

Family Name Name Student ID (Matricola):

Solve the problems adding to the replies short and essential explanations. Please write the solutions in the designed areas. NO EXTRA SHEETS WILL BE ACCEPTED. 1 Problem = 4 marks. Duration: 2 hours. No questions allowed in the first hour and in the last 20 minutes.

1	2	3	4	5	6	7	8	TOTAL

1. Compute $\gcd(1380, 1110)$ using the Extended Euclidean Algorithm and deduce a Bezout Identity.
2. Compute the 7-adic valuation $v_7(100!)$.
3. Let μ be the Möbius function and denote by $*$ the Dirichlet convolution of arithmetic functions. Prove that k -folded iterated convolution of μ satisfies:

$$(\mu * \mu * \cdots * \mu)(n) = \prod_p \binom{k}{v_p(n)} (-1)^{v_p(n)}$$

where for $a \in \mathbf{Z}$ and $b \in \mathbf{N}$, $\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b!}$ is the binomial coefficient.
(suggestion: try first to prove the formula for $k = 2, 3, \dots$)

4. After having stated Gauss Theorem of existence of primitive roots modulo integers, compute all primitive roots modulo 686.
5. Find all integers X in the interval $[-10, 200]$ such that
$$\begin{cases} X \equiv 3 \pmod{4} \\ X \equiv 2 \pmod{5} \\ X \equiv 4 \pmod{7} \end{cases}$$
6. After having stated the important properties of the Legendre–Jacobi Symbols, compute $\left(\frac{3073}{2919}\right)_J$.

7. Prove that

$$\left(\frac{-7}{p}\right)_J = \begin{cases} 1 & \text{if } p \equiv 1, 2, 4 \pmod{7} \\ 0 & \text{if } p = 7 \\ -1 & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

8. Let $p \geq 3$ be a prime and let $k \in \mathbf{N}$. Prove that
 - i) the equation $X^k \equiv 1 \pmod{p}$ has $\gcd(k, p-1)$ solutions,
 - ii) the equation $X^k \equiv 1 \pmod{p^\alpha}$ has $\gcd(k, p-1)$ solutions if $p \nmid k$.