Hahn and Jordan Decomposition Theorems

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Here's a complete proof of Hahn decomposition theorem and Jordan Decomposition theorem that we did not prove in class:

Theorem. ***(Hahn Decomposition Theorem)*** *If ν is a signed measure on $(X, \mathcal{M})$, there exist a positive set P and a negative set N for ν such taht $P \cup N = X$ and $P \cap N = \emptyset$. If P' and N' is another such pair, then $P \triangle P' (= N' \triangle N)$ is null for ν.*

Proof. Without loss of generality, we assume that ν does not assume the value $+\infty$ (otherweise consider $-\nu$). Let m be the supremum of $\nu(E)$ as E ranges over all positive sets; thus there is a sequence $\{P_j\}$ of positive sets such that $\nu(P_j) \to m$. Let $P = \bigcup_{j=1}^{P} {}_j$. Since the union of any countable family of positive sets is positive, P is positive. Additionally, let $P_n = \bigcup_{j=1}^{n} P_j$. Then P_n is an increasing sequence and $\nu(P) = \nu(\bigcup P_j) = \lim_{j\to\infty} \nu(P_j) = m$. We claim that $N = X \setminus P$ is negative.

Step 1: N is negative

Before we show that N is negative, here are some observations:

Observation 1: N does not contain any nonnull positive sets

If $E \subset N$ and E is positive (i.e. $\nu(E) > 0$), then $E \cup P$ is positive and $\nu(E \cup P) = \nu(E) + \nu(P) > m$ which contradicts that m is the supremum of $\nu(E)$. Hence N does not contain any nonnull positive sets.

Observation 2: If $A \subset N$ and $\nu(A) > 0$, then $\exists B \subset A$ such that $\nu(B) > \nu(A)$

Since A cannot be positive, $\exists C \subset A$ with $\nu(C) < 0$; thus if $B = A \setminus C$ we have $\nu(B) = \nu(A) - \nu(C) > \nu(A)$.

Now back to the proof:
Suppose N is not negative (i.e. $\neg(\nu(F) \leq 0$ for all $F \subset N))$. Since we have shown that N does not contain any positive sets, N must only contain sets that are neither positive nor negative. Hence, we can specify a sequence of subsets $\{A_j\}$ of N and a sequence $\{n_j\}$ of positive integers as follows: n_1 is the smallest integer for which there exists a set $B \subset N$ with $\nu(B) > n_1^{-1}$, and A_1 is such a set (i.e. we want to find the set whose measure is the largest which will have the largest lower bound, hence the smallest n_1).

Proceeding inductively, n_j is the smallest integer for which there exists a set $B \subset A_{j-1}$ with $\nu(B) > \nu(A_{j-1}) + n_j^{-1}$, and A_j is such a set. (e.g.

- $j = 1$, want to find $A_1 \subset N$ such that $\nu(A_1) > \frac{1}{n_1}$ maximizes $\frac{1}{n_1}$ (or minimizes n_1)
- $j = 2$, want to find $A_2 \subset A_1$ such that $\nu(A_2) >> \nu(A_1) + \frac{1}{n_2}$ maximizes $\frac{1}{n_2}$ (or minimizes n_2)
- $j = 3$, ...

) Notice that $A_1 \supset A_2 \supset \cdots \supset A_j \supset \ldots$. Then let $A = \bigcap_{j=1}^{\infty} A_j$. then $\infty > \nu(A) = \lim \nu(A_0) > \sum_{j=1}^{\infty} \frac{1}{n_j}$ so $n_j \to \infty$ as $j \to \infty$. However, we can find $A' \subset A$ such that $\nu(A') > \nu(A) + \frac{1}{n}$ for some n. Since $n_j \to \infty$, for some sufficiently large j, $n < n_j$ and $A' \subset A_{j-1}$ ($A \subset A_j$ for all j). However, this contradicts the construction since we claimed that n_j is the smallest integer for which there exists a set A_j (but since $n < n_j$, ,n_j is not the smallest). Hence N must be negative.

Step 2: $(P \triangle P')$ is ull for ν

If P', N' is another pair of sets as in the statement of the theorem, let $E \subset P \setminus P'$. Since $P \setminus P' \subset P$, $\nu(E) \geq 0$. In addition, we see that $P \setminus P' \subset N'$ so $\nu(E) \leq 0$. Hence $\nu(E) = 0$. Similarly, $(P' \setminus P) \subset P'$ so for $E \subset P' \setminus P$, $\nu(E) \geq 0$. Furthermore, $P' \setminus P \subset N$, so $\nu(E) \leq 0$. Hence $\nu(E) = 0$.

$\square$

Having proved Hahn decomposition theorem, jordan decomposition theorem can be proved easily:

Theorem. ***Jordan Decomposition Theorem*** *If ν is a signed measure, there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ + \nu^-$ and $\nu^+ \perp \nu^-$.*

Proof. Let $X = P \cup N$ be a Hahn decomposition for ν, and define $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$. Clearly, $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. If also $\nu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$, let $E, F \in \mathcal{M}$ be such that $E \cap F = \emptyset$, $E \cup F = X$, and $\mu^+(F) = \mu^-(E) = 0$. Then $X = E \cup F$ is another Hahan decomposition for ν, so $P \triangle E$ is ν-null. Therefore, for any $A \in \mathcal{M}$, $\mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \mu^+(A)$, and likewise, $\nu^- = \mu^-$. $\square$

References