Notes on Continued Fractions
for Math 4400

1. Continued fractions.

The continued fraction expansion converts a positive real number α into a sequence of natural numbers. Conversely, a sequence of natural numbers:

$$a_0, a_1, a_2, a_3, \ldots$$

is converted into a sequence of *rational numbers* via:

$$(*) \qquad a_0, \quad a_0 + \frac{1}{a_1}, \quad a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, \quad a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3}}}, \quad \cdots$$

Thus, for example, the sequence $3, 7, 15, 1$ is converted into:

$$3 = \frac{3}{1}, \quad 3 + \frac{1}{7} = \frac{22}{7}, \quad 3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106}, \quad 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = \frac{355}{113}$$

which are excellent approximations of π.

We will find some useful general properties of continued fractions by replacing numbers $a_0, a_1, a_2, a_3, \ldots$ with a sequence of *variables*:

$$q_0, q_1, q_2, q_3, \ldots$$

and considering the sequence of rational *functions* in many variables:

$$q_0 = \frac{q_0}{1}, \; q_0 + \frac{1}{q_1} = \frac{q_0 q_1 + 1}{q_1}, \; q_0 + \frac{1}{q_1 + \frac{1}{q_2}} = \frac{q_0 q_1 q_2 + q_0 + q_2}{q_1 q_2 + 1}$$

Definition. Define polynomials f_n and g_n by:

$$q_0 + \frac{1}{q_1 + \frac{1}{\ddots + \frac{1}{q_n}}} = \frac{f_n(q_0, \ldots, q_n)}{g_n(q_1, \ldots, q_n)}$$

(in lowest terms).

Examples. (o) $f_0(q_0) = q_0$ and $g_0 = 1$.

(i) $f_1(q_0, q_1) = q_0 q_1 + 1$ and $g_1(q_1) = q_1$.

(ii) $f_2(q_0, q_1, q_2) = q_0 q_1 q_2 + q_0 + q_2$ and $g_2(q_1, q_2) = q_1 q_2 + 1$.

Proposition 1.

(a) (Each g is an f) $g_n(q_1, \cdots, q_n) = f_{n-1}(q_1, \ldots, q_n)$.

(b) (Recursion) $f_n(q_0, \cdots, q_n) = q_0 f_{n-1}(q_1, \cdots, q_n) + f_{n-2}(q_2, \cdots, q_n)$.

Proof. By definition of the polynomials f_{n-1} and g_{n-1}, we have:

$$q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots + \frac{1}{q_n}}} = \frac{f_{n-1}(q_1, \dots, q_n)}{g_{n-1}(q_2, \dots, q_n)}$$

hence:

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \frac{1}{q_n}}} = q_0 + \frac{1}{\left(\frac{f_{n-1}(q_1,\dots,q_n)}{g_{n-1}(q_2,\dots,q_n)}\right)} = q_0 + \frac{g_{n-1}(q_2, \dots, q_n)}{f_{n-1}(q_1, \dots, q_n)}$$

$$= \frac{q_0 f_{n-1}(q_1, \dots, q_n) + g_{n-1}(q_2, \dots, q_n)}{f_{n-1}(q_1, \dots, q_n)}$$

and the denominator gives us (a), and the numerator gives us:

$$f_n(q_0, \dots, q_n) = q_0 f_{n-1}(q_1, \dots, q_n) + g_{n-1}(q_2, \dots, q_n)$$

which, with $g_{n-1}(q_2, \dots, q_n) = f_{n-2}(q_2, \dots, q_n)$ from (a), gives us (b).

Using the recursion, we can get a few more of these polynomials:

Example.

(iii) $f_3(q_0, q_1, q_2, q_3) = q_0(q_1q_2q_3 + q_1 + q_3) + (q_2q_3 + 1)$
$$= q_0q_1q_2q_3 + q_0q_1 + q_0q_3 + q_2q_3 + 1$$

(iv) $f_4(q_0, q_1, q_2, q_3, q_4) =$
$$= q_0(q_1q_2q_3q_4 + q_1q_2 + q_1q_4 + q_3q_4 + 1) + q_2q_3q_4 + q_2 + q_4$$
$$= q_0q_1q_2q_3q_4 + q_0q_1q_2 + q_0q_1q_4 + q_0q_3q_4 + q_2q_3q_4 + q_0 + q_2 + q_4$$

Notice that the first term is always the product of all the q's. In fact:

Euler's Continued Fraction Criterion:

$$f_n = q_0 \cdots q_n + \sum_i \frac{q_0 \cdots q_n}{q_i q_{i+1}} + \sum_{|i-j|>1} \frac{q_0 \cdots q_n}{q_i q_{i+1} q_j q_{j+1}} + \cdots$$

i.e. the other terms are obtained by removing *consecutive* pairs of q's from the product of all the q's.

Example.

$$f_5 = q_0 \cdots q_5 + q_2q_3q_4q_5 + q_0q_3q_4q_5 + q_0q_1q_4q_5 + q_0q_1q_2q_5 + q_0q_1q_2q_3 +$$
$$+ q_4q_5 + q_2q_5 + q_2q_3 + q_0q_5 + q_0q_3 + q_0q_1 + 1$$

Proof. The criterion is true for $f_0(q_0) = q_0$ and $f_1(q_0q_1) = q_0q_1 + 1$. By the recursive formula (b) above:

$$f_n(q_0, \dots, q_n) = q_0 f_{n-1}(q_1, \dots, q_n) + f_{n-2}(q_2, \dots, q_n)$$

By induction, f_{n-1} and f_{n-2} may be assumed to satisfy the criterion, and it then follows from the formula that f_n also satisfies the criterion!

Corollary 1. (The palindrome corollary)

$$f_n(q_0, \ldots, q_n) = f_n(q_n, \ldots, q_0)$$

Proof. Euler's criterion defines the same polynomial when the order of the variables is reversed.

The most important corollary is now the following.

Corollary 2. (An even better recursion)

$$f_n(q_0, \ldots, q_n) = q_n f_{n-1}(q_0, \ldots, q_{n-1}) + f_{n-2}(q_0, \ldots, q_{n-2})$$

Proof. Using Corollary 1 and Proposition 1 (b), we have:

$$\begin{aligned} f_n(q_0, \ldots, q_n) = f_n(q_n, \ldots, q_0) &= q_n f_{n-1}(q_{n-1}, \ldots, q_0) + f_{n-2}(q_{n-2}, \ldots, q_0) = \\ &= q_n f_n(q_0, \ldots, q_{n-1}) + f_{n-2}(q_0, \ldots, q_{n-2}) \end{aligned}$$

This corollary, together with the initial conditions:

$$f_0(q_0) = q_0, \;\; f_1(q_0, q_1) = q_0 q_1 + 1$$

is going to turn out to be extremely useful.

Back to Numbers. We now apply the polynomial results to continued fractions associated to natural numbers.

Definition. Given a sequence of natural numbers $a_0, a_1, a_2, \ldots,$ let:

(a) $A_n := f_n(a_0, a_1, \ldots, a_n)$ and (b) $B_n := g_n(a_1, \ldots, a_n) = f_{n-1}(a_1, \ldots, a_n)$

Gathering together what we have done with polynomials, we have:

Proposition 2. (a) The sequence of rational numbers (*) coming from the sequence of natural numbers $a_0, a_1, a_2, \ldots$ is:

$$\frac{A_0}{B_0} = \frac{a_0}{1}, \quad \frac{A_1}{B_1} = \frac{a_0 a_1 + 1}{a_1}, \quad \frac{A_2}{B_2}, \quad \frac{A_3}{B_3}, \quad \ldots$$

(b) The numbers A_n and B_n satisfy the Fibonacci-like rule:

$$A_n = a_n A_{n-1} + A_{n-2} \;\; \text{and} \;\; B_n = a_n B_{n-1} + B_{n-2} \;\; \text{for } n \geq 2$$

Proof. (a) is from the definition, and (b) follows from Corollary 2.

Example. Taking the sequence $3, 7, 15, 1$ again, we have:

$$A_0 = 3, \quad A_1 = 22, \quad A_2 = 15 \cdot 22 + 3 = 333, \quad A_3 = 1 \cdot 333 + 22 = 355$$

$$B_0 = 1, \quad B_1 = 7, \quad B_2 = 15 \cdot 7 + 1 = 106, \quad B_3 = 1 \cdot 106 + 6 = 113$$

giving the sequence of rational approximations to π that we saw earlier.

Another Example. Take the sequence $1, 1, 1, 1, \ldots$. We have:

$A_0 = 1,\ \ A_1 = 2,\ \ A_2 = 2+1 = 3,\ \ A_3 = 3+2 = 5,\ \ A_4 = 5+3 = 8, \ldots$

$B_0 = 1,\ \ B_1 = 1,\ \ B_2 = 1+1 = 2,\ \ B_3 = 2+1 = 3,\ \ B_4 = 3+2 = 5, \ldots$

which are two copies of the Fibonacci sequence offset by one.

The associated sequence of rational numbers:

$$1,\quad 2,\quad \frac{3}{2},\quad \frac{5}{3},\quad \frac{8}{5},\quad \frac{13}{8},\quad \frac{21}{13}, \ldots$$

converges pretty rapidly to the golden mean $\phi = (1+\sqrt{5})/2$.

2. Convergence. We now have some powerful tools for analyzing the sequence of rational numbers that arise from a continued fraction.

Proposition 3. Let $a_0, a_1, a_2, \ldots$ be a sequence of natural numbers, and let A_n and B_n be the natural numbers from Proposition 2. Then:

(a) Each quadruple of numbers $A_n, B_n, A_{n+1}, B_{n+1}$ satisfies:

$$A_{n+1}B_n - A_nB_{n+1} = (-1)^n$$

(b) Each pair (A_n, B_n) is relatively prime so A_n/B_n is in lowest terms.

(c) If the sequence $a_0, a_1, a_2, \ldots$ is infinite, then there is a limit:

$$\lim_{n\to\infty} \frac{A_n}{B_n} = \alpha \quad \text{and each} \quad \left|\alpha - \frac{A_n}{B_n}\right| < \frac{1}{B_nB_{n+1}} < \frac{1}{B_n^2}$$

Proof. We prove (a) by induction. First of all:

$$A_1B_0 - A_0B_1 = (a_0a_1 + 1) - a_0a_1 = (-1)^0 = 1$$

Using Proposition 2 (b), we have:

$$A_{n+2}B_{n+1} - A_{n+1}B_{n+2} = (a_{n+2}A_{n+1} + A_n)B_{n+1} - A_{n+1})(a_{n+2}B_{n+1} + B_n)$$
$$= A_nB_{n+1} - A_{n+1}B_n = (-1)(A_{n+1}B_n - A_nB_{n+1})$$

which proves it. Then (b) follows directly from (a), since any common factor of A_n and B_n would be a factor of $(-1)^n$. Finally, for (c), we notice that (a) also gives us:

$$\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \frac{(-1)^n}{B_nB_{n+1}}$$

which means that the differences of consecutive terms in the sequence:

$$\frac{A_0}{B_0},\quad \frac{A_1}{B_1},\quad \frac{A_2}{B_2},\quad \frac{A_3}{B_3},\quad \ldots$$

are alternating in sign and decreasing to zero. This implies that the sequence has a limit, and that the limit is between any two consecutive terms, which gives (c).

Example. The sequence $1, 2, 2, 2, \ldots$ gives:

$$A_0 = 1,\ A_1 = 3,\ A_2 = 7,\ A_3 = 17,\ A_4 = 41$$
$$B_0 = 1,\ B_1 = 2,\ B_2 = 5,\ B_3 = 12,\ B_4 = 29$$

which give rational numbers:

$$\frac{A_0}{B_0} = 1,\ \frac{A_1}{B_1} = 1.5,\ \frac{A_2}{B_2} = 1.4,\ \frac{A_3}{B_3} \approx 1.417,\ \frac{A_4}{B_4} \approx 1.414$$

that alternate above and below $\sqrt{2}$, and seem to converge to it.

Getting to the point (finally) which is to convert α into a_i's.

Case 1.

$$\alpha = \frac{r_0}{r_1} \text{ is a rational number} > 1 \text{ in lowest terms}$$

Apply the Euclidean algorithm to the (relatively prime) pair (r_0, r_1):

$$\begin{array}{ll} r_0 = a_0 r_1 + r_2; & \dfrac{r_0}{r_1} = a_0 + \dfrac{r_2}{r_1} \\ r_1 = a_1 r_2 + r_3; & \dfrac{r_1}{r_2} = a_1 + \dfrac{r_3}{r_2} \\ \vdots & \\ r_{n-1} = a_{n-1} \cdot r_n + 1; & \dfrac{r_{n-1}}{r_n} = a_{n-1} + \dfrac{1}{r_n} \\ r_n = a_n \cdot 1 + 0; & r_n = a_n \end{array}$$

But the right column gives us:

$$\alpha = a_0 + \frac{1}{\frac{r_1}{r_2}} = a_0 + \frac{1}{a_1 + \frac{1}{\frac{r_2}{r_3}}} = \cdots = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}$$

which **exactly** tells us that

$$\alpha = \frac{r_0}{r_1} = \frac{A_n}{B_n}$$

for the (finite!) sequence $a_0, a_1, \ldots, a_n$ of natural numbers.

Bonus! From Proposition 3(a), we get: $r_0 B_{n-1} - r_1 A_{n-1} = (-1)^{n-1}$ which tells us precisely how to solve the equation:

$$r_0 x + r_1 y = 1$$

with integers (x, y).

Example. $\alpha = 61/48$.
$$\begin{aligned} 61 &= 1 \cdot 48 + 13 \\ 48 &= 3 \cdot 13 + 9 \\ 13 &= 1 \cdot 9 + 4 \\ 9 &= 2 \cdot 4 + 1 \\ 4 &= 4 \cdot 1 + 0 \end{aligned}$$

The algorithm terminates with an output of: $1, 3, 1, 2, 4$. Rebuilding:
$$\frac{A_0}{B_0} = \frac{1}{1}, \ \frac{A_1}{B_1} = \frac{4}{3}, \ \frac{A_2}{B_2} = \frac{5}{4}, \ \frac{A_3}{B_3} = \frac{14}{11}, \ \frac{A_4}{B_4} = \frac{61}{48}$$
gives the bonus equation $61 \cdot 11 - 48 \cdot 14 = -1$.

Case 2.
$$\alpha > 1 \ \text{ is irrational.}$$

In this case there is no Euclidean algorithm, but we may define:

- $[\alpha]$ is the *round down* of α to the nearest integer, and
- $\{\alpha\} = \alpha - [\alpha]$ is the *fractional part* of α.

Then we get an infinite sequence of natural numbers $a_0, a_1, a_2, \ldots$:
$$\begin{aligned} \alpha &= a_0 + \frac{1}{\alpha_1}, \text{ letting } a_0 = [\alpha] \text{ and } \alpha_1 = \frac{1}{\{\alpha\}} \\ \alpha_1 &= a_1 + \frac{1}{\alpha_2}, \text{ letting } a_1 = [\alpha_1] \text{ and } \alpha_2 = \frac{1}{\{\alpha_1\}} \\ \alpha_2 &= a_2 + \frac{1}{\alpha_3}, \text{ letting } a_2 = [\alpha_2] \text{ and } \alpha_3 = \frac{1}{\{\alpha_2\}} \text{ etc.} \end{aligned}$$

Claim. The sequence of rational numbers $\{A_n/B_n\}$ coming from the sequence $\{a_n\}$ converges to the number α that we started with.

Proof of Claim. From the definitions above, we have:
$$\alpha = a_0 + \frac{1}{\alpha_1} = a_0 + \frac{1}{a_1 + \frac{1}{\alpha_2}} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\alpha_3}}} = \cdots$$
and our polynomial results tell us that:
$$\alpha = \frac{f_{n+1}(a_0, \ldots, a_n, \alpha_{n+1})}{f_n(a_1, \ldots, a_n, \alpha_{n+1})}$$
which, by Corollary 2, tells us that:
$$\alpha = \frac{\alpha_{n+1} f_n(a_0, \ldots, a_n) + f_{n-1}(a_0, \ldots, a_{n-1})}{\alpha_{n+1} f_{n-1}(a_1, \ldots, a_n) + f_{n-2}(a_1, \ldots, a_{n-1})} = \frac{\alpha_{n+1} A_n + A_{n-1}}{\alpha_{n+1} B_n + B_{n-1}}$$
But it is easy to check that this number is **between** $\frac{A_n}{B_n}$ and $\frac{A_{n-1}}{B_{n-1}}$, and since this is true for all n, it follows that α is the limit!

3. Periodic Continued Fractions. A *purely periodic* continued fraction is associated to a sequence of natural numbers of the form:

$$a_0, a_1, \ldots, a_n, a_0, a_1, \ldots, a_n, a_0, a_1, \ldots, a_n, \ldots$$

If we let:

$$\alpha = \lim_{n\to\infty} \frac{A_n}{B_n}$$

then we get:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_n + \frac{1}{\alpha}}}}$$

from which we may conclude, as in the proof of the claim above, that:

$$\alpha = \frac{\alpha A_n + A_{n-1}}{\alpha B_n + B_{n-1}}$$

so α is a root of the quadratic equation:

$$B_n x^2 + (B_{n-1} - A_n)x - A_{n-1} = 0$$

and one can solve for α with the quadratic formula.

Examples.

(i) Expansions of the form:

$$a, a, a, a, a, \ldots$$

give $\alpha = a + 1/\alpha$ so α is a root of the equation $x^2 - ax - 1 = 0$ and since $\alpha > 1$, we conclude that:

$$\alpha = \frac{a + \sqrt{a^2+4}}{2}$$

(a) When $a = 1$ we get the golden mean, $\alpha = (1+\sqrt{5})/2$.

(b) When $a = 2k$ is even, we get $\alpha = k + \sqrt{k^2+1}$ so:

$$k, 2k, 2k, 2k, \ldots \text{ is the expansion for } \sqrt{k^2+1}$$

(ii) Expansions of the form:

$$2k, k, 2k, k, \ldots$$

give α which is a root of: $0 = B_1x^2 + (B_0 - A_1)x - A_0 = k(x^2 - 2kx - 2)$ and therefore:

$$\alpha = \frac{2k + \sqrt{4k^2+8}}{2} = k + \sqrt{k^2+2}$$

Thus it follows that:

$$k, k, 2k, k, 2k, \ldots \text{ is the expansion of } \sqrt{k^2+2}$$

(iii) Expansions of the form:

$$2k, 1, 2k, 1, 2k, 1 \ldots$$

give α which is a root of $x^2 - 2kx - 2k = 0$, so:

$$\alpha = \frac{2k + \sqrt{4k^2 + 8k}}{2} = k + \sqrt{k^2 + 2k}$$

and square roots $\sqrt{k^2 + 2k} = \sqrt{(k+1)^2 - 1}$ have expansions:

$$k, 1, 2k, 1, 2k, \ldots$$

(iv) Expansions of the form:

$$2k, 2, 2k, 2, 2k, \ldots$$

give α which is a root of $2(x^2 - 2kx - k) = 0$ so:

$$\alpha = \frac{2k + \sqrt{4k^2 + 4k}}{2} = k + \sqrt{k^2 + k}$$

and square roots of the form $\sqrt{k^2 + k}$ have expansions:

$$k, 2, 2k, 2, 2k, \ldots$$

Question. Which $\alpha > 1$ have purely periodic continued fractions?

We know each such α is an irrational root of a quadratic equation.

Definition. Suppose $ax^2 + bx + c = 0$ is a quadratic equation, and

$$b^2 - 4ac \text{ is not a perfect square}$$

Then we will say that the roots are a conjugate pair $(\alpha, \overline{\alpha})$.

Example. (a) If $b^2 - 4ac < 0$, then $\alpha, \overline{\alpha}$ are complex numbers and they are conjugates in the ordinary sense.

(b) The conjugate of the golden mean $\alpha = (1+\sqrt{5})/2$ is $(1-\sqrt{5})/2$.

(c) The conjugate of $\sqrt{k}$ (when k is not a perfect square) is $-\sqrt{k}$.

Theorem. The $\alpha > 1$ with purely periodic continued fractions:

(i) Are irrational numbers, which

(ii) Are roots of a quadratic $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{Z}$ and

(iii) Have a conjugate root $\overline{\alpha}$ that satisfies $-1 < \overline{\alpha} < 0$.

Example. Any α of the form:

$$\alpha = k + \sqrt{k^2 + m} \text{ with } 0 < m \leq 2k$$

is irrational, a root of the quadratic equation: $x^2 - 2kx - m = 0$ and has conjugate root $\overline{\alpha} = k - \sqrt{k^2 + m} < 0$ satisfying $-1 < \overline{\alpha} < 0$.

4. Pell's Equation. We seek a solution to an equation of the form:

$$x^2 - dy^2 = 1$$

where $d > 0$ is a natural number that is not itself a perfect square.

Strategy. Such a solution satisfies $(x - y\sqrt{d})(x + y\sqrt{d}) = 1$ hence:

$$x - y\sqrt{d} = \frac{1}{x + y\sqrt{d}} \quad \text{and} \quad 0 < \frac{x}{y} - \sqrt{d} = \frac{1}{y(x + y\sqrt{d})} < \frac{1}{y^2}$$

Such good approximations of an irrational ($\sqrt{d}$) by a rational (x/y) are precisely what continued fraction expansions produce. For example, the continued fraction expansion of $\sqrt{7}$ is

$$2, 1, 1, 1, 4, 1, 1, 1, 4, \ldots$$

and the associated sequence of rational numbers:

$$\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \ldots$$

satisfy, respectively:

$$\begin{array}{rcr} 2^2 - 7 \cdot 1^2 & = & -3 \\ 3^2 - 7 \cdot 1^2 & = & 2 \\ 5^2 - 7 \cdot 2^2 & = & -3 \\ 8^2 - 7 \cdot 3^2 & = & 1 \end{array}$$

and $(8, 3)$ is a solution to Pell's equation.

A more involved example is $\sqrt{13}$, whose expansion is:

$$3, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, \ldots$$

producing the sequence:

$$\frac{3}{1}, \frac{4}{1}, \frac{7}{2}, \frac{11}{3}, \frac{18}{5},$$

which satisfy:

$$\begin{array}{rcr} 3^2 - 13 \cdot 1^2 & = & -4 \\ 4^2 - 13 \cdot 1^2 & = & 3 \\ 7^2 - 13 \cdot 2^2 & = & -3 \\ 11^2 - 13 \cdot 3^2 & = & 4 \\ 18^2 - 13 \cdot 5^2 & = & -1 \end{array}$$

and although $(18, 5)$ only solves $x^2 - 13y^2 = -1$, we know how to convert that to a solution to Pell's equation by squaring:

$$(18 - 5\sqrt{13})^2 = (324 + 25 \cdot 13) - 180\sqrt{13} = 649 - 180\sqrt{13}$$

giving the solution $(649, 180)$.

Proposition 4. Let:

$$a_0 - k, a_1, \ldots, a_n, a_0, a_1, \ldots, a_n, \ldots$$

be the continued fraction expansion of $\sqrt{k^2+m}$ $(m \leq 2k)$.

Then either:

(a) n is even, and:

$$A_n^2 - (k^2+m) \cdot B_n^2 = -1$$

or else

(b) n is odd, and:

$$A_n^2 - (k^2+m) \cdot B_n^2 = 1$$

In the latter case, Pells' equation is solved, and in the former:

$$(A_n - B_n\sqrt{k^2+m})^2 = (A_n^2 + (k^2+m)B_n^2) - 2A_nB_n\sqrt{k^2+m}$$

solves it with the pair $(A_n^2 + (k^2+m)B_n^2, 2A_nB_n)$.

Proof. $\alpha = k + \sqrt{k^2+m}$ has periodic expansion:

$$a_0, , a_1, \ldots, a_n, a_0, \ldots, a_n, \ldots$$

Let $\beta = \sqrt{k^2+m}$. Then:

$$\beta = \frac{\alpha A_n + A_{n-1}}{\alpha B_n + B_{n-1}} = \frac{(\beta+k)A_n + A_{n-1}}{(\beta+k)B_n + B_{n-1}}$$

from which it follows that β is a root of the polynomial:

$$B_n x^2 - (kB_n + B_{n-1} - A_n)x - (kA_n + A_{n-1}) = 0$$

But $\beta = \sqrt{k^2+m}$, so β is a root of the polynomial:

$$x^2 - (k^2+m) = 0$$

and it follows that:

(i) $kB_n + B_{n-1} - A_n = 0$ and

(ii) $kA_n + A_{n-1} = (k^2+m)B_n$.

Multiplying (i) through by A_n gives us:

$$kA_nB_n + A_nB_{n-1} - A_n^2 = 0$$

and Proposition 3(a) gives us $A_nB_{n-1} = A_{n-1}B_n + (-1)^{n-1}$ hence:

$$kA_nB_n + A_{n-1}B_n + (-1)^{n-1} - A_n^2 = (kA_n + A_{n-1})B_n - A_n^2 + (-1)^{n-1} = 0$$

and then (ii) gives:

$$(k^2+m)B_n^2 - A_n^2 = (-1)^n$$

which is exactly what we needed to prove.

Final Remarks. Our examples from §3 give us the following expansions:

$\sqrt{k^2+m}$	(k,m)	expansion
$\sqrt{2}$	$(1,1)$	$1,\overline{2}$
$\sqrt{3}$	$(1,2)$	$1,\overline{1,2}$
$\sqrt{5}$	$(2,1)$	$2,\overline{4}$
$\sqrt{6}$	$(2,2)$	$2,\overline{2,4}$
$\sqrt{7}$	$(2,3)$	not from the examples
$\sqrt{8}$	$(2,4)$	$2,\overline{1,4}$
$\sqrt{10}$	$(3,1)$	$3,\overline{6}$
$\sqrt{11}$	$(3,2)$	$3,\overline{3,6}$
$\sqrt{12}$	$(3,3)$	$3,\overline{2,6}$
$\sqrt{13}$	$(3,4)$	not from the examples
$\sqrt{14}$	$(3,5)$	not from the examples
$\sqrt{15}$	$(3,6)$	$3,\overline{1,6}$

and while *these* expansions are predictable, the others are mysterious:

(1) $\sqrt{7}$ has expansion $2,\overline{1,1,1,4}$

(2) $\sqrt{13}$ has expansion $3,\overline{1,1,1,1,6}$

(3) $\sqrt{14}$ has expansion $3,\overline{1,2,1,6}$

and some really long periods appear for small numbers. For example:

$$\sqrt{46} \text{ has expansion } 6,\overline{1,3,1,1,2,6,2,1,1,3,1,12}$$

leading to an enormous smallest solution to Pell's equation of $(24335, 3588)$.