Values of the Euler function free of k-th powers

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Abstract

We establish an asymptotic formula for the number of positive integers $n \leq x$ for which $\varphi(n)$ is free of k-th powers.

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1 Introduction

Let $\varphi(n)$ denote the *Euler function*, which is defined for all $n \in \mathbb{N}$ by

$$\varphi(n) := \#(\mathbb{Z}/n\mathbb{Z})^\times = \prod_{p^a \| n} p^{a-1}(p-1).$$

Here $p^a \| n$ means that $p^a \mid n$ but $p^{a+1} \nmid n$. We recall that an integer m is called *k-free* if $p^k \nmid m$ for any prime number p. In this paper, we study the set of integers $n \in \mathbb{N}$ for which $\varphi(n)$ is k-free.

In the special case $k = 2$, it is easy to see that if $m = \varphi(n)$ is *squarefree*, then the following properties hold:

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- If a prime p divides n, then $p-1$ is squarefree;
- $p^3 \nmid n$ for any prime p;
- If $4 \,|\, n$, then $p \nmid n$ for any odd prime p (thus, $n = 4$);
- If $4 \nmid n$, then $p \,|\, n$ for at most one odd prime p.

These properties imply that $n \in \{p, 2p, p^2, 2p^2\}$ for some prime p for which $p-1$ is squarefree, hence the problem of estimating the number of positive integers $n \le x$ for which $\varphi(n)$ is squarefree reduces to that of estimating the number of primes $p \le x$ for which $p-1$ is squarefree. These questions have been previously investigated. For example, Mirsky [9] (see also [7,14]) has shown that for any constant $C > 0$, the asymptotic relation

$$\#\{p \le x : p-1 \text{ is squarefree}\} = \alpha_2\, \pi(x) + O\left(\frac{x}{\log^C x}\right)$$

holds, and in [14], this fact is used to establish the formula

$$\#\{n \le x : \varphi(n) \text{ is squarefree}\} = \frac{3\alpha_2}{2}\, \pi(x) + O\left(\frac{x}{\log^C x}\right). \tag{1}$$

Here, α_2 is the *Artin constant*:

$$\alpha_2 := \prod_p \left(1 - \frac{1}{p(p-1)}\right) = 0.3739558136\cdots, \tag{2}$$

and the implied constant in the Landau symbol depends only on C.

Here, we study the same question for an arbitrary (but fixed) integer $k \ge 3$. Our main result is an asymptotic formula for the counting function $\#\mathcal{A}_k(x)$ of the set

$$\mathcal{A}_k(x) := \{n \le x : \varphi(n) \text{ is } k\text{-free}\}.$$

Clearly, since $2^k \nmid \varphi(n)$ for any $n \in \mathcal{A}_k(x)$, every such n can have at most $k-1$ distinct odd prime factors. Moreover, it is natural to expect that integers with precisely $k-1$ distinct prime factors make the largest contribution to $\#\mathcal{A}_k(x)$. In view of the well-known result of Landau [8] on the number of integers $n \le x$ with ℓ prime factors (stated as Theorem 2.6 below), the rough estimate

$$\#\mathcal{A}_k(x) \asymp \frac{x\,(\log\log x)^{k-2}}{\log x}$$

is not too surprising and can probably be established using simpler methods than those presented here. The problem of determining a precise asymptotic formula for $\#\mathcal{A}_k(x)$, however, is rather delicate.

We remark that the tools used to derive the precise estimate (1) in the case $k = 2$, for which one has a very simple description of the set $\mathcal{A}_k(x)$, are no longer available once $k > 2$. Also, in contrast to the case $k = 2$, when $k \ge 3$

there is a significant contribution to $\#\mathcal{A}_k(x)$, of order $\#\mathcal{A}_k(x)/\log\log x$, which comes from integers with precisely $k-2$ distinct prime factors.

In order to state our main result, let us define

$$\beta_{k,p} := \sum_{i=0}^{k-2}\sum_{j=0}^{k-2-i}\binom{k-1}{i}\binom{k-1+j}{j}\left(\frac{1}{p-1}\right)^i\left(1-\frac{1}{p-1}\right)^j$$

for every integer $k \geq 2$ and prime $p \geq 2$ (in the case $p = 2$, we adopt the convention that $0^0 = 1$, and thus $\beta_{k,2} = 2^{k-1} - 1$), and put

$$\alpha_k := \prod_p \left(1 - \frac{\beta_{k,p}}{p^{k-1}(p-1)}\right).$$

Theorem 1.1. *For every fixed integer $k \geq 3$, the asymptotic formula*

$$\#\mathcal{A}_k(x) = \frac{3\alpha_k}{2}\frac{x\,(\log\log x)^{k-2}}{(k-2)!\,\log x}\,(1+o(1))$$

holds as $x \to \infty$, where the function implied by $o(1)$ depends only on k.

Remarks. Taking $k = 2$ in the statement of Theorem 1.1, one recovers the asymptotic formula (1) (with an imprecise error term). Our proof of Theorem 1.1 uses *elementary methods*, and in several instances, we have followed closely certain arguments from the book [11] by Nathanson. We also remark that, as is clear from our proof, we do not need the full strength of some of the known results that are recalled in Section 2. In the proof, we show that the $o(1)$ error term in the statement of the theorem is of size

$$O_k\left(\frac{(\log\log\log x)^{2(k+1)^{2k-4}-1}}{(\log\log x)^{1-1/k}}\right),$$

which is reasonably close to the expected error $E_k(x) \asymp_k 1/\log\log x$ when k is large; however, our main focus in this paper is the determination of the main term for $\#\mathcal{A}_k(x)$. Finally, using the method of Moree [10] and the Pari program, we have computed the fifty decimal digits of each constant α_k with $2 \leq k \leq 10$:

$$\alpha_2 = 0.37395\,58136\,19202\,28805\,47280\,54346\,41641\,51116\,29248\,60615\cdots$$
$$\alpha_3 = 0.18984\,91224\,20235\,31991\,66681\,67621\,86073\,62451\,02880\,08604\cdots$$
$$\alpha_4 = 0.09671\,30507\,43055\,56792\,23408\,99706\,16695\,80481\,99713\,96223\cdots$$
$$\alpha_5 = 0.04914\,98625\,18789\,68323\,37383\,93355\,96691\,22084\,26605\,56681\cdots$$
$$\alpha_6 = 0.02492\,28833\,35419\,10300\,95723\,08364\,04557\,22962\,18064\,16898\cdots$$
$$\alpha_7 = 0.01261\,64236\,57195\,48181\,66980\,78776\,82881\,81757\,58940\,73078\cdots$$

$$\alpha_8 = 0.00637\,83900\,14315\,82338\,85155\,40430\,85389\,84736\,02454\,15212\cdots$$
$$\alpha_9 = 0.00322\,14087\,53892\,82515\,40641\,09868\,03144\,90880\,21993\,85574\cdots$$
$$\alpha_{10} = 0.00162\,56470\,18001\,48232\,26934\,50203\,11279\,49745\,37231\,54564\cdots$$

Hundreds of decimal digits can be determined using the same method.

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2 Preparations

Here we collect several results that are needed in the sequel.
Theorem 2.1. (Siegel-Walfisz) *There exists an absolute constant $c_1 > 0$ such that for any constant $c > 0$ and uniformly for*

$$x \geq 3, \qquad 1 \leq m \leq \log^c x \qquad \textit{and} \quad (a, m) = 1,$$

one has

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod m}} \Lambda(n) = \frac{x}{\varphi(m)} + O_c\left(xe^{-c_1\sqrt{\log x}}\right),$$

where $\Lambda(n)$ is the von Mangoldt function:

$$\Lambda(n) := \begin{cases} \log p & \textit{if } n \textit{ is a power of the prime } p, \\ 0 & \textit{otherwise.} \end{cases}$$

The reader can find a detailed proof of Theorem 2.1 in the book by Huxley [6] or in that by Ellison and Mendès-France [4].

Let $\mathcal{P} \subset \mathbb{N}$ denote the set of prime numbers, and for any real number $x \geq 1$, let $\mathcal{P}(x)$ be the set of primes such that $p \leq x$.
Corollary 2.1. *There exists an absolute constant $c_1 > 0$ such that for any*

constant $c > 0$, the relation

$$\sum_{\substack{p \in \mathcal{P}(x) \\ p \equiv 1 \pmod m}} \log p = \frac{x}{\varphi(m)} + O_c\left(x e^{-c_1\sqrt{\log x}}\right)$$

holds uniformly for $x \geq 3$ and $1 \leq m \leq \log^c x$.

Proof. If $\pi(y)$ denotes the number of primes $p \leq y$, one has the estimate

$$\sum_{\substack{p \in \mathcal{P},\ a \geq 2 \\ p^a \leq x}} \log p \leq \sum_{p \in \mathcal{P}(x^{1/2})} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \leq \pi(x^{1/2}) \log x = O(x^{1/2}),$$

and the result follows immediately from Theorem 2.1. $\square$

The following result is due to Norton [12]; see also Theorem 1 in [13].
Theorem 2.2. *The relation*

$$\sum_{\substack{p \in \mathcal{P}(x) \\ p \equiv 1 \pmod m}} \frac{1}{p} = \frac{\log \log x}{\varphi(m)} + O(1)$$

holds uniformly for $x \geq 3$ and $m \geq 1$, where the implied constant is absolute.

We also need the Brun-Titchmarsh theorem; for example, see Theorem 3.7 in Chapter 3 of [5]. Let $\pi(x; m, a)$ denote the number of primes $p \leq x$ such that $p \equiv a \pmod m$.
Theorem 2.3. (Brun-Titchmarsh) *For integers m, a with $m \geq 1$ and for all $x \geq m$, the bound*

$$\pi(x; m, a) \ll \frac{x}{\varphi(m)\big(1 + \log(x/m)\big)}$$

holds, where the implied constant is absolute.

We also need the following well-known result (see for example Theorem 9 in §I.1.5 of [15]):
Theorem 2.4. (Mertens) *There exists an absolute constant c such that for all $x \geq 2$, one has*

$$\sum_{p \in \mathcal{P}(x)} \frac{1}{p} = \log \log x + c + O\left(\frac{1}{\log x}\right).$$

For any integer $\ell \in \mathbb{N}$, let $\mathcal{P}^\ell$ denote the set of ordered ℓ-tuples of primes. For any real number $x \geq 1$, let $\mathcal{P}^\ell(x)$ be the set of ordered ℓ-tuples $(p_1, \ldots, p_\ell)$ in $\mathcal{P}^\ell$ such that $p_1 \cdots p_\ell \leq x$.

Theorem 2.5. (Landau) *For fixed $\ell \in \mathbb{N}$, the estimates*

$$S_\ell(x) := \sum_{(p_1,\ldots,p_\ell)\in\mathcal{P}^\ell(x)} \frac{1}{p_1\cdots p_\ell} \sim (\log\log x)^\ell$$

and

$$\vartheta_\ell(x) := \sum_{(p_1,\ldots,p_\ell)\in\mathcal{P}^\ell(x)} \log(p_1\cdots p_\ell) \sim \ell x(\log\log x)^{\ell-1}$$

hold as $x \to \infty$.

For an elementary proof of Theorem 2.5, we refer the reader to Theorems 9.7 and 9.8 in the book by Nathanson [11].

Now let $\omega(n)$ denote the number of distinct prime divisors of an integer $n \in \mathbb{N}$. Then we have the following well-known result of Landau [8] (see also Theorem 9.9 in [11]):

Theorem 2.6. (Landau) *For every fixed integer $\ell \in \mathbb{N}$, the asymptotic formula*

$$\#\{n \le x : \omega(n) = \ell\} \sim \frac{x\,(\log\log x)^{\ell-1}}{(\ell-1)!\,\log x}$$

holds as $x \to \infty$.

Let $\mu(n)$ be the Möbius function, and let $\tau(n)$ be the number of positive integral divisors of $n \in \mathbb{N}$.

Lemma 2.1. *For any positive integer t, the following estimate holds:*

$$\sum_{n\le y} \tau(n)^t \ll_t y(\log y)^{2^t-1}$$

For any integers $t, r \in \mathbb{N}$, we also have

$$\sum_{n>y} \frac{\mu^2(n)\tau(n)^t}{n^r\varphi(n)} \ll_t \frac{(\log y)^{2^t-1}}{y^r}.$$

Proof. The first estimate is well-known and follows directly from the theorem of Wirsing [16] which states that if $f(n)$ is a non-negative multiplicative arithmetic function such that:

(i) $f(p^\nu) \le c_1 c_2^\nu$ holds for any prime p and integer $\nu \ge 2$, where c_1, c_2 are positive constants with $c_2 < 2$;

(ii) $\sum_{p\le x} f(p) \sim \tau x/\log x$ holds as $x \to \infty$, where τ is a positive constant;

then, as $x \to \infty$, one has

$$\sum_{n\le x} f(n) \sim \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p\le x}\left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right)$$

where γ is the Euler-Mascheroni constant, and $\Gamma(s)$ is the Euler Γ-function. Indeed, applying Wirsing theorem with the function $f(n) = \tau(n)^t$, we obtain

$$\sum_{n\le y} \tau(n)^t \sim \frac{e^{-\gamma 2^t}}{\Gamma(2^t)} \frac{y}{\log y} \prod_{p\le y} \left(1 + \frac{2^t}{p} + \frac{3^t}{p^2} + \cdots\right),$$

and the first estimate follows from the observation that

$$\prod_{p\le y} \left(1 + \frac{2^t}{p} + \frac{3^t}{p^2} + \cdots\right) \ll \prod_{p\le y} \left(1 - \frac{1}{p}\right)^{-2^t} \ll (\log y)^{2^t},$$

where we have used Merten's theorem in the last step.

For the second estimate, we first apply Wirsing's theorem with the function

$$f(n) = \frac{\mu^2(n) n \tau(n)^t}{\varphi(n)},$$

which implies

$$\mathcal{D}(x) = \sum_{n\le x} \frac{\mu^2(n) n \tau(n)^t}{\varphi(n)} \sim \frac{e^{-\gamma 2^t}}{\Gamma(2^t)} \frac{x}{\log x} \prod_{p\le x} \left(1 + \frac{2^t}{p-1}\right) \ll x(\log x)^{2^t - 1}.$$

Then, by partial summation, we have

$$\sum_{n>y} \frac{\mu^2(n) \tau(n)^t}{n^r \varphi(n)} = -\frac{\mathcal{D}(y)}{y^{r+1}} + (r+1) \int_y^\infty \frac{\mathcal{D}(s)}{s^{r+2}}\, ds \ll \frac{(\log y)^{2^t - 1}}{y^r},$$

and this completes the proof. □

Finally, we need the following estimate, which is a simplified and weakened form of Lemma 2 from [2]:
Lemma 2.2. *Let $T(x, w, q)$ denote the number of positive integers $n \le x$ such that $\omega(n) \le w$ and $\varphi(n) \equiv 0 \pmod q$. Then*

$$T(x, w, q) \ll_w \frac{x(\log\log x)^{w-1}}{q^{2/5}}.$$

Proof. By Lemma 2 from [2],

$$T(x, w, q) \ll x(c \log\log x)^{w-1} \left(\frac{\tau_w(q)\tau(q)}{q}\right)^{1/2}$$

for some absolute constant $c > 0$, where $\tau_w(q)$ is the number of representations of n as an ordered product of w positive integers. It is well-known that $\tau_w(q) = O_\varepsilon(q^\varepsilon)$ for any fixed $\varepsilon > 0$; in particular, $\tau_w(q)\tau(q) \le q^{1/5}$ for all sufficiently large values of q (depending on w), and the lemma follows. □

3 The Seven Hills of Rome

The proof of Theorem 1.1 consists of seven individual steps (which, to commemorate the visit of the first author to Rome, we chose to name after the seven hills); these results are combined in the next section. The first step deals with the problem of expressing the constant α_k as an Euler product. The second step addresses issues related to the convergence of the series that defines α_k. The following three steps present an adaptation of the method of Landau for counting integers with a fixed number of prime divisors, which is applicable to the present situation. The last two steps concern our use of the inclusion-exclusion principle to eliminate integers n for which $\varphi(n)$ is not k-free.

$\sim$ *Viminal* $\sim$

Lemma 3.1. *For every $\ell \in \mathbb{N}$, let $\Delta_\ell(m)$ be the arithmetical function defined inductively by*

$$\Delta_\ell(m) := \begin{cases} \dfrac{1}{\varphi(m)} & \text{if } \ell = 1, \\[2ex] \displaystyle\sum_{\substack{d \mid m \\ e \mid (m/d)}} \frac{\mu(e)}{\varphi(ed)}\, \Delta_{\ell-1}\Big(\frac{m}{d}\Big) & \text{if } \ell \geq 2. \end{cases}$$

Then $\Delta_\ell(m)$ is multiplicative. For every integer $a \geq 1$ and prime $p \geq 2$, we have

$$\Delta_\ell(p^a) = \frac{1}{\varphi(p^a)} \sum_{i=0}^{\ell-1} \sum_{j=0}^{\ell-1-i} \binom{a-1}{i} \binom{a-1+j}{j} \left(\frac{1}{p-1}\right)^i \left(1 - \frac{1}{p-1}\right)^j. \quad (3)$$

Proof. One can verify directly that if $\{\Delta_\ell : \ell \in \mathbb{N}\}$ are the multiplicative functions defined on prime powers by (3), then these functions also satisfy the stated inductive property. In this proof, however, we show how to deduce (3) directly from the inductive definition using the method of *generating functions* since the results we obtain along the way are useful for estimating Δ_ℓ in Lemma 3.2 below. We remark that, for the special case $p = 2$, the formula (3) simplifies to

$$\Delta_\ell(2^a) = \frac{1}{\varphi(2^a)} \sum_{i=0}^{\ell-1} \binom{a-1}{i}.$$

To show that $\Delta_\ell(m)$ is multiplicative, we use induction on ℓ, the case $\ell = 1$ being obvious. Suppose that $\Delta_{\ell-1}(m)$ is multiplicative, and let $m = m_1 m_2$, where m_1 and m_2 are coprime. For any divisors $d\,|\,m$ and $e\,|\,(m/d)$, let $d_j = (d, m_j)$ and $e_j = (e, m_j)$, $j = 1, 2$. Then

$$\begin{aligned}
\Delta_\ell(m_1m_2) &= \sum_{\substack{d|m_1m_2\\ e|(m_1m_2/d)}} \frac{\mu(e)}{\varphi(ed)}\,\Delta_{\ell-1}\Big(\frac{m_1m_2}{d}\Big)\\
&= \sum_{\substack{d_1|m_1\\ e_1|(m_1/d_1)}} \sum_{\substack{d_2|m_2\\ e_2|(m_2/d_2)}} \frac{\mu(e_1e_2)}{\varphi(e_1e_2d_1d_2)}\,\Delta_{\ell-1}\Big(\frac{m_1m_2}{d_1d_2}\Big)\\
&= \sum_{\substack{d_1|m_1\\ e_1|(m_1/d_1)}} \frac{\mu(e_1)}{\varphi(e_1d_1)}\,\Delta_{\ell-1}\Big(\frac{m_1}{d_1}\Big) \sum_{\substack{d_2|m_2\\ e_2|(m_2/d_2)}} \frac{\mu(e_2)}{\varphi(e_2d_2)}\,\Delta_{\ell-1}\Big(\frac{m_2}{d_2}\Big)\\
&= \Delta_\ell(m_1)\Delta_\ell(m_2),
\end{aligned}$$

which shows that $\Delta_\ell(m)$ is multiplicative.

Suppose that $m = p^a$ where p is an odd prime and $a \geq 0$. Write $d = p^b$ for each divisor of m and $e = p^c$ for each divisor of m/d. Then the inductive formula for $\ell \geq 2$ becomes

$$\Delta_\ell(p^a) = \sum_{b=0}^{a}\sum_{c=0}^{a-b} \frac{\mu(p^c)}{\varphi(p^{b+c})}\Delta_{\ell-1}(p^{a-b}).$$

For the inner sum, since $\mu(n)$ is supported on squarefree integers, we see that

$$\sum_{c=0}^{a-b} \frac{\mu(p^c)}{\varphi(p^{b+c})}\Delta_{\ell-1}(p^{a-b}) = \begin{cases} \dfrac{1}{\varphi(p^a)} & \text{if } b = a,\\[2ex] \Big(1 - \dfrac{1}{p-1}\Big)\,\Delta_{\ell-1}(p^a) & \text{if } b = 0,\\[2ex] \dfrac{1}{p^b}\Delta_{\ell-1}(p^{a-b}) & \text{otherwise.} \end{cases}$$

Thus, if we define $D_\ell(a) := \varphi(p^a)\Delta_\ell(p^a)$, it follows that

$$D_\ell(a) = \begin{cases} 1 & \text{if } a = 0 \text{ or } \ell = 1,\\ \dfrac{-1}{p-1}D_{\ell-1}(a) + \displaystyle\sum_{b=0}^{a} D_{\ell-1}(b) & \text{otherwise.} \end{cases} \tag{4}$$

Now for each $a \geq 0$, let $E_a(x)$ be the generating function given by

$$E_a(x) := \sum_{\ell=1}^{\infty} D_\ell(a)x^\ell.$$

Using (4) it is easy to see that

$$E_a(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } a = 0, \\ \\ x + x\left(1 - \dfrac{1}{p-1}\right) E_a(x) + x \displaystyle\sum_{b=0}^{a-1} E_b(x) & \text{otherwise.} \end{cases}$$

By induction on a, one immediately verifies that

$$E_a(x) = \frac{x\left(1 + \frac{x}{p-1}\right)^{a-1}}{(1-x)\left(1 + \left(\frac{1}{p-1} - 1\right)x\right)^a}.$$

for every $a \geq 1$. Extracting the coefficient of x^ℓ from this expression, we find that

$$D_\ell(a) = \sum_{\substack{i,j,m \geq 0 \\ i+j+m=\ell-1}} \binom{a-1}{i}\binom{-a}{j}\left(\frac{1}{p-1}\right)^i \left(\frac{1}{p-1} - 1\right)^j,$$

and the result follows from this using standard algebraic manipulations. The case $p = 2$ is similar and somewhat easier since $1 - 1/(p-1) = 0$; the details are omitted. $\square$

∼ *Palatine* ∼

As usual, we denote by $\Omega(n)$ the number of prime factors of $n \geq 2$ counted with multiplicity, and we denote by $\omega(n)$ the number of distinct prime divisors of n; we also put $\Omega(1) = \omega(1) = 0$.

Lemma 3.2. *Let $\Delta_\ell(m)$ be the arithmetical function defined in Lemma 3.1. Then the following estimate holds for all $m \in \mathbb{N}$:*

$$\Delta_\ell(m) \leq \frac{2^{\Omega(m)+\omega(m)\ell}}{\varphi(m)}.$$

Furthermore, for $k \geq 3$ one has

$$\sum_{m \leq y} \mu(m)\Delta_{k-1}(m^k) = \alpha_k + O_k\left(\frac{(\log y)^{2^{2k-1}-1}}{y^{k-1}}\right),$$

where α_k is the constant defined in Theorem 1.1.

Proof. We first consider the case when $m = p^a$ for some prime $p \geq 3$ and $a \geq 1$, the cases $p = 2$ and $a = 0$ being obvious. As in the proof of Lemma 3.1, we have the formal relation:

$$\sum_{\ell=1}^{\infty} \varphi(p^a)\Delta_\ell(p^a)x^\ell = \frac{x\left(1 + \frac{x}{p-1}\right)^{a-1}}{(1-x)\left(1 + \left(\frac{1}{p-1} - 1\right)x\right)^a}$$

which is an identity of analytic functions whenever $|x| < (p-1)/(p-2)$. Taking $x = 1/2$ we deduce that

$$\varphi(p^a)\Delta_\ell(p^a) \le 2^{a+\ell}$$

which yields the stated bound in this case. For general $m \in \mathbb{N}$, we have

$$\Delta_\ell(m) = \prod_{p^a \| m} \Delta_\ell(p^a) \le \prod_{p^a \| m} \frac{2^{a+\ell}}{\varphi(p^a)} \frac{2^{\Omega(m)+\omega(m)\ell}}{\varphi(m)}.$$

For the second statement of the lemma, we first observe that

$$\sum_{m \le y} \mu(m)\Delta_{k-1}(m^k) = \sum_{m=1}^{\infty} \mu(m)\Delta_{k-1}(m^k) + O\left(\sum_{m>y} \mu^2(m)\Delta_{k-1}(m^k)\right).$$

Using the multiplicativity of Δ_{k-1}, we immediately deduce that

$$\sum_{m=1}^{\infty} \mu(m)\Delta_{k-1}(m^k) = \prod_{p \in \mathcal{P}} \left(1 - \Delta_{k-1}(p^k)\right) = \prod_{p \in \mathcal{P}} \left(1 - \frac{\beta_{k,p}}{p^{k-1}(p-1)}\right) = \alpha_k,$$

where $\{\beta_{k,p} : p \in \mathcal{P}\}$ are the constants defined in Theorem 1.1. If m is squarefree, we also have $\Omega(m^k) = k\omega(m)$, hence by the results above:

$$\Delta_{k-1}(m^k) \le \frac{2^{(2k-1)\omega(m)}}{\varphi(m^k)} = \frac{\tau(m)^{2k-1}}{m^{k-1}\varphi(m)}.$$

Therefore, by the second part of Lemma 2.1, it follows that

$$\sum_{m>y} \mu^2(m)\Delta_{k-1}(m^k) \le \sum_{m>y} \frac{\mu(m)^2\tau(m)^{2k-1}}{m^{k-1}\varphi(m)} \ll_k \frac{(\log y)^{2^{2k-1}-1}}{y^{k-1}},$$

which completes the proof. $\square$

$\sim$ *Aventine* $\sim$

Let $F(n)$ be the completely multiplicative function defined for all $n \in \mathbb{N}$ by

$$F(n) := \prod_{p^a \| n} (p-1)^a.$$

Observe that $F(n) = \varphi(n)$ whenever n is squarefree.

Recall that for any integer $\ell \in \mathbb{N}$, $\mathcal{P}^\ell$ denotes the set of ordered ℓ-tuples of primes, and for any real number $x \ge 1$, $\mathcal{P}^\ell(x)$ is the set of ordered ℓ-tuples $(p_1, \ldots, p_\ell)$ in $\mathcal{P}^\ell$ such that $p_1 \cdots p_\ell \le x$. For every $m \in \mathbb{N}$, we now put

$$\mathcal{P}^\ell(x; m) := \left\{(p_1, \ldots, p_\ell) \in \mathcal{P}^\ell(x) : m | F(p_1 \cdots p_\ell)\right\}.$$

The following result is not needed for our proof of Theorem 1.1, but we believe it to be of independent interest.

Lemma 3.3. *For all $\ell, m \in \mathbb{N}$ and $x \geq 1$, let*

$$S_\ell(x;m) := \sum_{(p_1,\ldots,p_\ell)\in\mathcal{P}^\ell(x;m)} \frac{1}{p_1\cdots p_\ell}.$$

Then for any constant $c > 0$ and any fixed $\ell \in \mathbb{N}$, and uniformly for $x \geq e^{2^{\ell+1}}$ and $1 \leq m \leq \log^c x$, we have

$$S_\ell(x;m) = (\log\log x)^\ell \left(\Delta_\ell(m) + O_{c,\ell}\left(\frac{\tau(m)^{2\ell-2}}{\log\log x}\right)\right),$$

where $\Delta_\ell(m)$ is the arithmetical function considered before.

Proof. To simplify the notation, let us write $\mathcal{L}(x) := \log\log x$ in what follows.

If $\ell = 1$, then by Theorem 2.2 we have for $x \geq 3$:

$$S_1(x;m) = \sum_{\substack{p\in\mathcal{P}(x)\\ p\equiv 1 \pmod m}} \frac{1}{p} = \frac{\mathcal{L}(x)}{\varphi(m)} + O(1) = \mathcal{L}(x)\left(\Delta_1(m) + O\left(\mathcal{L}(x)^{-1}\right)\right),$$

and the lemma is proved in this case.

Now suppose that $\ell \geq 2$ and that the lemma has been proved for $\ell - 1$. If $(p_1,\ldots,p_\ell)$ is any ℓ-tuple in $\mathcal{P}^\ell(x;m)$, then $m|(p_1-1)\cdots(p_\ell-1)$. Collecting together terms with $(p_\ell - 1, m) = d$ for each divisor d of m, we see that

$$\begin{aligned}
S_\ell(x;m) &= \sum_{d|m} \sum_{\substack{p_\ell\in\mathcal{P}(x)\\ (p_\ell-1,m)=d}} \frac{1}{p_\ell} \sum_{(p_1,\ldots,p_{\ell-1})\in\mathcal{P}^{\ell-1}(x/p_\ell;m/d)} \frac{1}{p_1\cdots p_{\ell-1}}\\
&= \sum_{d|m} \sum_{\substack{p\in\mathcal{P}^1(x;d)\\ ((p-1)/d,m/d)=1}} \frac{1}{p} S_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big)\\
&= \sum_{d|m} \sum_{p\in\mathcal{P}^1(x;d)} \frac{1}{p} \left(\sum_{e|((p-1)/d,m/d)} \mu(e)\right) S_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big)\\
&= \sum_{\substack{d|m\\ e|(m/d)}} \sum_{p\in\mathcal{P}^1(x;de)} \frac{\mu(e)}{p} S_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big).
\end{aligned}$$

By induction, we can assume that

$$S_{\ell-1}(y;n) = \mathcal{L}(y)^{\ell-1}\left(\Delta_{\ell-1}(n) + O_{c,\ell}\left(\frac{\tau(n)^{2\ell-4}}{\mathcal{L}(y)}\right)\right)$$

holds uniformly for all $y \geq e^{2^\ell}$ and $1 \leq n \leq \log^{2c} y$. In particular, for primes $p \leq x^{1/2}$ we have $x/p \geq x^{1/2} \geq e^{2^\ell}$ since $x \geq e^{2^{\ell+1}}$, and

$$1 \leq \frac{m}{d} \leq m \leq \log^c x \leq \log^{2c}\left(\frac{x}{p}\right);$$

therefore,

$$S_{\ell-1}\Big(\frac{x}{p}; \frac{m}{d}\Big) = \mathcal{L}\Big(\frac{x}{p}\Big)^{\ell-1}\left(\Delta_{\ell-1}\Big(\frac{m}{d}\Big) + O_{c,\ell}\left(\frac{\tau(m/d)^{2\ell-4}}{\mathcal{L}(x/p)}\right)\right)$$

for each divisor d of m. Moreover, since

$$\mathcal{L}\Big(\frac{x}{p}\Big) = \mathcal{L}(x) + O\left(\frac{\log p}{\log x}\right) = \mathcal{L}(x) + O(1)$$

for such primes p, we obtain the uniform estimate

$$S_{\ell-1}\Big(\frac{x}{p}; \frac{m}{d}\Big) = \mathcal{L}(x)^{\ell-1}\left(\Delta_{\ell-1}\Big(\frac{m}{d}\Big) + O_{c,\ell}\left(\frac{\tau(m)^{2\ell-4}}{\mathcal{L}(x)}\right)\right).$$

Consequently,

$$\begin{aligned}
&\sum_{\substack{d\mid m\\ e\mid(m/d)}} \sum_{p\in\mathcal{P}^1(x^{1/2};de)} \frac{\mu(e)}{p}\, S_{\ell-1}\Big(\frac{x}{p}; \frac{m}{d}\Big)\\
&= \mathcal{L}(x)^{\ell-1} \sum_{\substack{d\mid m\\ e\mid(m/d)}} \mu(e) S_1(x^{1/2}; de)\left(\Delta_{\ell-1}\Big(\frac{m}{d}\Big) + O_{c,\ell}\left(\frac{\tau(m)^{2\ell-4}}{\mathcal{L}(x)}\right)\right).
\end{aligned}$$

Since $x^{1/2} \geq e^{2^\ell} \geq e^4$ and

$$1 \leq de \leq m \leq \log^c x \leq \log^{2c}(x^{1/2}),$$

we have

$$\begin{aligned}
S_1(x^{1/2}; de) &= \mathcal{L}(x^{1/2})\left(\Delta_1(de) + O_c\left(\mathcal{L}(x^{1/2})^{-1}\right)\right)\\
&= \mathcal{L}(x)\left(\frac{1}{\varphi(de)} + O_c\left(\mathcal{L}(x)^{-1}\right)\right).
\end{aligned}$$

Recalling that

$$\sum_{\substack{d\mid m\\ e\mid(m/d)}} \frac{\mu(e)}{\varphi(de)}\, \Delta_{\ell-1}\Big(\frac{m}{d}\Big) = \Delta_\ell(m)$$

and using the naive estimate

$$\sum_{\substack{d|m\\ e|(m/d)}} \frac{1}{\varphi(de)} \le \tau(m)^2$$

together with the fact that $\Delta_{\ell-1}(m/d) = O_\ell(1)$ by Lemma 3.2, we obtain:

$$\sum_{\substack{d|m\\ e|(m/d)}} \sum_{p\in\mathcal{P}^1(x^{1/2};de)} \frac{\mu(e)}{p} S_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big) = \mathcal{L}(x)^\ell \left(\Delta_\ell(m) + O_{c,\ell}\left(\frac{\tau(m)^{2\ell-2}}{\mathcal{L}(x)}\right)\right).$$

To complete the proof, it suffices to show that

$$\left|\sum_{\substack{d|m\\ e|(m/d)}} \sum_{\substack{p\in\mathcal{P}^1(x;de)\\ p>x^{1/2}}} \frac{\mu(e)}{p} S_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big)\right| = O_\ell\Big(\tau(m)^2\mathcal{L}(x)^{\ell-1}\Big). \tag{5}$$

For this, we first apply Theorem 2.5 to obtain the crude estimate:

$$S_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big) \le S_{\ell-1}(x) = O_\ell\Big(\mathcal{L}(x)^{\ell-1}\Big).$$

By Theorem 2.4 we also have

$$\sum_{\substack{p\in\mathcal{P}^1(x;de)\\ p>x^{1/2}}} \frac{1}{p} \le \sum_{\substack{p\in\mathcal{P}(x)\\ p>x^{1/2}}} \frac{1}{p} = \sum_{p\in\mathcal{P}(x)} \frac{1}{p} - \sum_{p\in\mathcal{P}(x^{1/2})} \frac{1}{p} = O(1),$$

and (5) follows immediately. □

$\sim$ *Capitoline* $\sim$

Lemma 3.4. *For all $\ell, m \in \mathbb{N}$ and $x \ge 1$, let*

$$\vartheta_\ell(x;m) := \sum_{(p_1,\ldots,p_\ell)\in\mathcal{P}^\ell(x;m)} \log(p_1\cdots p_\ell).$$

Then for any constant $c>0$ and any fixed $\ell\in\mathbb{N}$, and uniformly for $x \ge e^{2^{\ell+1}}$ and $1 \le m \le \log^c x$, we have

$$\vartheta_\ell(x;m) = \ell x(\log\log x)^{\ell-1}\left(\Delta_\ell(m) + O_{c,\ell}\left(\frac{\tau(m)^{2\ell-2}}{\log\log x}\right)\right).$$

Proof. By Corollary 2.1 we have for $x \ge 3$ and $1 \le m \le \log^c x$:

$$
\begin{aligned}
\vartheta_1(x;m) \quad \sum_{\substack{p\in\mathcal{P}(x)\\ p\equiv 1 \pmod m}} \log p &= \frac{x}{\varphi(m)} + O_c\left(xe^{-c_1\sqrt{\log x}}\right)\\
&= x\left(\Delta_1(m) + O_c\left(\mathcal{L}(x)^{-1}\right)\right),
\end{aligned}
$$

where we use the notation $\mathcal{L}(x) := \log\log x$ as in Lemma 3.3, and the lemma is proved in this case.

Now suppose that $\ell \geq 2$ and that the lemma has been proved for $\ell - 1$. We claim that

$$
\vartheta_\ell(x;m) = \frac{\ell}{\ell-1} \sum_{\substack{d\mid m\\ e\mid(m/d)}} \sum_{p\in\mathcal{P}^1(x;de)} \mu(e)\vartheta_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big).
$$

Indeed, let

$$
p_1\cdots\widehat{p_j}\cdots p_\ell := \prod_{\substack{1\leq i\leq \ell\\ i\neq j}} p_i
$$

for $j = 1,\ldots,\ell$. Then

$$
\begin{aligned}
(\ell-1)\vartheta_\ell(x;m) &= \sum_{\substack{(p_1,\ldots,p_\ell)\in\mathcal{P}^\ell(x)\\ m\mid(p_1-1)\cdots(p_\ell-1)}} (\ell-1)\log(p_1\cdots p_\ell)\\
&= \sum_{\substack{(p_1,\ldots,p_\ell)\in\mathcal{P}^\ell(x)\\ m\mid(p_1-1)\cdots(p_\ell-1)}} \sum_{j=1}^{\ell} \log(p_1\cdots\widehat{p_j}\cdots p_\ell)\\
&= \sum_{j=1}^{\ell} \sum_{\substack{(p_1,\ldots,p_\ell)\in\mathcal{P}^\ell(x)\\ m\mid(p_1-1)\cdots(p_\ell-1)}} \log(p_1\cdots\widehat{p_j}\cdots p_\ell)\\
&= \sum_{j=1}^{\ell}\sum_{d\mid m} \sum_{\substack{p_j\in\mathcal{P}(x)\\ (p_j-1,m)=d}} \sum_{(p_1,\ldots,\widehat{p_j},\ldots,p_\ell)\in\mathcal{P}^{\ell-1}(x/p_j;m/d)} \log(p_1\cdots\widehat{p_j}\cdots p_\ell)\\
&= \ell\sum_{d\mid m} \sum_{\substack{p\in\mathcal{P}^1(x;d)\\ (p-1,m)=d}} \sum_{(p_1,\ldots,p_{\ell-1})\in\mathcal{P}^{\ell-1}(x/p;m/d)} \log(p_1\cdots p_{\ell-1})\\
&= \ell\sum_{d\mid m} \sum_{p\in\mathcal{P}^1(x;d)} \left(\sum_{e\mid((p-1)/d,m/d)} \mu(e)\right) \vartheta_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big)\\
&= \ell \sum_{\substack{d\mid m\\ e\mid(m/d)}} \sum_{p\in\mathcal{P}^1(x;de)} \mu(e)\vartheta_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big),
\end{aligned}
$$

which proves the claim.

Now, by induction, we can assume that

$$\vartheta_{\ell-1}(y;n) = (\ell-1)y\mathcal{L}(y)^{\ell-2}\left(\Delta_{\ell-1}(n) + O_{c,\ell}\left(\frac{\tau(n)^{2\ell-4}}{\mathcal{L}(y)}\right)\right).$$

holds uniformly for all $y \geq e^{2^\ell}$ and $1 \leq n \leq \log^{2c} y$. In particular, for primes $p \leq x^{1/2}$ we have $x/p \geq x^{1/2} \geq e^{2^\ell}$ since $x \geq e^{2^{\ell+1}}$, and

$$1 \leq \frac{m}{d} \leq m \leq \log^c x \leq \log^{2c}\left(\frac{x}{p}\right).$$

Therefore,

$$\vartheta_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big) = (\ell-1)\,\frac{x}{p}\,\mathcal{L}\Big(\frac{x}{p}\Big)^{\ell-2}\left(\Delta_{\ell-1}\Big(\frac{m}{d}\Big) + O_{c,\ell}\left(\frac{\tau(m/d)^{2\ell-4}}{\mathcal{L}(x/p)}\right)\right).$$

for each divisor d of m. Since

$$\mathcal{L}\Big(\frac{x}{p}\Big) = \mathcal{L}(x) + O\left(\frac{\log p}{\log x}\right) = \mathcal{L}(x) + O(1)$$

for such primes p, we obtain the uniform estimate

$$\vartheta_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big) = (\ell-1)\,\frac{x}{p}\,\mathcal{L}(x)^{\ell-2}\left(\Delta_{\ell-1}\Big(\frac{m}{d}\Big) + O_{c,\ell}\left(\frac{\tau(m)^{2\ell-4}}{\mathcal{L}(x)}\right)\right).$$

Consequently,

$$\begin{aligned}
&\frac{\ell}{\ell-1}\sum_{\substack{d\mid m\\ e\mid(m/d)}}\ \sum_{p\in\mathcal{P}^1(x^{1/2};de)} \mu(e)\,\vartheta_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big)\\
&= \ell x\mathcal{L}(x)^{\ell-2}\sum_{\substack{d\mid m\\ e\mid(m/d)}}\ \sum_{p\in\mathcal{P}^1(x^{1/2};de)} \frac{\mu(e)}{p}\left(\Delta_{\ell-1}\Big(\frac{m}{d}\Big) + O_{c,\ell}\left(\frac{\tau(m)^{2\ell-4}}{\mathcal{L}(x)}\right)\right)\\
&= \ell x\mathcal{L}(x)^{\ell-2}\sum_{\substack{d\mid m\\ e\mid(m/d)}} \mu(e)S_1(x^{1/2};de)\left(\Delta_{\ell-1}\Big(\frac{m}{d}\Big) + O_{c,\ell}\left(\frac{\tau(m)^{2\ell-4}}{\mathcal{L}(x)}\right)\right).
\end{aligned}$$

Proceeding from here as in proof of Lemma 3.3, we obtain that

$$\begin{aligned}
&\frac{\ell}{\ell-1}\sum_{\substack{d\mid m\\ e\mid(m/d)}}\ \sum_{p\in\mathcal{P}^1(x^{1/2};de)} \mu(e)\vartheta_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big)\\
&\qquad = \ell x\mathcal{L}(x)^{\ell-1}\left(\Delta_\ell(m) + O_{c,\ell}\left(\frac{\tau(m)^{2\ell-2}}{\mathcal{L}(x)}\right)\right).
\end{aligned}$$

To complete the proof, it suffices to show that

$$\left| \sum_{\substack{d|m \\ e|(m/d)}} \sum_{\substack{p\in\mathcal{P}^1(x;de) \\ p>x^{1/2}}} \mu(e)\vartheta_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big) \right| = O_\ell\Big(x\mathcal{L}(x)^{\ell-2}\tau(m)^2\Big). \tag{6}$$

For this, we first apply Theorem 2.5 to obtain the crude estimate:

$$\vartheta_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big) \le \vartheta_{\ell-1}\Big(\frac{x}{p}\Big) = O\left(\frac{x\mathcal{L}(x)^{\ell-2}}{p}\right).$$

As in the proof of Lemma 3.3, we also have

$$\sum_{\substack{p\in\mathcal{P}^1(x;de) \\ p>x^{1/2}}} \frac{1}{p} = O(1),$$

and the estimate (6) follows. $\square$

$\sim$ *Quirinal* $\sim$

Lemma 3.5. *For all $\ell, m \in \mathbb{N}$ and $x \ge 1$, let*

$$\mathcal{B}_\ell(x;m) := \{n \le x : \Omega(n) = \omega(n) = \ell \text{ and } m|\varphi(n)\}.$$

Then for any constant $c > 0$ and any fixed $\ell \in \mathbb{N}$, and uniformly for $x \ge e^{2^{\ell+1}}$ and $1 \le m \le \log^c x$, we have

$$\#\mathcal{B}_\ell(x;m) = \frac{x(\log\log x)^{\ell-1}}{(\ell-1)!\,\log x}\left(\Delta_\ell(m) + O_{c,\ell}\left(\frac{\tau(m)^{2\ell-2}}{\log\log x}\right)\right).$$

Proof. We recall that $F(n) = \varphi(n)$ if n is squarefree, i.e., when $\Omega(n) = \omega(n)$. In other words,

$$\mathcal{B}_\ell(x;m) = \{n \le x : \Omega(n) = \omega(n) = \ell \text{ and } m|F(n)\}.$$

Let $r_\ell(n)$ denote the number of representations of n as an ordered product of ℓ primes, i.e.,

$$r_\ell(n) = \sum_{\substack{(p_1,\dots,p_\ell)\in\mathcal{P}^\ell \\ p_1\cdots p_\ell = n}} 1.$$

Then

$$\vartheta_\ell(x;m) = \sum_{(p_1,\dots,p_\ell)\in\mathcal{P}^\ell(x;m)} \log(p_1\cdots p_\ell) = \sum_{\substack{n\le x \\ m|F(n)}} r_\ell(n)\log n.$$

Since $0 \le r_\ell(n) \le \ell!$ for all $n \in \mathbb{N}$,

$$\#\mathcal{P}^\ell(x;m) = \sum_{\substack{n \le x \\ m \mid F(n)}} r_\ell(n) = O_\ell(x),$$

and we obtain by partial summation:

$$\vartheta_\ell(x;m) = \#\mathcal{P}^\ell(x;m)\log x - \int_1^x \frac{\#\mathcal{P}^\ell(t;m)}{t}\,dt = \#\mathcal{P}^\ell(x;m)\log x + O_\ell(x).$$

By Lemma 3.4, this gives

$$\#\mathcal{P}^\ell(x;m) = \frac{\ell x(\log\log x)^{\ell-1}}{\log x}\left(\Delta_\ell(m) + O_{c,\ell}\left(\frac{\tau(m)^{2\ell-2}}{\log\log x}\right)\right). \tag{7}$$

On the other hand, by Theorem 2.6,

$$\begin{aligned}\#\mathcal{P}^\ell(x;m) &= \sum_{\substack{n \le x \\ \Omega(n)=\omega(n)=\ell \\ m \mid F(n)}} \ell! + \sum_{\substack{n \le x \\ \omega(n)<\Omega(n)=\ell \\ m \mid F(n)}} O_\ell(1) \\ &= \#\mathcal{B}(x;m)\,\ell! + O_\ell\left(\frac{x\,(\log\log x)^{\ell-2}}{\log x}\right).\end{aligned}$$

This estimate combined with (7) yields the desired result. □

∼ *Esquiline* ∼

The following result may be viewed as an analogue of the Brun-Titchmarsh theorem (Theorem 2.3).

Lemma 3.6. *For any integers $m, \ell \in \mathbb{N}$ and any real constant $C > 0$, we have*

$$\#\mathcal{B}_\ell(x;m) \ll_{\ell,C} \frac{\tau(m)^{2\ell-2}}{\varphi(m)}\,\frac{x(\log\log x)^{\ell-1}}{\log x}$$

if $m \le \log^C x$ and x is sufficiently large, depending only on ℓ and A.

Proof. The proof proceeds by induction on ℓ. First, suppose that $\ell = 1$. We have by Theorem 2.3:

$$\#\mathcal{B}_1(x;m) = \pi(x;m,1) \ll \frac{x}{\varphi(m)\big(1+\log(x/m)\big)}.$$

Since $m \le \log^C x$, the lemma follows in this case.

Now suppose the lemma is true up to $\ell - 1$, where $\ell \ge 2$. We remark that $\Omega(n) = \omega(n)$ implies that n is squarefree, and any squarefree $n \le x$ with

$\omega(n) = \ell$ can be expressed in the form $n = pn'$ where p is the smallest prime divisor of n, $p \le x^{1/\ell}$, $n' \le x/p$, and $\varphi(n) = (p-1)\varphi(n')$. Therefore,

$$\begin{aligned}
\#\mathcal{B}_\ell(x;m) = \sum_{\substack{n\le x\\ \Omega(n)=\omega(n)=\ell\\ m|\varphi(n)}} 1 &\le \sum_{p\le x^{1/\ell}} \sum_{\substack{n'\le x/p\\ \Omega(n')=\omega(n')=\ell-1\\ m|(p-1)\varphi(n')}} 1 \\
&\le \sum_{d|m} \sum_{\substack{p\le x^{1/\ell}\\ (m,p-1)=d}} \sum_{\substack{n'\le x/p\\ \Omega(n')=\omega(n')=\ell-1\\ (m/d)|\varphi(n')}} 1 \\
&\le \sum_{d|m} \sum_{\substack{p\le x^{1/\ell}\\ p\equiv 1 \pmod d}} \#\mathcal{B}_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big).
\end{aligned}$$

To estimate each term $\#\mathcal{B}_{\ell-1}(x/p;m/d)$, since $x/p \ge x^{1-1/\ell}$, we have

$$\frac{m}{d} \le m \le \log^C x \le \frac{\ell^C}{(\ell-1)^C} \log^C\Big(\frac{x}{p}\Big) \le \log^{2C}\Big(\frac{x}{p}\Big)$$

if x is sufficiently large; we can therefore apply the inductive hypothesis (with C replaced by $2C$), obtaining:

$$\begin{aligned}
\#\mathcal{B}_{\ell-1}\Big(\frac{x}{p};\frac{m}{d}\Big) &\ll_{\ell,C} \frac{\tau(m/d)^{2\ell-4}}{\varphi(m/d)} \frac{(x/p)(\log\log(x/p))^{\ell-2}}{\log(x/p)} \\
&\ll_\ell \frac{\tau(m)^{2\ell-4}}{\varphi(m/d)} \frac{x(\log\log x)^{\ell-2}}{\log x} \frac{1}{p}.
\end{aligned}$$

Since each $d \le \log^C x$, Theorem 2.2 implies that

$$\sum_{\substack{p\le x^{1/\ell}\\ p\equiv 1 \pmod d}} \frac{1}{p} \ll_{\ell,C} \frac{\log\log x}{\varphi(d)},$$

and therefore

$$\#\mathcal{B}_\ell(x;m) \ll_{\ell,C} \tau(m)^{2\ell-4} \frac{x(\log\log x)^{\ell-1}}{\log x} \sum_{d|m} \frac{1}{\varphi(m/d)\varphi(d)}.$$

Noting that

$$\varphi\Big(\frac{m}{d}\Big)\varphi(d) = \varphi(m) \prod_{p|(d,m/d)} (1-p^{-1}) \ge \frac{\varphi(m)}{2^{\omega((d,m/d))}} \ge \frac{\varphi(m)}{2^{\omega(d)}} \ge \frac{\varphi(m)}{\tau(d)} \ge \frac{\varphi(m)}{\tau(m)},$$

the lemma follows. □

Lemma 3.7. *For any fixed integer $k \geq 3$ and any real number $x \geq 1$, let*

$$\mathcal{C}_k(x) := \{\textit{odd } n \leq x : \Omega(n) = \omega(n) = k-1 \textit{ and } \varphi(n) \textit{ is } k\textit{-free}\}.$$

Then,

$$\#\mathcal{C}_k(x) = \alpha_k \, \frac{x\,(\log\log x)^{k-2}}{(k-2)!\,\log x}\left(1 + O_k\left(\frac{(\log\log\log x)^{2(k+1)^{2k-4}-1}}{(\log\log x)^{1-1/k}}\right)\right),$$

where α_k is the constant defined in Theorem 1.1.

Proof. Since the set

$$\{\text{even } n \leq x : \Omega(n) = \omega(n) = k-1 \text{ and } \varphi(n) \text{ is } k\text{-free}\}$$

is a subset of

$$\{n \leq x/2 : \omega(n) = k-2\},$$

and the latter set is of size

$$O_k\left(\frac{x\,(\log\log x)^{k-3}}{\log x}\right)$$

by Theorem 2.6, it suffices to estimate the number of elements in

$$\mathcal{D}_k(x) := \{n \leq x : \Omega(n) = \omega(n) = k-1 \text{ and } \varphi(n) \text{ is } k\text{-free}\}.$$

Now the characteristic function χ_k of k-free integers $n \in \mathbb{N}$ can be defined in terms of the Möbius function via the formula

$$\chi_k(n) := \sum_{m^k \mid n} \mu(m).$$

Thus for any real parameters y, z with $y < z < x^{1/k}$, we have

$$\begin{aligned}\#\mathcal{D}_k(x) &= \sum_{\substack{n \leq x \\ \Omega(n)=\omega(n)=k-1}} \sum_{m^k \mid \varphi(n)} \mu(m) \\ &= \sum_{m \leq x^{1/k}} \mu(m)\#\mathcal{B}_{k-1}(x; m^k) = \Sigma(0, y) + \Sigma(y, z) + \Sigma(z, x^{1/k}),\end{aligned}$$

where

$$\Sigma(a, b) := \sum_{a < m \leq b} \mu(m)\#\mathcal{B}_{k-1}(x; m^k).$$

To evaluate the main term $\Sigma(0,y)$, we apply Lemma 3.5 with $c = k$ and $\ell = k-1$; for any $y \le \log x$, we obtain that

$$\Sigma(0,y) = \frac{x(\log\log x)^{k-2}}{(k-2)!\,\log x}\sum_{m\le y}\mu(m)\left(\Delta_{k-1}(m^k) + O_k\left(\frac{\tau(m^k)^{2k-4}}{\log\log x}\right)\right).$$

By Lemma 3.2, we have

$$\sum_{m\le y}\mu(m)\Delta_{k-1}(m^k) = \alpha_k + O_k\left(\frac{(\log y)^{2^{2k-1}-1}}{y^{k-1}}\right),$$

while by the first part of Lemma 2.1,

$$\sum_{m\le y}\mu^2(m)\tau(m^k)^{2k-4} \le \sum_{m\le y}\tau(m)^{\lceil(2k-4)\log_2(k+1)\rceil} \ll_k y(\log y)^{2(k+1)^{2k-4}-1}.$$

Therefore, if $\theta = 2(k+1)^{2k-4}-1$, we see that

$$\Sigma(0,y) = \alpha_k\,\frac{x(\log\log x)^{k-2}}{(k-2)!\,\log x}\left(1 + O_k\left(\log^\theta y\left(\frac{1}{y^{k-1}} + \frac{y}{\log\log x}\right)\right)\right).$$

Choosing $y = (\log\log x)^{1/k}$ (which balances the two terms in this estimate) and noting that $y \le \log x$ if x is sufficiently large, it follows that

$$\Sigma(0,y) = \alpha_k\,\frac{x\,(\log\log x)^{k-2}}{(k-2)!\,\log x}\left(1 + O_k\left(\frac{(\log\log\log x)^{2(k+1)^{2k-4}-1}}{(\log\log x)^{1-1/k}}\right)\right).$$

Next, we take $z = \log^6 x$ and estimate $\Sigma(y,z)$ using Lemma 3.6 (with $C = 6$):

$$\begin{aligned}
\Sigma(y,z) &\le \sum_{y<m\le z}\mu^2(m)\#\mathcal{B}_{k-1}(x;m^k)\\
&\ll_k \frac{x(\log\log x)^{k-2}}{\log x}\sum_{y<m\le z}\mu^2(m)\frac{\tau(m^k)^{2k-4}}{\varphi(m^k)}\\
&\le \frac{x(\log\log x)^{k-2}}{\log x}\sum_{y<m\le z}\mu^2(m)\frac{\tau(m)^{\lceil(2k-4)\log_2(k+1)\rceil}}{m^{k-1}\varphi(m)}.
\end{aligned}$$

By the second part of Lemma 2.1,

$$\sum_{m>y}\mu^2(m)\frac{\tau(m)^{\lceil(2k-4)\log_2(k+1)\rceil}}{m^{k-1}\varphi(m)} \ll_k \frac{\log^\theta y}{y^{k-1}},$$

where θ is as before, and it follows that

$$\Sigma(y,z) = O_k\left(\frac{x\,(\log\log x)^{k-2}}{\log x}\,\frac{(\log\log\log x)^{2(k+1)^{2k-4}-1}}{(\log\log x)^{1-1/k}}\right).$$

Finally, we estimate $\Sigma(z, x^{1/k})$ using Lemma 2.2:

$$\begin{aligned}\Sigma(z, x^{1/k}) &\le \sum_{z<m\le x^{1/k}} \#\mathcal{B}_{k-1}(x; m^k) \le \sum_{z<m\le x^{1/k}} T(x, k-1, m^k) \\ &\ll_k \sum_{z<m\le x^{1/k}} \frac{x(\log\log x)^{k-2}}{m^{2k/5}} \ll \frac{x(\log\log x)^{k-2}}{z^{(2k-5)/5}}.\end{aligned}$$

Since $k \ge 3$,

$$z^{(2k-5)/5} \ge z^{1/5} = (\log x)^{6/5},$$

and therefore

$$\Sigma(z, x^{1/k}) = O_k\left(\frac{x\,(\log\log x)^{k-2}}{(\log x)^{6/5}}\right).$$

Putting together our estimates for $\Sigma(0, y)$, $\Sigma(y, z)$ and $\Sigma(z, x^{1/k})$, the proof is completed. $\square$

4 Proof of the Main Theorem

In view of Lemma 3.7, to prove Theorem 1.1 it suffices to show that

$$\#\mathcal{A}_k(x) = \frac{3}{2}\,\#\mathcal{C}_k(x) + O_k\left(\frac{x(\log\log x)^{k-3}}{\log x}\right).$$

Now for any real number $x \ge 1$, let us define the set

$$\mathcal{E}_k(x) := \{n \in \mathcal{A}_k(x) : n \text{ is odd and } \omega(n) = k-1\}.$$

If $n \in \mathcal{A}_k(x)$ is odd, then $2^{\omega(n)} | \varphi(n)$ since $2|(p-1)$ for each prime divisor p of n; as $\varphi(n)$ is k-free, it follows that $\omega(n) \le k-1$. On the other hand, from Theorem 2.6 it follows that

$$\#\{n \le x : \omega(n) \le k-2\} = O_k\left(\frac{x\,(\log\log x)^{k-3}}{\log x}\right).$$

Thus,

$$\#\{n \in \mathcal{A}_k(x) : n \text{ is odd}\} = \#\mathcal{E}_k(x) + O_k\left(\frac{x\,(\log\log x)^{k-3}}{\log x}\right).$$

Next, suppose that $n \in \mathcal{A}_k(x)$ is even and that $2\|n$. Then $n = 2m$ where m is odd, $m \le x/2$, and $\varphi(m) = \varphi(n)$ is k-free; conversely, if m has these properties, then $n = 2m$ lies in $\mathcal{A}_k(x)$ and $2\|n$. Arguing as before, we also see that $\omega(m) \leq k-1$, and therefore

$$\#\{n \in \mathcal{A}_k(x) : 2\|n\} = \#\mathcal{E}_k\left(\frac{x}{2}\right) + O_k\left(\frac{x\,(\log\log x)^{k-3}}{\log x}\right).$$

Finally, suppose that $n \in \mathcal{A}_k(x)$ and that $4|n$. If $a \geq 2$ is such that $2^a\|n$, then $2^{a-1+\omega(m)}|\varphi(n)$; since $\varphi(n)$ is k-free, it follows that

$$a - 1 + \omega(m) \leq k - 1,$$

which implies that $a \leq k$ and $\omega(m) \leq k-2$. Using Theorem 2.6 again, we conclude that

$$\#\{n \in \mathcal{A}_k(x) : 4|n\} = O_k\left(\frac{x\,(\log\log x)^{k-3}}{\log x}\right).$$

Putting everything together, we see that

$$\#\mathcal{A}_k(x) = \#\mathcal{E}_k(x) + \#\mathcal{E}_k\left(\frac{x}{2}\right) + O_k\left(\frac{x(\log\log x)^{k-3}}{\log x}\right),$$

hence it suffices to show that

$$\#\mathcal{E}_k(x) = \#\mathcal{C}_k(x) + O_k\left(\frac{x(\log\log x)^{k-3}}{\log x}\right). \tag{8}$$

We argue as follows. First, notice that $\mathcal{C}_k(x) \subset \mathcal{E}_k(x)$. Now if n lies in $\mathcal{E}_k(x)$ but not in $\mathcal{C}_k(x)$, then $\Omega(n) > \omega(n)$, thus n is divisible by some prime power p^a with $2 \leq a \leq k$. Moreover, $n \neq p^a$ since $\omega(n) = k-1 \geq 2$. But the number of such integers is bounded above by

$$\begin{aligned}
&\sum_{2<p\leq(x/2)^{1/2}} \sum_{a=2}^{k} \#\{\text{odd } n \leq x : \omega(n) = k-1,\ p^a\|n\} \\
&\qquad \ll_k \sum_{p\leq(x/2)^{1/2}} \#\{m \leq x/p^2 : \omega(m) = k-2\} \\
&\qquad \ll_k \sum_{p\leq(x/2)^{1/2}} \frac{x(\log\log x)^{k-3}}{p^2\log(x/p^2)},
\end{aligned}$$

where the last estimate follows from Theorem 2.6. Since the last sum is bounded above by

$$\sum_{p\leq x^{1/3}} \frac{x(\log\log x)^{k-3}}{p^2\log x} + \sum_{x^{1/3}<p\leq(x/2)^{1/2}} x^{1/3}(\log\log x)^{k-3} \ll \frac{x(\log\log x)^{k-3}}{\log x},$$

we obtain (8), completing the proof.

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