Math 104: Introduction to Analysis

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The notes scribed here follow the textbook Elementary Analysis by Kenneth A. Ross. Some additional remarks may be included based on the lecture teachings of Professor Wei Fan. I have slightly reordered some ideas - namely I began my introduction to topology before beginning the discussion of sequences. This is just so that all concepts can be presented in terms of both topology and the real number system.

Contents

1 Introduction

Dummy text

1.1 $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$

1.2 Completeness

1.3 Constructing $\mathbb{R}$ from $\mathbb{Q}$

1.4 Solved Exercises

2 A Brief Remark on Topology

In this section, will introduce the notion of a metric space. While we will not delve very deep, it is important to start considering how the absolute value distance function we will use on the real numbers can actually be abstracted, and that many results we prove on real numbers are much more general.

When working with the set of real numbers, we are able to perform much of our analysis through the use of a distance function, namely $d(x, y) = |x - y|$. As seen in (insert chapter 1 reference), d satisfies the triangle inequality, which we will make important use of very soon in our discussion of sequences. However, it turns out that many results on the real numbers that involve d make use of a small set of fundamental properties that the absolute value function demonstrates. As such, we could perform the same analysis with other functions that also demonstrate these same properties so long as we are able to write them out explicitly. We begin this abstraction process by defining the concepts of a metric, and metric space.

Definition 2.1: Metric Space

Let S be a set and suppose d is a function defined on all pairs (x, y). We call d a metric (or distance function) on S iff it satisfies the following conditions:

$$d(x, x) = 0 \ \forall x \in S \tag{1}$$

$$d(x, y) > 0 \text{ for distinct } x, y \in S \tag{2}$$

$$d(x, y) = d(y, x) \ \forall x, y \in S \tag{3}$$

$$d(x, z) \leq d(x, y) + d(y, z) \ \forall x, y, z \in S \tag{4}$$

A *metric space* M is a pair (S, d) where d is a metric on S. Note that there can be multiple valid metrics on any given set S, as we shall see soon.

As you read the coming section on sequences, please consider the situations in which it may be possible to replace the operations we perform on $|x - y|$ with the more general $d(x, y)$.

3 Sequences and Series

3.1 Limits of Sequences

Definition of a convergent sequence in a metric space:

3.2 Monotone Sequences and Cauchy Sequences

3.3 Lim Sup and Lim Inf

3.4 Series

4 Topology Part 2

4.1 Open and Closed Sets

4.2 Closure

4.3 Compact Sets

See 3.1

5 Continuity

5.1 Continuous Functions

5.2 Uniform Continuity

5.3 Convergence of Sequences of Functions

The weakest form of convergence that we will analyze in this course is pointwise convergence.

Definition 5.1: Pointwise Convergence

Let X be a set , and (f_n) be a sequence of functions $f_n : X \to \mathbb{R}$. We claim that $(f_n) \to f$ converges pointwisely to a function $f : X \to \mathbb{R}$ if $\forall x_0 \in X$, we have $\lim_{n\to\infty} f_n(x_0) = f(x_0)$ We can also write this equivalently as (insert epsilon definition of pointwise convergence).

Note that a sequence of continuous functions can converge pointwisely to a non continuous function (consider $(f_n = x^n)$). (Insert triangle function example with constant area of 1/2 that converges to f(x) = 0, but does not converge in integral). We can now define a stronger notion of convergence that deals with the issues of failure to converge in continuity and in integration.

Definition 5.2: Uniform Convergence

We claim that $(f_n) \to f$ converges uniformly to a function $f : X \to \mathbb{R}$ if $\forall \epsilon > 0, \exists N > 0$ such that $\forall x_0 \in X, \forall n > N, |f_n(x_0) - f(x_0)| < \epsilon$. The key difference between this definition of convergence and the pointwise definition of convergence is that uniform convergence requires N to be dependent only on ϵ, where as N can be dependent on both ϵ and x_0 in the definition of pointwise convergence.

Let us now see how exactly uniform convergence avoids some of the issues with pointwise convergence. (Insert theorem that a sequence of continuous functions that converge uniformly implies that their convergent function is also continuous)

Theorem 5.1

If (f_n) is a sequence of continuous functions on some set X, and $(f_n) \to f$ uniformly, then f is continuous on X.

Proof. Since $(f_n) \to f$ uniformly, we know that $\exists N$ s.t. c

$$n > N \implies |f(x) - f_n(x)| < \frac{\epsilon}{3}$$

Now, we pick some $n > N$. Since f_n is continuous, we know that $\exists \delta > 0$ s.t.

$$|x - x_0| < \delta \implies |f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}$$

$\forall x_0, x \in X, \forall \epsilon > 0$,
$|x - x_0| < \delta \implies$

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Thus, f is continuous on X. $\square$

(Insert theorem that if we have a sequence of continuous functions that converge to f uniformly, the limit of the integral of f_n is the same as the integral of f)