

Under, Over and Critical Damping

1. Response to Damping

As we saw, the unforced damped harmonic oscillator has equation

$$m\ddot{x} + b\dot{x} + kx = 0, \quad (1)$$

with $m > 0$, $b \geq 0$ and $k > 0$. It has characteristic equation

$$ms^2 + bs + k = 0$$

with characteristic roots

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \quad (2)$$

There are three cases depending on the sign of the expression under the square root:

- i) $b^2 < 4mk$ (this will be **underdamping**, b is small relative to m and k).
- ii) $b^2 > 4mk$ (this will be **overdamping**, b is large relative to m and k).
- iii) $b^2 = 4mk$ (this will be **critical damping**, b is just between over and underdamping).

Mathematically, the easiest case is overdamping because the roots are real. However, most people think of the oscillatory behavior of a damped oscillator. Since this is connected to underdamping we start with that case.

Case (i) Underdamping (non-real complex roots)

If $b^2 < 4mk$ then the term under the square root is negative and the characteristic roots are not real. In order for $b^2 < 4mk$ the damping constant b must be relatively small.

First we use the roots (2) to solve equation (1).

Let $\omega_d = \sqrt{|b^2 - 4mk|}/2m$. Then we have

characteristic roots: $\frac{-b}{2m} \pm i\omega_d$. leading to

complex exponential solutions: $e^{(-b/2m + i\omega_d)t}$, $e^{(-b/2m - i\omega_d)t}$.

The basic real solutions are $e^{-bt/2m} \cos(\omega_d t)$ and $e^{-bt/2m} \sin(\omega_d t)$.

The general real solution is found by taking linear combinations of the two basic solutions, that is:

$$x(t) = c_1 e^{-bt/2m} \cos(\omega_d t) + c_2 e^{-bt/2m} \sin(\omega_d t)$$

or

$$x(t) = e^{-bt/2m}(c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t)) = Ae^{-bt/2m} \cos(\omega_d t - \phi). \quad (3)$$

Let's analyze this physically. When $b = 0$ the response is a sinusoid. Damping is a frictional force, so it generates heat and dissipates energy. When the damping constant b is small we would expect the system to still oscillate, but with decreasing amplitude as its energy is converted to heat. Over time it should come to rest at equilibrium. This is exactly what we see in (3). The factor $\cos(\omega_d t - \phi)$ shows the oscillation. The exponential factor $e^{-bt/2m}$ has a negative exponent and therefore gives the decaying amplitude. As $t \rightarrow \infty$, the exponential goes asymptotically to 0, so $x(t)$ also goes asymptotically to its equilibrium position $x = 0$.

We call ω_d the **damped angular (or circular) frequency** of the system. This is sometimes called a **pseudo-frequency** of $x(t)$. We need to be careful to call it a pseudo-frequency because $x(t)$ is not periodic and only periodic functions have a frequency. Nonetheless, $x(t)$ does oscillate, crossing $x = 0$ twice each pseudo-period.

Example 1. Show that the system $\ddot{x} + 1\dot{x} + 3x = 0$ is underdamped, find its damped angular frequency and graph the solution with initial conditions $x(0) = 1, \dot{x}(0) = 0$.

Solution. Characteristic equation: $s^2 + s + 3 = 0$.

Characteristic roots: $-1/2 \pm i\sqrt{11}/2$.

Basic real solutions: $e^{-t/2} \cos(\sqrt{11} t/2), e^{-t/2} \sin(\sqrt{11} t/2)$.

General solution:

$$x(t) = e^{-t/2}(c_1 \cos(\sqrt{11} t/2) + c_2 \sin(\sqrt{11} t/2)) = Ae^{-t/2} \cos(\sqrt{11} t/2 - \phi).$$

Since the roots have nonzero imaginary part, the system is underdamped.

The damped angular frequency is $\omega_d = \sqrt{11}/2$.

The initial conditions are satisfied when $c_1 = 1$ and $c_2 = 1/\sqrt{11}$. So,

$$\begin{aligned} x(t) &= e^{-t/2} \left(\cos(\sqrt{11} t/2) + \frac{1}{\sqrt{11}} \sin(\sqrt{11} t/2) \right) \\ &= \frac{\sqrt{12}}{\sqrt{11}} e^{-t/2} \cos(\sqrt{11} t/2 - \phi), \end{aligned}$$

where $\phi = \tan^{-1}(1/\sqrt{11})$.

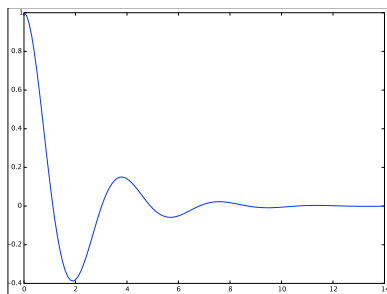


Figure 1: The damped oscillation for example 1.

Case (ii) Overdamping (distinct real roots)

If $b^2 > 4mk$ then the term under the square root is positive and the characteristic roots are real and distinct. In order for $b^2 > 4mk$ the damping constant b must be relatively large.

One extremely important thing to notice is that in this case **the roots are both negative**. You can see this by looking at the formula (2). The term under the square root is positive by assumption, so the roots are real. Since $b^2 - 4mk < b^2$ the square root is less than b and therefore the root $-b + \sqrt{b^2 - 4mk} < 0$. The other root is clearly negative.

Now we use the roots to solve equation (1) in this case.

$$\text{Characteristic roots: } r_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}.$$

$$\text{Exponential solutions: } e^{r_1 t}, e^{r_2 t}.$$

General solution:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Let's analyze this physically. When the damping is large the frictional force is so great that the system can't oscillate. It might sound odd, but an unforced overdamped harmonic oscillator does not oscillate. Since both exponents are negative every solution in this case goes asymptotically to the equilibrium $x = 0$.

At the top of many doors is a spring to make them shut automatically. The spring is damped to control the rate at which the door closes. If the damper is strong enough, so that the spring is overdamped, then the door just settles back to the equilibrium position (i.e. the closed position) without oscillating—which is usually what is wanted in this case.

Example 2. Show that the system $\ddot{x} + 4\dot{x} + 3x = 0$ is overdamped and graph the solution with initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. Which root controls how fast the solution returns to equilibrium?

Solution. Characteristic equation: $s^2 + 4s + 3 = 0$.

Characteristic roots: (this factors) $-1, -3$.

Exponential solutions: e^{-t}, e^{-3t} .

General solution:

$$x(t) = c_1 e^{-t} + c_2 e^{-3t}.$$

Because the roots are real and different, the system is overdamped.

The initial conditions are satisfied when $c_1 = 3/2, c_2 = -1/2$. So, $x(t) = 3e^{-t}/2 - e^{-3t}/2$.

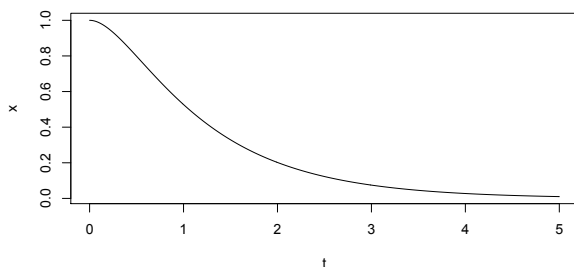


Figure 2: The overdamped graph for example 2.

Because e^{-t} goes to 0 more *slowly* than $e^{-3t/2}$ it controls the rate at which x goes to 0. (Remember, it is the term that goes to zero slowest term that controls the rate.)

Case (iii) Critical Damping (repeated real roots)

If $b^2 = 4mk$ then the term under the square root is 0 and the characteristic polynomial has repeated roots, $-b/2m, -b/2m$.

Now we use the roots to solve equation (1) in this case. We have only one exponential solution, so we need to multiply it by t to get the second solution.

Basic solutions: $e^{-bt/2m}, te^{-bt/2m}$.

General solution:

$$x(t) = e^{-bt/2m}(c_1 + c_2 t).$$

As in the overdamped case, this does not oscillate. It is worth noting that for a fixed m and k , choosing b to be the critical damping value gives the fastest return of the system to its equilibrium position. In engineering design this is often a desirable property. This can be seen by considering the roots, but we will not go through the algebra that shows this. (See figure (4).)

Example 3. Show that the system $\ddot{x} + 4\dot{x} + 4x = 0$ is critically damped and graph the solution with initial conditions $x(0) = 1, \dot{x}(0) = 0$.

Solution. Characteristic equation: $s^2 + 4s + 4 = 0$.

Characteristic roots: (this factors) $-2, -2$.

Exponential solutions: (only one) e^{-2t} .

General solution:

$$x(t) = e^{-2t}(c_1 + c_2 t).$$

Because the roots are repeated, the system is critically damped.

The initial conditions are satisfied when $c_1 = 1, c_2 = 2$. So,

$$x(t) = e^{-2t}(1 + 2t).$$

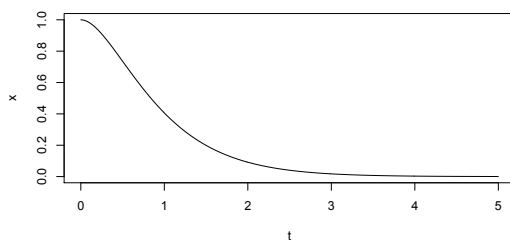


Figure 3: The critically damped graph for example 3.

Notice that qualitatively the graphs for the overdamped and critically damped cases are similar.

The following figure shows plots for solutions to $\ddot{x} + b\dot{x} + x = 0$ with initial conditions $x(0) = 1, \dot{x}(0) = 0$. The three plots are $b = 1$ underdamped; $b = 2$ critically damped (dashed line); $b = 3$ overdamped. Notice that the critically damped curve has the fastest decay.

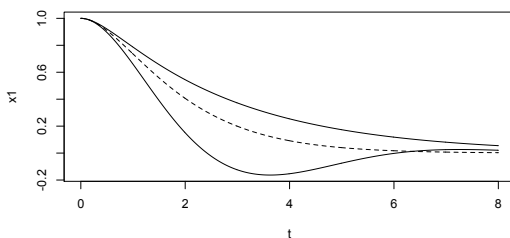


Figure 4: Plots of solutions to $\ddot{x} + b\dot{x} + x = 0$.