# A goal-oriented approach to classical negation.

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joint work with
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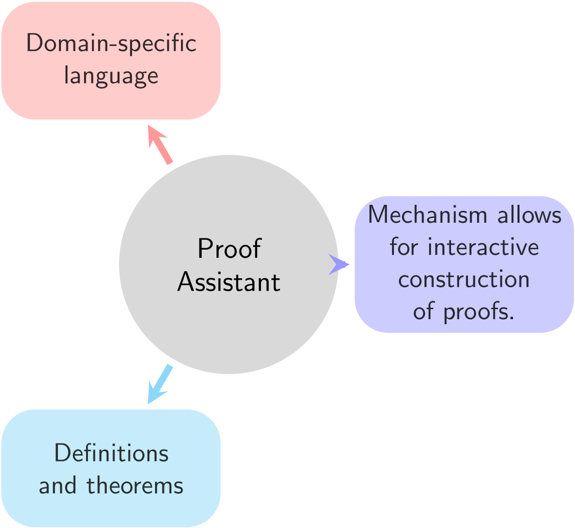
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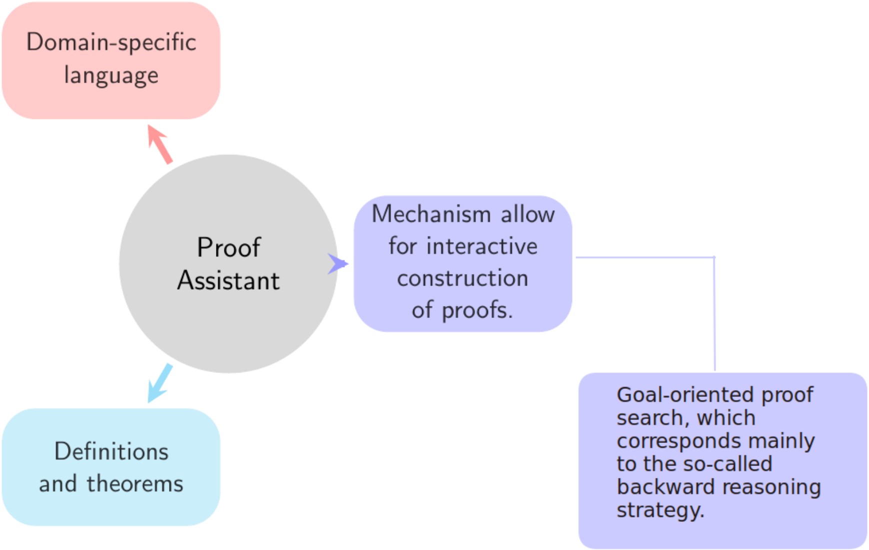
Proof Assistant
Mechanism allows for interactive construction of proofs.
Definitions and theorems


Domain-specific language
Proof Assistant
Mechanism allow for interactive construction of proofs.
Goal-oriented proof search, which corresponds mainly to the so-called backward reasoning strategy.
Definitions and theorems

**Forward reasoning**: To show that $\Gamma \vdash B$ is derivable, we start with axioms and work forwards towards the desired judgement.

$$\begin{array}{ccc}
 & \boldsymbol{\Gamma_1, A \vdash A} & \\
\text{Forward}\ \Bigg\downarrow & \vdots & \Bigg\uparrow\ \text{Backward} \\
 & \boldsymbol{\Gamma_i \vdash A_i} & \\
 & \vdots & \\
 & \boldsymbol{\Gamma \vdash B} &
\end{array}$$

**Backward reasoning**: To show that $\Gamma \vdash B$ is derivable, we start by analyzing the judgement and work backwards towards the axioms.

# Heuristics

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma,\ H1 : A \vdash C \qquad \Gamma,\ H2 : B \vdash C}{\Gamma \vdash C}\ (\vee\ \mathrm{E})$$

To prove $\Gamma, H : A \vee B \vdash C$ it suffices to prove

$$\Gamma, H1 : A \vdash C \quad \text{and} \quad \Gamma, H2 : B \vdash C.$$

$$\frac{\Gamma,\ H1 : A \vdash C \qquad \Gamma,\ H2 : B \vdash C}{\Gamma,\ H : A \vee B \vdash C}\ (\vee\ \mathrm{P})$$

`destruct`: $\Gamma, H : A \vee B \vdash C; \mathcal{S} \quad \triangleright \quad \Gamma, H1 : A \vdash C\ \mathbf{;}\ \Gamma, H2 : B \vdash C\ \mathbf{;}\ \mathcal{S}$

## Heuristics

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$$\frac{\Gamma,\ H1 : A \vdash C \qquad \Gamma,\ H2 : B \vdash C}{\Gamma,\ H : A \vee B \vdash C}\ (\vee\ \mathrm{P})$$

### Tactic

`destruct`: $\Gamma, H : A \vee B \vdash C; \mathcal{S} \quad \triangleright \quad \Gamma, H1 : A \vdash C\ ; \Gamma, H2 : B \vdash C\ ; \mathcal{S}$

Our proposed system is depicted as follows:

$$\frac{}{\Gamma, H : A \vdash A}\ (\textsc{Hyp})$$

$$\frac{\Gamma, H : A \vdash B}{\Gamma \vdash A \to B}\ (\to \mathrm{I}) \qquad \frac{\Gamma,\ H : A \to B \vdash A}{\Gamma, H : A \to B \vdash B}\ (\to \mathrm{P})$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge\ \mathrm{I}) \qquad \frac{\Gamma,\ H1 : A,\ H2 : B \vdash C}{\Gamma,\ H : A \wedge B \vdash C}\ (\wedge\ \mathrm{P})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ (\vee\ \mathrm{I}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ (\vee\ \mathrm{I})$$

$$\frac{\Gamma,\ H1 : A \vdash C \qquad \Gamma,\ H2 : B \vdash C}{\Gamma,\ H : A \vee B \vdash C}\ (\vee\ \mathrm{P})$$

$$\frac{\Gamma \vdash A \quad x \notin FV(\Gamma)}{\Gamma \vdash \forall x A}\ (\forall\ \mathrm{I}) \qquad \frac{}{\Gamma,\ H : \forall x A \vdash A[x := t]}\ (\forall\ \mathrm{P})$$

$$\frac{\Gamma \vdash A[x := t]}{\Gamma \vdash \exists x A}\ (\exists\ \mathrm{I}) \qquad \frac{\Gamma,\ H : A \vdash C \quad x \notin FV(\Gamma, C)}{\Gamma,\ H : \exists x A \vdash C}\ (\exists\ \mathrm{P})$$

The **transition system of tactics** $\mathcal{T}$ is defined by the following rules, called **tactics**:

- **Inversion of Introduction Rules :**
  - `intro`: $\Gamma \vdash A \to B; \mathcal{S} \rhd \Gamma, A \vdash B; \mathcal{S}$
  - `split`: $\Gamma \vdash A \wedge B; \mathcal{S} \rhd \Gamma \vdash A; \Gamma \vdash B; \mathcal{S}$
  - `left`: $\Gamma \vdash A \vee B; \mathcal{S} \rhd \Gamma \vdash A; \mathcal{S}$
  - `right`: $\Gamma \vdash A \vee B; \mathcal{S} \rhd \Gamma \vdash B; \mathcal{S}$
  - `intro`: $\Gamma \vdash \forall x A; \mathcal{S} \rhd \Gamma \vdash A; \mathcal{S}$ where w.l.o.g., $x \notin FV(\Gamma)$
  - `exists`: $\Gamma \vdash \exists x A; \mathcal{S} \rhd \Gamma \vdash A[x := t]; \mathcal{S}$

- **Inversion of P Rules:**
  - apply: $\Gamma, A \to B \vdash B; \mathcal{S} \ \triangleright \Gamma, A \to B \vdash A; \mathcal{S}$
  - destruct: $\Gamma, A \wedge B \vdash C; \mathcal{S} \triangleright \Gamma, A, B \vdash C; \mathcal{S}$
  - destruct: $\Gamma, A \vee B \vdash C; \mathcal{S} \triangleright \Gamma, A \vdash C; \Gamma, B \vdash C; \mathcal{S}$
  - destruct: $\Gamma, \exists x A \vdash C; \mathcal{S} \ \triangleright \Gamma, A \vdash C; \mathcal{S}$ where w.l.o.g. $x \notin FV(\Gamma)$

- **Discarding tactics:**
  - trivial: $\Gamma, A \vdash A; \mathcal{S} \triangleright \mathcal{S}$.
  - apply: $\Gamma, \forall x A \vdash A[x := t]; \mathcal{S} \triangleright \mathcal{S}$

**Theorem (Equivalence of $\vdash$ and $\triangleright^{+}$)**

*For any sequent* $\Gamma \vdash A$

$$\Gamma \vdash A \triangleright^{+} \square \quad \textit{if and only if} \quad \Gamma \vdash A \textit{ is provable.}$$

# Classical logic

$$\textit{Classical Logic} = \textit{Intuitionistic Logic} + \begin{cases} \textit{excluded middle} \\ \textit{elimination of double negation} \\ \textit{reductio ad absurdum} \end{cases}$$

Our purpose is to develop an interactive proof-search methodology for the kind of classical logic employed in daily mathematical reasoning, where a contradiction must be shown in an explicit way and not by deriving the $\perp$ constant.

Thus, we consider classical logic as the extension of minimal logic with the following rule of proof by contradiction:

$$\frac{\Gamma, \neg A \vdash B \qquad \Gamma, \neg A \vdash \neg B}{\Gamma \vdash A}\ (\mathrm{RAA})$$

# Classical logic

In order to make the proof construction process easier, it is convenient to have more rules about negation at hand.

The following rules are admissible:

$$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A}\ (\mathrm{I}\neg\neg) \qquad \frac{\Gamma \vdash \neg A \qquad \Gamma \vdash \neg B}{\Gamma \vdash \neg(A \vee B)}\ (\mathrm{I}\neg\vee) \qquad \frac{\Gamma \vdash \neg A \vee \neg B}{\Gamma \vdash \neg(A \wedge B)}\ (\mathrm{I}\neg\wedge)$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash \neg B}{\Gamma \vdash \neg(A \to B)}\ (\mathrm{I}\neg\to) \qquad \frac{\Gamma \vdash \forall x \neg A}{\Gamma \vdash \neg\exists x A}\ (\mathrm{I}\neg\exists) \qquad \frac{\Gamma \vdash \exists x \neg A}{\Gamma \vdash \neg\forall x A}\ (\mathrm{I}\neg\forall)$$

# Classical logic

The following rules are admissible:

$$\frac{\Gamma, A \vdash B}{\Gamma, \neg\neg A \vdash B}\ (\mathrm{P}\neg\neg) \qquad \frac{\Gamma, \neg A \vee \neg B \vdash C}{\Gamma, \neg(A \wedge B) \vdash C}\ (\mathrm{P}\neg\wedge)$$

$$\frac{\Gamma, \neg A,\ \neg B \vdash C}{\Gamma, \neg(A \vee B) \vdash C}\ (\mathrm{P}\neg\vee) \qquad \frac{\Gamma, A,\ \neg B \vdash C}{\Gamma, \neg(A \to B) \vdash C}\ (\mathrm{P}\neg\to)$$

$$\frac{\Gamma, \exists x \neg A \vdash C}{\Gamma, \neg\forall x A \vdash C}\ (\mathrm{P}\neg\forall) \qquad \frac{\Gamma, \forall x \neg A \vdash C}{\Gamma, \neg\exists x A \vdash C}\ (\mathrm{P}\neg\exists)$$

# Classical logic

Schemes for mathematical reasoning are useful but they do not belong to the traditional natural deduction systems, forcing us to reason about negation artificially.

To ease this issue we extend our system with some admissible rules :

$$\frac{\Gamma, \neg A \vdash B}{\Gamma, \neg B \vdash A}\ (\textsc{cp}3) \qquad \frac{\Gamma, \neg B \vdash \neg A}{\Gamma, A \vdash B}\ (\textsc{cp}4) \qquad \frac{\Gamma,\ \neg A \vdash B}{\Gamma \vdash A \vee B}\ (\mathrm{C}{\vee}\mathrm{I})$$

$$\frac{\Gamma,\ A \vdash B}{\Gamma \vdash \neg A \vee B}\ (\mathrm{C}{\vee}\mathrm{I}) \qquad \frac{\Gamma,\ A \vdash B \qquad \Gamma,\ \neg A \vdash B}{\Gamma \vdash B}\ (\textsc{Dich})$$

$$\frac{\Gamma,\ A \vee B \vdash \neg A}{\Gamma,\ A \vee B \vdash B}\ \text{DS-P} \qquad \frac{\Gamma,\ \neg A \vee B \vdash A}{\Gamma,\ \neg A \vee B \vdash B}\ \text{DS-P} \qquad \frac{\Gamma,\ A \to B \vdash \neg B}{\Gamma,\ A \to B \vdash \neg A}\ \text{MT-P}$$

# Classical logic

**Tactics for classical negation**

- `RAA (A)`: $\Gamma \vdash B\,;\,\mathcal{S} \rhd \Gamma,\ H:\neg B \vdash A;\ \Gamma,\ H:\neg B \vdash \neg A\,;\mathcal{S}$

Inversion of Introduction rules:

- `neg_neg:` $\Gamma \vdash \neg\neg A\,;\,\mathcal{S} \rhd \Gamma \vdash A;\mathcal{S}$
- `neg_and:` $\Gamma \vdash \neg(A \land B)\,;\,\mathcal{S} \rhd \Gamma \vdash \neg A \lor \neg B;\mathcal{S}$
- `neg_or:` $\Gamma \vdash \neg(A \lor B)\,;\,\mathcal{S} \rhd \Gamma \vdash \neg A\,;\ \Gamma \vdash \neg B;\mathcal{S}$
- `neg_imp:` $\Gamma \vdash \neg(A \to B)\,;\,\mathcal{S} \rhd \Gamma \vdash A\,;\ \Gamma \vdash \neg B;\mathcal{S}$
- `neg_ex`: $\Gamma \vdash \neg\exists x A\,;\,\mathcal{S} \rhd \Gamma \vdash \forall x \neg A;\mathcal{S}$
- `neg_fa`: $\Gamma \vdash \neg\forall x A\,;\,\mathcal{S} \rhd \Gamma \vdash \exists x \neg A;\mathcal{S}$

# Classical logic

**Tactics for classical negation**

Inversion of P-rules:

- `destruct_neg H:` $\Gamma, H : \neg\neg A \vdash B\,;\, \mathcal{S} \triangleright \Gamma, H : A \vdash B; \mathcal{S}$
- `destruct_neg H:` $\Gamma, H : \neg(A \wedge B) \vdash C\,;\, \mathcal{S} \triangleright \Gamma, H : \neg A \vee \neg B \vdash C; \mathcal{S}$
- `destruct_neg H:`
  $\Gamma, H : \neg(A \vee B) \vdash C\,;\, \mathcal{S} \triangleright \Gamma, H : \neg A, H1 : \neg B \vdash C; \mathcal{S}$
- `destruct_neg H:`
  $\Gamma, H : \neg(A \to B) \vdash C\,;\, \mathcal{S} \triangleright \Gamma, H : A, H1 : \neg B \vdash C; \mathcal{S}$
- `destruct_neg H:` $\Gamma, H : \neg\forall x A \vdash C\,;\, \mathcal{S} \triangleright \Gamma, H : \exists x \neg A \vdash C; \mathcal{S}$
- `destruct_neg H:` $\Gamma, H : \neg\exists x A \vdash C\,;\, \mathcal{S} \triangleright \Gamma, H : \forall x \neg A \vdash C; \mathcal{S}$

# Classical logic

**Tactics for classical negation**

Mathematical rules:

- `cdisj H:` $\Gamma \vdash A \vee B\,;\mathcal{S} \triangleright \Gamma, H : \neg A \vdash B; \mathcal{S}$
- `cdisj H:` $\Gamma \vdash \neg A \vee B\,;\mathcal{S} \triangleright \Gamma, H : A \vdash B; \mathcal{S}$
- `contrapositive H:` $\Gamma, H : \neg B \vdash A\,;\mathcal{S} \triangleright \Gamma, H : \neg A \vdash B; \mathcal{S}$
- `contrapositive H:` $\Gamma, H : B \vdash A\,;\mathcal{S} \triangleright \Gamma, H : \neg A \vdash \neg B; \mathcal{S}$
- `dichotomy (A)`: $\Gamma \vdash B\,;\mathcal{S} \triangleright \Gamma,\ H : A \vdash B;\ \Gamma,\ H : \neg A \vdash \neg B; \mathcal{S}$
- `disjsyll H`: $\Gamma,\ H : A \vee B \vdash B\,;\mathcal{S} \triangleright \Gamma,\ H : A \vee B \vdash \neg A; \mathcal{S}$
- `disjsyll H`: $\Gamma,\ H : \neg A \vee B \vdash B\,;\mathcal{S} \triangleright \Gamma,\ H : \neg A \vee B \vdash A; \mathcal{S}$
- `mtapply H`: $\Gamma,\ H : A \rightarrow B \vdash \neg A\,;\mathcal{S} \triangleright \Gamma,\ H : A \rightarrow B \vdash \neg B; \mathcal{S}$

Let $\Gamma = \{H : \neg(p \to \neg\neg(q \to \neg r))\}$. The following is a *derivation by tactics* of $\Gamma \vdash \neg(\neg(p \to \neg q) \to \neg r)$ :

$H : \neg(p \to \neg\neg(q \to \neg r)) \vdash \neg(\neg(\mathbf{p} \to \neg\mathbf{q}) \to \neg\mathbf{r})$ ▷ `contrapositive`

$H : \neg(p \to \neg q) \to \neg r \vdash \mathbf{p} \to \neg\neg(\mathbf{q} \to \neg\mathbf{r})$ ▷ `intro H1`

$H : \neg(p \to \neg q) \to \neg r,\ H_1 : p \vdash \neg\neg(\mathbf{q} \to \neg\mathbf{r})$ ▷ `neg_neg`

$H : \neg(p \to \neg q) \to \neg r,\ H_1 : p \vdash \mathbf{q} \to \neg\mathbf{r}$ ▷ `intro H2`

$H : \neg(p \to \neg q) \to \neg r,\ H_1 : p,\ H_2 : q \vdash \neg\mathbf{r}$ ▷ `apply H`

$H : \neg(p \to \neg q) \to \neg r,\ H_1 : p,\ H_2 : q \vdash \neg(\mathbf{p} \to \neg\mathbf{q})$ ▷ `neg_imp`

$H : \neg(p \to \neg q) \to \neg r,\ H_1 : p,\ H_2 : q \vdash \mathbf{p}\ ;$

$H : \neg(p \to \neg q) \to \neg r,\ H_1 : p,\ H_2 : q \vdash \neg\neg\mathbf{q}$ ▷ `trivial`

$H : \neg(p \to \neg q) \to \neg r,\ H_1 : p,\ H_2 : q \vdash \neg\neg\mathbf{q}$ ▷ `neg_neg`

$H : \neg(p \to \neg q) \to \neg r,\ H_1 : p,\ H_2 : q \vdash \mathbf{q}$ ▷ `trivial`

$\square$

## Closing Remarks

- The motivation of this work came out from the obstacles we found while trying to verify, with teaching aims, some logical arguments and simple mathematical proofs involving classical negation in the Coq proof assistant.
- Classical negation must be treated as a primitive operator and not as the usual constructive definition.
- We have extended our previous goal-oriented Natural Deduction system with suitable rules for classical logic; the inversion of these inference rules yields a transition system which results equivalent to the natural deduction system for classical logic.
- The next step in our project is to implement our transition system as a Coq library for classical logic, one that would be more adequate to perform classical logical reasoning than those already available.

**Gracias.**