{\cos y}{e} \; ; \; v = -\frac{\sin y}{e}$

$\Rightarrow e^2(u^2 \mp v^2) = 1$

$\Rightarrow u^2 \mp v^2 = \frac{1}{e^2}$

$y = \pi/4 ; 0 \le x \le 1$

$w = e^{-(x + i\pi/4)}$

$\Rightarrow w = \frac{1}{\sqrt{2}} e^{-x}(1 \mp i) \; ; -u = v$

$x = 0 ; 0 \le y \le \pi/4$

$w = e^{-(0 + iy)}$

$\Rightarrow w = \cos y \mp i \sin y$

$u = \cos y$

$v = -\sin y$

$\Rightarrow u^2 \mp v^2 = 1$

Yes, this mapping is conformal

ii. $z = 1, i, -1$; $w = 2, i, -2$

$$\frac{(z-1)(i+1)}{(z+1)(i-1)} = \frac{(w-2)(i+2)}{(\overset{w}{\cancel{z}}+2)(i-2)}$$

$$\Rightarrow (-3-i)(z-1)(w+2) = (-3+i)(z+1)(w-2)$$

$$\Rightarrow (-3-i)(zw - w + 2z - 2) = (-3+i)(zw + w - 2z - 2)$$

$$\Rightarrow -2izw + 6w - 12z + 4i = 0$$

$$\Rightarrow -izw + 3w - 6z + 2i = 0$$

$$\Rightarrow z(-iw - 6) = -3w \cancel{+} - 2i$$

$$\Rightarrow z = \frac{-3w - 2i}{-iw - 6}$$

$$\Rightarrow z = \frac{3w + 2i}{iw + 6}$$

Invariant points

$$z = \frac{3z + 2i}{iz + 6}$$

$$\Rightarrow iz^2 + 6z = 3z + 2i \Rightarrow iz^2 + 6\,3z - 2i$$

$$\Rightarrow z = \frac{-3 \pm \sqrt{9 - 4(i)(-2i)}}{2i} = \frac{-3 \pm \sqrt{9 - 8}}{2i} = i \text{ or } 2i$$

3i. $f(z) = \dfrac{1}{(z-3)(z+1)^2}$, $1 < |z-2| < 3$

$A(z+1)^2 + B(z-3)(z+1) + C(z-3) = 1$

$C(-4) = 1$

$\Rightarrow C = -1/4$

$A = 1/16$

$A + B = 0$

$\Rightarrow B = -1/16$

$\therefore f(z) = \dfrac{1/16}{z-3} - \dfrac{1/16}{z+1} - \dfrac{1/4}{(z+1)^2}$

$= \dfrac{1/16 \times 1/3}{(z/3) - 1} - \dfrac{1/16}{z+1} - \dfrac{1/4}{(z+1)^2}$

$= -\dfrac{1}{48} \sum_{n=0}^{\infty} (-1)^n \left(-\dfrac{z}{3}\right)^n - \dfrac{1}{16} \sum_{n=0}^{\infty} (-z)^n - \dfrac{1}{4} \sum_{n=0}^{\infty} (n+1)(-z)^n$

$= -\sum_{n=0}^{\infty} \left(\dfrac{1}{48}\left(\dfrac{z}{3}\right)^n - \dfrac{(-z)^n}{16} - \dfrac{(n+1)(-z)^n}{4} \right)$

ii. $\displaystyle\int_{-\infty} \frac{\cos x}{x^2-10x+9}\,dx$

$$= \int_{-\infty}^{\infty} \frac{\cos x \; dx}{(x-\overset{9}{\cancel{10}})(x-1)}$$

$$f(x) = \frac{1}{(x-9)(x-1)}$$

$$f(z) = \frac{1}{(z-9)(z-1)} \quad , \text{ singularities } = 9, 1$$

$$\oint_C f(z)e^{iz}\,dz = \int_{C_R} f(z)e^{iz}\,dz + \int_{-R}^{R} f(x)e^{ix}\,dx$$

<u>By Cauchy's Residue Theorem:</u>

$$\oint_C f(z)e^{iz}\,dz = 2\pi i \,(\text{sum of residues})$$

$$\text{Res}\,(f(z);\, z=9) = \lim_{z\to 9} (z-9)f(z)e^{iz}$$
$$= \frac{e^{9i}}{8}$$

$$\text{Res}\,(f(z);\, z=1) = -\frac{e^{i}}{8}$$

$$\therefore \oint_C f(z)e^{iz}\,dz = 2\pi i\left(\frac{e^{9i}-e^{i}}{8}\right)$$
$$= \frac{\pi i}{4}(e^{9i}-e^{i})$$

$$\therefore \frac{\pi i}{4}\left(e^{9i}-e^{i}\right) = \int_{C_R} f(z)e^{iz}\,dz + \int_{-R}^{R} f(x)e^{ix}\,dx$$

AS $R \to \infty$

$$\int_{C_R} f(z)\,e^{iz}\,dz \longrightarrow 0$$

$$\therefore \frac{\pi i}{4}\left(e^{9i}-e^{i}\right) = \int_{-\infty}^{\infty} f(x)\,e^{ix}\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{(x-9)(x-1)}\,dx = \frac{\pi i}{4}\left(\cos 9 + i\sin 9 - \cos 1 - i\sin 1\right)$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x}{(x-9)(x-1)}\,dx = \frac{(\sin 1 - \sin 9)\,\pi}{4}$$

$$= 2\cos 5\sin(-4)\,\frac{\pi}{4}$$

$$= -\frac{\pi}{2}\cos 5\sin 4$$

4. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$\text{Trace}(A) = 4$

$\text{Det}(A) = 2 + 2 + 2 - 1 - 8 - 1$

$= -4$

Sum of minors of prin diag:

$$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$= 1 + 1 - 3$

$= -1$

$-\lambda^3 + 4\lambda^2 + \lambda - 4 = 0$

$\therefore \cancel{A^3 = 4A^2 + A - 4}.$

$\Rightarrow A^3 = 4A^2 + A - 4$

$$\Rightarrow A^{-1} = \frac{A^3 - 4A^2 - A}{-4A}$$

$$\Rightarrow A^{-1} = \frac{-1}{4}\left(A^2 - 4A - I_3\right)$$

$$A^2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 7 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{4}\left(\begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$= -\frac{1}{4}\begin{bmatrix} 1 & -3 & 1 \\ -5 & 1 & 1 \\ 1 & 1 & -3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1/4 & 3/4 & -1/4 \\ 5/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

$$A^4 = 4A^3 + A^2 - 4A$$

$$= 4(4A^2 + A - 4) + A^2 - 4A$$

$$= 17A^2 - 16$$

$$\therefore A^4 = \begin{bmatrix} 86 & 69 & 69 \\ 69 & 86 & 69 \\ 69 & 69 & 86 \end{bmatrix}$$

Eigen values of A:

$$\begin{array}{r} -\lambda^2 + 3\lambda + 4 \\ \lambda - 1 \,\big)\, -\lambda^3 + 4\lambda^2 + \lambda - 4 \\ -\lambda^3 + \lambda^2 \\ \hline 3\lambda^2 + \lambda \\ 3\lambda^2 - 3\lambda \\ \hline 4\lambda - 4 \\ \hline 0 \end{array}$$

$\lambda = 1, -1, 4$

$\therefore$ Eigen values of

A^{-1}: $1, -1, 1/4$

A^4: $1, 1, 256$

5.

i. $A = \begin{bmatrix} 2 & 0 & 3 & 1 & 3 \\ -3 & 1 & -2 & -4 & -1 \\ 5 & 4 & 9 & 1 & 13 \\ 7 & 6 & 13 & 1 & 19 \end{bmatrix}$

$R(A) = \{r_1, r_2, r_3, r_4\}$

$C(A) = \{c_1, c_2, c_3, c_4, c_5\}$

$$RREF(A) = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for $R(A) = \{r_1, r_2, r_3\}$

$C(A) = \{c_1, c_2, c_3\}$

$\therefore \ Dim(R(A)) = 3 = Dim(C(A))$

<u>For null space:</u>

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \ x_1 + 2x_4 = 0$

$x_2 + x_5 = 0$

$x_3 - x_4 + x_5 = 0$

x_4, x_5 are free variables

let $x_4 = p$, $x_5 = q$

$\therefore$ $x_1 = -2p$

$x_2 = -q$

$x_3 = p - q$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} p + \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} q$$

$$\therefore N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\therefore$ $\text{Dim}(N(A)) = 2$

$\therefore$ Nullity$(A) = 2$

ii. $T(x, y, z) = (2x + y + z,\ x + y - 2z,\ x - y)$

$$T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ for kernel

$$T = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$R_1 \to R_1/2$

$$\sim \left[\begin{array}{ccc|c} \cancel{2}\,1 & 1/2 & 1/2 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & -5/2 & 0 \\ 0 & -3/2 & -1/2 & 0 \end{array}\right]$$

$R_3 \to R_3 + 3R_1$; $R_1 \to R_1 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1/2 & -5/2 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right]$$

$R_3 \to R_3/-8$

$R_1 \to R_1 + 3R_3$

$R_2 \to R_2 - \frac{5}{2} R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\therefore$ Kernel = span $\{\ \}$

$\therefore$ Nullity $(T) = 0$

$\therefore$ Rank $(T) = 3$

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$T(e_1) = (2, 1, 1)$

$T(e_2) = (1, 1, -1)$

$T(e_3) = (1, -2, 0)$

$$\therefore \ _\beta[T]_\alpha = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\therefore$ ~~[illegible]~~

$$a(1, 0, -1) + b(1, 2, 0) + c(1, 1, 1) = (2, 1, 1)$$

$\Rightarrow a = 2/3$

$b = -1/3$

$c = 5/3$

From the other 2 equations:

$a = 0$	$a = 4/3$
$b = 1$	$b = -5/3$
$c = 0$	$c = 4/3$

$$\therefore [T]_\alpha^\beta = \begin{pmatrix} 2/3 & 0 & 4/3 \\ -1/3 & 1 & -5/3 \\ 5/3 & 0 & 4/3 \end{pmatrix}$$