

On the Equation $\tau(\lambda(n)) = \omega(n) + k$

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1. Introduction

For every positive integer n , the function $\tau(n)$ counts the number of divisors of n , the function $\omega(n)$ counts the number of distinct prime divisors of n , and the Carmichael function $\lambda(n)$ is the exponent of the multiplicative group of the invertible congruence classes modulo n . The value of the function $\lambda(n)$ can be computed as follows:

$$\lambda(n) = \begin{cases} 1 & \text{if } n = 1; \\ 2^{\alpha-2} & \text{if } n = 2^\alpha, \alpha > 2; \\ p^{\alpha-1}(p-1) & \text{if } n = p^\alpha \text{ and } \begin{array}{l} p \geq 3 \text{ or} \\ p = 2, \alpha \leq 2; \end{array} \\ [\lambda(p_1^{\alpha_1}), \dots, \lambda(p_s^{\alpha_s})] & \text{if } n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}. \end{cases}$$

In [6], Erdős, Pomerance, and Schmutz proved a number of fundamental properties of λ . In the process of proving the lower bound $\lambda(n) > (\log n)^{c_0 \log \log \log n}$ for all large n (provided $c_0 < 1/\log 2$), they proved the inequality

$$n \leq (4\lambda(n))^{3\tau(\lambda(n))}.$$

Numerical calculations suggest that the stronger inequality

$$n \leq \lambda(n)^{\tau(\lambda(n))} \tag{1}$$

holds except for $n = 2, 6, 8, 12, 24, 80, 120, 240$. This will be proved in Corollary 1. One of the tools for proving (1) is the inequality $\tau(\lambda(n)) > \omega(n)$, which holds except for $n = 2, 6, 12, 24, 30, 60, 120, 240$; we will prove this in Proposition 1 and Proposition 2.

This motivates us to compare $\tau(\lambda(n))$ with $\omega(n)$. Since $\tau(\lambda(n)) \geq \omega(n)$ holds for all positive integers n (see Proposition 1), we can write $\tau(\lambda(n)) = \omega(n) + k$, where k is some nonnegative integer depending on n . We then fix $k \geq 0$ and investigate the positive integers n such that $\tau(\lambda(n)) = \omega(n) + k$.

Throughout this paper, we use x to denote a positive real number. We also use the Landau symbols O and o and the Vinogradov symbols \gg and \ll with their

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usual meanings. We write $\log x$ for the maximum between 1 and the natural logarithm of x . For a set \mathcal{A} of positive integers we write $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. We write p and q with or without subscripts for prime numbers.

Let us set

$$\mathcal{A}_k = \{n : \tau(\lambda(n)) = \omega(n) + k\}.$$

We will show in Theorem 3 that if k is a positive integer and

$$b_k = 2(k+1)^2 + 3 + \lfloor \log_2(2(k+1)^2 + k+1) \rfloor,$$

then the upper bound

$$\#\mathcal{A}_k(x) \ll_k \frac{x(\log \log x)^{b_k}}{(\log x)^2}$$

holds as $x \rightarrow \infty$. Here, $\log_2 a$ stands for the base 2 logarithm of the positive number a . Furthermore, in Theorem 2, we will show that if $k > 4$ then the lower bound

$$\#\mathcal{A}_k(x) \gg_k \frac{x}{(\log x)^2}$$

holds as $x \rightarrow \infty$. We will also give complete descriptions of the sets \mathcal{A}_0 , \mathcal{A}_1 , and \mathcal{A}_2 (Proposition 2, Proposition 3, and Proposition 4). We will show that \mathcal{A}_0 contains eight integers and that the infiniteness of \mathcal{A}_1 and \mathcal{A}_2 would follow if it were known that there exist infinitely many primes of the form $2q+1$ with q also prime. Finally, in Proposition 5 we deal with the cases $k=3, 4$ and prove that, if either \mathcal{A}_3 or \mathcal{A}_4 are infinite, then there exists an even positive integer c such that the set of primes of the form $p = cq^\beta + 1$ (with q prime and $\beta \leq 4$) is infinite. This explains the difficulty of proving the infiniteness of \mathcal{A}_k for $k=1, 2, 3, 4$.

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2. Determining \mathcal{A}_k for Small Values of k

PROPOSITION 1. *For any positive integer n , we have*

$$\tau(\lambda(n)) \geq \omega(n).$$

More precisely,

$$\tau(\lambda(n)) \geq \omega(n/(2^\infty, n)) + \tau(\lambda^o(n')),$$

where n' is the product of the primes dividing n and where $\lambda^o(m)$ denotes the odd part of $\lambda(m)$. That is, $\lambda^o(m) = \lambda(m)/(2^\infty, \lambda(m))$.

Proof. Let us first note that, if $n \mid m$, then $\lambda(n) \mid \lambda(m)$ and therefore $\tau(\lambda(n)) \leq \tau(\lambda(m))$. Thus, we can assume that n is square-free (indeed, if n' is the product of the distinct primes dividing n , then $\omega(n) = \omega(n')$ and $\tau(\lambda(n)) \geq \tau(\lambda(n'))$).

Suppose that n is odd and $n = p_1 p_2 \cdots p_r$, where $p_1 < \cdots < p_r$ are primes. Let $2 < q_2 < \cdots < q_s$ be all the odd prime factors of $\lambda(n)$ and write

$$p_1 - 1 = 2^{\alpha_{11}} q_2^{\alpha_{12}} \cdots q_s^{\alpha_{1s}},$$

$$p_2 - 1 = 2^{\alpha_{21}} q_2^{\alpha_{22}} \cdots q_s^{\alpha_{2s}},$$

$$\vdots$$

$$p_r - 1 = 2^{\alpha_{r1}} q_2^{\alpha_{r2}} \cdots q_s^{\alpha_{rs}}.$$

If $A_i = \max\{\alpha_{1i}, \dots, \alpha_{ri}\}$ for $i = 1, \dots, s$, then

$$\tau(\lambda(n)) = \tau([p_1 - 1, \dots, p_r - 1]) = (A_1 + 1)(A_2 + 1) \cdots (A_s + 1).$$

Consider now the matrix

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1s} \\ \vdots & & \vdots \\ \alpha_{r1} & \cdots & \alpha_{rs} \end{pmatrix}.$$

We know that the entries of the matrix consist of nonnegative integers. The elements in the first column are positive and less than or equal to A_1 . For each $i = 1, \dots, r$, the elements of the i th column are nonnegative integers less than or equal to A_i .

Furthermore, for each fixed natural number s , the number of rows r is less than or equal to the maximum number of distinct s -tuples (a_1, \dots, a_s) with $a_1 \in [1, A_1]$ and $a_i \in [0, A_i]$ for $i = 2, \dots, s$. This follows because $(2^{\alpha_{ij}} \prod_{j=2}^s q_j^{\alpha_{ij}})_{i=1, \dots, s}$ are distinct positive integers. Hence,

$$r \leq A_1(A_2 + 1) \cdots (A_s + 1).$$

From the foregoing discussion we deduce that

$$\begin{aligned} \tau(\lambda(n)) &= (A_1 + 1)(A_2 + 1) \cdots (A_s + 1) \\ &\geq r + \tau(\lambda^o(n)) = \omega(n) + \tau(\lambda^o(n)), \end{aligned}$$

where $\lambda^o(n) = \lambda(n)/(2^\infty, \lambda(n))$ is the largest odd divisor of $\lambda(n)$. As a result, if n is square-free and odd then

$$\tau(\lambda(n)) \geq \omega(n) + 1,$$

but if n is square-free and even then

$$\tau(\lambda(n)) = \tau(\lambda(n/2)) \geq \omega(n/2) + 1 = \omega(n);$$

this concludes the proof. \square

Lemma 1 is the main tool we use to determine the set \mathcal{A}_k for $k \leq 2$.

PROPOSITION 2. $\mathcal{A}_0 = \{2, 6, 12, 24, 30, 60, 120, 240\}$.

Proof. Let $n \in \mathcal{A}_0$. Applying Lemma 1, we obtain that if n is odd then $\tau(\lambda(n)) > \omega(n)$, which is impossible.

If n is even then, by Lemma 1, the condition $\tau(\lambda(n)) = \omega(n)$ implies that

$$\tau(\lambda^o(n')) = 1.$$

This is possible only if $\lambda(n') = 2^\alpha$ for some $\alpha \in \mathbb{N}$. If $n = 2^\gamma$ and $\tau(\lambda(2^\gamma)) = 1$, then $\gamma = 1$ and so $n = 2$.

Assume now that n is not a power of 2 and write

$$n = 2^{\gamma_0}(2^{2^{\alpha_1}} + 1)^{\gamma_1} \cdots (2^{2^{\alpha_r}} + 1)^{\gamma_r},$$

where $\gamma_j \geq 1$ ($j = 0, \dots, r$), $0 \leq \alpha_1 < \dots < \alpha_r$, and the numbers $2^{2^{\alpha_i}} + 1$ are primes for each $i = 1, \dots, r$. Plugging our expression for n into the identity $\tau(\lambda(n)) = \omega(n)$ yields

$$\max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1\} \cdot \gamma_1 \cdots \gamma_r = r + 1,$$

which is satisfied only for $r = 1$ or $r = 2$ (because we can now gather that $r + 1 \geq 2^{\alpha_r} + 1 \geq 2^{r-1} + 1$).

If $r = 2$ then necessarily $\alpha_2 = 1$. This forces $\alpha_1 = 0$, $\gamma_1 = \gamma_2 = 1$, and $1 \leq \gamma_0 \leq 4$, which correspond to the four values for n of 30, 60, 120, and 240. Finally, if $r = 1$ then $\alpha_1 = 0$; this forces $\gamma_1 = 1$ and $1 \leq \gamma_0 \leq 3$, which correspond to the three values for n of 6, 12, and 24. \square

We are now ready to prove the motivating inequality (1).

COROLLARY 1. *Let φ denote the Euler function. Excepting only $n = 2, 6, 8, 12, 24, 80, 120, 240$, we have*

$$n \leq \lambda(n)^{\tau(\lambda(n))}.$$

Furthermore, $\varphi(n) \leq \lambda(n)^{\tau(\lambda(n))}$ except for $n = 24$. Finally, the inequality $\varphi(n) \leq \lambda(n)^{\omega(n)}$ holds unless n is a power of 2 times a product of distinct Fermat primes.

Proof. Let $v_p(m)$ be the exponent of the prime p in the factorization of the positive integer m . We know that $\lambda(n)$ divides $\varphi(n)$. We also know that if p odd then

$$\begin{aligned} v_p(\varphi(n)) &= \sum_{l^\beta \parallel n} v_p(l^{\beta-1}(l-1)) \\ &\leq \omega(n) \left(\max_{l^\beta \parallel n} \{v_p(l^{\beta-1}(l-1))\} \right) \leq v_p(\lambda(n)^{\omega(n)}), \end{aligned}$$

while $v_2(\varphi(n)) = v_2(n) - 1 + \sum_{l \mid n} v_2(l-1) \leq 1 + \omega(n)v_2(\lambda(n))$.

Necessarily, then, $\varphi(n) \mid 2\lambda(n)^{\omega(n)}$. Furthermore, the only circumstances in which $\varphi(n) = 2\lambda(n)^{\omega(n)}$ is when $\varphi(n)$ is a power of 2. If this happens, then n is necessarily a power of 2 times a product of distinct Fermat primes. In all other cases we have $\varphi(n) \leq \lambda(n)^{\omega(n)}$, and this proves the third inequality.

In order to prove the second, it is enough to notice that $\tau(\lambda(n)) \geq \omega(n)$ by Proposition 1; hence we need only show that $\varphi(n) \leq \lambda(n)^{\tau(\lambda(n))}$ when $\varphi(n) = 2^\alpha$ and $n \neq 24$. Observe that the latter is certainly true when n is a power of 2, since $\varphi(2^\alpha) = 2^{\alpha-1} \leq 2^{(\alpha-2)(\alpha-1)} = \lambda(2^\alpha)^{\tau(\lambda(2^\alpha))}$ for $\alpha > 2$. In the other cases, if we write

$$n = 2^{\alpha_0} \cdot (2^{2^{\alpha_1}} + 1) \cdots (2^{2^{\alpha_r}} + 1)$$

with $\alpha_1 < \dots < \alpha_r$, then

$$\begin{aligned}\varphi(n) &= 2^{2^{\alpha_1} + \dots + 2^{\alpha_r} + \max\{\alpha_0 - 1, 0\}} \\ &\leq 2^{2^{\alpha_r}(1+1/2+\dots+1/2^{r-1}) + \max\{\alpha_0 - 1, 0\}} \leq 2^{3M+1},\end{aligned}$$

where $M = \max\{\log_2(\lambda(2^{\alpha_0})), 2^{\alpha_r}\}$. Similarly,

$$\lambda(n)^{\tau(\lambda(n))} = 2^{M(M+1)}.$$

Finally, $3M + 1 \leq M(M + 1)$ for $M > 2$ while for $M \leq 2$ we have $r \leq 2$ and so $n \in \{3, 6, 12, 24, 48, 5, 10, 20, 40, 80, 15, 30, 60, 120, 240\}$; the only value of n from this set that does not satisfy the inequality $\varphi(n) \leq \lambda(n)^{\tau(\lambda(n))}$ is $n = 24$. This completes the proof of the second statement.

Observe that for $n \in \mathcal{A}_0$ the first statement holds if and only if $n \in \{30, 60\}$. So we can assume that $n \notin \mathcal{A}_0$ and thus $\tau(\lambda(n)) \geq \omega(n) + 1$. This implies that

$$\lambda(n)^{\tau(\lambda(n))} \geq \lambda(n)\varphi(n)$$

unless $\varphi(n)$ is a power of 2. In order to conclude the proof we must verify that the statement holds when $\varphi(n)$ is a power of 2 and $n \neq 2, 8$, and we must also show that

$$\lambda(n)\varphi(n) \geq n.$$

We claim that this inequality holds unless $n \in \{2, 3, 6, 12, 24\}$ (values for which the statement is verified directly). Indeed, let p be the greatest prime divisor of n . If $p \geq 5$, then

$$\frac{n}{\varphi(n)} = \prod_{l|n} \frac{l}{l-1} \leq \frac{3}{4}p \leq p-1 \leq \lambda(n).$$

Similarly, if $p = 3$, then $n/\varphi(n) \leq 3 \leq \lambda(n)$ unless $n \in \{3, 6, 12, 24\}$. Finally, if $p = 2$, then $n/\varphi(n) = 2 \leq \lambda(n)$ unless $n = 2$.

If $\varphi(n)$ is a power of 2, we proceed as in the proof of the second inequality. Observe that if $n = 2^{\alpha_0}$ then $n \leq \lambda(n)^{\tau(\lambda(n))}$ unless $\alpha_0 = 1, 3$. If $n = 2^{\alpha_0} \cdot (2^{\alpha_1} + 1) \cdots (2^{\alpha_r} + 1)$ with $\alpha_1 < \dots < \alpha_r$ and if $M = \max\{\log_2(\lambda(2^{\alpha_0}), 2^{\alpha_r})\}$ so that $2^{M(M+1)} = \lambda(n)^{\tau(\lambda(n))}$, then

$$n \leq 2^{2(2^{\alpha_1} + \dots + 2^{\alpha_r}) + \alpha_0} \leq 2^{5M+2}.$$

Since $5M + 2 \leq M(M + 1)$ for $M > 5$, we are left with checking the statement for integers that divide $2^7 \cdot 3 \cdot 5 \cdot 17$, and this is done by a short calculation. \square

PROPOSITION 3.

$$\begin{aligned}\mathcal{A}_1 &= \{1, 3, 4, 8, 10, 15, 20, 40, 48, 80, 126, 252, 480, 504, \\ &\quad 510, 1020, 2040, 2730, 4080, 5460, 8160, 8190, 10920, \\ &\quad 16320, 16380, 21840, 32760, 65520, 6q, 12q, 24q\},\end{aligned}$$

where $q = 2p + 1$ is prime with $p > 2$ also prime.

Proof. We follow the same method as in the proof of Proposition 2.

If $n > 1$ is odd then, by Lemma 1, $\lambda^o(n') = 1$. This implies that $\lambda(n') = 2^\alpha$ for some $\alpha \geq 0$. Thus,

$$n = (2^{2^{\alpha_1}} + 1)^{\gamma_1} \cdots (2^{2^{\alpha_r}} + 1)^{\gamma_r},$$

where $\gamma_j \geq 1$ ($j = 1, \dots, r$), $0 \leq \alpha_1 < \dots < \alpha_r$, and again $2^{2^{\alpha_i}} + 1$ is prime for $i = 1, \dots, r$.

The equation $\tau(\lambda(n)) = \omega(n) + 1$ is equivalent to

$$(2^{\alpha_r} + 1)\gamma_1 \cdots \gamma_r = r + 1.$$

Since $\alpha_r \geq r - 1$, the preceding equality is satisfied only if $r = 1$ or $r = 2$. If $r = 1$ then necessarily $\alpha_1 = 0$ and $\gamma_1 = 1$, so $n = 3$. If $r = 2$ then we have $\alpha_1 = 0$, $\alpha_2 = 1$, and $\gamma_1 = \gamma_2 = 1$, so $n = 15$.

Assume now that n is even. If $n = 2^r$, then $\tau(\lambda(n)) = 2$ is satisfied only for $n = 4$ or $n = 8$.

If n is not a power of 2, then Lemma 1 yields $\tau(\lambda^o(n')) \leq 2$. This can happen only if $\lambda(n') = 2^a$ or $\lambda(n') = 2^a p$ with p an odd prime. If $\lambda(n') = 2^a$ then

$$n = 2^{\gamma_0} \cdot (2^{2^{\alpha_1}} + 1)^{\gamma_1} \cdots (2^{2^{\alpha_r}} + 1)^{\gamma_r},$$

where $\gamma_j \geq 1$ ($j = 0, \dots, r$), $0 \leq \alpha_1 < \dots < \alpha_r$, and again $2^{2^{\alpha_i}} + 1$ is prime for $i = 1, \dots, r$.

If we plug the preceding expression for n into the identity $\tau(\lambda(n)) = \omega(n) + 1$, we obtain

$$\max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1\} \cdot \gamma_1 \cdots \gamma_r = r + 2,$$

which can only be satisfied for $r \leq 3$ because $r + 2 \geq 2^{\alpha_r} + 1 \geq 2^{r-1} + 1$. A quick computation shows that $\gamma_j = 1$ for all $j \geq 1$, and we have only the following possibilities:

| r | $(\alpha_1, \dots, \alpha_r)$ | n |
|-----|-------------------------------|------------------------------------|
| 1 | (0) | 48 |
| | (1) | 10, 20, 40, 80 |
| 2 | — | — |
| 3 | (0, 1, 2) | 510, 1020, 2040, 4080, 8160, 16320 |

The next case to consider is when $\lambda(n') = 2^a p$, so that each odd prime dividing n is of the form $2^{2^\alpha} + 1$ or of the form $2^{\beta} p + 1$. Hence,

$$n = 2^{\gamma_0} \cdot (2^{2^{\alpha_1}} + 1)^{\gamma_1} \cdots (2^{2^{\alpha_r}} + 1)^{\gamma_r} \cdot (2^{\beta_1} p + 1)^{\gamma_{r+1}} \cdots (2^{\beta_s} p + 1)^{\gamma_{r+s}},$$

where $\gamma_j \geq 1$ ($j = 0, \dots, r+s$), $0 \leq \alpha_1 < \dots < \alpha_r$, $2^{2^{\alpha_i}} + 1$ is prime for $i = 1, \dots, r$, $1 < \beta_1 < \dots < \beta_s$, and $2^{\beta_k} p + 1$ is prime for $k = 1, \dots, s$.

We now distinguish two more subcases: $p^2 \mid n$ and $p^2 \nmid n$. If $p^2 \mid n$, then the equation $\tau(\lambda(n)) = \omega(n) + 1$ translates into

$$\max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1, \beta_s + 1\} \cdot \gamma_1 \cdots \gamma_{r+s} = r + s + 2. \quad (2)$$

In this case, there exists a $j \leq r$ such that $\gamma_j \geq 2$; since $\max\{a, b\} \geq (a+b)/2$, the LHS of (2) is greater than or equal to $2^{\alpha_r} + 1 + \beta_s + 1$. Using that $\alpha_r \geq r - 1$

and $\beta_s \geq s$, we once again obtain $2^{r-1} + 1 \leq r + 1$, which implies that $r = 1$ or $r = 2$.

If $r = 1$, then necessarily $\alpha_1 = 0$, $\gamma_1 = 2$, $s = 1$, and $\beta_1 = \gamma_2 = 1$. This implies that $n = 2^{\gamma_0} \cdot 3^2 \cdot 7$ and $\gamma_0 = 1, 2, 3$. If $r = 2$, then necessarily $\alpha_1 = 0$, $\alpha_2 = 1$, and $s \leq 2$ (since the LHS of (2) is greater than or equal to $2s + 2$). Checking all possibilities, we find that $n = 2^{\gamma_0} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ and $\gamma_0 = 1, 2, 3, 4$.

For the other subcase, if $p^2 \nmid n$ then the equation $\tau(\lambda(n)) = \omega(n) + 1$ translates into

$$2 \cdot \max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1, \beta_s + 1\} \cdot \gamma_1 \cdots \gamma_{r+s} = r + s + 2. \quad (3)$$

For the same reasons as before, it follows that $r = 1$ or $r = 2$ and $s = 1$ or $s = 2$.

If $r = s = 1$ then we have the family of solutions $n = 2^{\gamma_0} \cdot 3 \cdot (2p + 1)$, where $\gamma_0 = 1, 2, 3$ and $2p + 1$ is prime with $p \geq 3$. If $r = s = 2$ then we have the solutions $n = 2^{\gamma_0} \cdot 3 \cdot 5 \cdot 7 \cdot 13$, where $\gamma_0 = 1, 2, 3, 4$. The remaining cases ($r = 1$, $s = 2$; $r = 2$, $s = 1$) produce for the RHS of (3) a value equal to 5 and so do not lead to any more solutions. \square

PROPOSITION 4. *We have that $\mathcal{A}_2 = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$, where:*

$$\mathcal{F}_1 = \left\{ \begin{array}{l} 5, 2^4 \cdot 2^5 \cdot 3, 2^5 \cdot 5, 2^\beta \cdot 3^2, 2^6 \cdot 3 \cdot 5, \\ 2^\alpha \cdot 3 \cdot 17, 2^\alpha \cdot 5 \cdot 17, 3 \cdot 5 \cdot 17, \\ 2^7 \cdot 3 \cdot 5 \cdot 17 \end{array} \middle| \begin{array}{l} 1 \leq \alpha \leq 6, \\ 1 \leq \beta \leq 3 \end{array} \right\},$$

$$\mathcal{F}_2 = \left\{ \begin{array}{l} 2^\alpha \cdot 3^\beta \cdot 5 \cdot 7, 3^\beta \cdot 7, 3^\beta \cdot 5 \cdot 7 \cdot 13 \\ 2^\alpha \cdot 3^\beta \cdot 5 \cdot 13, 2^\alpha \cdot 3^\beta \cdot 7 \cdot 13, \\ 2^\alpha \cdot 5 \cdot 7 \cdot 13 \end{array} \middle| \begin{array}{l} 1 \leq \alpha \leq 4, \\ \beta = 1, 2 \end{array} \right\},$$

$$\mathcal{F}_3 = \{2^\alpha \cdot 3 \cdot 5^2 \cdot 11 \mid 1 \leq \alpha \leq 4\},$$

$$\mathcal{F}_4 = \left\{ \begin{array}{l} 2^\delta \cdot 3^\beta \cdot 7 \cdot 19, \\ 2^\alpha \cdot 3^\beta \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 37 \end{array} \middle| \begin{array}{l} 1 \leq \alpha \leq 4, \\ 1 \leq \beta, \delta \leq 3 \end{array} \right\};$$

$$\mathcal{I}_1 = \{2^\alpha \cdot (2p + 1) \mid 2p + 1, p \geq 3 \text{ primes}, 1 \leq \alpha \leq 3\},$$

$$\mathcal{I}_2 = \{3 \cdot (2p + 1) \mid 2p + 1, p \geq 3 \text{ primes}\},$$

$$\mathcal{I}_3 = \left\{ 2^\alpha \cdot 3 \cdot 5 \cdot (2^\beta p + 1) \mid \begin{array}{l} 2^\beta p + 1, p \geq 3 \text{ primes}, \\ 1 \leq \alpha \leq 4, \beta = 1, 2 \end{array} \right\}.$$

Proof. We follow the same approach as in the previous results and obtain that, in order for n to satisfy $\tau(\lambda(n)) = \omega(n) + 2$, we must have $\lambda(n') = 2^\alpha p^\beta$ with $\alpha \geq 0$ and $\beta = 0, 1, 2$. This implies that n should be of the form

$$n = 2^{\gamma_0} \cdot A \cdot B \cdot C,$$

where A , B , and C are either 1 or of the respective forms

$$A = (2^{2^{\alpha_1}} + 1)^{\gamma_1} \cdots (2^{2^{\alpha_r}} + 1)^{\gamma_r},$$

$$B = (2^{\beta_1} p + 1)^{\gamma_{r+1}} \cdots (2^{\beta_s} p + 1)^{\gamma_{r+s}},$$

$$C = (2^{\delta_1} p^2 + 1)^{\gamma_{r+s+1}} \cdots (2^{\delta_t} p^2 + 1)^{\gamma_{r+s+t}}.$$

Here we assume the following conditions: $\gamma_j \geq 1$ for $j = 0, \dots, r+s+t$, $0 \leq \alpha_1 < \dots < \alpha_r$, $2^{2\alpha_i} + 1$ is prime for $i = 1, \dots, r$, $1 < \beta_1 < \dots < \beta_s$, $2^{\beta_k} p + 1$ is prime for $k = 1, \dots, s$, $1 < \delta_1 < \dots < \delta_t$, and $2^{\delta_l} p^2 + 1$ is prime for $l = 1, \dots, t$. We allow any one of r, s, t, γ_0 to be zero with the obvious meaning.

The equation $\tau(\lambda(n)) = \omega(n) + 2$ is equivalent to

$$\Theta \cdot \Lambda \cdot \gamma_1 \cdots \gamma_{r+s+t} = r + s + t + \min\{1, \gamma_0\} + 2, \quad (4)$$

where:

$$\Theta = \begin{cases} 1 & \text{if } (s+t > 0 \text{ and } p^3 \mid n) \text{ or } (s+t=0 \\ & \text{or } (t=0, s>0 \text{ and } p^2 \parallel n); \\ 3/2 & \text{if } t > 0 \text{ and } p^2 \parallel n; \\ 2 & \text{if } t=0, s>0, \text{ and } p^2 \nmid n; \\ 3 & \text{if } t > 0 \text{ and } p^2 \nmid n; \end{cases}$$

and $\Lambda = \max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1, \beta_s + 1, \delta_t + 1\}$. The terms $\beta_s + 1$ (resp. $\delta_t + 1$) should be omitted if $s = 0$ (resp. $t = 0$).

If $s = t = 0$, then these remarks imply that $r \leq 3$ and

$$n = 2^{\delta_0} \cdot 3^{\delta_1} \cdot 5^{\delta_2} \cdot 17^{\delta_3}.$$

In this case, all possible solutions of $\tau(\lambda(n)) = \omega(n) + 2$ are

| r | $(\delta_0, \delta_1, \delta_2, \delta_3)$ | n |
|-----|--|---|
| 0 | (4, 0, 0, 0) | 2^4 |
| 1 | (0, 0, 1, 0) | 5 |
| | $(\delta, 2, 0, 0), \delta = 1, 2, 3$ | $2 \cdot 3^2, 2^2 \cdot 3^2, 2^3 \cdot 3^2$ |
| | (5, 1, 0, 0) | $2^5 \cdot 3$ |
| | (5, 0, 1, 0) | $2^5 \cdot 5$ |
| 2 | (6, 1, 1, 0) | $2^6 \cdot 3 \cdot 5$ |
| | $(\delta, 1, 0, 1), 1 \leq \delta \leq 6$ | $2^\delta \cdot 3 \cdot 17, 1 \leq \delta \leq 6$ |
| | $(\delta, 0, 1, 1), 1 \leq \delta \leq 6$ | $2^\delta \cdot 5 \cdot 17, 1 \leq \delta \leq 6$ |
| 3 | (0, 1, 1, 1) | $3 \cdot 5 \cdot 17$ |
| | (7, 1, 1, 1) | $2^7 \cdot 3 \cdot 5 \cdot 17$ |

These solutions are exactly the 22 elements of \mathcal{F}_1 .

When $t = 0$ and $s \neq 0$, equation (4) simplifies to

$$\Theta \cdot \Lambda \cdot \gamma_1 \cdots \gamma_{r+s} = r + s + \min\{1, \gamma_0\} + 2, \quad (5)$$

where:

$$\Theta = \begin{cases} 1 & \text{if } p^2 \mid n, \\ 2 & \text{if } p^2 \nmid n; \end{cases}$$

$\Lambda = \max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1, \beta_s + 1\}$, and the middle term is omitted if $r = 0$. In such a case, we have $p \nmid n$ and $s \leq \beta_s \leq (s + \min\{1, \gamma_0\})/2$. This is possible only for n even and $s = \beta_s = 1$. This implies that $n = 2^{\gamma_0}(2p+1)$ with $\gamma_0 = 1, 2, 3$, which are exactly the elements of \mathcal{I}_1 .

If $r > 0$ then the LHS of (5) is greater than or equal to $2^{\alpha_r} + \beta_s + 2$, which implies that $2^{\alpha_r} \leq r + \min\{1, \gamma_0\}$. From this inequality it follows that $r \leq 2 + \min\{1, \gamma_0\}$.

We distinguish the two subcases $p = 3$ and $p > 3$. In the first subcase, $s \leq r + \min\{1, \gamma_0\}$ and $\beta_s \leq (r + s + \min\{1, \gamma_0\})/2$. This implies that

$$n = 2^{\delta_0} \cdot 3^{\delta_1} \cdot 5^{\delta_2} \cdot 7^{\delta_3} \cdot 13^{\delta_4}.$$

In this subcase, all possible solutions of $\tau(\lambda(n)) = \omega(n) + 2$ are

| (r, s) | $(\delta_0, \delta_1, \delta_2, \delta_3, \delta_4)$ | n |
|----------|--|---------------------------------------|
| (1, 1) | $(0, \delta, 0, 1, 0), \delta = 1, 2$ | $3 \cdot 7, 3^2 \cdot 7$ |
| (1, 2) | $(\delta, 1, 0, 1, 1), 1 \leq \delta \leq 4$ | $2^\delta \cdot 3 \cdot 7 \cdot 13$ |
| | $(\delta, 2, 0, 1, 1), 1 \leq \delta \leq 4$ | $2^\delta \cdot 3^2 \cdot 7 \cdot 13$ |
| | $(\delta, 0, 1, 1, 1), 1 \leq \delta \leq 4$ | $2^\delta \cdot 5 \cdot 7 \cdot 13$ |
| (2, 2) | $(0, 1, 1, 1, 1)$ | $3 \cdot 5 \cdot 7 \cdot 13$ |
| | $(0, 2, 1, 1, 1)$ | $3^2 \cdot 5 \cdot 7 \cdot 13$ |
| (2, 1) | $(\delta, 1, 1, 1, 0), 1 \leq \delta \leq 4$ | $2^\delta \cdot 3 \cdot 5 \cdot 7$ |
| | $(\delta, 2, 1, 1, 0), 1 \leq \delta \leq 4$ | $2^\delta \cdot 3^2 \cdot 5 \cdot 7$ |
| | $(\delta, 1, 1, 0, 1), 1 \leq \delta \leq 4$ | $2^\delta \cdot 3 \cdot 5 \cdot 13$ |
| | $(\delta, 2, 1, 0, 1), 1 \leq \delta \leq 4$ | $2^\delta \cdot 3^2 \cdot 5 \cdot 13$ |

These solutions are exactly the 32 elements of \mathcal{F}_2 .

In the subcase where $r > 0, s > 0, t = 0$, and $p > 3$, we have $\beta_s \geq 2s - 1$. Thus,

$$2^{\alpha_r} + 2s + 1 \leq 2^{\alpha_r} + \beta_s + 2 \leq r + s + \min\{\gamma_0, 1\} + 2$$

and $s \leq r + 1 + \min\{\gamma_0, 1\} - 2^{\alpha_r} \leq 1$, which implies that $s = 1$ and $\beta_1 \leq 2$. Note that $\alpha_r \leq 1$ and note also that r cannot be 3 since this would imply $s = 1, 2^{\alpha_r} + 1 \geq 5$, $\tau(\lambda(n)) \geq 10$, and $\omega(n) \geq 8$, which is impossible because $\omega(n) \leq r + s + 3 \leq 7$.

Therefore,

$$n = 2^{\delta_0} \cdot 3^{\delta_1} \cdot 5^{\delta_2} \cdot (2^{\beta_1} p + 1)^{\delta_3}$$

with $p > 3$. If $5^2 \mid n$ then we have the solutions $n = 2^\alpha \cdot 3 \cdot 5^2 \cdot 11$ ($\alpha = 1, 2, 3, 4$), which are exactly the elements of \mathcal{F}_3 . If $5^2 \nmid n$ then we have the solutions $n = 3 \cdot (2p + 1)$, which are elements of \mathcal{I}_2 , and $n = 2^\alpha \cdot 3 \cdot 5 \cdot (2^\beta + 1)$ with $\alpha = 1, 2, 3, 4$ and $\beta = 1, 2$, which are elements of \mathcal{I}_3 .

The last case to consider is when $t > 0$, so that there is a prime dividing n of the form $2^\beta \cdot p^2 + 1$. Now equation $\tau(\lambda(n)) = \omega(n) + 2$ is equivalent to

$$\Theta \cdot \Lambda \cdot \gamma_1 \cdots \gamma_{r+s+t} = r + s + t + \min\{1, \gamma_0\} + 2, \quad (6)$$

where

$$\Theta = \begin{cases} 1 & \text{if } p^3 \mid n, \\ 3/2 & \text{if } p^2 \parallel n, \\ 3 & \text{if } p^2 \nmid n, \end{cases}$$

and $\Lambda = \max\{\tau(\lambda(2^{\gamma_0})), 2^{\alpha_r} + 1, \beta_s + 1, \delta_t + 1\}$. Here the terms $2^{\alpha_r} + 1$ (resp. $\beta_s + 1$) are to be omitted if $r = 0$ (resp. $s = 0$).

We claim that $r, s \neq 0$ (and will show this later). Hence, from (6) we may deduce that

$$2^{\alpha_r} + \beta_s + \delta_t + 3 \leq r + s + t + \min\{1, \gamma_0\} + 2.$$

On one hand, this relation implies that $2^{\alpha_r} \leq r - 1 + \min\{1, \gamma_0\}$, so that $\gamma_0 \geq 1$ and either $r = 1$ and $\alpha_1 = 0$ or $r = 2, \alpha_2 = 1$, and $\alpha_1 = 0$. On the other hand, the same relation implies that $s + t \leq \beta_s + \delta_t \leq 2r$.

If $r = 1$ then $s = t = \beta_s = \delta_t = 1$, and since $2p^2 + 1$ is prime we necessarily have $p = 3$. Therefore,

$$n = 2^{\gamma_0} \cdot 3^{\gamma_1} \cdot 7^{\gamma_2} \cdot 19^{\gamma_3},$$

and the only solutions of $\tau(\lambda(n)) = 6$ of this form are the first nine elements of \mathcal{F}_4 .

If $r = 2$, then $4 \leq s + t \leq \beta_s + \delta_t \leq 4$. This implies that $s = t = 2$ and $(\beta_1, \beta_2, \delta_1, \delta_2) = (1, 2, 1, 2)$; hence $p = 3$,

$$n = 2^{\gamma_0} \cdot 3^{\gamma_1} \cdot 5^{\gamma_2} \cdot 7^{\gamma_3} \cdot 13^{\gamma_4} \cdot 19^{\gamma_5} \cdot 37^{\gamma_6},$$

and the only solutions of $\tau(\lambda(n)) = 9$ of this form are the last twelve elements of \mathcal{F}_4 .

Finally, we must prove the claim for $r, s \neq 0$. If $r = 0$ and $s \neq 0$ then we deduce from (6) that

$$3(s + t + 2)/2 \leq 3(\beta_s + \delta_t + 2)/2 \leq s + t + 3,$$

which implies $s + t \leq 0$ —a contradiction. A similar argument rules out the possibility $r = 0$ and $s = 0$. Lastly, if $r \neq 0$ and $s = 0$, then from (6) and from $\delta_t \geq t$ we deduce that

$$3(2^{\alpha_r} + t + 2)/2 \leq r + t + 3,$$

which is also a contradiction and thus ends the proof of the proposition. \square

3. Lower Bounds on the Counting Functions of \mathcal{A}_k

THEOREM 1. *\mathcal{A}_k is nonempty for all nonnegative integers k .*

Proof. Let $p_1 = 3, p_2 = 5, p_3 = 13$, and $p_4 = 31$. Then, for each $m \geq 3$ and for each $t \in \{4, 5, 6, 7\}$, the number $n = 2^{m+1} \cdot 7 \cdot 11 \cdot p_1 \cdots p_{t-3}$ satisfies $\omega(n) = t$ and $\tau(\lambda(n)) = \tau(2^{m-1} \cdot 3 \cdot 5) = 4m$. This means that $\tau(\lambda(n)) - \omega(n) = 4(m-1) - (t-4)$ can assume all possible values ≥ 8 . Finally, $3 \in \mathcal{A}_0, 4 \in \mathcal{A}_1, 5 \in \mathcal{A}_2, 7 \in \mathcal{A}_3, 17 \in \mathcal{A}_4, 13 \in \mathcal{A}_5, 62 \in \mathcal{A}_6$, and $31 \in \mathcal{A}_7$. This completes the proof. \square

In what follows, we show that if k is sufficiently large, then \mathcal{A}_k contains “many” elements.

THEOREM 2. *For all $k \neq 0, 1, 2, 3, 4$, we have the lower bound*

$$\#\mathcal{A}_k(x) \gg_k \frac{x}{(\log x)^2} \text{ as } x \rightarrow \infty.$$

Proof. The proof uses the famous Theorem of Chen that we state in the following form (see also [4; 7, Lemma 1.2; 8, Chap. 11]).

LEMMA 1. *Let $a \in \mathbb{N}$ be an even number. There exists a constant $c = c(a)$ such that, if $x > x_0(a)$, then the number of primes $p \in [x/2, x]$ such that $p \equiv 1 \pmod{a}$ and $(p-1)/a$ has at most two prime factors, each of which exceeds $x^{1/10}$, is at least $c_a x / (\log x)^2$.*

We write $k = 4s + r$ for $s \geq 1$ and $r \in \{0, 1, 2, 3\}$, distinguishing two cases as follows:

Case 1. $r \neq 3$;

Case 2. $r = 3$.

In Case 1, we apply Chen's theorem with the choice $a = 2^s$ and obtain that there are either (a) at least $M_a \gg_a x / (\log x)^2$ primes $p \leq x/42$ with $p-1 = 2^s q$ and q prime or (b) at least $N_a \gg_a x / (\log x)^2$ primes $p \leq x/42$ with $p-1 = 2^s q_1 q_2$, where q_1 and q_2 are distinct primes exceeding $x^{1/10}$.

For (a), consider the M_a integers $n \leq x$ of the form $n = 7pT$, where

$$T = \begin{cases} 1 & \text{if } r = 2, \\ 2 & \text{if } r = 1, \\ 6 & \text{if } r = 0. \end{cases}$$

These choices yield $\omega(n) = 4 - r$, $\lambda(n) = 2^s \cdot 3 \cdot q$, and $\tau(\lambda(n)) = 4(s+1)$; therefore, $\tau(\lambda(n)) - \omega(n) = 4s + r = k$.

For (b), consider the N_a integers $n \leq x$ of the form $n = 2pT$, where

$$T = \begin{cases} 1 & \text{if } r = 2, \\ 3 & \text{if } r = 1, \\ 15 & \text{if } r = 0. \end{cases}$$

For $s \geq 2$ and for $s = 1$ with $r \neq 0$, we have $\omega(n) = 4 - r$, $\lambda(n) = 2^s \cdot q_1 \cdot q_2$, and $\tau(\lambda(n)) = 4(s+1)$, so that again $\tau(\lambda(n)) - \omega(n) = k$.

In Case 2, we apply Chen's theorem with the choice $a = 2^{s+1}$ and obtain that there are either (a) at least $M_a \gg_a x / (\log x)^2$ primes $p \leq x/510$ with $p-1 = 2^{s+1}q$ and q prime or (b) at least $N_a \gg_a x / (\log x)^2$ primes $p \leq x/510$ with $p-1 = 2^{s+1}q_1 q_2$, where q_1 and q_2 are distinct primes that exceed $x^{1/10}$.

Given (a), consider the M_a integers $n \leq x$ of the form $n = 210p$. For $s \geq 1$ we have $\omega(n) = 5$ and $\tau(\lambda(n)) = 4(s+2)$, so $\tau(\lambda(n)) - \omega(n) = 4s + 3 = k$. Given (b), consider the N_a integers $n \leq x$ of the form $n = 510p$. For $s \geq 3$ we have $\omega(n) = 5$ and $\tau(\lambda(n)) = 4(s+2)$, so again $\tau(\lambda(n)) - \omega(n) = 4s + 3 = k$.

Next assume that $k = 7$. Then we apply Chen's theorem with the choice $a = 2$ and obtain that there are either (a) at least $M_a \gg_a x / (\log x)^2$ primes $p \leq x/192$ with $p-1 = 2q$ and q prime or (b) at least $N_a \gg_a x / (\log x)^2$ primes $p \leq x/192$ with $p-1 = 2q_1 q_2$, where q_1 and q_2 are distinct primes exceeding $x^{1/10}$.

For (a), consider the M_a integers $n \leq x$ of the form $n = 2^6 3p$. We have $\omega(n) = 3$ and $\tau(\lambda(n)) = \tau(2^4 p)$, so $\tau(\lambda(n)) - \omega(n) = 10 - 3 = 7$. For (b), consider the N_a integers $n \leq x$ of the form $n = p = 2q_1 q_2 + 1$. We have $\omega(n) = 1$ and $\tau(\lambda(n)) = 8$, so again $\tau(\lambda(n)) - \omega(n) = 8 - 1 = 7$.

Finally, we treat the case $k = 11$. Here we apply Chen's theorem with the choice $a = 4$ and deduce that either (a) there exist $M \gg x/(\log x)^2$ primes $p \leq x/4510$ such that $p - 1 = 4q$ with q prime or (b) there exist $N \gg x/(\log x)^2$ primes $p \leq x/4510$ such that $p - 1 = 4q_1 q_2$, where q_1 and q_2 are distinct primes that exceed $x^{1/10}$.

Given (a), we note that, for large x , all M positive integers $n = 2 \cdot 5 \cdot 11 \cdot 41 \cdot p = 4510p$ (where $p \leq x$ is of the form $4q + 1$) are $\leq x$ and satisfy both $\omega(n) = 5$ and $\lambda(n) = 2^3 \cdot 5 \cdot q$; hence $\tau(\lambda(n)) = 16 = \omega(n) + 11$. Given (b), it follows that, for large x , the N positive integers $n = p$ (where $p \leq x$ is such that $p - 1 = 4q_1 q_2$, with distinct primes q_1 and q_2 that exceed $x^{1/10}$) have the property that $\tau(\lambda(n)) = \tau(4q_1 q_2) = 12 = \omega(n) + 11$. Thus, $\#\mathcal{A}_{11}(x) \geq \max\{M, N\} \gg x/(\log x)^2$, which completes the proof of this theorem. \square

The remaining cases are $k = 0, 1, 2, 3, 4$ and must be treated separately. Propositions 2, 3, and 4 address the first three cases, and certainly there is no hope of showing even that \mathcal{A}_k is infinite for $k = 1, 2$. Although the next result is not as precise a characterization of \mathcal{A}_k for $k = 3, 4$ as Propositions 2–4 for the smaller values of k , its aim is to demonstrate the impossibility of showing that either one of these two sets is infinite.

PROPOSITION 5. *Assume that $\mathcal{A}_3 \cup \mathcal{A}_4$ is infinite. Then there exists an even positive integer c such that the set of primes of the form $p = cq^\beta + 1$, with q prime and $\beta \leq 4$, is infinite.*

Proof. Assume that $n \in \mathcal{A}_3 \cup \mathcal{A}_4$. Then $\tau(\lambda(n)) \leq \omega(n) + 4$. Write $m = \lambda(n)$ and note that $\omega(n)$ is at most the number of divisors of m of the form $p - 1$ for some prime p . Hence, m can have at most four divisors d such that $d + 1$ is composite. Write $m = 2^\alpha \ell$, where ℓ is odd. If $\alpha \geq 9$, then $2^3, 2^5, 2^6, 2^7, 2^9$ are five divisors of m and none of the form $p - 1$ for some prime p . Therefore, $\alpha \leq 8$. If $\tau(\ell) \geq 6$ then ℓ (and hence m) has at least five odd divisors > 1 , and certainly none of them is of the form $p - 1$ for some prime p . Thus $\tau(\ell) \leq 5$, which shows that either $\ell = q^\beta$ for some prime q and some $\beta \leq 4$ or $\ell = q_1 q_2$, where q_1 and q_2 are distinct primes.

Assume that $\ell = q^\beta$ holds for infinitely many n . Then there exist infinitely many primes p of the form $p - 1 = 2^{\alpha_0} q^\beta$ for some $\alpha_0 \in \{1, \dots, 9\}$ and $\beta \in \{1, \dots, 4\}$, which implies the conclusion of the proposition.

Assume now that $\ell = q_1 q_2$ holds for infinitely many n . Suppose further that $q_1 < q_2$. We then distinguish two cases. The first case is when q_1 remains bounded for infinitely many such n . Then $2^\alpha q_1$ can take only finitely many values. Since we have infinitely many values for n , there must exist some fixed even positive integer c (an even divisor of a number of the form $2^9 q_1$ over all the finitely many possibilities for q_1) such that $p - 1 = cq_2$ holds for infinitely many primes p , which implies the conclusion of the proposition. The second case is when q_1 tends to infinity as n tends to infinity in $\mathcal{A}_3 \cup \mathcal{A}_4$. If for infinitely many such n we have that either $2q_1 + 1$ or $2q_2 + 1$ is prime, then the conclusion of the proposition follows with $c = 2$. Assuming this is not the case, we derive a contradiction.

Observe first that $\alpha \leq 3$, for otherwise $2^3, q_1, q_2, 2q_1, 2q_2$ are five divisors of n , none of which is of the form $p - 1$ for some odd prime p . Assume now that $\alpha = 1$. Then $\tau(\lambda(n)) = \tau(2q_1q_2) = 8$ and so $\omega(n) \geq 4$. Since the only prime factors of n are in $\{2, 3, 2q_1+1, 2q_2+1, 2q_1q_2+1\}$, we deduce that one of $2q_1+1$ and $2q_2+1$ must be prime—a contradiction. Finally, if $\alpha = 2$ then $\tau(\lambda(n)) = \tau(4q_1q_2) = 12$, so $\omega(n) \geq 8$. Because all the prime factors of n belong to $\{2, 3, 5, 2q_1+1, 2q_2+1, 4q_1+1, 4q_2+1, 2q_1q_2+1, 4q_1q_2+1\}$, again it follows that one of $2q_1+1$ or $2q_2+1$ must be a prime, which is the final contradiction. \square

4. Upper Bounds on the Counting Functions of \mathcal{A}_k

Our first result in this section shows that, for numbers $n \in \mathcal{A}_k$, $\omega(n)$ is bounded in terms of k .

PROPOSITION 6. *If $n \in \mathcal{A}_k$, then $\omega(n) \leq 2(k+1)^2 + 1$.*

Proof. We use the same idea and notation as in the proof of Proposition 5. Let $n \in \mathcal{A}_k$ and put $m = \lambda(n) = 2^\alpha \ell$, where α is a nonnegative integer and ℓ is odd. If $\alpha \geq 2k+3$, then $2^3, 2^5, \dots, 2^{2k+3}$ are $k+1$ divisors of m and none of the form $p - 1$ for some prime p , which is a contradiction. If $\tau(\ell) \geq k+2$ then ℓ (and hence m) has $k+1$ odd divisors > 1 , and obviously none of them is of the form $p - 1$ for some prime p , which is again a contradiction. Hence $\alpha \leq 2k+2$ and $\tau(\ell) \leq k+1$, so

$$\begin{aligned} \omega(n) &= \tau(\lambda(n)) - k = \tau(2^\alpha \ell) - k = (\alpha+1)\tau(\ell) - k \\ &\leq (2k+3)(k+1) - k = 2(k+1)^2 + 1. \end{aligned} \quad \square$$

An upper bound for the counting function $\#\mathcal{A}_k(x)$ of \mathcal{A}_k follows from Proposition 6 with a little extra work. Let us set

$$b_k = 2(k+1)^2 + 3 + \lfloor \log_2(2(k+1)^2 + k+1) \rfloor.$$

We then have the following result.

THEOREM 3. *For all nonnegative integers k we have the upper bound*

$$\#\mathcal{A}_k(x) \ll_k \frac{x(\log \log x)^{b_k}}{(\log x)^2} \text{ as } x \rightarrow \infty.$$

Proof. Let $K \geq 2$ be any fixed positive integer. Let $\pi_K(x)$ be the number of primes $p \leq x$ such that $\omega(p-1) \leq K$. We begin with the following lemma.

LEMMA 2. *There exists an absolute constant c_0 such that the following estimate holds:*

$$\pi_K(x) \ll \frac{x(\log \log x + c_0)^{K+1}}{(K-1)! (\log x)^2} \text{ as } x \rightarrow \infty.$$

Proof. Let $\mathcal{P}(x) = \{p \leq x : \omega(p-1) \leq K\}$. Put $y = x^{1/\log \log x}$ and $u = \log x / \log y = \log \log x$. For a positive integer n we write $P(n)$ for the largest prime factor of n . Let

$$\Psi(x, y) = \{n \leq x : P(n) \leq y\}.$$

By a result of de Bruijn ([2]; see also [3; 9, Cor. 3; 13, Chap. III.5]), the bound

$$\#\Psi(x, y) \leq x \exp(-(1 + o(1))u \log u) < \frac{x}{(\log x)^2} \quad (7)$$

holds as $u \rightarrow \infty$, where $u = \log x / \log y$ and provided that $u \leq y^{1/2}$, which is satisfied for our choice of y .

Therefore, if $\mathcal{P}_1(x) = \mathcal{P}(x) \cap \Psi(x, y)$ then

$$\#\mathcal{P}_1(x) \ll \frac{x}{(\log x)^2}.$$

Now let $\mathcal{P}_2(x) = \{p \leq x : q^2 \mid p - 1 \text{ for some } q \geq y\}$. For a fixed $q \geq y$, the number of $1 < n \leq x$ such that $q^2 \mid n - 1$ and is $\leq x/q^2$. Thus,

$$\#\mathcal{P}_2(x) \leq \sum_{q \geq y} \frac{x}{q^2} \ll x \int_y^\infty \frac{dt}{t^2} \ll \frac{x}{y} = o\left(\frac{x}{(\log x)^2}\right).$$

Put $\mathcal{P}_3(x) = \mathcal{P}(x) \setminus (\mathcal{P}_1(x) \cup \mathcal{P}_2(x))$. Write $p - 1 = Pm$, where $P = P(p - 1)$. Since $P > y$ and $p \notin \mathcal{P}_2(x)$, we deduce that $P(m) < P$. Thus, $\omega(m) \leq K - 1$. Fix m . By Brun's sieve (see e.g. [8, Thm. 2.3]), the number of primes $p \leq x$ such that $p - 1 = mP$ for some prime P is

$$\ll \frac{x}{\varphi(m)} \frac{1}{(\log x/m)^2} \ll \frac{x}{\varphi(m)(\log y)^2} \ll \frac{x(\log \log x)^2}{\varphi(m)(\log x)^2}.$$

Summing now over all the acceptable values of m , we obtain

$$\begin{aligned} \#\mathcal{P}_3(x) &\ll \frac{x(\log \log x)^2}{(\log x)^2} \sum_{\substack{m \leq x \\ \omega(m) \leq K-1}} \frac{1}{\varphi(m)} \\ &\leq \frac{x(\log \log x)^2}{(\log x)^2} \sum_{k=1}^{K-1} \sum_{\substack{m \leq x \\ \omega(m)=k}} \frac{1}{\varphi(m)} \\ &\leq \frac{x(\log \log x)^2}{(\log x)^2} \sum_{k=1}^{K-1} \frac{1}{k!} \left(\sum_{p^\alpha \leq x} \frac{1}{p^{\alpha-1}(p-1)} \right)^k \\ &\ll \frac{x(\log \log x)^2}{(\log x)^2} \sum_{k=1}^{K-1} \frac{1}{k!} \left(\sum_{p \leq x} \frac{1}{p-1} + O(1) \right)^k \\ &\ll \frac{x(\log \log x)^2}{(\log x)^2} \sum_{k=1}^{K-1} \frac{1}{k!} (\log \log x + c_0)^{k-1}. \end{aligned}$$

It remains only to observe that the last term dominates as x tends to infinity, which finishes the proof of Lemma 2. \square

We are now ready to prove Theorem 3. Assume that $k \geq 3$, since otherwise the result follows immediately (from Propositions 2–4 and Brun's sieve) even with a smaller b_k (i.e., $b_0 = 0$, $b_1 = 1$, and $b_2 = 1$).

Now note that if $p \mid n$ and $n \in \mathcal{A}_k$, then

$$2^{\omega(p-1)} \leq \tau(p-1) \leq \tau(\lambda(n)) = \omega(n) + k \leq 2(k+1)^2 + k + 1$$

(by Proposition 6) and so $\omega(p-1) \leq K = \lfloor \log_2(2(k+1)^2 + k + 1) \rfloor$. Lemma 2 shows that

$$\#\{p \leq x : \omega(p-1) \leq K\} \ll_k \frac{x(\log \log x)^{K+1}}{(\log x)^2}. \quad (8)$$

We put $\mathcal{A}_{k,1}(x)$ for the set of $n \in \mathcal{A}_k(x)$ such that either $P \leq y = x^{1/\log \log x}$ or P^2 divides n . As in the proof of Lemma 2,

$$\#\mathcal{A}_{k,1} \ll \frac{x}{(\log x)^2}. \quad (9)$$

Let $\mathcal{A}_{k,2}(x)$ stand for the complement of $\mathcal{A}_{k,1}(x)$ in $\mathcal{A}_k(x)$. Now write $n \in \mathcal{A}_{k,2}(x)$ as $n = Pm$, where $P = P(n)$. Hence $P > y = x^{1/\log \log x}$, P^2 does not divide n , and $\omega(m) = \omega(n) - 1 \leq 2(k+1)^2$. Fixing m , the number of values for $P \leq x/m$ such that $\omega(P-1) \leq K$ is, by (8),

$$\begin{aligned} \pi_K(x/m) &\ll_k \frac{x(\log \log(x/m))^{K+1}}{m(\log(x/m))^2} \ll_k \frac{x(\log \log x)^{K+1}}{m(\log y)^2} \\ &\ll_k \frac{x(\log \log x)^{K+3}}{m(\log x)^2}. \end{aligned}$$

If we sum the preceding inequality over all the values of $m \leq x$ with $\omega(m) \leq 2(k+1)^2$, then it follows that the number of possibilities is

$$\begin{aligned} \#\mathcal{A}_{k,2}(x) &\ll_k \frac{x(\log \log x)^{K+3}}{(\log x)^2} \sum_{\substack{m \leq x \\ \omega(m) \leq 2(k+1)^2}} \frac{1}{m} \\ &\ll_k \frac{x(\log \log x)^{K+3}}{(\log x)^2} \sum_{\ell=0}^{2(k+1)^2} \frac{1}{\ell!} \left(\sum_{p^\alpha \leq x} \frac{1}{p^\alpha} \right)^\ell \\ &\ll_k \frac{x(\log \log x)^{K+3+2(k+1)^2}}{(\log x)^2}; \end{aligned}$$

this, together with (9), completes the proof of Theorem 3. \square

A more careful analysis (along the lines of the proof of [1, Thm. 4.1]) shows that Theorem 3 holds with a somewhat smaller b_k . Furthermore, it is clear that one could write down a formula for the implied constant in terms of k . We do not enter into such details.

5. A More General Statement

Let $f(x) \geq 1$ be any function that tends to infinity with n and that is monotonically decreasing for $x > x_0$. Let

$$\mathcal{B}_f = \{n : \tau(\lambda(n)) - \omega(n) < \exp((\log \log n)/f(n))\}. \quad (10)$$

We can then show the following result.

THEOREM 4. *If \mathcal{B}_f is the set appearing in (10), then*

$$\#\mathcal{B}_f(x) \leq \frac{x}{(\log x)^{2+o(1)}} \text{ as } x \rightarrow \infty.$$

We start by proving the following lemma.

LEMMA 3. *Let $\mathcal{P}_f = \{p : \omega(p-1) < 2(\log \log p)/\sqrt{f(p)}\}$. Then*

$$\#\mathcal{P}_f(x) \leq \frac{x}{(\log x)^{2+o(1)}} \text{ as } x \rightarrow \infty. \quad (11)$$

Proof. Let x be large, put $y = x^{1/\log \log x}$, and let

$$\mathcal{P}_2(x) = \{p \in \mathcal{P}_f(x) : p-1 \notin \Psi(x, y)\}.$$

If $p \in \mathcal{P}_2(x)$ then $p-1 = Qm$, where $Q = P(p-1) > y$ and $m \leq x/y$. Fix m . By Brun's method, the number of primes $Q \leq x/m$ such that $p = Qm + 1$ is also prime is

$$\ll \frac{x}{\varphi(m)(\log(x/m))^2} \leq \frac{x}{\varphi(m)(\log y)^2} \leq \frac{x(\log \log x)^2}{\varphi(m)(\log x)^2}.$$

Using the minimal order $\varphi(m)/m \gg 1/\log \log x$ of the Euler function in the interval $[1, x]$, we get that if m is fixed then the number of acceptable primes $p \in \mathcal{P}_2(x)$ with $(p-1)/P(p-1) = m$ is

$$\ll \frac{x(\log \log x)^3}{m(\log x)^2}.$$

Let $\mathcal{M}(x)$ be the set of acceptable values for m . Since $\omega(p-1) \leq 2(\log \log p)/\sqrt{f(p)}$, f is increasing for large x , and $p > y$ for all $p \in \mathcal{P}_2(x)$, it follows that

$$z = \max\{2(\log \log p)/\sqrt{f(p)} : p \in \mathcal{P}_2(x)\} \leq \frac{2 \log \log x}{\sqrt{f(y)}} = o(\log \log x) \quad (12)$$

as $x \rightarrow \infty$. Furthermore, $\mathcal{M}(x) \subseteq \{m \leq x : \omega(m) \leq z\}$. As a result,

$$\#\mathcal{P}_2(x) \ll \frac{x(\log \log x)^3}{(\log x)^2} \sum_{m \in \mathcal{M}(x)} \frac{1}{m} \ll \frac{x(\log \log x)^3}{(\log x)^2} \sum_{k \leq z} \sum_{\substack{m \leq x \\ \omega(m)=k}} \frac{1}{m}. \quad (13)$$

Put

$$\mathcal{S}_k(x) = \sum_{\substack{m \leq x \\ \omega(m)=k}} \frac{1}{m}.$$

Then unique factorization, the multinomial formula, and Stirling's formula imply that

$$\mathcal{S}_k(x) \leq \frac{1}{k!} \left(\sum_{p \leq x} \sum_{\alpha \geq 1} \frac{1}{p^\alpha} \right)^k \leq \left(\frac{e \log \log x + O(1)}{k} \right)^k,$$

where we have used the obvious fact that

$$\sum_{p \geq 2} \sum_{\alpha \geq 1} \frac{1}{p^\alpha} = O(1)$$

together with Mertens's formula

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + O(1).$$

For every fixed value of $A > 1$, the function $(eA/t)^t$ is increasing for $t < A$ and so

$$\begin{aligned} \mathcal{S}_k(x) &\leq \left(\frac{e \log \log x + O(1)}{z} \right)^z = \exp(z \log(e(\log \log x + O(1))/z)) \\ &\leq \exp\left(\frac{2 \log \log x}{\sqrt{f(y)}} \log(O(\sqrt{f(y)})) \right) = \exp(o(\log \log x)) \\ &= (\log x)^{o(1)} \quad \text{for } k \leq z. \end{aligned} \tag{14}$$

Hence, the inequalities (12) and (13) together with the estimate (14) yield

$$\begin{aligned} \#\mathcal{P}_2(x) &\ll \frac{x(\log \log x)^3}{(\log x)^2} \sum_{k \leq z} \mathcal{S}_k(x) \\ &\ll \frac{x(\log \log x)^4}{(\log x)^2} \max\{\mathcal{S}_k(x) : k \leq z\} = \frac{x}{(\log x)^{2+o(1)}}, \end{aligned}$$

combining this with estimate (7) implies inequality (11) and completes the proof of Lemma 3. \square

Now partial summation immediately yields our next result.

COROLLARY 2. *If \mathcal{P}_f is the set of primes appearing in Lemma 3, then*

$$\sum_{p \in \mathcal{P}_f} \frac{1}{p} = O(1).$$

Proof of Theorem 4. Let again $y = x^{1/\log \log x}$, $w = x/(\log x)^2$, and

$$\mathcal{B}_1(x) = \{n \leq w\} \cup \Psi(x, y).$$

By (7) we have

$$\#\mathcal{B}_1(x) \leq \frac{2x}{(\log x)^2} \tag{15}$$

once x is large. Let $\mathcal{B}_2(x) = \{n \leq x : \omega(n) > 10 \log \log x\}$. It follows from results of Norton [11; 12] that

$$\#\mathcal{B}_2(x) \ll \frac{x}{(\log x)^\lambda},$$

where $\lambda = 1 + 10 \log(10/e) > 2$; therefore,

$$\#\mathcal{B}_2(x) < \frac{x}{(\log x)^2}. \tag{16}$$

Now put

$$\mathcal{B}_3(x) = \mathcal{B}_f(x) \setminus (\mathcal{B}_1(x) \cup \mathcal{B}_2(x)),$$

and assume that $n \in \mathcal{B}_3(x)$. Replacing $f(x)$ with $\min\{f(x), \log \log \log x\}$, we may assume that $f(x) \leq \log \log \log x$. Then $p - 1 \mid \lambda(n)$ for all prime factors p of n and so

$$\begin{aligned} 2^{\omega(p-1)} &\leq \tau(\lambda(n)) \leq \omega(n) + \exp((\log \log n)/f(n)) \\ &< 10 \log \log x + \exp((\log \log x)/f(w)) \\ &< \exp\left(\frac{1.1(\log \log x)}{f(w)}\right), \end{aligned}$$

so

$$\omega(p-1) < \frac{1.6(\log \log x)}{\sqrt{f(w)}}, \quad (17)$$

where we used the fact that $1.1/\log 2 < 1.6$. Let $\mathcal{B}_4(x) = \{n \in \mathcal{B}_3(x) : P(n) > w\}$. Since $w \geq p/(\log p)^2$ holds for all $p \in [w, x]$ once x is large, it follows that if $p = P(n)$ for $n \in \mathcal{B}_4(x)$ then

$$\omega(p-1) < \frac{1.6(\log \log x)}{f(p/(\log p)^2)} < \frac{2(\log \log p)}{\sqrt{g(p)}}$$

holds for large x . Here g is the function $g(t) = (f(t/(\log t)^2))^2$, which is increasing for large t . Thus, $p \in \mathcal{P}_g$. Let us now write $n = Pm$, where $m < x/p < (\log x)^2$, and let us fix m . Then $p \in \mathcal{P}_g(x/m)$ and, by Lemma 3, the number of such choices for p is

$$\#\mathcal{P}_g(x/m) \leq \frac{x}{m(\log x/m)^{2+o(1)}} = \frac{x}{m(\log x)^{2+o(1)}}.$$

Summing this inequality for $m \leq (\log x)^2$, we have

$$\begin{aligned} \#\mathcal{B}_4(x) &\leq \sum_{m \leq (\log x)^2} \#\mathcal{P}_g\left(\frac{x}{m}\right) \\ &\leq \frac{x}{(\log x)^{2+o(1)}} \sum_{m \leq (\log x)^2} \frac{1}{m} \\ &= \frac{x}{(\log x)^{2+o(1)}}, \end{aligned} \quad (18)$$

because

$$\sum_{m \leq (\log x)^2} \frac{1}{m} \ll \log \log x = (\log x)^{o(1)}.$$

From now on we assume that $n \in \mathcal{B}_5(x) = \mathcal{B}_3(x) \setminus \mathcal{B}_4(x)$. Let $n = Pm$, where $P = P(n) \in [y, w]$. Since $1.6 \log \log x < 2 \log \log y \leq 2 \log \log P$ for large x and since $f(w) \geq f(P)$, it follows that

$$\omega(P-1) < \frac{1.6(\log \log x)}{f(w)} < \frac{2(\log \log P)}{f(P)}.$$

In particular, $P \in \mathcal{P}_{f^2}$. By Lemma 3, if $m \leq x/y$ is fixed then the number of choices for P is at most

$$\#\mathcal{P}_{f^2}(x/m) \leq \frac{x}{m(\log(x/m))^{2+o(1)}} \leq \frac{x}{m(\log y)^{2+o(1)}} \leq \frac{x}{m(\log x)^{2+o(1)}},$$

where we have used that $x/m \geq y$ and $\log y = \log x / \log \log x = (\log x)^{1+o(1)}$. Let $\mathcal{M}(x)$ be the set of acceptable values of m . Then

$$\#\mathcal{B}_5(x) \leq \sum_{m \in \mathcal{M}(x)} \frac{x}{m(\log x)^{2+o(1)}} \leq \frac{x}{(\log x)^{2+o(1)}} \sum_{m \in \mathcal{M}(x)} \frac{1}{m}. \quad (19)$$

Let $\mathcal{Q}(x)$ be the set of primes dividing some $m \in \mathcal{M}(x)$, and note that $\mathcal{Q}(x)$ consists of the primes $q \leq x$ satisfying inequality (17). We put

$$v = \exp\left(\exp\left(\frac{\log \log x}{\sqrt{f(w)}}\right)\right)$$

and split the primes in \mathcal{Q} into two subsets as follows:

$$\mathcal{Q}_1 = \{q \leq v\} \cap \mathcal{Q};$$

$$\mathcal{Q}_2 = \mathcal{Q} \cap [v, w].$$

Observe that if $q \in \mathcal{Q}_2$ then

$$\frac{2 \log \log q}{\sqrt{f(q)}} \geq \frac{2 \log \log x}{\sqrt{f(q)f(w)}} \geq \frac{2 \log \log x}{f(w)} > \omega(q-1);$$

therefore, $\mathcal{Q}_2 \subset \mathcal{P}_f$. This argument shows that

$$\begin{aligned} \sum_{m \in \mathcal{M}(x)} \frac{1}{m} &\leq \prod_{q \in \mathcal{Q}_1 \cup \mathcal{Q}_2} \left(\sum_{\alpha \geq 0} \frac{1}{q^\alpha} \right) \\ &\leq \exp\left(\sum_{q \in \mathcal{Q}_1} \frac{1}{q} + \sum_{q \in \mathcal{Q}_2} \frac{1}{q} + o\left(\sum_{q \geq 2} \sum_{\alpha \geq 2} \frac{1}{q^\alpha} \right) \right). \end{aligned} \quad (20)$$

Since

$$\sum_{q \in \mathcal{Q}_1} \frac{1}{q} \leq \sum_{q \leq v} \frac{1}{q} = \log \log v + O(1) = o(\log \log x)$$

(by Mertens's formula),

$$\sum_{q \in \mathcal{Q}_2} \frac{1}{q} \leq \sum_{q \in \mathcal{P}_f} \frac{1}{q} = O(1)$$

(by Corollary 2), and

$$\sum_{q \geq 2} \sum_{\alpha \geq 2} \frac{1}{q^\alpha} = O(1),$$

it follows from (20) that

$$\sum_{m \in \mathcal{M}(x)} \frac{1}{m} \leq \exp(o(\log \log x)) = (\log x)^{o(1)},$$

which together with (19) gives

$$\#\mathcal{B}_5(x) \leq \frac{x}{(\log x)^{2+o(1)}}. \quad (21)$$

Since $\mathcal{B}_3(x) \subseteq \mathcal{B}_4(x) \cup \mathcal{B}_5(x)$, by estimates (18) and (21) we have that

$$\#\mathcal{B}_3(x) < \frac{x}{(\log x)^{2+o(1)}}, \quad (22)$$

which together with estimates (15) and (16) completes the proof of Theorem 4. \square

6. Average and Normal Orders of $\tau(\lambda(n)) - \omega(n)$

Our last result addresses average and normal orders of the function

$$h(n) = \tau(\lambda(n)) - \omega(n).$$

THEOREM 5. (i) *There exist positive constants c_0, c_1 such that the inequalities*

$$\exp\left(c_0\sqrt{\frac{\log x}{\log \log x}}\right) \leq \frac{1}{x} \sum_{n \leq x} h(n) \leq \exp\left(c_1\sqrt{\frac{\log x}{\log \log x}}\right) \quad (23)$$

hold for all $x \geq 1$.

(ii) *The inequality*

$$h(n) = 2^{0.5(1+o(1))(\log \log n)^2}$$

holds for almost all positive integers n .

Proof. (i) In [10] it is shown that inequalities (23) hold with some constants c_0 and c_1 for the function $\tau(\lambda(n)) = h(n) + \omega(n)$. Since the average value of $\omega(n)$ is $\log \log x = \exp(o(\sqrt{\log x / \log \log x}))$, the required inequality follows.

(ii) In [5] it is shown that the normal order of both $\omega(\varphi(n))$ and $\Omega(\varphi(n))$ is $0.5(\log \log n)^2$. Since $\omega(\lambda(n)) = \omega(\varphi(n))$ and $\Omega(\lambda(n)) \leq \Omega(\varphi(n))$, it follows that the normal order of both $\omega(\lambda(n))$ and $\Omega(\lambda(n))$ is also $0.5(\log \log n)^2$. Finally, since

$$2^{\omega(\lambda(n))} \leq \tau(\lambda(n)) \leq 2^{\Omega(\lambda(n))}$$

and since the normal order of $\omega(n)$ is $\log \log n = 2^{o((\log \log n)^2)}$, the desired inequalities follow. \square

7. Remarks

We suspect that for every $k \geq 1$ there exist constants $a_k > 0$ and $c_k \geq 0$ such that

$$\#\mathcal{A}_k(x) = a_k(1+o(1)) \frac{x(\log \log x)^{c_k}}{(\log x)^2} \quad \text{as } x \rightarrow \infty. \quad (24)$$

Widely believed conjectures concerning the distribution of Sophie Germain primes p together with Proposition 3 seem to support conjecture (24) at $k = 1$ (with $c_1 = 0$ and some $a_1 > 0$). Note that an upper bound for this shape is given in Theorem 3.

We would like to leave this conjecture as an open problem.

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