Paco GOZE TP n°4 - Improve the Metropolis-Hastings algorithm

Bayesian analysis of one-way random effects model:

① $p(X,\mu,\sigma^2,\tau^2|Y) = p(Y|X,\mu,\sigma^2,\tau^2)\,p(X,\mu,\sigma^2,\tau^2)/p(Y)$

$= C\,p(Y|X,\mu,\sigma^2,\tau^2)\,p(X|\mu,\sigma^2,\tau^2)\,\pi_{prior}(\mu,\sigma^2,\tau^2)$

We have:

$p(Y|X,\mu,\sigma^2,\tau^2) = \prod_{i=1}^{N}\prod_{j=1}^{k_i} f_{X_i,\tau^2}(y_{ij})$

$p(X|\mu,\sigma^2,\tau^2) = \prod_{i=1}^{N} f_{\mu,\sigma^2}(X_i)$

$\pi_{prior}(\mu,\sigma^2,\tau^2) = K\,\frac{1}{\sigma^{2(1+\alpha)}}e^{-\beta/\sigma^2}\,\frac{1}{\tau^{2(1+\gamma)}}e^{-\beta/\tau^2}$, K a constant.

With f_{μ,σ^2} the density of a $\mathcal{N}(\mu,\sigma^2)$ law

Therefore:

$p(X,\mu,\sigma^2,\tau^2|Y) \propto \prod_{i=1}^{N}\left[f_{\mu,\sigma^2}(X_i)\prod_{j=1}^{k_i} f_{X_i,\tau^2}(y_{ij})\right]\pi_{prior}(\mu,\sigma^2,\tau^2)$

② To perform Gibbs sampling, we need to compute the conditionnale distributions of each parameter.

$p(X|\sigma^2,\tau^2,\mu,Y) = \frac{p(X,Y,\mu,\sigma^2,\tau^2)}{p(\sigma^2,\tau^2,\mu,Y)} = \frac{p(X,\mu,\sigma^2,\tau^2|Y)\,p(Y)}{p(\sigma^2,\tau^2,\mu,Y)}$ $\perp\!\!\!\perp X$

$\propto p(X,\mu,\sigma^2,\tau^2|Y)$

So:

$$p(X \mid \mu, \sigma^2, \tau^2, Y) \propto \prod_{i=1}^{N} \left[\prod_{j=1}^{k_i} f_{x_i, \tau^2}(y_{ij}) \, f_{\mu, \sigma^2}(x_i) \right]$$

$$p(x_i \mid \mu, \sigma^2, \tau^2, \{y_{ij}\}_{j \in [\![1, k_i]\!]}) \propto \underbrace{\prod_{j=1}^{k_i} f_{x_i, \tau^2}(y_{ij}) \, f_{\mu, \sigma^2}(x_i)}_{(*)}$$

$$(*) = \prod_{j=1}^{k_i} \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{1}{2}\left(\frac{y_{ij} - x_i}{\tau}\right)^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2}$$

$$= \left(\frac{1}{2\pi\sigma^2 (2\pi\tau^2)^{k_i}} \right)^{1/2} e^{-\frac{1}{2}\left[\left(\frac{x_i - \mu}{\sigma}\right)^2 + \sum_{j=1}^{k_i} \left(\frac{y_{ij} - x_i}{\tau}\right)^2 \right]}$$

$$\left(\frac{x_i - \mu}{\sigma}\right)^2 + \sum_{j=1}^{k_i} \left(\frac{y_{ij} - x_i}{\tau}\right)^2 = \frac{1}{\sigma^2}(x_i^2 - 2\mu x_i + \mu^2) + \frac{1}{\tau^2}\left(\sum_{j=1}^{k_i} y_{ij}^2 - 2x_i y_{ij} + x_i^2 \right)$$

$$= x_i^2 \left(\frac{1}{\sigma^2} + \frac{k_i}{\tau^2} \right) - 2x_i \left(\frac{\mu}{\sigma^2} + \frac{1}{\tau^2} \sum_{j=1}^{k_i} y_{ij} \right) + \text{cste}$$

$$= \left(\frac{1}{\sigma^2} + \frac{k_i}{\tau^2} \right) \left[x_i^2 - 2x_i \left(\frac{\mu}{\sigma^2} + \frac{1}{\tau^2} \sum_{j=1}^{k_i} y_{ij} \right) \left(\frac{1}{\sigma^2} + \frac{k_i}{\tau^2} \right)^{-1} \right] + \text{cste}$$

$$= \underbrace{\left(\frac{1}{\sigma^2} + \frac{k_i}{\tau^2} \right)}_{1/\sigma_i^2} \left[x_i - \underbrace{\left(\frac{\mu}{\sigma^2} + \frac{1}{\tau^2} \sum_{j=1}^{k_i} y_{ij} \right) \left(\frac{1}{\sigma^2} + \frac{k_i}{\tau^2} \right)^{-1}}_{\mu_i} \right]^2 + \text{cste}$$

Therefore:

$$(X_i \mid \mu, \sigma^2, \tau^2, \{y_{ij}\}_{j \in [\![1, k_i]\!]}) \sim \mathcal{N}(\mu_i, \sigma_i^2)$$

with $$\mu_i = \left(\frac{\mu}{\sigma^2} + \frac{1}{\tau^2} \sum_{j=1}^{k_i} y_{ij} \right) \sigma_i^2$$

$$\sigma_i^2 = \frac{\sigma^2 \tau^2}{\tau^2 + k_i \sigma^2}$$

The X_i are iid and follow a $\mathcal{N}(\mu, \sigma^2)$

Therefore: $\mu \mid X, \sigma^2, \tau^2, Y \sim \mathcal{N}\left(\bar{X}_n, \frac{\sigma^2}{N}\right)$

With similar computations, we obtain $\sigma^2 \mid X, \mu, \tau^2, Y$ and $\tau^2 \mid \mu, X, \sigma^2, Y$, from $p(X, \mu, \sigma^2, \tau^2 \mid Y)$

$$\sigma^2 \mid X, Y, \mu, \tau^2 \sim \Gamma^{-1}\left(\alpha + \frac{N}{2}, \beta + \frac{1}{2}\sum_{i=1}^{N}(X_i - \mu)^2\right)$$

$$\tau^2 \mid X, Y, \mu, \sigma^2 \sim \Gamma^{-1}\left(\gamma + \frac{K}{2}, \beta + \sum_{i=1}^{N}\sum_{j=1}^{k_i}\frac{1}{2}(Y_{ij} - X_i)^2\right)$$

These distributions being known, we can implement a Gibbs sampler which updates in turn σ^2, τ^2, μ, X.

③ We have to determine the distribution of $(X, \mu) \mid \sigma^2, \tau^2, Y$.

$$\begin{aligned}
p(X, \mu \mid \sigma^2, \tau^2, Y) &= p(X, \mu, \sigma^2, \tau^2, Y) \,/\, p(\sigma^2, \tau^2, Y) \\
&= p(Y \mid X, \mu, \sigma^2, \tau^2)\, p(X, \mu, \sigma^2, \tau^2) \,/\, p(\sigma^2, \tau^2, Y) \\
&= p(Y \mid X, \mu, \sigma^2, \tau^2)\, p(X \mid \mu, \sigma^2, \tau^2) \underbrace{\frac{p(\mu, \sigma^2, \tau^2)}{p(\sigma^2, \tau^2, Y)}}_{\text{cste of } (X, \mu)\,!} \\
&\propto p(Y \mid X, \mu, \sigma^2, \tau^2)\, p(X \mid \mu, \sigma^2, \tau^2)
\end{aligned}$$

And we have:

$$p(X \mid \mu, \sigma^2, \tau^2) = \prod_{i=1}^{N} f_{\mu, \sigma^2}(X_i)$$

$$p(Y \mid X, \mu, \sigma^2, \tau^2) = \prod_{i=1}^{N}\prod_{j=1}^{k_i} f_{X_i, \tau^2}(Y_{ij})$$

Therefore:

$$p(X,\mu \mid \sigma^2,\tau^2,Y) \propto \prod_{i=1}^{N} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2} \prod_{j=1}^{k_i} e^{-\frac{1}{2}\left(\frac{y_{ij}-x_i}{\tau}\right)^2}$$

$$\log\left(p(X,\mu \mid \sigma^2,\tau^2,Y)\right) = \sum_{i=1}^{N}\left[\frac{-1}{2\sigma^2}(x_i-\mu)^2 + \sum_{j=1}^{k_i}\frac{-1}{2\tau^2}(y_{ij}-x_i)^2\right] + cste$$

$$= \frac{-1}{2}\sum_{i=1}^{N}\left[\frac{1}{\sigma^2}(x_i^2-2x_i\mu+\mu^2) + \frac{1}{\tau^2}\sum_{j=1}^{k_i} y_{ij}^2 - 2x_i y_{ij} + x_i^2\right] + cste$$

$$= \frac{-1}{2}\sum_{i=1}^{N}\left[x_i^2\left(\frac{1}{\sigma^2}+\frac{k_i}{\tau^2}\right) + x_i\left(\frac{-2\mu}{\sigma^2} - \frac{2}{\tau^2}\sum_{j=1}^{k_i} y_{ij}\right) + \frac{1}{\tau^2}\sum_{j=1}^{k_i} y_{ij}^2\right] - \frac{N\mu^2}{2\sigma^2} + cste.$$

$$= -\frac{1}{2}\frac{N\mu^2}{\sigma^2} - \frac{1}{2}\sum_{i=1}^{N}\left[x_i^2\left(\frac{1}{\sigma^2}+\frac{k_i}{\tau^2}\right) - 2x_i\left(\frac{\mu}{\sigma^2} + \frac{1}{\tau^2}\sum_{j=1}^{k_i} y_{ij}\right)\right] + cste$$

$\perp\!\!\!\perp$ de (X,μ)

By identification:

$X,\mu \mid \sigma^2,\tau^2,Y \sim \mathcal{N}(M_U, \Sigma)$, with

$$\Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots & 0 & -1/\sigma^2 \\ 0 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \frac{1}{\sigma_N^2} & -1/\sigma^2 \\ -1/\sigma^2 & \cdots & \cdots & -1/\sigma^2 & N/\sigma^2 \end{pmatrix}$$

$$M_U = \frac{1}{\tau^2}\Sigma \begin{pmatrix} \sum_{j=1}^{k_1} y_{1,j} \\ \vdots \\ \sum_{j=1}^{k_N} y_{N,j} \\ 0 \end{pmatrix}$$