FAMILY NAME .... *NAME* .... *MATRICOLA* ....

Solve as many problems as possible by giving clear and essential explenations. *Write each solution in the appropriate space. SOLUTIONS IN OTHER SHEETS WILL NOT BE ACCEPTED.* 1 Problem = 4 points. Time: 2 hours.

SIGNATURE	1	2	3	4	5	6	7	8	TOT.
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1. Let p be a prime number, let $\mathbf{F}_{p^n}$ be a finite field with p^n elements, let $f \in \mathbf{F}_p[x]$ and let $\alpha \in \mathbf{F}_{p^n}$ be a root of f.
   a. Show that also α^p is a root of f.
   b. Show that for every positive integer k, α^{p^k} is a root of f.
   c. Show that if f is irriducibile and $n = \deg f$, then $\alpha, \alpha^p, \cdots, \alpha^{p^{n-1}}$ are all distinct.
   d. Deduce that every finite field with p^n elements is a normal extension of $\mathbf{F}_p$.

2. Give the definition of an algebraic closed field and of the algebraic closure of a field.

3. Determine the degree of the splitting field of $(x^3 - 2)(x^3 - 5)(x^2 + x + 1)$ over $\mathbf{Q}$.

4. Show that if $(x, y) \in \mathbf{C}$ is costructible, then $\mathbf{Q}(x, y)/\mathbf{Q}$ is finite and $[\mathbf{Q}(x, y) : \mathbf{Q}]$ is a power of 2.

5. Let $K = \mathbf{Q}(\sqrt{3}, \sqrt{5})$
   a. Compute $[K : \mathbf{Q}]$ and show that $K = \mathbf{Q}(\sqrt{3} + \sqrt{5})$
   a. Compute minimum polynomial of $\sqrt{3} + \sqrt{5}$ over $\mathbf{Q}$ and over $\mathbf{Q}(\sqrt{3})$
   a. After having shown that $\mathbf{Q}(\sqrt{15}) \subseteq K$, describe the monomorphisms $K \to \mathbf{C}$ that fix $\mathbf{Q}(\sqrt{15})$.

6. Consider the cyclotomic field $\mathbf{Q}(\zeta_{15})$ ($\zeta_{15} = e^{2\pi/15}$).
   a. Compute the mimimal polynomial of ζ_{15} over $\mathbf{Q}$
   b. Compute the mimimal polynomial of ζ_{15} over $\mathbf{Q}(\zeta_3)$ and over $\mathbf{Q}(\zeta_5)$
   c. Determine all the automorphisms of $\mathbf{Q}(\zeta_{15})$ that fix $\mathbf{Q}(\zeta_3)$

7. After having shown that it is algebraic, compute the minimal polynomial of $\cos 2\pi/15$ over $\mathbf{Q}$. **(hint: if $\theta = 2\pi/15$, consider the $\cos(5\theta)$ and apply the classical formulas from trigonometry)**

8. State and prove the "multiplicativity of degrees Theorem" (if $K \subseteq L \subseteq M$, then $[M : K] = [M : L][L : K]$).