On the Calculation of Moments of Polygons

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Abstract—This paper describes a method to calculate various shape features, namely the area, center of gravity, and second order moments, from polygons that enclose a region of the two-dimensional space. Contrary to most approaches, which compute these features from discrete pixel data, here they are calculated by solely using the points of the enclosing polygon, which may be sub-pixel accurate. This has two main advantages. Firstly, since the precise points of the boundary of the shape are used the resulting features will be computed with maximum accuracy. Furthermore, since the boundary typically consists of much less points than the whole region the procedure is computationally more efficient.

1 Introduction

A problem that frequently occurs in image processing tasks is to calculate the area, centroid (center of gravity), and (centralized) second order moments of a region in an image. These are important shape features in itself. For example, they can be used to fit elliptic curve segments to extracted contours [8]. Often they are also used to compute the dominant directions and approximate diameters of a region. These in turn are important for camera calibration tasks, for example, where they can be used to determine approximate values for some parameters of the exterior orientation, and thus to obtain good starting values for the calibration procedure [6].

The area of an arbitrary region R is given by

$$a = \iint_R 1 \, dx \, dy \ , \tag{1}$$

its centroid by

$$\alpha_x = \frac{1}{a} \iint_R x \, dx \, dy \tag{2}$$

$$\alpha_y = \frac{1}{a} \iint_R y \, dx \, dy \ , \tag{3}$$

and its second order moments by

$$\alpha_{xx} = \frac{1}{a} \iint_R x^2 \, dx \, dy \tag{4}$$

$$\alpha_{xy} = \frac{1}{a} \iint_R xy \, dx \, dy \tag{5}$$

$$\alpha_{yy} = \frac{1}{a} \iint_R y^2 \, dx \, dy \ . \tag{6}$$

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If the first order moments α_x and α_y are known, the centralized second order moments are given by

$$\mu_{xx} = \frac{1}{a} \iint\limits_R (x - \alpha_x)^2 \, dx \, dy \qquad (7)$$

$$\mu_{xy} = \frac{1}{a} \iint\limits_R (x - \alpha_x)(y - \alpha_y) \, dx \, dy \qquad (8)$$

$$\mu_{yy} = \frac{1}{a} \iint\limits_R (y - \alpha_y)^2 \, dx \, dy \ . \qquad (9)$$

It is unnecessary to compute these explicitly, however, since they can easily be obtained from the normal second order moments by

$$\mu_{xx} = \alpha_{xx} - \alpha_x^2 \qquad (10)$$

$$\mu_{xy} = \alpha_{xy} - \alpha_x \alpha_y \qquad (11)$$

$$\mu_{yy} = \alpha_{yy} - \alpha_y^2 \ . \qquad (12)$$

Usually, the region R will be discrete, i.e., consist of a set of pixels, each of which has an area of 1. The integrals in the equations above can then simply be calculated by summation over the region R. However, if regions are extracted by a sub-pixel precise feature extraction algorithm, e.g., by the line detector given in [7], only the closed boundary b of the region is known, usually as a sub-pixel precise contour, which can be regarded as a polygon. Therefore, the equations above cannot be applied to calculate the moments. There are two obvious solutions to this problem. The first one is to discretize the region R to the pixel raster, which is undesirable since the sub-pixel accuracy is lost. The second one is to triangulate the polygon, and to calculate the moments by computing them for each triangle, which can be done easily, and to add up the results. However, triangulation is a costly operation, and therefore a scheme that only uses the points on the polygon p to compute the moments is highly desirable.

2   Mathematical Tools

A way to solve the problem stated above is to apply the powerful result of Green's theorem, sometimes also referred to as the Green formula, the Gauss formula, or the Stokes formula. This theorem lets us compute the integral of a function over a sub-domain R of the two-dimensional space by reducing it to a curve integral over the border b of R [1, Section 3.1.13.1]. More formally, it can be stated as follows: Let $P(x, y)$ and $Q(x, y)$ be two continuously differentiable functions on the two-dimensional region R, and let $b(t)$ be the boundary of R. If b is piecewise differentiable and oriented such that it is traversed in positive direction (counterclockwise), an integral over the region R can be reduced to a curve integral over the boundary b of R in the following manner:

$$\iint\limits_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx \, dy = \int\limits_b P \, dx + Q \, dy \ . \qquad (13)$$

It is obvious that to compute an integral of an arbitrary function $F(x, y)$ over R, as is the case for the moments, $F(x, y)$ will somehow have to be decomposed into $\partial P/\partial y$ and $\partial Q/\partial x$. It is also

obvious that this decomposition cannot be unique. Therefore, the choice of the decomposition is essentially arbitrary.

The integral on the right side of (13) is a general curve integral. It is defined as follows [1, Section 3.1.8.4]: Let $b(t) = (x(t), y(t))$, $t \in [t_1, t_2]$ be a curve, and let $f(x, y)$ be a continuous function. Then the following two curve integrals exist and can be transformed to definite integrals:

$$\int_b f(x, y)\, dx = \int_{t_1}^{t_2} f(x(t), y(t)) x'(t)\, dt \tag{14}$$

$$\int_b f(x, y)\, dy = \int_{t_1}^{t_2} f(x(t), y(t)) y'(t)\, dt \ . \tag{15}$$

Furthermore, let $P(x, y)$ and $Q(x, y)$ be two continous functions. The general curve integral is then given by [1, Section 3.1.8.3]:

$$\int_b P(x, y)\, dx + Q(x, y)\, dy = \int_b P(x, y)\, dx + \int_b Q(x, y)\, dy \ . \tag{16}$$

Curve integrals have some important properties, which will be useful in later sections: Let $b_1(t)$, $t \in [t_1, t_2]$ and $b_2(t)$, $t \in [t_2, t_3]$ be two curves with $b_1(t_2) = b_2(t_2)$, and let $b = b_1 \cup b_2$. Then the curve integral over b can be calculated as follows:

$$\int_b f(x, y)\, dx = \int_{b_1} f(x, y)\, dx + \int_{b_2} f(x, y)\, dy \ . \tag{17}$$

Furthermore, if the direction of the curve is reversed, i.e., if b' is the reverse of b, the sign of the integral changes:

$$\int_b f(x, y)\, dx = - \int_{b'} f(x, y)\, dx \ . \tag{18}$$

3   Application to Closed Polygons

We now have all the tools at hand to compute the moments of R by just using the points on its border b. Before we can apply these tools, however, we need to consider how we can parameterize $b(t)$.

A closed polygon p with points $p_i = (x_i, y_i)$, $i \in \{0, \ldots, n\}$, and $p_0 = p_n$ bounding a region R in the two-dimensional plane can be regarded as a piecewise linear curve b, which in turn can be regarded as the union of n line segments

$$b(t) = \bigcup_{i=1}^{n} b_i(t) \ , \tag{19}$$

where $b_i(t)$, $t \in [0, 1]$ is given by

$$b_i(t) = p_{i-1} + t(p_i - p_{i-1}) \ . \tag{20}$$

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Hence it follows that the coordinate functions and their derivatives needed to calculate the curve integral are given by

$$
\begin{aligned}
x_i(t) &= x_{i-1} + t(x_i - x_{i-1}) && (21)\\
y_i(t) &= y_{i-1} + t(y_i - y_{i-1}) && (22)\\
x_i'(t) &= x_i - x_{i-1} && (23)\\
y_i'(t) &= y_i - y_{i-1} \;. && (24)
\end{aligned}
$$

Therefore, any general curve integral along a polygon $p = b(t)$ can be calculated in the following manner:

$$
\int_b P\,dx + Q\,dy = \sum_{i=1}^{n} \int_{b_i} P\,dx + Q\,dy \;. \tag{25}
$$

4   Calculation of Moments

The area of a region is given by (1). In order to apply (13) we need to decompose 1 into $\partial Q/\partial x$ and $\partial P/\partial y$. Purely for reasons of symmetry we choose

$$
\frac{\partial Q}{\partial x} = \frac{1}{2} \quad \text{and} \quad \frac{\partial P}{\partial y} = -\frac{1}{2} \;. \tag{26}
$$

Hence $P(x,y) = -y/2$ and $Q(x,y) = x/2$. Therefore, the area a of a region R can be calculated as follows:

$$
\begin{aligned}
a &= \iint_R 1\,dx\,dy = \iint_R \frac{1}{2} - \left(-\frac{1}{2}\right) dx\,dy\\
&= \int_b -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy = \frac{1}{2}\int_b x\,dy - y\,dx\\
&= \frac{1}{2}\int_b x\,dy - \frac{1}{2}\int_b y\,dx \;. \tag{27}
\end{aligned}
$$

By (25), each of the two integrals in (27) can be calculated as the sum over the curve integrals along the line segments of the polygon. For the first integral each term of the sum is given by:

$$
\begin{aligned}
\int_{b_i} x\,dy &= \int_0^1 x_i(t)y_i'(t)\,dt\\
&= \int_0^1 (x_{i-1} + t(x_i - x_{i-1}))(y_i - y_{i-1})\,dt\\
&= (y_i - y_{i-1})\left(tx_{i-1} + \frac{1}{2}t^2(x_i - x_{i-1})\Big|_{t=0}^{1}\right)\\
&= (y_i - y_{i-1})\left(x_{i-1} + \frac{1}{2}(x_i - x_{i-1})\right)\\
&= \frac{1}{2}(x_{i-1} + x_i)(y_i - y_{i-1}) \;. \tag{28}
\end{aligned}
$$

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By a similar calculation, each term of the second integral is given by:

$$\int_{b_i} y\,dx = \int_0^1 y_i(t)x_i'(t)\,dt$$

$$= \int_0^1 (y_{i-1} + t(y_i - y_{i-1}))(x_i - x_{i-1})\,dt$$

$$= \cdots$$

$$= \frac{1}{2}(y_{i-1} + y_i)(x_i - x_{i-1}) \ . \tag{29}$$

Hence, each term of the sum is given by:

$$\int_{b_i} x\,dy - y\,dx = \frac{1}{2}((x_{i-1} + x_i)(y_i - y_{i-1}) - (y_{i-1} + y_i)(x_i - x_{i-1}))$$

$$= \frac{1}{2}(x_{i-1}y_i - x_{i-1}y_{i-1} + x_iy_i - x_iy_{i-1}$$

$$- x_iy_{i-1} + x_{i-1}y_{i-1} - x_iy_i + x_{i-1}y_i)$$

$$= x_{i-1}y_i - x_iy_{i-1} \ . \tag{30}$$

Therefore, by (25) and (30), the area of a polygon can be calculated as

$$a = \frac{1}{2}\sum_{i=1}^n x_{i-1}y_i - x_iy_{i-1} \ . \tag{31}$$

It should be noted that this equation only holds if the polygon p encloses the region R counterclockwise. However, from (18) and from the antisymmetry of (31) it is obvious that only the sign of the area changes if p encloses R clockwise. Hence we have a simple criterion to decide whether the result of (31) is valid, namely the sign of a. If a is negative every calculated moment needs to be multiplied by -1.

We now turn to the calculation of the first moments, or the centroid of the region R. Because the calculations become more lengthy for these features, the exact derivation is given in Appendix A. We will only present the final results here.

According to (2), for the first moment α_x there seems to be a preferred direction. Therefore, we decompose x into $\partial Q/\partial x$ and $\partial P/\partial y$ as follows:

$$\frac{\partial Q}{\partial x} = 0 \quad \text{and} \quad \frac{\partial P}{\partial y} = -x \ . \tag{32}$$

Hence $P(x, y) = -xy$ and $Q(x, y) = 0$. Therefore,

$$\alpha_x = \frac{1}{a}\iint_R x\,dx\,dy = \frac{1}{a}\int_b -xy\,dx$$

$$= \frac{1}{6a}\sum_{i=1}^n (x_{i-1} + x_i)(x_{i-1}y_i - x_iy_{i-1}) \ . \tag{33}$$

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Analogously, for α_y we choose $P(x, y) = 0$ and $Q(x, y) = xy$, and obtain:

$$\alpha_y = \frac{1}{a} \iint_R y \, dx \, dy = \frac{1}{a} \int_b xy \, dy$$

$$= \frac{1}{6a} \sum_{i=1}^{n} (y_{i-1} + y_i)(x_{i-1}y_i - x_iy_{i-1}) \quad . \tag{34}$$

The final features to calculate are the second moments. Again, only the final results are given here. The detailed calculations can be found in Appexdix B. Let us first consider α_{xx}. Analogously to (32), we choose the following decomposition of x^2:

$$\frac{\partial Q}{\partial x} = 0 \quad \text{and} \quad \frac{\partial P}{\partial y} = -x^2 \quad . \tag{35}$$

Hence, $P(x, y) = -x^2y$ and $Q(x, y) = 0$. Therefore,

$$\alpha_{xx} = \frac{1}{a} \iint_R x^2 \, dx \, dy = \frac{1}{a} \int_b -x^2 y \, dx$$

$$= \frac{1}{12a} \sum_{i=1}^{n} (x_{i-1}^2 + x_{i-1}x_i + x_i^2)(x_{i-1}y_i - x_iy_{i-1}) \quad . \tag{36}$$

By the same line of reasoning we choose $P(x, y) = 0$ and $Q(x, y) = xy^2$ for α_{yy}, and obtain:

$$\alpha_{yy} = \frac{1}{a} \iint_R y^2 \, dx \, dy = \frac{1}{a} \int_b xy^2 \, dy$$

$$= \frac{1}{12a} \sum_{i=1}^{n} (y_{i-1}^2 + y_{i-1}y_i + y_i^2)(x_{i-1}y_i - x_iy_{i-1}) \quad . \tag{37}$$

For the mixed moment α_{xy} there is no preferred direction, similarly to (26). Therefore, we choose

$$\frac{\partial Q}{\partial x} = \frac{1}{2}xy \quad \text{and} \quad \frac{\partial P}{\partial y} = -\frac{1}{2}xy \tag{38}$$

as the decomposition of xy, and have $P(x, y) = -xy^2/4$ and $Q(x, y) = x^2y/4$. Therefore, the mixed second order moment is given by

$$\alpha_{xy} = \frac{1}{a} \iint_R xy \, dx \, dy = -\frac{1}{4a} \int_b xy^2 \, dx + \frac{1}{4a} \int_b x^2 y \, dy$$

$$= \frac{1}{24a} \sum_{i=1}^{n} (2x_{i-1}y_{i-1} + x_{i-1}y_i + x_iy_{i-1} + 2x_iy_i)(x_{i-1}y_i - x_iy_{i-1}) \quad . \tag{39}$$

5 Examples

Let us now demonstrate that the moments calculated by only considering the points p_i of the enclosing polygon yield correct results. Consider the rectangle given by $p_0 = (2, 0)$, $p_1 =$
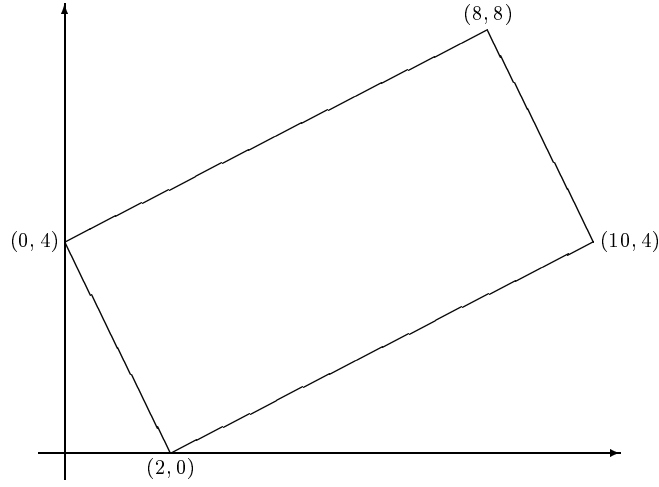
Figure 1: The rectangle used in the example

$(10, 4)$, $p_2 = (8, 8)$, and $p_3 = (0, 4)$. In order to be able to compute the moments by the formulas drived above, we need to introduce an additional point $p_4 = p_0$ to close the polygon. Figure 1 displays this rectangle.

From the geometry of this rectangle it is obvious that $a = 40$, and that the centroid is $(\alpha_x, \alpha_y) = (5, 4)$. Hence, we will not derive these values by integration. The second moment α_{xx} can be calculated as follows:

$$
\begin{aligned}
\alpha_{xx} &= \frac{1}{a} \iint_R x^2 \, dx \, dy \\
&= \frac{1}{a} \left(\int_0^2 \int_{4-2x}^{x/2+4} x^2 \, dx \, dy + \int_2^8 \int_{x/2-1}^{x/2+4} x^2 \, dx \, dy + \int_8^{10} \int_{x/2-1}^{24-2x} x^2 \, dx \, dy \right) \\
&= \frac{1}{40} \left(\int_0^2 \left(x^2 y \Big|_{y=4-2x}^{x/2+4} \right) dx + \int_2^8 \left(x^2 y \Big|_{y=x/2-1}^{x/2+4} \right) dx + \int_8^{10} \left(x^2 y \Big|_{y=x/2-1}^{24-2x} \right) dx \right) \\
&= \frac{1}{40} \left(\int_0^2 x^2 (\tfrac{1}{2}x + 4) - x^2(4 - 2x) \, dx + \int_2^8 x^2(\tfrac{1}{2}x + 4) - x^2(\tfrac{1}{2}x - 1) \, dx \right. \\
&\qquad \left. + \int_8^{10} x^2(24 - 2x) - x^2(\tfrac{1}{2}x - 1) \, dx \right) \\
&= \frac{1}{40} \left(\left(\tfrac{5}{8} x^4 \Big|_0^2 \right) + \left(\tfrac{5}{3} x^3 \Big|_2^8 \right) + \left(-\tfrac{5}{8} x^4 + \tfrac{25}{3} x^3 \Big|_8^{10} \right) \right) \\
&= 30\frac{2}{3} \ .
\end{aligned}
\tag{40}
$$

Similarly, we obtain

$$
\alpha_{xy} = 22 \tag{41}
$$

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$$\alpha_{yy} = 18\frac{2}{3} . \tag{42}$$

We now calculate a by using (31):

$$a = \frac{1}{2}(2 \cdot 4 - 10 \cdot 0 + 10 \cdot 8 - 8 \cdot 4 + 8 \cdot 4 - 0 \cdot 8 + 0 \cdot 0 - 2 \cdot 4) = 40 . \tag{43}$$

Therefore, the area computed by using (31) yields the correct result with much less computational effort. Furthermore, we can see that the rectangle is indeed oriented counterclockwise since $a > 0$.

For α_x and α_y, by (33) and (34) we have

$$\begin{aligned}
\alpha_x &= \frac{1}{6a}((2+10) \cdot (2 \cdot 4 - 10 \cdot 0) + (10+8) \cdot (10 \cdot 8 - 8 \cdot 4) + \\
&\quad (8+0) \cdot (8 \cdot 4 - 0 \cdot 8) + (0+2) \cdot (0 \cdot 0 - 2 \cdot 4)) \\
&= 5 \tag{44} \\
\alpha_y &= \cdots \\
&= 4 . \tag{45}
\end{aligned}$$

Again, (33) and (34) yield the correct results with much less computational burden.

To calculate α_{xx} and α_{yy} we use (36) and (37):

$$\begin{aligned}
\alpha_{xx} &= \frac{1}{12a}((2^2 + 2 \cdot 10 + 10^2) \cdot (2 \cdot 4 - 10 \cdot 0) + (10^2 + 10 \cdot 8 + 8^2) \cdot (10 \cdot 8 - 8 \cdot 4) + \\
&\quad (8^2 + 8 \cdot 0 + 0^2) \cdot (8 \cdot 4 - 0 \cdot 8) + (0^2 + 0 \cdot 2 + 2^2) \cdot (0 \cdot 0 - 2 \cdot 4)) \\
&= 30\frac{2}{3} \tag{46} \\
\alpha_{yy} &= \cdots \\
&= 18\frac{2}{3} . \tag{47}
\end{aligned}$$

For α_{xy} we obtain by (39):

$$\begin{aligned}
\alpha_{xy} &= \frac{1}{24a}((2 \cdot 2 \cdot 0 + 2 \cdot 4 + 10 \cdot 0 + 2 \cdot 10 \cdot 4) \cdot (2 \cdot 4 - 10 \cdot 0) + \\
&\quad (2 \cdot 10 \cdot 4 + 10 \cdot 8 + 8 \cdot 4 + 2 \cdot 8 \cdot 8) \cdot (10 \cdot 8 - 8 \cdot 4) + \\
&\quad (2 \cdot 8 \cdot 8 + 8 \cdot 4 + 0 \cdot 8 + 2 \cdot 0 \cdot 4) \cdot (8 \cdot 4 - 0 \cdot 8) + \\
&\quad (2 \cdot 0 \cdot 4 + 0 \cdot 0 + 2 \cdot 4 + 2 \cdot 2 \cdot 0) \cdot (0 \cdot 0 - 2 \cdot 4)) \\
&= 22 . \tag{48}
\end{aligned}$$

Again, for the second order moments (36), (37), and (39) yield the correct results. The centralized second order moments are given by:

$$\begin{aligned}
\mu_{xx} &= \alpha_{xx} - \alpha_x^2 = 30\frac{2}{3} - 25 = \frac{17}{3} \tag{49} \\
\mu_{xy} &= \alpha_{xy} - \alpha_x\alpha_y = 22 - 20 = 2 \tag{50} \\
\mu_{yy} &= \alpha_{yy} - \alpha_y^2 = 18\frac{2}{3} - 16 = \frac{8}{3} . \tag{51}
\end{aligned}$$

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From these, according to [2, Appendix A] we can compute the parameters of an ellipse with the same second order moments by calculating the eigenvalues and eigenvectors of the following matrix:

$$\frac{1}{4(\mu_{xx}\mu_{yy} - \mu_{xy}^2)} \begin{pmatrix} \mu_{yy} & -\mu_{xy} \\ -\mu_{xy} & \mu_{xx} \end{pmatrix} = \frac{3}{400} \begin{pmatrix} 8 & -6 \\ -6 & 17 \end{pmatrix} \ . \tag{52}$$

The eigenvalues are given by the solutions of

$$\begin{vmatrix} 8 - \lambda & -6 \\ -6 & 17 - \lambda \end{vmatrix} = (8 - \lambda)(17 - \lambda) - 36 = \lambda^2 - 25\lambda + 100 \ . \tag{53}$$

Hence, $\lambda_1 = 5$ and $\lambda_2 = 20$. Therefore, the corresponding major axes of the ellipse have the following lengths: $a = 2/(\sqrt{5 \cdot 3/400}) = 8\sqrt{5/3}$ and $b = 2/(\sqrt{20 \cdot 3/400}) = 4\sqrt{5/3}$. The directions of the major axes a and b are given by $(2, 1)$ and $(1, -2)$, respectively, as is easily obtainable by calculating the corresponding eigenvectors. Thus, the dominant directions of this region were obtained correctly. Obviously, this can be done much easier for a rectangle. However, the approach is valid for arbitrary shapes, as the next example shows.

Figure 2(a) displays a calibration target and Fig. 2(b) the upper-rightmost calibration mark [6]. From this mark edges were extracted with sub-pixel precision by extracting bright lines in the gradient image [7]. Figure 3(a) displays the resulting edges. In this example, 103 edge points were found, leading to a closed polygon with 102 line segments. From these edge points, the moments of the extracted shape can be calculated as

$$
\begin{aligned}
a &= 566.32474 \\
\alpha_r &= 26.40761 \\
\alpha_c &= 28.17205 \\
\alpha_{rr} &= 770.99165 \\
\alpha_{rc} &= 729.00953 \\
\alpha_{cc} &= 824.30414 \\
\mu_{rr} &= 73.62978 \\
\mu_{rc} &= -14.94700 \\
\mu_{cc} &= 30.63971 \ ,
\end{aligned}
$$

where r and c denote the row and column axis, respectively. According to [2, Appendix A], the corresponding ellipse with the same moments has a major axis of length $a = 35.39849$, a minor axis of $b = 20.37789$, and the angle of the major axis to the column axis is given by $\varphi = 72.59321°$. Figure 3(b) displays this ellipse superimposed onto the extracted edge points. As can be seen, the difference is hardly noticeable.

6 Conclusions

This paper has presented an explicit method to calculate the moments of arbitrary closed polygons. Contrary to most implementations, which obtain the moments from discrete pixel data,
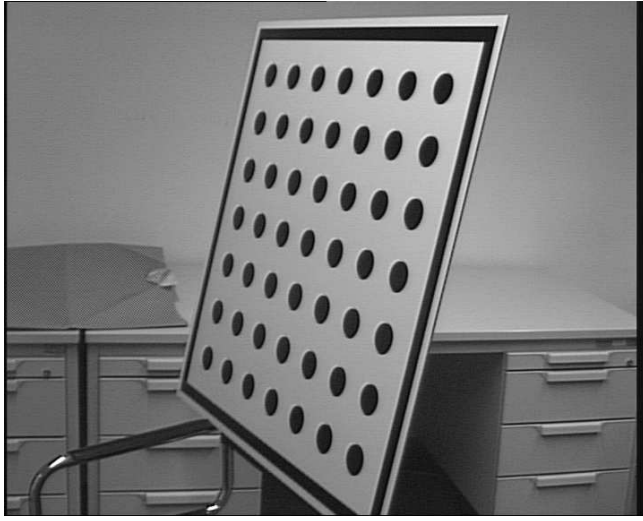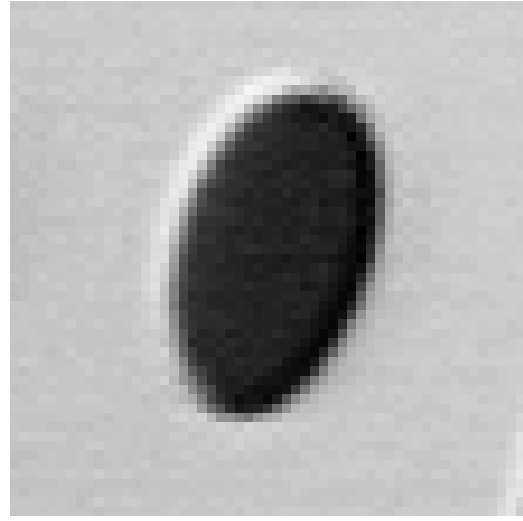
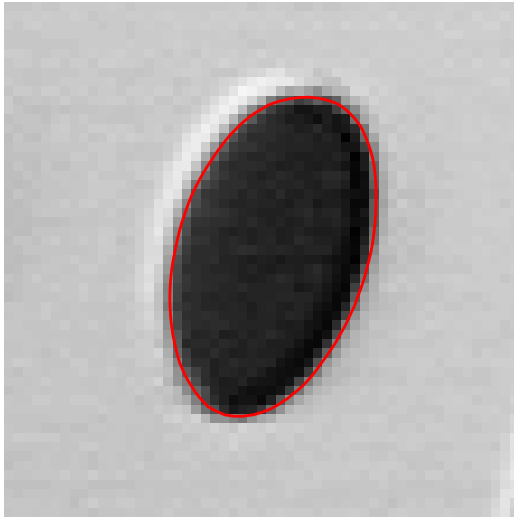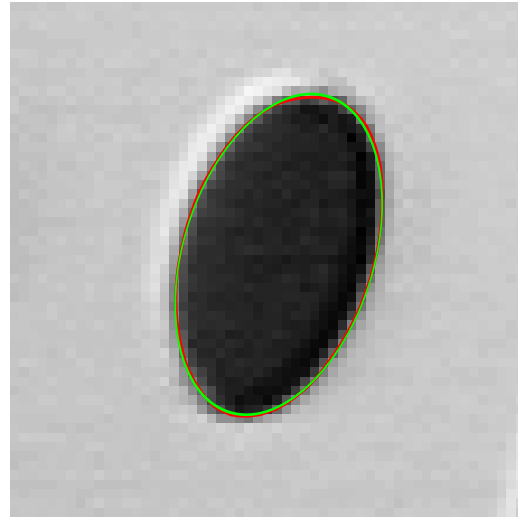(a)   (b)

Figure 2: A calibration target (a) and one calibration mark (b).

(a)   (b)

Figure 3: Extracted edges (a) and ellipse with the same moments (b).

in this approach moments are calculated by solely using the border of a region. This means that moments of sub-pixel precise features can be computed without loss of accuracy. Furthermore, since no explicit region needs to be constructed and because the border of a region usually consists of much less points than the entire region, the approach is very efficient. The presented algorithm will be used in the vision system described in [3, 4, 5] to approximate edges by straight lines and ellipse segments using an approach similar to [8].

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A   Calculation of the First Order Moments

The integral for α_x in (33) can be decomposed into a sum by (25). Each term is given by:

$$
\begin{aligned}
\int_{b_i} -xy \, dx &= -\int_0^1 x_i(t) y_i(t) x_i'(t) \, dt \\[2ex]
&= -\int_0^1 (x_{i-1} + t(x_i - x_{i-1}))(y_{i-1} + t(y_i - y_{i-1}))(x_i - x_{i-1}) \, dt \\[2ex]
&= -(x_i - x_{i-1}) \int_0^1 x_{i-1} y_{i-1} + t(x_{i-1}(y_i - y_{i-1}) + y_{i-1}(x_i - x_{i-1})) \\
&\qquad + t^2 (x_i - x_{i-1})(y_i - y_{i-1}) \, dt \\[2ex]
&= -(x_i - x_{i-1}) \Big(t x_{i-1} y_{i-1} + \frac{1}{2} t^2 (x_i y_{i-1} - x_{i-1} y_{i-1} + x_{i-1} y_i - x_{i-1} y_{i-1}) \\
&\qquad + \frac{1}{3} t^3 (x_{i-1} y_{i-1} - x_{i-1} y_i - x_i y_{i-1} + x_i y_i) \Big|_{t=0}^{1} \Big) \\[2ex]
&= -(x_i - x_{i-1}) \Big(x_{i-1} y_{i-1} - x_{i-1} y_{i-1} + \frac{1}{2} x_{i-1} y_i + \frac{1}{2} x_i y_{i-1} \\
&\qquad + \frac{1}{3} x_{i-1} y_{i-1} - \frac{1}{3} x_{i-1} y_i - \frac{1}{3} x_i y_{i-1} + \frac{1}{3} x_i y_i \Big) \\[2ex]
&= -(x_i - x_{i-1}) \Big(\frac{1}{3} x_{i-1} y_{i-1} + \frac{1}{6} x_{i-1} y_i + \frac{1}{6} x_i y_{i-1} + \frac{1}{3} x_i y_i \Big) \\[2ex]
&= -\frac{1}{6} (x_i - x_{i-1})(x_{i-1}(2 y_{i-1} + y_i) + x_i (y_{i-1} + 2 y_i)) \ . \qquad (54)
\end{aligned}
$$

Therefore, we have

$$
\alpha_x = -\frac{1}{6a} \sum_{i=1}^n (x_i - x_{i-1})(x_{i-1}(2 y_{i-1} + y_i) + x_i (y_{i-1} + 2 y_i)) \ . \qquad (55)
$$

If we take a closer look at this equation we see that some terms in each term of the sum cancel, while some terms telescope, i.e., cancel in consequtive terms of the sum. Thus, the equation can be simplified to:

$$
\alpha_x = \frac{1}{6a} \sum_{i=1}^n (x_{i-1} + x_i)(x_{i-1} y_i - x_i y_{i-1}) \ . \qquad (56)
$$

The calculation of α_y proceeds in a completly analogous manner:

$$
\begin{aligned}
\int_{b_i} xy \, dy &= \int_0^1 x_i(t) y_i(t) y_i'(t) \, dt \\[2ex]
&= -\int_0^1 (x_{i-1} + t(x_i - x_{i-1}))(y_{i-1} + t(y_i - y_{i-1}))(y_i - y_{i-1}) \, dt \\[1ex]
&= \cdots \\[1ex]
&= \frac{1}{6} (y_i - y_{i-1})(y_{i-1}(2 x_{i-1} + x_i) + y_i (x_{i-1} + 2 x_i)) \ . \qquad (57)
\end{aligned}
$$

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Therefore, after eliminating canceling and telescoping terms, we have

$$\alpha_y = \frac{1}{6a} \sum_{i=1}^{n} (y_{i-1} + y_i)(x_{i-1}y_i - x_i y_{i-1}) \quad . \tag{58}$$

B   Calculation of the Second Order Moments

According to (36), we have

$$
\begin{aligned}
\int_{b_i}^{1} -x^2 y\, dx &= -\int_0^1 x_i^2(t) y_i(t) x_i'(t)\, dt \\[2mm]
&= -\int_0^1 (x_{i-1} + t(x_i - x_{i-1}))^2 (y_{i-1} + t(y_i - y_{i-1}))(x_i - x_{i-1})\, dt \\[2mm]
&= -(x_i - x_{i-1}) \int_0^1 (x_{i-1}^2 + 2t x_{i-1}(x_i - x_{i-1}) + t^2(x_i - x_{i-1})^2) \\
&\qquad\qquad (y_{i-1} + t(y_i - y_{i-1}))\, dt \\[2mm]
&= -(x_i - x_{i-1}) \int_0^1 x_{i-1}^2 y_{i-1} + t x_{i-1}^2(y_i - y_{i-1}) + 2t x_{i-1}(x_i - x_{i-1}) y_{i-1} \\
&\qquad\qquad + 2t^2 x_{i-1}(x_i - x_{i-1})(y_i - y_{i-1}) + t^2(x_i - x_{i-1})^2 y_{i-1} \\
&\qquad\qquad + t^3(x_i - x_{i-1})^2(y_i - y_{i-1})\, dt \\[2mm]
&= -(x_i - x_{i-1}) \int_0^1 x_{i-1}^2 y_{i-1} + t(x_{i-1}^2 y_i - x_{i-1}^2 y_{i-1} + 2x_{i-1}x_i y_{i-1} - 2x_{i-1}^2 y_{i-1}) \\
&\qquad\qquad + t^2(2x_{i-1}x_i y_i - 2x_{i-1}x_i y_{i-1} - 2x_{i-1}^2 y_i + 2x_{i-1}^2 y_{i-1} \\
&\qquad\qquad\qquad + x_i^2 y_{i-1} - 2x_{i-1}x_i y_{i-1} + x_{i-1}^2 y_{i-1}) \\
&\qquad\qquad + t^3(x_i^2 y_i - 2x_{i-1}x_i y_i + x_{i-1}^2 y_i - x_i^2 y_{i-1} + 2x_{i-1}x_i y_{i-1} \\
&\qquad\qquad\qquad - x_{i-1}^2 y_{i-1})\, dt \\[2mm]
&= -(x_i - x_{i-1}) \Big(t x_{i-1}^2 y_{i-1} + \frac{1}{2} t^2(-3x_{i-1}^2 y_{i-1} + x_{i-1}^2 y_i + 2x_{i-1}x_i y_{i-1}) \\
&\qquad\qquad + \frac{1}{3} t^3 (3x_{i-1}^2 y_{i-1} - 2x_{i-1}^2 y_i - 4x_{i-1}x_i y_{i-1} + 2x_{i-1}x_i y_i \\
&\qquad\qquad\qquad + x_i^2 y_{i-1}) \\
&\qquad\qquad + \frac{1}{4} t^4 (-x_{i-1}^2 y_{i-1} + x_{i-1}^2 y_i + 2x_{i-1}x_i y_{i-1} - 2x_{i-1}x_i y_i \\
&\qquad\qquad\qquad - x_i^2 y_{i-1} + x_i^2 y_i) \Big|_{t=0}^{1} \Big) \\[2mm]
&= -(x_i - x_{i-1}) \Big(x_{i-1}^2 y_{i-1} - \frac{3}{2} x_{i-1}^2 y_{i-1} + \frac{1}{2} x_{i-1}^2 y_i + x_{i-1}x_i y_{i-1} + x_{i-1}^2 y_{i-1} \\
&\qquad\qquad - \frac{2}{3} x_{i-1}^2 y_i - \frac{4}{3} x_{i-1}x_i y_{i-1} + \frac{2}{3} x_{i-1}x_i y_i + \frac{1}{3} x_i^2 y_{i-1}
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\frac{1}{4}x_{i-1}^2 y_{i-1} + \frac{1}{4}x_{i-1}^2 y_i + \frac{1}{2}x_{i-1}x_i y_{i-1} - \frac{1}{2}x_{i-1}x_i y_i \\
&\qquad \left. -\frac{1}{4}x_i^2 y_{i-1} + \frac{1}{4}x_i^2 y_i \right) \\
&= -(x_i - x_{i-1})\left(\frac{1}{4}x_{i-1}^2 y_{i-1} + \frac{1}{12}x_{i-1}^2 y_i + \frac{1}{6}x_{i-1}x_i y_{i-1} + \frac{1}{6}x_{i-1}x_i y_i \right. \\
&\qquad \left. + \frac{1}{12}x_i^2 y_{i-1} + \frac{1}{4}x_i^2 y_i \right) \\
&= -\frac{1}{12}(x_i - x_{i-1})\big(x_{i-1}^2(3y_{i-1} + y_i) + 2x_{i-1}x_i(y_{i-1} + y_i) \\
&\qquad + x_i^2(y_{i-1} + 3y_i)\big) \ .
\end{aligned}
\tag{59}
$$

As was the case above for the first order moments, some of the terms cancel and some telescope. Therefore, we have

$$
\alpha_{xx} = \frac{1}{12a}\sum_{i=1}^{n}(x_{i-1}^2 + x_{i-1}x_i + x_i^2)(x_{i-1}y_i - x_i y_{i-1}) \ .
\tag{60}
$$

By a completely analogous derivation we have

$$
\begin{aligned}
\int_{b_i} xy^2 \, dy &= \int_0^1 x_i(t)y_i^2(t)y_i'(t)\, dt \\
&= \int_0^1 (x_{i-1} + t(x_i - x_{i-1}))(y_{i-1} + t(y_i - y_{i-1}))^2 (y_i - y_{i-1})\, dt \\
&= \cdots \\
&= \frac{1}{12}(y_i - y_{i-1})\big(y_{i-1}^2(3x_{i-1} + x_i) + 2y_{i-1}y_i(x_{i-1} + x_i) \\
&\qquad + y_i^2(x_{i-1} + 3x_i)\big) \ .
\end{aligned}
\tag{61}
$$

Therefore, again after removing all canceling and telescoping terms, we have

$$
\alpha_{yy} = \frac{1}{12a}\sum_{i=1}^{n}(y_{i-1}^2 + y_{i-1}y_i + y_i^2)(x_{i-1}y_i - x_i y_{i-1}) \ .
\tag{62}
$$

In order to calculate α_{xy} we need to compute the following terms:

$$
-\int_{b_i} xy^2 \, dx \quad \text{and} \quad \int_{b_i} x^2 y \, dy \ .
\tag{63}
$$

Obviously, the first term is just the negative of (61) with $(y_i - y_{i-1})$ substituted by $(x_i - x_{i-1})$. Analogously, the second term is the negative of (59) with $(x_i - x_{i-1})$ substituted by $(y_i - y_{i-1})$. Thus, we have

$$
\begin{aligned}
&\frac{1}{4}\int_{b_i} -xy^2 \, dx + x^2 y \, dy \\
&= \frac{1}{48}\big(3x_{i-1}^2 y_{i-1}^2 + x_{i-1}x_i y_{i-1}^2 - 3x_{i-1}x_i y_{i-1}^2 - x_i^2 y_{i-1}^2 + 2x_{i-1}^2 y_{i-1}y_i + 2x_{i-1}x_i y_{i-1}y_i
\end{aligned}
$$

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$$\begin{aligned}
&\quad - 2x_{i-1}x_iy_{i-1}y_i - 2x_i^2y_{i-1}y_i + x_{i-1}^2y_i^2 + 3x_{i-1}x_iy_i^2 - x_{i-1}x_iy_i^2 - 3x_i^2y_i^2 \\
&\quad + 3x_{i-1}^2y_{i-1}y_i + x_{i-1}^2y_i^2 - 3x_{i-1}^2y_{i-1}^2 - x_{i-1}^2y_{i-1}y_i + 2x_{i-1}x_iy_{i-1}y_i + 2x_{i-1}x_iy_i^2 \\
&\quad - 2x_{i-1}x_iy_{i-1}^2 - 2x_{i-1}x_iy_{i-1}y_i + x_i^2y_{i-1}y_i + 3x_i^2y_i^2 - x_i^2y_{i-1}^2 - 3x_i^2y_{i-1}y_i) \\
&= \frac{1}{48}(4x_{i-1}^2y_{i-1}y_i + 2x_{i-1}^2y_i^2 - 4x_{i-1}x_iy_{i-1}^2 + 4x_{i-1}x_iy_i^2 - 2x_i^2y_{i-1}^2 - 4x_i^2y_{i-1}y_i) \\
&= \frac{1}{24}(2x_{i-1}y_{i-1} + x_{i-1}y_i + x_iy_{i-1} + 2x_iy_i)(x_{i-1}y_i - x_iy_{i-1}) \ .
\end{aligned} \tag{64}$$

Therefore,

$$\alpha_{xy} = \frac{1}{24a}\sum_{i=1}^{n}(2x_{i-1}y_{i-1} + x_{i-1}y_i + x_iy_{i-1} + 2x_iy_i)(x_{i-1}y_i - x_iy_{i-1}) \ . \tag{65}$$

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