Implementation of the Black-Scholes Equation Using Finite Difference Methods *

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1 Introduction

The Black-Scholes Model is a mathematical model published by Fischer Black and Myron Scholes in 1973 in their seminal paper, *The Pricing of Options and Corporate Liabilities*. The model focuses on the pricing and valuing of 'European' options. An option is a contract between two parties, an option writer (seller) and an option holder (buyer), which grants the option holder the right to call (buy) or put (sell) a financial asset such as a security at an agreed price, called the strike price [1]. The contract grants the holder this right over a specific period of time or date, i.e. the expiration date. There are two main types of options, American and European, whose major difference occurs over the right to exercise. The American option allows the holder to exercise their right any time up to and including the expiration date, while the European option restricts the holder to exercise their right only on the expiration date [2]. The goal of the Black-Scholes model is to determine optimal prices at which to buy European options.

*The code used to produce the graphs in this paper can be found at https://github.com/papifane/APAM-4301—Final-Project

[1] http://www.investopedia.com/terms/o/option.asp

[2] http://www.investopedia.com/articles/optioninvestor/08/american-european-options.asp

2 Derivation of the Black-Scholes Model

2.1 Assumptions of the Model [3]

The Black-Scholes model is based under 'ideal conditions' for the market concerning the stock and respective options. These conditions are stipulated to be:

1. The short-term interest rate is known and constant through time.

2. The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price.

3. The stock pays no dividends or other distributions.

4. The option is European, i.e. it can only be exercised when it matures.

5. There are no transactions costs in buying or selling the stock or the option.

6. It is possible to borrow any fraction of the price of a security to buy or hold it, at the short-term interest rate.

7. There are no penalties to short-selling. A seller who does not own a security will simply accept the price of the security from a buyer and will agree to settle with the buyer on some future date by paying an amount equal to the price of the security on that date.

One reason for these assumptions is that they restrict the value of an option to depend solely on the price of the stock, time, and on known constant or fixed variables, such as interest rate. In an equation, this implies:

$$w \equiv w(S, t),$$

where w is the price of the option, S is the stock price and t is the time.

Another reason for taking on these assumptions is that, in restricting the number of variables, they allow one to created a hedged position, i.e. an investment which seeks to minimize risk, independent of the price of the stock. The specific strategy consists of a 'long' position in stocks, which expects the price of stocks to rise over time, and a 'short' position in options, which expects the price of stocks to vary and at some point drop due to the variation in the short term. Very importantly, the independence of the equity or value of the hedged position from the price of the stock allows one to create a risk-free portfolio.

2.2 Black-Scholes Equation [4]

The price of the stock S is expected to vary as follows

$$dS = \mu S dt + \sigma S dz,$$

[3] Black and Scholes, *The Pricing of Options and Corporate Liabilities*

[4] John C. Hull, Futures, Options and Other Derivatives, Chapter 14

where μ is stock's expected rate of return, σ is the volatility or standard deviation of the stock price, and z is an stochastic variable which varies randomly and can be thought of as the risk variable.

The price of the option w varies according to the stock price and time. According to Ito's Lemma, a variation of the price of the option is given by:

$$dw = \left(\frac{\partial w}{\partial S}\mu S + \frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 w}{\partial S^2} \right) dt + \sigma S \frac{\partial w}{\partial S} dz$$

Ito's Lemma effectively expands a variation on the option price as a Taylor expansion up to first order in time (second term) and second order in stock price (3rd term); the first and fourth terms are a consequence of the first order expansion in stock price.

The equity value Π of the hedged portfolio described in the previous section is given by the following for a single stock in the long position:

$$\Pi = S - w\frac{\partial S}{\partial w}.$$

A variation in the value of portfolio yields

$$d\Pi = dS - \frac{\partial S}{\partial w} dw.$$

Plugging in the differential (though really variational) equations found above for dS and dw, the variation is found to be:

$$\begin{aligned} d\Pi &= \mu S dt + \sigma S dz - \frac{\partial S}{\partial w} \left(\left(\mu S \frac{\partial w}{\partial S} + \frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 w}{\partial S^2} \right) dt + \sigma S \frac{\partial w}{\partial S} dz \right) \\ &= \mu S dt + \sigma S dz - \frac{\partial S}{\partial w} \left(\mu S \frac{\partial w}{\partial S} + \frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 w}{\partial S^2} \right) dt - \sigma S dz \\ &= -\frac{\partial S}{\partial w} \left(\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 w}{\partial S^2} \right) dt. \end{aligned}$$

Of notable importance is the fact that the stochastic term dependent on z drops out of the equations, as it implies that the portfolio is riskless.

On the other hand, the variation in the value of the portfolio should be proportional to the interest rate times the value itself:

$$d\Pi = r\Pi dt = r\left(S - w\frac{\partial S}{\partial w} \right) dt.$$

Finally, putting together the two equations for variation in the value of the portfolio:

$$\begin{aligned} -\frac{\partial S}{\partial w} \left(\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 w}{\partial S^2} \right) dt &= r\left(S - w\frac{\partial S}{\partial w} \right) dt \\ -\frac{\partial w}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 w}{\partial S^2} &= rS\frac{\partial w}{\partial S} - rw \\ \frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 w}{\partial S^2} + rS\frac{\partial w}{\partial S} - rw = 0 & \end{aligned}$$

The last equation is known as the Black-Scholes equation and its solutions define the price of an option based on the stock price and time variables. The boundary and terminal conditions for the Black-Scholes equation in the case of a European call are given by:

$$\begin{aligned} w(0,t) &= 0 \\ w(x,T) &= S-K, \quad S>K \\ &= 0, \qquad\quad S\le K, \end{aligned}$$

where T is the expiration date and K is the strike price.

For a European option, the boundary and terminal conditions become:

$$\begin{aligned} w(0,t) &= 0 \\ w(x,T) &= K-S, \quad S<k \\ &= 0, \qquad\quad S\ge K, \end{aligned}$$

2.3 Black-Scholes Formula [5]

Finding the Black-Scholes Formula is a matter of solving the above PDE with the respective boundary and terminal condition. The derivation below pertains to the Black-Scholes Formula for European call options. To begin, a change of variable transformation is needed:

$$\begin{aligned} \tau &= T-t \\ u(x,\tau) &= w(S,t)e^{r\tau} \\ x &= \ln\frac{S}{K} + \left(r-\frac{\sigma^2}{2}\right)\tau \end{aligned}$$

Using these, one can find the equations for the partial differentials in the Black-Scholes equations:

$$\begin{aligned} w &= ue^{-r\tau} \\ \frac{\partial w}{\partial t} &= \frac{\partial}{\partial \tau}(ue^{-rt})\frac{d\tau}{dt} + \frac{\partial}{\partial x}(ue^{-rt})\frac{dx}{dt} = rue^{-rt} - \frac{\partial u}{\partial \tau}e^{-rt} - (r-\frac{\sigma^2}{2})\frac{\partial u}{\partial x}e^{-rt} \\ \frac{\partial w}{\partial S} &= \frac{\partial}{\partial \tau}(ue^{-rt})\frac{d\tau}{dS} + \frac{\partial}{\partial x}(ue^{-rt})\frac{dx}{dS} = \frac{\partial u}{\partial x}e^{-rt}\frac{1}{S} \\ \frac{\partial^2 w}{\partial S^2} &= \frac{\partial}{\partial \tau}(\frac{\partial u}{\partial x}e^{-rt}\frac{1}{S})\frac{d\tau}{dS} + \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}e^{-rt}\frac{1}{S})\frac{dx}{dS} + \frac{\partial}{\partial S}(\frac{\partial u}{\partial x}e^{-rt}\frac{1}{S}) \\ &= \frac{\partial^2 u}{\partial x^2}e^{-rt}\frac{1}{S^2} - \frac{\partial u}{\partial x}e^{-rt}\frac{1}{S^2} \end{aligned}$$

[5] Wikipedia Article on Black-Scholes Equation

Plugging these expressions into the Black-Scholes Equation, we find:

$$
\begin{aligned}
0 &= \frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial^2 w}{\partial S^2}\sigma^2 S^2 + r\frac{\partial w}{\partial S}S - rw \\
&= rue^{-rt} - \frac{\partial u}{\partial \tau}e^{-rt} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x}e^{-rt} \\
&\quad + \frac{1}{2}\sigma^2 S^2\left(\frac{\partial^2 u}{\partial x^2}e^{-rt}\frac{1}{S^2} - \frac{\partial u}{\partial x}e^{-rt}\frac{1}{S^2}\right) + rS\left(\frac{\partial u}{\partial x}e^{-rt}\frac{1}{S}\right) - rue^{-r\tau} \\
\frac{\partial u}{\partial \tau} &= \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}
\end{aligned}
$$

which is the familiar diffusion equation. The boundary and terminal conditions now become:

$$
\begin{aligned}
u(-\infty,\tau) &= 0 \\
u(x,0) &= S - K = K(e^x - 1), \quad && x \geq 0 \\
&= 0, && x < 0
\end{aligned}
$$

The above heat equation has solutions of the form:

$$
u(x,\tau) = \frac{1}{\sqrt{2\pi\tau\sigma^2}}\int_{-\infty}^{\infty} u(y,0)e^{\frac{-(x-y)^2}{2\sigma^2\tau}}dy
$$

Resulting in the following equations:

$$
\begin{aligned}
u(x,\tau) &= Ke^{x+\frac{\sigma^2}{2}\tau}N(d_1) - KN(d_2) \\
d_1 &= \frac{1}{\sqrt{\tau\sigma^2}}\left(x + \sigma^2\tau\right) \\
d_2 &= \frac{1}{\sqrt{\tau\sigma^2}}x
\end{aligned}
$$

where N is the cumulative normal density function.

Finally, reverse the transformations to find the Black-Scholes Formula for European call options:

$$
\begin{aligned}
w(S,t) &= u(x,\tau)e^{-r\tau} \\
&= Ke^{x+\frac{\sigma^2}{2}\tau - r\tau}N(d_1) - KN(d_2)e^{-\tau} \\
&= SN(d_1) - KN(d_2)e^{-r(T-t)} \\
d_1 &= \frac{1}{\sqrt{(T-t)\sigma^2}}\left(\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right) \\
d_2 &= \frac{1}{\sqrt{(T-t)\sigma^2}}\left(\ln\frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)
\end{aligned}
$$

The Black Scholes formula for European put options is given by [6]:

$$
w(S,t) = Ke^{-r(T-t)}SN(-d_2) - SN(-d_1)
$$

[6] John C. Hull, Futures, Options and Other Derivatives, Chapter 14

3 Background on Numerical Methods: LTE and Stability Analysis

The Black-Scholes problem is given by:

$$\frac{\partial w}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 w}{\partial S^2} - rS\frac{\partial w}{\partial S} + rw$$

$$w(0,t) = 0$$

$$w(x,T) = \begin{cases} S-K, & S > K \\ 0, & S \leq K \end{cases}$$

The two methods that will be used to study the problem are the forward Euler and Crank-Nicholson methods. The reason why we use these methods is because the Black-Scholes equation is parabolic, and these two methods are suited to exploring these type of equation. In addition, in place of w, U will now be used as the variable representative of the option cost.

3.1 Forward Euler [7]

The forward Euler method is given by:

$$U_i^{n+1} = U_i^n + \Delta t f(t_n, U_i^n)$$

In the case of the Black-Scholes equation, $f(t_n, U_i^n)$ represents the right hand side of the PDE above. Second order centered discretizations with respect to S will be used to approximate the partial derivatives. Thus, the full discretization of the Black-Scholes equation is:

$$U_i^{n+1} = U_i^n + \Delta t \left(-\frac{1}{2}\sigma^2 S^2 \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta S^2} - rS\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta S} + rU_i^n \right).$$

The local truncation error (LTE) of the method is given by finding the remnant terms from the Taylor series expansion of:

$$LTE = \frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{1}{2}\sigma^2 S^2 \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta S^2} + rS\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta S} - rU_i^n.$$

The Taylor expansions of the elements above is given by:

$$U(t+\Delta t, S) = U_i^n + \Delta t U_t(t,S) + \frac{\Delta t^2}{2} U_{tt}(t,S) + \mathcal{O}(\Delta t^3)$$

$$U(t, S \pm \Delta S) = U_i^n \pm \Delta S U_S(t,S) + \frac{\Delta S^2}{2} U_{SS}(t,S) \pm \frac{\Delta S^3}{6} U_{SSS}(t,S) + \frac{\Delta S^4}{24} U_{SSSS}(t,S) + \mathcal{O}(\Delta S^5).$$

[7] Class notes on Parabolic Equations

So that the discretizations become:

$$\begin{aligned}
\frac{U_i^{n+1} - U_i^n}{\Delta t} &= U_t(t,S) + \frac{\Delta t}{2} U_{tt}(t,S) \\
\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta S^2} &= U_{SS}(t,S) + \frac{\Delta S^2}{12} U_{SSSS}(t,S) \\
\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta S} &= U_S(t,S) + \frac{\Delta S^2}{6} U_{SSS}(t,S).
\end{aligned}$$

Putting them all together in the LTE equation, we find:

$$\begin{aligned}
LTE = U_t(t,S) &+ \frac{\Delta t}{2} U_{tt}(t,S) + \frac{1}{2}\sigma^2 S^2 \left(U_{SS}(t,S) + \frac{\Delta S^2}{12} U_{SSSS}(t,S) \right) \\
&+ rS \left(U_S(t,S) + \frac{\Delta S^2}{6} U_{SSS}(t,S) \right) - rU(t,S) \\
&= \left(U_t(t,S) + \frac{1}{2}\sigma^2 S^2 U_{SS}(t,S) + rSU_S(t,S) - rU(t,S) \right) + \frac{\Delta t}{2} U_{tt}(t,S) \\
&+ \frac{1}{2}\sigma^2 S^2 \frac{\Delta S^2}{12} U_{SSSS}(t,S) + rS\frac{\Delta S^2}{6} U_{SSS}(t,S) \\
&= \frac{\Delta t}{2} U_{tt}(t,S) + \frac{\Delta S^2}{6} \left[\frac{1}{4}\sigma^2 S^2 U_{SSSS}(t,S) + rSU_{SSS}(t,S) \right].
\end{aligned}$$

Given the above LTE, the Euler Method is found to be first order with respect to time and second order with respect to stock price. This turned out as expected, because a first order finite difference in time and a second order finite difference in stock price were used.

Next, the stability of the forward Euler method is examined using Von-Neumann analysis. The main component of Von-Neumann analysis lies in the assumption that the system can be Fourier transformed into a set of algebraic equations. To do so, one uses the following relation:

$$U_j^n = e^{ij\Delta x \xi}.$$

Applying this transform into the discretization of forward Euler, the following is obtained:

$$\begin{aligned}
U_i^{n+1} &= U_i^n + \Delta t \left(-\frac{1}{2}\sigma^2 S^2 \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta S^2} - rS\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta S} + rU_i^n \right) \\
&= U_i^n + \Delta t \left(-\frac{1}{2}\sigma^2 S^2 \frac{e^{i\Delta x\xi}U_i^n - 2U_i^n + e^{-i\Delta x\xi}U_i^n}{\Delta S^2} - rS\frac{e^{i\Delta x\xi}U_i^n - e^{-i\Delta x\xi}U_i^n}{2\Delta S} + rU_i^n \right) \\
&= \left[1 + \Delta t \left(-\frac{1}{2}\sigma^2 S^2 \frac{e^{i\Delta x\xi} - 2 + e^{-i\Delta x\xi}}{\Delta S^2} - rS\frac{e^{i\Delta x\xi} - e^{-i\Delta x\xi}}{2\Delta S} + r \right) \right] U_i^n \\
&= \left[1 + \Delta t \left(-\sigma^2 S^2 \frac{\cos \Delta S\xi - 1}{\Delta S^2} - rS\frac{\cos \Delta S\xi}{\Delta S} + r \right) \right] U_i^n \\
&= (1+z)U_i^n
\end{aligned}$$

where z is defined as

$$\begin{aligned}
z(\xi) &= \Delta t\left(-\sigma^2 S^2 \frac{\cos \Delta S\xi - 1}{\Delta S^2} - rS\frac{\cos \Delta S\xi}{\Delta S} + r\right) \\
&= \Delta t\left(-\frac{\sigma^2 S^2}{\Delta S^2}\cos \Delta S\xi + \frac{\sigma^2 S^2}{\Delta S^2} - \frac{rS}{\Delta S}\cos \Delta S\xi + r\right) \\
&= -\Delta t\left(\frac{\sigma^2 S^2}{\Delta S^2} + \frac{rS}{\Delta S}\right)\cos \Delta S\xi + \sigma^2 S^2 \frac{\Delta t}{\Delta S^2} + r\Delta t.
\end{aligned}$$

Because of the fact that there is a terminal, as opposed to initial conditions, the iterations will run in reverse. Consequently, the amplification factor is given by:

$$g(z) = \frac{1}{1+z}$$

.

For the iteration to be stable, the amplification factor must satisfy $|g(z)| < 1$. As a result, there bounds that z must satisfy are $z > 0$ or $z < -2$. Because the free variable in z appears inside the cosine function, for all ξ, $z(\xi)$ is bounded by:

$$r\Delta t\left(1 - \frac{S}{\Delta S}\right) \leq z(\xi) \leq rS\frac{\Delta t}{\Delta S} + 2\sigma^2 S^2 \frac{\Delta t}{\Delta S^2} + r\Delta t.$$

Combining the two sets of bounds, the following stability conditions are obtained:

$$rS\frac{\Delta t}{\Delta S} + 2\sigma^2 S^2 \frac{\Delta t}{\Delta S^2} + r\Delta t > z > 0$$

$$r\Delta t\left(1 - \frac{S}{\Delta S}\right) < z < -2$$

The first inequality will always be true, given that all parameters are greater than zero. The second inequality will remain true as long as:

$$\frac{S}{\Delta S} - 1 > \frac{2}{r\Delta t}.$$

This condition is stable as long as Δt does not become too small. But there is an additional consideration: since $\frac{S}{\Delta S}$ depends on the stock price and $\frac{S}{\Delta S}$ is equivalent to the grid position, the condition becomes unstable for elements located on the lower end of the grid.

3.2 Crank-Nicholson Method [8]

The Crank-Nicholson method is given by:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{1}{2}(f(U_i^n) + f(U_i^{n+1}))$$

[8] Class notes on Parabolic Equations

In the case of the Black-Scholes equation, $f(t_n, U_i^n)$ represents the right hand side of the PDE above. Second order centered discretizations with respect to S will be used to approximate the partial derivatives. Thus, the full discretization of the Black-Scholes equation is:

$$U_i^{n+1} - \frac{\Delta t}{2} f(U_i^{n+1}) = U_i^n + \frac{\Delta t}{2} f(U_i^n)$$

$$f(U_i^n) = -\frac{1}{2}\sigma^2 S^2 \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta S^2} - rS\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta S} + rU_i^n.$$

The local truncation error (LTE) of the method is given by finding the remnant terms from the Taylor series expansion of:

$$LTE = \frac{U_i^{n+1} - U_i^n}{\Delta t} - \frac{1}{2}(f(U_i^n) + f(U_i^{n+1})).$$

The Taylor expansions of the elements above is given by:

$$\begin{aligned}
U(t+\Delta t, S) &= U(t,S) + \Delta t U_t(t,S) + \frac{\Delta t^2}{2} U_{tt}(t,S) + \frac{\Delta t^3}{6} U_{ttt}(t,S) + \frac{\delta t^4}{24} U_{tttt}(t,S) + \mathcal{O}(\Delta t^5) \\
U(t, S \pm \Delta S) &= U(t,S) \pm \Delta S U_S(t,S) + \frac{\Delta S^2}{2} U_{SS}(t,S) \pm \frac{\Delta S^3}{6} U_{SSS}(t,S) + \frac{\Delta S^4}{24} U_{SSSS}(t,S) + \mathcal{O}(\Delta S \\
U(t+\Delta t, S \pm \Delta S) &= U(t,S) + \Delta t U_t(t,S) \pm \Delta S U_S(t,S) + \frac{1}{2}\left(\Delta t^2 U_{tt}(t,S) \pm 2\Delta t \Delta S U_{St}(t,S) + \Delta S^2 U_{SS}(t,\right. \\
&\quad + \frac{1}{6}\left(\Delta t^3 U_{ttt}(t,S) \pm 3\Delta t^2 \Delta S U_{ttS}(t,S) + 3\Delta t \Delta S^2 U_{tSS}(t,S) \pm \frac{\Delta S^3}{6} U_{SSS}(t,S)\right) \\
&\quad + \frac{1}{24}\left(\Delta t^4 U_{tttt}(t,S) \pm 4\Delta t^3 \Delta S U_{tttS}(t,S) + 6\Delta t^2 \Delta S^2 U_{ttSS}(t,S)\right. \\
&\quad \left.\pm 4\Delta t \Delta S^3 U_{tSSS}(t,S) + \Delta S^4 U_{SSSS}(t,S)\right).
\end{aligned}$$

So that the discretizations become:

$$\begin{aligned}
\frac{U_i^{n+1} - U_i^n}{\Delta t} &= U_t(t,S) + \frac{\Delta t}{2} U_{tt}(t,S) + \frac{\Delta t^2}{6} U_{ttt}(t,S) \\
\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta S^2} &= U_{SS}(t,S) + \frac{\Delta S^2}{12} U_{SSSS}(t,S) \\
\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta S} &= U_S(t,S) + \frac{\Delta S^2}{6} U_{SSS}(t,S) \\
\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta S^2} &= U_{SS}(t,S) + \Delta t U_{tSS}(t,S) + \frac{\Delta t^2}{2} U_{ttSS}(t,S) + \frac{\Delta S^2}{12} U_{SSSS}(t,S) \\
\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2\Delta S} &= U_S(t,S) + \Delta t U_{tS}(t,S) + \frac{\Delta t^2}{2} U_{ttS}(t,S) + \frac{\Delta S^2}{6} U_{SSS}(t,S)
\end{aligned}$$

Putting them all together in the LTE equation, we find:

$$
\begin{aligned}
LTE = {} & U_t(t,S) + \frac{\Delta t}{2}U_{tt}(t,S) + \frac{\Delta t^2}{6}U_{ttt}(t,S) \\
& + \frac{1}{2}\left[\frac{1}{2}\sigma^2 S^2 \left(U_{SS}(t,S) + \frac{\Delta S^2}{12}U_{SSSS}(t,S)\right) + rS\left(U_S(t,S) + \frac{\Delta S^2}{6}U_{SSS}(t,S)\right) - rU(t,S)\right] \\
& + \frac{1}{2}\left[\frac{1}{2}\sigma^2 S^2 \left(U_{SS}(t,S) + \Delta t U_{tSS}(t,S) + \frac{\Delta t^2}{2}U_{ttSS} + \frac{\Delta S^2}{12}U_{SSSS}(t,S)\right)\right. \\
& + rS\left(U_S(t,S) + \Delta t U_{tS}(t,S) + \frac{\Delta t^2}{2}U_{ttS}(t,S) + \frac{\Delta S^2}{6}U_{SSS}(t,S)\right) \\
& \left. - r\left(U(t,S) + \Delta t U_t(t,S) + \frac{\Delta t^2}{2}U_{tt}(t,S)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
LTE = {} & \left(U_t(t,S) + \frac{1}{2}\sigma^2 S^2 U_{SS}(t,S) + rSU_S(t,S) - rU(t,S)\right) \\
& + \frac{\Delta t}{2}U_{tt}(t,S) + \frac{\Delta t^2}{6}U_{ttt}(t,S) + \frac{1}{2}\left[\frac{1}{2}\sigma^2 S^2 \frac{\Delta S^2}{12}U_{SSSS}(t,S) + rS\frac{\Delta S^2}{6}U_{SSS}(t,S)\right] \\
& + \frac{1}{2}\left[\frac{1}{2}\sigma^2 S^2 \left(\Delta t U_{tSS}(t,S) + \frac{\Delta t^2}{2}U_{ttSS}(t,S) + \frac{\Delta S^2}{12}U_{SSSS}(t,S)\right)\right. \\
& \left. + rS\left(\Delta t U_{tS}(t,S) + \frac{\Delta t^2}{2}U_{ttS}(t,S) + \frac{\Delta S^2}{6}U_{SSS}(t,S)\right) - r\left(\Delta t U_t(t,S) + \frac{\Delta t^2}{2}U_{tt}(t,S)\right)\right] \\
= {} & \frac{\Delta t}{2}\left[U_{tt}(t,S) + \frac{1}{2}\sigma^2 S^2 U_{tSS}(t,S) + rSU_{tS}(t,S) - rU_t(t,S)\right] \\
& + \frac{\Delta t^2}{2}\left[\frac{1}{3}U_{ttt}(t,S) + \frac{1}{4}\sigma^2 S^2 U_{ttSS}(t,S) + \frac{1}{2}rSU_{ttS} - \frac{1}{2}rU_{tt}(t,S)\right] \\
& + \frac{\Delta S^2}{12}\left[\frac{1}{2}\sigma^2 S^2 U_{SSSS}(t,S) + 2rSU_{SSS}(t,S)\right] \\
= {} & \frac{\Delta t^2}{2}\left[\frac{1}{3}U_{ttt}(t,S) + \frac{1}{4}\sigma^2 S^2 U_{ttSS}(t,S) + \frac{1}{2}rSU_{ttS}(t,S) - \frac{1}{2}rU_{tt}(t,S)\right] \\
+ {} & \frac{\Delta S^2}{12}\left[\frac{1}{2}\sigma^2 S^2 U_{SSSS}(t,S) + 2rSU_{SSS}(t,S)\right].
\end{aligned}
$$

For the last step, the Δt term equals zero, as this term is the partial time derivative of the original PDE, which itself equals zero. Given the above LTE, the Crank-Nicholson methods is found to be second order with respect to both time and stock price. This turned out as anticipated, since Crank-Nicholson is meant to produce second order accurate results with regard to time.

Next, the stability of the Crank-Nicholson method is examined using Von-Neumann analysis. The main component of Von-Neumann analysis lies in the assumption that the system can be Fourier transformed into a set of algebraic equations. To do so, one uses the following relation:

$$U_j^n = e^{ij\Delta x\xi}.$$

Applying this transform to the discretization $f(U_i^n)$:

$$\begin{aligned}
f(U_i^n) &= -\frac{1}{2}\sigma^2 S^2 \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta S^2} - rS\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta S} + rU_i^n \\
&= \left[-\sigma^2 S^2 \frac{\cos \Delta x\xi - 1}{\Delta S^2} - rS\frac{\cos \Delta x\xi}{\Delta S} + r\right] U_i^n \\
&= \frac{z}{\Delta t} U_i^n,
\end{aligned}$$

where

$$\begin{aligned}
z(\xi) &= \Delta t \left[-\sigma^2 S^2 \frac{\cos \Delta x\xi - 1}{\Delta S^2} - rS\frac{\cos \Delta x\xi}{\Delta S} + r\right] \\
&= -\Delta t \left(\frac{\sigma^2 S^2}{\Delta S^2} + \frac{rS}{\Delta S}\right) \cos \Delta S\xi + \sigma^2 S^2 \frac{\Delta t}{\Delta S^2} + r\Delta t.
\end{aligned}$$

Putting this together with the Crank-Nicholson discretization:

$$\begin{aligned}
U_i^{n+1} - \frac{\Delta t}{2} f(U_i^{n+1}) &= U_i^n + \frac{\Delta t}{2} f(U_i^{n+1}) \\
U_i^{n+1} - \frac{z}{2} U_i^{n+1} &= U_i^n + \frac{z}{2} U_i^n \\
U_i^{n+1} &= \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}} U_i^n
\end{aligned}$$

Once again, since time condition is terminal, the amplification factor is given by:

$$g(z) = \frac{1 - \frac{z}{2}}{1 + \frac{z}{2}}.$$

For the amplification factor to satisfy $|g(z)| < 1$, then z must be bounded by $z > 0$. For all ξ, $z(\xi)$ is bounded by:

$$r\Delta t \left(1 - \frac{S}{\Delta S}\right) \leq z(\xi) \leq rS\frac{\Delta t}{\Delta S} + 2\sigma^2 S^2 \frac{\Delta t}{\Delta S^2} + r\Delta t.$$

Combining the two conditions together implies that the method will remain stable as long as the following is true

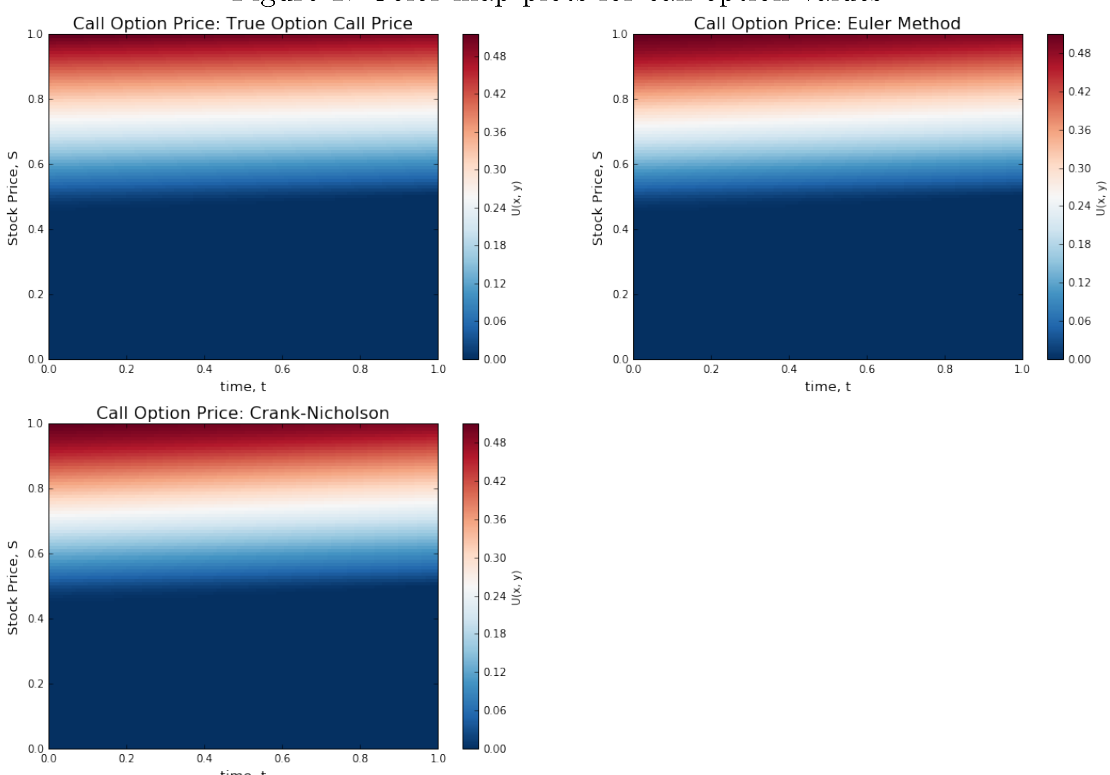
$$0 < z(\xi) \leq rS\frac{\Delta t}{\Delta S} + 2\sigma^2 S^2 \frac{\Delta t}{\Delta S^2} + r\Delta t.$$

4 Convergence Testing [9]

Before beginning the discussion on the convergence of the Forward Euler and the Crank-Nicholson Methods, three things must be pointed out. First, only the testing for call options will be discussed and analyzed, given that the there is no real difference between testing one or the other. Second, Black-Scholes formula is treated as the true solution and is the point of comparison for finding the error of each method. Third, the functions used to build the Forward-Euler and Crank-Nicholson estimations were written to accept two boundary conditions at the extrema of the stock price; the boundary conditions were provided by the valuation of the Black-Scholes formula at these edges. This contradicts the fact that there is only one real boundary condition for the problem. However, the additional boundary is necessary in order to perform the methods.

To begin, the color-map plots for the true and estimated call option price values are displayed in figure 1, in order to qualitatively demonstrate the ability of both methods to find the true solution under 'good' conditions. These conditions are defined as $r = 0$, $\sigma = 0.05$, $K = 0.5$ and $T = 1.0$ over the domain $S = [0.0, 1.0] \times t = [0, 1.0]$ with 100 partitions for each direction. These conditions produce good estimates and hence do not breach the stability conditions. When these values are taken to extremes, both methods become highly unstable and are not suitable for convergence testing.

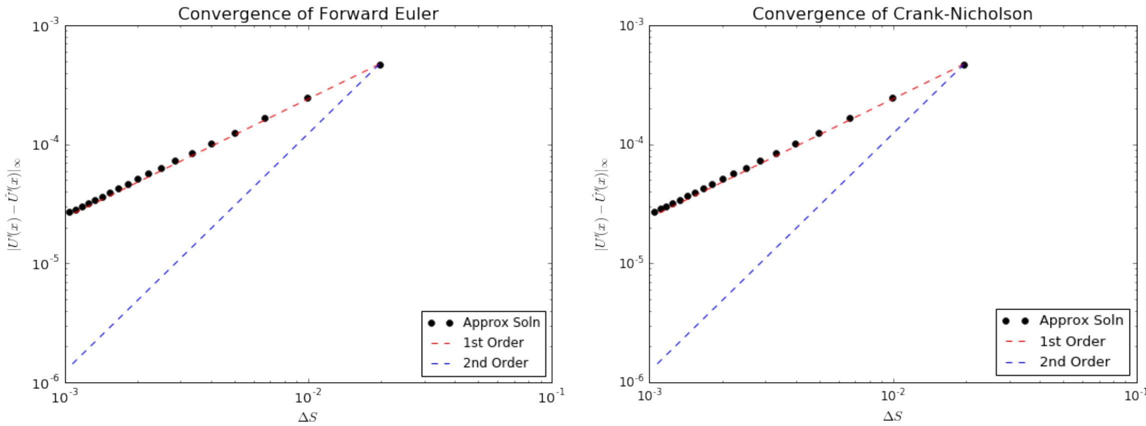
Figure 1: Color-map plots for call-option values



[9]The code used to test the convergence of the methods can be found at https://github.com/papifane/APAM-4301—Final-Project

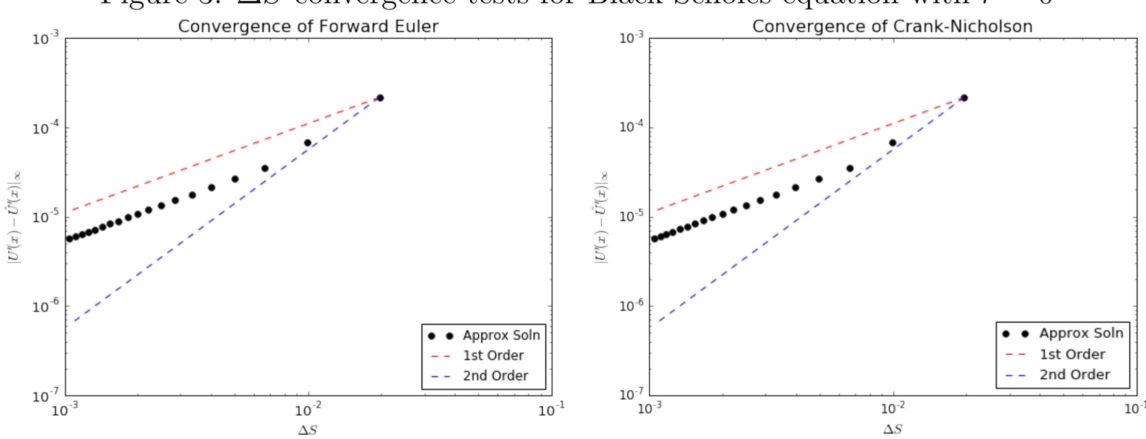
Using the above conditions, convergence tests for the ΔS parameter were done and are shown below in figure 2. Evidently, the plots demonstrate that both methods converge as $\mathcal{O}(\Delta S)$. This is indeed an odd result, given that both methods are expected to converge as $\mathcal{O}(\Delta S^2)$.

Figure 2: ΔS convergence tests for full Black-Scholes equation



The reason for this discrepancy may in fact lie in the limited stability of the methods themselves. Evidence for this hypothesis is evident in figure 3, which shows the convergence test of the methods in the limit that the interest rate is set to zero. Looking at the Black Scholes equation, setting the interest rate to zero results in the traditional heat equation and both methods simplify and solve for the heat equation. For both methods, two convergence rates are visible and imply that for larger ΔS the methods converge as $(O)\Delta S^2)$ while for smaller ΔS they converge as $\mathcal{O}(\Delta S)$. Because of this rate change, it is evident that the system has undergone some sort of transformation in its stability regime.

Figure 3: ΔS convergence tests for Black-Scholes equation with $r = 0$



5 Conclusion

Given that overall the two Forward Euler and Crank-Nicholson Methods described above failed to converge at their expected rates and are not very versatile in terms of the parameters under which they are useful, other ways of solving the Black-Scholes equation should be devised. The first and most obvious suggestion is to explore the use of methods designed to be used backwards such as the Backwards Euler method. The second suggestion is to attempt using the transformed PDE:

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2},$$

which was found on the way to solve for the Black-Scholes formula in section 2.3. The numerical methods for this PDE are easier to implement and their stability conditions simpler to solve for. In addition, the transformation converts the original terminal condition into an initial condition, which implies that familiar forward methods are convenient to use.