Hadamard's factorization theorem for entire (integral) functions of order 1

Prerequisites.

1. *Theorem A (existence of the logarithm)* Let f be a nowhere vanishing holomorphic function in a simply connected region Ω. Then it is possible to define a holomorphic function $g\colon \Omega \to \mathbb{C}$ such that $e^{g(z)} = f(z)$. In other words, it is possible to determine a single valued branch of $\log(f(z))$. In the particular case of $\Omega = D_1(0)$ and $f(z) = 1 - z$ we can define $g(z)$ as $\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$.

2. *Theorem B (mean value theorem for harmonic functions)* Let g be a holomorphic function in the disk $D_R(z_0)$. Then
$$\Re g(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \Re g(z_0 + re^{i\theta})d\theta \quad \forall\ 0 < r < R.$$

3. *Theorem C (sufficient condition for the convergence of an infinite product)* Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on the open set Ω. Suppose that there exists a sequence $(c_n)_{n\in\mathbb{N}}$ of nonnegative numbers such that $|1 - F_n(s)| \le c_n$ for all $s \in \Omega$ and $n \in \mathbb{N}$. If $\sum_n c_n < +\infty$ then
   (a) $\prod_n F_n(s)$ converges uniformly to a holomorphic function F in Ω;
   (b) F vanishes at z_0 if and only if at least one factor F_n vanishes at z_0.

Exercises

1. Determine the Hadamard factorization of
   (a) $\cos \pi z$;
   (b) $e^z - 1$.

2. Show that for $|z| < 1$ we have
$$\prod_{n=0}^{\infty}(1 + z^{2^n}) = \sum_{n=0}^{\infty} z^n.$$

3. Determine a sequence of complex numbers $(a_n)_{n\in\mathbb{N}}$ such that $\prod_n(1 + a_n)$ converges but $\sum_n a_n$ diverges.

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5. Let f be an (integral) entire function of finite order. Deduce from the characterization of nowhere vanishing functions that if f misses two distinct values then f is constant (the conclusion of the theorem remains valid under the hypothesis that f is an entire function).