*Family Name* ...................... *Name* ......................*Student ID(Matricola):* ......................
Solve the problems adding to the replies short and essential explenations. *Please write the solutions in the designed areas.*
*NO EXTRA SHEETS WILL BE ACCEPTED.* 1 Problem = 4 marks. Duration: 2 hours. No questions allowed in the first hour and in the last 20 minutes.

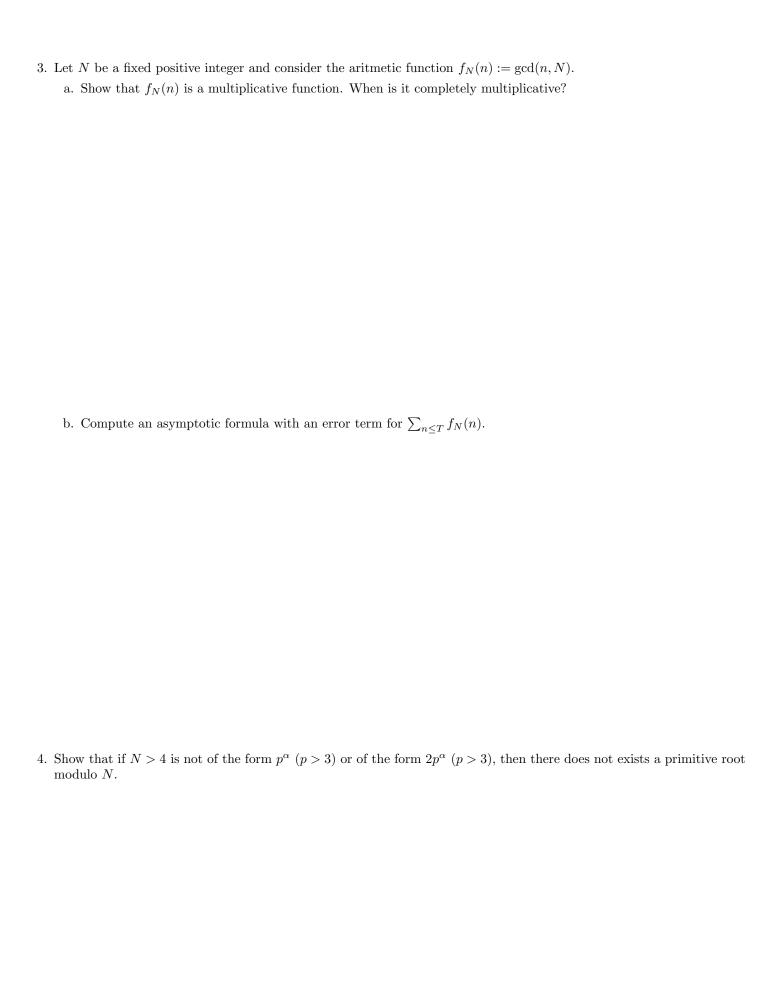| 1 | 2 | 3a | 3b | 4 | 5 | 6 | 7 | 8 | TOTAL | |
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|   |   |    |    |   |   |   |   |   |       |   |

1. Determine the number of solutions modulo $3^3 \times 5^3$ of the congruence $X^{20} - 1 \equiv 0 (\mathrm{mod} 3^2 \times 5^2)$ .

2. Prove that a real number admits a finite continued fraction if and only if it is rational.

3. Let $N$ be a fixed positive integer and consider the aritmetic function $f_N(n) := \gcd(n, N)$.

   a. Show that $f_N(n)$ is a multiplicative function. When is it completely multiplicative?

   b. Compute an asymptotic formula with an error term for $\sum_{n \leq T} f_N(n)$.

4. Show that if $N > 4$ is not of the form $p^\alpha$ ($p > 3$) or of the form $2p^\alpha$ ($p > 3$), then there does not exists a primitive root modulo $N$.

5. Determine all primitive roots modulo $2 \times 19^4$,

6. Determine all integers $n$ such that $\sigma(n) = 16$.

7. Determine a formula for the number of pairs in $(x, y) \in \mathbf{Z}^2$ such that $x^2 + y^2 = 2^a 5^c 13^d$.

8. Give a complete proof of Lagrange Theorem that states that every integer can be written as the sum of at most four squares.