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Equity prices as a simple harmonic oscillator with noise

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Abstract

The centred return on the London Stock Exchange's FTSE All Share Index is modelled as a simple harmonic oscillator with noise over the period from 1 January, 1994 until 30 June 2006. Our empirical results are compatible with the hypothesis that there is a period in the FTSE All Share Index of between two and two and one half years. This means the centred return will on average continue to increase for about a year after reaching the minimum in its oscillatory cycle; alternatively, it will continue on average to decline for about a year after reaching a maximum. Our analysis also shows that there is potential to exploit the harmonic nature of the returns process to earn abnormal profits. Extending our analysis to the low energy states of a quantum harmonic oscillator is also suggested.

Keywords: Brownian motion; Classical mechanics; Econophysics; Harmonic oscillator

1. Introduction

In recent years a large volume of empirical evidence has emerged in the finance literature which suggests that a simple strategy of buying short term equity stock "winners" and selling short term equity stock "losers" leads to significant abnormal profits over the medium term. Most of this empirical work is motivated by the early North American study of Jegadeesh and Titman [1] which contends that in the medium term equity securities with abnormal positive returns over the previous 3–12 months continue to outperform equity securities with negative abnormal returns over the same 3–12 month period. Furthermore, more recent studies show that these medium term "momentum" profits have continued beyond the 1990s and are also present in a number of stock markets outside of North America [2–4]. Against this, DeBondt and Thaler [5,6] and others summarise empirical evidence compatible with the hypothesis that momentum profits for US equity securities tend to reverse over the longer term; that is, cumulative abnormal profits on medium term winners gradually decay away and become negative over the longer term. Moreover, medium term losers tend to return abnormal profits over the longer term [1,3,7]. Taken together the DeBondt and Thaler [5,6] and Jegadeesh and Titman [1] results suggest that stock returns evolve harmonically in which case momentum has an important role to play in the generation of stock prices. The finance literature provides a number of explanations as to

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why this momentum phenomenon might exist, the most popular of which is that investors initially under-react to the release of firm specific information. This in turn means that over the medium term prices gradually drift upwards in the case of ‘‘good’’ news (downwards in the case of ‘‘bad’’ news) to more appropriate levels [8,9]. Recently, however, momentum theories of the evolution of stock prices have also been advanced in the newly emerging econophysics literature although the approach taken here rests more on concepts from statistical mechanics than the intuitive approaches normally encountered in the finance literature [10–14]. Given this, our purpose here is to contribute to this emerging econophysics literature by assessing whether the evolution of equity prices is compatible with that of a simple harmonic oscillator with Gaussian noise. Our empirical analysis, which is based on the London Stock Exchange’s FTSE All Share Index, is compatible with the hypothesis that there is a period in equity prices of between two and two and one half years. Moreover, there is potential to exploit the harmonic nature of this pricing process and thereby earn abnormal profits.

2. An harmonic oscillator with Gaussian noise

Andersen et al. [10] have noted that ‘‘analogies with classical mechanics are deeply ingrained within both modern economics and technical analysis …’’ and they exploit these commonalities by devising trading rules defined in terms of an asset’s price momentum and its price acceleration for various equity, bond and foreign exchange markets. Likewise, Wang and Pandy [14] observe that ‘‘the analogy between … stock dynamics and the classical mechanics of particles … is an appealing concept to pursue …’’ and they too determine momentum trading strategies for the 30 stocks comprising the Dow-Jones Industrial Average based on classical mechanics concepts. Moreover, Roehner [11] and Ausloos and Ivanova [12] suggest that equity prices can often be modelled in terms of relatively simple second (and perhaps, higher) order dynamical systems.

Given this, suppose one defines $P(t)$ to be the market price of a unit investment in an equity security at time t. Furthermore, let μ be the expected long run return (per unit time) on the given equity security. It then follows that $x(t) = \log[P(t)/P(0)] - \mu t$ is the ‘‘accumulated abnormal’’ or ‘‘centred’’ return on the given security over the period from time zero until time t. Now, suppose the equity security’s centred return evolves as a simple harmonic oscillator with Gaussian noise; that is, the centred return *accelerates* towards its long run (or normalised) value of zero at a rate which (apart from a Gaussian noise component) is proportional to the centred return’s current magnitude, or:

$$x''(t) + \lambda^2 x(t) = W'(t), \tag{1}$$

where λ^2 is the speed of adjustment coefficient which measures the intensity with which the centred return accelerates towards its normal (long run) value of zero whilst $W'(t)$ is a Brownian motion with a variance parameter of σ^2. In other words, positive abnormal returns are eliminated through negative acceleration; negative abnormal returns are eliminated through positive acceleration. Under the boundary condition $x(0) = 0$ all solutions of the differential Eq. (1) take the form [15]:

$$x(t) = k\,\sin(\lambda t) + \frac{1}{\lambda}\int_0^t \sin(\lambda(t-s))\,W'(s)\,\mathrm{d}s, \tag{2}$$

where k is a constant. Thus, $x(t)$ has a mean of $k\sin(\lambda t)$ so that in expectations, the abnormal return oscillates harmonically with an amplitude of k and a period of $2\pi/\lambda$. This has the important implication that over the middle part of the oscillatory cycle, $(\pi/2\lambda,\ 3\pi/2\lambda)$, the equity security will on average return ‘‘momentum’’ induced abnormal profits; that is, once the centred return begins to increase in magnitude it will continue to do so until it reaches its maximum amplitude at which point it will then go into decline. Superimposed on this harmonic component, however, are Gaussian distributed random fluctuations that possess a mean of zero and a variance of $\sigma^2/2\lambda^2[t - \sin(2\lambda t/2\lambda)]$ [15]. Hence, the centred return’s variance is also a harmonic function of time.

The harmonic nature of the centred return’s mean and variance pose significant difficulties for parameter estimation [16]. Fortunately, one can stabilise both the mean and variance by taking a kind of second difference across Eq. (2), namely:

$$x(t+2\Delta t)-\beta x(t+\Delta t)+x(t)=\frac{1}{\lambda}\int_{t+\Delta t}^{t+2\Delta t}\sin(\lambda(t+2\Delta t-s))W'(s)\,\mathrm{d}s-\frac{1}{\lambda}\int_{t}^{t+\Delta t}\sin(\lambda(t-s))W'(s)\,\mathrm{d}s, \tag{3}$$

where $\beta=2\cos(\lambda\Delta t)$. It is not hard to show that this modified second difference operator has a mean of zero and a variance of $(\sigma^2/2\lambda^2)$ $[\Delta t-\sin(2\lambda\Delta t/2\lambda)]$ [15]. In other words, if the centred return is observed at equally spaced intervals then both the mean and variance of the modified second difference operator are stationary in time. Given this, one can use modified second differences computed from non-overlapping and equally spaced time periods in conjunction with the maximum likelihood technique to estimate the parameters of the model. A detailed description of the estimation procedure is given in the Appendix. For now it suffices to note that if there is no period (that is, no momentum) in the centred returns data then it will be the case that $\lambda=0$ and $\beta=2$. Given this, our empirical analysis is based on a Student's "t" test which assesses whether the maximum likelihood estimate of β is significantly different from the "no momentum" hypothesis of $\beta=2$. We now apply this testing procedure to returns computed from the United Kingdom FTSE All Share Index.

3. United Kingdom stock index data and analysis

Compatible with previous studies appearing in the finance literature [1,4,17] our data consist of the monthly continuously compounded returns implied by the FTSE All Share Index over the period from 1 January 1994 until 30 June 2006. Prior to this period, UK index compilers "underestimated" the importance of dividends in the returns calculation and it was not until mid-1993 that the FTSE indices were first computed on a "total returns" basis [18]. Hence, it is only from this point onwards that one can have confidence in the accuracy of returns calculated from the FTSE All Share Index and this explains why we have chosen this particular period on which to base our empirical analysis.

Our empirical analysis is thus based on the 75 observations of the modified second difference formula (3) available over the period from 1 January 1994 to 30 June 2006. Given this, Table 1 provides summary information about the fitting of our modified second difference model (3) to this data set. The first row gives the assumed long run (expected) return on the FTSE All Share Index. Thus, if one follows the usual convention of assuming that over the long run the expected return on well diversified indices like the FTSE All Share Index is in the vicinity of 10% (per annum) [19], then the maximum likelihood estimate of the beta coefficient in our modified second difference model turns out to be $\hat{\beta}=1.9331$. Furthermore, under the null

Table 1
FTSE All Share Index—estimated parameters and diagnostic statistics for a simple harmonic oscillator with gaussian noise

μ	0.06	0.08	0.10	0.12	0.14	0.16
$\hat{\beta}$	1.9758	1.9285	1.9331	1.9547	1.9683	1.9763
$t(\hat{\beta})$	−0.8172	-2.1299^{θ}	-2.5952^{*}	-2.4747^{*}	-2.3147^{θ}	-2.1941^{θ}
$\hat{\lambda}$	0.1558	0.2682	0.2593	0.2131	0.1782	0.1542
Period (months)	40.3373	23.4311	24.2300	29.4779	35.2522	40.7494
Jarque–Bera	6.6513^{θ}	4.2938	3.5065	3.8688	4.3211	4.6627
Ljung–Box	5.8004	6.3921	4.5175	3.8111	3.6127	3.5402
Parameter stability: likelihood ratio	5.3973	7.4296	6.8185	5.0361	4.1434	3.7240

This table provides summary information about the empirical estimation of our momentum model. The first row gives the assumed long run (expected) return on the FTSE All Share Index. The second and third rows give the estimates of $\beta=2\cos(\lambda\Delta t)$ and their associated $t(\hat{\beta})$ statistics. The "largest" t statistic, $t(\hat{\beta})=-2.5954$, occurs when the assumed expected long run return on the FTSE All Share Index is $\mu=10.0505\%$. The fourth row gives the parameter $\hat{\lambda}=\cos^{-1}(\hat{\beta}/2)/\Delta t$ implied by the estimate of β whilst the fifth row uses $\hat{\beta}$ to determine the estimated period in the centred returns data, $2\pi/\hat{\lambda}$. The sixth row gives the Jarque–Bera statistic which, under the null hypothesis that our data are drawn from a Gaussian distribution, is asymptotically distributed as a Chi-square variate with two degrees of freedom. The seventh row summarises Ljung–Box autocorrelation statistic which is asymptotically distributed as a Chi-square variate with five degrees of freedom. The final row summarises the parameter stability statistic which is asymptotically distributed as a Chi-square variate with four degrees of freedom. *signifies that the relevant statistic falls in the top (bottom) 1% of the area under the relevant probability density; θ signifies that the relevant statistic falls in the top (bottom) 5% of the area under the relevant probability density.

hypothesis that $\beta = 2$ this estimate returns a Student's statistic of $t(\hat{\beta}) = -2.5952$ and this is significantly different from zero at the one percentage point level. Moreover, one can solve the equation $\hat{\beta} = 2\cos(\hat{\lambda}\Delta t) = 1.9331$ and thereby show that $\hat{\lambda} = 0.2593$ or that the estimated period in the centred returns data is $2\pi/\hat{\lambda} = 2\pi/0.2593 = 24.23$ months. In other words, our empirical results are compatible with the hypothesis that the centred return fluctuates randomly around a sinusoidal trend with a period of approximately two years.

Standard diagnostics also show that our regression procedures are well specified. The Ljung and Box [20] test was used to assess whether our parameter estimates have been contaminated by serial correlation in the residuals. The test is based on the statistic $Q(\hat{r}) = n(n+2)\sum_{k=1}^{m} \hat{r}_k^2/(n-k)$ where $n = 75$ is the number of modified second differences (residuals) on which our estimation procedures are based, $\hat{r}_k$ is the kth order autocorrelation coefficient between the residuals obtained from Eq. (3) and $m = 6$ is the highest order autocorrelation coefficient on which the test is based. Under the null hypothesis that there is no serial correlation between the residuals Ljung and Box [20] show that $Q(\hat{r})$ is asymptotically distributed as a Chi-square with $(m-1)$ degrees of freedom. Table 1 shows that our modified second difference model (3) returned a test statistic of $Q(\hat{r}) = 4.5175$ with five degrees of freedom when the expected long run return on the FTSE All Share Index is assumed to be $\mu = 10\%$. This is well below generally accepted significance benchmarks and so the test statistic is compatible with the hypothesis that there is little evidence of autocorrelation in the residuals of our model.

The Jarque–Bera test [21] assesses whether the residuals from our modified second difference model are compatible with a Gaussian distribution. The test statistic is based on the sum of the standardised skewness and kurtosis measures and under the null hypothesis that the residuals are drawn from a Gaussian distribution, is asymptotically distributed as a Chi-square variate with two degrees of freedom. Table 1 shows that the residuals from our modified second difference model (3) return a test statistic of $\chi_2^2 = 3.5065$ when the expected long run return on the FTSE All Share Index is assumed to be $\mu = 10\%$. This is well below generally accepted significance benchmarks and thereby indicates that the residuals from our modified second difference model (3) are consistent with having been drawn from a Gaussian distribution. This is a particularly encouraging result given the continuing failure of virtually all first order approximations of the returns process to converge to a Gaussian distribution [22].

Finally, in the Appendix we summarise a likelihood ratio test for assessing whether the parameters of our model are stationary in time. Here we show that if one assumes that the parameters of our modified second difference model (3) change on (integral) k occasions over the period covered by the data, then the likelihood ratio is asymptotically distributed as a Chi-square variate with $2k-2$ degrees of freedom. We implement this test by dividing the 75 modified second differences available from our data into $k = 3$ groups of 25 observations each. This means that under the null hypothesis that the parameters of our model are stationary the likelihood ratio is based on the Chi-square distribution with $(2k-2 =)$ four degrees of freedom. Our calculations return a test statistic of $\chi_4^2 = 6.8185$ which does not exceed any of the usual significance benchmarks and thereby indicates that parameter stability is a reasonable maintained hypothesis.

The crucial assumption in our empirical analysis, however, is that the expected long run return on the FTSE All Share Index is $\mu = 10\%$ (per annum). To assess the sensitivity of our empirical results to variations in this assumption Table 1 also summarises results from replicating our regression procedure using expected long run returns which vary from $\mu = 6\%$ to $\mu = 16\%$. The table shows that whilst changing μ has only a small impact on most parameters/statistics, it does have a significant effect on $t(\hat{\beta})$, the t statistic which indicates whether or not there might be a period in our centred returns data. Once μ falls below 7% the $t(\hat{\beta})$ statistic declines sharply, thereby indicating that below this level of return there is little evidence of a period in the data. This is of particular importance in light of the fact that the average return on the FTSE All Share Index over the period from 1 January 1994 to 30 June 2006 amounts to 7.79% (per annum). Whilst at this expected long run return $t(\hat{\beta}) = -2.0125$ is still significantly different from zero, at marginally lower levels of μ the $t(\hat{\beta})$ statistic quickly falls below generally accepted significance benchmarks. This means that whether or not a period exists in the FTSE All Share centred returns data hinges crucially on the assumption made about μ.

Fortunately, at generally accepted levels of μ (that is, for μ between 8% and 12% per annum) Table 1 shows that our estimates of λ all return "t" statistics that are significantly different from zero at generally accepted benchmarks and that the period in the centred return lies somewhere between two and two and one half years. This in turn means that the centred return will on average decline for (approximately) the first six months of its period but then gradually increase over the ensuing year before falling away again over the final six months of its period. Hence, if one makes an investment in the FTSE All Share Index some nine months to a year after the beginning of a period (that is, three to six months into the centred return's upward phase), on average there will still be six to nine months during which the centred return will continue to increase; in other words, the abnormal returns earned over the previous three to six months will continue to be earned over the ensuing six to nine months. Likewise, dis-investing some three to six months into the downward phase of the oscillatory cycle will on average reduce the losses that would otherwise have been incurred over the ensuing six to nine months.

Note also that this investment strategy is also consistent with the momentum trading strategies identified in the finance literature [1,3,4]. However, viewing equity prices in terms of a simple harmonic oscillator provides a more rigorous basis for understanding why such strategies appear to be profitable than is currently available from the finance literature. Recall that the harmonic nature of the returns process means there is a period of about a year when the centred return on the FTSE All Share Index gradually increases and one can exploit this to earn abnormal returns. One can illustrate this investment strategy by going long in the FTSE All Share Index when the centred return has been positive for (say) four months, liquidating this long investment (say) seven months later and then deferring any further investment decisions until the centred return becomes negative again. Likewise, "short" the FTSE All Share Index when the centred return has been negative for four months, cover this short investment by buying the index back seven months later and then defer any further investment decisions until the centred return becomes positive again. Following this investment strategy over the period from 1 January 1994 until 30 June 2006 returns an average abnormal return (per investment cycle) of 5.14% if one assumes the expected return on the FTSE All Share Index is $\mu = 0.08$ or 8%. An investment cycle begins when the investor purchases (shorts) the FTSE All Share Index and is completed when the investor sells (buys back) the investment made in the index. However, the average abnormal return (per investment cycle) declines sharply to 0.78% and 0.80%, respectively, when $\mu = 0.10$ and 0.12. In these latter two instances it is doubtful if the profit margins are sufficient to cover the transactions costs associated with implementing the investment strategy. This again emphasises the importance of being able to make accurate assessments about the long run return, μ, if one is to successfully exploit the potential profits available from the (apparent) harmonic movements in the FTSE All Share Index.

4. Summary and conclusions

The centred (or abnormal) return on the FTSE All Share Index is modelled as a simple harmonic oscillator over the period from 1 January 1994 until 30 June 2006. Our analysis is based on the assumption that the centred return *accelerates* towards its long run (or normalised) value of zero at a rate which (apart from a Gaussian noise term) is proportional to the current magnitude of the centred return. Our empirical results are compatible with the hypothesis that there is a period in the FTSE All Share Index of between two and two and one half years. This means that the centred return will on average continue to increase for about a year after reaching the minimum in its oscillatory cycle; alternatively, it will continue on average to decline for about a year after reaching a maximum. Our analysis also shows that there is potential to exploit the harmonic nature of the returns process in order to earn abnormal profits. The ability to do so, however, hinges on one's skill in making an accurate assessment of the long run return, μ, on the FTSE All Share Index over any given period.

There are a variety of ways in which our analysis might be further developed. First, a number of authors have suggested mechanisms for classifying price patterns based on the "Froude" index which, in the present context, takes the following form [10]:

$$F(t) = \frac{[P'(t)]^2}{P(t)P''(t)},$$

where $P(t)$ is the price of the given equity security, $P'(t)$ is its price velocity and $P''(t)$ is its price acceleration, all at time t. If, for example, $F(t) > 0$ then it indicates the given equity security is in the early stages of the upward or downward phase of its oscillatory cycle, depending on the sign of $P''(t)$. Now, from (1) and (2) above the price of the equity security will be:

$$P(t) = P(0)\exp\left[\mu t + k\,\sin(\lambda t) + \frac{1}{\lambda}\int_0^t \sin(\lambda(t-s))\,W'(s)\,\mathrm{d}s\right]$$

and this means that the Froude index evolves stochastically in time. Information about the strength and direction of trends in the price of the equity security will be of considerable importance to investors and given this, it would be useful to know the distributional properties of the Froude index. Unfortunately, the distributional properties of the Froude Index are not easily determined when the price of the equity security is an exponential function of a simple harmonic oscillator with Gaussian noise, as it is here. However, the issue is of such practical importance for rational decision making that it is an issue that ought to be investigated further.

One could also let the centred return evolve as a quantum harmonic oscillator [23] instead of the classical harmonic oscillator on which most of our empirical analysis is based. The ''correspondence principle'' [23] means that for high energy states the quantum and classical harmonic oscillators will more than likely give similar results, although one must be careful to note that the simple harmonic oscillator employed in our empirical work is augmented by a Gaussian noise term. What is worth exploring, however, is whether solutions of the steady state form of the Schrödinger equation for low energy states [23] lead to a better description of the centred return's distributional and time series properties than the classical harmonic oscillator employed in our empirical work. There are already suggestions of this approach in the finance literature [24] but few, if any, have explored the possibility of modelling the evolution of equity returns in this way in detail.

Appendix

We initialise our analysis at $t = 0$ in which case it follows: $\varepsilon(2j\Delta t) = x(2j\Delta t) - \beta x((2j-1)\Delta t) + x((2j-2)\Delta t)$ for $j = 1,2,\ldots,m$ will be normally distributed with a mean of zero and variance of δ^2. One can then define the following ''restricted'' version of the log-likelihood function:

$$\begin{aligned}\log[L_{\mathrm{R}}(\beta,\delta^2)] = & -\frac{m}{2}\cdot\log(2\pi) - \frac{m}{2}\cdot\log(\delta^2)\\ & -\frac{1}{2\delta^2}\sum_{j=1}^{m}[x(2j\Delta t) - \beta x((2j-1)\Delta t) + x((2j-2)\Delta t)]^2,\end{aligned}$$

where $\beta = 2\cos(\lambda\Delta t)$ and $\delta^2 = (\sigma^2/2\lambda^2)[\Delta t - \sin(2\lambda\Delta t)/2\lambda]$. Differentiating through the log-likelihood function returns the following maximum likelihood estimates for the parameters β and δ^2:

$$\hat{\beta} = \frac{\sum_{j=1}^{m} x((2j-1)\Delta t)[x(2j\Delta t) + x((2j-2)\Delta t)]}{\sum_{j=1}^{m} x^2((2j-1)\Delta t)},$$

$$\hat{\delta}^2 = \frac{1}{m}\sum_{j=1}^{m}[x(2j\Delta t) - \hat{\beta}x((2j-1)\Delta t) + x((2j-2)\Delta t)]^2.$$

Substituting the maximum likelihood estimates into the likelihood function shows that the maximised value of the likelihood function will be:

$$L_R(\hat{\beta},\hat{\delta}^2) = (2\pi)^{-m/2}\cdot\hat{\delta}^{-m}\cdot e^{-m/2}.$$

Moreover, one can use the fact that $\varepsilon(2j\Delta t) = x(2j\Delta t) - \beta x((2j-1)\Delta t) + x((2j-2)\Delta t)$ to show that the expression for $\hat{\beta}$ reduces to:

$$\hat{\beta} = \frac{\sum_{j=1}^{m} x((2j-1)\Delta t)[\varepsilon(2j\Delta t) + \beta x((2j-1)\Delta t)]}{\sum_{j=1}^{m} x^2((2j-1)\Delta t)} = \beta + \frac{\sum_{j=1}^{m} x((2j-1)\Delta t)\varepsilon(2j\Delta t)}{\sum_{j=1}^{m} x^2((2j-1)\Delta t)}.$$

Taking expectations across this latter expression shows that $\hat{\beta}$ is normally distributed with a mean of β and a variance of $\delta^2/\sum_{j=1}^{m} x^2((2j-1)\Delta t)$. It then follows that $(\hat{\beta}-\beta)\sqrt{\sum_{j=1}^{m} x^2((2j-1)\Delta t)}/\delta$ is distributed as a standard normal variate.

Simple algebraic manipulation also shows:

$$\frac{\sum_{j=1}^{m}[x(2j\Delta t)-\hat{\beta}x((2j-1)\Delta t)+x((2j-2)\Delta t)]^2}{\delta^2}=\frac{\sum_{j=1}^{m}\varepsilon^2(2j\Delta t)}{\delta^2}-\frac{(\hat{\beta}-\beta)^2\sum_{j=1}^{m}x^2((2j-1)\Delta t)}{\delta^2},$$

Now, standard results show $\sum_{j=1}^{m}\varepsilon^2(2j\Delta t)/\delta^2$ is distributed as a Chi-square variate with m degrees of freedom [16]. Moreover, previous analysis implies $(\hat{\beta}-\beta)^2\sum_{j=1}^{m}x^2((2j-1)\Delta t)/\delta^2$ is distributed as a Chi-square variate with one degree of freedom. Finally, $\sum_{j=1}^{m}\varepsilon^2(2j\Delta t)/\delta^2$ and $(\hat{\beta}-\beta)^2\sum_{j=1}^{m}x^2((2j-1)\Delta t)/\delta^2$ are stochastically independent [16] in which case it follows $\sum_{j=1}^{m}[x(2j\Delta t)-\hat{\beta}x((2j-1)\Delta t)+x((2j-2)\Delta t)]^2/\delta^2$ is distributed as a Chi-square variate with $(m-1)$ degrees of freedom. It then follows that $(m-1)/m\cdot(\hat{\beta}-\beta)\sqrt{\sum_{j=1}^{m}x^2((2j-1)\Delta t)}/\hat{\delta}$ is distributed as a Student's variate with $(m-1)$ degrees of freedom [16].

To test for parameter stationarity, divide the sample into k sub-samples and define the following "unrestricted" version of the log-likelihood function:

$$\begin{aligned}\log[L_{\mathrm{U}}(\beta_i,\delta_i^2)]= & -\frac{m}{2}\cdot\log(2\pi)-\frac{m_1}{2}\cdot\log(\delta_1^2)-\frac{m_2}{2}\cdot\log(\delta_2^2)-\cdots-\frac{m_k}{2}\cdot\log(\delta_k^2)\\ & -\sum_{i=1}^{k}\frac{1}{2\delta_i^2}\sum_{j=1}^{m_i}[x_i(2j\Delta t)-\beta_i x_i((2j-1)\Delta t)+x_i((2j-2)\Delta t)]^2,\end{aligned}$$

where m_i is the number of modified second difference observations in the ith sub-sample, $\sum_{i=1}^{k}m_i=m$ is the total number of modified second difference observations across all k sub-samples, δ_i^2 is the variance of the modified second differences for the ith sub-sample and $\beta_i=2\cos(\lambda_i\Delta t)$ is the parameter defining the period, $2\pi/\lambda_i$ for the ith sub-period. Differentiating through the log-likelihood function returns the following maximum likelihood estimates for the parameters β_i and δ_i^2:

$$\hat{\beta}_i=\frac{\sum_{j=1}^{m_i}x_i((2j-1)\Delta t)[x_i(2j\Delta t)+x_i((2j-2)\Delta t)]}{\sum_{j=1}^{m_i}x_i^2((2j-1)\Delta t)},$$

$$\hat{\delta}_i^2=\frac{1}{m_i}\sum_{j=1}^{m_i}[x_i(2j\Delta t)-\hat{\beta}_i x_i((2j-1)\Delta t)+x_i((2j-2)\Delta t)]^2$$

for $i=1,2,\ldots,k$. Thus, under this non-stationary model there are $2k$ parameters to be estimated. Substituting the maximum likelihood estimates into the likelihood function shows that the maximised value of the likelihood function will be:

$$L_{\mathrm{U}}(\hat{\beta}_i,\hat{\delta}_i^2)=(2\pi)^{-m/2}\cdot\hat{\delta}_1^{-m_1}\cdot\hat{\delta}_2^{-m_2}\cdot\cdots\cdot\hat{\delta}_k^{-m_k}\cdot e^{-m/2}.$$

Now, the likelihood ratio for the restricted (R) and unrestricted (U) models takes the form $\Lambda=L_{\mathrm{R}}(\hat{\beta},\hat{\delta}^2)/L_{\mathrm{U}}(\hat{\beta}_i,\hat{\delta}_i^2)=\hat{\delta}^{-m}/\hat{\delta}_1^{-m_1}\cdot\hat{\delta}_2^{-m_2}\cdot\cdots\cdot\hat{\delta}_k^{-m_k}$. Moreover, since the unrestricted model is based on $2k$ parameters and the restricted model is based on just two parameters, standard results dictate that $-2\log(\Lambda)=m\log(\hat{\delta}^2)-\sum_{i=1}^{k}m_i\log(\hat{\delta}_i^2)$ is asymptotically distributed as a Chi-square variate with $2k-2$ degrees of freedom [16].

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