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Fractional driven-damped oscillator and its general closed form exact solution

Michael Berman [a,*], Lorenz S. Cederbaum [b]

[a] Computer Science, Hadassah Academic College, Jerusalem, Israel
[b] Theoretische Chemie, University of Heidelberg, Heidelberg, Germany

HIGHLIGHTS

- The oscillator with Caputo derivatives is solved in closed form.
- The exact solution is expressed in terms of generalized Mittag-Leffler functions.
- The standard driven-damped Harmonic Oscillator is recovered as a special case.
- The solution is shown to decay algebraically with a finite number of zeros.
- Momentum–position phase-plane diagrams and loss of energy are discussed.

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ABSTRACT

New questions in fundamental physics and in other fields, which cannot be formulated adequately using traditional integral and differential calculus emerged recently. Fractional calculus was shown to describe phenomena where conventional approaches have been unsatisfactory. The driven damped fractional oscillator entails a rich set of important features, including loss of energy to the environment and resonances. In this paper, this oscillator with *Caputo* fractional derivatives is solved analytically in closed form. The exact solution is expressed in terms of generalized *Mittag-Leffler* functions. The standard driven-damped Harmonic Oscillator is recovered as a special case of non-fractional derivatives. In contradistinction to the standard oscillator, the solution of the fractional oscillator is shown to *decay algebraically* and to possess a *finite number* of zeros. Several decay patterns are uncovered and are a direct consequence of the asymptotic properties of the generalized *Mittag-Leffler* functions. Other interesting properties of the fractional oscillator like the momentum–position phase-plane diagrams and the time dependence of the energy terms are discussed as well.

1. Introduction

The driven damped mechanical Harmonic Oscillator is studied in numerous physics text books [1,2] and obeys the following differential equation

$$m\frac{d^2x\left(t\right)}{dt^2}+b\frac{dx\left(t\right)}{dt}+k\,x\left(t\right)=F\left(t\right),\quad x^{(n)}\left(0\right)=x_0{}^{(n)}\quad\left(n=0,\,1\right),\tag{1}$$

* Corresponding author.
E-mail addresses: michael@hadassah.ac.il (M. Berman), Lorenz.Cederbaum@pci.uni-heidelberg.de (L.S. Cederbaum).

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which is a statement of Newton's second law, balancing the frictional force $F_f = -b\frac{\mathrm{dx}(t)}{\mathrm{dt}}$ proportional to the speed with which the object moves, the stiffness force (Hooke's law) $F_s = -k\, x\,(t)$, the driving force $F\,(t)$ and the force related to the acceleration $m\frac{d^2x(t)}{dt^2}$. Here m is the mass, b is the proportionality coefficient of the frictional force, and k is the stiffness coefficient of the oscillator's spring. $x^{(n)}\,(0)$ are the nth derivatives at $t = 0$, and ${x_0}^{(n)}$ are the initial conditions.

Despite its simplicity, the driven damped Harmonic Oscillator entails a rich set of important features, including loss of energy to the environment and resonances. Eq. (1) can be solved in closed form with various methods, thereby enabling insight into such phenomena. This equation is very important also in branches of physics beyond mechanics, as well as in other areas. For example, in electrical engineering, Eq. (1) is used to describe electric circuits of capacitance C, resistance R, and inductance L, connected in a series, under an applied voltage $V\,(t)$, namely the following equation for the electric charge $q\,(t)$ [2],

$$L\frac{d^2q\,(t)}{dt^2} + R\frac{dq\,(t)}{dt} + \frac{1}{C}\,q\,(t) = V\,(t)\,. \tag{2}$$

A similar equation can be obtained when connecting the components in parallel, rather than in a series.

In addition to the above mechanical and electrical cases, there are many circumstances in nature in which something is "oscillating" and in which energy is lost to the environment and/or the resonance phenomenon occurs. For example, another system that shows the same features is the atmosphere of the whole earth. If the atmosphere, which we suppose surrounds the earth evenly on all sides, is pulled to one side by the moon or, rather, squashed prolate into a double tide, and if we could then let it go, it would move like a driven damped oscillator. It turned out that the frequency of the phenomenon could be measured, resulting in a very interesting and important solved problem [2].

Another area is the physics of musical instruments [3], in which forced and damped oscillations play a major role. The history of musical instruments is nearly as old as the history of civilization itself. The role of acoustical science in this context is an interesting one. Indeed, the choice of the authors of this reasonably balanced treatment of the whole field of acoustics of musical instruments devoted a significant attention to forced damped oscillations.

Yet another area of interest is the occurrence of oscillatory and resonance phenomena in neural networks as well as in robotics systems [4], such as in stabilizing tremor.

The driven damped Harmonic Oscillator of Eqs. (1) and (2) relate to classical dynamics. However, this model is also very useful in the context of quantum mechanics. The wave equations for the forced, damped, and forced and damped oscillators were solved in closed form for an arbitrary forcing function [5], and exact calculations for a quantum oscillator linearly coupled to a thermal reservoir were presented in [6].

More involved physical situations are studied by forcing the oscillator with a random driving force. Directly connected with this problem is the foundation of the Langevin equation. In fact, randomness may be introduced alternatively in the mass of the first term, in the damping term or in the stiffness terms of Eq. (1) [7]. The concept of a noisy oscillator was introduced to science by Einstein in studies of Brownian motion. As noted by Gitterman [7] and references therein, multiplicative noise can have a twofold impact on a harmonic oscillator. The case of a random frequency has received the most study, while a random damping parameter has been discussed recently in connection with velocity fluctuations in convective media. A very recent study of an under-damped stochastic harmonic oscillator has recently appeared [8].

A recent historical survey [9] attributes the birth of fractional calculus to N. H. Abel. In his article [10] Abel introduced fractional order integration in the form that is currently known as the *Riemann–Liouville* fractional integral, and fractional-order differentiation in the form that is currently known as the *Caputo* fractional derivative both defined below in Section 2. The latter is the derivative form used in the derivation of the fractional driven damped oscillator in the present paper. An application-oriented exposition using differential operators of *Caputo* type [11] surveys a long list of application areas in which this derivative plays an important role. Applications in the area of viscoelasticity are reviewed in [12], where results on various fractional mechanical models are reported. A comprehensive account of the field is covered in [13] with an early survey of applications in chapter 10 therein. Fractional calculus was shown to describe physical phenomena where conventional approaches have been unsatisfactory. Anomalous diffusion is one such intensively studied and recently studied field [14]. Related to that is the fractional advection dispersion equation (FADE), which was introduced to model anomalous super diffusion. A very interesting and recent clarification of a space–time duality in this model has been reported [15]. Applications of fractional calculus are currently actively pursued. Applications of fractional calculus in physics can be found in [16]. The Role of fractional calculus in modeling biological phenomena has been recently reviewed [17]. A recent review of fractional derivatives and their applications in reservoir engineering problems appeared in [18], covering issues related to the advantages over the classic reservoir engineering based approaches. Applications of fractional calculus in Earth system dynamics were recently reviewed [19], focusing on some challenges for the future applications in this area. Applications that have appeared in the literature of fractional pharmacokinetics with a focus on optimal control methodologies were recently reviewed [20], alluding to the design of dosing regimens suitable to individuals and to populations. A recent book [21] reported image processing applications resulting in a stable algorithm for 3D-shape recovery in confocal microscopy, folded potentials in cluster physics, related to smooth transition from Coulomb to Yukawa-potentials, infrared spectroscopy, with a first practical application of the fractional quantum harmonic oscillator to describe rot-vib spectra in diatomic molecules from a generalized point of view. This book also reports on new questions in fundamental physics, which cannot be formulated adequately using traditional differential calculus. It is further stated therein, that it is of fundamental importance to study the fractional pendant of the harmonic oscillator and to discuss the specific properties of its solutions.

Recent progress in the theory and mathematical aspects in the field of fractional calculus have appeared in a series of papers, Special Issue of the Journal of Fractional Calculus and Applied Analysis [22]. The Special Issue contains both survey and research papers on Special Functions and Integral Transforms, both very valuable to the field of fractional calculus. Another collection of very recent papers encompasses most of the important areas of fractional calculus research and applications [23]. Many references on the mathematical background are of course included in the above selected literature on applications.

A fractional version of Eq. (1) is the subject of the present work. To the best of our knowledge, a closed-form solution of the driven-damped fractional oscillator was not reported in the literature.

2. Fractional calculus

We review below some basic concepts of fractional calculus, focusing only on the functional relations used in the derivations of this paper. It is instructive to start by inspecting the well-known integration formula of standard calculus [11–13]

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_{n-1}} dx_n f(x_n) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt \equiv {}_aJ_x^n f(x)\,. \tag{3}$$

This equation can be proved by induction. ${}_aJ_x^0 = I$ and n-fold differentiation of this integral recovers the original function, namely $\frac{d^n}{dx^n}[{}_aJ_x^n f(x)] = f(x)$.

Inspired by the above standard formula, one now introduces the fractional analog.

2.1. *Definition of fractional integrals*

Let α be a positive real number. The operator ${}_aJ_x^\alpha$ defined by

$${}_aJ_x^\alpha f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt \tag{4}$$

for $a \le x \le b$, is called the *Riemann–Liouville* fractional integral operator of order α [11–13]. For $\alpha = 0$, one sets ${}_aJ_x^0 = I$, the identity operator. Here $\Gamma(\alpha)$ is the *Gamma* function, with the special case when $\alpha = n+1$, $n = 1, 2, \ldots \in \mathbb{N}$, an integer, for which $\Gamma(n+1) = n!$

2.2. Riemann–Liouville *fractional derivative*

Let α be as for the integral a positive real number and $n = [\alpha]+1$. Here we denote $\alpha = [\alpha]+\{\alpha\}$, where $[\alpha]$ is the integer part of α and $\{\alpha\}$ is the remainder. The operator ${}_aD_x^\alpha$ defined by [11–13]

$${}_aD_x^\alpha f(x) \equiv \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt \tag{5}$$

for $a \le x \le b$, is called the *Riemann–Liouville* differential operator of order α. For $\alpha = 0$, i.e $n = 1$, one sets ${}_aD_x^0 = I$, the identity operator.

As a first consequence of this definition we note, that if $\alpha \in \mathbb{N}$ (is an integer), the operator ${}_aD_x^\alpha$ coincides with the classical differential operator D^n, $n \in \mathbb{N}$. Indeed, for $\alpha = k$, $n = k+1$, $k, n \in \mathbb{N}$ we have ${}_aD_x^k f(x) = \left(\frac{d}{dx}\right)^k f(x)$.

For $0 < \alpha < 1$; $[\alpha] = 0$ for which $n = 1$ the derivative is

$${}_aD_x^\alpha f(x)\big|_{0<\alpha<1} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)\, dt\,.$$

2.3. Caputo *fractional derivative*

With the same definitions of α, n, a, b, f, x, and by placing the derivative $\left(\frac{d}{dx}\right)^n$ within the integral in the expression for the *Riemann–Liouville* derivative, one defines the *Caputo* fractional derivative by [11–13]

$${}_a^CD_x^\alpha f(x) \equiv \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n f(t)\, dt\,. \tag{6}$$

In terms of the *Riemann–Liouville* integral, Eq. (4), the *Caputo* fractional derivative of Eq. (6) can be written as

$${}_a^CD_x^\alpha f(x) = {}_aJ_x^{n-\alpha} \left[\left(\frac{d}{dx}\right)^n f(x)\right].$$

As for the *Riemann–Liouville* derivative, the *Caputo* derivative for $\alpha = k,\ n = k + 1,\ k, n \in \mathbb{N}$ reduces to the standard non-fractional derivative, i.e. one has ${}_{a}^{C}D_{x}^{k}f\,(x) = \left(\frac{d}{dx}\right)^{k} f\,(x)$. The advantage of the *Caputo* fractional derivative is that it enables standard boundary conditions (initial values) for differential equations. This is not the case for the *Riemann–Liouville* derivative. However, *Caputo* fractional derivative requires that $\left(\frac{d}{dt}\right)^{n} f\,(t)$ exists for all n, while *Riemann–Liouville* derivative only requires that $f\,(t)$ is continuous.

2.3.1. Laplace transform of the Caputo fractional derivative

The Laplace transform and inverse Laplace transform of the *Caputo* derivative are used intensively in the derivation of the closed-form solution of the fractional oscillator. The Laplace transform $\mathcal{L}f$ of a function f is defined as $\mathcal{L}[f(t); s] \equiv \int_0^\infty e^{-st} f(t)dt = \tilde{f}(s)$, $s \in \mathbb{C}$. Assume that the Laplace transform $\mathcal{L}f$ of a function f exists. Define $\alpha > 0$ and again the closest next integer $n = [\alpha] + 1$, and define $D_{*}^{\alpha} \equiv {}_{a}^{C}D_{x}^{\alpha}$ as a shorthand notation for the *Caputo* fractional derivative. Then, with the Laplace transform variable s we have [11–13]

$$\mathcal{L}D_{*}^{\alpha}f\,(s) = s^{\alpha}\mathcal{L}f\,(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}\,(0)\,. \tag{7}$$

A particular case of Eq. (7) for $0 < \alpha < 1$, i.e. $n = 1$ reads

$$\mathcal{L}D_{*}^{\alpha}f\,(s) = s^{\alpha}\mathcal{L}f\,(s) - s^{\alpha-1}f\,(0)\,, \quad 0 < \alpha \le 1. \tag{8}$$

2.4. Mittag-Leffler function — The queen of fractional calculus

The generalized *Mittag-Leffler* function is defined by [24,25]

$$E_{\alpha,\beta}\,(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\,(\alpha k + \beta)}, \quad Real\,\alpha,\ \beta > 0,\ k \in \mathbb{N},\ z \in \mathbb{C}. \tag{9}$$

Below we review selected properties of this important function. Its properties and functional relations summarized below are used in the derivation of the closed form solution of the fractional oscillator.

2.4.1. Functional relations of the Mittag-Leffler function

Elementary exponential, cosine and reciprocal functions [26] can be obtained as special cases of the values of the pair α, β:

$$\begin{aligned} &E_{1,1}(z) = e^z, \quad E_{2,1}(-z^2) = cos(z), \\ &E_{2,2}(-z^2) = \frac{sin(z)}{z}, \quad z \in \mathbb{C}; \\ &E_{0,1}(z) = \frac{1}{1-z}, \quad |z| < 1, \quad z \in \mathbb{C}. \end{aligned} \tag{10}$$

Useful recursion relations are:

$$E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z), \tag{11}$$

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z E_{\alpha,\beta+\alpha}(z), \tag{12}$$

and the duplication formula reads:

$$E_{\alpha,\beta}(z^2) = \frac{1}{2}[E_{\alpha/2,\beta}(+z) + E_{\alpha/2,\beta}(-z)]. \tag{13}$$

2.4.2. Laplace transform of the Mittag-Leffler function

The following pair of Laplace and Inverse Laplace transforms is used in the derivation of the closed form fractional oscillator. We have, using (5.1.8) of Ref. [24]

$$\mathcal{L}^{-1}\left[\frac{s^{\alpha-1}}{(s^{\alpha} - a)}\right] = E_{\alpha,1}\,(at^{\alpha})\,. \tag{14}$$

Eq. (14) plays an important role in the derivations of this paper. Therefore, we give a proof of a more general relation, which will be employed later. We note that the Laplace transform of the argument of the *Mittag-Leffler* function on the right-hand side of Eq. (14) is $\mathcal{L}[at^{\alpha k+\beta-1}] = \frac{a^k \Gamma(\alpha k+\beta)}{s^{\alpha k+\beta}}$. Therefore, according to the definition (9) of the *Mittag-Leffler* function and using the fact that the *Mittag-Leffler* function is an entire function (allowing the change of order of summation and Laplace transformation), one finds

$$\mathcal{L}[t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})] = \sum_{k=0}^{\infty} \frac{\mathcal{L}[t^{\beta-1}(at^{\alpha})^k]}{\Gamma(\alpha k+\beta)} = \sum_{k=0}^{\infty} \frac{\frac{a^k \Gamma(\alpha k+\beta)}{s^{\alpha k+\beta}}}{\Gamma(\alpha k+\beta)} = \sum_{k=0}^{\infty} \frac{a^k}{s^{\alpha k+\beta}} = \sum_{k=0}^{\infty} \frac{1}{s^{\beta}}\left(\frac{a}{s^{\alpha}}\right)^k$$
$$= \frac{1}{s^{\beta}} \sum_{k=0}^{\infty} \left(\frac{a}{s^{\alpha}}\right)^k = \frac{1}{s^{\beta}} \frac{1}{(1-\frac{a}{s^{\alpha}})} = \frac{s^{\alpha-\beta}}{(s^{\alpha}-a)},$$

which completes the proof for the case $\frac{a}{s^{\alpha}} < 1$ (this is (5.1.7) of Ref. [24], $\beta = 1$ in Eq. (14)).

2.4.3. *Asymptotic expansion of the Mittag-Leffler function*

The *Mittag Leffler* functions possess important properties. First, each function possesses a finite number of zeros. Second, the functions decay algebraically. This is apparent in Theorem 7.6 of Ref. [27] and from Ref. [28]:

$$E_{\alpha,1}(z)\big|_{z\to\infty} = -\frac{1}{z\Gamma(1-\alpha)}\left(1+O\left(|z|^{-1}\right)\right), \quad |\arg(-z)| < (1-\frac{1}{2}\alpha)\pi. \tag{15}$$

For the special case $z = at^{\alpha}$, one obtains

$$E_{\alpha,1}(at^{\alpha})\big|_{t\to\infty} = -\frac{t^{-\alpha}}{a\Gamma(1-\alpha)}\left(1+o\left(|at^{\alpha}|^{-1}\right)\right), \quad |\arg(-at^{\alpha})| < (1-\frac{1}{2}\alpha)\pi. \tag{16}$$

Using the Euler reflection formula , $\Gamma(1-\alpha)\,\Gamma(\alpha) = \pi/\sin(\pi\alpha)$ [29], Eq. (16) can be expressed in the form

$$E_{\alpha,1}(at^{\alpha})\big|_{t\to\infty} = -\frac{\Gamma(\alpha)\,t^{-\alpha}\sin(\pi\alpha)}{a\pi}\left(1+O\left(|at^{\alpha}|^{-1}\right)\right), \quad |\arg(-at^{\alpha})| < (1-\frac{1}{2}\alpha)\pi. \tag{17}$$

We note that $\sin(\pi\alpha)$ is positive for $0 < \alpha \le 1$, so that expression (17) approaches zero algebraically from below or from above, depending on the sign of a. The above asymptotic behavior is used in studying the long time decay of the fractional oscillator studied in this paper.

3. Fractional damped oscillator and its closed form solution

We are looking at the following damped and driven fractional oscillator

$$D_*^{2\gamma}x(t) + 2\beta D_*^{\gamma}x(t) + \omega^2 x(t) = q(t), \quad x^{(k)}(0) = {x_0}^{(k)} \quad (k = 0, 1, \ldots, n-1), \tag{18}$$

where the source term $q \in C$ is a given function. $x(t)$ is the position of the oscillator, β the damping factor and ω the frequency of the oscillator ($\omega, \beta \in \mathbb{R}$). Throughout this paper we denote by *Greek* letters (such as γ or α) the fractional order of derivatives, and by *Roman* letters (such as n or k) integer indices. The fractional derivatives for the oscillator in Eq. (18) are determined by γ and 2γ, $0 < \gamma \le 1$. The integer $n = [2\gamma] + 1$ determines the $0, 1, \ldots, n-1$ required initial values. In Eq. (18) $n = 2$, so there are two required initial conditions ${x_0}^{(k)}$, $k = 0, 1$. This is consistent with the number of initial conditions of the integer (*non-fractional*) differential equations, in which a first order equation requires one initial condition and a second order equation requires two. $D_*^{\alpha} \equiv {}_a^C D_t^{\alpha}$ is the shorthand notation for the *Caputo* fractional derivative with respect to the time t. We stress that when using the *Caputo* fractional derivative D_*^{α}, standard initial conditions are suitable, i.e $x^{(k)}(0)$ in Eq. (18) are standard derivatives, just as in Eq. (1)! The special case of the *non-damped* oscillator, $\beta = 0$, $q(t) \neq 0$ was solved in the past, see e.g. [30,12,31]. In this work we solve the general case $\beta \neq 0$.

3.0.4. *Solving the driven-damped fractional oscillator using Laplace transforms*

In solving the driven and damped fractional oscillator in Eq. (18), it is useful to Laplace transform the differential equation. Using Eq. (7), we can calculate the Laplace transforms of the *Caputo* fractional derivatives appearing in Eq. (18). Namely, for $\mathcal{L}D_*^{2\gamma}x(s)$ with $0 \le \gamma \le 1$,

$$\mathcal{L}D_*^{2\gamma}x(s) = s^{2\gamma}\mathcal{L}x(s) - \sum_{k=1}^{2} s^{2\gamma-k}x^{(k-1)}(0) = s^{2\gamma}\mathcal{L}x(s) - s^{2\gamma-2}x^{(1)}(0) - s^{2\gamma-1}x(0)$$

and for $\mathcal{L}D_*^{\gamma}x(s)$ with $0 \le \gamma \le 1$, we find

$$\mathcal{L}D_*^{\gamma}x(s) = s^{\gamma}\mathcal{L}x(s) - \sum_{k=1}^{1} s^{\gamma-k}x^{(k-1)}(0) = s^{\gamma}\mathcal{L}x(s) - s^{\gamma-1}x(0).$$

Defining $\widetilde{x}(s) \equiv \mathcal{L}x(s)$ and $\widetilde{q}(s) \equiv \mathcal{L}q(s)$ and with the above Laplace transforms of the *Caputo* derivatives, Eq. (18) in the Laplace space takes the form

$$\left[s^{2\gamma}\widetilde{x}(s) - s^{2\gamma-2}x^{(1)}(0) - s^{2\gamma-1}x(0)\right] + 2\beta\left[s^{\gamma}\widetilde{x}(s) - s^{\gamma-1}x(0)\right] + \omega^2\,\left[\widetilde{x}(s)\right] = \widetilde{q}(s)\,, \tag{19}$$

resulting in

$$\widetilde{x}(s) = \frac{s^{2\gamma-2}x^{(1)}(0) + s^{2\gamma-1}x(0) + 2\beta s^{\gamma-1}x(0) + \widetilde{q}(s)}{s^{2\gamma} + 2\beta s^{\gamma} + \omega^2}. \tag{20}$$

In order to facilitate the calculation of the inverse Laplace transform of Eq. (20) to arrive at $x(t)$, we look for the roots of the denominator $s^{2\gamma} + 2\beta s^{\gamma} + \omega^2 = 0$. Defining $u = s^{\gamma}$, the denominator reads $u^2 + 2\beta u + \omega^2 = 0$, with the roots:

$$u_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega^2} = -\beta \pm i\omega_d;\quad \omega_d \equiv \sqrt{\omega^2 - \beta^2}. \tag{21}$$

In this work we focus on the more-interesting under-damped case, namely on $|\omega| > |\beta|$. The critically-damped $|\omega| = |\beta|$ and the over-damped $|\omega| < |\beta|$ cases can be treated in a similar way. The under-damped case is more interesting, as it corresponds to oscillatory behavior. We are looking for the Inverse Laplace transform of Eq. (20), namely at

$$x(t) = \mathcal{L}^{-1}\left[\frac{s^{2\gamma-2}x^{(1)}(0) + s^{2\gamma-1}x(0) + 2\beta s^{\gamma-1}x(0) + \widetilde{q}(s)}{s^{2\gamma} + 2\beta s^{\gamma} + \omega^2}\right].$$

In Appendix A we show (see Eq. (A.1)) that the above can be expanded using partial-fraction decomposition as

$$\begin{aligned} x(t) = {} & \frac{1}{(u_1 - u_2)}\mathcal{L}^{-1}\left[\frac{x(0)\,s^{2\gamma-1}}{(s^{\gamma} - u_1)} + \frac{x(0)\,2\beta s^{\gamma-1}}{(s^{\gamma} - u_1)} + \frac{x^{(1)}(0)\,s^{2\gamma-2}}{(s^{\gamma} - u_1)} + \frac{\widetilde{q}(s)}{(s^{\gamma} - u_1)} - \frac{s^{2\gamma-1}}{(s^{\gamma} - u_2)}\right. \\ & \left. - \frac{2\beta s^{\gamma-1}}{(s^{\gamma} - u_2)} - \frac{x^{(1)}(0)\,s^{2\gamma-2}}{(s^{\gamma} - u_2)} - \frac{\widetilde{q}(s)}{(s^{\gamma} - u_2)}\right]. \end{aligned} \tag{22}$$

3.0.5. Inverse Laplace transforms: The exact solution

We now calculate explicitly the Inverse Laplace transforms appearing in Eq. (22). The basic inverse Laplace transform, proven below Eq. (14) , reads,

$$\mathcal{L}^{-1}\left[\frac{s^{\alpha-\beta}}{(s^{\alpha} - a)}\right] = t^{\beta-1}E_{\alpha,\beta}\left(at^{\alpha}\right). \tag{23}$$

The expressions that require inverse Laplace transforms are: $\frac{s^{\gamma-1}}{(s^{\gamma}-a)}, \frac{s^{2\gamma-1}}{(s^{\gamma}-a)}, \frac{s^{2\gamma-2}}{(s^{\gamma}-a)}, \frac{1}{(s^{\gamma}-a)}$. Using Eq. (23) these terms read

$$\begin{aligned} \alpha = \gamma, \beta = 1 &: \mathcal{L}^{-1}\left[\frac{s^{\gamma-1}}{(s^{\gamma} - a)}\right] = E_{\gamma,1}\left(at^{\gamma}\right), \\ \alpha = \gamma, \beta = 1-\gamma &: \mathcal{L}^{-1}\left[\frac{s^{2\gamma-1}}{(s^{\gamma} - a)}\right] = t^{-\gamma}E_{\gamma,1-\gamma}\left(at^{\gamma}\right), \\ \alpha = \gamma, \beta = 2-\gamma &: \mathcal{L}^{-1}\left[\frac{s^{2\gamma-2}}{(s^{\gamma} - a)}\right] = t^{1-\gamma}E_{\gamma,2-\gamma}\left(at^{\gamma}\right), \\ \alpha = \gamma, \beta = \gamma &: \mathcal{L}^{-1}\left[\frac{1}{(s^{\gamma} - a)}\right] = t^{\gamma-1}E_{\gamma,\gamma}\left(at^{\gamma}\right). \end{aligned} \tag{24}$$

Returning to Eq. (22), we get using Eq. (24) the final time-dependent solution as

$$x(t) = \frac{x(0)\,A + x^{(1)}(0)\,B + C}{(u_1 - u_2)} \tag{25}$$

$$\begin{aligned} A &= t^{-\gamma}E_{\gamma,1-\gamma}\left(u_1t^{\gamma}\right) + 2\beta E_{\gamma,1}\left(u_1t^{\gamma}\right) - t^{-\gamma}E_{\gamma,1-\gamma}\left(u_2t^{\gamma}\right) - 2\beta E_{\gamma,1}\left(u_2t^{\gamma}\right), \\ B &= t^{-\gamma+1}E_{\gamma,-\gamma+2}\left(u_1t^{\gamma}\right) - t^{-\gamma+1}E_{\gamma,-\gamma+2}\left(u_2t^{\gamma}\right), \\ C &= \left[t^{\gamma-1}E_{\gamma,\gamma}\left(u_1t^{\gamma}\right)\right] * q(t) - \left[t^{\gamma-1}E_{\gamma,\gamma}\left(u_2t^{\gamma}\right)\right] * q(t)\,, \end{aligned}$$

where $u_{1,2}$ are given by Eq. (21) and the Laplace convolution is defined by:

$$f(t) * q(t) \equiv \int_0^t q(t-\tau)f(\tau)d\tau. \tag{26}$$

A more compact form of the main solution as compared to the above is derived in Appendix B (Eq. (B.2)), using the definition of $u_{1,2}$ in Eq. (21), and reads

$$\begin{aligned} x(t) &= \frac{x(0)\,A + x^{(1)}(0)\,B + C}{2i\omega_d} \\ A &= (\beta + i\omega_d)\,E_{\gamma,1}\left[(-\beta + i\omega_d)\,t^{\gamma}\right] - (\beta - i\omega_d)\,E_{\gamma,1}\left[(-\beta - i\omega_d)\,t^{\gamma}\right], \\ B &= t\,(-\beta + i\omega_d)\,E_{\gamma,2}\left[(-\beta + i\omega_d)\,t^{\gamma}\right] - t\,(-\beta - i\omega_d)\,E_{\gamma,2}\left[(-\beta - i\omega_d)\,t^{\gamma}\right], \\ C &= \left\{t^{\gamma-1}E_{\gamma,\gamma}\left[(-\beta + i\omega_d)\,t^{\gamma}\right]\right\} * q(t) - \left\{t^{\gamma-1}E_{\gamma,\gamma}\left[(-\beta - i\omega_d)\,t^{\gamma}\right]\right\} * q(t). \end{aligned} \tag{27}$$

Eqs. (25) and (27) are the main results of this manuscript. As we show shortly, the *Mittag-Leffler* functions in this main solution replace the role of elementary functions in the solution of the standard driven-damped Harmonic Oscillator. Another important observation is that the long-time asymptotic behavior of the solution is entirely determined by the asymptotic properties of the *Mittag-Leffler* functions.

As a check on the results in Eqs. (25) and (27), a numerical Laplace transform of Eq. (20), i.e. a computation that refrains from resorting to the *Mittag-Leffler* functions of the exact solution, is included in Appendix C. The numerical computations agree with the closed-form ones as can be seen in Figs. C.1 and C.2!

We briefly discuss the results for the simple case of the forcing term. For an impulse, $q(t) = \delta(t)$ the convolution Eq. (26) becomes

$$\int_0^t q(t-\tau)\,\tau^{\gamma-1}E_{\gamma,\gamma}\left(u_{1,2}\tau^{\gamma}\right)d\tau = \int_0^t \delta(t-\tau)\,\tau^{\gamma-1}E_{\gamma,\gamma}\left(u_{1,2}\tau^{\gamma}\right)d\tau = t^{\gamma-1}E_{\gamma,\gamma}\left(u_{1,2}t^{\gamma}\right),$$

and the result $G(t) = \left. x(t)\right|_{q(t)=\delta(t)}$ is the Greens function of Eq. (18). Oscillatory forcing term was studied in [32,33].

3.1. Special case I of the main result: No damping

Recalling the definitions of $u_{1,2}$, a special case of Eqs. (25) or (27) is obtained for no damping, namely for $\beta = 0$. With $\left. u_{1,2}\right|_{\beta=0} = \pm i\omega$ in Eqs. (25) or (27), one gets for the case of vanishing driving $q(t)$

$$\left. x(t)\right|_{x^{(1)}(0)=0,\ q(t)=0,\ \beta=0} = \frac{x(0)}{2}\left\{E_{\gamma,1}\left(i\omega t^{\gamma}\right) + E_{\gamma,1}\left(-i\omega t^{\gamma}\right)\right\}. \tag{28}$$

Using the duplication formula Eq. (13), Eq. (28) becomes

$$\left. x(t)\right|_{x^{(1)}(0)=0,\ q(t)=0,\ \beta=0} = x(0)\,E_{2\gamma,1}\left(-\omega^2 t^{2\gamma}\right), \tag{29}$$

which is a well-known expression for the non-damped fractional oscillator, see Ref. [13], p. 140, and [31] in terms of the *Mittag-Leffler* function. An example of the driving term $q(t) \neq 0$ in the special case of no explicit damping, i.e. $\beta = 0$, fractional oscillator was studied in Ref. [32].

3.2. Special case II of the main result: The standard oscillator

The standard text-book oscillator is obtained from Eq. (18) by setting $\gamma = 1$. The general solution in Eq. (27) of course reduces to that of the standard oscillator by putting $\gamma = 1$. We would like to show the reduction for an example. For $\gamma = 1$, no forcing source $q(t) = 0$, and using again the recursion Eq. (12), Eq. (27) reads

$$\begin{aligned} \left. x(t)\right|_{q(t)=0,\gamma=1} &= \frac{x(0)}{2i\omega_d}\Big\{(\beta + i\omega_d)\,E_{1,1}\left[(-\beta + i\omega_d)\,t\right] - (\beta - i\omega_d)\,E_{1,1}\left[(-\beta - i\omega_d)\,t\right]\Big\} \\ &+ \frac{x^{(1)}(0)}{2i\omega_d}\Big\{E_{1,1}\left[(-\beta + i\omega_d)\,t\right] - E_{1,1}\left[(-\beta - i\omega_d)\,t\right]\Big\}, \end{aligned} \tag{30}$$

which becomes using the special values according to Eq. (10)

$$\begin{aligned} \left. x(t)\right|_{q(t)=0,\gamma=1} &= e^{-\beta t}\left\{\frac{x(0)}{2i\omega_d}\Big[\beta\left(e^{i\omega_d t} - e^{-i\omega_d}\right) + i\omega_d\left(e^{i\omega_d t} + e^{-i\omega_d t}\right)\Big]\right. \\ &\left. + \frac{x^{(1)}(0)\,e^{-\beta t}}{2i\omega_d}\left(e^{i\omega_d t} - e^{-i\omega_d}\right)\right\}. \end{aligned}$$

Using the Euler identity, we arrive at

$$\left. x(t)\right|_{\gamma=1,q(t)=0} = e^{-\beta t}\left\{x(0)\left[\frac{\beta}{\omega_d}\sin(\omega_d t) + \cos(\omega_d t)\right] + \frac{x^{(1)}(0)}{\omega_d}\sin(\omega_d t)\right\}.$$

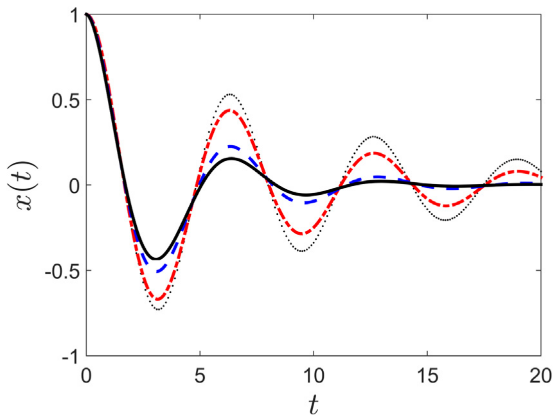

Fig. 1. (Color online) Position $x(t)$ as a function of time t of the fractional damped oscillator with damping $\beta = 0.1$ and fractional order $\gamma = 1, 0.98, 0.92, 0.89, \ldots$ *dotted line (black), -.-. dashed–dotted line (red), - - dashed line (blue), — full line (black)*, respectively. The order $\gamma = 1$ corresponds to the special case of the damped standard Harmonic Oscillator. Other quantities are a vanishing source term $q(t) = 0$, initial conditions $x(0) = 1$, $x^{(1)}(0) = 0$ and frequency $\omega = 1$.

With

$$\phi := atan(\frac{x^{(1)}(0)}{\beta\omega_d x(0)} + \frac{\beta}{\omega_d}) \text{ and } A := \frac{x(0)}{cos(\phi)},$$

we arrive at

$$x\,(t)\,|_{\gamma=1,q(t)=0} = A\,e^{-\beta t}\,cos\,(\omega_d t - \phi)\,, \tag{31}$$

which is the well-known expression for the standard, i.e. textbook, Damped Harmonic Oscillator.

4. Numerical results

In this section we present numerical examples of the exact solution using Eq. (27) and the corresponding special cases, Eqs. (28) and (31). We further study the long-time asymptotic behavior of these equations. Subsequently, we define the fractional momentum of the oscillator and study its time evolution, as well as its long-time asymptotes. We then compute examples of phase-plane diagrams of the momentum versus the position, thereby demonstrating visually the decay of these quantities over time. Finally, we present calculations of the energy of the fractional oscillator, demonstrating how it looses its energy.

4.1. Time dependence of the position $x(t)$

Fig. 1 shows the position $x(t)$ of the damped non-driven fractional oscillator as a function of time t. The dotted curve denotes the standard damped Harmonic Oscillator. Clearly the oscillations of this curve are symmetric around zero and, as is well known, decay exponentially, see Eq. (31).

The fractional oscillator curves for zero initial velocity and zero driving source term have a different decay pattern than the standard oscillator as will be studied shortly. We note that the decay is more pronounced the smaller the values of the fractional derivative order γ are.

Fig. 2 shows the position $x(t)$ as a function of time t for no damping according to Eq. (27) with $\beta = 0$, i.e. according to the special case I, Eq. (29). The dotted curve denotes the standard non-damped Harmonic Oscillator. Clearly the oscillations of this curve are symmetric around zero and, as is well known, do not decay, see Eq. (31). The fractional oscillator curves, for zero initial velocity and zero driving source term, Eq. (27), have a different decay pattern even for the no damping $\beta = 0$ case. We note also here that the decay is less pronounced for values of the fractional derivative order γ closer to 1. This decay is, however, less pronounced than the corresponding curves of the explicitly damped fractional oscillator displayed in Fig. 1.

4.2. Numerical long-time asymptotes of the position $x(t)$

Fig. 3 shows the long-time asymptote of the position $x(t)$ as a function of time t. The dotted curve denotes the standard damped Harmonic Oscillator. Clearly the oscillations of this curve are symmetric around zero and, as is well known, decay exponentially, see Eq. (31).

The fractional oscillator curves for zero initial velocity and zero driving source term have a different decay pattern. They possess a *finite number* of zeros and *decay algebraically* from *above zero*, as can be seen for times larger than the largest finite

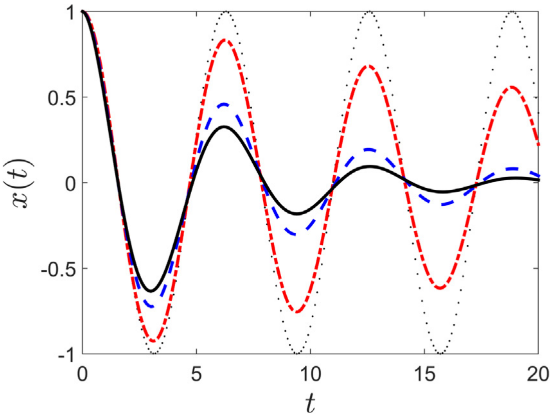

Fig. 2. (Color online) Position $x(t)$ as a function of time t of the fractional damped oscillator with no damping ($\beta = 0$) and fractional order $\gamma = 1, 0.98, 0.92, 0.89$, *... dotted line (black), -.-. dashed–dotted line (red), - - dashed line (blue), — full line (black)*, respectively. The order $\gamma = 1$ corresponds to the special case of the non-damped Harmonic Oscillator. Other quantities are a vanishing source term $q(t) = 0$, initial conditions $x(0) = 1,\ x^{(1)}(0) = 0$ and frequency $\omega = 1$.

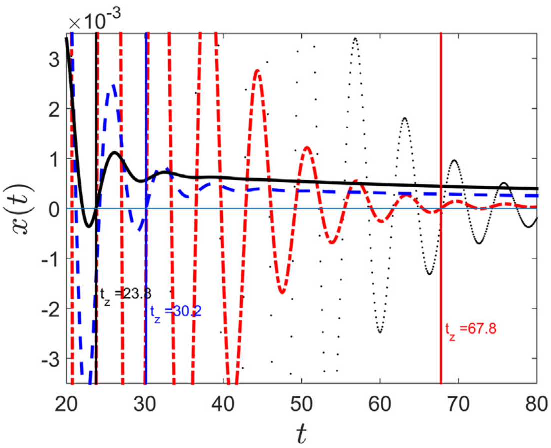

Fig. 3. (Color online) Long-time asymptote of the position $x(t)$ as a function of time t of the fractional damped oscillator with damping $\beta = 0.1$ and fractional order $\gamma = 1, 0.98, 0.92, 0.89$, *... dotted line (black), -.-. dashed–dotted line (red), - - dashed line (blue), — full line (black)*, respectively. The order $\gamma = 1$ corresponds to the special case of the damped Harmonic Oscillator. t_z are the largest zeros, i.e. the times after which the position curves do not cross the zero line. For $\gamma = 1$ this number is infinity. Other quantities are a vanishing source term $q(t) = 0$, initial conditions $x(0) = 1,\ x^{(1)}(0) = 0$ and frequency $\omega = 1$.

zero t_z for each curve. This algebraic decay is consistent with the long-time asymptotes of the *Mittag-Leffler* functions, Eq. (17).

We now calculate this long-time behavior explicitly. Using Eq. (17) for the asymptotic formula of the *Mittag-Leffler* functions, and Eq. (27) for the compact exact solution, we may write

$$x(t)\,|_{x^{(1)}(0)=0,q(t)=0}|_{t\to\infty} = -\frac{x(0)}{2i\omega_d}\frac{sin(\pi\gamma)t^{-\gamma}}{\pi}\left\{(\beta+i\omega_d)\left[\frac{1}{-\beta+i\omega_d}\right]\right.$$
$$\left.-\,(\beta-i\omega_d)\left[\frac{1}{-\beta-i\omega_d}\right]\right\} = x(0)\,\frac{\beta}{\beta^2+\omega_d^2}\frac{sin(\pi\gamma)t^{-\gamma}}{\pi} \tag{32}$$

Clearly, the expression on the right-hand side of Eq. (32) is positive for $\beta > 0,\ 0 < \gamma < 1$, as $sin(\pi\gamma)$ is positive for $0 < \gamma < 1$, confirming the observed decay from above in Fig. 3. The long-time behavior is compared to the asymptotic formula in Fig. 4 for even longer times and another value of β.

For comparison, Fig. 5 shows the long-time asymptote of the position $x(t)$ as a function of time t for no damping according to Eq. (27) with $\beta = 0$, i.e. according to special case I, Eq. (29). The fractional oscillator curves possess a finite number of zeros and, in contra-distinction to the non-zero explicit damping case, decay algebraically from below zero, as can be seen for times larger than the largest finite zero t_z. Also here, this algebraic decay is consistent with the long-time asymptotes of the *Mittag-Leffler* functions as we prove now. For this case, Eq. (32) can no longer be used, as this asymptotic formula

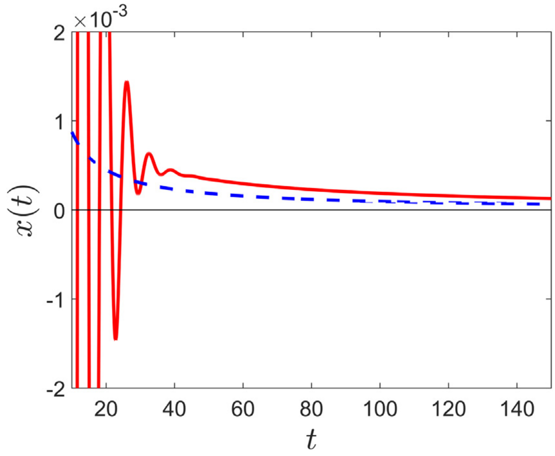

Fig. 4. (Color online) Long time asymptote of the position $x(t)$ as a function of time t of the fractional damped oscillator with damping $\beta = 0.2$ and fractional order $\gamma = 0.96$, — *full (red)* line. The decay follows the asymptotic formula, Eq. (32), - - *dashed (blue)* line. Clearly, the decay approaches zero from above. Other quantities are a vanishing source term $q(t) = 0$, initial conditions $x(0) = 1,\ x^{(1)}(0) = 0$ and frequency $\omega = 1$.

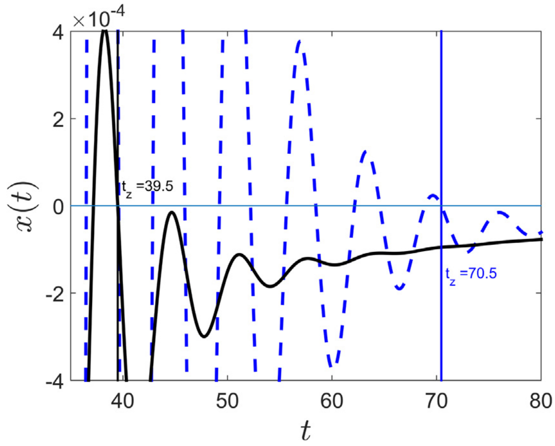

Fig. 5. (Color online) Long-time asymptote of the position $x(t)$ versus time t of the fractional oscillator with no damping $\beta = 0$ and fractional order $\gamma = 0.92, 0.89$, - - *dashed (blue) line*, — *full (black) line*, respectively. t_z are the largest zeros, i.e. the times after which the position curves do not cross the zero line. Other quantities are source term $q(t) = 0$, initial conditions $x(0) = 1,\ x^{(1)}(0) = 0$ and frequency $\omega = 1$.

vanishes for $\beta = 0$. Therefore, one has to include the next term of the asymptotic expansion of the *Mittag-Leffler* functions, namely Eq. (17) has to be replaced by

$$E_{\alpha,1}(z)\big|_{z\to\infty} = -\frac{1}{z\Gamma(1-\alpha)} - \frac{1}{z^2\Gamma(1-2\alpha)} + O\left(|z|^{-2}\right), \quad |\arg(-z)| < (1-\frac{1}{2}\alpha)\pi. \tag{33}$$

Using the Euler reflection formula , $\Gamma(1-\alpha)\,\Gamma(\alpha) = \pi/\sin(\pi\alpha)$ [29], Eq. (33) can be expressed in the form

$$E_{\alpha,1}(at^\alpha)\big|_{t\to\infty} = -\frac{\Gamma(\alpha)\,t^{-\alpha}\sin(\pi\alpha)}{a\pi} - \frac{\Gamma(2\alpha)\,t^{-2\alpha}\sin(2\pi\alpha)}{a^2\pi} + O\left(|at^\alpha|^{-2}\right), \quad |\arg(-at^\alpha)| < (1-\frac{1}{2}\alpha)\pi. \tag{34}$$

Using Eq. (34) and the fact that for $\beta = 0$ $x(t)$ in Eq. (32) vanishes, we get

$$x(t)\,|_{x^{(1)}(0)=0,q(t)=0,\beta=0}|_{t\to\infty} = -\frac{x(0)}{2i\omega_d}\frac{\sin(2\pi\gamma)t^{-2\gamma}}{\pi}\left\{(i\omega_d)\left[\frac{1}{(i\omega_d)^2}\right] - (-i\omega_d)\left[\frac{1}{(-i\omega_d)^2}\right]\right\} = x(0)\frac{\sin(2\pi\gamma)t^{-2\gamma}}{\pi\omega_d^2}. \tag{35}$$

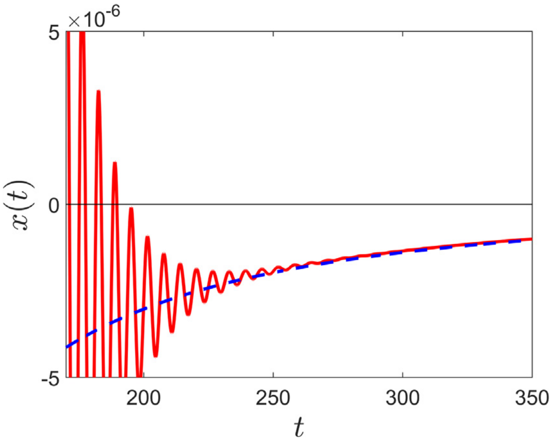

Fig. 6. (Color online) Long-time asymptote of the position $x(t)$ as a function of time t of the fractional oscillator with no damping $\beta = 0$ and fractional order $\gamma = 0.96$, *—full (red)* line. The asymptotic formula, Eq. (35), *- - dashed (blue)* line, follows this decay. Clearly, the decay approaches zero from below. Other quantities are vanishing source $q(t) = 0$, initial conditions $x(0) = 1,\ x^{(1)}(0) = 0$ and frequency $\omega = 1$.

Clearly, the expression on the right-hand side of Eq. (35) is negative, as $sin(2\pi\gamma)$ is negative for $\frac{1}{2} < \gamma < 1$, confirming the observed decay from below in Fig. 5. We further compare the long-time decay at much longer times to the analytic expression Eq. (35) in Fig. 6.

To the best of the authors' knowledge, this distinction between the asymptotic behavior of the two mechanisms, namely, the explicit damping ($\beta \neq 0$) and the implicit damping ($\beta = 0$, damping comes from $\gamma \neq 1$) of the fractional oscillator, is new. This result points at the possibility of designing an experiment that would distinguish between these two mechanisms. Measuring the long-time decay and verifying whether the decay approaches the zero line from above or from below would identify the mechanism.

4.3. Momentum and energy of the fractional oscillator

Following the suggestion of Ref. [30], we define the momentum of the fractional oscillator as

$$p\,(t) := m\, D_*^{\gamma}[x\,(t)], \tag{36}$$

with a Laplace transform according to Eq. (8)

$$\tilde{p}\,(s) = m\mathcal{L}[D_*^{\gamma} x\,(t)](s) = m[s^{\gamma}\tilde{x}\,(s) - s^{\gamma-1}x\,(0)]. \tag{37}$$

We note that m in these equations does not have the dimension of mass [30]. We also mention a different definition of the momentum in [34] in terms of stable distributions. As a preparation, the *Caputo* derivative of the *Mittag-Leffler* functions with $1 < \alpha, \gamma \leq 1$ is calculated using Eq. (8) and reads

$$D_*^{\gamma}\left[E_{\alpha,1}\,(at^{\alpha})\right]\Big|_{\alpha=\gamma} = \mathcal{L}^{-1}\left[s^{\gamma}\frac{s^{\alpha-1}}{s^{\alpha}-a} - s^{\gamma-1}\right]_{\alpha=\gamma} = \mathcal{L}^{-1}\left[\frac{as^{\gamma-1}}{s^{\gamma}-a}\right] = aE_{\gamma,1}\,(at^{\gamma})\,, \tag{38}$$

and

$$\begin{aligned} D_*^{\gamma}\left(t^{1-\alpha}\,E_{\alpha,2-\alpha}\,(at^{\alpha})\right)\Big|_{\alpha=\gamma} &= \mathcal{L}^{-1}\left[s^{\gamma}\frac{s^{\alpha-2}}{s^{\alpha}-a} - s^{\gamma-1}\right]_{\alpha=\gamma} = \mathcal{L}^{-1}\left[\frac{s^{2\gamma-2}}{s^{\gamma}-a} - s^{\gamma-1}\right] \\ &= t^{1-\gamma}E_{\gamma,2-\gamma}\,(at^{\gamma}) - \frac{t^{-\gamma}}{\Gamma(1-\gamma)}. \end{aligned} \tag{39}$$

The convolution term is obtained in a similar way, namely, we look at

$$\begin{aligned} D_*^{\gamma}\left\{\left[t^{\alpha-1}E_{\alpha,\alpha}\,(at^{\alpha})\right] * q\,(t)\right\}\Big|_{\alpha=\gamma} &= \mathcal{L}^{-1}\left[\frac{s^{\gamma}\tilde{q}(s)}{s^{\alpha}-a} - s^{\gamma-1}R(0,a)\right]_{\alpha=\gamma} \\ &= [t^{-1}E_{\gamma,0}\,(at^{\gamma})] * q\,(t) - \frac{t^{-\gamma}R(0,a)}{\Gamma(1-\gamma)}, \end{aligned} \tag{40}$$

where $R(0,a) = \left[t^{\alpha-1}E_{\alpha,\alpha}\,(at^{\alpha})\right] * q\,(t)\,\Big|_{t=0}$.

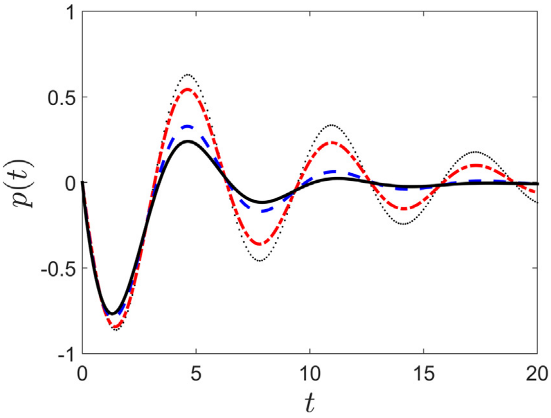

Fig. 7. (Color online) Momentum $p(t)$ as a function of time t of the fractional damped oscillator with damping $\beta = 0.1$ and fractional order $\gamma = 1, 0.98, 0.92, 0.89, \ldots$ *dotted line (black), -.-. dashed–dotted line (red), - - dashed line (blue), — full line (black)*, respectively. The order $\gamma = 1$ corresponds to the special case of the damped Harmonic Oscillator. Other quantities are source term $q(t) = 0$, initial conditions $x(0) = 1$, $x^{(1)}(0) = 0$, frequency $\omega = 1$ and mass $m = 1$.

The momentum is obtained using the fractional derivatives, Eqs. (38)–(40) of the *Mittag-Leffler* functions as

$$p(t) = m\frac{x(0)D + x^{(1)}(0)F + H}{u_1 - u_2} \tag{41}$$

$$D = -u_2 u_1\left\{E_{\gamma,1}[u_1 t^{\gamma}] - E_{\gamma,1}[u_2 t^{\gamma}]\right\},$$

$$F = t^{-\gamma+1}\left\{u_1 E_{\gamma,-\gamma+2}[u_1 t^{\gamma}] - u_2 E_{\gamma,-\gamma+2}[u_2 t^{\gamma}]\right\},$$

$$H = [t^{-1}E_{\gamma,0}(u_1 t^{\gamma})] * q(t) - [t^{-1}E_{\gamma,0}(u_2 t^{\gamma})] * q(t) + \frac{t^{-\gamma}[-R(0,u_1) + R(0,u_2)]}{\Gamma(1-\gamma)}.$$

Fig. 7 shows the momentum $p(t)$ as a function of time t according to Eq. (41) for the case of no driving source, $q(t) = 0$. The dotted curve denotes the standard damped Harmonic Oscillator, i.e fractional order $\gamma = 1$. Clearly the oscillations of this curve are symmetric around zero and, as is well known, decay exponentially. The fractional oscillator curves for zero driving source term have a different decay pattern as discussed also for the position decay curves. We note also here that the decay is more pronounced the smaller the values of the fractional derivative order γ are. We further note the time-shift of the peaks of the momentum in Fig. 7 as compared to the peaks of the position curves in Fig. 1. Appendix C.2 of Appendix C compares the evaluation of the closed from expression to a numerical computation of the inverse Laplace transform of Eq. (37) for the momentum.

It is instructive to calculate the special case of no damping $\beta = 0$ studied in Ref. [30]. For $\beta = 0$ Eq. (41) simplifies to

$$p(t)\big|_{q(t)=0,\beta=0} = \frac{m\,x(0)}{2i\omega_d}\left\{(i\omega_d)^2 E_{\gamma,1}[i\omega_d t^{\gamma}] - (-i\omega_d)^2 E_{\gamma,1}[-i\omega_d t^{\gamma}]\right\}$$
$$+ \frac{m\,x^{(1)}(0)}{2i\omega_d} t^{-\gamma+1}\left\{(i\omega_d)E_{\gamma,-\gamma+2}[i\omega_d t^{\gamma}] - (-i\omega_d)E_{\gamma,-\gamma+2}[-i\omega_d t^{\gamma}]\right\}.$$

In order to proceed, we use the recursion relation Eq. (12) $E_{\gamma,1}(i\omega_d t^{\gamma}) = i\omega_d t^{\gamma} E_{\gamma,1+\gamma}(i\omega_d t^{\gamma}) + \frac{1}{\Gamma(1-\gamma)}$ and obtain

$$p(t)\big|_{q(t)=0,\beta=0} = \frac{m\,x(0)}{2} t^{\gamma}\left\{-\omega_d^2 E_{\gamma,1+\gamma}[i\omega_d t^{\gamma}] - \omega_d^2 E_{\gamma,1+\gamma}[-i\omega_d t^{\gamma}]\right\}$$
$$+ \frac{m\,x^{(1)}(0)}{2} t^{1+\gamma}\left\{E_{\gamma,-\gamma+2}[i\omega_d t^{\gamma}] + E_{\gamma,-\gamma+2}[-i\omega_d t^{\gamma}]\right\}.$$

Recalling that $\omega_d|_{\beta=0} = \omega$ and using the duplication formula Eq. (13), this becomes

$$p(t)\big|_{x^{(1)}(0)=0,q(t)=0,\beta=0} = -m\,x(0)\,t^{\gamma}E_{2\gamma,1+\gamma}\left(-\omega^2 t^{2\gamma}\right) + m\,x^{(1)}(0)\,t^{1+\gamma}E_{2\gamma,-\gamma+2}\left(-\omega^2 t^{\gamma}\right). \tag{42}$$

The first term of Eq. (42) coincides with the reported momentum of Ref. [30] - they assumed $x^{(1)}(0) = 0$. We further note that for $\gamma = 1$, and using the special value of $E_{2,2}$ in Eq. (10), the first term of the momentum of the Harmonic Oscillator, $-mx(0)\omega^2 t E_{2,2}(-\omega^2 t^2) = -mx(0)\omega sin(\omega t)$ is recovered.

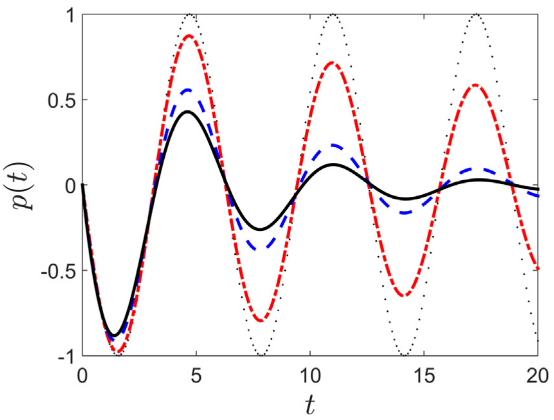

Fig. 8. (Color online) Momentum $p(t)$ as a function of time t of the fractional oscillator with no damping ($\beta = 0$) and fractional order $\gamma = 1, 0.98, 0.92, 0.89$, *... dotted line (black), -.-. dashed–dotted line (red), - - dashed line (blue), — full line (black)*, respectively. The order $\gamma = 1$ corresponds to the special case of the undamped Harmonic Oscillator. Other quantities are source term $q(t) = 0$, initial conditions $x(0) = 1,\ x^{(1)}(0) = 0$, frequency $\omega = 1$ and mass $m = 1$.

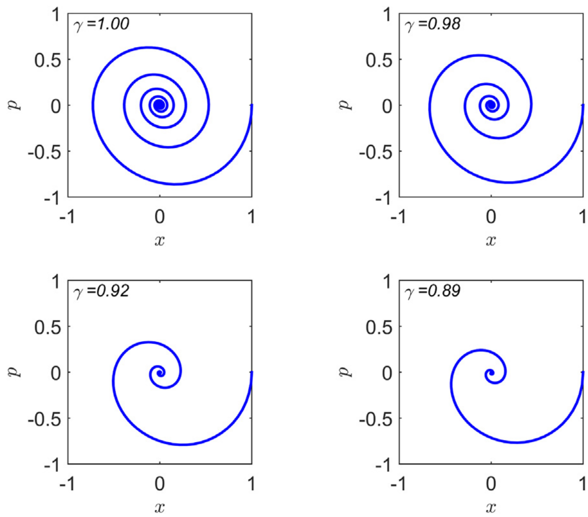

Fig. 9. (Color online) Momentum $p(t)$ as a function of position $x(t)$ of the fractional damped oscillator with damping $\beta = 0.1$ and fractional order $\gamma = 1, 0.98, 0.92, 0.89$, *top-left, top-right,bottom-left,bottom-right*, respectively. The fractional order $\gamma = 1$ corresponds to the special case of the standard damped Harmonic Oscillator. Other quantities are $q(t) = 0$, initial conditions $x(0) = 1,\ x^{(1)}(0) = 0$, frequency $\omega = 1$ and mass $m = 1$.

Fig. 8 shows the momentum $p(t)$ as a function of time t according to Eq. (41) with $q(t) = 0$, but now without damping. The intrinsic damping due to the fractional nature is very pronounced in sharp contrast to the Harmonic Oscillator. Also here we note the time-shift of peaks of the momentum in Fig. 8 as compared to peaks of the position in Fig. 2.

The correspondence between the position and momentum becomes clearer when displayed in a phase diagram (Momentum versus Position). Fig. 9 shows the momentum $p(t)$ as a function of position $x(t)$ according to Eqs. (27) and (41), respectively, with $\beta = 0.1$. For comparison the phase diagram of the standard damped Harmonic Oscillator ($\gamma = 1$) is also displayed. The appearance of spiral non-closed curves in the figures indicates that the system is open and, therefore, does not conserve energy. We note also here that the energy loss is more pronounced the smaller the values of the fractional derivative order γ.

For the sake of completeness and comparison with the reported results of Ref. [30], the correspondence between the position and momentum for the no damping case $\beta = 0$ is also produced. We see in Fig. 10 that for the case $\beta = 0, \gamma = 1$, the standard Harmonic Oscillator, the curve is closed, indicating conservation of energy. For values of $\gamma < 1$ the curves show a spiral pattern even though $\beta = 0$. The energy loss is, however, weaker than for $\beta \neq 0$ discussed above.

We define the total energy of the fractional oscillator as

$$E_{Tot} := \frac{1}{2m}[p(t)]^2 + \frac{1}{2}k[x(t)]^2, \tag{43}$$

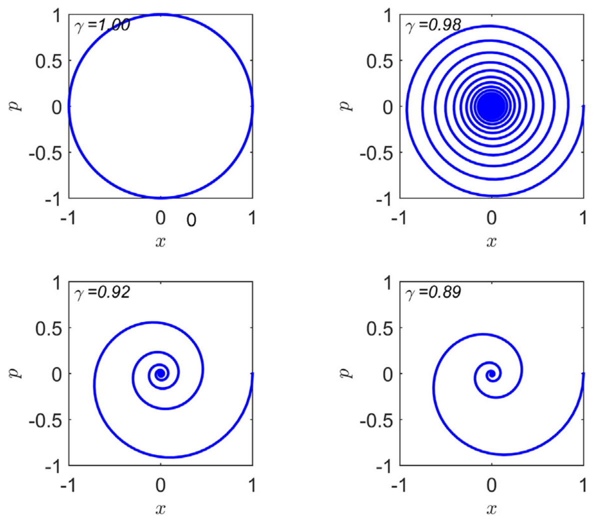

Fig. 10. (Color online) Momentum $p(t)$ as a function of position $x(t)$ of the fractional oscillator with no damping ($\beta = 0$) and fractional order $\gamma = 1, 0.98, 0.92, 0.89$, *top-left,top-right,bottom-left,bottom-right*, respectively. The fractional order $\gamma = 1$ corresponds to the special case of the standard undamped Harmonic Oscillator. Other quantities are $q(t) = 0$, initial conditions $x(0) = 1,\ x^{(1)}(0) = 0$, frequency $\omega = 1$ and mass $m = 1$.

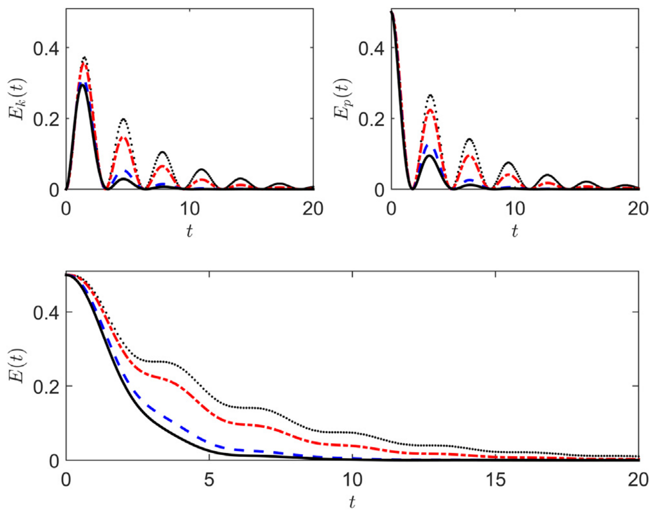

Fig. 11. (Color online) Energy as a function of time t of the fractional damped oscillator with damping $\beta = 0.1$. *Top-left* panel: Kinetic energy $E_k(t)$. *Top-right* panel: Potential energy $E_p(t)$. *Bottom* panel: Total energy $E(t)$. Results are shown for $\gamma = 1, 0.98, 0.92, 0.89$, *... dotted line (black), -.-. dashed–dotted line (red), - - dashed line (blue), — full line (black)*, respectively. The quantities used are $q(t) = 0$, initial conditions $x(0) = 1,\ x^{(1)}(0) = 0$, frequency $\omega = 1$ and mass $m = 1$.

where $k = m\omega^2$ is the stiffness coefficient.

Using Eqs. (27) and (41), the total energy in Eq. (43) can be evaluated. Fig. 11 shows the kinetic energy $E_k(t)$ (top-left panel), the potential energy $E_p(t)$ (top-right panel) and total energy $E_{Tot}(t)$ (bottom panel) as a function of time t of the fractional damped oscillator with damping $\beta = 0.1$. As the system is open, the total energy $E_{Tot}(t)$ is not conserved, i.e. is not a constant of motion. This is consistent with the picture reported in Fig. 9. As is well known, this is also true for the special case $\gamma = 1$ of the standard damped Harmonic Oscillator,

Fig. 12 depicts the same quantities, but for vanishing damping ($\beta = 0$). Also here the system is open and energy is lost as a function of time, except for the special case $\gamma = 1$ of the standard Harmonic Oscillator for which the total energy, of course, is conserved.

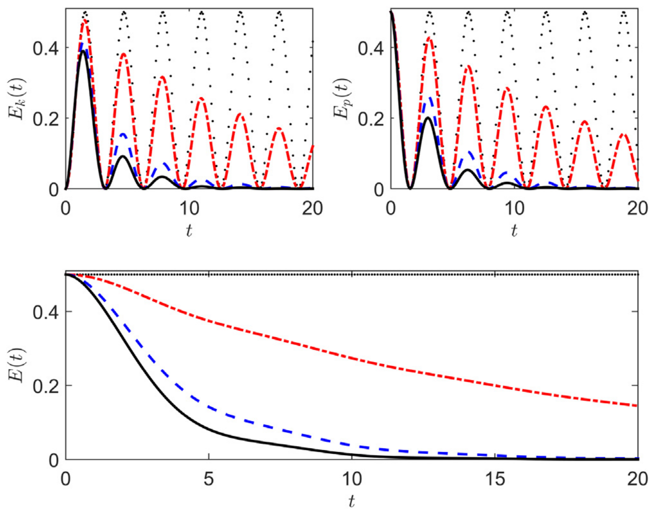

Fig. 12. (Color online) Energy as a function of time t of the fractional oscillator with no damping ($\beta = 0$). *Top-left* panel: Kinetic energy $E_k(t)$. *Top-right* panel: Potential energy $E_p(t)$. *Bottom* panel: Total energy $E(t)$. Results are shown for $\gamma = 1, 0.98, 0.92, 0.89, \ldots$ *dotted line (black), -.-. dashed–dotted line (red), - - dashed line (blue), — full line (black)*, respectively. The quantities used are $q(t) = 0$, initial conditions $x(0) = 1$, $x^{(1)}(0) = 0$, frequency $\omega = 1$ and mass $m = 1$.

5. Conclusions

The exact solution of the fractional driven-damped oscillator is derived and given in closed form in terms of generalized Mittag-Leffler functions. While the solution is a natural generalization of the standard driven-damped Harmonic Oscillator, its properties are very different from that of the latter. The differences are particularly manifested in the long-time decay pattern of the solution. While the standard oscillator solution possesses an infinite number of zeros and is exponentially decaying exhibiting symmetric oscillations around zero, the fractional oscillator solution possesses a finite number of zeros and decays to zero from either above or from below, depending on the underlying damping mechanism. In the case of explicit non-zero damping, the decay to zero is algebraic from above. When the damping parameter is zero, there is still damping present due to the fractional nature of the oscillator. In this non-explicit damping case, the decay is also algebraic but now the decay is from below. These unique properties constitute a platform for analyzing and describing phenomena that cannot be modeled by the standard driven-damped Harmonic Oscillator. Moreover, the availability of an exact closed-form solution for the fractional oscillator sets the stage for in-depth understanding of even more involved problems, for which fractional calculus can be suitable, but for which a closed-form solution is not feasible.

Appendix A. Partial-fraction decomposition

Consider the identity $R(x) = \frac{f(x)}{g(x)} \equiv \frac{f(x)}{P(x)Q(x)}$ where f and g are pseudo polynomials. There exist $C(x)$ and $D(x)$ such that

$$CP + DQ = 1;\ \frac{1}{g} = \frac{CP + DQ}{PQ} = \frac{C}{Q} + \frac{D}{P};\ \ R = \frac{f}{g} = \frac{Df}{P} + \frac{Cf}{Q}.$$

making the correspondence:

$$f = s^{2\gamma-1} + 2\beta s^{\gamma-1}, P = u - u_1, Q = u - u_2$$

one gets

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(u-u_1)(u-u_2)}[s^{2\gamma-2}x^{(1)}(0) + s^{2\gamma-1}x(0) + 2\beta s^{\gamma-1}x(0) + \widetilde{q}(s)]\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{D}{u-u_1}\left[s^{2\gamma-2}x^{(1)}(0) + s^{2\gamma-1}x(0) + 2\beta s^{\gamma-1}x(0) + \widetilde{q}(s)\right] + \frac{C}{u-u_2}\left[s^{2\gamma-2}x^{(1)}(0)\right.\right. \\ &\quad \left.\left. + s^{2\gamma-1}x(0) + 2\beta s^{\gamma-1}x(0) + \widetilde{q}(s)\right]\right\}. \end{aligned}$$

As $CP + DQ = 1$ we have

$$C(u - u_1) + D(u - u_2) = 1.$$

Therefore,

$$D = 1/(u_1 - u_2) \text{ and } C = 1/(u_2 - u_1)\,.$$

Indeed one can check that

$$\frac{1}{(u_2 - u_1)}(u - u_1) + \frac{1}{(u_1 - u_2)}(u - u_2) = \frac{(u - u_1) - (u - u_2)}{(u_2 - u_1)} = \frac{(u_2 - u_1)}{(u_2 - u_1)} = 1.$$

Therefore, we have

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left[\frac{s^{2\gamma-2}x^{(1)}(0) + s^{2\gamma-1}x(0) + 2\beta s^{\gamma-1}x(0) + \widetilde{q}(s)}{(u - u_1)(u - u_2)}\right] \\ &= \mathcal{L}^{-1}\left[\frac{s^{2\gamma-2}x^{(1)}(0) + s^{2\gamma-1}x(0) + 2\beta s^{\gamma-1}x(0) + \widetilde{q}(s)}{(u - u_1)(u_1 - u_2)} \right. \\ &\quad \left. - \frac{s^{2\gamma-2}x^{(1)}(0) + s^{2\gamma-1}x(0) + 2\beta s^{\gamma-1}x(0) + \widetilde{q}(s)}{(u - u_2)(u_1 - u_2)}\right]. \end{aligned}$$

So finally, the above identities leads to

$$\begin{aligned} x(t) = \frac{1}{(u_1 - u_2)}\mathcal{L}^{-1}\Bigg\{&\frac{x(0)\,s^{2\gamma-1}}{(s^\gamma - u_1)} + \frac{x(0)\,2\beta s^{\gamma-1}}{(s^\gamma - u_1)} + \frac{x^{(1)}(0)\,s^{2\gamma-2}}{(s^\gamma - u_1)} + \frac{\widetilde{q}(s)}{(s^\gamma - u_1)} - \frac{s^{2\gamma-1}}{(s^\gamma - u_2)} \\ &- \frac{2\beta s^{\gamma-1}}{(s^\gamma - u_2)} - \frac{x^{(1)}(0)\,s^{2\gamma-2}}{(s^\gamma - u_2)} - \frac{\widetilde{q}(s)}{(s^\gamma - u_2)}\Bigg\}. \end{aligned} \tag{A.1}$$

Appendix B. Simplification of the main result

Eq. (25) reads

$$\begin{aligned} x(t) &= \frac{x(0)A + x^{(1)}(0)B + C}{(u_1 - u_2)}, \quad \text{where} \\ A &= t^{-\gamma}E_{\gamma,1-\gamma}(u_1 t^\gamma) + 2\beta E_{\gamma,1}(u_1 t^\gamma) - t^{-\gamma}E_{\gamma,1-\gamma}(u_2 t^\gamma) - 2\beta E_{\gamma,1}(u_2 t^\gamma) \\ B &= t^{-\gamma+1}E_{\gamma,-\gamma+2}(u_1 t^\gamma) - t^{-\gamma+1}E_{\gamma,-\gamma+2}(u_2 t^\gamma) \\ C &= \left[t^{\gamma-1}E_{\gamma,\gamma}(u_1 t^\gamma)\right] * q(t) - \left[t^{\gamma-1}E_{\gamma,\gamma}(u_2 t^\gamma)\right] * q(t). \end{aligned} \tag{B.1}$$

With the recursion $E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\beta+\alpha}(z)$, see Eq. (12), we can replace the following *Mittag-Leffler* functions with others in a useful manner:

$$t^{-\gamma}E_{\gamma,1-\gamma}(u_1 t^\gamma) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)} + t^{-\gamma}u_1 t^\gamma E_{\gamma,1}(u_1 t^\gamma) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)} + u_1 E_{\gamma,1}(u_1 t^\gamma),$$

$$t^{-\gamma}E_{\gamma,1-\gamma}(u_2 t^\gamma) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)} + t^{-\gamma}u_2 t^\gamma E_{\gamma,1}(u_2 t^\gamma) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)} + u_2 E_{\gamma,1}(u_2 t^\gamma),$$

$$t^{-\gamma+1}E_{\gamma,-\gamma+2}(u_1 t^\gamma) = \frac{t^{-\gamma+1}}{\Gamma(2-\gamma)} + t^{-\gamma+1}u_1 t^\gamma E_{\gamma,2}(u_1 t^\gamma) = \frac{t^{-\gamma+1}}{\Gamma(2-\gamma)} + tu_1 E_{\gamma,2}(u_1 t^\gamma),$$

$$t^{-\gamma+1}E_{\gamma,-\gamma+2}(u_2 t^\gamma) = \frac{t^{-\gamma+1}}{\Gamma(2-\gamma)} + t^{-\gamma+1}u_2 t^\gamma E_{\gamma,2}(u_2 t^\gamma) = \frac{t^{-\gamma+1}}{\Gamma(2-\gamma)} + tu_2 E_{\gamma,2}(u_2 t^\gamma).$$

Substituting these in (B.1) and collecting terms one obtains

$$\begin{aligned} x(t) = \frac{1}{(u_1 - u_2)}\Big\{&x(0)\,u_1 E_{\gamma,1}(u_1 t^\gamma) + x(0)\,2\beta E_{\gamma,1}(u_1 t^\gamma) + x^{(1)}(0)\,tu_1 E_{\gamma,2}(u_1 t^\gamma) \\ &+ \left[t^{\gamma-1}E_{\gamma,\gamma}(u_1 t^\gamma)\right] * q(t) - x(0)\,u_2 E_{\gamma,1}(u_2 t^\gamma) - x(0)\,2\beta E_{\gamma,1}(u_2 t^\gamma) \\ &- x^{(1)}(0)\,tu_2 E_{\gamma,2}(u_2 t^\gamma) - \left[t^{\gamma-1}E_{\gamma,\gamma}(u_2 t^\gamma)\right] * q(t)\Big\}. \end{aligned}$$

Using $u_1 = (-\beta + i\omega_d)\,;\;\; u_2 = (-\beta - i\omega_d)$ this reads

$$\begin{aligned}
x(t) &= \frac{x(0)\,A + x^{(1)}(0)\,B + C}{2i\omega_d} \\
A &= (\beta + i\omega_d)\,E_{\gamma,1}\left[(-\beta + i\omega_d)\,t^{\gamma}\right] - (\beta - i\omega_d)\,E_{\gamma,1}\left[(-\beta - i\omega_d)\,t^{\gamma}\right] \\
B &= t\,(-\beta + i\omega_d)\,E_{\gamma,2}\left[(-\beta + i\omega_d)\,t^{\gamma}\right] - t\,(-\beta - i\omega_d)\,E_{\gamma,2}\left[(-\beta - i\omega_d)\,t^{\gamma}\right] \\
C &= \left\{t^{\gamma-1}E_{\gamma,\gamma}\left[(-\beta + i\omega_d)\,t^{\gamma}\right]\right\} * q(t) - \left\{t^{\gamma-1}E_{\gamma,\gamma}\left[(-\beta - i\omega_d)\,t^{\gamma}\right]\right\} * q(t)\,.
\end{aligned} \tag{B.2}$$

which is the compact form of the main result of this manuscript.

Appendix C. Consistency checking of the main solution

C.1. Comparison to numerical computation of the inverse Laplace transform — position

It is instructive to compare the exact closed form results to a numerical computation of the inverse Laplace transform of Eq. (20). This comparison is made in Fig. C.1, in which the solid line represents the exact closed-form solution, Eq. (B.2), and the stars represent the numerical computation of the inverse Laplace transform using the Euler method [35]. In other words, the numerical results are obtained with no reference to the *Mittag-Leffler* functions appearing in the exact solution, nor to the algebraic steps we were taking in order to reach the closed-form solution. The numerical computations agree with the closed-form ones!

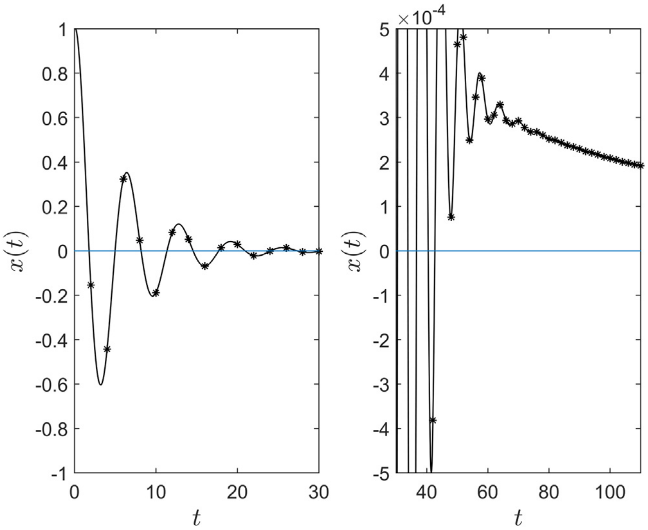

Fig. C.1. (Color online) Position $x(t)$ as a function of time t of the fractional damped oscillator with damping $\beta = 0.1$ and fractional order $\gamma = 0.96$. The solid line displays Eq. (B.2) with $q(t) = 0$. The stars (*) display the numerical computation of the inverse Laplace transform of Eq. (20) with $\tilde{q}(s) = 0$. The *left* panel displays the short-time and the *right* panel the long-time tail. The initial conditions are $x(0) = 1$, $x^{(1)}(0) = 0.1$ and the frequency $\omega = 1$.

C.2. Comparison to numerical computation of the inverse Laplace transform — momentum

It is also instructive to compare the exact closed form expression for the momentum, Eq. (41), to a numerical computation of the inverse Laplace transform of Eq. (37). This comparison is made in Fig. C.2, in which the solid line represents the exact closed-form solution, and the stars represent the numerical computation of the inverse Laplace transform using the Euler method [35]. Also here, the numerical results are obtained with no reference to the *Mittag-Leffler* functions appearing in the exact solution, nor to the algebraic steps we were taking in order to reach the closed-form solution. The numerical computations agree with the closed-form ones!

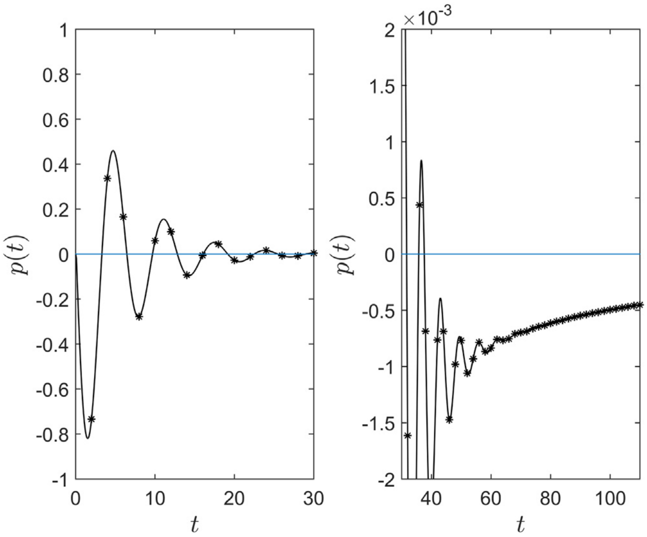

Fig. C.2. (Color online) Momentum $p(t)$ as a function of time t of the fractional damped oscillator with damping $\beta = 0.1$ and fractional order $\gamma = 0.96$. The solid line displays Eq. (41), with $q(t) = 0$. The stars (*) display the numerical computation of the inverse Laplace transform of Eq. (37) with $\tilde{q}(s) = 0$. The *left* panel displays the short-time and the *right* panel the long-time tail. The initial conditions are $x(0) = 1$, $x^{(1)}(0) = 0.1$ and the frequency $\omega = 1$.

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