Math 104: Introduction to Analysis

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The notes scribed here follow the textbook Elementary Analysis by Kenneth A. Ross. Some additional remarks may be included based on the lecture teachings of Professor Wei Fan. I have slightly reordered some ideas - namely I began my introduction to topology before beginning the discussion of sequences. This is just so that all concepts can be presented in terms of both topology and the real number system.

Contents

1 Introduction

Dummy text

1.1 $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$

1.2 Completeness

1.3 Constructing $\mathbb{R}$ from $\mathbb{Q}$

1.4 Solved Exercises

2 A Brief Remark on Topology

In this section, will introduce the notion of a metric space. While we will not delve very deep, it is important to start considering how the absolute value distance function we will use on the real numbers can actually be abstracted, and that many results we prove on real numbers are much more general.

When working with the set of real numbers, we are able to perform much of our analysis through the use of a distance function, namely $d(x, y) = |x - y|$. As seen in (insert chapter 1 reference), d satisfies the triangle inequality, which we will make important use of very soon in our discussion of sequences. However, it turns out that many results on the real numbers that involve d make use of a small set of fundamental properties that the absolute value function demonstrates. As such, we could perform the same analysis with other functions that also demonstrate these same properties so long as we are able to write them out explicitly. We begin this abstraction process by defining the concepts of a metric, and metric space.

Definition 2.1: Metric Space

Let S be a set and suppose d is a function defined on all pairs (x, y). We call d a metric (or distance function) on S iff it satisfies the following conditions:

$$d(x, x) = 0 \ \forall x \in S \tag{1}$$
$$d(x, y) > 0 \text{ for distinct } x, y \in S \tag{2}$$
$$d(x, y) = d(y, x) \ \forall x, y \in S \tag{3}$$
$$d(x, z) \leq d(x, y) + d(y, z) \ \forall x, y, z \in S \tag{4}$$

A *metric space* M is a pair (S, d) where d is a metric on S. Note that there can be multiple valid metrics on any given set S, as we shall see soon.

As you read the coming section on sequences, please consider the situations in which it may be possible to replace the operations we perform on $|x - y|$ with the more general $d(x, y)$.

3 Sequences and Series

3.1 Limits of Sequences

> **Definition 3.1: Convergent Sequence in Metric Space**
>
> Let (S, d) be a metric space. Let (a_n) be a sequence of points such that each $a_n \in S$. $\lim_{n \to \infty} a_n = a \in S$ if:
>
> $$\text{for each } \epsilon > 0, \text{ there exists } N > 0 \text{ such that}$$
> $$n > N \implies d(a_n, a) < \epsilon$$
>
> In this case, we call (a_n) convergent. This is another example of extending ideas we develop on $\mathbb{R}$ to work on a general metric space (S, d).

3.2 Monotone Sequences and Cauchy Sequences

3.3 Lim Sup and Lim Inf

3.4 Series

4 Topology Part 2 - Electric Boogaloo

We will now expand upon the concept of metric spaces that we covered in 2. The basis of most topological concepts we will cover are derived from the definitions of open and closed sets in the following section.

4.1 Open Sets

A helpful tool for understanding open and closed sets will be the idea of an 'open ball'. This name will be justified after we have defined what an open set is.

Definition 4.1: Open Ball

For some metric space (S, d), $x \in S$, $r > 0$,

$$B_r(x) := \{y \in S : d(x, y) < r\} \subseteq S$$

I will sometimes refer to this concept as an 'epsilon ball', since we will often be considering very small values of r. Thus, we can have an equivalent definition by replacing r with ϵ.

Example 4.1

$(\mathbb{R}, d_{std})$: $B_1(3) = (2, 4)$ where $(2, 4)$ is the open interval between 2 and 4.

Example 4.2

$(\mathbb{R}^2, d_{std})$: $B_2((0, 0)) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$. Note that this is a circle of radius 2 centered at the origin.

The reason we call this a ball is because of how we can visualize this set in the metric spaces of $(\mathbb{R}^2, d_{std})$ and $(\mathbb{R}^3, d_{std})$. In $(\mathbb{R}^2, d_{std})$, we can visualize the epsilon ball as the circle of radius r centered at x. Similarly, in $\mathbb{R}^3$ we can visualize this as a sphere of radius r centered at x. In general, one can think of the open ball in $\mathbb{R}^k$ as being a k-dimensional sphere. Another detail to note is that we need not be working with a metric space of the form $(\mathbb{R}^k, d_{std})$ for the open ball to be useful. We simply reference $\mathbb{R}^k$ in examples since it is easiest to visualize.

Definition 4.2: Open Set

Let (S, d) be a metric space. Let $E \subseteq S$.
$E \subseteq S$ is open if for all $x \in E$, there **exists** some $r > 0$ such that $B_r(x) \subseteq E$.

The important take away from the above definition is that for any point $x \in E$, we just need to be able to find 1 value of r that can be as small as we like such that the ball centered at x of radius r is contained entirely within E.

Example 4.3

$[0, 1] \subseteq \mathbb{R}$ is not open. Consider the epsilon ball $B_r(0) = (-r, r) \not\subseteq \mathbb{R}$ since $-r < 0$ and the interval $[0, 1]$ contains no numbers less than 0.

Example 4.4

$(0,1) \subseteq \mathbb{R}$ is open. To show that this is open, consider any point $x \in (0,1)$. Your task is to find some r small enough such that $B_r(x)$ is entirely contained within $(0,1)$. Remember that r can be expressed in terms of x. It may also be helpful to consider the fact that $B_r(x) = (x - r, x + r)$.

Example 4.5

Let (S, d) be a metric space. S is open, and $\emptyset$ is open (reminder that $\emptyset$ is the empty set). To see why S is open, consider any point $x \in S$. The epsilon ball $B_r(x)$ can only contain points in defining set of the metric space S, and so $B_r(x) \subseteq S$. Thus, S is open. We say that the empty set is trivially open because there are no elements in it, and so no such epsilon balls can be constructed on the empty set.

The following theorem will be proven very explicitly to help build some intuition with open sets. Subsequent proofs will not be broken down as deeply.

Theorem 4.1

Let (S, d) be a metric space. For any $x \in S$, and for any $r > 0$, $B_r(x)$ is an open subset in S.

To prove this statement, we must show that for any $y \in B_r(x)$, we can find some $s > 0$ such that $B_s(y) \subseteq B_r(x)$. We can think of this intuitively as follows: the point y must be at some distance $d < r$ away from x. Thus, we know that we can find points between y and the border of $B_r(x)$ by considering those points that are of distance $r - d$ away from y. So, we let $s = d - r$. Thus, the maximum distance of any point in $B_s(y)$ from x will be $< d + s = d + (r - d) = r$, and so this point will also lie in $B_r(x)$. Remember that to show $B_s(y) \subseteq B_r(x)$, it is sufficient to show that $a \in B_s(y) \implies a \in B_r(x)$. We can formalize our intuition by using the triangle inequality and the following diagram:

Proof. For any $y \in S$, define $D := d(x, y) < r$, and $s := r - D > 0$. Consider some $z \in B_s(y)$.
$$d(x, z) \leq d(x, y) + d(y, z) = D + d(y, z) < D + s = r$$

This implies that $B_s(y) \subseteq B_r(x)$, and since $y \in S$ is arbitrary, we have shown that $B_r(x)$ is open.

$\square$

> **Theorem 4.2**
>
> Let (S, d) be a metric space. Let U be a collection of open sets (of potentially infinite size): $U = \{u_i : i \in I\}$. Then,
> $$\bigcup_{i \in I} u_i \text{ is open.}$$
> In other words, the union of any collection of open sets is an open set.
>
> Intuitively, we can consider the fact that for any element x in one of the open sets u_i, there will exist $r > 0$ such that the epsilon ball of radius r is also contained in u_i since it is given that each u_i is open. Thus, since the epsilon ball is contained in u_i, that same epsilon ball must also be contained in the union of all the u_i. This is true since no elements are lost when unioning a collection of sets. Thus, since an epsilon ball exists for any element in the union of the u_i, it follows by definition that the the union of the u_i is also an open set.
>
> *Proof.* Consider any $x \in \bigcup_{i \in I} u_i$. We can infer that there exists some i such that $x \in u_i$. Since u_i is open, there exists $r > 0$ such that $B_r(x) \subseteq u_i$. Thus, it follows that $B_r(x) \subseteq \bigcup_{i \in I} u_i$. Since our choice of x was arbitrary, this logic must hold for all $x \in \bigcup_{i \in I} u_i$. Thus, $\bigcup_{i \in I} u_i$ is an open set. $\square$

The following is also worth noting:

> **Example 4.6**
>
> In the metric space $(\mathbb{R}, d_{std})$:
>
> - For $a \leq b \in \mathbb{R}$, the open interval (a, b) is an open set.
>
> - For $a \in \mathbb{R}$, the open intervals $(-\infty, a)$, and (a, ∞) are open sets.
>
> - For $a \leq b \in \mathbb{R}$, the intervals $[a, b)$, $(a, b]$, and $[a, b]$ are **not** open sets.
>
> These results can easily be proven by using similar logic to that of example 3. Try thinking them through!

4.2   Closed Sets

Naively, one may think that a closed set is simply a set that is not open. However, the definition of a closed set differs slightly from this, which may seem a bit counter intuitive. Instead, we chose to call a set closed if its *complement* is open. This means that it is possible to have a set that is both open and closed, or a set that is not open and not closed (some examples will be discussed later).

> **Definition 4.3: Closed Set**
>
> Let (S, d) be a metric space. Let $E \subseteq S$.
> E is **closed** if E^C is *open*.
>
> Remember that E^C is called the *complement* of E, and is defined as $E^c := \{x \in S : x \notin E\}$.

> **Example 4.7**
>
> $[0, 1] \subseteq \mathbb{R}$ is closed. Since a closed set is defined by whether its complement is open, we must consider $[0, 1]^C = (-\infty, 0) \cup (1, \infty)$. As seen in example 5, both $(-\infty, 0)$ and $(1, \infty)$ are open, and by theorem 4.2, the union of open sets is also an open set. Thus, $[0, 1]^C$ is open, and so $[0, 1]$ is closed. Similarly, it can be shown that for any $a < b \in \mathbb{R}$, the closed interval $[a, b]$ is also a closed set.

> **Example 4.8**
>
> $(0, 1) \subseteq \mathbb{R}$ is not closed. $(0, 1)^C = (-\infty, 0] \cup [1, \infty)$, which is not open (seen by considering an epsilon ball centered at 0 or 1: $B_r(0)$ or $B_r(1)$). Since $(0, 1)^C$ is not open, $(0, 1)$ is not closed.

> **Example 4.9**
>
> S is both open **and** closed and $\emptyset$ is both open **and** closed. We already showed in example 4.5 that S is open and $\emptyset$ is open. Note that $S^C = \emptyset$ and $\emptyset^C = S$. Thus, it follows that S and $\emptyset$ are also closed.

> **Example 4.10**
>
> $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is not closed.
>
> *Proof.* $0 \notin E \implies 0 \in E^C$. For any $r > 0$, there must exist $x \in E$ such that $x \in B_r(0)$. This follows because the sequence $(\frac{1}{n})$ converges to 0, and so the sequence will get within ϵ of 0 for any $\epsilon > 0$. In other words, $B_r(0) \cap E \neq \emptyset$. This implies that $B_r(0)$ is not entirely contained within E^C, and so E^C is not open. Thus, E is not closed. $\qquad \square$

4.3 Closure

Please carefully review example 4.10, as this will motivate our coming discussion of limit points. Note what we observed: we had a sequence contained within E that converged to a value that was not in E. This led us to conclude that E was not closed, since any epsilon ball centered at the convergent point must contain a point in E by definition of convergent sequence. Thus, no epsilon ball centered at the convergent point can be contained entirely inside E^C, which means E^C cannot be an open set. This shows that E is not closed.

Definition 4.4: Limit Point

Let (S, d) be a metric space. Let $E \subseteq S$. We say $x \in S$ is a *limit point* of E if for all $r > 0$, $B_r(x) \cap E$ contains some point $y \in S$ such that $x \neq y$. Here are some equivalent formulations of this idea:

- E contains points arbitrarily close to x that are not equal to x.

- For any $\epsilon > 0$, one can find $y \in E$ such that $y \in B_\epsilon(x)$ and $x \neq y$

- There exists a sequence (s_n) where each $s_n \in E$ that converges to x and s_n is not constant valued (such as $s_n = x$ for all n).

The following two remarks are also important to keep in mind:

- It is possible for a limit point of E to not be an element of E. For example, $E = (0, 1) \subseteq \mathbb{R}$ has a limit point of 0 and a limit point of 1, but neither is contained within E.

- It is possible that a point in E is **not** a limit point of E. For example, consider example 4.10. $1 \in E$, but 1 is not a limit point of E since $B_{\frac{1}{4}} \cap E = \{1\}$, and so since the intersection does not contain any values besides 1, we know that 1 is not a limit point of E by definition.

Definition 4.5: Closure

Let (S, d) be a metric space. Let $E \subseteq S$. The *closure* of E is written as $\overline{E}$.

$$\overline{E} := E \cup \{\text{limit points of E}\}$$

Theorem 4.3

Let (S, d) be a metric space. Let $E \subseteq S$.

E is closed $\iff E = \overline{E}$

For intuition of the forward direction, we notice that if a subset is closed but doesn't contain all of its limit points, we will be able to arrive at a contradiction by the same reasoning above. Thus, we will be able to show that a closed set must contain all of its limit points, and thus be equal to its closure.

For the reverse direction, consider why we began our discussion of limit points. We saw that if there were any limit points of our subset that were not contained in our subset, then the complement of our subset would never be open, and so our subset would never be closed. However, if our subset contained all of its limit points, then the complement should be open, meaning that our subset will be closed. We will now formalize both of these proofs.

Forward Direction:

Proof. Assume E is a closed subset. Showing that E is equal to its closure is equivalent to showing that E contains all of its limit points. This is equivalent to showing that no limit points of E are in E^C. Consider some limit point of E, $x \in S$. By definition of limit point, $B_r(x) \cap E \neq \emptyset$ for all $r > 0$. Since E is closed, E^C is open. Since E^C is open, it cannot contain the point x since $B_r(x) \not\subseteq E^C$ for any $r > 0$. Finally, $x \notin E^C \implies x \in E$. Since the limit point x is arbitrary, E contains all of its limit points. $\square$

Reverse Direction:

Proof. Assume $E = \overline{E}$. To show E is closed, we must show E^C is open. Consider any $x \in E^C$. Since E contains all of its limit points, we know that x is not a limit point of E. This implies that $B_r(x) \cap E = \emptyset$ or $\{x\}$ for some $r > 0$. However, by assumption $x \notin E$, so $B_r(x) \cap E = \emptyset$. Thus, $B_r(x)$ must be contained entirely in E^C, and so E^C is open. This implies E is closed. $\square$

Example 4.11

$E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$ is closed. We first notice that the only limit point of $\{\frac{1}{n} : n \in \mathbb{N}\}$ is 0, and so E contains all of its limit points. Thus, E is closed.

4.4   Compact Subsets

Definition 4.6: Open Covers and Finite Subcovers

Let (S, d) be a metric space. Let $E \subseteq S$. Let $\{u_i : i \in I\}$ be a collection of open subsets of S.

- $\{u_i : i \in I\}$ is called an **open cover** of E if:

$$E \subseteq \bigcup_{i \in I} u_i$$

  Equivalently, for any $x \in E$, there exists $i \in I$ such that $x \in u_i$. In words: the union of the open sets contains all of E.

- We say an open cover has a **finite subcover** if there exist finitely many $i_1, i_2, \ldots, i_n$ such that $E \subseteq \bigcup_{k=1}^{n} u_{i_k}$.

Definition 4.7: Compact Subset

Let (S, d) be a metric space. Let $E \subseteq S$.
E is a **compact** subset if *every* open cover of E has a *finite subcover*.

This definition should make it easy to show that a subset is not compact. We only have to find 1 example of an open cover of E that has **no** finite subcover to show that E is not compact.

Example 4.12

$\mathbb{N} = \{1, 2, 3, \ldots\} \subseteq \mathbb{R}$ is not compact. We can show this by finding some open cover of $\mathbb{N}$ that has no finite subcover. Since $\mathbb{N}$ is of infinite size, it shouldn't be hard to create an open cover that requires infinitely many sets to cover all of $\mathbb{N}$. For example, consider $\{u_n = (n - \frac{1}{2}, n + \frac{1}{2}) : n \in N\}$. Each u_n is open, and for any $x \in \mathbb{N}$, we know that $x \in u_x$ by construction (if this is not immediately clear, try plugging in x to u_x). However, any finite collection of these u_n must have some largest value of n. This means that u_{n+1} is not in the finite collection, and so $n + 1$ is not in the union of the finite collection. Thus, no finite cover can exist for this open cover.

Example 4.13

$E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$ is compact. Consider any open cover $\{u_i : i \in I\}$. Since $0 \in E$, there must exist $i \in I$ such that $0 \in u_i$. Since u_i is open, there exists $r > 0$ such that $B_r(0) \subseteq u_i$. Equivalently, u_i contains the set $\{\frac{1}{n} : n > \frac{1}{r}\}$ (this follows from the fact that the values of $\frac{1}{n}$ that are in $B_r(0)$ satisfy $d(\frac{1}{n}, 0) < r$ which means $\frac{1}{n} < r$). Note that $\{\frac{1}{n} : n \in \mathbb{N}\} = \{\frac{1}{n} : 1 \leq n \leq \frac{1}{r}\} \cup \{\frac{1}{n} : n > \frac{1}{r}\}$, and the term on the left of the union has finitely many values of n (since $r > 0$ is finite), and the term on the right of the union takes on infinitely many values of n. We now want to make use of the fact that there are only finitely many values of $\frac{1}{n}$ for $1 \leq n \leq \frac{1}{r}$. Since there are finitely many values, and each value must be covered by at least one u_k, it follows that there must be a finite number of u_k that cover these values. Finally, we can construct a finite subcover by taking each u_k along with the u_i we chose earlier. Thus, any open cover of E has a finite subcover, and so E is compact.

Definition 4.8: Sequentialy Compact

Let (S, d) be a metric space. A subset $E \subseteq S$ is called **sequentially compact** if for all sequences of points in E, there is a subsequence that converges to a point in E.

Note that this definition sounds somewhat similar to the definition of a closet subset, but we must be very careful to understand the difference. Earlier we showed that a subset E is closed if it contains all of its limit points. In other words, every sequence of points in E that converges (if any exist) must converge to a point in E. However, the definition of sequentially compact is even stronger. It requires that **any** sequence of points in E (whether or not the sequence converges is irrelevant) **MUST** have a subsequence that converges to a point in E. Soon, we will prove that sequentially compact is a stronger statement than closedness: i.e. sequentially compact $\implies$ closed.

Example 4.14

Consider $E = (0, 1) \subseteq \mathbb{R}$. E is **not** sequentially compact. Consider the sequence $(1, \frac{1}{2}, \frac{1}{3}, \dots)$. Since this sequence converges to 0, all of its subsequences must also converge to 0. However, $0 \notin E$, so E is not sequentially compact.

See definition 3.1

5 Continuity

5.1 Continuous Functions

5.2 Uniform Continuity

5.3 Convergence of Sequences of Functions

The weakest form of convergence that we will analyze in this course is pointwise convergence.

Definition 5.1: Pointwise Convergence

Let X be a set , and (f_n) be a sequence of functions $f_n : X \to \mathbb{R}$. We claim that $(f_n) \to f$ converges pointwisely to a function $f : X \to \mathbb{R}$ if $\forall x_0 \in X$, we have $\lim_{n \to \infty} f_n(x_0) = f(x_0)$ We can also write this equivalently as (insert epsilon definition of pointwise convergence).

Note that a sequence of continuous functions can converge pointwisely to a non continuous function (consider $(f_n = x^n)$). (Insert triangle function example with constant area of $1/2$ that converges to f(x) = 0, but does not converge in integral). We can now define a stronger notion of convergence that deals with the issues of failure to converge in continuity and in integration.

Definition 5.2: Uniform Convergence

We claim that $(f_n) \to f$ converges uniformly to a function $f : X \to \mathbb{R}$ if $\forall \epsilon > 0, \exists N > 0$ such that $\forall x_0 \in X, \forall n > N, |f_n(x_0) - f(x_0)| < \epsilon$. The key difference between this definition of convergence and the pointwise definition of convergence is that uniform convergence requires N to be dependent only on ϵ, where as N can be dependent on both ϵ and x_0 in the definition of pointwise convergence.

Let us now see how exactly uniform convergence avoids some of the issues with pointwise convergence. (Insert theorem that a sequence of continuous functions that converge uniformly implies that their convergent function is also continuous)

> **Theorem 5.1**
>
> If (f_n) is a sequence of continuous functions on some set X, and $(f_n) \to f$ uniformly, then f is continuous on X.
>
> *Proof.* Since $(f_n) \to f$ uniformly, we know that $\exists N$ s.t. c
>
> $$n > N \implies |f(x) - f_n(x)| < \frac{\epsilon}{3}$$
>
> Now, we pick some $n > N$. Since f_n is continuous, we know that $\exists \delta > 0$ s.t.
>
> $$|x - x_0| < \delta \implies |f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}$$
>
> $\forall x_0, x \in X, \forall \epsilon > 0$,
> $|x - x_0| < \delta \implies$
>
> $$|f(x) - f(x_0)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$
> $$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$
> $$= \epsilon$$
>
> Thus, f is continuous on X. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

(Insert theorem that if we have a sequence of continuous functions that converge to f uniformly, the limit of the integral of f_n is the same as the integral of f)