Coherence for Categorical Groups and Symmetrization of Categorical Groups

Dissertation

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Abstract

Categorical groups are monoidal categories in which every object admits a weak inverse. In this document, we establish and prove a coherence theorem for categorical groups. Furthermore, we show that any categorical group is equivalent, via categorical group functors, to a strict categorical group. Central to our approach is the introduction of a new algebraic structure, the dashed monoid, for which we provide a complete and formal description of the free dashed monoid generated by a set. We conclude by presenting a construction that produces a symmetric categorical group from a given categorical group.

This thesis is dedicated to my parents for their unwavering support throughout my journey.

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Contents

Chapter 1: Introduction

The thesis is organized into three parts: I. Foundation, II. Formal Constructions and Main Results, and III. Examples and Applications. Below, we provide a brief overview of these sections.

Foundation

In this part, we introduce the foundational concepts necessary to present the statement of coherence for categorical groups.

In Chapter 2, we provide an overview of categories and key topics in category theory. Of particular interest is the notion of thin categories. A category is called *thin* if, for every pair of objects, the set of morphisms between them is either empty or a singleton.

Chapter 3 presents an overview of monoidal categories. A monoidal category is a category equipped with a bifunctor, called the *tensor product* or *monoidal product*, which is associative up to a natural isomorphism known as the *associator*. Additionally, there is a distinguished *unit* object, which acts as a unit for the bifunctor up to natural isomorphisms called the *left unitor* and *right unitor*. These natural isomorphisms, collectively referred to as the structure isomorphisms of monoidal categories, satisfy coherence axioms that ensure compatibility among the monoidal structures.

A strict monoidal category is a monoidal category in which all structure morphisms are identities.

In Section 3.2, we construct the free monoidal category generated by a set. The free monoidal category generated by a set S, denoted $\mathcal{MonCat}\langle S\rangle$, is a category whose objects are given by $\mathcal{M}[S]$, the free magma generated by S. The associators and unitors generate the morphisms, while the coherence axioms serve as relations. For any monoidal category $\mathcal{M}$ with objects $M = \mathrm{Obj}(\mathcal{M})$, there exists a strict monoidal functor

$$\mathcal{EV} : \mathcal{MonCat}\langle M\rangle \longrightarrow \mathcal{M}.$$

A diagram $D : \mathcal{D} \longrightarrow \mathcal{M}$ is called a formal diagram in $\mathcal{M}$ if it lifts to the free monoidal category $\mathcal{MonCat}\langle M\rangle$.

Mac Lane's coherence theorem for monoidal categories states that the free monoidal category generated by a set is thin. Consequently, every formal diagram in a monoidal category commutes. Another important consequence is that every monoidal category is monoidally equivalent to a strict monoidal category.

Section 3.3 presents an alternative approach to coherence for monoidal categories. We consider the induced functor

$$Q_{\mathcal{MonCat}} : \mathcal{MonCat}\langle S\rangle \longrightarrow \mathcal{Mon}\langle S\rangle$$

where $\mathcal{Mon}\langle S\rangle$ is the free monoid generated by S. As a consequence of the coherence theorem, there is a unique morphism $f : x \longrightarrow y$ in the free monoidal category $\mathcal{MonCat}\langle S\rangle$ if and only if $Q_{\mathcal{M}}(x) = Q_{\mathcal{M}}(y)$. This observation motivates an alternative construction of the free monoidal category, in which morphisms are instead generated by this condition. This approach shifts the focus of the coherence theorem from showing that the free monoidal category is thin to verifying that this already thin category satisfies the universal property. The advantage of this construction is that it provides a concise criterion for morphisms in the free monoidal category.

Mac Lane discusses this alternative construction in Exercise 3 on page 170 of [ML78]. In fact, Mac Lane uses this construction to build the free monoidal category generated by a singleton, which plays a central role in proving coherence for categorical groups (see Theorem 1 on page 166 of [ML78]). This alternative approach motivates the concepts and ideas in categorical groups developed in the subsequent chapters.

In Chapter 4, we formally introduce categorical groups. A *categorical group $\mathscr{M}$* is a monoidal category in which every object x has a weak inverse x', called the negator of x, with dedicated cancellation isomorphisms

$$\eta_x : I \longrightarrow x \otimes x' \qquad \text{and} \qquad \epsilon_x : x' \otimes x \longrightarrow I$$

that satisfy coherence axioms. The associators, unitors, and cancellation isomorphisms are collectively referred to as the *structure isomorphisms* of the categorical group $\mathscr{M}$. A *semi-strict categorical group* is a categorical group whose underlying monoidal category is strict, while a *strict categorical group* is one in which all structure isomorphisms are identities.

In Section 4.2, we show that the underlying monoidal functors uniquely determine categorical group functors between semi-strict categorical groups. In Section 4.3, we use coherence for categorical groups to show that every categorical group is equivalent to a semi-strict categorical group via categorical group functors.

In Section 4.4, we construct the free categorical group generated by a set. The free categorical group generated by a set S, denoted $\mathcal{CatGrp}\langle S\rangle$, is a category whose objects are freely generated by S under multiplication and a unary operation called *dash*. Denote the set of objects of $\mathcal{CatGrp}\langle S\rangle$ by $\mathcal{DMul}\langle S\rangle$. The associators, unitors, and cancellation isomorphisms generate the morphisms of $\mathcal{CatGrp}\langle S\rangle$, and the coherence axioms serve as relations. For a categorical group $\mathscr{M}$ with objects $M = \mathrm{Obj}(\mathscr{M})$, there is a strict categorical group functor $\mathcal{EV} : \mathcal{CatGrp}\langle M\rangle \longrightarrow \mathscr{M}$ from the free categorical group generated by M to $\mathscr{M}$. A diagram $D : \mathscr{D} \longrightarrow \mathscr{M}$ is called a formal diagram in $\mathscr{M}$ if it lifts to the free categorical group $\mathcal{CatGrp}\langle M\rangle$.

The coherence theorem for categorical groups states that the free categorical group generated by a set is thin. As a result, every formal diagram in a categorical group commutes. This theorem is established in [Lap83]. In this thesis, we adopt an alternative approach inspired by the discussion earlier, which enables us to prove a stronger statement: the induced functor

$$\mathcal{Q}_{\mathcal{CatGrp}} : \mathcal{CatGrp}\langle S\rangle \longrightarrow \mathcal{Grp}\langle S\rangle$$

is an equivalence, where $\mathcal{Grp}\langle S\rangle$ denotes the free group generated by S. As a further consequence of this approach, we show that every categorical group is equivalent, via categorical group functors, to a strict categorical group.

In this approach, we focus on the intermediary structure of semi-strict categorical groups. We show that the free categorical group over a set is equivalent to the free semi-strict categorical group over the same set via categorical group functors. Thus, the main goal is reduced to showing that the free semi-strict categorical group generated by a set is thin.

We construct the free semi-strict categorical group generated by a set S, denoted $\mathcal{SSCatGrp}\langle S\rangle$, as follows: The set of objects, denoted $\mathcal{DMon}\langle S\rangle$, is freely generated by S under associative multiplication and a unary operation called *dash*. (We will discuss this general structure in detail in later chapters.) Consider the induced function

$$Q_{\mathcal{DMon}} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathcal{Grp}\langle S\rangle$$

where $\mathcal{Grp}\langle S\rangle$ is the free group generated by S. For objects x, y in $\mathcal{DMon}\langle S\rangle$, there is a unique morphism $\star_{x,y} : x \longrightarrow y$ if and only if $Q_{\mathcal{DMon}}(x) = Q_{\mathcal{DMon}}(y)$. Thus, the category $\mathcal{SSCatGrp}\langle S\rangle$ is a thin category. The main goal is now reduced to showing that this construction satisfies the appropriate universal property of the free semi-strict categorical group.

The subsequent chapters are devoted to analyzing and formalizing this construction.

Formal Constructions and Main Results

In Chapter 5, we construct the *distribution and reduction natural isomorphisms* in a semi-strict categorical group and establish the commutativity of compatibility diagrams involving these isomorphisms. These constructions serve as essential components in the proof of coherence for semi-strict categorical groups.

Chapter 6 provides a concise overview of monoid theory, including a formal construction of the free monoid generated by a set S. We use the notion of a *monoid basis*: A subset $X \subseteq M$ such that every element of M can be uniquely expressed as a product of elements from X. We demonstrate that the existence of a monoid basis is equivalent to M being a free monoid.

In Chapter 7, we introduce the algebraic structure of dashed monoids: monoids equipped with a unary operation, *dash*, which need not interact with the monoid multiplication. The motivation for this structure arises from the observation that the objects of a semi-strict categorical group naturally form a dashed monoid. In

Section 7.2, we state the universal property of the free dashed monoid and define the concept of a *dashed monoid basis*. Unlike the case for monoids, the structure of a dashed monoid basis is more intricate, and we devote Sections 7.3 through 7.5 to a detailed analysis. In Sections 7.6 and 7.7, we construct the free dashed monoid generated by a set. The technical developments in these sections culminate in a complete characterization of dashed monoid bases, formally established in Section 7.8. For the remainder of the thesis, only the results from Section 7.8 are required; the preceding sections serve as technical groundwork.

It is worth noting that the content of Chapter 7 has been formally encoded using the Lean 4 programming language and theorem prover. Lean 4 enables the formalization and verification of mathematical statements, including theorems and their proofs. The corresponding formalization is available at [Par25].

Chapter 8 details the construction of the free semi-strict categorical group generated by S, denoted $\mathcal{SSCatGrp}\langle S\rangle$. In Section 8.2, we define three nested subsets of $\mathcal{DMon}\langle S\rangle$,

$$\mathtt{A} \subseteq \mathtt{B} \subseteq \mathtt{C} \subseteq \mathcal{DMon}\langle S\rangle\,,$$

which reflects increasing complexity in the combination of multiplication and dash operations. These subsets are used to classify morphisms in $\mathcal{SSCatGrp}\langle S\rangle$. For any $x \in \mathcal{DMon}\langle S\rangle$, there exist unique morphisms

$$x \xrightarrow{\text{Type C}} x_{\mathtt{C}} \xrightarrow{\text{Type B}} x_{\mathtt{B}} \xrightarrow{\text{Type A}} x_{\mathtt{A}}$$

with $x_{\mathtt{C}} \in \mathtt{C}$, $x_{\mathtt{B}} \in \mathtt{B}$, and $x_{\mathtt{A}} \in \mathtt{A}$. Moreover, if there is a morphism $f : x \longrightarrow y$ then $x_{\mathtt{A}} = y_{\mathtt{A}}$ and $f : x \longrightarrow y$ factors through $x_{\mathtt{A}} = y_{\mathtt{A}}$ via these morphisms. In Sections 8.3

through 8.5, we show that for any function

$$u : S \longrightarrow \mathrm{Obj}(\mathscr{M})$$

into the objects of a semi-strict categorical group $\mathscr{M}$, one can assign unique morphisms in $\mathscr{M}$ to each of the Type C, B, and A morphisms.

In Chapter 9, we prove the coherence theorem for categorical groups. In Section 9.1, we show that the function

$$u : S \longrightarrow \mathrm{Obj}(\mathscr{M})$$

lifts to a strict functor

$$F : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathscr{M}.$$

In Section 9.2, we construct a collection N of morphisms between functors

$$F, G : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathscr{M}.$$

The collection N is given by

$$N = \{(x, f) \mid x \in \mathcal{DMon}\langle S\rangle,\ f : F(x) \longrightarrow G(x)\}.$$

We show that N itself forms a dashed monoid. Thus, any function $\widehat{\phi} : S \longrightarrow N$ induces a categorical group natural transformation $\phi : F \longrightarrow G$. This shows the existence and uniqueness of categorical group natural transformation required by the universal property.

Section 9.3 summarizes these findings and presents a formal proof of the coherence theorem for categorical groups.

Examples and Applications

In Chapter 10, we demonstrate that every semi-strict categorical group is equivalent to a strict categorical group by embedding a semi-strict categorical group into a strict categorical group whose objects are given by the free group generated by the objects of the semi-strict categorical group. The proof relies only on results from Section 9.3 and basic categorical concepts, making this chapter largely self-contained. Together with the previously established equivalence between categorical groups and semi-strict categorical groups, this shows that every categorical group is equivalent, via categorical group functors, to a strict categorical group.

In Chapter 11, we apply the coherence theorem to formalize the concept of canonical natural transformations in categorical groups. This chapter depends only on results from Section 9.3 and basic categorical concepts. Thus, this chapter is also largely self-contained. We introduce the notion of *iterates* in categorical groups: functors constructed as finite compositions of tensor and negator functors. To formalize this, we first define the pointed functor category $\mathcal{F}ct_n(\mathcal{M})$, consisting of all functors from $\mathcal{M}^n$ to $\mathcal{M}$. We then construct a categorical group functor

$$\mathtt{Can} : \mathcal{C}at\mathcal{G}rp\langle n\rangle \longrightarrow \mathcal{F}ct_n(\mathcal{M}).$$

The images of objects in $\mathcal{C}at\mathcal{G}rp\langle n\rangle$ under this functor are called iterates in n variables, while the images of morphisms are referred to as *canonical natural transformations* in n variables. This framework provides a precise formalization of canonical natural transformations, which we illustrate with several examples.

In Chapter 12, we construct a symmetric categorical group associated with a given categorical group. We begin by defining symmetric categorical groups and stating the coherence theorem for such structures.

Given a set S, consider the induced functor

$$R : \mathcal{CatGrp}\langle S\rangle \longrightarrow \mathcal{AbGrp}\langle S\rangle$$

where $\mathcal{AbGrp}\langle S\rangle$ denotes the free abelian group generated by S. We focus on the subcategorical group $\ker(R)$, consisting of objects and morphisms that map to the zero element under R.

For a categorical group $\mathcal{M}$, the *commutator subcategory* $[\mathcal{M},\mathcal{M}]$ is generated by those objects and morphisms that lift to $\ker(R)$. For example, objects of the form $((x\otimes y)\otimes x')\otimes y'$ and morphisms of the form $((f\otimes g)\otimes f')\otimes g'$ belong to $[\mathcal{M},\mathcal{M}]$, reflecting the usual notion of commutators.

Utilizing the canonical natural transformations developed previously, we demonstrate that $[\mathcal{M},\mathcal{M}]$ forms a $\mathcal{M}$-crossed module as described in [CGV06]. Following the quotient construction from [CGV06], we define the categorical group

$$\mathtt{Sym}(\mathcal{M}) := \mathcal{M}/[\mathcal{M},\mathcal{M}].$$

We further define a braiding on $\mathtt{Sym}(\mathcal{M})$ and show that it admits the structure of a symmetric categorical group.

Foundation

Chapter 2: Topics in Category Theory

In this chapter, we review foundational concepts in category theory. We present definitions of categories, functors, natural transformations, categorical equivalences, and the universal property of free objects.

2.1 Grothendieck Universes and Categories

In this section, we establish the basic set-theoretic conventions used throughout our discussion. We then introduce the basic definitions of categories, functors, and natural transformations, along with essential categorical constructions.

Definition 2.1.1. A *Grothendieck universe*, or simply a *universe*, is a set $\mathscr{U}$ satisfying:

i. If $x \in \mathscr{U}$ and $y \in x$ then $y \in \mathscr{U}$.

ii. If $I \in \mathscr{U}$ and $x_i \in \mathscr{U}$ for each $i \in I$, then the union $\cup_{i \in I} x_i \in \mathscr{U}$.

iii. If $x \in \mathscr{U}$, then the power set of x, denoted by $P(X)$, is an element of $\mathscr{U}$.

iv. The set of finite ordinals $\mathbb{N} \in \mathscr{U}$. ◇

Remark 2.1.2. In some literature, the finite ordinals property (iv) is replaced by the following: $(iv)'$ if $x, y \in \mathscr{U}$, then $\{x, y\} \in \mathscr{U}$. This allows for universes to be empty. For non-empty universes, the two definitions are equivalent. $\diamond$

Definition 2.1.3. Let $\mathscr{U}$ and $\mathscr{V}$ be two Grothendieck universes. We say that the universe $\mathscr{U}$ belongs to the universe $\mathscr{V}$ if $\mathscr{U} \in \mathscr{V}$. $\diamond$

Convention 2.1.4. We assume the

Axiom of Universes: Every set belongs to some universe.

Consequently, for every universe $\mathscr{U}$ there exists a universe $\mathscr{V}$ such that $\mathscr{U} \in \mathscr{V}$. $\diamond$

Convention 2.1.5. We fix a universe $\mathscr{U}$. An element in $\mathscr{U}$ is called a *small set* or simply a *set*, and a subset of $\mathscr{U}$ is called a *class*. $\diamond$

Definition 2.1.6. Assume Convention 2.1.5. A *category* $\mathscr{C}$ in the universe $\mathscr{U}$ has the following data:

- A class of *objects* denoted $\mathrm{Obj}(\mathscr{C})$;
- for any objects $x, y \in \mathrm{Obj}(\mathscr{C})$, a set of *morphisms with domain x and codomain y*, denoted $\mathrm{Hom}_{\mathscr{C}}(x, y)$. We use the notation $f : x \longrightarrow y$ to mean $f \in \mathrm{Hom}_{\mathscr{C}}(x, y)$;
- an *identity* morphism $1_x : x \longrightarrow x$ for every object $x \in \mathrm{Obj}(\mathscr{C})$;
- an assignment called the *composition*

$$\mathrm{Hom}_{\mathscr{C}}(y, z) \times \mathrm{Hom}_{\mathscr{C}}(c, y) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}(x, z),$$

  for objects $x, y, z \in \mathrm{Obj}(\mathscr{C})$. We use the notation $f \circ g$ to mean $\circ(f, g)$.

These data satisfy the following axioms:

Unity: For every morphism $f : x \longrightarrow y$, the equalities

$$1_y \circ f = f = f \circ 1_x \tag{2.1.7}$$

hold in $\mathrm{Hom}_{\mathscr{C}}(x, y)$.

Associativity: For morphisms

$$f : x \longrightarrow y,\ g : y \longrightarrow z,\ \text{ and }\ h : z \longrightarrow w,$$

the equlity

$$h \circ (g \circ f) = (h \circ g) \circ f. \tag{2.1.8}$$

holds in $\mathrm{Hom}_{\mathscr{C}}(x, w)$. ◇

Definition 2.1.9. A *small category* is a category in which the class of objects forms a set. ◇

Definition 2.1.10. Let $\mathscr{C}$ be a category. The class of all morphism in $\mathscr{C}$, denoted $\mathrm{Mor}(\mathscr{C})$, is defined as follows:

$$\mathrm{Mor}(\mathscr{C}) := \bigsqcup_{(x,y)\in \mathrm{Obj}(\mathscr{C})^2} \mathrm{Hom}_{\mathscr{C}}(x, y). \tag{2.1.11}$$

A morphism $f : x \longrightarrow y$ in $\mathscr{C}$ is called an *isomorphism* if there exists a morphism $g : y \longrightarrow x$ in $\mathscr{C}$ such that $g \circ f = 1_x$ and $f \circ g = 1_y$. If such a morphism g exists, then it is unique. The morphism $g : y \longrightarrow x$ is called the *inverse* of f, denoted by f^{-1}. Two morphisms $f, g \in \mathrm{Mor}(\mathscr{C})$ are called *parallel morphisms* if they share the same domain and codomain. That is, $f, g \in \mathrm{Hom}_{\mathscr{C}}(x, y)$ for some $x, y \in \mathrm{Obj}(\mathscr{C})$. ◇

Definition 2.1.12. A category $\mathscr{C}$ is called:

- *Discrete* if every morphism in $\mathscr{C}$ is an identity morphism.
- A *groupoid* if every morphism in $\mathscr{C}$ is an isomorphism.
- A *thin category* if for any objects x, y in $\mathscr{C}$, the set of morphisms $\mathrm{Hom}_{\mathscr{C}}(x, y)$ contains at most one element. $\diamond$

Example 2.1.13. The *category of small sets* or simply the *category of sets* in the universe $\mathscr{U}$, denoted $\mathcal{S}et_{\mathscr{U}}$ or simply $\mathcal{S}et$, is defined as follows:

- The objects of $\mathcal{S}et_{\mathscr{U}}$ are the sets in $\mathscr{U}$, that is,

$$\mathrm{Obj}(\mathcal{S}et_{\mathscr{U}}) = \mathscr{U}.$$

- For any $x, y \in \mathscr{U}$, the set of morphisms $\mathrm{Hom}_{\mathcal{S}et_{\mathscr{U}}}(x, y)$ is the set of all functions from x to y:

$$\mathrm{Hom}_{\mathcal{S}et_{\mathscr{U}}}(x, y) := y^x.$$

- The identity morphism for each $x \in \mathscr{U}$ is the identity function 1_x.
- Composition of morphisms is given by composition of functions.

The composition of functions is associative and unital. Thus, $\mathcal{S}et_{\mathscr{U}}$ satisfies the axioms of a category. Note that, $\mathcal{S}et_{\mathscr{U}}$ is *not* a small-category as described in Definition 2.1.9.

Now, from the axiom of universe Convention 2.1.4, we get a universe $\mathscr{V}$ such that $\mathscr{U} \in \mathscr{V}$. When viewed from this larger universe $\mathscr{V}$, $\mathcal{S}et_{\mathscr{U}}$ is a small category. $\diamond$

Definition 2.1.14. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A *functor* $F : \mathscr{C} \longrightarrow \mathscr{D}$ has the following data:

- a function *on objects*

$$\mathrm{Obj}(\mathscr{C}) \longrightarrow \mathrm{Obj}(\mathscr{D}), \qquad x \longmapsto F(x);$$

- a function *on morphisms*

$$\mathrm{Hom}_{\mathscr{C}}(x,y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(x),F(y)), \qquad f \longmapsto F(f)$$

  for objects $x, y \in \mathscr{C}$.

These data satisfy the following functor conditions:

Identity: For every object $x \in \mathscr{C}$, the equality

$$F(1_x) = 1_{F(x)} \tag{2.1.15}$$

in $\mathrm{Hom}_{\mathscr{D}}(F(x),F(x))$ holds.

Composition: For morphisms $f : x \longrightarrow y$ and $g : y \longrightarrow z$, the equality

$$F(g \circ f) = F(g) \circ F(f) \tag{2.1.16}$$

in $\mathrm{Hom}_{\mathscr{D}}(F(x),F(z))$ holds. $\diamond$

Definition 2.1.17. Let $\mathscr{C}$ be a category. Define the *identity functor* $\mathrm{Id}_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathscr{C}$ as follows:

- For an object $x \in \mathscr{C}$ define

$$\mathrm{Id}_{\mathscr{C}}(x) := x. \tag{2.1.18}$$

- For a morphism $f : x \longrightarrow y$ in $\mathscr{C}$ define

$$\mathrm{Id}_{\mathscr{C}}(f) := f. \tag{2.1.19}$$

Above assignment satisfies the identity condition since

$$\begin{aligned} \mathrm{Id}_{\mathscr{C}}(1_x) &= 1_x && (2.1.19) \\ &= 1_{\mathrm{Id}_{\mathscr{C}}(x)}. && (2.1.18) \end{aligned}$$

The composition condition is satisfied since

$$\begin{aligned} \mathrm{Id}_{\mathscr{C}}(g \circ f) &= g \circ f && (2.1.19) \\ &= \mathrm{Id}_{\mathscr{C}}(g) \circ \mathrm{Id}_{\mathscr{C}}(f). && (2.1.19) \end{aligned}$$

◇

Definition 2.1.20. Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be categories and

$$F : \mathscr{A} \longrightarrow \mathscr{B} \quad \text{and} \quad G : \mathscr{B} \longrightarrow \mathscr{C}$$

be functors. We define *functor composition* denoted $G \circ F : \mathscr{A} \longrightarrow \mathscr{C}$ as follows:

- For an object $x \in \mathscr{A}$ define

$$G \circ F\ (x) := G(F(x)). \tag{2.1.21}$$

- For a morphism $f : x \longrightarrow y$ in $\mathscr{A}$ define

$$G \circ F\ (f) := G(F(f)). \tag{2.1.22}$$

These data satisfy the identity condition since

$$\begin{aligned} F \circ G\ (1_x) &= F(G(1_x)) && (2.1.22) \\ &= F(1_{G(x)}) && (2.1.15) \\ &= 1_{F(G(x))} && (2.1.15) \\ &= 1_{F \circ G(x)}. && (2.1.21) \end{aligned}$$

The composition condition is satisfied from the following

$$
\begin{aligned}
F \circ G\ (f \circ g) &= F(G(f \circ g)) && (2.1.22)\\
&= F(G(f) \circ G(g)) && (2.1.16)\\
&= F(G(f)) \circ F(G(f)) && (2.1.16)\\
&= \big(F \circ G\ (f)\big) \circ \big(F \circ G\ (g)\big). && (2.1.22)
\end{aligned}
$$

$\diamond$

Definition 2.1.23. Let $\mathscr{C}$ be category and $\mathscr{D}$ be a small category. A $\mathscr{D}$ shaped *diagram* in $\mathscr{C}$ is a functor $D : \mathscr{C} \longrightarrow \mathscr{D}$. We say that the *diagram D commutes* if it factors through a thin category. That is, there exists a thin category $\mathscr{B}$ and functors $\widehat{D} : \mathscr{C} \longrightarrow \mathscr{B}$ and $\overline{D} : \mathscr{B} \longrightarrow \mathscr{D}$ such that

$$\overline{D} \circ \widehat{D} = D.$$

Equivalently, a diagram D commutes if and only if for every pair of parallel morphisms $f, g : x \longrightarrow y$ in $\mathscr{C}$, we have $D(f) = D(g)$. $\diamond$

Definition 2.1.24. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor. A *natural transformation* $\phi : F \Rightarrow G$ has the following data:

- For every object $x \in \mathscr{C}$, a morphism $\phi_x : F(x) \longrightarrow G(x)$.

These morphisms satisfy the following condition:

<u>Naturality</u>: For every morphism $f : x \longrightarrow y$, the diagram

$$
\begin{tikzcd}
F(x) \arrow[r, "F(f)"] \arrow[d, "\phi_x"'] & F(y) \arrow[d, "\phi_y"] \\
G(x) \arrow[r, "G(f)"'] & G(x)
\end{tikzcd}
\tag{2.1.25}
$$

commutes. $\diamond$

Definition 2.1.26. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and suppose $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor. Define the identity natural transformation $\mathrm{Id}_F : F \Rightarrow F$ as follows:

- For an object $x \in \mathscr{C}$ define

$$\mathrm{Id}_F\,(x) := 1_{F(x)}. \tag{2.1.27}$$

The naturality condition is satisfied since for $f : x \longrightarrow y$ in $\mathscr{C}$ we have

$$\begin{aligned}
\mathrm{Id}_F(y) \circ F(f) &= 1_{F(y)} \circ F(f) && (2.1.27)\\
&= F(f) \circ 1_{F(x)} && (2.1.7)\\
&= F(f) \circ \mathrm{Id}_F(x). && (2.1.27)
\end{aligned}$$

$\diamond$

Definition 2.1.28. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F, G, H : \mathscr{C} \longrightarrow \mathscr{D}$ be functors. Suppose $\phi : F \Rightarrow G$ and $\psi : G \Rightarrow H$ are natural transformations.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G, near end}] \arrow[r, bend right=60, "H"'{name=H}] & \mathscr{D}
\arrow[Rightarrow, from=F, to=G, "\phi"'] \arrow[Rightarrow, from=G, to=H, "\psi"']
\end{tikzcd}$$

Then, the *vertical composition* denoted $\psi \circ \phi : F \Rightarrow H$ defined as follows:

- For an object $x \in \mathscr{C}$ define

$$(\psi \circ \phi)_x := \psi_x \circ \phi_x. \tag{2.1.29}$$

For a morphism $f : y \longrightarrow y$ in $\mathscr{C}$ we get

$$\begin{aligned}
(\psi \circ \phi)_y \circ F(f) &= \psi_y \circ \phi_y \circ F(f) && (2.1.29)\\
&= \psi_y \circ G(f) \circ \phi_x && (2.1.25)\\
&= H(f) \circ \psi_x \circ \phi_x && (2.1.25)\\
&= H(f) \circ (\psi \circ \phi)_x. && (2.1.29)
\end{aligned}$$

Thus, the above assignment satisfies the naturality condition (2.1.25). ◇

Definition 2.1.30. Let $F,H : \mathscr{A} \longrightarrow \mathscr{B}$ and $G,K : \mathscr{B} \longrightarrow \mathscr{D}$ be functors. Suppose $\phi : F \Rightarrow H$ and $\psi : G \Rightarrow K$ are natural transformations.

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathscr{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathscr{C}. \end{tikzcd}$$

Then, the *horizontal composition* denoted $\psi \star \phi : G \circ F \Rightarrow K \circ H$ is defined as follows:

- For an object $x \in \mathscr{A}$, the morphism $(\psi \star \phi)_x$ is defined as the any one of the following equal morphisms:

$$(\psi \star \phi)_x := \psi_{H(x)} \circ G(\phi_x) \tag{2.1.31}$$

$$= K(\phi_x) \circ \psi_{F(x)}. \qquad (2.1.25)$$

For a morphism $f : x \longrightarrow y$ in $\mathscr{C}$, we get

$$\begin{aligned}
(\psi \star \phi)_y \circ GF(f) &= \psi_{H(y)} \circ G(\phi_y) \circ GF(f) && (2.1.31)\\
&= \psi_{H(y)} \circ G(\phi_y \circ F(f)) && (2.1.16)\\
&= \psi_{H(x)} \circ G(H(f) \circ \phi_x) && (2.1.25)\\
&= \psi_{H(x)} \circ GH(f) \circ G(\phi_x) && (2.1.16)\\
&= KH(f) \circ \psi_{H(x)} \circ G(\phi_x) && (2.1.25)\\
&= KH(f) \circ (\psi \star \phi)_x. && (2.1.31)
\end{aligned}$$

Thus, the above assignment satisfies the naturality condition (2.1.25). ◇

Notation 2.1.32. Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ be categories. Let $H : \mathscr{A} \longrightarrow \mathscr{B}$, $F,G : \mathscr{B} \longrightarrow \mathscr{C}$, and $K : \mathscr{C} \longrightarrow \mathscr{D}$ be functors. For a natural transformation $\phi : F \Rightarrow G$ the composition $\mathrm{Id}_K \star \phi$ is denoted by $K \star \phi$. Similarly, the composition $\phi \star \mathrm{Id}_H$ is denoted by $\phi \star H$. ◇

2.2 Adjoint Functors and the Universal Property of Free Objects

In this section, we introduce the concepts of adjoint functors (Definition 2.2.1) and the universal property of a free object in a category (Definition 2.2.7). We also discuss the relationship between these notions, as established in Theorems 2.2.11 and 2.2.12.

Definition 2.2.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. An *adjunction* from C to D, denoted as a triple (L,R,ϕ) consists of:

- A functor $L : \mathscr{C} \longrightarrow \mathscr{D}$ called the *left adjoint*.
- A functor $R : \mathscr{D} \longrightarrow \mathscr{C}$ called the *right adjoint*.
- A family of bijections

$$\mathrm{Hom}_{\mathscr{D}}(L(x), y) \xrightarrow[\cong]{\phi_{x,y}} \mathrm{Hom}_{\mathscr{C}}(x, R(y))$$

  that is natural in the objects $x \in \mathscr{C}$ and $y \in \mathscr{D}$. ◇

Notation 2.2.2. An adjunction (L,R,ϕ) is also called an *adjoint pair* with L the *left adjoint* and R the *right adjoint*. This is denoted by $L \dashv R$. ◇

Remark 2.2.3. Suppose (L,R,ϕ) is an adjunction from $\mathscr{C}$ to $\mathscr{D}$. By setting $y = L(x)$ we can get a natural transformation $\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow R \circ L$ defined as

$$\eta_x := \phi_{x,L(x)}(1_{L(x)}).$$

This natural transformation is called the *unit* of the adjunction. Similarly, setting $x = R(y)$ yeilds a natural transformation $\epsilon : L \circ R \Rightarrow \mathrm{Id}_{\mathscr{D}}$ given by

$$\epsilon_y := \phi^{-1}_{R(y),y}(1_{R(y)}).$$

This natural transformation is called the *counit* of the adjunction. The unit and counit satisfy the following *triangle identities*. For objects $x \in \mathscr{C}$ and $y \in \mathscr{D}$ the composites

$$R(y) \xrightarrow{(\eta \star R)_y} RLR(y) \xrightarrow{(R \star \epsilon)_y} R(y)$$

and

$$L(x) \xrightarrow{(L \star \eta)_x} LRL(x) \xrightarrow{(\epsilon \star L)_x} L(y)$$

are equal to $1_{R(y)}$ and $1_{L(x)}$ respectively. That is

$$(R \star \epsilon) \circ (\eta \star R) = \mathrm{Id}_R \tag{2.2.4}$$

and

$$(\epsilon \star L) \circ (L \star \eta) = \mathrm{Id}_L. \tag{2.2.5}$$

In fact, the adjunction (L, R, ϕ) is completely determined by unit and counit that satisfy the triangle identities. $\diamond$

Definition 2.2.6. A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is called an *isomorphism* if there exists a functor $G : \mathscr{D} \longrightarrow \mathscr{C}$ such that

$$\mathrm{Id}_{\mathscr{C}} = G \circ F \qquad \text{and} \qquad F \circ G = \mathrm{Id}_{\mathscr{D}}.$$

A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is called an *equivalence* if there exists a functor $G : \mathscr{D} \longrightarrow \mathscr{C}$ and natural isomorphisms

$$\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow G \circ F \qquad \text{and} \qquad \epsilon : F \circ G \Rightarrow \mathrm{Id}_{\mathscr{D}}.$$

If an equivalence (F, G, η, ϵ) is an adjunction then the equivalence is known as *adjoint equivalence.* $\diamond$

Definition 2.2.7. Let $\mathscr{C}$ be a category, $U : \mathscr{C} \longrightarrow \mathcal{S}et$ be a functor and $S \in \mathcal{S}et$ be a set. Let $A \in \mathscr{C}$ be an object in $\mathscr{C}$ and $\phi : S \longrightarrow U(A)$ be a function. We say that the pair

$$\big(A,\ \phi : S \longrightarrow U(A)\big)$$

satisfies the *universal property of a free object* over U if the following conditions are satisfied:

Existence: For an object $X \in \mathscr{C}$ and a function $u : S \longrightarrow U(X)$ there exists a morphism $f_u : A \longrightarrow X$ in $\mathscr{C}$ such that

$$U(f_u) \circ \phi = u. \tag{2.2.8}$$

That is, the diagram

$$\begin{tikzcd} S \arrow[d, "\phi"'] \arrow[dr, "u"] & \\ U(A) \arrow[r, "U(f_u)"'] & U(X) \end{tikzcd}$$

commutes.

Uniqueness: For an object $X \in \mathscr{C}$ and parallel morphisms $f, g : A \longrightarrow X$, the following implication holds:

$$U(f) \circ \phi = U(g) \circ \phi \qquad \text{implies} \qquad f = g. \tag{2.2.9}$$

◇

Convention 2.2.10. Fix a set S. If a pair (A, ϕ) satisfies the universal property of a free object over U, then the pair (A, ϕ) is unique up to an isomorphism. That is, if $(A, \phi : S \longrightarrow U(A))$ and $(B, \psi : S \longrightarrow U(B))$ both satisfy universal property of a free object over U then, there exists a unique isomorphism $\gamma : A \longrightarrow B$ such that

$$U(\gamma) \circ \phi = \psi.$$

With this understanding we say A is the free object generated by set S. $\diamond$

Theorem 2.2.11. *Let $\mathscr{C}$ be a category and $U : \mathscr{C} \longrightarrow \mathcal{S}et$ be a functor. Suppose U has a left adjoint $F : \mathcal{S}et \longrightarrow \mathscr{C}$ with unit*

$$\Phi : \mathrm{Id}_{\mathcal{S}et} \Rightarrow U \circ F$$

and counit

$$\Psi : F \circ U \longrightarrow \mathrm{Id}_{\mathscr{C}}.$$

Then, for a set S, it is true that $(F(S),\ \Phi_S)$ satisfies the universal property of a free object over U.

Proof. We will show the existence and uniqueness conditions of the universal property.

Existence: Let $X \in \mathrm{Obj}(\mathscr{C})$ and $u : S \longrightarrow U(X)$ be a set map. Consider the morphism $f_u : F(S) \longrightarrow X$ given by

$$f_u := \Psi_X \circ F(u). \tag{i}$$

We get

$$\begin{aligned} U(f_u) \circ \Phi_S &= U(\Psi_X) \circ UF(u) \circ \Phi_S && (i) \\ &= U(\Psi_X) \circ \Phi_{U(X)} \circ u && (2.1.25) \\ &= (U \star \Psi)_X \circ (\Phi \star U)_X \circ u && (2.1.31) \\ &= u && (2.2.4) \end{aligned}$$

as required.

Uniqueness: Let $X \in \mathrm{Obj}(\mathscr{C})$ and let $f, g : F(S) \longrightarrow X$. Suppose we have

$$U(f) \circ \Phi_S = U(g) \circ \Phi_S. \tag{ii}$$

Consider the following diagrams:

$$\begin{tikzcd}
F(S) \arrow[d, "F(\Phi_S)"'] \arrow[dr, "F(U(f)\circ\Phi_S)"] & \\
FUF(S) \arrow[r, "FU(f)"] \arrow[d, "\Psi_{F(S)}"'] & FU(X) \arrow[d, "\Psi_X"] \\
F(S) \arrow[r, "f"'] & X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F(S) \arrow[d, "F(\Phi_S)"'] \arrow[dr, "F(U(g)\circ\Phi_S)"] & \\
FUF(S) \arrow[r, "FU(g)"] \arrow[d, "\Psi_{F(S)}"'] & FU(X) \arrow[d, "\Psi_X"] \\
F(S) \arrow[r, "g"'] & X.
\end{tikzcd}$$

The squares in above diagrams commute since they are naturality squares for Ψ. The triangles are composition triangles. Since F, U are adjoint pairs with Φ and Ψ as unit and co-unit respectively, we get that the left vertical compositions in both diagrams are identities. Thus, we get

$$\begin{aligned}
f &= \Psi_X \circ F(U(f)\circ\Phi_S) \\
&= \Psi_X \circ F(U(g)\circ\Phi_S) && (ii) \\
&= g
\end{aligned}$$

as required. □

Theorem 2.2.12. *Let $U : \mathscr{C} \longrightarrow \mathscr{Set}$ be a functor. Suppose for any set S we have an object $F_S \in \mathscr{C}$ and a set map $\phi_S : S \longrightarrow U(F_S)$ such that the pair (F_S, ϕ_S) satisfies the universal property of a free object. Then, the functor U has a left adjoint.*

Proof. We will divide the proof into the following parts: At Definition 2.2.13 we will define a functor $F : \mathscr{Set} \longrightarrow \mathscr{C}$ that will serve as a left adjoint. At Definitions 2.2.17 and 2.2.19 we will define the unit and co-unit natural transformations respectively. Finally, at Proposition 2.2.22 we will show that (F, U, Φ, Ψ) form an adjoint.

Definition 2.2.13. Consider the same hypothesis as Theorem 2.2.12. Define a functor $F : \mathcal{S}et \longrightarrow \mathscr{C}$ as follows:

- Suppose $S \in \mathcal{S}et$. From the hypothesis, we get $F_S \in \mathscr{C}$ and a set map $\phi_S : S \longrightarrow U(F_S)$ such that the pair $(F_S,\ \phi_S)$ satisfies the universal property of a free object. Define

$$F(S) := F_S. \tag{2.2.14}$$

- Suppose $u : S \longrightarrow T$ is a set map. We get a pair

$$(F_S,\ \phi_S : S \longrightarrow U(F_S))$$

  that satisfies the universal property of a free object as above. Similarly, we get

$$(F_T,\ \phi_T : T \longrightarrow U(F_T)).$$

  Using the universal property of $(F_S,\ \phi_S)$, we get a unique morphism $f_u : F_S \longrightarrow F_T$ such that

$$U(f_u) \circ \phi_S = \phi_T \circ u. \tag{2.2.15}$$

  Define

$$F(u) := f_u. \tag{2.2.16}$$

We will show that the above assignment satisfies the identity condition. We have

$$\begin{aligned} U(f_{1_S}) \circ \phi_S &= \phi_S \circ 1_S && (2.2.15) \\ &= U(1_{F_S}) \circ \phi_S. \end{aligned}$$

Thus, from the uniqueness of the universal property of (F_S, ϕ_S), we get

$$f_{1_S} = 1_{F_S}.$$

Therefore, we get

$$F(1_S) = 1_{F(S)}$$

as required.

Next, we will show the composition condition. Let $u : S \longrightarrow T$ and $v : T \longrightarrow R$ be functions. We have

$$\begin{aligned} U(f_{v \circ u}) \circ \phi_S &= \phi_R \circ v \circ u && (2.2.15) \\ &= U(f_v) \circ \phi_T \circ u && (2.2.15) \\ &= U(f_v) \circ U(f_u) \circ \phi_S && (2.2.15) \\ &= U(f_v \circ f_u) \circ \phi_S. \end{aligned}$$

Again, from the uniqueness of the universal property of (F_S, ϕ_S), we get

$$f_{v \circ u} = f_v \circ f_u.$$

Therefore, we get

$$F(v \circ u) = F(v) \circ F(u)$$

as required. ⋄

Definition 2.2.17. Consider the same hypothesis as Theorem 2.2.12 and let $F : \mathcal{S}et \longrightarrow \mathscr{C}$ be the functor as in Definition 2.2.13. Define a natural transformation

$$\Phi : \mathrm{Id}_{\mathcal{S}et} \Rightarrow UF$$

as follows:

- For a set $S \in \mathcal{S}et$ define

$$\Phi_S := \phi_S. \tag{2.2.18}$$

The naturality condition follows from the defining condition (2.2.15) for the functor F. $\diamond$

Definition 2.2.19. Consider the same hypothesis as Theorem 2.2.12 and let $F : \mathcal{S}et \longrightarrow \mathscr{C}$ be the functor as in Definition 2.2.13. Define a natural transformation

$$\Psi : FU \Rightarrow \mathrm{Id}_{\mathscr{C}}$$

as follows:

- Suppose $C \in \mathscr{C}$. From the hypothesis we get $FU(C) \in \mathscr{C}$ and a function $\phi_{U(C)} : U(C) \longrightarrow FU(C)$ such that $(FU(C), \phi_{U(C)})$ satisfies the universal property of a free object. Thus, we get a unique morphism $\psi_C : FU(C) \longrightarrow C$ such that

$$U(\psi_C) \circ \phi_{U(C)} = 1_{U(C)}. \tag{2.2.20}$$

  Define

$$\Psi_C := \psi_C. \tag{2.2.21}$$

Let $f : C \longrightarrow D$ be a morphism in $\mathscr{C}$. We will show that the naturality square

$$\begin{tikzcd}
FU(C) \arrow[r, "FU(f)"] \arrow[d, "\psi_C"'] & FU(D) \arrow[d, "\psi_D"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

commutes. Observe that,

$$\begin{aligned}
U(\psi_D \circ FU(f)) \circ \phi_{U(C)} &= U(\psi_D) \circ UFU(f) \circ \phi_{U(C)} && (2.1.16)\\
&= U(\psi_D) \circ \phi_{U(D)} \circ U(f) && (2.2.15)\\
&= U(f) && (2.2.20)\\
&= U(f) \circ U(\psi_C) \circ \phi_{U(C)} && (2.2.20)\\
&= U(f \circ \psi_C) \circ \phi_{U(C)}. && (2.1.16)
\end{aligned}$$

From the uniqueness of the universal property of $(F_{U(C)}, \phi_{U(C)})$, we get

$$\psi_D \circ FU(f) = f \circ \psi_C$$

as required. $\diamond$

Proposition 2.2.22. *The quadruple* (F, G, Φ, Ψ) *form an adjoint.*

Proof. We will show that the pair (F, U) is an adjoint pair where $F : \mathcal{S}et \longrightarrow \mathscr{C}$ is as in Definition 2.2.13 by exibiting that Φ and Ψ as in Definitions 2.2.17 and 2.2.19 satisfy the triangle identities (2.2.4) and (2.2.5).

The equality

$$U(\Psi_C) \circ \Phi_{U(C)} = 1_{U(C)}.$$

follows from the defining condition (2.2.20) of the counit Ψ.

It remains to show that

$$\Psi_{F(S)} \circ F(\Phi_S) = 1_{F(S)}$$

which follows from the following calculation:

$$\begin{aligned}
U(\Psi_{F(S)} \circ F(\Phi_S)) \circ \phi_S &= U(\Psi_{F(S)}) \circ U(F(\Phi_S)) \circ \phi_S && (2.1.16) \\
&= U(\Psi_{F(S)}) \circ \Phi_{UF(S)} \circ \phi_S && (2.2.15) \\
&= 1_{UF(S)} \circ \phi_S && (2.2.20) \\
&= U(1_{F(S)}) \circ \phi_S. && (2.1.15)
\end{aligned}$$

From the uniqueness of universal property of (F_S, ϕ_S), we get

$$\Psi_{F(S)} \circ F(\Phi_S) = 1_{F(S)}$$

as required. $\square$

This completes the proof of Theorem 2.2.12. $\square$

Chapter 3: Topics in Monoidal Categories

In this chapter, we introduce monoidal categories and discuss coherence for monoidal categories. We present several approaches to coherence. This chapter serves as motivation for the concepts and ideas developed in the following chapters.

3.1 Definition of a Monoidal Category

In this section, we will define monoidal categories, monoidal functors, and monoidal natural transformations.

Definition 3.1.1. A *monoidal category*

$$(\mathscr{M}, I, \otimes, \lambda, \rho, \alpha)$$

has the following data:

- A category $\mathscr{M}$ called the *underlying category*.
- A distinguished object $I \in \mathrm{Obj}(\mathscr{M})$ called the *unit object*.
- A bifunctor

  $$\otimes : \mathscr{M} \times \mathscr{M} \longrightarrow \mathscr{M}$$

  called the *tensor product* (also known as the *monoidal product*).

- A natural isomorphism

$$\lambda := \big(\lambda_x : I \otimes x \longrightarrow x\big)_{x \in \mathscr{M}}$$

called the *left unitor*.

- A natural isomorphism

$$\rho := \big(\rho_x : x \otimes I \longrightarrow x\big)_{x \in \mathscr{M}}$$

called the *right unitor*.

- A natural isomorphism

$$\alpha := \big(\alpha_{x,y,z} : x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z\big)_{(x,y,z) \in \mathscr{M}^3}$$

called the *associator*.

These data satisfy the following coherence axioms:

Pentagon axiom: For objects $x, y, z, w \in \mathscr{M}$, the diagram

$$\begin{tikzcd}
x \otimes (y \otimes (z \otimes w)) \arrow[r, "\alpha_{x,y,z\otimes w}"] \arrow[d, "1_x \otimes \alpha_{y,z,w}"'] & (x \otimes y) \otimes (z \otimes w) \arrow[r, "\alpha_{x\otimes y,z,w}"] & ((x \otimes y) \otimes z) \otimes w \\
x \otimes ((y \otimes z) \otimes w) \arrow[rr, "\alpha_{x,y\otimes z,w}"'] & & (x \otimes (y \otimes z)) \otimes w \arrow[u, "\alpha_{x,y,z} \otimes 1_w"']
\end{tikzcd} \tag{3.1.2}$$

commutes.

Unit axiom: For objects $x, y \in \mathscr{M}$, the diagram

$$\begin{tikzcd}
x \otimes (I \otimes y) \arrow[rr, "\alpha_{x,I,y}"] \arrow[dr, "1_x \otimes \lambda_y"'] & & (x \otimes I) \otimes y \arrow[dl, "\rho_x \otimes 1_y"] \\
& x \otimes y &
\end{tikzcd} \tag{3.1.3}$$

commutes. ◇

Notation 3.1.4. Let

$$\left(\mathcal{M}, I, \otimes, \lambda, \rho, \alpha\right)$$

be a monoidal category. For objects $x, y, z \in \mathcal{M}$, the morphisms

$$\alpha_{x,y,z} : x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z, \qquad \lambda_x : I \otimes x \longrightarrow x,$$

$$\rho_x : x \otimes I \longrightarrow x, \qquad 1_x : x \longrightarrow x$$

and their inverses are called the *structure morphisms* of the monoidal category $\mathcal{M}$. ◇

Remark 3.1.5. As a consequence of the coherence axioms, the equality

$$\lambda_I = \rho_I : I \otimes I \longrightarrow I \tag{3.1.6}$$

holds. This result is proven as Theorem 5 in [Kel64]. ◇

Definition 3.1.7. A monoidal category

$$(\mathcal{M}, I, \otimes, \alpha, \lambda, \rho)$$

is called a *strict monoidal category* if the natural isomorphisms α, λ and ρ are all equal to identity natural transformations. ◇

Convention 3.1.8. Whenever convenient, we will omit the explicit notation for the monoidal category $(\mathcal{M}, I, \otimes, \lambda, \rho, \alpha)$ and simply write $\mathcal{M}$ or $(\mathcal{M}, I, \otimes)$. ◇

Definition 3.1.9. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ and $\mathcal{N} = (\mathcal{N}, J, \bullet)$ be two monoidal categories. A *monoidal functor*

$$(F, F_0, F_2) : \mathcal{M} \longrightarrow \mathcal{N}$$

has the following data:

- A functor
$$F : \mathcal{M} \longrightarrow \mathcal{N}$$
between the underlying categories called the *underlying functor*.

- A morphism
$$F_0 : J \longrightarrow F(I)$$
called the *unit structure morphism* of the functor F.

- A natural isomorphism
$$F_2 := \big(F_2(x,y) : F(x) \bullet F(y) \longrightarrow F(x \otimes y)\big)_{(x,y) \in \mathcal{M}^2}$$
called the *tensor structure morphism* of the functor F.

These data satisfy the following conditions:

Associator condition: For objects $x, y, z \in \mathcal{M}$ the diagram

$$\begin{tikzcd}
F(x) \bullet (F(y) \bullet F(z)) \arrow[r, "\alpha^N_{F(x),F(y),F(z)}"] \arrow[d, "1_{F(x)} \bullet F_2(y,z)"'] & (F(x) \bullet F(y)) \bullet F(z) \arrow[d, "F_2(x,y) \bullet 1_{F(z)}"] \\
F(x) \bullet F(y \otimes z) \arrow[d, "F_2(x, y \otimes z)"'] & F(x \otimes y) \bullet F(z) \arrow[d, "F_2(x \otimes y, z)"] \\
F(x \otimes (y \otimes z)) \arrow[r, "F(\alpha^M_{x,y,z})"'] & F((x \otimes y) \otimes z).
\end{tikzcd} \tag{3.1.10}$$

commutes.

Left unitor condition: For an object $x \in \mathcal{M}$ the diagram

$$\begin{tikzcd}
J \bullet F(x) \arrow[r, "F_0 \bullet 1_{F(x)}"] \arrow[d, "\lambda^N_{F(x)}"'] & F(I) \bullet F(x) \arrow[d, "F_2(I,x)"] \\
F(x) & F(I \otimes x). \arrow[l, "F(\lambda^M_x)"]
\end{tikzcd} \tag{3.1.11}$$

commutes.

Right unitor condition: For an object $x \in \mathscr{M}$ the diagram

$$\begin{tikzcd} F(x) \bullet J \arrow[r, "1_{F(x)} \bullet F_0"] \arrow[d, "\rho^N_{F(x)}"'] & F(x) \bullet F(I) \arrow[d, "F_2(x,I)"] \\ F(x) & F(x \otimes I). \arrow[l, "F(\rho^M_x)"] \end{tikzcd} \tag{3.1.12}$$

commutes. ◇

Definition 3.1.13. Let $\mathscr{M} = (\mathscr{M}, I, \otimes)$ and $\mathscr{N} = (\mathscr{N}, J, \bullet)$ be two monoidal categories. A monoidal functor

$$(F, F_0, F_2) : \mathscr{M} \longrightarrow \mathscr{N}$$

is called a *strict monoidal functor* if it satisfies the following conditions:

Unit condition: The equality

$$F_0 = 1_J : J \longrightarrow F(I) \tag{3.1.14}$$

holds.

Tensor condition: For objects $x, y \in \mathscr{M}$ the equality

$$F_2(x, y) = 1_{F(x) \bullet F(y)} : F(x) \bullet F(y) \longrightarrow F(x \otimes y) \tag{3.1.15}$$

holds. ◇

Remark 3.1.16. Note that, a monoidal functor $F : \mathscr{M} \longrightarrow \mathscr{N}$ is a strict monoidal functor if and only if the equalities

$$F(-) \otimes F(-) = F(- \otimes -) \qquad \text{and} \tag{3.1.17}$$

$$F(\mathrm{Const}_I) = \mathrm{Const}_J \tag{3.1.18}$$

of functors hold. ◇

Definition 3.1.19. Let $\mathscr{M} = (\mathscr{M}, I, \otimes)$ be a monoidal category. Define the identity monoidal functor $\mathrm{Id}_{\mathscr{M}} : \mathscr{M} \longrightarrow \mathscr{M}$ as follows:

- The underlying functor $\mathrm{Id}_{\mathscr{M}} : \mathscr{M} \longrightarrow \mathscr{M}$ is equal to the identity functor as in Definition 2.1.17.

- The unit structure morphism, $(\mathrm{Id}_{\mathscr{M}})_0 : I \longrightarrow \mathrm{Id}_{\mathscr{M}}(I)$, is given by

$$(\mathrm{Id}_{\mathscr{M}})_0 := 1_I. \tag{3.1.20}$$

  This is well-defined since $\mathrm{Id}_{\mathscr{M}}(I) = I$.

- For objects $x, y \in \mathscr{M}$, the tensor structure morphism, $(\mathrm{Id}_{\mathscr{M}})_2(x,y) : \mathrm{Id}_{\mathscr{M}}(x) \otimes \mathrm{Id}_{\mathscr{M}}(y) \longrightarrow \mathrm{Id}_{\mathscr{M}}(x \otimes y)$ is given by

$$(\mathrm{Id}_{\mathscr{M}})_2(x,y) := 1_{x \otimes y}. \tag{3.1.21}$$

  This is well-defined since $\mathrm{Id}_{\mathscr{M}}(w) = w$ for $w \in \mathrm{Obj}(\mathscr{M})$.

Since the structure morphisms are all equal to identities, the associator condition and the unitor conditions are satisfied. Note that the identity monoidal functor is a strict monoidal functor. $\diamond$

Definition 3.1.22. Let $\mathscr{M} = (\mathscr{M}, I, \otimes)$, $\mathscr{N} = (\mathscr{N}, J, \bullet)$, and $\mathscr{P} = (\mathscr{P}, K, \cdot)$ be monoidal categories. Let

$$F : \mathscr{M} \longrightarrow \mathscr{N} \qquad \text{and} \qquad G : \mathscr{N} \longrightarrow \mathscr{P}$$

be monoidal functors. Define the composition of monoidal functors

$$G \circ F : \mathscr{M} \longrightarrow \mathscr{P}$$

as follows: For this definition, we will denote $G \circ F$ by GF.

- The underlying functor $GF : \mathscr{M} \longrightarrow \mathscr{P}$ is equal to the composition of functors as in Definition 2.1.20.

- The unit structure morphism, $(GF)_0 : K \longrightarrow GF(I)$, is given by

$$(GF)_0 := G(F_0) \circ G_0. \tag{3.1.23}$$

- For objects $x, y \in \mathscr{M}$, the tensor structure morphism,

$$(GF)_2(x,y) : GF(x) \cdot GF(y) \longrightarrow GF(x \otimes y)$$

is given by

$$(GF)_2(x,y) := G(F_2(x,y)) \circ G_2(F(x), F(y)). \tag{3.1.24}$$

Now, we will show that the associator condition is satisfied. Let $x, y, z \in \mathrm{Obj}(\mathscr{M})$. Consider the following diagram

$$\begin{tikzcd}
GF(x) \cdot (GF(y) \cdot GF(z)) \arrow[r, "\alpha^P_{GF(x),GF(y),GF(z)}"] \arrow[d, "1_{GF(x)} \cdot G_2(Fy,Fz)"'] & (GF(x) \cdot GF(y)) \cdot GF(z) \arrow[d, "G_2(Fx,Fy) \cdot 1_{GF(z)}"] \\
GF(x) \cdot G(Fy \bullet Fz) \arrow[d, "G_2(Fx, Fy \bullet Fz)"'] & G(Fx \bullet Fy) \cdot GF(z) \arrow[d, "G_2(Fx \bullet Fy, Fz)"] \\
G(Fx \bullet (Fy \bullet Fz)) \arrow[r, "G(\alpha^N_{Fx,Fy,Fz})"] \arrow[d, "G(1_{Fx} \bullet F_2(y,z))"'] & G((Fx \bullet Fy) \bullet Fz) \arrow[d, "G(F_2(x,y) \bullet 1_{Fz})"] \\
G(Fx \bullet F(y \otimes z)) \arrow[d, "G(F_2(x, y \otimes z))"'] & G(F(x \otimes y) \bullet Fz) \arrow[d, "G(F_2(x \otimes y, z))"] \\
GF(x \otimes (y \otimes z)) \arrow[r, "GF(\alpha^M_{x,y,z})"'] & GF((x \otimes y) \otimes z).
\end{tikzcd}$$

The top rectangle commutes due to the associator condition of G and the bottom rectangle commutes due to the associator condition of F.

From naturality of G_2, we get that the equality

$$G(1_{Fx} \bullet F_2(y,z)) \circ G_2(Fx, Fy \bullet Fz) = G_2(Fx, F(y \otimes z)) \circ 1_{GF(x)} \cdot G(F_2(y,z))$$

holds. Therefore, the left vertical composite is equal to

$$\begin{aligned} &G(F_2(x, y \otimes z)) \circ G_2(Fx, F(y \otimes z)) \circ 1_{GF(x)} \cdot G(F_2(y,z)) \\ &\qquad \circ 1_{GF(x)} \cdot G_2(Fy, Fz) \\ &\qquad = (GF)_2(x, y \otimes z) \circ 1_{GF(x)} \cdot (GF)_2(y,z). \qquad (3.1.24) \end{aligned}$$

From naturality of G_2, the equality

$$G(F_2(x,y) \bullet 1_{Fz}) \circ G_2(Fx \bullet Fy, Fz) = G_2(F(x \otimes y), Fz) \circ G(F_2(x,y)) \cdot 1_{GF(z)}$$

holds. Thus, the right vertical composition is equal to

$$\begin{aligned} &G(F_2(x \otimes y, z) \circ G_2(F(x \otimes y, Fz))) \circ G(F_2(x,y)) \cdot 1_{GF(z)} \\ &\qquad \circ G_2(Fx, Fy) \cdot 1_{GF(z)} \\ &\qquad = (GF)_2(x \otimes y, z) \circ (GF)_2(x,y) \cdot 1_{GF(z)}. \qquad (3.1.24) \end{aligned}$$

This shows that the associator condition (3.1.10) is satisfied.

Next, we will show that the left unitor condition (3.1.11) is satisfied. Let $x \in \mathscr{M}$ be an object. The diagram

$$\begin{tikzcd}
K \cdot GF(x) \arrow[r, "G(J) \cdot 1_{GF(x)}"] \arrow[d, "\lambda^{\mathscr{P}}_{GF(x)}"'] & G(J) \cdot F(x) \arrow[d, "G_2(J, Fx)"] \\
GF(x) & G(J \bullet Fx) \arrow[l, "G(\lambda^{\mathscr{N}}_{Fx})"] \arrow[d, "G(F_0 \bullet 1_{Fx})"] \\
GF(I \otimes x) \arrow[u, "GF(\lambda^{\mathscr{M}}_x)"] & G(FI \bullet Fx) \arrow[l, "G(F_2(I,x))"]
\end{tikzcd}$$

commutes since the top square is the left unitor condition of G whereas the bottom square commutes is the left unitor condition of F. From the naturality of G_2, the right vertical composite is equal to

$$G(F_0 \bullet 1_{Fx}) \circ G_2(J, Fx) = G_2(F(I), F(x)) \circ G(F_0) \cdot 1_{GF(x)}.$$

Using above, the top-right-bottom composite is given by

$$\begin{aligned}\big(G(F_2(I,x)) \circ G_2(F(I), Fx)\big) \circ \big(G(F_0) \cdot 1_{GF(x)} \circ G_0 \cdot 1_{GF(x)}\big) \\ = (GF)_2(I,x) \circ (GF)_0 \cdot 1_{GF(x)}. \qquad \text{(3.1.23) and (3.1.24)}\end{aligned}$$

Since the above diagram commutes, the left unitor condition is satisfied.

Finally, we will show the right unitor condition (3.1.12). Let $x \in \mathscr{M}$ be an object. The diagram

$$\begin{tikzcd}
GF(x) \cdot K \arrow[rr, "1_{GF(x)} \cdot G(J)"] \arrow[d, "\rho^{\mathscr{P}}_{GF(x)}"'] & & GF(x) \cdot G(J) \arrow[d, "G_2(Fx, J)"] \\
GF(x) & & G(Fx \bullet J) \arrow[ll, "G(\rho^{\mathscr{N}}_{Fx})"] \arrow[d, "G(1_{Fx} \bullet F_0)"] \\
GF(x \otimes I) \arrow[u, "GF(\rho^{\mathscr{M}}_x)"] & & G(Fx \bullet FI) \arrow[ll, "G(F_2(x,I))"]
\end{tikzcd}$$

commutes since the top square is the right unitor condition of G whereas the bottom square is the right unitor condition of F. From naturality of G_2, the right vertical composite is equal to

$$G(1_{Fx} \bullet F_0) \circ G_2(Fx, J) = G_2(F(x), F(I)) \circ 1_{GF(x)} \cdot G(F_0).$$

Using above, the top-right-bottom composite is given by

$$\begin{aligned}\big(G(F_2(x,I)) \circ G_2(Fx, F(I))\big) \circ \big(1_{GF(x)} \cdot G(F_0) \circ 1_{GF(x)} \cdot G_0\big) \\ = (GF)_2(x,I) \circ 1_{GF(x)} \cdot (GF)_0. \qquad \text{(3.1.23) and (3.1.24)}\end{aligned}$$

Since the above diagram commutes, the right unitor condition is satisfied. $\diamond$

Definition 3.1.25. Let $\mathscr{M} = (\mathscr{M}, I, \otimes)$ and $\mathscr{N} = (\mathscr{N}, J, \bullet)$ be two monoidal categories. Let $F, G : \mathscr{M} \longrightarrow \mathscr{N}$ be two monoidal functors. A *monoidal natural transformation*

$$\phi : F \Rightarrow G$$

has the following data:

- A natural transformation

$$\phi : F \Rightarrow G$$

  called the *underlying natural transformation.*

Above natural transformation satisfies the following conditions.

Unit condition: The diagram

$$\begin{tikzcd} & F(I) \arrow[d, "\phi_I"] \\ J \arrow[ur, "F_0"] \arrow[r, "G_0"'] & G(I) \end{tikzcd} \tag{3.1.26}$$

commutes.

Tensor condition: For $x, y \in \mathrm{Obj}(\mathscr{M})$ the diagram

$$\begin{tikzcd} F(x) \bullet F(y) \arrow[r, "F_2(x,y)"] \arrow[d, "\phi_x \bullet \phi_y"'] & F(x \otimes y) \arrow[d, "\phi_{x \otimes y}"] \\ G(x) \bullet G(y) \arrow[r, "G_2(x,y)"'] & G(x \otimes y). \end{tikzcd} \tag{3.1.27}$$

commutes. $\diamond$

Definition 3.1.28. Let $\mathscr{M}$ and $\mathscr{N}$ be two monoidal categories. A *monoidal equivalence*

$$\big(G,\ F, \phi,\ \psi\big)$$

between $\mathscr{M}$ and $\mathscr{N}$ has the following data:

- Monoidal functors

$$F : \mathscr{N} \longrightarrow \mathscr{M} \qquad \text{and} \qquad G : \mathscr{M} \longrightarrow \mathscr{N}$$

  that witness the monoidal equivalence.

- Monoidal natural transformation

$$\phi : G \circ F \Rightarrow \mathrm{Id}_{\mathscr{N}} \qquad \text{and} \qquad \psi : F \circ G \Rightarrow \mathrm{Id}_{\mathscr{M}}$$

  that witness the monoidal equivalence. $\diamond$

Definition 3.1.29. Let $\mathscr{M}$ and $\mathscr{N}$ be two monoidal categories. A monoidal equivalence

$$(G : \mathscr{M} \longrightarrow \mathscr{N},\ F : \mathscr{N} \longrightarrow \mathscr{M},\ \phi : G \circ F \Rightarrow \mathrm{Id}_{\mathscr{N}},\ \psi : F \circ G \Rightarrow \mathrm{Id}_{\mathscr{M}})$$

is called a *monoidal adjoint equivalence* if it satisfies the following conditions:

Triangle condition (right): For $x \in \mathscr{M}$ the equality of morphisms

$$\phi_{G(x)} \circ G(\psi_x^{-1}) = 1_{G(x)} \tag{3.1.30}$$

holds. That is, the composition

$$G(x) \xrightarrow{G(\psi_x^{-1})} GFG(x) \xrightarrow{\phi_{G(x)}} G(x)$$

is equal to the identity morphism.

Triangle condition (left): For $a \in \mathrm{Obj}(\mathscr{N})$ the equality of morphisms

$$F(\phi_a) \circ \psi_{F(a)}^{-1} = 1_{F(a)} \tag{3.1.31}$$

holds. That is, the composition

$$F(a) \xrightarrow{\psi_{F(a)}^{-1}} FGF(a) \xrightarrow{F(\phi_a)} F(a)$$

is equal to the identity morphisms. $\diamond$

Definition 3.1.32. Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ and $\mathcal{N} = (\mathcal{N}, J, \bullet)$ be two monoidal categories. A monoidal equivalence

$$(G : \mathcal{M} \longrightarrow \mathcal{N},\ F : \mathcal{N} \longrightarrow \mathcal{M},\ \phi : G \circ F \Rightarrow \mathrm{Id}_{\mathcal{N}},\ \psi : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{M}})$$

is called a *special monoidal equivalence* from $\mathcal{M}$ to $\mathcal{N}$ if it satisfies the following conditions:

<u>Strictness of G</u>: The monoidal functor G is a strict monoidal functor.

<u>Unit condition for F</u>: The equality of morphisms

$$F_0 = 1_J : J \longrightarrow F(I) \tag{3.1.33}$$

holds.

<u>Counit condition</u>: The equality of monoidal natural transformations

$$\psi = \mathrm{Id}_{\mathrm{Id}_{\mathcal{M}}} : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{M}} \tag{3.1.34}$$

holds. $\diamond$

Motivation 3.1.35. We will see in Theorem 3.2.13 that every monoidal functor is equivalent to a strict monoidal functor via a special monoidal equivalence. $\diamond$

Proposition 3.1.36. *Let $\mathcal{M} = (\mathcal{M}, I, \otimes)$ be a strict monoidal category. For $f : x \longrightarrow y$ in $\mathcal{M}$ we have*

$$1_I \otimes f = f \tag{3.1.37}$$

$$f \otimes 1_I = f. \tag{3.1.38}$$

Proof. Consider the following diagrams:

$$\begin{tikzcd} I \otimes x \arrow[r, "\lambda_x"] \arrow[d, "1_I \otimes f"'] & x \arrow[d, "f"] \\ I \otimes y \arrow[r, "\lambda_y"'] & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x \otimes I \arrow[r, "\rho_x"] \arrow[d, "f \otimes 1_I"'] & x \arrow[d, "f"] \\ y \otimes I \arrow[r, "\rho_y"'] & y. \end{tikzcd}$$

The diagrams commute since λ and ρ are natural transformations. Since $\mathscr{M}$ is a strict monoidal category we get that λ and ρ are identity natural transformations. This gives the required equations. $\square$

Proposition 3.1.39. *Let $\mathscr{M} = (\mathscr{M}, I, \otimes)$ be a monoidal category and $\mathscr{N} = (\mathscr{N}, J, \bullet)$ be a strict monoidal category. Let $F : \mathscr{M} \longrightarrow \mathscr{N}$ be a strict monoidal functor. Then $x, y, z \in \mathrm{Obj}(\mathscr{M})$ we have following:*

$$F(\alpha^{\mathscr{M}}_{x,y,z}) = 1_{F(x)\bullet F(y)\bullet F(z)} \tag{3.1.40}$$

$$F(\lambda^{\mathscr{M}}_{x}) = 1_{F(x)} \tag{3.1.41}$$

$$F(\rho^{\mathscr{M}}_{x}) = 1_{F(x)}. \tag{3.1.42}$$

Proof. Since F is a monoidal functor, it satisfies the associator condition (3.1.10), the left unitor condition (3.1.11), and the right unitor condition (3.1.12). These conditions with the fact that F is a strict monoidal functor and $\mathscr{N}$ is strict monoidal category gives the required equations. $\square$

3.2 Coherence for Monoidal Categories

The coherence theorem for monoidal categories comes in various forms. These include

- Every diagram in the free monoidal category generated by a set commutes.

- The free monoidal category over a set is equivalent to the free strict monoidal category over the same set.

- Every monoidal category is monoidally equivalent to a strict monoidal category.

These statements ultimately lead to the slogan for the coherence theorem: "Every formal diagram in a monoidal category commutes."

In this section, we will define universal property of a free monoidal category generated by a set. We will give the most common construction of the free monoidal category. We will define formal diagrams in a monoidal category and state the coherence theorem for monoidal categories.

Definition 3.2.1. Let S be a set, $\mathscr{M}$ be a monoidal category, and $i : S \longrightarrow \mathrm{Obj}(\mathscr{M})$ be a function. We say that the pair

$$(\mathscr{M}, i : S \longrightarrow \mathrm{Obj}(\mathscr{M}))$$

satisfies the *universal property of a free monoidal category* if the following conditions are satisfied:

Existence (strict functor): For a monoidal category $\mathscr{N}$, and a set map $f : S \longrightarrow \mathrm{Obj}(\mathscr{N})$ there exists a strict monoidal functor

$$F : \mathscr{M} \longrightarrow \mathscr{N}$$

such that the equality

$$\mathrm{Obj}(F) \circ i = f \tag{3.2.2}$$

holds.

Existence (monoidal natural transformation): Given a monoidal category $\mathscr{N}$, a pair of monoidal functors

$$F, G : \mathscr{M} \longrightarrow \mathscr{N},$$

and a collection of isomorphisms

$$\widehat{\phi} = \left(\widehat{\phi}_a : F(i(a)) \longrightarrow G(i(a))\right)_{a \in S}$$

there exists a monoidal natural isomorphism

$$\phi : F \Rightarrow G$$

such that for an element $a \in S$ the equality

$$\phi_{i(a)} = \widehat{\phi}_a : F(i(a)) \longrightarrow G(i(a)) \tag{3.2.3}$$

holds.

Uniqueness: Given a monoidal category $\mathscr{N}$, a pair of monoidal functors

$$F, G : \mathscr{M} \longrightarrow \mathscr{N},$$

and a pair of monoidal natural isomorphisms

$$\phi, \psi : F \Rightarrow G$$

the following implication holds: If for every element $a \in S$ the equality

$$\phi_{i(a)} = \psi_{i(a)} : F(i(a)) \longrightarrow G(i(a))$$

of morphisms is satisfied then we get the equality

$$\phi = \psi : F \Rightarrow G \tag{3.2.4}$$

of natural isomorphisms. $\diamond$

Remark 3.2.5. Let S be a set, and suppose $(\mathscr{M}, i)$ and $(\mathscr{N}, j)$ are two pairs that each satisfy the universal property of the free monoidal category generated by S. Then, by the universal property, there exist strict monoidal functors

$$F : \mathscr{M} \longrightarrow \mathscr{N} \qquad \text{and} \qquad G : \mathscr{N} \longrightarrow \mathscr{M}$$

such that $F \circ G = \mathrm{Id}_{\mathscr{N}}$ and $G \circ F = \mathrm{Id}_{\mathscr{M}}$. Therefore, it is justified to refer to *the* free monoidal category on S. ◇

Remark 3.2.6. Suppose the forgetful 2-functor $U : \mathcal{MonCat} \longrightarrow \mathcal{Set}$ has a left strict 2-adjoint $F : \mathcal{Set} \longrightarrow \mathcal{MonCat}$, then for every $S \in \mathcal{Set}$ the pair $(F(S), i : S \longrightarrow F(S))$ satisfies the universal property for the free monoidal category generated by S. ◇

Construction 3.2.7. We will give a construction of the free monoidal category generated by a set S. This construction will describe the objects as freely generated under tensor and morphisms will be freely generated by the structure morphisms. This construction is described on page 25 of [JS93]. We will now summarize the construction:

Let S be a set. Define a monoidal category denoted $\mathcal{MonCat}\langle S \rangle$ as follows: The objects of $\mathcal{MonCat}\langle S \rangle$ are constructed inductively by requiring that they formally include elements of S, a new distinguished object J, and an object $a \otimes b$ for any two objects a, b in $\mathcal{MonCat}\langle S \rangle$. The morphisms of $\mathcal{MonCat}\langle S \rangle$ are formally generated from the structure morphisms

$$1_x : x \longrightarrow x, \qquad \alpha_{x,y,z} : x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z,$$

$$\lambda_x : J \otimes x \longrightarrow x, \qquad \rho_x : x \otimes J \longrightarrow x$$

by tensoring, substituting, inverting, and composing, subject to the equivalence relation generated by the axioms of monoidal category. The equivalence relation ensures that the category $\mathcal{M}on\mathcal{C}at\,\langle S\rangle$ is a monoidal category.

Let $i : S \longrightarrow \mathrm{Obj}(\mathcal{M}on\mathcal{C}at\,\langle S\rangle)$ be the function that sends an element $a \in S$ to the corresponding element in $\mathrm{Obj}(\mathcal{M}on\mathcal{C}at\,\langle S\rangle)$. Then, the pair $(\mathcal{M}on\mathcal{C}at\,\langle S\rangle, i)$ satisfies the universal property of the free monoidal category. $\diamond$

Remark 3.2.8. Recall that for a category $\mathscr{C}$, a diagram in $\mathscr{C}$ is a functor $D : \mathscr{D} \longrightarrow \mathscr{C}$ where $\mathscr{D}$ is a small category. Next, we will define formal diagrams in monoidal categories. $\diamond$

Definition 3.2.9. Let $\mathscr{M}$ be a monoidal category and

$$M := \mathrm{Obj}(\mathscr{M})$$

be the set of objects in $\mathscr{M}$. Consider the free monoidal category, $\mathcal{M}on\mathcal{C}at\,\langle M\rangle$, generated by M. Let

$$\mathcal{EV} : \mathcal{M}on\mathcal{C}at\,\langle M\rangle \longrightarrow \mathscr{M}$$

be the strict monoidal functor that we get from the universal property. A *formal diagram* in $\mathscr{M}$ is a diagram $D : \mathscr{D} \longrightarrow \mathscr{M}$ that lifts to $\mathcal{M}on\mathcal{C}at\,\langle M\rangle$. That is, a diagram $D : \mathscr{D} \longrightarrow \mathscr{M}$ is called a formal diagram in $\mathscr{M}$ if there exists a functor $\widehat{D} : \mathscr{D} \longrightarrow \mathcal{M}on\mathcal{C}at\,\langle M\rangle$ such that the diagram

$$\begin{tikzcd} & \mathcal{M}on\mathcal{C}at\,\langle M\rangle \arrow[d, "\mathcal{EV}"] \\ \mathscr{D} \arrow[ur, "\widehat{D}"] \arrow[r, "D"'] & \mathscr{M} \end{tikzcd}$$

commutes giving the equality

$$\mathcal{EV} \circ \widehat{D} = D. \qquad \diamond$$

Remark 3.2.10. In the above definition, since the functor $\mathcal{EV}$ is a strict monoidal functor, the structure morphisms in the free monoidal category maps to the corresponding structure morphism in $\mathscr{M}$. Thus, the formal diagrams in $\mathscr{M}$ are exclusively the diagram that are made up of the structure morphisms. In practice, most diagrams that involve the structure morphisms are formal diagrams. Hence, a theorem that states every formal diagram commutes is extremely powerful.

It is important to note that not every diagram built from structure morphisms is necessarily a formal diagram. For example, in a monoidal category, we could have $a \otimes (b \otimes c) = (a \otimes b) \otimes c$ as objects, it does not follow that $\alpha_{a,b,c} = 1_{a\otimes(b\otimes c)}$. A concrete example illustrating this, due to J. R. Isbell, can be found on page 164 of [ML78]. $\diamond$

Theorem 3.2.11. *Let S be a set. The free monoidal category over S is a thin category.*

Proof. This result follows from Corollary 1.6 on page 29 of [JS93]. $\square$

Corollary 3.2.12. *Every formal diagram in monoidal category commutes.*

Proof. Let $\mathscr{M}$ be a monoidal category with M as the set of objects. Let $D : \mathscr{D} \longrightarrow \mathscr{M}$ be a formal diagram in $\mathscr{M}$. Thus, there exists a lift $\widehat{D} : D \longrightarrow \mathcal{MonCat}\langle M\rangle$ such that

$$\mathcal{EV} \circ \widehat{D} = D$$

where $\mathcal{EV} : \mathcal{MonCat}\langle M\rangle \longrightarrow \mathscr{M}$ is the strict monoidal functor as in Definition 3.2.9. From Theorem 3.2.11 we get that $\mathcal{MonCat}\langle M\rangle$ is a thin category. Since D factors through a thin category, the diagram D commutes. $\square$

Theorem 3.2.13. *Let $\mathscr{M} = (\mathscr{M}, I, \otimes)$ be a monoidal category. Then one can construct a strict monoidal category $\mathscr{S} = (\mathscr{S}, J, \bullet)$ such that there is a special monoidal equivalence from $\mathscr{M}$ to $\mathscr{S}$.*

Proof. This result is established as Theorem 1 on page 257 of [ML78]. The proof constructs a larger strict monoidal category $\mathscr{S}$ where the objects are given by the free monoid generated by the objects of $\mathscr{M}$, and then embeds $\mathscr{M}$ inside $\mathscr{S}$.

Alternatively, this theorem can be deduced from Corollary 1.4 on page 27 of [JS93], where the authors realize $\mathscr{M}$ as a subcategory of the strict monoidal category of endofunctors on $\mathscr{M}$, equipped with a right translation natural isomorphism. $\square$

3.3 An alternate approach to coherence for monoidal categories

Let S be a set and let $\mathcal{MonCat}\langle S\rangle$ be the free monoidal category on S as in Construction 3.2.7. The set of objects of $\mathcal{MonCat}\langle S\rangle$, which we denote by M, is the set freely generated by a binary operation $-\bullet-$ on S together with a unit object J. In other words, M is the free magma on $S \sqcup \{J\}$.

Let $\mathcal{Mon}\langle S\rangle$ be the free monoid generated by S (see Chapter 6 for details). We regard $\mathcal{Mon}\langle S\rangle$ as a discrete category, which is strict monoidal due to its associative multiplication and unit. By the existence condition (3.2.2) of the universal property, there exists a strict monoidal functor

$$Q : \mathcal{MonCat}\langle S\rangle \longrightarrow \mathcal{Mon}\langle S\rangle$$

such that $Q(a) = a$ for every $a \in S$. This functor Q is unique: if

$$R : \mathcal{MonCat}\langle S\rangle \longrightarrow \mathcal{Mon}\langle S\rangle$$

is any monoidal functor with $R(a) = a$ for all $a \in S$, then by the existence condition (3.2.3) of the universal property, there exists a monoidal natural isomorphism $\phi : Q \Rightarrow R$. Since $\mathcal{Mon}\langle S\rangle$ is discrete, ϕ must be the identity, so $Q = R$.

The functor Q satisfies

$$Q(x \bullet y) = Q(x)Q(y) \qquad \text{and} \qquad Q(J) = e,$$

where J and e are the units in $\mathcal{MonCat}\langle S\rangle$ and $\mathcal{Mon}\langle S\rangle$, respectively. Together with $Q(a) = a$ for $a \in S$, these equations completely determine Q on objects.

Since $\mathcal{Mon}\langle S\rangle$ is discrete, for any morphism $f : x \longrightarrow y$ in $\mathcal{MonCat}\langle S\rangle$, we have $Q(f) = 1_{Q(x)}$. Moreover, for objects $x, y \in \mathcal{MonCat}\langle S\rangle$, if $Q(x) = Q(y)$, then there exists a morphism $f : x \longrightarrow y$ in $\mathcal{MonCat}\langle S\rangle$. To see this, one can construct a morphism $f_1 : x \longrightarrow \overline{x}$ by repeatedly applying associators to move all parentheses to the left, and then use the unitors to remove all occurrences of J. Similarly, construct $f_2 : y \longrightarrow \overline{y}$. Note that $\overline{x} = \overline{y}$ if and only if $Q(x) = Q(y)$. Thus, we obtain a morphism $f = f_2^{-1} \circ f_1 : x \longrightarrow y$ built from associators and unitors, factoring through $\overline{x} = \overline{y}$. By the coherence Theorem 3.2.11, this morphism f is unique.

This motivates the following alternative construction of the free monoidal category $\mathscr{M}$ generated by S. The set of objects M is the free magma on $S \sqcup \{J\}$. Define $Q : M \longrightarrow \mathcal{Mon}\langle S\rangle$ by

$$Q(a) = a, \qquad Q(x \bullet y) = Q(x)Q(y), \qquad Q(J) = e$$

for $a \in S$ and $x, y \in M$. For $x, y \in M$, there is a unique morphism $f : x \longrightarrow y$ in $\mathscr{M}$ if and only if $Q(x) = Q(y)$. Mac Lane hints at this construction in Exercise 3 on page 170 of [ML78].

This approach does not trivialize the proof of the coherence theorem for monoidal categories; rather, it shifts the focus from showing that the free category is thin to verifying that this already thin category satisfies the universal property. Theorem

1 on page 166 of [ML78] shows that the free monoidal category on a singleton, constructed in this way, satisfies the universal property. The advantage of this alternative construction is that it encapsulates the structure morphisms and their relations in the single function Q.

While we will not prove here that the alternative construction above satisfies the universal property of the free monoidal category, we will use a similar approach in Chapter 8 to formally construct the free categorical group and demonstrate that it satisfies the universal property of the free categorical group.

Chapter 4: Topics in Categorical Groups

In this chapter, we introduce categorical groups, categorical group functors, and categorical group natural transformations. We demonstrate that, up to equivalence, it suffices to study categorical groups whose underlying monoidal structure is strict (semi-strict categorical groups). Furthermore, we show that the essential data of categorical group functors and natural transformations is determined by their underlying monoidal functors and monoidal natural transformations, respectively.

We will also present the universal property of the free categorical group generated by a set, and state the coherence theorem for categorical groups.

4.1 Definition of a Categorical Group

Definition 4.1.1. A *categorical group*

$$\big(\mathscr{M}, I, (-)', \otimes, \lambda, \rho, \eta, \epsilon, , \alpha\big)$$

consists of the following data:

- A monoidal category

  $$(\mathscr{M}, I, \otimes, \alpha, \lambda, \rho)$$

  called the underlying monoidal cateogry.

- A functor

$$(-)' : \mathscr{M} \longrightarrow \mathscr{M}$$

called the *negator*.

- A natural isomorphism,

$$\epsilon := \left(\epsilon_x : x' \otimes x \longrightarrow I\right)_{x \in \mathscr{M}}$$

called the *left cancellation isomorphism*.

- A natural isomorphism,

$$\eta := \left(\eta_x : I \longrightarrow x \otimes x'\right)_{x \in \mathscr{M}}$$

called the *right cancellation isomorphism*.

These satisfy the following coherence axioms:

Groupoid axiom: The underlying category $\mathscr{M}$ is a groupoid.

Unit axiom: The equalities

$$I' = I \qquad \text{and} \tag{4.1.2}$$

$$\epsilon_I = \rho_I : I' \otimes I \longrightarrow I \tag{4.1.3}$$

hold.

Cancellation axioms: For every $x \in \mathrm{Obj}(\mathscr{M})$ the diagrams

$$\begin{tikzcd} x \arrow[r, "\lambda_x^{-1}"] \arrow[d, equal] & I \otimes x \arrow[r, "\eta_x \otimes 1_x"] & (x \otimes x') \otimes x \arrow[d, "\alpha^{-1}_{x,x',x}"] \\ x & x \otimes I \arrow[l, "\rho_x"] & x \otimes (x' \otimes x) \arrow[l, "1_x \otimes \epsilon_x"] \end{tikzcd} \tag{4.1.4}$$

and

$$
\begin{tikzcd}
x' \arrow[r, "\rho_{x'}^{-1}"] \arrow[d, equal] & x' \otimes I \arrow[r, "1_{x'} \otimes \eta_x"] & x' \otimes (x \otimes x') \arrow[d, "\alpha_{x',x,x'}"] \\
x' & I \otimes x' \arrow[l, "\lambda_{x'}"'] & (x' \otimes x) \otimes x' \arrow[l, "\epsilon_x \otimes 1_{x'}"']
\end{tikzcd}
\tag{4.1.5}
$$

commute. ◇

Notation 4.1.6. Let

$$(\mathscr{M}, I, (-)', \otimes, \lambda, \rho, \epsilon, \eta, \alpha)$$

be a categorical group. For objects $x, y, z \in \mathscr{M}$, the morphisms

$$
\begin{aligned}
&\alpha_{x,y,z} : x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z, && \lambda_x : I \otimes x \longrightarrow x, \\
&\rho_x : x \otimes I \longrightarrow x, && \epsilon_x : x' \otimes x \longrightarrow I, \\
&\eta_x : I \longrightarrow x \otimes x', && 1_x : x \longrightarrow x
\end{aligned}
$$

and their inverses are called the *structure morphisms* of the categorical group. The morphisms

$$
\begin{aligned}
&\alpha_{x,y,z} : x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z, && \lambda_x : I \otimes x \longrightarrow x, \\
&\rho_x : x \otimes I \longrightarrow x, && 1_x : x \longrightarrow x
\end{aligned}
$$

and their inverses are called the *monoidal structure morphisms* of the categorical group. The morphisms

$$\epsilon_x : x' \otimes x \longrightarrow I, \qquad \eta_x : I \longrightarrow x \otimes x'$$

and their inverses are called the *cancellation isomorphisms* of the categorical group.◇

Notation 4.1.7. Let $\mathscr{M}$ be a categorical group and let $k \in \mathbb{N}$. Denote the k-times iterated dash functor $(-)' : \mathscr{M} \longrightarrow \mathscr{M}$ by

$$(-)^{(k)} : \mathscr{M} \longrightarrow \mathscr{M}.$$

That is,

$$(-)^{(k)} := \begin{cases} \mathrm{Id}_{\mathscr{M}} & \text{if } k = 0 \\ (-)' \circ (-)^{(k-1)} & \text{if } k \geq 1. \end{cases}$$ ◇

Remark 4.1.8. As a consequence of the equation (3.1.6), unit axioms (4.1.2) and (4.1.3), equalities

$$\lambda_I = \rho_I = \epsilon_I = \eta_I^{-1} : I \otimes I \longrightarrow I \tag{4.1.9}$$

hold. ◇

Definition 4.1.10. A categorical group

$$(\mathscr{M}, I, (-)', \otimes, \lambda, \rho, \epsilon, \eta, \alpha)$$

is called a *semi-strict categorical group* if the natural isomorphisms λ, ρ, and α are equal to identity natural transformations. That is, a semi-strict categorical group is a categorical group with strict monoidal structure. ◇

Remark 4.1.11. In Lemma 5.3.13, we will see that for semi-strict categorical groups, the natural transformation η is entirely determined by the natural transformation ϵ. For the general case of categorical groups, see Proposition 4.1 on page 439 of [BL04].◇

Definition 4.1.12. A categorical group

$$(\mathscr{M}, I, (-)', \otimes, \lambda, \rho, \epsilon, \eta, \alpha)$$

is called a *strict categorical group* if the natural isomorphisms λ, ρ, ϵ, η, and α are all equal to identity natural transformations. ◇

Convention 4.1.13. Whenever convenient, we will omit the explicit notation for the categorical group

$$(\mathscr{M}, I, (-)', \otimes, \lambda, \rho, \alpha)$$

and simply write $\mathscr{M}$ or $(\mathscr{M}, I, (-)', \otimes)$. ◇

Remark 4.1.14. In a semi-strict categorical group $(\mathscr{M}, I, (-)', \otimes, \epsilon, \eta)$ the unit conditions (4.1.2) and (4.1.3) will reduce to the following equalities:

$$I' = I, \tag{4.1.15}$$

$$\epsilon_I = 1_I : I \longrightarrow I, \qquad \text{and} \tag{4.1.16}$$

$$\lambda_I = 1_I : I \longrightarrow I. \qquad (3.1.6) \tag{4.1.17}$$

Similarly, the cancellation conditions (4.1.4) and (4.1.5) will reduce to the following equalities: For $x \in \mathrm{Obj}(\mathscr{M})$,

$$(1_x \otimes \epsilon_x) \circ (\eta_x \otimes 1_x) = 1_x \tag{4.1.18}$$

$$(\epsilon_x \otimes 1_{x'}) \circ (1_{x'} \otimes \eta_x) = 1_{x'}. \tag{4.1.19}$$

These equalities are described by the following commuting diagrams:

$$\begin{tikzcd} x \arrow[r, "\eta_x \otimes 1_x"] \arrow[dr, equal] & xx'x \arrow[d, "1_x \otimes \epsilon_x"] \\ & x \end{tikzcd} \tag{4.1.20}$$

and

$$\begin{tikzcd} x' \arrow[r, "1_{x'} \otimes \eta_x"] \arrow[dr, equal] & x'xx' \arrow[d, "\epsilon_x \otimes 1_{x'}"] \\ & x'. \end{tikzcd} \tag{4.1.21}$$

◇

Definition 4.1.22. Let $\mathscr{M} = (\mathscr{M}, I, (-)', \otimes)$ and $\mathscr{N} = (\mathscr{N}, J, (-)^\dagger, \bullet)$ be two categorical groups. A *categorical group functor*

$$(F, F_0, F_1, F_2) : \mathscr{M} \longrightarrow \mathscr{N}$$

has the following data:

- A monoidal functor

  $$(F, F_0, F_2) : \mathscr{M} \longrightarrow \mathscr{N}$$

  on the underlying monoidal categories.

- A natural isomorphism,

  $$F_1 := \left(F_1(x) : F(x)^\dagger \longrightarrow F(x')\right)_{x \in \mathscr{M}},$$

  called the *negator structure isomorphism* of the functor F.

These satisfy following condition:

Left cancellation condition: For every $x \in \mathrm{Obj}(\mathscr{M})$ the diagram

$$\begin{tikzcd}
F(x)^\dagger \bullet F(x) \arrow[r, "\epsilon^{\mathscr{N}}_{F(x)}"] \arrow[d, "F_1(x) \bullet 1_{F(x)}"'] & J \arrow[r, "F_0"] & F(I) \\
F(x') \bullet F(x) \arrow[rr, "F_2(x', x)"'] & & F(x' \otimes x) \arrow[u, "F\left(\epsilon^{\mathscr{M}}_x\right)"']
\end{tikzcd} \tag{4.1.23}$$

commutes.

Right cancellation condition: For every $x \in \mathrm{Obj}(\mathscr{M})$ the diagram

$$\begin{tikzcd}
J \arrow[r, "F_0"] \arrow[d, "\eta^{\mathscr{N}}_{F(x)}"'] & F(I) \arrow[r, "F\left(\eta^{\mathscr{M}}_x\right)"] & F(x \otimes x') \\
F(x) \bullet F(x)^\dagger \arrow[rr, "1_{F(x)} \bullet F_1(x)"'] & & F(x) \bullet F(x') \arrow[u, "F_2(x, x')"']
\end{tikzcd} \tag{4.1.24}$$

commutes. ◇

Remark 4.1.25. Let $\mathscr{M}$ and $\mathscr{N}$ be categorical groups, and let $F : \mathscr{M} \longrightarrow \mathscr{N}$ be a monoidal functor between their underlying monoidal categories. Then F extends uniquely to a categorical group functor. For the case of semi-strict categorical groups, this is shown in Theorems 4.2.16 and 4.2.17. For the general case, see Theorem 6.1 on page 446 of [BL04]. ◇

Definition 4.1.26. Let $\mathscr{M} = (\mathscr{M}, I, (-)', \otimes)$ and $\mathscr{N} = (\mathscr{N}, J, (-)^{\dagger}, \bullet)$ be two categorical groups. A categorical group functor

$$(F, F_0, F_1, F_2) : \mathscr{M} \longrightarrow \mathscr{N}$$

is called a *strict categorical group functor* if the following conditions are satisfied:

Unit condition: The equality

$$F_0 = 1_J : J \longrightarrow F(I) \tag{4.1.27}$$

holds.

Negator condition: For $x \in \mathrm{Obj}(\mathscr{M})$, the equality

$$F_1(x) = 1_{F(x)^{\dagger}} : F(x)^{\dagger} \longrightarrow F(x') \tag{4.1.28}$$

holds.

Tensor condition: For $x, y \in \mathrm{Obj}(\mathscr{M})$, the equality

$$F_2(x, y) = 1_{F(x) \bullet F(y)} : F(x) \bullet F(y) \longrightarrow F(x \otimes y) \tag{4.1.29}$$

holds. ◇

Remark 4.1.30. From Remark 4.1.25, it follows that a categorical group functo $F : \mathscr{M} \longrightarrow \mathscr{N}$ is a strict categorical group functor if and only if the underlying monoidal functor is a strict monoidal functor. ◇

Definition 4.1.31. Let $\mathscr{M} = (\mathscr{M}, I, (-)', \otimes)$ be a categorical group. Define the identity categorical group functor $\mathrm{Id}_{\mathscr{M}} : \mathscr{M} \longrightarrow \mathscr{M}$ as follows:

- The underlying monoidal functor $\mathrm{Id}_{\mathscr{M}} : \mathscr{M} \longrightarrow \mathscr{M}$ is equal to the identity monoidal functor as in Definition 3.1.19.
- For an object $x \in \mathscr{M}$, the negator structure isomorphism,

$$(\mathrm{Id}_{\mathscr{M}})_1(x) : (\mathrm{Id}_{\mathscr{M}}(x))' \longrightarrow \mathrm{Id}_{\mathscr{M}}(x'),$$

  is given by

$$(\mathrm{Id}_{\mathscr{M}})_1(x) := 1_{x'}. \tag{4.1.32}$$

  This is well-defined since $\mathrm{Id}_{\mathscr{M}}(x) = x$.

Since the structure isomorphisms are all equal to identities, the left and right cancellation conditions are satisfied. $\diamond$

Definition 4.1.33. Let

$$\mathscr{M} = (\mathscr{M}, I, (-)', \otimes), \qquad \mathscr{N} = (\mathscr{N}, J, (-)^{\dagger}, \bullet), \qquad \text{and} \qquad \mathscr{P} = (\mathscr{P}, K, (-)^{\flat}, \cdot)$$

be categorical groups. Let

$$F : \mathscr{M} \longrightarrow \mathscr{N} \qquad \text{and} \qquad G : \mathscr{N} \longrightarrow \mathscr{P}$$

be categorical group functors. Define the composition of categorical group functors

$$G \circ F : \mathscr{M} \longrightarrow \mathscr{P}$$

as follows: For this definition, we will denote $G \circ F$ by GF.

- The underlying monoidal functor $GF : \mathscr{M} \longrightarrow \mathscr{P}$ is equal to the composition of monoidal functors as in Definition 3.1.22.

- For an object $x \in \mathscr{M}$, the negator structure isomorphism,

$$(GF)_1(x) : (GF(x))^\flat \longrightarrow GF(x')$$

  is given by

$$(GF)_1(x) := G(F_1(x)) \circ G_1(F(x)). \tag{4.1.34}$$

Now, we will show that the left cancellation condition is satisfied. For $x \in \mathrm{Obj}(\mathscr{M})$ the diagram

$$\begin{tikzcd}
(GF(x))^\flat \cdot GF(x) \arrow[r, "G_1(Fx)\cdot 1_{GF(x)}"] \arrow[d, "\epsilon^{\mathscr{P}}_{GF(x)}"'] & G((Fx)^\dagger)\cdot GF(x) \arrow[ddd, "G_2((Fx)^\dagger, Fx)"'] \arrow[dr, bend left, "G(F_1(x))\cdot 1_{GF(x)}"] & \\
K \arrow[dd, "G_0"'] & & \\
 & & GF(x')\cdot GF(x) \arrow[ddl, bend left, "G_2(F(x'), Fx)"] \\
G(J) \arrow[dd, "G(F_0)"'] & G((Fx)^\dagger \bullet (Fx)) \arrow[l, "G(\epsilon^{\mathscr{N}}_{Fx})"'] \arrow[d, "G(F_1(x)\bullet 1_{Fx})"'] & \\
 & G(F(x')\bullet(Fx)) \arrow[d, "G(F_2(x',x))"] & \\
GF(I) & GF(x'\otimes x) \arrow[l, "GF(\epsilon^{\mathscr{M}}_x)"] &
\end{tikzcd}$$

commutes. This is true since the top rectangle is the left cancellation condition for G, the bottom rectangle is the left cancellation condition for F and the right semi-circle is the naturality of G_2. From the definition of unit structure morphisms, $(GF)_0$, $(GF)_1$, and $(GF)_2$ as in equations (3.1.23), (3.1.24), and (4.1.34) respectively, the outer diagram is the left negator condition for GF.

Next, we will show that the right cancellation condition is satisfied. For $x \in \mathrm{Obj}(\mathscr{M})$ the diagram

$$\begin{tikzcd}
GF(x)\cdot(GF(x))^{\flat} \arrow[r, "1_{GF(x)}\cdot G_1(Fx)"] \arrow[d, "\left(\epsilon^{\mathscr{P}}_{GF(x)}\right)^{-1}"'] & GF(x)\cdot G((Fx)^{\dagger}) \arrow[ddd, "G_2(Fx,(Fx)^{\dagger})"'] \arrow[ddr, bend left, "1_{GF(x)}\cdot G(F_1(x))"] & \\
K \arrow[dd, "G_0"'] & & \\
 & & GF(x)\cdot GF(x') \arrow[ddl, bend left, "G_2(Fx,F(x'))"] \\
G(J) \arrow[dd, "G(F_0)"'] & G((Fx)\bullet(Fx)^{\dagger}) \arrow[l, "\left(G(\epsilon^{\mathscr{N}}_{Fx})\right)^{-1}"'] \arrow[d, "G(1_{Fx}\bullet F_1(x))"'] & \\
 & G((Fx)\bullet F(x')) \arrow[d, "G(F_2(x,x'))"] & \\
GF(I) & GF(x\otimes x') \arrow[l, "\left(GF(\epsilon^{\mathscr{M}}_{x})\right)^{-1}"] &
\end{tikzcd}$$

commutes. This is true since the top rectangle is the right cancellation condition for G, the bottom rectangle is the right cancellation condition for F and the right semi-circle is the naturality of G_2. From the definition of unit structure morphisms, $(GF)_0$, $(GF)_1$, and $(GF)_2$ as in equations (3.1.23), (3.1.24), and (4.1.34) respectively, the outer diagram is the right negator condition for GF. $\diamond$

Definition 4.1.35. Let $\mathscr{M} = (\mathscr{M}, I, (-)', \otimes)$ and $\mathscr{N} = (\mathscr{N}, J, (-)^{\dagger}, \bullet)$ be two categorical groups and $F, G : \mathscr{M} \longrightarrow \mathscr{N}$ be categorical group functors. A *categorical group natural transformation*

$$\phi : F \Rightarrow G$$

has the following data:

- A monoidal natural transformation

$$\phi : F \Rightarrow G$$

 on the underlying monoidal functors

such that the diagram

$$\begin{tikzcd} F(x)^\dagger \arrow[r, "F_1(x)"] \arrow[d, "\phi_x^\dagger"'] & F(x') \arrow[d, "\phi_{x'}"] \\ G(x)^\dagger \arrow[r, "G_1(x)"'] & G(x'). \end{tikzcd} \tag{4.1.36}$$

commutes for every $x \in \mathrm{Obj}(\mathscr{M})$. This condition is called the *negator condition* for a categorical group natural transformation. ⋄

Remark 4.1.37. Let $\mathscr{M}$ and $\mathscr{N}$ be categorical groups, $F, G : \mathscr{M} \longrightarrow \mathscr{N}$ be monoidal functors between their underlying monoidal categories, and $\phi : F \Rightarrow G$ be a monoidal natural transformation. From Remark 4.1.25, the monoidal functors F, G extend uniquely to categorical group functors. Moreover, the monoidal functor $\phi : F \Rightarrow G$ extends uniquely to a categorical group natural transformation. For the case of semi-strict categorical groups, this is shown in Theorem 4.2.20. ⋄

Remark 4.1.38. The categorical groups, categorical group functors, and categorical group natural transformations together form a 2-category, denoted by $\mathcal{CatGrp}$. Likewise, semi-strict categorical groups, categorical group functors between them, and categorical group natural transformations form a 2-category denoted by $\mathcal{SSCatGrp}$. ⋄

Definition 4.1.39. Let $\mathscr{M}$ and $\mathscr{N}$ be two categorical groups. A *categorical group equivalence* from $\mathscr{M}$ to $\mathscr{N}$ has the following data:

- Categorical group functors

$$F : \mathcal{N} \longrightarrow \mathcal{M} \qquad \text{and} \qquad G : \mathcal{M} \longrightarrow \mathcal{N}.$$

- Categorical group natural isomorphisms

$$\phi : G \circ F \Rightarrow \mathrm{Id}_{\mathcal{N}} \qquad \text{and} \qquad \psi : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{M}}.$$

Definition 4.1.40. Let $\mathcal{M} = (\mathcal{M}, I, \otimes, (-)')$ and $\mathcal{N} = (\mathcal{N}, J, \bullet, (-)^{\dagger})$ be two categorical groups. A *special categorical group equivalence* from $\mathcal{M}$ to $\mathcal{N}$ has the following data:

- A categorical group equivalence (F, G, ϕ, ψ) from $\mathcal{M}$ to $\mathcal{N}$.

These satisfy the following conditions:

Special monoidal equivalence: The underlying monoidal equivalence is a special monoidal equivalence as in Definition 3.1.32.

Strictness of G: The categorical group functor G is a strict functor as in Definition 4.1.26.

Negator condition for F: For every $a \in \mathrm{Obj}(\mathcal{N})$ the equality of morphisms

$$F_1(a) = 1_{F(a)'} : F(a)' \longrightarrow F(a^{\dagger}) \tag{4.1.41}$$

holds.

Counit condition: The equality of categorical group natural transformations

$$\psi = \mathrm{Id}_{\mathrm{Id}_{\mathcal{M}}} : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{M}} \tag{4.1.42}$$

holds. ◇

Motivation 4.1.43. We will see in Theorem 4.3.28 that a categorical group is equivalent to a semi-strict categorical group via a special categorical group equivalence. ◇

4.2 Semi-Strict Categorical Groups

In this section, we will see that a monoidal functor between semi-strict categorical groups extends uniquely to a categorical group functor. That is, given a monoidal functor

$$(F, F_0, F_2) : \mathscr{M} \longrightarrow \mathscr{N}$$

from a semi-strict categorical group $\mathscr{M}$ to a semi-strict categorical group $\mathscr{N}$, we can uniquely construct a natural transformation

$$F_1(x) : F(x)' \longrightarrow F(x')$$

such that it satisfies the left cancellation condition (4.1.23) and the right cancellation condition (4.1.24). As a consequence, we get that

$$(F, F_0, F_1, F_2) : \mathscr{M} \longrightarrow \mathscr{N}$$

is a categorical group functor.

Framework 4.2.1. Throughout this section, let

$$\mathscr{M} = (\mathscr{M}, I, (-)', \cdot) \qquad \text{and} \qquad \mathscr{N} = (\mathscr{N}, J, (-)^{\dagger}, \bullet)$$

be semi-strict categorical groups. Let

$$(F, F_0, F_2) : \mathscr{M} \longrightarrow \mathscr{N}$$

be a monoidal functor between the underlying monoidal categories. Note that, since $\mathscr{M}$ and $\mathscr{N}$ are semi-strict categorical groups, the tensor functor is strictly associative. Therefore, we will omit brackets to show the order of multiplication. $\diamond$

Definition 4.2.2. Define a natural transformation l, with components

$$l(x) : J \longrightarrow F(x') \bullet F(x)$$

for $x \in \mathrm{Obj}(\mathscr{M})$, as follows:

- For $x \in \mathrm{Obj}(\mathscr{M})$ define

$$l(x) := F_2(x', x)^{-1} \circ F(\epsilon_x^{-1}) \circ F_0. \tag{4.2.3}$$

That is, $l(x)$ is the composition of the following morphisms.

$$l(x) : J \xrightarrow{\quad F_0 \quad} F(I) \xrightarrow{\quad F(\epsilon_x^{-1}) \quad} F(x'x) \xrightarrow{F_2(x',x)^{-1}} F(x') \bullet F(x).$$

Since each component of $l(x)$ is a natural transformation, we get that l is a natural transformation. $\diamond$

Definition 4.2.4. Define a natural transformation r, with components

$$r(x) : J \longrightarrow F(x) \bullet F(x')$$

for $x \in \mathrm{Obj}(\mathscr{M})$, as follows:

- For $x \in \mathrm{Obj}(\mathscr{M})$ define

$$r(x) := F_2(x, x')^{-1} \circ F(\eta_x) \circ F_0. \tag{4.2.5}$$

That is, $r(x)$ is the composition of the following morphisms.

$$r(x) : J \xrightarrow{\quad F_0 \quad} F(I) \xrightarrow{\quad F(\eta_x) \quad} F(xx') \xrightarrow{F_2(x,x')^{-1}} F(x) \bullet F(x').$$

Since each component of $r(x)$ is a natural transformation, we get that r is a natural transformation. $\diamond$

Proposition 4.2.6. *Let $x \in \mathrm{Obj}(\mathscr{M})$. Then, the equality of morphisms*

$$1_{F(x)} \bullet l(x) = r(x) \bullet 1_{F(x)} : F(x) \longrightarrow F(x) \bullet F(x') \bullet F(x) \tag{4.2.7}$$

holds. That is, the diagram

$$\begin{tikzcd}
F(x) \arrow[r, "1_{F(x)} \bullet F_0"] \arrow[d, "F_0 \bullet 1_{F(x)}"'] & F(x) \bullet F(I) \arrow[d, "1_{F(x)} \bullet F(\epsilon_x^{-1})"] \\
F(I) \bullet F(x) \arrow[d, "F(\eta_x) \bullet 1_{F(x)}"'] & F(x) \bullet F(x'x) \arrow[d, "1_{F(x)} \bullet F_2(x',x)^{-1}"] \\
F(xx') \bullet F(x) \arrow[r, "F_2(x,x')^{-1} \bullet 1_{F(x)}"'] & F(x) \bullet F(x') \bullet F(x).
\end{tikzcd}$$

commutes.

Proof. Consider the following diagram.

$$\begin{tikzcd}
F(x) \arrow[rr, "1_{F(x)} \bullet F_0"] \arrow[dr, "F(\eta_x 1_x)"] \arrow[dd, "F_0 \bullet 1_{F(x)}"'] & & F(x) \bullet F(I) \arrow[dr, "1_{F(x)} \bullet F(\epsilon_x^{-1})"] & \\
& F(xx'x) \arrow[rr, "F_2(x,x'x)^{-1}"] \arrow[dd, "F_2(xx',x)^{-1}"] & & F(x) \bullet F(x'x) \arrow[dd, "1_{F(x)} \bullet F_2(x',x)^{-1}"] \\
F(I) \bullet F(x) \arrow[dr, "F(\eta_x) \bullet 1_{F(x)}"'] & & & \\
& F(xx') \bullet F(x) \arrow[rr, "F_2(x,x')^{-1} \bullet 1_{F(x)}"'] & & F(x) \bullet F(x') \bullet F(x).
\end{tikzcd}$$

First we will show that, in the above diagram, the top square commutes. We have

$$1_{F(x)} \bullet F_0 = F_2(x, I) : F(x) \longrightarrow F(x) \bullet F(I)$$

from the monoidal functor condition (3.1.12). We have

$$\eta_x 1_x = 1_x \epsilon_x^{-1} : x \longrightarrow xx'x$$

from the cancellation condition (4.1.4). Thus, the top square becomes

$$\begin{tikzcd}
F(x) \arrow[rr, "1_{F(x)} \bullet F_0 = F_2(x,I)"] \arrow[dr, "F(\eta_x 1_x) = F(1_x \epsilon_x^{-1})"'] & & F(x) \bullet F(I) \arrow[dr, "1_{F(x)} \bullet F(\epsilon_x^{-1})"] & \\
& F(xx'x) \arrow[rr, "F_2(x,x'x)^{-1}"'] & & F(x) \bullet F(x'x).
\end{tikzcd}$$

This diagram commutes from naturality of F_2. Thus, the top square commutes.

For the left square, note that we have

$$F_0 = 1_{F(x)} = F_2(I,x) : F(x) \longrightarrow F(I) \bullet F(x)$$

from the monoidal functor condition (3.1.11). Thus, the left square becomes

$$\begin{tikzcd}
F(x) \arrow[dd, "F_0 \bullet 1_{F(x)} = F_2(I,x)"'] \arrow[dr, "F(\eta_x 1_x)"] & \\
& F(xx'x) \arrow[dd, "F_2(xx',x)^{-1}"] \\
F(I) \bullet F(x) \arrow[dr, "F(\eta_x) \bullet 1_{F(x)}"'] & \\
& F(xx') \bullet F(x)
\end{tikzcd}$$

Again, this diagram commutes from the naturality of F_2. Thus, the left square commutes.

Finally, the front square commutes from condition (3.1.10). Therefore, the entire diagram commutes.

Now, the top right composite is given by

$$1_{F(x)} \bullet l(x)$$

and the left bottom composite is given by

$$r(x) \bullet 1_{F(x)}.$$

Thus, we get

$$1_{F(x)} \bullet l(x) = r(x) \bullet 1_{F(x)}$$

as required. $\square$

Proposition 4.2.8. *Let $x \in \mathrm{Obj}(\mathscr{M})$. Then, the following diagram commutes.*

$$\begin{tikzcd}
F(x)^\dagger \arrow[r, "1_{F(x)^\dagger} \bullet r(x)"] \arrow[d, "l(x) \bullet 1_{F(x)^\dagger}"'] & F(x)^\dagger \bullet F(x) \bullet F(x') \arrow[d, "\epsilon_{F(x)} \bullet 1_{F(x')}"] \\
F(x') \bullet F(x) \bullet F(x)^\dagger \arrow[r, "1_{F(x')} \bullet \eta^{-1}_{F(x)}"'] & F(x').
\end{tikzcd} \tag{4.2.9}$$

Proof. Consider the following diagram.

$$\begin{tikzcd}
F(x)^\dagger \bullet F(x) \bullet F(x)^\dagger \arrow[rr, "1_{F(x)^\dagger} \bullet \eta^{-1}_{F(x)}"] \arrow[dd, "\epsilon_{F(x)} \bullet 1_{F(x)^\dagger}"'] \arrow[dr, "\langle 1_{F(x)} \bullet l(x) \rangle = \langle r(x) \bullet 1_{F(x)} \rangle"] & & F(x)^\dagger \arrow[d, "1_{F(x)^\dagger} \bullet r(x)"] \\
& F(x)^\dagger \bullet F(x) \bullet F(x') \bullet F(x) \bullet F(x)^\dagger \arrow[r, "\langle \eta^{-1}_{F(x)} \rangle"] \arrow[d, "\langle \epsilon_{F(x)} \rangle"'] & F(x)^\dagger \bullet F(x) \bullet F(x') \arrow[d, "\epsilon_{F(x)} \bullet 1_{F(x')}"] \\
F(x)^\dagger \arrow[r, "l(x) \bullet 1_{F(x)^\dagger}"'] & F(x') \bullet F(x) \bullet F(x)^\dagger \arrow[r, "1_{F(x')} \bullet \eta^{-1}_{F(x)}"'] & F(x')
\end{tikzcd}$$

In the above diagram, we get the equality, $1_{F(x)} \bullet l(x) = r(x) \bullet 1_{F(x)}$, at the diagonal morphism from Proposition 4.2.6. All the squares commute because of functoriality of the tensor.

Now, the right vertical composite is precisely the top-right composite of the diagram (4.2.11) and the bottom horizontal composite is the left-bottom composite of the same diagram (4.2.11). It remains to show that the composition of inverse of the top arrow with the left arrow is identity. This follows from the cancellation condition (4.1.5). □

Definition 4.2.10. Define a natural transformation with components

$$F_1(x) : (Fx)^\dagger \Rightarrow F(x')$$

as follows:

- Let $x \in \mathrm{Obj}(\mathscr{M})$. The morphism

  $$F_1(x) : F(x)^\dagger \longrightarrow F(x')$$

  is defined as either of the following two equal composite:

  $$\begin{aligned} F_1(x) &:= \left(\epsilon_{F(x)} \bullet 1_{F(x')}\right) \circ \left(1_{F(x)^\dagger} \bullet r(x)\right) \\ &= \left(1_{F(x')} \bullet \eta^{-1}_{F(x)}\right) \circ \left(l(x) \bullet 1_{F(x)^\dagger}\right). \qquad (4.2.9) \end{aligned} \tag{4.2.11}$$

  This is demonstrated by the following diagram:

  $$\begin{tikzcd} F(x)^\dagger \arrow[r, "1_{F(x)^\dagger} \bullet r(x)"] \arrow[d, "l(x) \bullet 1_{F(x)^\dagger}"'] & F(x)^\dagger \bullet F(x) \bullet F(x') \arrow[d, "\epsilon_{F(x)} \bullet 1_{F(x')}"] \\ F(x') \bullet F(x) \bullet F(x)^\dagger \arrow[r, "1_{F(x')} \bullet \eta^{-1}_{F(x)}"'] & F(x'). \end{tikzcd}$$

  Since each morphism in the above composite is a natural transformation, it follows that F_1 is a natural transformation. ◇

Lemma 4.2.12. *Let $x \in \mathrm{Obj}(\mathscr{M})$. Then, the following diagram commutes.*

$$\begin{tikzcd} F(x)^\dagger \bullet F(x) \arrow[r, "\epsilon_{F(x)}"] \arrow[d, "F_1(x) \bullet 1_{F(x)}"'] & J \arrow[r, "F_0"] & F(I) \arrow[d, "F\left(\epsilon_x^{-1}\right)"] \\ F(x') \bullet F(x). & & F(x'x) \arrow[ll, "F_2(x', x)^{-1}"] \end{tikzcd} \tag{4.2.13}$$

Proof. First, note that the composition starting with F_0 at the top, with left and bottom morphism is

$$l(x): J \longrightarrow F(x') \bullet F(x). \tag{4.2.3}$$

Also, we have

$$F_1(x) = \left(1_{F(x')} \bullet \eta^{-1}_{F(x)}\right) \circ \left(l(x) \bullet 1_{F(x)^\dagger}\right). \tag{4.2.11}$$

Thus, we need to show the diagram

$$\begin{tikzcd}
F(x)^\dagger \bullet F(x) \arrow[r, "\epsilon_{F(x)}"] \arrow[d, "l(x) \bullet 1_{F(x)^\dagger} \bullet 1_{F(x)}"'] & J \arrow[d, "l(x)"] \\
F(x') \bullet F(x) \bullet F(x)^\dagger \bullet F(x) \arrow[r, "1_{F(x')} \bullet \eta^{-1}_{F(x)} \bullet 1_{F(x)}"'] & F(x') \bullet F(x)
\end{tikzcd}$$

commutes. From the cancellation condition (4.1.4) we get that the bottom morphism is equal to

$$1_{F(x')} \bullet 1_{F(x)} \bullet \epsilon_{F(x)} : F(x') \bullet F(x) \bullet F(x)^\dagger \bullet F(x) \longrightarrow F(x') \bullet F(x).$$

Thus, from the functoriality of the tensor, we know that the diagram commutes. □

Lemma 4.2.14. *Let $x \in \mathrm{Obj}(\mathcal{M})$. Then, the following diagram commutes.*

$$\begin{tikzcd}
J \arrow[r, "F_0"] \arrow[d, "\eta_{F(x)}"'] & F(I) \arrow[r, "F(\eta_x)"] & F(xx') \arrow[d, "F_2(x,x')^{-1}"'] \\
F(x) \bullet F(x)^\dagger \arrow[rr, "1_{F(x)} \bullet F_1(x)"'] & & F(x) \bullet F(x')
\end{tikzcd} \tag{4.2.15}$$

Proof. First, note that the composition of the top morphisms and the right morphism is

$$r(x): J \longrightarrow F(x) \bullet F(x'). \tag{4.2.5}$$

Also, we have

$$F_1(x) = \left(\epsilon_{F(x)} \bullet 1_{F(x')}\right) \circ \left(1_{F(x)^\dagger} \bullet r(x)\right). \tag{4.2.11}$$

Thus, we need to show that the diagram

$$\begin{tikzcd}
J \arrow[rr, "r(x)"] \arrow[d, "\eta_{F(x)}"'] & & F(x) \bullet F(x') \\
F(x) \bullet F(x)^\dagger \arrow[r, "1_{F(x)} \bullet 1_{F(x)^\dagger} \bullet r(x)"'] & F(x) \bullet F(x)^\dagger \bullet F(x) \bullet F(x') \arrow[ur, "1_{F(x)} \bullet \epsilon_{F(x)} \bullet 1_{F(x')}"'] &
\end{tikzcd}$$

commutes. From the cancellation condition (4.1.4) we get that the right vertical morphism is equal to

$$\eta^{-1}_{F(x)} \bullet 1_{F(x)} \bullet 1_{F(x')} : F(x) \bullet F(x)^\dagger \bullet F(x) \bullet F(x') \longrightarrow F(x) \bullet F(x').$$

Thus, from the functoriality of the tensor, we know that the diagram commutes. □

Theorem 4.2.16. *Suppose we are in the setting of Framework 4.2.1. Let $F_1 : F(-)^\dagger \Rightarrow F((-)')$ be the natural transformation defined in Definition 4.2.10. Then*

$$(F, F_0, F_1, F_2) : \mathscr{M} \longrightarrow \mathscr{N}$$

is a categorical group functor.

Proof. Lemmas 4.2.12 and 4.2.14 show the required left cancellation condition (4.1.23) and the right cancellation conditions (4.1.24). □

Theorem 4.2.17. *Let*

$$\mathscr{M} = (\mathscr{M}, I, (-)', \cdot) \qquad \text{and} \qquad \mathscr{N} = (\mathscr{N}, J, (-)^\dagger, \bullet)$$

be semi-strict categorical groups, and let

$$(F, F_0, F_1, F_2) : \mathscr{M} \longrightarrow \mathscr{N}$$

and

$$(G,G_0,G_1,G_2):\mathscr{M} \longrightarrow \mathscr{N}$$

be categorical group functors such that

$$F = G, \qquad F_0 = G_0, \qquad \text{and} \qquad F_2 = G_2.$$

Then, the equality of natural transformations

$$F_1 = G_1$$

holds.

Proof. It is enough to show that, for $x \in \mathrm{Obj}(\mathscr{M})$ we have

$$F_1(x) = G_1(x) : F(x)^\dagger \longrightarrow F(x').$$

Let

$$l_1(x) : J \longrightarrow F(x') \bullet F(x) \qquad \text{and} \qquad l_2(x) : J \longrightarrow G(x') \bullet G(x)$$

be as in Definition 4.2.2. Since $F = G$, $F_0 = G_0$, and $F_2 = G_2$, we get

$$l_1(x) = l_2(x).$$

From the left cancellation condition (4.1.23) of categorical groups, we know that the diagrams

$$\begin{tikzcd}
F(x)^\dagger \bullet F(x) \arrow[r, "\epsilon_{F(x)}"] \arrow[d, "F_1(x)\bullet 1_{F(x)}"'] & J \arrow[dl, "l_1(x)"] \\
F(x') \bullet F(x) &
\end{tikzcd}
\qquad
\begin{tikzcd}
G(x)^\dagger \bullet G(x) \arrow[r, "\epsilon_{G(x)}"] \arrow[d, "G_1(x)\bullet 1_{G(x)}"'] & J \arrow[dl, "l_2(x)"] \\
G(x') \bullet G(x) &
\end{tikzcd}$$

commute.

Now, consider the diagram

$$\begin{tikzcd}
F(x)^{\dagger} \arrow[r, "1_{F(x)^{\dagger}} \bullet \eta_{F(x)}"] \arrow[d, "F_1(x)"'] & F(x)^{\dagger} \bullet F(x) \bullet F(x)^{\dagger} \arrow[r, "\epsilon_{F(x)} \bullet 1_{F(x)^{\dagger}}"] \arrow[d, "F_1(x) \bullet 1_{F(x)} \bullet 1_{F(x)^{\dagger}}"'] & F(x)^{\dagger} \arrow[dl, "l_1(x)"] \\
F(x') \arrow[r, "1_{F(x')} \bullet \eta_{F(x)}"'] & F(x') \bullet F(x) \bullet F(x)^{\dagger} &
\end{tikzcd}$$

Here, from the functoriality of the tensor, the left square commutes. Whereas, the triangle commutes from the previous diagram. From the cancellation condition (4.1.5), we get that the top horizontal composite is equal to

$$1_{F(x)^{\dagger}} : F(x)^{\dagger} \longrightarrow F(x)^{\dagger}.$$

Thus, we have

$$F_1(x) = \big(1_{F(x')} \bullet \eta_{F(x)}\big) \circ l_1(x).$$

Similarly, we get

$$G_1(x) = \big(1_{G(x')} \bullet \eta_{G(x)}\big) \circ l_2(x).$$

Finally, since $F = G$ and $l_1 = l_2$ we get

$$F_1(x) = G_1(x)$$

as required. □

Lemma 4.2.18. *Let $\mathscr{M} = (\mathscr{M}, I, (-)', \cdot)$ a be semi-strict categorical group. Let $x, y \in$ Obj$(\mathscr{M})$ and $f : x \longrightarrow x$ be a morphism. Suppose that*

$$f 1_y = 1_{xy} : xy \longrightarrow xy$$

then, we have

$$f = 1_x : x \longrightarrow x.$$

Proof. Consider the following diagram:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "1_x\eta_y"'] & x \arrow[d, "1_x\eta_y"] \\ xyy' \arrow[r, "f1_{yy'}"'] & xyy' \end{tikzcd}$$

This diagram commutes due to the functoriality of the multiplication. Observe that

$$\begin{aligned} f1_{yy'} &= f1_y1_{y'} \\ &= 1_{xy}1_{y'} && \text{by assumption} \\ &= 1_{xyy'}. \end{aligned}$$

Here, the first and the third equalities hold since $\mathscr{M}$ has strict monoidal structure. Therefore, we get

$$f = 1_x$$

as required. $\square$

Theorem 4.2.19. *Let*

$$\mathscr{M} = (\mathscr{M}, I, (-)', \cdot) \qquad \text{and} \qquad \mathscr{N} = (\mathscr{N}, J, (-)^{\dagger}, \bullet)$$

be semi-strict categorical groups, and

$$(F, F_0, F_1, F_2) : \mathscr{M} \longrightarrow \mathscr{N}$$

be categorical group functor such that the underlying monoidal functor $(F, F_0, F_2) : \mathscr{M} \longrightarrow \mathscr{N}$ is a strict monoidal functor. Suppose we have

$$F(x)^{\dagger} = F(x')$$

and

$$F(\epsilon_x) = \epsilon_{F(x)} : F(x)^\dagger \bullet F(x) \longrightarrow J$$

for every $x \in \mathrm{Obj}(\mathscr{M})$. *Then, we get*

$$F_1(x) = 1_{F(x)^\dagger} : F(x)\dagger \longrightarrow F(x')$$

for every $x \in \mathrm{Obj}(\mathscr{M})$. *Consequently,* F *is a strict categorical group functor.*

Proof. Let $x \in \mathrm{Obj}(\mathscr{M})$. By assumption, $F(\epsilon_x) = \epsilon_{F(x)}$ and $F(x)^\dagger = F(x')$. Since F is a strict monoidal functor, the left cancellation condition (4.1.23) gives

$$F_1(x) \bullet 1_{F(x)} = 1_{F(x)^\dagger \bullet F(x)} : F(x)^\dagger \bullet F(x) \longrightarrow F(x') \bullet F(x).$$

Applying Lemma 4.2.18, it follows that

$$F_1(x) = 1_{F(x)^\dagger} : F(x)^\dagger \longrightarrow F(x').$$

Therefore, F is a strict categorical group functor, since its underlying monoidal functor is strict and $F_1(x)$ is the identity for all x. □

Theorem 4.2.20. *Let* $\mathscr{M} = (\mathscr{M}, I, (-)', \cdot)$ *and* $\mathscr{N} = (\mathscr{N}, J, (-)^\dagger, \bullet)$ *be semi-strict categorical groups, and* $F, G : \mathscr{M} \longrightarrow \mathscr{N}$ *be categorical group functors. Let*

$$\phi : F \Rightarrow G$$

be a monoidal natural transformation. Then, ϕ *satisfies the negator condition (4.1.36). Consequently,* $\phi : F \Rightarrow G$ *is a categorical group natural transformation.*

Proof. For $x \in \mathrm{Obj}(M)$, we want to show that the diagram

$$\begin{tikzcd}
F(x)^\dagger \arrow[r, "F_1(x)"] \arrow[d, "\phi_x^\dagger"'] & F(x') \arrow[d, "\phi_{x'}"] \\
G(x)^\dagger \arrow[r, "G_1(x)"'] & G(x')
\end{tikzcd}$$

commutes. From Theorems 4.2.16 and 4.2.17 we know that the natural transformation F_1 of a categorical group functor is uniquely determined by the underlying monoidal functor. Thus, the morphism $F_1(x) : F(x)^\dagger \longrightarrow F(x')$ is equal to the morphism

$$F_1(x) = \left(\epsilon_{F(x)} \bullet 1_{F(x')}\right) \circ \left(1_{F(x)^\dagger} \bullet r(x)\right)$$

Now, consider the diagram

$$\begin{tikzcd}
F(x)^\dagger \arrow[r, "\phi(x)^\dagger"] \arrow[d, "1_{F(X)^\dagger} \bullet F_0"'] & G(X)^\dagger \arrow[r, equal] \arrow[d, "1_{G(x)^\dagger} \bullet F_0"] & G(x)^\dagger \arrow[d, "1_{G(x)^\dagger} \bullet F_0"] \\
F(x)^\dagger \bullet F(I) \arrow[r, "\left\langle \phi(x)^\dagger \right\rangle"'] \arrow[d, "\langle F(\eta_x) \rangle"'] & G(x)^\dagger \bullet F(I) \arrow[r, "\langle \phi(I) \rangle"'] \arrow[d, "\langle F(\eta_x) \rangle"] & G(x)^\dagger \bullet G(I). \arrow[d, "\langle G(\eta_x) \rangle"] \\
F(x)^\dagger \bullet F(xx') \arrow[r, "\left\langle \phi(x)^\dagger \right\rangle"'] \arrow[d, "\langle F_2(x,x')^{-1} \rangle"'] & G(x)^\dagger \bullet F(xx') \arrow[r, "\langle \phi(xx') \rangle"'] \arrow[d, "\langle F_2(x,x')^{-1} \rangle"] & G(x)^\dagger \bullet G(xx') \arrow[d, "\langle G_2(x,x')^{-1} \rangle"] \\
F(x)^\dagger \bullet F(x) \bullet F(x') \arrow[r, "\left\langle \phi(x)^\dagger \right\rangle"'] \arrow[d, "\langle \epsilon_{F(x)} \rangle"'] & G(x)^\dagger \bullet F(x) \bullet F(x') \arrow[r, "\langle \phi(x) \bullet \phi(x') \rangle"'] & G(x)^\dagger \bullet G(x) \bullet G(x') \arrow[d, "\langle \epsilon_{G(x)} \rangle"] \\
F(x') \arrow[rr, "\phi(x')"'] & & G(x')
\end{tikzcd} \tag{i}$$

Here, the squares on the left (not considering the bottom rectangle) commute because of functoriality of the tensor. The top right square commutes from the unit condition (3.1.26) of the monoidal natural transformation ϕ. From the naturality of

ϕ, we get that the middle right square (below the top square) commutes. The bottom right square (below the middle right) commutes from the tensor condition (3.1.27) of the monoidal natural transformation ϕ.

The left and right vertical composites are precisely the morphisms

$$F_1(x)\colon F(x)^\dagger \longrightarrow F(x') \qquad \text{and} \qquad G_1(x)\colon G(x)^\dagger \longrightarrow G(x')$$

respectively, as discussed earlier. Now, it remains to show that the bottom rectangle commutes.

Note that, from the functoriality of the tensor, we see that the composite

$$F(x)^\dagger \bullet F(x) \bullet F(x') \xrightarrow{\left\langle \phi(x)^\dagger \right\rangle} G(x)^\dagger \bullet F(x) \bullet F(x') \xrightarrow{\langle \phi(x) \bullet \phi(x') \rangle} G(x)^\dagger \bullet G(x) \bullet G(x')$$

is equal to the composite

$$F(x)^\dagger \bullet F(x) \bullet F(x') \xrightarrow{\left\langle \phi(x)^\dagger \bullet \phi(x) \right\rangle} G(x)^\dagger \bullet G(x) \bullet F(x') \xrightarrow{\langle \phi(x') \rangle} G(x)^\dagger \bullet G(x) \bullet G(x').$$

Thus, the bottom rectangle of the diagram (i), after flipping it vertically, becomes

$$\begin{tikzcd}
F(x)^\dagger \bullet F(x) \bullet F(x') \arrow[r, "\epsilon_{F(x)} \bullet 1_{F(x')}"] \arrow[d, "\phi(x)^\dagger \bullet \phi(x) \bullet 1_{F(x')}"'] & F(x') \arrow[d, equal] \\
G(x)^\dagger \bullet G(x) \bullet F(x') \arrow[r, "\epsilon_{G(x)} \bullet 1_{F(x')}"'] \arrow[d, "1_{G(x)^\dagger} \bullet 1_{G(x)} \bullet \phi(x)"'] & F(x') \arrow[d, "\phi(x)"] \\
G(x)^\dagger \bullet G(x) \bullet G(x') \arrow[r, "\epsilon_{G(x)} \bullet 1_{G(x')}"'] & G(x)'.
\end{tikzcd}$$

Here, the top square commutes from the naturality of the cancellation natural transformation ϵ and the bottom square commutes from the functoriality of the tensor. This completes the proof. □

4.3 Equivalence of Categorical Groups and Semi-Strict Categorical Groups

The aim of this section is to show that every categorical group is categorically equivalent to a semi-strict categorical group via a special categorical group equivalence. We proceed in two main steps. First, by applying Mac Lane's coherence theorem for monoidal categories, we establish that every categorical group is monoidally equivalent to a strict monoidal category through suitable monoidal functors. Second, we use this monoidal equivalence to endow the strict monoidal category with a categorical group structure, thereby upgrading the monoidal equivalence to a categorical group equivalence. We begin by demonstrating how a monoidal equivalence between a categorical group and a strict monoidal category allows us to promote the strict monoidal category to a semi-strict categorical group.

Framework 4.3.1. Throughout this section, let

$$\mathscr{M} = (\mathscr{M}, I, \otimes, (-)', \alpha, \lambda, \rho, \epsilon, \eta)$$

be a categorical group, and

$$\mathscr{N} = (\mathscr{N}, J, \bullet)$$

be a strict monoidal category. Assume that (F, G, ϕ, ψ) is a special monoidal equivalence from $\mathscr{M}$ to $\mathscr{N}$ as described in Definition 3.1.32. ⋄

Definition 4.3.2. Define a functor $(-)^\dagger : \mathscr{N} \longrightarrow \mathscr{N}$ as follows:

$$a^\dagger := G\left((F(a))'\right) \qquad \text{on objects} \qquad (4.3.3)$$

$$f^\dagger = G\left((F(f))'\right) \qquad \text{on morphisms.} \qquad (4.3.4)$$

Since $(-)^\dagger$ is a composition of functors, it is a functor.

Define a natural isomorphism $\hat{\epsilon}$, with components $\hat{\epsilon}_a : a^\dagger \bullet a \longrightarrow J$, as follows:

$$\widehat{\epsilon} := \left(\hat{\epsilon}_a := G(\epsilon_{F(a)}) \circ \left(1_{a^\dagger} \bullet \phi_a^{-1}\right) : a^\dagger \bullet a \longrightarrow J\right)_{a \in \mathcal{N}}. \tag{4.3.5}$$

That is,

$$\begin{tikzcd}
\hat{\epsilon}_a : a^\dagger \bullet a \arrow[r, "1_{a^\dagger} \bullet \phi_a^{-1}"] & a^\dagger \bullet GF(a) \arrow[r, equal, "(4.3.3)"] & G(F(a)') \bullet GF(a) \arrow[d, equal, "(3.1.15)"] \\
J \arrow[r, equal, "(3.1.14)"'] & G(I) & G\left(F(a)' \otimes F(a)\right). \arrow[l, "G\left(\epsilon_{F(a)}\right)"]
\end{tikzcd}$$

Since $\hat{\epsilon}$ is a composition and whiskering of natural isomorphisms, it is a natural isomorphism.

Define a natural isomorphism $\hat{\eta}$, with components $\hat{\eta}_a : J \longrightarrow a \bullet a^\dagger$, as follows:

$$\widehat{\eta} := \left(\hat{\eta}_a := \left(\phi_a \bullet 1_{a^\dagger}\right) \circ G(\eta_{F(a)}) : J \longrightarrow a \bullet a^\dagger\right)_{a \in \mathcal{N}}. \tag{4.3.6}$$

That is,

$$\begin{tikzcd}
\hat{\eta}_a : J \arrow[r, equal, "(3.1.14)"] & G(I) \arrow[r, "G\left(\eta_{F(a)}\right)"] & G\left(F(a) \otimes F(a)'\right) \arrow[d, equal, "(3.1.15)"] \\
a \bullet a^\dagger & GF(a) \bullet a^\dagger \arrow[l, "\phi_a \bullet 1_{a^\dagger}"'] & GF(a) \bullet G(F(a)'). \arrow[l, equal, "(4.3.3)"']
\end{tikzcd}$$

Since $\hat{\eta}$ is a composition and whiskering of natural isomorphisms, it is a natural isomorphism. $\diamond$

Lemma 4.3.7. *We have the following equalities:*

$$J^\dagger = J, \tag{4.3.8}$$

and

$$\hat{\epsilon}_J = 1_J. \tag{4.3.9}$$

Proof. Observe that

$$
\begin{aligned}
J^\dagger &= G\left(F(J)'\right) && (4.3.3)\\
&= G(I') && (3.1.33)\\
&= G(I) && (4.1.2)\\
&= J. && (3.1.14)
\end{aligned}
$$

Also, we get

$$
\begin{aligned}
\hat{\epsilon}_J &= G(\epsilon_{F(J)}) \circ \left(1_{J^\dagger} \bullet \phi_J^{-1}\right) && (4.3.5)\\
&= G(\epsilon_{F(J)}) \circ \left(1_{J^\dagger} \bullet \phi_{G(I)}^{-1}\right) && (3.1.33)\\
&= G(\epsilon_{F(J)}) && (3.1.30) \text{ and } (3.1.34)\\
&= G(\epsilon_I) && (3.1.33)\\
&= G(\rho_I) && (4.1.3)\\
&= 1_{G(I)} && (3.1.42)\\
&= 1_J. && (3.1.14)
\end{aligned}
$$

□

Lemma 4.3.10. *For $a \in \mathrm{Obj}(\mathcal{N})$, the equality*

$$(1_a \bullet \hat{\epsilon}_a) \circ (\hat{\eta}_a \bullet 1_a) = 1_a \tag{4.3.11}$$

holds. In other words, the composition

$$a \xrightarrow{\hat{\eta}_a \bullet 1_a} a \bullet a^\dagger \bullet a \xrightarrow{1_a \bullet \hat{\epsilon}_a} a$$

is equal to the identity morphism, 1_a.

Proof. Let $f : F(a) \longrightarrow (F(a) \otimes F(a)') \otimes F(a)$ be the following morphism in $\mathscr{M}$:

$$f := \left(\eta_{F(a)} \otimes 1_{F(a)}\right) \circ \lambda_{F(a)}^{-1}. \tag{i}$$

That is,

$$f : F(a) \xrightarrow{\lambda_{F(a)}^{-1}} I \otimes F(a) \xrightarrow{\eta_{F(a)} \otimes 1_{F(a)}} (F(a) \otimes F(a)') \otimes F(a)$$

Let $g : (F(a) \otimes F(a)') \otimes F(a) \longrightarrow F(a)$ be the following morphism in $\mathscr{M}$:

$$g := \rho_{F(a)} \circ \left(1_{F(a)} \bullet \epsilon_{F(a)}\right) \circ \alpha_{F(a),F(a)',F(a)}^{-1}. \tag{ii}$$

That is,

$$\begin{tikzcd}
g : (F(a) \otimes F(a)') \otimes F(a) \arrow[r, "\alpha_{F(a),F(a)',F(a)}^{-1}"] & F(a) \otimes (F(a)' \otimes F(a)) \arrow[d, "1_{F(a)} \cdot \epsilon_{F(a)}"] \\
F(a) & F(a) \otimes I \arrow[l, "\rho_{F(a)}"']
\end{tikzcd}$$

From (4.1.4) we get that

$$g \circ f = 1_{F(a)}. \tag{iii}$$

Now we will do some calculations:

$$\begin{aligned}
G(f) &= G\left(\left(\eta_{F(a)} \otimes 1_{F(a)}\right) \circ \lambda_{F(a)}^{-1}\right) && (i) \\
&= G\left(\eta_{F(a)} \otimes 1_{F(a)}\right) \circ G\left(\lambda_{F(a)}^{-1}\right) && G \text{ is a functor} \\
&= \left(G\left(\eta_{F(a)}\right) \bullet G\left(1_{F(a)}\right)\right) \circ G\left(\lambda_{F(a)}^{-1}\right) \\
&= \left(G\left(\eta_{F(a)}\right) \bullet 1_{GF(a)}\right) \circ G\left(\lambda_{F(a)}\right)^{-1} && G \text{ is a functor} \\
&= G\left(\eta_{F(a)}\right) \bullet 1_{GF(a)}. && (3.1.41)
\end{aligned} \tag{iv}$$

The third equality follows from the fact that G is a strict monoidal functor, this in turn follows from the fact that (F, G, ϕ, ψ) is a special monoidal equivalence. Also, we have

$$
\begin{aligned}
G(g) &= G\left(\rho_{F(a)} \circ \left(1_{F(a)} \otimes \epsilon_{F(a)}\right) \circ \alpha^{-1}_{F(a),F(a)',F(a)}\right) && (ii) \qquad \text{(v)}\\
&= G\left(\rho_{F(a)}\right) \circ G\left(1_{F(a)} \otimes \epsilon_{F(a)}\right) \circ G\left(\alpha^{-1}_{F(a),F(a)',F(a)}\right) && G \text{ is a functor}\\
&= G\left(1_{F(a)} \otimes \epsilon_{F(a)}\right) && (3.1.40) \text{ and } (3.1.42)\\
&= G\left(1_{F(a)}\right) \bullet G\left(\epsilon_{F(a)}\right) &&\\
&= 1_{GF(a)} \bullet G\left(\epsilon_{F(a)}\right). && G \text{ is a functor}
\end{aligned}
$$

The fourth equality follows from the fact that G is a strict monoidal functor. Next, we have

$$
\begin{aligned}
1_a \bullet \hat{\epsilon}_a &= 1_a \bullet \left(G(\epsilon_{F(a)}) \circ 1_{a^\dagger} \bullet \phi_a^{-1}\right) && (4.3.5) \qquad \text{(vi)}\\
&= \left(\phi_a \circ 1_{GF(a)} \circ \phi_a^{-1}\right) \bullet \left(1_J \circ G(\epsilon_{F(a)}) \circ 1_{a^\dagger} \bullet \phi_a^{-1}\right) &&\\
&= \left(\phi_a \bullet 1_J\right) \circ \left(1_{GF(a)} \bullet G(\epsilon_{F(a)})\right) \circ \left(\phi_a^{-1} \bullet 1_{a^\dagger} \bullet \phi_a^{-1}\right) &&\\
&= \phi_a \circ G(g) \circ \left(\phi_a^{-1} \bullet 1_{a^\dagger} \bullet \phi_a^{-1}\right). && (3.1.38) \text{ and } (v)
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
\hat{\eta}_a \bullet 1_a &= \left(\phi_a \bullet 1_{a^\dagger} \circ G(\eta_{F(a)})\right) \circ 1_a && (4.3.6) \qquad \text{(vii)}\\
&= \left(\phi_a \bullet 1_{a^\dagger} \circ G(\eta_{F(a)}) \circ 1_J\right) \bullet \left(\phi_a \circ 1_{GF(a)} \circ \phi_a^{-1}\right) &&\\
&= \left(\phi_a \bullet 1_{a^\dagger} \bullet \phi_a\right) \circ \left(G(\eta_{F(a)}) \bullet 1_{GF(a)}\right) \circ \left(1_J \bullet \phi_a^{-1}\right) &&\\
&= \left(\phi_a \bullet 1_{a^\dagger} \bullet \phi_a\right) \circ G(f) \circ \phi_a^{-1}. && (3.1.37) \text{ and } (iv)
\end{aligned}
$$

Combining above we get,

$$\begin{aligned}(1_a \bullet \hat{\varepsilon}_a) \circ (\hat{\eta}_a \bullet 1_a) &= \phi_a \circ G(g) \circ \left(\phi_a^{-1} \bullet 1_{a^\dagger} \bullet \phi_a^{-1}\right) \circ \\ &\quad \left(\phi_a \bullet 1_{a^\dagger} \bullet \phi_a\right) \circ G(f) \circ \phi_a^{-1} && \text{(vi) and (vii)} \\ &= \phi_a \circ G(g) \circ G(f) \circ \phi_a^{-1} \\ &= \phi_a \circ G(g \circ f) \circ \phi_a^{-1} \\ &= \phi_a \circ \phi_a^{-1} && (iii) \\ &= 1_a \end{aligned}$$

as required. □

Lemma 4.3.12. *For $a \in \mathrm{Obj}(\mathscr{N})$, the equality*

$$(\hat{\varepsilon}_a \bullet 1_{a^\dagger}) \circ (1_{a^\dagger} \bullet \hat{\eta}_a) = 1_{a^\dagger} \tag{4.3.13}$$

holds. In other words, the composition

$$a^\dagger \xrightarrow{1_{a^\dagger} \bullet \hat{\eta}_a} a^\dagger \bullet a \bullet a^\dagger \xrightarrow{\hat{\varepsilon}_a \bullet 1_{a^\dagger}} a$$

is equal to the identity morphism, $1_{a^\dagger}$.

Proof. Let $f : F(a)' \longrightarrow (F(a)' \otimes F(a)) \otimes F(a)'$ be the following morphism in $\mathscr{M}$:

$$f := \alpha_{F(a)',F(a),F(a)'} \circ \left(1_{F(a)'} \otimes \eta_{F(a)}\right) \circ \rho^{-1}_{F(a)'}. \tag{i}$$

That is,

$$\begin{tikzcd} f : F(a)' \arrow[rr, "\rho^{-1}_{F(a)'}"] & & F(a)' \otimes I \arrow[d, "1_{F(a)'} \otimes \eta_{F(a)}"] \\ (F(a) \otimes F(a)') \otimes F(a) & & F(a)' \otimes (F(a) \otimes F(a)') \arrow[ll, "\alpha_{F(a)',F(a),F(a)'}"'] \end{tikzcd}$$

Let $g : (F(a)' \otimes F(a)) \otimes F(a)' \longrightarrow F(a)'$ be the following morphism in $\mathscr{M}$:

$$g := \lambda_{F(a)'} \circ \left(\epsilon_{F(a)} \otimes 1_{F(a)'}\right). \tag{ii}$$

That is, the composition

$$g : (F(a)' \otimes F(a)) \otimes F(a)' \xrightarrow{\epsilon_{F(a)} \otimes 1_{F(a)'}} I \otimes F(a)' \xrightarrow{\lambda_{F(a)'}} F(a)'$$

is equal to g. From (4.1.5) we get that

$$g \circ f = 1_{F(a)'}. \tag{iii}$$

Now, we will do some calculations:

$$\begin{aligned}
G(f) &= G\left(\alpha_{F(a)',F(a),F(a)'} \circ \left(1_{F(a)'} \otimes \eta_{F(a)}\right) \circ \rho^{-1}_{F(a)'}\right) && (i) \\
&= G\left(1_{F(a)'} \otimes \eta_{F(a)}\right) && (3.1.40) \text{ and } (3.1.42) \\
&= G\left(1_{F(a)'}\right) \bullet G\left(\eta_{F(a)}\right) \\
&= 1_{G(F(a)')} \bullet G\left(\eta_{F(a)}\right) && G \text{ is a functor} \\
&= 1_{a^\dagger} \bullet G(\eta_{F(a)}) && (4.3.3)
\end{aligned} \tag{iv}$$

The third equality follows from the fact that G is a strict monoidal functor, this in turn follows from the fact that (F, G, ϕ, ψ) is a special monoidal equivalence. Also, we have

$$\begin{aligned}
G(g) &= G\left(\lambda_{F(a)'} \circ \left(\epsilon_{F(a)} \otimes 1_{F(a)'}\right)\right) && (ii) \\
&= G\left(\epsilon_{F(a)} \otimes 1_{F(a)'}\right) && (3.1.41) \\
&= G\left(\epsilon_{F(a)}\right) \bullet G\left(1_{F(a)'}\right) \\
&= G\left(\epsilon_{F(a)}\right) \bullet 1_{G(F(a)')} && G \text{ is a functor} \\
&= G\left(\epsilon_{F(a)}\right) \bullet 1_{a^\dagger}. && (4.3.3)
\end{aligned} \tag{v}$$

The third equality follows from the fact that G is a strict monoidal functor. Next, we have

$$\begin{aligned}
1_{a^\dagger} \bullet \hat{\eta}_a &= 1_{a^\dagger} \bullet \left(\phi_a \bullet 1_{a^\dagger} \circ G(\eta_{F(a)})\right) && (4.3.6) \\
&= \left(1_{a^\dagger} \circ 1_{a^\dagger}\right) \bullet \left(\phi_a \bullet 1_{a^\dagger} \circ G(\eta_{F(a)})\right) \\
&= \left(1_{a^\dagger} \bullet \phi_a \bullet 1_{a^\dagger}\right) \circ \left(1_{a^\dagger} \bullet G(\eta_{F(a)})\right) \\
&= \left(1_{a^\dagger} \bullet \phi_a \bullet 1_{a^\dagger}\right) \circ G(f). && (iv)
\end{aligned} \tag{vi}$$

Similarly, we get

$$\begin{aligned}
\hat{\epsilon}_a \bullet 1_{a^\dagger} &= \left(G(\epsilon_{F(a)}) \circ 1_{a^\dagger} \bullet \phi_a^{-1}\right) \bullet 1_{a^\dagger} && (4.3.5) \\
&= \left(G(\epsilon_{F(a)}) \circ 1_{a^\dagger} \bullet \phi_a^{-1}\right) \bullet \left(1_{a^\dagger} \circ 1_{a^\dagger}\right) \\
&= \left(G(\epsilon_{F(a)}) \bullet 1_{a^\dagger}\right) \circ \left(1_{a^\dagger} \bullet \phi_a^{-1} \bullet 1_{a^\dagger}\right) \\
&= G(g) \circ \left(1_{a^\dagger} \bullet \phi_a^{-1} \bullet 1_{a^\dagger}\right). && (v)
\end{aligned} \tag{vii}$$

Combining above we get,

$$\begin{aligned}
(\hat{\epsilon}_a \bullet 1_{a^\dagger}) \circ (1_{a^\dagger} \bullet \hat{\eta}_a) &= G(g) \circ \left(1_{a^\dagger} \bullet \phi_a^{-1} \bullet 1_{a^\dagger}\right) \circ \\
&\qquad \left(1_{a^\dagger} \bullet \phi_a \bullet 1_{a^\dagger}\right) \circ G(f) && \text{(vi) and (vii)} \\
&= G(g) \circ G(f) \\
&= G(g \circ f) \\
&= G(1_{F(a)'}) && (iii) \\
&= 1_{G(F(a)')} \\
&= 1_{a^\dagger} && (4.3.3)
\end{aligned}$$

as required. □

Construction 4.3.14. The strict monoidal category $\mathscr{N}$ can be equipped with a structure of semi-strict categorical group as follows:

- Let $\mathscr{N}$ be the underlying monoidal category. Since $\mathscr{M}$ is a categorical group, it is a groupoid. The special monoidal equivalence gives a categorical equivalence between $\mathscr{M}$ and $\mathscr{N}$. Thus, $\mathscr{N}$ is also a groupoid.

- The functor

$$(-)^{\dagger} : \mathscr{N} \longrightarrow \mathscr{N}$$

as in Definition 4.3.2 is the negator functor.

- The natural isomorphisms

$$\hat{\epsilon} \qquad \text{and} \qquad \hat{\eta}$$

as in Definition 4.3.2 are the cancellation isomorphisms.

Above assignements satisfy the unit condition from Lemma 4.3.7 and the cancellation conditions from Lemmas 4.3.10 and 4.3.12. ◇

Framework 4.3.15. We have shown that a strict monoidal category can be elevated to a semi-strict categorical group using a special monoidal equivalence (see Definition 3.1.32). Next, we will show that the special monoidal equivalence itself can be upgraded to a special categorical group equivalence (see Definition 4.1.40).

Now, for the rest of this section let

$$\mathscr{N} = (\mathscr{N}, J, (-)^{\dagger}, \bullet, \hat{\epsilon}, \hat{\eta})$$

be the induced semi-strict categorical group as in Construction 4.3.14. ◇

Definition 4.3.16. Define a natural isomorphism G_1, with components

$$G_1(x) : G(x)^\dagger \longrightarrow G(x')$$

as

$$G_1(x) := 1_{G(x)^\dagger} : G(x)^\dagger \longrightarrow G(x'). \tag{4.3.17}$$

That is, G_1 is the identity natural transformation. This assignement is well-defined since we have

$$\begin{aligned} (G(x))^\dagger &= G\left((FG(x))'\right) && \text{Definition 4.3.2} \\ &= G(x'). && (3.1.34) \end{aligned}$$

◇

Lemma 4.3.18. *For $x \in \mathrm{Obj}(\mathcal{M})$ the diagram*

$$\begin{tikzcd} G(x)^\dagger \bullet G(x) \arrow[r, "\hat{\epsilon}_{G(x)}"] \arrow[d, "G_1(x) \bullet 1_{G(x)}"'] & J \arrow[r, "G_0"] & G(I) \\ G(x') \bullet G(x) \arrow[rr, "G_2(x', x)"'] & & G(x' \otimes x) \arrow[u, "G(\epsilon_x)"'] \end{tikzcd} \tag{4.3.19}$$

commutes.

Proof. The special monoidal equivalence (F, G, ϕ, ψ) ensures that G is a strict monoidal functor. Consequently, G_0, $G_1(x)$, and $G_2(x, y)$ are identity morphisms for $x, y \in \mathrm{Obj}(\mathcal{M})$. Thus, it remains to show that

$$\hat{\epsilon}_{G(x)} = G(\epsilon_x).$$

Let $x \in \mathrm{Obj}(\mathscr{M})$. We get

$$\begin{aligned}
\hat{\epsilon}_{G(x)} &= G(\epsilon_{FG(x)}) \circ \left(1_{G(x)^\dagger} \bullet \phi^{-1}_{G(x)}\right) && (4.3.5)\\
&= G(\epsilon_{FG(x)}) \circ \left(1_{G(x)^\dagger} \bullet (G(\psi_x) \circ 1_{G(x)})\right) && (3.1.34)\\
&= G(\epsilon_{FG(x)}) \circ \left(1_{G(x)^\dagger} \bullet 1_{G(x)}\right) && (3.1.30)\\
&= G(\epsilon_x). && (3.1.34)
\end{aligned}$$

This completes the proof. □

Lemma 4.3.20. *For $x \in \mathrm{Obj}(\mathscr{M})$, the diagram*

$$\begin{tikzcd}
J \arrow[r, "G_0"] \arrow[d, "\hat{\eta}_{G(x)}"'] & G(I) \arrow[r, "G(\eta_x)"] & G(x \otimes x') \\
G(x) \bullet G(x)^\dagger \arrow[rr, "1_{G(x)} \bullet G_1(x)"'] & & G(x) \bullet G(x') \arrow[u, "G_2(x,x')"']
\end{tikzcd} \tag{4.3.21}$$

commutes.

Proof. The special monoidal equivalence (F, G, ϕ, ψ) ensures that G is a strict monoidal functor. Consequently, G_0, $G_1(x)$, and $G_2(x, y)$ are identity morphisms for $x, y \in \mathrm{Obj}(\mathscr{M})$. Thus, it remains to show that

$$\hat{\eta}_{G(x)} = G(\eta_x).$$

Let $x \in \mathrm{Obj}(\mathscr{M})$. We get

$$\begin{aligned}
\hat{\eta}_{G(x)} &= \left(\phi_{G(x)} \bullet 1_{G(x)^\dagger}\right) \circ G(\eta_{FG(x)}) && (4.3.6)\\
&= \left((\phi_{G(x)} \circ G(\psi_x^{-1})) \bullet 1_{G(x)^\dagger}\right) \circ G(\eta_{FG(x)}) && (3.1.34)\\
&= \left(1_{G(x)} \bullet 1_{G(x)^\dagger}\right) \circ G(\eta_{FG(x)}) && (3.1.30)\\
&= G(\eta_x). && (3.1.34)
\end{aligned}$$

This completes the proof. □

Definition 4.3.22. Define a natural isomorphism F_1, with components

$$F_1(a): F(a)' \longrightarrow F(a^\dagger)$$

as

$$F_1(a) = 1_{F(a)'} : F(a)' \longrightarrow F(a^\dagger). \tag{4.3.23}$$

That is, F_1 is the identity natural transformation. This assignements is well-defined since we have

$$\begin{aligned} (F(a))' &= FG((Fa)') && (3.1.34) \\ &= F(a^\dagger). && (4.3.3) \end{aligned}$$
◇

Lemma 4.3.24. *For $a \in \mathrm{Obj}(\mathcal{N})$, the diagram*

$$\begin{tikzcd} F(a)' \otimes F(a) \arrow[r, "\epsilon_{F(a)}"] \arrow[d, "F_1(a) \otimes 1_{F(a)}"'] & I \arrow[r, "F_0"] & F(J) \\ F(a^\dagger) \otimes F(a) \arrow[rr, "F_2(a^\dagger, a)"'] & & F(a^\dagger \bullet a) \arrow[u, "F(\hat{\epsilon}_a)"'] \end{tikzcd} \tag{4.3.25}$$

commutes.

Proof. Given that (F, G, ϕ, ψ) is a special monoidal equivalence, equation (3.1.33) implies that $F_0 : I \longrightarrow F(J)$ is the identity morphism. From equation (4.3.23), we conclude that $F_1(a)$ is the identity morphism for $a \in \mathrm{Obj}(\mathcal{N})$. Thus, it remains to show

$$\epsilon_{F(a)} = F(\hat{\epsilon}_a) \circ F_2(a^\dagger, a).$$

The naturality of F_2 ensures that the diagram

$$\begin{tikzcd} F(a^\dagger) \otimes F(a) \arrow[r, "F_2(a^\dagger, a)"] \arrow[d, "F(1_{a^\dagger}) \otimes F(\phi_a^{-1})"'] & F(a^\dagger \bullet a) \arrow[d, "F(1_{a^\dagger} \bullet \phi_a^{-1})"] \\ F(a^\dagger) \otimes FGF(a) \arrow[r, "F_2(a^\dagger, GF(a))"'] & F(a^\dagger \bullet GF(a)) \end{tikzcd}$$

commutes. Thus, we get

$$
\begin{aligned}
& F\left(1_{a^\dagger} \bullet \phi_a^{-1}\right) \circ F_2(a^\dagger, a) && \text{(i)} \\
&= F_2(a^\dagger, GF(a)) \circ \left(F(1_{a^\dagger}) \otimes F(\phi_a^{-1})\right) && \text{from above} \\
&= F_2(a^\dagger, GF(a)) \circ \left(F(1_{a^\dagger}) \otimes (\psi_{F(a)} \circ F(\phi_a^{-1}))\right) && (3.1.34) \\
&= F_2(a^\dagger, GF(a)) \circ (1_{F(a^\dagger)} \otimes 1_{F(a)}) && (3.1.31) \\
&= F_2(G(F(a)'), GF(a)) && (4.3.3) \\
&= F(G_2((Fa)', Fa)) \circ F_2(G(F(a)'), GF(a)) && (3.1.15) \\
&= (FG)_2(F(a)', F(a)) && (3.1.24) \\
&= 1_{F(a)' \otimes F(a)}. && (3.1.34)
\end{aligned}
$$

Here, the fourth equality follows from the fact that G is a strict monoidal functor. From above, we get

$$
\begin{aligned}
F(\hat{\epsilon}_a) \circ F_2(a^\dagger, a) &= FG(\epsilon_{F(a)}) \circ F\left(1_{a^\dagger} \bullet \phi_a^{-1}\right) \circ F_2(a^\dagger, a) && (4.3.5) \\
&= \epsilon_{F(a)} \circ F\left(1_{a^\dagger} \bullet \phi_a^{-1}\right) \circ F_2(a^\dagger, a) && (3.1.34) \\
&= \epsilon_{F(a)} && (i)
\end{aligned}
$$

as required. □

Lemma 4.3.26. *For $a \in \mathrm{Obj}(\mathcal{N})$, the diagram*

$$
\begin{tikzcd}
I \arrow[r, "F_0"] \arrow[d, "\eta_{F(a)}"'] & F(J) \arrow[r, "F\left(\hat{\eta}_a\right)"] & F(a \bullet a^\dagger) \\
F(a) \otimes F(a)' \arrow[rr, "1_{F(a)} \bullet F_1(a)"'] & & F(a) \otimes F(a^\dagger). \arrow[u, "F_2(a, a^\dagger)"']
\end{tikzcd}
\tag{4.3.27}
$$

commutes.

Proof. Given that (F, G, ϕ, ψ) is a special monoidal equivalence, equation (3.1.33) implies that $F_0 : I \longrightarrow F(J)$ is the identity morphism. From equation (4.3.23), we conclude that $F_1(a)$ is the identity morphism for $a \in \mathrm{Obj}(\mathcal{N})$. Thus, it remains to show

$$F(\hat{\eta}_a) = F_2(a, a^\dagger) \circ \eta_{F(a)}.$$

The naturality of F_2 ensures that the diagram

$$\begin{tikzcd} FGF(a) \otimes F(a^\dagger) \arrow[r, "F_2(GF(a), a^\dagger)"] \arrow[d, "F(\phi_a) \otimes F(1_{a^\dagger})"'] & F(GF(a) \bullet a^\dagger) \arrow[d, "F(\phi_a \bullet 1_{a^\dagger})"] \\ F(a) \otimes F(a^\dagger) \arrow[r, "F_2(a, a^\dagger)"'] & F(a \bullet a^\dagger) \end{tikzcd}$$

Thus, we get

$$\begin{aligned} F(\phi_a \bullet 1_{a^\dagger}) &= F_2(a, a^\dagger) \circ \big(F(\phi_a) \otimes F(1_{a^\dagger})\big) \circ F_2^{-1}(GF(a), a^\dagger) && \text{from above} \quad \text{(i)} \\ &= F_2(a, a^\dagger) \circ \big((F(\phi_a) \circ \psi_{F(a)}) \otimes F(1_{a^\dagger})\big) \circ F_2^{-1}(GF(a), a^\dagger) && (3.1.34) \\ &= F_2(a, a^\dagger) \circ \big(1_{F(a)} \otimes 1_{F(a^\dagger)}\big) \circ F_2^{-1}(GF(a), a^\dagger) && (3.1.31) \\ &= F_2(a, a^\dagger) \circ F_2(GF(a), G(F(a)')) && (4.3.3) \\ &= F_2(a, a^\dagger) \circ F(G_2(Fa, (Fa)')) \circ F_2(GF(a), G((Fa)')) && (3.1.15) \\ &= F_2(a, a^\dagger) \circ (FG)_2(F(a), F(a)') && (3.1.24) \\ &= F_2(a, a^\dagger). && (3.1.34) \end{aligned}$$

Here, the fifth equality follows from the fact that G is a strict monoidal functor. From above, we get

$$\begin{aligned} F(\hat{\eta}_a) &= F(\phi_a \bullet 1_{a^\dagger}) \circ FG(\eta_{F(a)}) && (4.3.6) \\ &= F_2(a, a^\dagger) \circ FG(\eta_{F(a)}) && (i) \\ &= F_2(a, a^\dagger) \circ \eta_{F(a)} && (3.1.34) \end{aligned}$$

as required. $\square$

Theorem 4.3.28. *Let*

$$\mathscr{M} = (\mathscr{M}, I, \otimes, \alpha, \lambda, \rho, \eta, \epsilon)$$

be a categorical group. Then, there exists a semi-strict categorical group

$$\mathscr{N} = (\mathscr{N}, J, \bullet, (-)^{\dagger}, \hat{\eta}, \hat{\epsilon})$$

such that there is a special categorical group equivalence from $\mathscr{M}$ to $\mathscr{N}$.

Proof. The underlying monoidal category of the categorical group $\mathscr{M}$, gives a strict monoidal category $\mathscr{N}$ and a special monoidal equivalence (F, G, ϕ, ψ) from $\mathscr{M}$ to $\mathscr{N}$ from Theorem 3.2.13. Induce a semi-strict categorical group structure on $\mathscr{N}$ using Construction 4.3.14.

Definition 4.3.16 and Lemmas 4.3.18 and 4.3.20 show that the monoidal functor G can be upgraded to a categorical group functor with G_1 equal to identity natural transformation. Similarly, Definition 4.3.22 and Lemmas 4.3.24 and 4.3.26 show that the monoidal functor F can be upgraded to a categorical group functor with F_1 equal to identity natural transformation. Thus, the special monoidal equivalence (F, G, ϕ, ψ) is in fact a special categorical group equivalence. Moreover, from Theorem 4.2.20, the monoida natural isomorphisms ϕ and ψ can be extended to categorical group natural isomorphisms isomorphisms. This completes the proof. $\square$

Motivation 4.3.29. We will use the above result to show that the coherence for categorical groups follows from the coherence from semi-strict categorical groups. $\diamond$

4.4 Free Categorical Groups

In this section, we define the universal properties of free categorical groups and free semi-strict categorical groups. In the next section we will provide specific constructions of free categorical groups and free semi-strict categorical groups and state the coherence theorem for categorical groups. This section lays the groundwork for the coherence theorems stated in the next section.

Definition 4.4.1. Let S be a set, $\mathscr{M}$ be a categorical group, and

$$i : S \longrightarrow \mathrm{Obj}(\mathscr{M})$$

be a function. We say that the pair

$$\big(\mathscr{M}, i : S \longrightarrow \mathrm{Obj}(\mathscr{M})\big)$$

satisfies the *universal property of a free categorical group* if the following conditions are satisfied:

Existence (strict functor): For a categorical group $\mathscr{N}$, and a set map $f : S \longrightarrow \mathrm{Obj}(\mathscr{N})$ there exists a strict categorical group functor

$$F : \mathscr{M} \longrightarrow \mathscr{N}$$

such that the equality

$$\mathrm{Obj}(F) \circ i = f \tag{4.4.2}$$

holds.

Existence (categorical group natural transformation): For a categorical group $\mathscr{N}$, a pair of categorical group functors

$$F, G : \mathscr{M} \longrightarrow \mathscr{N},$$

and a collection of isomorphisms

$$\widehat{\phi} = \big(\widehat{\phi}_a : F(i(a)) \longrightarrow G(i(a))\big)_{i \in S}$$

there exists a categorical group natural transformation

$$\phi : F \Rightarrow G$$

such that for every element $a \in S$, the equality

$$\phi_{i(a)} = \widehat{\phi}_a : F(i(a)) \longrightarrow G(i(a)) \tag{4.4.3}$$

holds.

Uniqueness: For a categorical group $\mathscr{N}$, a pair of categorical group functors

$$F, G : \mathscr{M} \longrightarrow \mathscr{N},$$

and a pair of categorical group natural isomorphisms

$$\phi, \psi : F \Rightarrow G$$

the following implication holds: If for every element $a \in S$ the equality

$$\phi_{i(a)} = \psi_{i(a)} : F(i(a)) \longrightarrow G(i(a))$$

of morphism is satisfied then the equality of natural transformations

$$\phi = \psi : F \Rightarrow G \tag{4.4.4}$$

holds. ◇

Notation 4.4.5. Given a set S, suppose

$$(\mathscr{M}, i : S \longrightarrow \mathrm{Obj}(\mathscr{M})) \qquad \text{and} \qquad (\mathscr{N}, j : S \longrightarrow \mathrm{Obj}(\mathscr{N}))$$

are two pairs, each satisfying the universal property of a free categorical group. Then, by the universal property, the categorical groups $\mathscr{M}$ and $\mathscr{N}$ are isomorphic to each other via strict categorical group functors. Thus, any pair

$$(\mathscr{M}, i : S \longrightarrow \mathrm{Obj}(\mathscr{M}))$$ ◇

satisfying the universal property is called *the free categorical group* generated by S.

Remark 4.4.6. Suppose the forgetful 2-functor $U : \mathcal{C}at\mathcal{G}rp \longrightarrow \mathcal{S}et$ has a left strict 2-adjoint $F : \mathcal{S}et \longrightarrow \mathcal{C}at\mathcal{G}rp$, then for every $S \in \mathcal{S}et$, the pair $(F(S), i : S \longrightarrow F(S))$ satisfies the universal property of the free categorical group generated by S. ◇

Definition 4.4.7. Let S be a set, $\mathscr{M}$ be a semi-strict categorical group, and

$$i : S \longrightarrow \mathrm{Obj}(\mathscr{M})$$

be a function. We say that the pair

$$\left(\mathscr{M}, i : S \longrightarrow \mathrm{Obj}(\mathscr{M})\right)$$

satisfies the *universal property of a free free semi-strict categorical group* if the following conditions are satisfied:

Existence (strict functor): For a semi-strict categorical group $\mathscr{N}$, and a set map $f : S \longrightarrow \mathrm{Obj}(\mathscr{N})$ there exists a strict categorical group functor

$$F : \mathscr{M} \longrightarrow \mathscr{N}$$

such that the equality

$$\mathrm{Obj}(F) \circ i = f \tag{4.4.8}$$

holds.

Existence (categorical group natural transformation): For a semi-strict categorical group $\mathcal{N}$, a pair of categorical group functors

$$F, G : \mathcal{M} \longrightarrow \mathcal{N},$$

and a collection of isomorphisms

$$\widehat{\phi} = \big(\widehat{\phi}_a : F(i(a)) \longrightarrow G(i(a))\big)_{a \in S}$$

there exists a categorical group natural transformation

$$\phi : F \Rightarrow G$$

such that for every element $a \in S$ the equality

$$\phi_{i(a)} = \widehat{\phi}_a : F(i(a)) \longrightarrow G(i(a)) \tag{4.4.9}$$

holds.

Uniqueness: For a semi-strict categorical group $\mathcal{N}$, a pair of categorical group functors

$$F, G : \mathcal{M} \longrightarrow \mathcal{N},$$

and a pair of categorical group natural isomorphisms

$$\phi, \psi : F \Rightarrow G$$

the following implication holds: If for every element $a \in S$ the equality

$$\phi_{i(a)} = \psi_{i(a)} : F(i(a)) \longrightarrow G(i(a))$$

of morphisms is satisfied then the equality

$$\phi = \psi : F \Rightarrow G \tag{4.4.10}$$

of natural transformation holds. $\diamond$

Notation 4.4.11. Given a set S, suppose

$$(\mathscr{M}, i : S \longrightarrow \mathrm{Obj}(\mathscr{M})) \qquad \text{and} \qquad (\mathscr{N}, j : S \longrightarrow \mathrm{Obj}(\mathscr{N}))$$

are two pairs, each satisfying the universal property of a free semi-strict categorical group. Then, by the universal property, the semi-strict categorical groups $\mathscr{M}$ and $\mathscr{N}$ are isomorphic to each other via strict categorical group functors. Thus, any pair

$$(\mathscr{M}, i : S \longrightarrow \mathrm{Obj}(\mathscr{M})) \qquad \diamond$$

satisfying the universal property is called *the free semi-strict categorical group* generated by S.

Remark 4.4.12. Suppose the forgetful 2-functor $U : \mathcal{SSCatGrp} \longrightarrow \mathcal{Set}$ has a left strict 2-adjoint $F : \mathcal{Set} \longrightarrow \mathcal{SSCatGrp}$, then for every $S \in \mathcal{Set}$, the pair $(F(S), i : S \longrightarrow F(S))$ satisfies the universal property of the free categorical group generated by S. $\diamond$

Theorem 4.4.13. *Let S be a set, $\mathscr{M}$. Suppose a pair*

$$\big(\mathscr{M}, \quad i : S \longrightarrow \mathrm{Obj}(\mathscr{M})\big)$$

satisfies the universal property of the free categorical group over S (see Definition 4.4.1) and a pair

$$\big(\mathscr{T}, \quad j : S \longrightarrow \mathrm{Obj}(\mathscr{T})\big)$$

satisfies the universal property of the free semi-strict categorical group over S (see Definition 4.4.7). Then, $\mathscr{M}$ is categorically equivalent to $\mathscr{T}$ via a categorical group equivalence.

Proof. Since $\mathscr{T}$ is a semi-strict categorical group, in particular, it is a categorical group. The universal property of the free categorical group provides a strict categorical group functor

$$P : \mathscr{M} \longrightarrow \mathscr{T}$$

such that

$$\mathrm{Obj}(P) \circ i = j. \qquad (4.4.2) \qquad \text{(i)}$$

Next, since $\mathscr{M}$ is a categorical group, Theorem 4.3.28 yields a semi-strict categorical group $\mathscr{N}$ and a special categorical group equivalence (F, G, ϕ, ψ), where

$$F : \mathscr{N} \longrightarrow \mathscr{M} \qquad \text{and} \qquad G : \mathscr{M} \longrightarrow \mathscr{N}$$

and

$$\phi : G \circ F \Rightarrow \mathrm{Id}_{\mathscr{N}} \qquad \text{and} \qquad \psi : F \circ G \Rightarrow \mathrm{Id}_{\mathscr{M}}.$$

Consider the set map

$$\mathrm{Obj}(G) \circ i : S \longrightarrow \mathrm{Obj}(\mathscr{N}).$$

By the universal property of the free semi-strict categorical group, there exists a strict categorical group functor

$$Q : \mathscr{T} \longrightarrow \mathscr{N}$$

such that

$$\mathrm{Obj}(Q) \circ j = \mathrm{Obj}(G) \circ i. \qquad (4.4.8) \qquad \text{(ii)}$$

Now, consider the following composition of categorical group functors

$$\mathscr{M} \xrightarrow{P} \mathscr{T} \xrightarrow{Q} \mathscr{N} \xrightarrow{F} \mathscr{M}.$$

Observe that

$$\begin{aligned}
\mathrm{Obj}(F \circ Q \circ P) \circ i &= \mathrm{Obj}(F) \circ \mathrm{Obj}(Q) \circ \mathrm{Obj}(P) \circ i \\
&= \mathrm{Obj}(F) \circ \mathrm{Obj}(Q) \circ j && \text{(i)} \\
&= \mathrm{Obj}(F) \circ \mathrm{Obj}(G) \circ i && \text{(ii)} \\
&= \mathrm{Obj}(F \circ G) \circ i \\
&= \mathrm{Obj}(\mathrm{Id}_{\mathscr{M}}) \circ i. && (3.1.34)
\end{aligned}$$

Consider the following collection of isomorphisms:

$$\widehat{\phi} := \left(\widehat{\phi}_a := 1_{i(a)} : (F \circ Q \circ P)\,(i(a)) \longrightarrow i(a)\right)_{a \in S}$$

By the existence and uniqueness of a natural transformation part of the universal property of $\mathscr{M}$, there exists a unique categorical group natural isomorphism

$$\Phi : (F \circ Q) \circ P \Rightarrow \mathrm{Id}_{\mathscr{M}}$$

such that

$$\Phi_{i(a)} = 1_{i(a)}.$$

Similarly, consider the following composition of categorical group functors.

$$\mathscr{T} \xrightarrow{Q} \mathscr{N} \xrightarrow{F} \mathscr{M} \xrightarrow{P} \mathscr{T}.$$

We have

$$\begin{aligned}
\mathrm{Obj}(P\circ F\circ Q)\circ j &= \mathrm{Obj}(P)\circ \mathrm{Obj}(F)\circ \mathrm{Obj}(Q)\circ j \\
&= \mathrm{Obj}(P)\circ \mathrm{Obj}(F)\circ \mathrm{Obj}(G)\circ i && \text{(ii)} \\
&= \mathrm{Obj}(P)\circ \mathrm{Obj}(F\circ G)\circ i \\
&= \mathrm{Obj}(P)\circ i && \text{(3.1.34)} \\
&= j && \text{(i)} \\
&= \mathrm{Obj}(\mathrm{Id}_{\mathscr{T}})\circ j.
\end{aligned}$$

Consider the following collection of isomorphisms:

$$\widehat{\psi} := \left(\widehat{\psi}_a := 1_{j(a)} : (P\circ F\circ Q)\,(j(a)) \longrightarrow j(a)\right)_{a\in S}.$$

By the existence and uniqueness of a natural transformation part of the universal property of $\mathscr{T}$, there exists a unique categorical group natural isomorphism

$$\Psi : P\circ (F\circ Q) \Rightarrow \mathrm{Id}_{\mathscr{T}}$$

such that

$$\Psi_{j(a)} = 1_{j(a)}.$$

Therefore, the tuple

$$\left(P, F\circ Q, \Phi, \Psi\right)$$

gives a categorical group equivalence between $\mathscr{M}$ and $\mathscr{T}$. □

4.5 Coherence Theorems for Categorical Groups: Statements

In this section, we present an explicit construction of the free categorical group generated by a set. Based on this construction, we state the coherence theorem for categorical groups.

Construction 4.5.1. Let S be a set. The *free dashed multiplicative set with unit* generated by S, denoted $\mathcal{DMul}\langle S\rangle$, is constructed inductively as follows:

- The unit symbol J and each element of S are elements of $\mathcal{DMul}\langle S\rangle$.
- If $x, y \in \mathcal{DMul}\langle S\rangle$, then $x \otimes y$ is also in $\mathcal{DMul}\langle S\rangle$.
- If $x \in \mathcal{DMul}\langle S\rangle$, then x' is also in $\mathcal{DMul}\langle S\rangle$.

These are subject to the relation

$$J' = J. \qquad \diamond$$

Construction 4.5.2. Let S be a set. The *free dashed monoid* generated by S, denoted $\mathcal{DMon}\langle S\rangle$, is constructed inductively as follows:

- The unit symbol J and each element of S are elements of $\mathcal{DMon}\langle S\rangle$.
- If $x, y \in \mathcal{DMon}\langle S\rangle$, then $x \bullet y$ is also in $\mathcal{DMon}\langle S\rangle$.
- If $x \in \mathcal{DMon}\langle S\rangle$, then x' is also in $\mathcal{DMon}\langle S\rangle$.

These are subject to the ralations

$$J' = J, \qquad J \bullet x = x = x \bullet J, \qquad \text{and} \qquad x \bullet (y \bullet z) = (x \bullet y) \bullet z. \qquad \diamond$$

Remark 4.5.3. We will spend Chapter 7 to understand the algebraic structure behind the above construction. In Section 7.2, we will present a criterion for identifying the free dashed monoid $\mathcal{DMon}\langle S\rangle$. An alternate, formal set-theoretic construction of the free dashed monoid $\mathcal{DMon}\langle S\rangle$ will be given in Section 7.6. Finally, in Section 7.7, we will verify that $\mathcal{DMon}\langle S\rangle$ satisfies the required universal property. $\diamond$

Construction 4.5.4. Let S be a set. The *free group generated by* S, denoted $\mathcal{G}rp\langle S\rangle$, consists of all finite words formed from elements of S and their formal inverses, with the condition that no element appears immediately adjacent to its own inverse. The empty word serves as the identity element, denoted by e. Multiplication is given by concatenation of words, followed by reduction: whenever an element and its inverse appear consecutively, they are removed. The inverse of a word is obtained by reversing the word and replacing each letter from S with its formal inverse and inverse of letter from S by the original letter. $\diamond$

Construction 4.5.5. Construct a morphism

$$P : \mathcal{DM}ul\langle S\rangle \longrightarrow \mathcal{DM}on\langle S\rangle$$

as follows:

$$P(J) = J, \tag{4.5.6}$$

$$P(a) = a \qquad \text{for } a \in S, \tag{4.5.7}$$

$$P(x \otimes y) = P(x) \bullet P(y) \qquad \text{for } x, y \in \mathcal{DM}ul\langle S\rangle\text{, and} \tag{4.5.8}$$

$$P(x') = P(x)' \qquad \text{for } x \in \mathcal{DM}ul\langle S\rangle. \tag{4.5.9}$$

Construct a morphism

$$Q_{\mathcal{DM}on} : \mathcal{DM}on\langle S\rangle \longrightarrow \mathcal{G}rp\langle S\rangle$$

as follows:

$$Q_{\mathcal{DM}on}(J) = e, \tag{4.5.10}$$

$$Q_{\mathcal{DM}on}(a) = a \qquad \text{for } a \in S, \tag{4.5.11}$$

$$Q_{\mathcal{DM}on}(x \bullet y) = Q_{\mathcal{DM}on}(x) \cdot Q_{\mathcal{DM}on}(y) \qquad \text{for } x, y \in \mathcal{DM}on\langle S\rangle\text{, and} \tag{4.5.12}$$

$$Q_{\mathcal{DM}on}(x') = Q_{\mathcal{DM}on}(x)^{-1} \qquad \text{for } x \in \mathcal{DM}on\langle S\rangle. \tag{4.5.13}$$

Construct a morphism

$$Q_{\mathcal{DMul}} : \mathcal{DMul}\langle S\rangle \longrightarrow \mathcal{Grp}\langle S\rangle$$

as follows:

$$Q_{\mathcal{DMul}}(J) = e, \tag{4.5.14}$$

$$Q_{\mathcal{DMul}}(a) = a \qquad \text{for } a \in S, \tag{4.5.15}$$

$$Q_{\mathcal{DMul}}(x \otimes y) = Q_{\mathcal{DMul}}(x) \cdot Q_{\mathcal{DMul}}(y) \qquad \text{for } x, y \in \mathcal{DMul}\langle S\rangle, \text{ and} \tag{4.5.16}$$

$$Q_{\mathcal{DMul}}(x') = Q_{\mathcal{DMul}}(x)^{-1} \qquad \text{for } x \in \mathcal{DMul}\langle S\rangle. \tag{4.5.17}$$

Observe that

$$Q_{\mathcal{DMon}} \circ P = Q_{\mathcal{DMul}}. \tag{4.5.18}$$

◇

Construction 4.5.19. Let S be a set. The *free categorical group* generated by S, denoted $\mathcal{CatGrp}\langle S\rangle$, is constructed as follows:

- The free dashed multiplicative set with unit $\mathcal{DMul}\langle S\rangle$ as in Construction 4.5.1 serves as the set of objects in $\mathcal{CatGrp}\langle S\rangle$.

- The morphisms in $\mathcal{CatGrp}\langle S\rangle$ are formally generated from the structure arrows

$$\begin{aligned} &1_x : x \longrightarrow x, && \tilde{\alpha}_{x,y,z} : x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z, \\ &\tilde{\lambda}_x : J \otimes x \longrightarrow x, && \tilde{\rho}_x : x \otimes J \longrightarrow x, \\ &\tilde{\eta}_x : J \longrightarrow x \otimes x', && \tilde{\epsilon}_x : x' \otimes x \longrightarrow J \end{aligned}$$

  for $x, y, z \in \mathcal{DMul}\langle S\rangle$ by tensoring, inverting, taking dash, and composing, subject to the equivalence relations generated by the axioms of a categorical group.

◇

Remark 4.5.20. The details of the above 'long, straightforward, and rather deceptive' construction of the free categorical group is provided on page 312 of [Lap83]. ◇

Remark 4.5.21. Recall that, for a category $\mathscr{C}$, a diagram in $\mathscr{C}$ is a functor $D : \mathscr{D} \longrightarrow \mathscr{C}$ where $\mathscr{D}$ is a small category. Next, we will define formal diagrams in a categorical group. ◇

Definition 4.5.22. Let $\mathscr{M}$ be a categorical group and

$$M := \mathrm{Obj}(\mathscr{M})$$

be the set of objects in $\mathscr{M}$. Consider the free categorical group, $\mathcal{CatGrp}\langle M\rangle$, generated by M. Let

$$\mathcal{EV} : \mathcal{CatGrp}\langle M\rangle \longrightarrow \mathscr{M}$$

be the strict categorical group functor that we get from the universal property. A *formal diagram* in $\mathscr{M}$ is a diagram $D : \mathscr{D} \longrightarrow \mathscr{M}$ that lifts to $\mathcal{CatGrp}\langle M\rangle$. That is, a diagram $D : \mathscr{D} \longrightarrow \mathscr{M}$ is called a formal diagram in $\mathscr{M}$ if there exists a functor $\widehat{D} : \mathscr{D} \longrightarrow \mathcal{CatGrp}\langle M\rangle$ such that the diagram

$$\begin{tikzcd} & \mathcal{CatGrp}\langle M\rangle \arrow[d, "\mathcal{EV}"] \\ \mathscr{D} \arrow[ur, "\widehat{D}"] \arrow[r, "D"'] & \mathscr{M} \end{tikzcd}$$

commutes giving the equality

$$\mathcal{EV} \circ \widehat{D} = D.$$ ◇

Definition 4.5.23. Let S be a set, $\mathcal{Grp}\langle S\rangle$ the free group generated by S, and $\mathcal{CatGrp}\langle S\rangle$ the free categorical group generated by S. We may view $\mathcal{Grp}\langle S\rangle$ as a discrete strict categorical group. By the universal property of $\mathcal{C}at\mathcal{G}rp\langle S\rangle$, there is a strict functor

$$Q_{\mathcal{C}at\mathcal{G}rp}\colon \mathcal{C}at\mathcal{G}rp\langle S\rangle \longrightarrow \mathcal{G}rp\langle S\rangle\,.$$

This functor on objects is given by $Q_{\mathcal{DM}ul}$ as in Construction 4.5.5. ◇

Remark 4.5.24. We now state several coherence theorems for categorical groups. Formal proofs are provided in the following chapters; here, we outline the main ideas behind the proofs. ◇

Theorem 4.5.25. *Let S be a set. Then, the free categorical group $\mathcal{C}at\mathcal{G}rp\langle S\rangle$ generated by S is a thin category.*

Outline of the proof. The proof proceeds by constructing the free semi-strict categorical group $\mathcal{SSC}at\mathcal{G}rp\langle S\rangle$ and showing that it is a thin category. By Theorem 4.4.13, it then follows that $\mathcal{C}at\mathcal{G}rp\langle S\rangle$ is also thin.

The necessary groundwork is developed in Chapters 5 through 7. The explicit construction and verification that $\mathcal{SSC}at\mathcal{G}rp\langle S\rangle$ satisfies the universal property of the free semi-strict categorical group are given in Chapters 8 and 9. The theorem is restated and formally proved in Theorem 9.3.2. □

Remark 4.5.26. The next theorem is a stronger version of the previous statement and serves as a central result for the developments in Chapters 10 through 12. ◇

Theorem 4.5.27. *Let S be a set, $\mathcal{G}rp\langle S\rangle$ the free group generated by S, and $\mathcal{C}at\mathcal{G}rp\langle S\rangle$ the free categorical group generated by S. Then, the induced functor*

$$Q_{\mathcal{C}at\mathcal{G}rp}\colon \mathcal{C}at\mathcal{G}rp\langle S\rangle \longrightarrow \mathcal{G}rp\langle S\rangle$$

as in Definition 4.5.23 is full, faithful, and essentially surjective.

Therefore, $Q_{\mathcal{C}at\mathcal{G}rp}$ is an equivalence of categories.

Outline of the proof. Follows from Theorem 4.5.25. The result is restated and proved in details in Theorem 9.3.5. □

Theorem 4.5.28. *Let $\mathcal{M}$ be a categorical group. Then, every formal diagram in $\mathcal{M}$ commutes.*

Outline of the proof. Follows from Definition 4.5.22 and Theorem 4.5.25. Restated and rigorously proved in Theorem 9.3.6. □

Theorem 4.5.29. *Let $\mathcal{M}$ be a categorical group. Then, there exists a strict categorical group $\mathcal{N}$ such that $\mathcal{M}$ is equivalent to $\mathcal{N}$ via categorical group equivalence.*

Outline of the proof. We will devote Chapter Chapter 10 to show that a semi-strict categorical group is equivalent to a strict categorical group. This result will follow from Theorem 4.3.28. □

Constructions and Main Results

Chapter 5: Diagrams Commute

In this chapter, we construct key natural isomorphisms in a semi-strict categorical group and prove that specific diagrams involving these isomorphisms commute. These constructions and commutative diagrams play a central role in establishing the coherence for semi-strict categorical groups.

5.1 Expanded Instance of a Morphism

In this section, we introduce the concept of an "expanded instance" of a morphism in a semi-strict categorical group. This notion allows us to concisely represent morphisms that arise from inserting a given morphism into an object.

Framework 5.1.1. Throughout this chapter, let

$$\left(\mathscr{M}, I, (-)', \cdot, \eta, \epsilon\right)$$

denote a semi-strict categorical group. As $\mathscr{M}$ is semi-strict, we will omit parentheses indicating the order of multiplication. Let M be the set of objects of $\mathscr{M}$:

$$M := \mathrm{Obj}(\mathscr{M}).$$

Recall Notation 4.1.7, we denote k-times iterated dash functor $(-)' : \mathscr{M} \longrightarrow \mathscr{M}$ be

$$(-)^{(k)} : \mathscr{M} \longrightarrow \mathscr{M}.$$

◇

Definition 5.1.2. Let $f : x \longrightarrow y$ be a morphism in $\mathscr{M}$. A morphism $g : z \longrightarrow w$ in $\mathscr{M}$ is called a *multiplicative expanded instance* or simply an *expanded instance* of f if g is an identity morphism or there are objects $a, b \in M$ such that

$$z = axb \qquad \text{and} \qquad w = ayb$$

and the equality

$$g = 1_a f 1_b : z \longrightarrow w \tag{5.1.3}$$

of morphisms in $\mathscr{M}$ holds.

If g is an identity morphism then we say g is a trivial expanded instance of f, otherwise we say g is a non-trivial expanded instance of f. We denote an expanded instance of f by $\langle f \rangle$. ◇

Motivation 5.1.4. The above notation allows us to write morphisms more concisely while preserving the essential information about them. This is demonstrated by the following example. ◇

Example 5.1.5. Let x, y be objects in $\mathscr{M}$. Consider the following morphism in $\mathscr{M}$:

$$1_{(xy)'x}\ \eta_y\ 1_{x'} : (xy)'xx' \longrightarrow (xy)'xyy'x'.$$

This morphism is an expanded instance of η_y and can be concisely denoted by

$$\langle \eta_y \rangle : (xy)'xx' \longrightarrow (xy)'xyy'x'.$$ ◇

5.2 Distribution Natural Isomorphism

Let $x, y \in M$. There are two canonical compositions that distribute the dash from $(xy)'$ to $y'x'$. These are

$$(xy)' \xrightarrow{\langle \eta_x \rangle} (xy)'xx' \xrightarrow{\langle \eta_y \rangle} (xy)'xyy'x' \xrightarrow{\langle \epsilon_{xy} \rangle} y'x'$$

and

$$(xy)' \xrightarrow{\left\langle \epsilon_y^{-1} \right\rangle} y'y(xy)' \xrightarrow{\left\langle \epsilon_x^{-1} \right\rangle} y'x'xy(xy)' \xrightarrow{\left\langle \eta_{xy}^{-1} \right\rangle} y'x'.$$

We will show in Proposition 5.2.4 that these two composite are equal. In this section, we will construct a natural isomorphism, denoted $C^k(x,y)$, which distributes dashes from $(xy)^{(k)}$ to either $x^{(k)}y^{(k)}$ or $y^{(k)}x^{(k)}$, depending on the parity of k. The explicit construction is given in Definition 5.2.8.

We generalize this construction to define a natural isomorphism that distributes dashes over the product of more than two objects. For instance, a composition with domain $(xyz)''$ and co-domain $x''y''z''$ may be represented as

$$(xyz)'' \longrightarrow (xy)''z'' \longrightarrow x''y''z''$$

where each morphism is an instance of the composition C^2.

Another such composition is

$$(xyz)'' \longrightarrow x''(yz)'' \longrightarrow x''y''z''$$

where again, the arrows are instances of the composition C^2. The general construction is provided in Definition 5.2.21.

We will prove that any such compositions yield the same composite morphism. This is formally established in Theorem 5.2.26. We will refer to such morphisms as *distribution morphisms*.

Definition 5.2.1. Let $k \in \mathbb{N}$. Define a functor $\square_k : \mathcal{M}^2 \longrightarrow \mathcal{M}$ as follows:

$$\square_k(x_1,x_2) = \begin{cases} x_1 x_m & \text{if } k \text{ is even} \\ x_2 x_1 & \text{if } k \text{ is odd} \end{cases} \qquad \text{on objects}$$

$$\square_k(f_1,f_2) = \begin{cases} f_1 f_2 & \text{if } k \text{ is even} \\ f_2 f_1 & \text{if } k \text{ is odd} \end{cases} \qquad \text{on morphisms.}$$

The composition and identity conditions follow from the functoriality of the multiplication in $\mathscr{M}$. $\diamond$

Proposition 5.2.2. *For $k \in \mathbb{N}$, the functor $\Box_k : \mathscr{M}^2 \longrightarrow \mathscr{M}$ as in Definition 5.2.1 is associative.*

Proof. Suppose k is even, then $\Box_k$ is equal to the multiplication of $\mathscr{M}$. Since $\mathscr{M}$ is a strict monoidal category, we conclude that $\Box_k$ is associative. Now suppose k is odd, then we get

$$\begin{aligned}
\Box_k(x_1, \Box_k(x_2, x_3)) &= \Box_k(x_1, x_3x_2) \\
&= (x_3x_2)x_1 \\
&= x_3(x_2x_1) && \text{since } \mathscr{M} \text{ is strict monoidal} \\
&= \Box_k(x_2x_1, x_3) \\
&= \Box_k(\Box_k(x_1, x_2), x_3)
\end{aligned}$$

for objects $x_1, x_2, x_3 \in \mathrm{Obj}(\mathscr{M})$. The same argument holds for morphisms. Thus, $\Box_k$ is associative in this case as well. $\Box$

Remark 5.2.3. Fix $k \in \mathbb{N}$, and let $\Box = \Box_k$. From above we know that $\Box$ is associative. We denote

$$x_1 \Box \cdots \Box x_m = \begin{cases} x_1 \cdots x_m & \text{if } k \text{ is even} \\ x_m \cdots x_1 & \text{if } k \text{ is odd} \end{cases} \qquad \text{on objects}$$

$$f_1 \Box \cdots \Box f_m = \begin{cases} f_1 \cdots f_m & \text{if } k \text{ is even} \\ f_m \cdots f_1 & \text{if } k \text{ is odd} \end{cases} \qquad \text{on morphisms.} \quad \diamond$$

Proposition 5.2.4. *Assume Framework 5.1.1. For $x, y \in M$ the diagram*

$$\begin{tikzcd}
(xy)' \arrow[r, "\langle \eta_x \rangle"] \arrow[d, "\langle \epsilon_y^{-1} \rangle"'] & (xy)'xx' \arrow[r, "\langle \eta_y \rangle"] & (xy)'xyy'x' \arrow[d, "\langle \epsilon_{xy} \rangle"] \\
y'y(xy)' \arrow[r, "\langle \epsilon_x^{-1} \rangle"'] & y'x'xy(xy)' \arrow[r, "\langle \eta_{xy}^{-1} \rangle"'] & y'x'
\end{tikzcd} \tag{5.2.5}$$

commutes.

Proof. Let $f, g : (xy)' \longrightarrow y'x'$ be the top-right and left-bottom composite of the diagram (5.2.5) respectively. That is,

$$f : (xy)' \xrightarrow{\langle \eta_x \rangle} (xy)'xx' \xrightarrow{\langle \eta_y \rangle} (xy)'xyy'x' \xrightarrow{\langle \epsilon_{xy} \rangle} y'x'$$
$$g : (xy)' \xrightarrow{\langle \epsilon_y^{-1} \rangle} y'y(xy)' \xrightarrow{\langle \epsilon_x^{-1} \rangle} \quad \xrightarrow{\langle \eta_{xy}^{-1} \rangle} y'x'.$$

We want to show $f = g$.

Consider the diagram

$$\begin{tikzcd}
(xy)' \arrow[r, "\langle \eta_x \rangle"] \arrow[d, "\langle \epsilon_y^{-1} \rangle"'] & (xy)'xx' \arrow[r, "\langle \eta_y \rangle"] \arrow[d, "\langle \epsilon_y^{-1} \rangle"'] & (xy)'xyy'x' \arrow[r, "\langle \epsilon_{xy} \rangle"] \arrow[d, "\langle \epsilon_y^{-1} \rangle"'] & y'x' \arrow[d, "\langle \epsilon_y^{-1} \rangle"] \\
y'y(xy)' \arrow[r, "\langle \eta_x \rangle"] \arrow[d, "\langle \epsilon_x^{-1} \rangle"'] & y'y(xy)'xx' \arrow[r, "\langle \eta_y \rangle"] \arrow[d, "\langle \epsilon_x^{-1} \rangle"'] & y'y(xy)'xyy'x' \arrow[r, "\langle \epsilon_{xy} \rangle"] \arrow[d, "\langle \epsilon_x^{-1} \rangle"'] & y'yy'x' \arrow[d, "\langle \epsilon_x^{-1} \rangle"] \\
y'x'xy(xy)' \arrow[r, "\langle \eta_x \rangle"] \arrow[d, "\langle \eta_{xy}^{-1} \rangle"'] & y'x'xy(xy)'xx' \arrow[r, "\langle \eta_y \rangle"] \arrow[d, "\langle \eta_{xy}^{-1} \rangle"'] & y'x'xy(xy)'xyy'x' \arrow[r, "\langle \epsilon_{xy} \rangle"] \arrow[d, "\langle \eta_{xy}^{-1} \rangle"'] & y'x'xyy'x' \arrow[dl, equal] \\
y'x' \arrow[r, "\langle \eta_x \rangle"'] & y'x'xx' \arrow[r, "\langle \eta_y \rangle"'] & y'x'xyy'x' &
\end{tikzcd}$$

where the top horizontal composite is equal to $f : (xy)' \longrightarrow y'x'$, the left vertical composite is equal to $g : (xy)' \longrightarrow y'x'$. Denote the bottom horizontal composite by $f_1 : y'x' \longrightarrow y'x'xyy'x'$ and the right vertical composite by $g_1 : y'x' \longrightarrow y'x'xyy'x'$.

The above diagram commutes because every square is a multiplication functoriality square and the bottom right triangle is the cancellation triangle (4.1.20). Thus, we get

$$g_1 \circ f = f_1 \circ g.$$

Now, consider the diagram

$$\begin{tikzcd}
y'x'. \arrow[r, "\langle \eta_x \rangle"] \arrow[dr, equal] & y'x'xx' \arrow[r, "\langle \eta_y \rangle"] \arrow[d, "\langle \epsilon_x \rangle"] & y'x'xyy'x' \arrow[d, "\langle \epsilon_x \rangle"] \\
& y'x' \arrow[r, "\langle \eta_y \rangle"] \arrow[dr, equal] & y'yy'x' \arrow[d, "\langle \epsilon_y \rangle"] \\
& & y'x'
\end{tikzcd}$$

where the top horizontal composite is equal to $f_1 : (y'x') \longrightarrow y'x'xyy'x'$ and the right vertical composite is equal to $g_1^{-1} : (xy)' \longrightarrow y'x'xyy'x'$. The diagram commutes since the top-right square is a multiplicaion functoriality square and the two triangles are the cancellation triangles (4.1.21). Therefore, we get

$$g_1^{-1} \circ f_1 = 1_{y'x'}.$$

Combining, we get

$$\begin{aligned}
f &= g_1^{-1} \circ g_1 \circ f \\
&= g_1^{-1} \circ f_1 \circ g && \text{showed earlier} \\
&= g && \text{from above}
\end{aligned}$$

as required. $\square$

Definition 5.2.6. Let $F : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ be the following functor:

$$F(x,y) = (xy)' \qquad \text{on objects}$$
$$F(f,g) = (fg)' \qquad \text{on morphisms.}$$

Let $G : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ be the following functor:

$$G(x,y) = y'x' \qquad \text{on objects}$$
$$G(f,g) = g'x' \qquad \text{on morphisms.}$$

Define a natural transformation

$$\delta : F \Rightarrow G$$

as follows:

- For objects $x, y \in \mathcal{M}$, define the map

  $$\delta_{x,y} : (xy)' \longrightarrow y'x'$$

  as follows:

  $$\delta_{x,y} := \left(\epsilon_{xy} 1_{y'x'}\right) \circ \left(1_{(xy)'x} \eta_y 1_{x'}\right) \circ \left(1_{(xy)'} \eta_x\right). \tag{5.2.7}$$

  That is, the morphism $\delta_{x,y}$ is described by the following composition:

  $$\delta_{x,y} : (xy)' \xrightarrow{\langle \eta_x \rangle} (xy)'xx' \xrightarrow{\langle \eta_y \rangle} (xy)'xyy'x' \xrightarrow{\langle \epsilon_{xy} \rangle} y'x'.$$

  We will show that the morphism $\delta_{x,y}$ is a natural transformation in variables x and y. In the above construction

  $$\langle \eta_x \rangle = 1_{(xy)'} \eta_x : (xy)' \longrightarrow (xy)'xx'$$

  is an expanded instance of the natural transformation η_x. Thus, $\langle \eta_{x,y} \rangle$ is a natural transformation in variables x and y.

Similarly,

$$\langle \eta_y \rangle = 1_{(xy)'x} \eta_y 1_{x'} : (xy)'xx' \longrightarrow (xy)'xyy'x'$$

is a natural transformation in variables x and y.

Finally, $\epsilon_{xy} : (xy)'xy \longrightarrow I$ is a natural transformation in variables x and y since it is a whiskering of the natural transformation ϵ with the multiplication functor of $\mathscr{M}$:

$$\left(\epsilon_{xy}\right)_{x,y \in \mathrm{Obj}(\mathscr{M})} = \epsilon * \otimes.$$

Thus,

$$\langle \epsilon_{xy} \rangle = \epsilon_{xy} 1_{y'x'} : (xy)'xyy'x' \longrightarrow y'x'$$

is a natural transformation in variables x and y.

Since each component morphism in (5.2.7) is a natural transformation in variables x and y, the morphism $\delta_{x,y}$ is also a natural transformation. $\diamond$

Definition 5.2.8. For $k \in \mathbb{N}$, let $F_k, G_k : \mathscr{M} \times \mathscr{M} \longrightarrow \mathscr{M}$ be the following functors:

$$F_k(x,y) = (xy)^{(k)} \qquad \text{on objects}$$

$$F_k(f,g) = (fg)^{(k)} \qquad \text{on morphisms}$$

and

$$G_k(x,y) = x^{(k)} \Box_k y^{(k)} \qquad \text{on objects}$$

$$G_k(f,g) = f^{(k)} \Box_k g^{(k)} \qquad \text{on morphisms.}$$

Define a natural transformation

$$C^k : F_k \Rightarrow G_k$$

inductively as follows:

Base case ($n=0$): Note that $F_0(x,y) = xy = G_0(x,y)$ and $F_0(f,g) = fg = G_0(f,g)$. It follows that $F_0 = G_0$. Define the natural transformation $C^0 : F_0 \Rightarrow G_0$ to be the identity natural transformation on F_0:

$$C^0 = \mathrm{Id}_{F_0}. \tag{5.2.9}$$

That is,

$$C^0(x,y) := \mathrm{Id}_{xy} \tag{5.2.10}$$

for $x, y \in \mathrm{Obj}(\mathscr{M})$.

Induction case ($k \geq 0$): Assume that we have defined the natural transformation $C^k : F_k \Rightarrow G_k$. We will define the natural transformation $C^{k+1} : F_{k+1} \Rightarrow G_{k+1}$ as follows:

- Let $x, y \in \mathrm{Obj}(\mathscr{M})$. The morphism

  $$C^{k+1}(x,y) : (xy)^{(k+1)} \longrightarrow x^{(k+1)} \square_{k+1} y^{(k+1)}$$

  is defined as follows:

  $$C^{k+1}(x,y) := C^k(y', x') \circ \delta^{(k)}_{x,y}. \tag{5.2.11}$$

  That is,

  $$C^{k+1}(x,y) : (xy)^{(k+1)} \xrightarrow{\delta^{(k)}_{x,y}} \left(y'x'\right)^{(k)} \xrightarrow{C^k(y',x')} y^{(k+1)} \square_k x^{(k+1)}.$$

  Note that

  $$y^{(k+1)} \square_k x^{(k+1)} = x^{(k+1)} \square_{k+1} y^{(k+1)}.$$

Now we will prove that the above assignement satisfies the *naturality* condition. From Definition 5.2.6 we know that $\delta_{x,y}$ is a natural transformation in variables x and y. The morphism $\delta^{(k)}_{x,y}$ is also a natural transformation in variables x and y

since it is a whiskering of the natural transformation δ with the functor $(-)^{(k)}$. From induction hypothesis we have that $C^k(x,y)$ is a natural transformation in variables x and y. Thus, $C^k(y',x')$ is also a natural transformation since it is a whiskering of the functor that maps (x,y) to (y',x') with the natural transformation $C^k(x,y)$. With this, we conclude that

$$C^{k+1}(x,y) := C^k(y',x') \circ \delta^{(k)}_{x,y}$$

is a natural transformation in variables x and y. ◇

Definition 5.2.12. Let $k \in \mathbb{N}$. Let $F_k, G_k : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ be the following functors:

$$F_k(x,y,z) = (xyz)^{(k)} \qquad \text{on objects}$$

$$F_k(f,g,h) = (fgh)^{(k)} \qquad \text{on morphisms}$$

and

$$G_k(x,y,z) = x^{(k)} \square y^{(k)} \square z^{(k)} \qquad \text{on objects}$$

$$G_k(f,g,h) = f^{(k)} \square g^{(k)} \square h^{(k)} \qquad \text{on morphisms.}$$

Here, $\square$ is stands for the functor $\square_k$. Define a natural transformation

$$L^k : F_k \Rightarrow G_k$$

as follows:

- For objects $x,y,z \in \mathrm{Obj}(\mathcal{M})$ the morphism

$$L^k(x,y,z) : (xyz)^{(k)} \longrightarrow x^{(k)} \square y^{(k)} \square z^{(k)}$$

is defined as

$$L^k(x,y,z) := \left(C^k(x,y)\Box 1_{z^{(k)}}\right) \circ C^k(xy,z). \tag{5.2.13}$$

This is described by the following composition:

$$L^k(x,y,z) : (xyz)^{(k)} \xrightarrow{C^k(xy,z)} (xy)^{(k)}\Box z^{(k)} \xrightarrow{\left\langle C^k(x,y)\right\rangle} x^{(k)}\Box y^{(k)}\Box z^{(k)}.$$

We will show that the above assignement satisfies the naturality condition: From Definition 5.2.8 we have that $C^k(x,y)$ is a natural transformation in variables x and y. Since $C^k(xy,z)$ is a wiskering of the natural transformation $C^k(x,y)$ with the tensor functor in the first component, it is a natural transformation in variables x, y, and z. It follows that, $C^k(x,y)\Box 1_{z^n}$ is a natural transformation in variables x, y, and z. Hence, the composition

$$L^k(x,y,z) := \left(C^k(x,y)\Box 1_{z^{(k)}}\right) \circ C^k(xy,z)$$

is a natural transformation in variables x, y, and z. ◇

Definition 5.2.14. For $k \in \mathbb{N}$ let $F_k, G_k : \mathcal{M}\times\mathcal{M}\times\mathcal{M} \longrightarrow \mathcal{M}$ be the following functor:

$$F_k(x,y,z) = (xyz)^{(k)} \qquad \text{on objects}$$

$$F_k(f,g,h) = (fgh)^{(k)} \qquad \text{on morphisms}$$

and

$$G_k(x,y,z) = x^{(k)}\Box y^{(k)}\Box z^{(k)} \qquad \text{on objects}$$

$$G_k(f,g,h) = f^{(k)}\Box g^{(k)}\Box h^{(k)} \qquad \text{on morphisms.}$$

Here, $\Box$ stands for the functor $\Box_k$. Define a natural transformation

$$R^k : F_k \Rightarrow G_k$$

as follows:

- For objects $x, y, z \in \mathrm{Obj}(\mathscr{M})$ the morphism

$$R^k(x,y,z) : (xyz)^{(k)} \longrightarrow x^{(k)} \square y^{(k)} \square z^{(k)}$$

is defined as

$$R^k(x,y,z) := \left(1_{x^{(k)}} \square C^k(y,z)\right) \circ C^k(x,yz). \tag{5.2.15}$$

This composition is described by the following diagram:

$$R^k(x,y,z) : (xyz)^{(k)} \xrightarrow{C^k(x,yz)} x^{(k)} \square (yz)^{(k)} \xrightarrow{\left\langle C^k(y,z)\right\rangle} x^{(k)} \square y^{(k)} \square z^{(k)}.$$

We will show that the above assignement satisfies the naturality condition: From Definition 5.2.8 we have that $C^k(x,y)$ is a natural transformation in variables x and y. Since $C^k(x,yz)$ is a wiskering of the natural transformation $C^k(x,y)$ with the tensor functor in the second component, it is a natural transformation in variables x, y, and z. It follows that $1_{x^n} \square C^k(y,z)$ is a natural transformation in variables x, y, and z. Hence, the morphism

$$R^k(x,y,z) := \left(1_{x^{(k)}} \square C^k(y,z)\right) \circ C^k(x,yz)$$

is a natural transformation in variables x, y, and z. ⋄

Lemma 5.2.16. *Let $x, y, z \in M$. Then the following diagram commutes.*

$$\begin{tikzcd} (xyz)' \arrow[r, "\delta_{xy,z}"] \arrow[d, "\delta_{x,yz}"'] & z'(xy)' \arrow[d, "\mathrm{Id}_{z'}\delta_{x,y}"] \\ (yz)'x' \arrow[r, "\delta_{y,z}\mathrm{Id}_{x'}"'] & z'y'x'. \end{tikzcd}$$

That is,

$$\left(\mathrm{Id}_{z'}\delta_{x,y}\right) \circ \delta_{xy,z} = \left(\delta_{y,z}\mathrm{Id}_{x'}\right) \circ \delta_{x,yz}. \tag{5.2.17}$$

Proof. Let l^1 and r^1 be the top-right and left-bottom compositions respectively. That is,

$$l^1 := \left(\mathrm{Id}_{z'}\delta_{x,y}\right) \circ \delta_{xy,z} \tag{i}$$

and

$$r^1 := \left(\delta_{y,z}\mathrm{Id}_{x'}\right) \circ \delta_{x,yz}. \tag{ii}$$

Let f be the following morphism in $\mathscr{M}$.

$$f : (xyz)' \xrightarrow{\langle\eta_x\rangle} (xyz)'xx' \xrightarrow{\langle\eta_y\rangle} (xyz)'xyy'x' \xrightarrow{\langle\eta_z\rangle} (xyz)'xyzz'y'x' \xrightarrow{\langle\epsilon_{xyz}\rangle} z'y'x'. \tag{iii}$$

We will show that

$$l^1 = f = r^1.$$

Consider following diagram:

$$\begin{tikzcd}
(xyz)' \arrow[r, "\langle\eta_x\rangle"] \arrow[d, "\langle\eta_{xy}\rangle"'] & (xyz)'xx' \arrow[r, "\langle\eta_y\rangle"] \arrow[d, "\langle\eta_{xy}\rangle"'] & (xyz)'xyy'x' \arrow[r, equal] \arrow[d, "\langle\eta_{xy}\rangle"'] \arrow[dr, phantom, "\mathtt{Canc}"] & (xyz)'xyy'x' \arrow[dd, "\langle\eta_z\rangle"] \\
(xyz)'xy(xy)' \arrow[r, "\langle\eta_x\rangle"'] \arrow[d, "\langle\eta_z\rangle"'] & (xyz)'xy(xy)'xx' \arrow[r, "\langle\eta_y\rangle"'] \arrow[d, "\langle\eta_z\rangle"'] & (xyz)'xy(xy)'xyy'x' \arrow[ur, "\langle\epsilon_{xy}\rangle"'] \arrow[d, "\langle\eta_z\rangle"'] & \\
(xyz)'xyzz'(xy)' \arrow[r, "\langle\eta_x\rangle"'] \arrow[d, "\langle\epsilon_{xyz}\rangle"'] & (xyz)'xyzz'(xy)'xx' \arrow[r, "\langle\eta_y\rangle"'] \arrow[d, "\langle\epsilon_{xyz}\rangle"'] & (xyz)'xyzz'(xy)'xyy'x' \arrow[r, "\langle\epsilon_{xy}\rangle"] \arrow[d, "\langle\epsilon_{xyz}\rangle"'] & (xyz)'xyzz'y'x' \arrow[d, "\langle\epsilon_{xyz}\rangle"] \\
z'(xy)' \arrow[r, "\langle\eta_x\rangle"'] & z'(xy)'xx' \arrow[r, "\langle\eta_y\rangle"'] & z'(xy)'xyy'x' \arrow[r, "\langle\epsilon_{xy}\rangle"'] & z'y'x'.
\end{tikzcd}$$

In the above diagram the triangle `Canc` commutes because of the cancellation condition (4.1.20) and every other square commutes due to the functoriality of the multiplication in $\mathscr{M}$. The left-bottom composition of the whole diagram is given by

$$
\begin{aligned}
\big(\langle\epsilon_{xy}\rangle\circ\langle\eta_y\rangle\circ\langle\eta_x\rangle\big)\circ\big(\langle\epsilon_{xyz}\rangle\circ\langle\eta_z\rangle\circ\langle\eta_{xy}\rangle\big) &= \langle\delta_{x,y}\rangle\circ\delta_{xy,z} && (5.2.7)\\
&= l^1. && (i)
\end{aligned}
$$

The top-right composition is

$$\langle\epsilon_{xyz}\rangle\circ\langle\eta_z\rangle\circ\langle\eta_y\rangle\circ\langle\eta_x\rangle = f. \qquad (iii)$$

Since the diagram commutes, we get

$$l^1 = f.$$

Now consider the following diagram

$$
\begin{tikzcd}
(xyz)' \arrow[d, "\langle\eta_x\rangle"'] & & \\
(xyz)'xx' \arrow[r, "\langle\eta_{yz}\rangle"] \arrow[d, "\langle\eta_y\rangle"'] & (xyz)'xyz(yz)'x' \arrow[r, "\langle\epsilon_{xyz}\rangle"] \arrow[d, "\langle\eta_y\rangle"] & (yz)'x' \arrow[d, "\langle\eta_y\rangle"] \\
(xyz)'xyy'x' \arrow[r, "\langle\eta_{yz}\rangle"] \arrow[d, "\langle\eta_z\rangle"'] & (xyz)'xyz(yz)'yy'x' \arrow[r, "\langle\epsilon_{xyz}\rangle"] \arrow[d, "\langle\eta_z\rangle"] & (yz)'yy'x' \arrow[d, "\langle\eta_z\rangle"] \\
(xyz)'xyzz'y'x' \arrow[r, "\langle\eta_{yz}\rangle"] \arrow[dr, equal] & (xyz)'xyz(yz)'yzz'y'x' \arrow[r, "\langle\epsilon_{xyz}\rangle"] \arrow[d, "\langle\epsilon_{yz}\rangle"] \arrow[dl, phantom, "\texttt{Canc}"] & (yz)'yzz'y'x' \arrow[d, "\langle\epsilon_{yz}\rangle"] \\
& (xyz)'xyzz'y'x' \arrow[r, "\langle\epsilon_{xyz}\rangle"'] & z'y'x'.
\end{tikzcd}
$$

In the above diagram the triangle `Canc` commutes because of the cancellation identity and every other square commutes because $\mathscr{M}$ is a monoidal category. The left-bottom composition is given by

$$\langle \epsilon_{xyz} \rangle \circ \langle \eta_z \rangle \circ \langle \eta_y \rangle \circ \langle \eta_x \rangle = f. \qquad (iii)$$

The top-right composition is

$$\begin{aligned}\big(\langle \epsilon_{yz} \rangle \circ \langle \eta_z \rangle \circ \langle \eta_y \rangle\big) \circ \big(\langle \epsilon_{xyz} \rangle \circ \langle \eta_{yz} \rangle \langle \eta_x \rangle\big) &= \langle \delta_{y,z} \rangle \circ \delta_{x,yz} && (5.2.7)\\ &= r^1. && (ii)\end{aligned}$$

Since the diagram commutes we get

$$f = r^1.$$

Thus, we get

$$l^1 = r^1$$

as required. □

Remark 5.2.18. The morphisms l^1 and r^1 in the proof above are, in fact, the natural transformations $L^1_{x,y,z}$ and $R^1_{x,y,z}$ respectively. ◇

Theorem 5.2.19. *Let $k \in \mathbb{N}$. Then, the natural transformations L^k and R^k are equal. That is, the equality*

$$L^k(x,y,z) = R^k(x,y,z) \tag{5.2.20}$$

of morphisms holds for every $x, y, z \in M$.

Proof. We will show this using the induction on k.

Base case ($k = 0$): For $x, y, z \in \mathrm{Obj}(\mathscr{M})$, we will show

$$L^0(x,y,z) = R^0(x,y,z).$$

We have

$$
\begin{aligned}
L^0(x,y,z) &= \left(C^0(x,y)1_z\right)\circ C^0(xy,z) && (5.2.13)\\
&= (1_{xy}1_z)\circ 1_{xyz} && (5.2.9)\\
&= 1_{xyz}\\
&= (1_x 1_{yz})\circ 1_{xyz}\\
&= \left(1_x C^0(y,z)\right)\circ C^0(x,yz) && (5.2.9)\\
&= R^0(x,y,z) && (5.2.15).
\end{aligned}
$$

Thus, we get

$$L^0 = R^0.$$

<u>Induction case</u> ($k \geq 0$): We assume that the natural transformations L^k and R^k are equal. We want to show that the natural transformations L^{k+1} and R^{k+1} are equal. Let $x,y,z \in \mathrm{Obj}(\mathscr{M})$. We will show that

$$L^{k+1}(x,y,z) = R^{k+1}(x,y,z).$$

Consider following diagram:

$$
\begin{tikzcd}
(xyz)^{(k+1)} \arrow[r, "\delta^{(k)}_{xy,z}"] \arrow[d, "\delta^{(k)}_{x,yz}"'] \arrow[dr, phantom, "\mathtt{Prev}"] & \left(z'(xy)'\right)^{(k)} \arrow[r, "C^k(z',(xy)')"] \arrow[d, "\langle\delta_{x,y}\rangle^{(k)}"] \arrow[dr, phantom, "\mathtt{Nat}"] & z^{(k+1)}\Box(xy)^{(k+1)} \arrow[d, "\left\langle\delta^{(k)}_{x,y}\right\rangle"] \\
\left((yz)'x'\right)^{(k)} \arrow[r, "\langle\delta_{y,z}\rangle^{(k)}"] \arrow[d, "C^k((yz)',x')"'] \arrow[dr, phantom, "\mathtt{Nat}"] & \left(z'y'x'\right)^{(k)} \arrow[r, "C^k(z',y'x')"] \arrow[d, "C^k(z'y',x')"] \arrow[dr, phantom, "\mathtt{Ind}"] & z^{(k+1)}\Box\left(y'x'\right)^{(k)} \arrow[d, "\left\langle C^k(y',x')\right\rangle"] \\
(yz)^{(k+1)}\Box x^{(k+1)} \arrow[r, "\left\langle\delta^{(k)}_{y,z}\right\rangle"'] & \left(z'y'\right)^{(k)}\Box x^{(k+1)} \arrow[r, "\left\langle C^k(z',y')\right\rangle"'] & z^{(k+1)}\Box y^{(k+1)}\Box x^{(k+1)}.
\end{tikzcd}
$$

Here, $\Box$ stands for the functor $\Box_k$. Note that, the right bottom object

$$z^{(k+1)}\Box y^{(k+1)}\Box x^{(k+1)} = x^{(k+1)}\Box_{k+1} y^{(k+1)}\Box_{k+1} z^{(k+1)}.$$

Consider the square `Prev` in the above diagram. We have

$$\begin{aligned}
\left(\mathrm{Id}_{z'}\delta_{x,y}\right)^{(k)} \circ \delta^{(k)}_{xy,z} &= \left(\left(\mathrm{Id}_{z'}\delta_{x,y}\right) \circ \delta_{xy,z}\right)^{(k)} \\
&= \left(\left(\delta_{y,z}\mathrm{Id}_{x'}\right) \circ \delta_{x,yz}\right)^{(k)} && (5.2.17) \\
&= \left(\delta_{y,z}\mathrm{Id}_{x'}\right)^{(k)} \circ \delta^{(k)}_{x,yz}
\end{aligned}$$

where the first and third equalities hold since $(-)^{(k)}$ is a functor. Thus, the square `Prev` commutes. Next, let's look at the square `Ind`. We have

$$\begin{aligned}
\left\langle C^k(y',x')\right\rangle \circ C^k(z',y'x') &= R^k(z',y',x') && (5.2.15) \\
&= L^k(z',y',x') && \text{induction} \\
&= \left\langle C^k(z',y')\right\rangle \circ C^k(z'y',x'). && (5.2.13)
\end{aligned}$$

Thus, the square `Ind` commutes as well.

The squares `Nat` commute because C^k is a natural transformation (Definition 5.2.8). Consequently, the entire diagram commutes.

Now, consider the top-right composition of the entire diagram. We have

$$\begin{aligned}
&\left(\left\langle C^k(y',x')\right\rangle \circ \left\langle \delta^{(k)}_{x,y}\right\rangle\right) \circ \left(C^k(z',(xy)') \circ \delta^{(k)}_{xy,z}\right) \\
&\qquad = \left\langle C^{k+1}(x,y)\right\rangle \circ C^{k+1}(xy,z) && (5.2.11) \\
&\qquad = L^{k+1}(x,y,z). && (5.2.13)
\end{aligned}$$

The left-bottom composition of the entire diagram is given by

$$\begin{aligned}
&\left(\left\langle C^k(z',y')\right\rangle \circ \left\langle \delta^{(k)}_{y,z}\right\rangle\right) \circ \left(C^k((yz)',x') \circ \delta^{(k)}_{x,yz}\right) \\
&\qquad = \left\langle C^{k+1}(y,z)\right\rangle \circ C^{k+1}(x,yz) && (5.2.11) \\
&\qquad = R^{k+1}(x,y,z). && (5.2.15)
\end{aligned}$$

Since the entire diagram commutes, we get

$$L^{k+1}(x,y,z) = R^{k+1}(x,y,z).$$

Now since $x,y,z \in \mathrm{Obj}(\mathscr{M})$ were arbitrarily chosen, we get that the natural transformations L^{k+1} and R^{k+1} are equal. Thus, from the mathematical induction we conclude that

$$L^k = R^k$$

for every $k \in \mathbb{N}$. $\square$

Definition 5.2.21. Let $k \in \mathbb{N}$. For $m \in \mathbb{N}$ consider the functors

$$H_m, K_m : \mathscr{M}^m \longrightarrow \mathscr{M}$$

given by

$$\begin{aligned} H_m(x_1,\ldots,x_m) &= (x_1 \cdots x_m)^{(k)} && \text{on objects} \\ H_m(f_1,\ldots,f_m) &= (f_1 \cdots f_m)^{(k)} && \text{on morphisms} \end{aligned}$$

and

$$\begin{aligned} K_m(x_1,\ldots,x_m) &= x_1^{(k)} \square \cdots \square x_m^{(k)} && \text{on objects} \\ K_m(f_1,\ldots,f_m) &= f_1^{(k)} \square \cdots \square f_m^{(k)} && \text{on morphisms.} \end{aligned}$$

Here, $\square$ stands for the functor $\square_k$. We will construct a collection, $\mathcal{D}ist_m^k$, of natural transformations from H_m to K_m, called the *distribution transformations* of order k over m variables. We will use induction on m.

<u>Base case</u> ($m = 0$): In this case, we have $H_0 = \mathrm{Const}_I = K_0$. Let $\mathrm{Id}_{\mathrm{Const}_I}$ be the only distribution transformation from H_0 to K_0. That is,

$$\mathcal{D}ist_0 := \{\mathrm{Id}_{\mathrm{Const}_I} : H_0 \Rightarrow K_0\}. \tag{5.2.22}$$

Base case ($m = 1$): In this case, we have $H_1 = (-)^{(k)} = K_1$. Let Id_{H_1} be the only distribution transformation from H_1 to K_1. That is,

$$\mathcal{D}ist_1 := \{\mathrm{Id}_{(-)^{(k)}} : H_1 \Rightarrow K_1\}. \tag{5.2.23}$$

Induction case ($m \geq 2$): Suppose, we have defined $\mathcal{D}ist_n$ for $n < m$. A natural transformation

$$\gamma : H_m \Rightarrow K_m \in \mathcal{D}ist_m$$

is constructed as follows: Let

$$1 \leq p < m, \qquad \alpha : H_p \Rightarrow K_p \in \mathcal{D}ist_p, \qquad \text{and} \qquad \beta : H_{m-p} \Rightarrow K_{m-p}.$$

Then, a natural transformation $\gamma : H_m \Rightarrow K_m$ is given as follows:

- For $X = (x_1, \ldots, x_m) \in \mathcal{M}^m$, let

$$Y = (x_1, \ldots, x_p), \qquad y = x_1 \cdots x_p,$$
$$Z = (x_{p+1}, \ldots, x_m), \qquad z = x_{p+1} \cdots x_m.$$

  The morphism

$$\gamma(X) : (x_1 \cdots x_m)^{(k)} \longrightarrow x_1^{(k)} \square \cdots \square x_m^{(k)}$$

  is given by

$$\gamma(X) := \big(\alpha(Y) \square \beta(Z)\big) \circ C^k(y,\ z) \tag{5.2.24}$$

  where C^k is as in Definition 5.2.8. That is,

$$\gamma(X) : (x_1 \cdots x_m)^{(k)} \xrightarrow{C^k(y,z)} (y)^{(k)} \square (z)^{(k)} \xrightarrow{\alpha(Y) \square \beta(Z)} x_1^{(k)} \square \cdots \square x_m^{(k)}.$$

Since α, β, and C^k are all natural transformations, it follows that γ is a natural transformation as well.

The collection $\mathcal{D}ist^k_m$ is defined as the set of all natural transformation

$$\gamma : H_m \longrightarrow K_m$$

that are constructed as above. That is,

$$\mathcal{D}ist^k_m := \left\{\gamma \mid 1 \leq p < m,\ \alpha \in \mathcal{D}ist^k_p,\ \text{ and }\ \beta \in \mathcal{D}ist^k_{m-p}\right\}. \tag{5.2.25}$$

◇

Theorem 5.2.26. *Let $k, m \in \mathbb{N}$ and $\alpha, \beta \in \mathcal{D}ist^k_m$ be distribution natural transformations. Then, we have $\alpha = \beta$. In other words, $\mathcal{D}ist^k_m$ is a singleton.*

We will denote this unique distribution natural transformation by

$$C^k_m \in \mathcal{D}ist^k_m.$$

Proof. We will prove this theorem using induction on m.

Base case ($m = 0$ or $m = 1$): From (5.2.22) and (5.2.23), we conclude that $\mathcal{D}ist_0$ and $\mathcal{D}ist_1$ are singletons.

Induction case ($m \geq 2$): Assume that, for $n < m$ we have that $\mathcal{D}ist_n$ is a singleton. Let $\gamma_n \in \mathcal{D}ist^k_n$ be the unique distribution natural transformation for $n < m$. Let $\alpha, \beta \in \mathcal{D}ist_m$. We will show that $\alpha = \beta$. That is, for $X = (x_1, \ldots, x_m) \in \mathcal{M}^m$, we wish to show that

$$\alpha(X) = \beta(X) : (x_1 \cdots x_m)^{(k)} \longrightarrow x_1^{(k)} \Box \cdots \Box x_m^{(k)}$$

where $\Box$ stands for the functor $\Box_k$.

Since $\alpha \in \mathcal{D}ist_m$ from Definition 5.2.21, there exists $1 \leq p < m$ such that

$$\alpha(X) = \left(\gamma_p(Y_p) \Box \gamma_{m-p}(Z_p)\right) \circ C^k(y_p,\ z_p),$$

where

$$Y_p = (x_1, \ldots, x_p), \qquad\qquad y_p = x_1 \cdots x_p,$$

$$Z_p = (x_{p+1}, \ldots, x_m), \qquad\qquad z_p = x_{p+1} \cdots x_m,$$

$$\gamma_p \in \mathcal{D}ist_p, \qquad \text{and} \qquad \gamma_{m-p} \in \mathcal{D}ist_{m-p}.$$

Similarly, there exists $1 \leq q < m$ such that

$$\beta(X) = \left(\gamma_q(Y_q) \Box \gamma_{m-q}(Z_q)\right) \circ C^k(y_q,\ z_q)$$

where

$$Y_q = (x_1, \ldots, x_q), \qquad\qquad y_q = x_1 \cdots x_q,$$

$$Z_q = (x_{q+1}, \ldots, x_m), \qquad\qquad z_q = x_{q+1} \cdots x_m,$$

$$\gamma_q \in \mathcal{D}ist_q, \qquad \text{and} \qquad \gamma_{m-q} \in \mathcal{D}ist_{m-q}.$$

If $p = q$, then we get $\alpha(X) = \beta(X)$ as required. Now, without the loss of generality assume $p < q$. Let

$$A := Y_p = (x_1, \ldots, x_p), \qquad B := (x_{p+1}, \ldots, x_q), \qquad D := Z_q = (x_{q+1}, \ldots, x_m)$$

$$a := y_p = x_1 \cdots x_p, \qquad b := x_{p+1} \cdots x_q, \qquad d := z_q = x_{q+1} \cdots x_m$$

Furthermore, let

$$AB := Y_q = (x_1, \cdots, x_q), \qquad BD := Z_p = (x_p, \cdots, x_m).$$

With the new notation, we have

$$\alpha(X) = \left(\gamma_p(A) \Box \gamma_{m-p}(BD)\right) \circ C^k(a,\ bd) \tag{i}$$

$$\beta(X) = \left(\gamma_q(AB) \Box \gamma_{m-q}(D)\right) \circ C^k(ab,\ d). \tag{ii}$$

From Definition 5.2.21, we get

$$\gamma_{m-p}(BD) = \big(\gamma_{q-p}(B)\Box\gamma_{m-q}(D)\big)\circ C^k(b,\ d) \tag{ii}$$

$$\gamma_q(AB) = \big(\gamma_p(A)\Box\gamma_{q-p}(B)\big)\circ C^k(a,\ b). \tag{iii}$$

Now, consider the following diagram:

$$\begin{tikzcd}
 & (abd)^{(k)} \arrow[dl, "C^k(ab,\ d)"'] \arrow[dr, "C^k(a,bd)"] & \\
(ab)^{(k)}\Box(d)^{(k)} \arrow[dr, "C^k(a,b)\Box 1_{d^{(k)}}"'] & & (a)^{(k)}\Box(bd)^{(k)} \arrow[dl, "1_{a^{(k)}}\Box C^k(b,d)"] \\
 & (a)^{(k)}\Box(b)^{(k)}\Box(d)^{(k)} \arrow[d, "\gamma_p(A)\Box\gamma_{q-p}(B)\Box\gamma_{m-q}(D)"] & \\
 & x_1^{(k)}\Box\cdots\Box x_m^{(k)} &
\end{tikzcd}$$

In the above diagram, the left composite of the diamond is given by

$$\Big(C^k(a,b)\Box 1_{d^{(k)}}\Big)\circ C^k(ab,d) = L^k(a,b,d). \tag{5.2.13}$$

Whereas, the right composite of the diagram is given by

$$\Big(1_{a^{(k)}\Box C^k(b,d)}\Big)\circ C^k(a,bd) = R^k(a,b,d). \tag{5.2.15}$$

From Theorem 5.2.19, we get

$$L^k(a,b,d) = R^k(a,b,d).$$

Therefore, the above diagram commutes. Now, the left composite of the entire diagram is given by

$$
\begin{aligned}
&\big(\gamma_p(A)\square\gamma_{q-p}(B)\square\gamma_{m-q}(D)\big)\circ\Big(C^k(a,b)\square 1_{d^{(k)}}\Big)\circ C^k(ab,d)\\
&=\Big(\Big(\big(\gamma_p(A)\square\gamma_{q-p}(B)\big)\circ C^k(a,b)\Big)\square\gamma_{m-q}(D)\Big)\ \circ\ C^k(ab,d)\\
&=\big(\gamma_q(AB)\square\gamma_{m-q}(D)\big)\circ C^k(ab,d) && (iii)\\
&=\beta(X). && (ii)
\end{aligned}
$$

Here, the first equality holds due to the functoriality of multiplication in $\mathscr{M}$. Similarly, the right composite of the entire diagram is given by

$$
\begin{aligned}
&\big(\gamma_p(A)\square\gamma_{q-p}(B)\square\gamma_{m-q}(D)\big)\circ\Big(1_{a^{(k)}}\square C^k(b,d)\Big)\circ C^k(a,bd)\\
&=\Big(\gamma_p(A)\square\Big(\big(\gamma_{q-p}(B)\square\gamma_{m-q}(D)\big)\circ C^k(b,d)\Big)\Big)\ \circ\ C^k(a,bd)\\
&=\big(\gamma_p(A)\square\gamma_{m-p}(BD)\big)\circ C^k(a,bd) && (ii)\\
&=\alpha(X). && (i)
\end{aligned}
$$

Since the entire diagram commutes, we get $\alpha(X)=\beta(X)$ as required. $\square$

Corollary 5.2.27. *Let $k\in\mathbb{N}$. For $m\in\mathbb{N}$ consider the functors*

$$H_m, K_m : \mathscr{M}^m \longrightarrow \mathscr{M}$$

as in Definition 5.2.21. Let

$$C^k_m : H_m \Rightarrow K_m \in \mathcal{D}ist^k_m,$$

be the unique distribution transformations of order k over m variables as in Theorem 5.2.26. Then, the following equalities hold:

$$C^k_0 = \mathrm{Id}_{\mathrm{Const}_I} : H_0 \Rightarrow K_0. \tag{5.2.28}$$

$$C^k_1 = \mathrm{Id}_{(-)^{(k)}} : H_1 \Rightarrow K_1. \tag{5.2.29}$$

For $m \geq 2$, $1 \leq p < m$, and $X = (x_1, \dots, x_m) \in \mathscr{M}^m$ we have

$$C^k_m(X) = \left(C^k_p(Y) \Box C^k_{m-p}(Z)\right) \circ C^k(y,\ z) \tag{5.2.30}$$

where

$$Y = (x_1, \dots, x_p), \qquad Z = (x_{p+1}, \dots, x_m)$$

$$y = x_1 \cdots x_p, \qquad z = x_{p+1} \cdots x_m,$$

and C^k is as in Definition 5.2.8.

Proof. Above equalities are direct consequence of Definition 5.2.21 and Theorem 5.2.26.□

Lemma 5.2.31. *For $k \in \mathbb{N}$, the following equalities hold:*

$$C^k_2 = C^k \tag{5.2.32}$$

$$C^1_2 = \delta \tag{5.2.33}$$

$$C^k_3 = L^k = R^k \tag{5.2.34}$$

where C^k is as in Definition 5.2.8, δ is as in Definition 5.2.6, L^k is as in Definition 5.2.12, and R^k is as in Definition 5.2.14.

Proof. For $x, y \in \mathrm{Obj}(\mathscr{M})$ observe that

$$\begin{aligned} C^k_2(x,y) &= \left(C^k_1(x) \Box C^k_1(y)\right) \circ C^k(x,y) && (5.2.30) \\ &= C^k(x,y). && (5.2.29) \end{aligned}$$

Next, we have

$$\begin{aligned} C_2^1(x,y) &= C^1(x,y) && (5.2.32) \\ &= C^0(y',x') \circ \delta_{x,y} && (5.2.11) \\ &= \delta_{x,y}. && (5.2.9) \end{aligned}$$

Finally, we have

$$\begin{aligned} C_3^k(x,y,z) &= \left(C_2^k(x,y) \Box C_1^k(z)\right) \circ C^k(xy,\ z) && (5.2.30) \\ &= \left(C^k(x,y) \Box 1_{z^{(k)}}\right) \circ C^k(xy,\ z) && (5.2.29) \text{ and } (5.2.32) \\ &= L^k(x,y,z) && (5.2.13) \\ &= R^k(x,y,z). && (5.2.20) \end{aligned}$$

$\square$

Proposition 5.2.35. *Let $k \in \mathbb{N}$. For $m \in \mathbb{N}$ let $x_1, \dots, x_m \in \mathscr{M}$. Then, the following diagram commutes.*

$$\begin{tikzcd} (x_1 \cdots x_m)^{(k+1)} \arrow[r, "\left(C_m^1(X)\right)^{(k)}"] \arrow[d, "C_m^{k+1}(X)"'] & (x'_m \cdots x'_1)^{(k)} \arrow[d, "C_m^k(X')"] \\ x_m^{(k+1)} \Box \cdots \Box x_1^{(k+1)} \arrow[r, equal] & x_m^{(k+1)} \Box \cdots \Box x_1^{(k+1)} \end{tikzcd} \tag{5.2.36}$$

where $X = (x_1, \dots, x_m)$, $X' = (x'_m, \dots, x'_1)$, and $\Box = \Box_k$.

Proof. We will show this using induction on m.

Base case ($m = 0$): In this case, the objects in diagram (5.2.36) are equal to I and the morphisms are 1_I from Corollary 5.2.27. Thus, the diagram commutes.

Base case ($m = 1$): In this case, we get

$$\begin{aligned} C_1^k(x_1') &= 1_{x_1^{(k+1)}} && (5.2.29) \\ (C_1^1(x_1))^{(k)} &= (1_{x_1'})^{(k)} && (5.2.29) \\ &= 1_{x_1^{(k+1)}} \\ C_1^{k+1}(x_1) &= 1_{x_1^{(k+1)}}. && (5.2.29) \end{aligned}$$

Thus, the diagram commutes.

Induction case ($m \geq 2$): Assume that the diagram (5.2.36) commutes for $n < m$. We want to show that the diagram commutes for m.

Let $Y = (x_1, \ldots, x_{m-1})$, $Y' = (x_{m-1}', \ldots, x_1')$, and $y = x_1 \cdots x_{m-1}$. Consider the following diagram.

$$\begin{tikzcd}
(x_1 \cdots x_m)^{(k+1)} \arrow[d, "\left(C_2^1(y,\, x_m)\right)^{(k)}"'] & \\
(x_m'(x_1 \cdots x_{m-1})')^{(k)} \arrow[r, "\left(1_{x_m'} C_{m-1}^1(Y)\right)^{(k)}"] \arrow[d, "C_2^k(x_m',\, y')"'] & (x_m' \cdots x_1')^{(k)} \arrow[d, "C_2^k(x_m',\, x_{m-1}' \cdots x_1')"] \\
x_m^{k+1} \square (x_1 \cdots x_{m-1})^{(k+1)} \arrow[r, "1_{x_m^{k+1}} \square \left(C_{m-1}^1(Y)\right)^{(k)}"] \arrow[d, "1_{x_m^{(k+1)}} \square C_{m-1}^{(k+1)}(Y)"'] & x_m^{k+1} \square (x_{m-1}' \cdots x_1')^{(k)} \arrow[d, "1_{x_m^{(k+1)}} \square C_{m-1}^k(Y')"] \\
x_m^{(k+1)} \square \cdots \square x_1^{(k+1)} \arrow[r, equal] & x_m^{(k+1)} \square \cdots \square x_1^{(k+1)}
\end{tikzcd}$$

Here, the top square commutes since C_2^k is a natural transformation. The bottom square commute because of the induction hypothesis.

First, consider the first two morphisms in the left vertical composition. We get,

$$\begin{aligned} C_2^k(x'_m, y') \circ \left(C_2^1(y,\ x_m)\right)^{(k)} &= C^k(x'_m, y') \circ (\delta(y,\ x_m))^{(k)} && \text{(5.2.32) and (5.2.33)} \\ &= C^{k+1}(y, x_m). && \text{(5.2.11)} \end{aligned}$$

Now, the left vertical composite is given by

$$\begin{aligned} &\left(1_{x_m^{(k+1)}} \square C_{m-1}^{(k+1)}(Y)\right) \circ C_2^k(x'_m, y') \circ \left(C_2^1(y,\ x_1)\right)^{(k)} \\ &\qquad = \left(C_1^{k+1}(x_m) \square C_{m-1}^{(k+1)}(Y)\right) \circ C^{k+1}(y, x_m) && \text{(5.2.29)} \\ &\qquad = C_m^{(k+1)}(X). && \text{(5.2.30)} \end{aligned}$$

The right vertical composite is given by

$$\begin{aligned} &\left(1_{x_m^{(k+1)}} \square C_{m-1}^k(Y')\right) \circ C_2^k(x'_m,\ x'_{m-1} \cdots x'_1) \\ &\qquad = \left(C_1^k(x'_m) \square C_{m-1}^k(Y')\right) \circ C_2^k(x'_m,\ x'_{m-1} \cdots x'_1) && \text{(5.2.29)} \\ &\qquad = \left(C_1^k(x'_m) \square C_{m-1}^k(Y')\right) \circ C^k(x'_m,\ x'_{m-1} \cdots x'_1) && \text{(5.2.32)} \\ &\qquad = C_m^k(X'). && \text{(5.2.30)} \end{aligned}$$

Finally, the composite of the top vertical arrow with top horizontal arrow is given by

$$\begin{aligned} \left(1_{x'_m} C_{m-1}^1(Y)\right)^{(k)} \circ \left(C_2^1(y,\ x_m)\right)^{(k)} &= \left(1_{x'_m} C_{m-1}^1(Y) \circ C_2^1(y,\ x_m)\right)^{(k)} \\ &= \left(C_1^1(x_m) C_{m-1}^1(Y) \circ C_2^1(y,\ x_m)\right)^{(k)} && \text{(5.2.29)} \\ &= \left(C_1^1(x_m) C_{m-1}^1(Y) \circ C^1(y,\ x_m)\right)^{(k)} && \text{(5.2.32)} \\ &= \left(C_m^1(X)\right)^{(k)}. && \text{(5.2.30)} \end{aligned}$$

Now, since the above diagram commutes, we get

$$C_m^k(X') \circ \left(C_m^1(X)\right)^{(k)} = C_m^{k+1}(X)$$

as required. □

5.3 Reduction Natural Isomorphism

Definition 5.3.1. Define a function

$$-\%2 : \mathbb{N} \longrightarrow \{0,1\}$$

as follows:

$$(n\%2) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases} \tag{5.3.2}$$

◇

Let $x \in M$. There are are two canonical compositions that reduce the dash from x'' to x. These are

$$x'' \xrightarrow{1_{x''}\,\epsilon_x^{-1}} x''x'x \xrightarrow{\epsilon_{x'}\,1_x} x$$

and

$$x'' \xrightarrow{\eta_x 1_{x''}} xx'x'' \xrightarrow{1_x \eta_x^{-1}} x.$$

We will show in Proposition 5.3.3 that these two composites are equal. Let's denote the above composite by B_x.

We will generalize this construction to define a natural isomorphism that reduces iterated dashes down to either a single or no dash. For instance, a composition with domain $x^{(4)}$ and co-domain x may be represented as

$$x^{(4)} \xrightarrow{B_{x''}} x'' \xrightarrow{B_x} x.$$

In this section, we will give a composition of canonical morphisms with domain $x^{(k)}$ and co-domain $x^{(k\%2)}$. We refer to such morphisms as *reduction morphisms*.

Proposition 5.3.3. *For $x \in \mathrm{Obj}(\mathscr{M})$ the following diagram commutes*

$$\begin{tikzcd}
x'' \arrow[r, "1_{x''}\epsilon_x^{-1}"] \arrow[d, "\eta_x 1_{x''}"'] & x''x'x \arrow[d, "\epsilon_{x'}1_x"] \\
xx'x'' \arrow[r, "1_x\eta_{x'}^{-1}"'] & x
\end{tikzcd}$$

Proof. Let $f, g : x'' \longrightarrow x$ be the top-right and left-bottom composite respectively. We want to show $f = g$.

Consider the following diagram

$$\begin{tikzcd}
x'' \arrow[r, "1_{x''}\epsilon_x^{-1}"] \arrow[d, "\eta_x 1_{x''}"'] & x''x'x \arrow[r, "\epsilon_{x'}1_x"] \arrow[d, "\eta_x 1_{x''x'x}"'] & x \arrow[d, "\eta_x 1_x"] \\
xx'x'' \arrow[r, "1_{xx'x''}\epsilon_x^{-1}"] \arrow[d, "1_x\eta_{x'}^{-1}"'] & xx'x''x'x \arrow[r, "1_{xx'}\epsilon_{x'}1_x"] \arrow[d, "1_x\eta_{x'}^{-1}1_{x'x}"'] & xx'x \arrow[dl, equal] \\
x \arrow[r, "1_x\epsilon_x^{-1}"'] & xx'x &
\end{tikzcd}$$

The diagram commutes since the squares are multiplication functoriality squares and the bottom-right triangle is the cancellation triangle. Observe that, the top composite is $f : x'' \longrightarrow x$ and the left composite is $g : x'' \longrightarrow x$. Since the diagram commutes, we get

$$\left(\eta_x 1_x\right) \circ f = \left(1_x \epsilon_x^{-1}\right) \circ g.$$

Thus, we get

$$
\begin{aligned}
g &= (1_x\epsilon_x)\circ\left(1_x\epsilon_x^{-1}\right)\circ g \\
&= (1_x\epsilon_x)\circ\left(\eta_x 1_x\right)\circ f \\
&= f. && (4.1.18)
\end{aligned}
$$

□

Definition 5.3.4. Define a natural transformation

$$B : (-)^{(2)} \Rightarrow \mathrm{Id}_{\mathscr{M}}$$

as follows:

Objects: For $x \in \mathrm{Obj}(\mathscr{M})$, define the map $B_x : x'' \longrightarrow x$ as

$$B_x := (\epsilon_{x'} 1_x)\circ\left(1_{x''}\epsilon_x^{-1}\right). \qquad (5.3.5)$$

That is,

$$B_x : x'' \xrightarrow{\langle\epsilon_x^{-1}\rangle} x''x'x \xrightarrow{\langle\epsilon_{x'}\rangle} x$$

Naturality: Since the composition morphism are themselves expanded instances of natural transformations, the composite B_x is a natural transformation. ◇

Definition 5.3.6. For $k \in \mathbb{N}$ let $F_k, G_k : \mathscr{M} \longrightarrow \mathscr{M}$ be the following functors:

$$F_k := (-)^{(k)} \qquad \text{and} \qquad G_k := (-)^{(k\%2)}.$$

Define a natural transformation

$$B^k : F_k \Rightarrow G_k,$$

called the *reduction transformation*, inductively as follows:

Base case ($k = 0$): In this case, we have $F_0 = \mathrm{Id}_{\mathscr{M}} = G_0$. Define the natural transformation $B^0 : F_0 \Rightarrow G_0$ as follows:

$$B^0 := \mathrm{Id}_{\mathrm{Id}_{\mathscr{M}}}. \tag{5.3.7}$$

Base case ($k = 1$): In this case, we have $F_1 = (-)' = G_1$. Define the natural transformation $B^1 : F_1 \Rightarrow G_1$ as follows:

$$B^1 := \mathrm{Id}_{(-)'}. \tag{5.3.8}$$

Induction case ($k \geq 0$): Assume that we have defined the natural transformations $B^k : F_k \Rightarrow G_k$. We will define the natural transformation $B^{k+2} : F_{k+2} \Rightarrow G_{k+2}$ as follows:

Objects: Let $x \in \mathrm{Obj}(\mathscr{M})$. The morphism $B^{k+2}_x : x^{(k+2)} \longrightarrow x^{(k\%2)}$ is given by

$$B^{k+2}_x := B^k_x \circ B_{x^{(k)}}. \tag{5.3.9}$$

That is,

$$B^{k+2}_x : x^{(k+2)} \xrightarrow{B_{x^{(k)}}} x^{(k)} \xrightarrow{B^k_x} x^{(k\%2)}$$

Naturality: We have assumed that B^k is a natural transformation. From Definition 5.3.4, we have that B is a natural transformation. Therefore, we get that B^{k+2} is a natural transformation. ◇

Definition 5.3.10. Let $m \in \mathbb{N}$ and $L = (k_1, \ldots, k_m) \in \mathbb{N}^m$. Let

$$F_m, G_m : \mathscr{M}^m \longrightarrow \mathscr{M}$$

be the following functors:

$$F_m(x_1, \ldots, x_m) = x_1^{(k_1)} \cdots x_m^{(k_m)} \qquad \text{on objects}$$

$$F_m(f_1, \ldots, f_m) = f_1^{(k_1)} \cdots f_m^{(k_m)} \qquad \text{on morphisms}$$

and

$$G_m(x_1,\dots,x_m) = x_1^{(k_1\%2)}\cdots x_m^{(k_m\%2)} \qquad \text{on objects}$$

$$G_m(f_1,\dots,f_m) = f_1^{(k_1\%2)}\cdots f_m^{(k_m\%2)} \qquad \text{on morphisms.}$$

Define a natural transformation

$$B_m^L : F_m \Rightarrow G_m$$

as

$$B_m^L := B^{k_1}\cdots B^{k_m}. \tag{5.3.11}$$

◇

Remark 5.3.12. We will take a little detour to show that using the above definition, we can express the natural transformation η entirely in terms of the natural transformation ϵ. ◇

Lemma 5.3.13. *For $x \in \mathrm{Obj}(\mathcal{M})$, the equality*

$$\eta_x = (B_x 1_{x'}) \circ \left(\epsilon_{x'}^{-1}\right) \tag{5.3.14}$$

holds. That is, the following diagram commutes:

$$\begin{tikzcd} & x''x' \arrow[dl, "\epsilon_{x'}"'] \arrow[d, "B_x 1_{x'}"] \\ I \arrow[r, "\eta_x"'] & xx' \end{tikzcd}$$

Proof. Consider the following diagram:

$$\begin{tikzcd} & x''x' \arrow[dl, equal] \arrow[d, "1_{x''}\epsilon_x^{-1}1_{x'}"] \\ x''x' \arrow[r, "1_{x''}1_{x'}\eta_x"'] \arrow[d, "\epsilon_{x'}"'] & x''x'xx' \arrow[d, "\epsilon_{x'}1_x1_{x'}"] \\ I \arrow[r, "\eta_x"'] & xx' \end{tikzcd}$$

The top triangle commutes due to the cancellation condition (4.1.5). The bottom square commutes since it is a multiplication functoriality square. The right vertical composition is given by

$$\begin{aligned}(\epsilon_{x'}1_x 1_{x'})\circ\left(1_{x''}\epsilon_x^{-1}1_{x'}\right) &= \left(\epsilon_{x'}1_x\circ 1_{x''}\epsilon_x^{-1}\right)1_{x'} \\ &= B_x 1_{x'}. \qquad (5.3.5)\end{aligned}$$

Since the above diagram commutes, we get

$$B_x 1_{x'} = \eta_x \circ \epsilon_{x'}.$$

Therefore, we get

$$\eta_x = (B_x 1_{x'})\circ\left(\epsilon_{x'}^{-1}\right)$$

as required. □

5.4 Compatibility with Cancellation

Let $\mathcal{M} = (\mathcal{M}, \cdot, I, (-)')$ be a semi-strict categorical group.

Proposition 5.4.1. *Let $x \in \mathcal{M}$. Then, we get*

$$\left(\eta_x^{-1}1_x\right) = (1_x\epsilon_x) : xx'x \longrightarrow x \tag{5.4.2}$$

and

$$(\epsilon_x 1_x) = \left(1_x\eta_x^{-1}\right) : x'xx' \longrightarrow x'. \tag{5.4.3}$$

Proof. Both follow from cancellation equations (4.1.18) and (4.1.19). □

Proposition 5.4.4. *Let $x, y \in \mathrm{Obj}(\mathcal{M})$. Then, the following diagram commutes*

$$\begin{tikzcd}
(xy)'xy \arrow[r, "\epsilon_{xy}"] \arrow[d, "\delta_{x,y}1_{xy}"'] & I \\
y'x'xy \arrow[r, "1_{y'}\epsilon_x 1_y"'] & y'y. \arrow[u, "\epsilon_y"']
\end{tikzcd}$$

Proof. Consider the following diagram

$$\begin{tikzcd}
(xy)'xy \arrow[r, "\epsilon_{xy}"] \arrow[d, "\left\langle \epsilon_y^{-1} \right\rangle"'] & I \arrow[d, "\epsilon_y^{-1}"] \\
y'y(xy)'xy \arrow[r, "\langle \epsilon_{xy} \rangle"] \arrow[d, "\langle \epsilon_x^{-1} \rangle"'] & y'y \arrow[d, "1_{y'}\epsilon_x^{-1}1_y"] \\
y'x'xy(xy)'xy \arrow[r, "\langle \epsilon_{xy} \rangle"'] & y'x'xy
\end{tikzcd}$$

In the above diagram, both squares commute because of functoriality of the tensor product. From Proposition 5.2.4 and Definition 5.2.6 we get that the left-bottom composition is

$$\delta_{x,y}1_{xy} : (xy)'xy \longrightarrow y'x'xy.$$

Thus, we get

$$\epsilon_{xy} = \epsilon_y \circ \left(1_{y'}\epsilon_y 1_y\right) \circ \left(\delta_{x,y}1_{xy}\right)$$

as required. □

Definition 5.4.5. Let $m \in \mathbb{N}$. Let $F_m, G_m : \mathscr{M}^m \longrightarrow \mathscr{M}$ be following functors.

$$F_m(x_1,\ldots,x_m) = x'_m \cdots x'_1 x_1 \cdots x_m \qquad \text{on objects}$$

$$F_m(f_1,\ldots,f_m) = f'_m \cdots f'_1 f_1 \cdots f_m \qquad \text{on morphisms}$$

and

$$G_m(x_1,\ldots,x_m) = I \qquad \text{on objects}$$

$$G_m(f_1,\ldots,f_m) = 1_I \qquad \text{on morphisms.}$$

Define a natural transformation

$$A_m : F_m \Rightarrow G_m$$

as follows:

For $m = 0$, we get $F_0 = \mathrm{Const}_I = G_0$. Define

$$A_0 := \mathrm{Id}_{\mathrm{Const}_I}. \tag{5.4.6}$$

For $m \geq 1$, we define A_m on objects as follows: Let $X = (x_1, \ldots, x_m) \in \mathscr{M}^m$. For $1 \leq i \leq m$, let

$$\langle \epsilon_{x_i} \rangle : x'_m \cdots x'_i x_i \cdots x_m \longrightarrow x'_m \cdots x'_{i+1} x_{i+1} \cdots x_m$$

be the morphism given by

$$\langle \epsilon_{x_i} \rangle := 1_{x'_m \cdots x'_{i+1}} \epsilon_{x_i} 1_{x_{i+1} \cdots x_m}. \tag{5.4.7}$$

Define

$$A_m(X) : x'_m \cdots x'_1 x_1 \cdots x_m \longrightarrow I$$

as the iterated cancellation morphisms:

$$A_m(X) := \langle \epsilon_{x_m} \rangle \circ \cdots \circ \langle \epsilon_{x_1} \rangle. \tag{5.4.8}$$

That is,

$$\begin{tikzcd}
A_m(X) : x'_m \cdots x'_1 x_1 \cdots x_m \arrow[d, "\langle \epsilon_{x_1} \rangle"] \\
x'_m \cdots x'_2 x_2 \cdots x_m \arrow[d] \\
\vdots \arrow[d] \\
x'_m x_m \arrow[d, "\epsilon_{x_m}"] \\
I.
\end{tikzcd} \tag{5.4.9}$$

Since, each component morphism in the composition of $A_m(X)$ is a natural transformation, $A_m(X)$ is a natural transformation. $\diamond$

Proposition 5.4.10. *For $m \in \mathbb{N}$ and $X = (x_1, \ldots, x_{m+1}) \in \mathscr{M}^{m+1}$ we get*

$$A_{m+1}(X) = \epsilon_{x_{m+1}} \circ \left(1_{x'_{m+1}} A_m(Y) 1_{x_{m+1}}\right) \tag{5.4.11}$$

where $Y = (x_1, \ldots, x_m)$.

Proof. For $m = 0$ we see

$$\begin{aligned}
\epsilon_{x_1} \circ \left(1_{x'_1} A_0(Y) 1_{x_1}\right) &= \epsilon_{x_1} \circ \left(1_{x'_1} 1_I 1_{x_1}\right) && (5.4.6) \\
&= \epsilon_{x_1} \\
&= A_1(X).
\end{aligned}$$

Now, for $m \geq 1$ and $1 \leq i \leq m$, let

$$\langle \epsilon_{x_i} \rangle_m = 1_{x'_m \cdots x'_{i+1}} \epsilon_{x_i} 1_{x_{i+1} \cdots x_m}$$

same as in (5.4.7). Then, we get

$$\begin{aligned}
1_{x'_{m+1}} \langle \epsilon_{x_i} \rangle_m 1_{x_{m+1}} &= 1_{x'_{m+1}} 1_{x'_m \cdots x'_{i+1}} \epsilon_{x_i} 1_{x_{i+1} \cdots x_m} 1_{x_{m+1}} \\
&= 1_{x'_{m+1} \cdots x'_{i+1}} \epsilon_{x_i} 1_{x_{i+1} \cdots x_{m+1}} \\
&= \langle \epsilon_{x_i} \rangle_{m+1}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\epsilon_{x_{m+1}} \circ \left(1_{x'_{m+1}} A_m(Y) 1_{x_{m+1}}\right) \\
&\quad = \epsilon_{x_{m+1}} \circ \left(1_{x'_{m+1}} \langle \epsilon_{x_m} \rangle_m 1_{x_{m+1}}\right) \circ \cdots \circ \left(1_{x'_{m+1}} \langle \epsilon_{x_1} \rangle_m 1_{x_{m+1}}\right) && (5.4.8) \\
&\quad = \epsilon_{x_{m+1}} \circ \langle \epsilon_{x_m} \rangle_{m+1} \circ \cdots \circ \langle \epsilon_{x_1} \rangle_{m+1} \\
&\quad = A_{m+1}(X) && (5.4.8)
\end{aligned}$$

This completes the proof. □

Definition 5.4.12. Let $m \in \mathbb{N}$. Define a functor

$$(-)' : \mathcal{M}^m \longrightarrow \mathcal{M}^m$$

as

$$(x_1, \dots, x_m)' = (x'_m, \dots, x'_1) \qquad \text{on objects}$$

$$(f_1, \dots, f_m)' = (f'_m, \dots, f'_1) \qquad \text{on morphisms.}$$ ⋄

For $k \in \mathbb{N}$ let

$$(-)^{(k)} : \mathcal{M}^m \longrightarrow \mathcal{M}^m$$

denote the functor that we get after composing $(-)'$ with itself k times. Note that

$$(-)^{(0)} = \mathrm{Id}_{\mathcal{M}^m} \qquad \text{and} \qquad (-)^{(1)} = (-)'.$$

Example 5.4.13. Let $X = (x, y) \in \mathcal{M}^2$. We get

$$A_2(X) = \epsilon_y \circ \left(1_{y'} \epsilon_x 1_y\right).$$

That is,

$$\begin{tikzcd} A_2(X) : y'x'xy \arrow[r, "\langle \epsilon_x \rangle"] & y'y \arrow[d, "\epsilon_y"] \\ & I. \end{tikzcd}$$

Whereas, we get

$$A_2(X') = \epsilon_{x'} \circ \left(1_{x''} \epsilon_{y'} 1_{x'}\right).$$

That is,

$$\begin{tikzcd} A_2(X') : x''y''y'x' \arrow[r, "\langle \epsilon_{y'} \rangle"] & x''x' \arrow[d, "\epsilon_{x'}"] \\ & I. \end{tikzcd}$$

In general, for $m, k \geq 0$ and $X = (x_1, \ldots, x_m) \in \mathcal{M}^m$, we get

$$A_m(X^{(k)}) : \left(x_m^{(k+1)} \Box_k \cdots \Box_k x_1^{(k+1)}\right)\left(x_1^{(k)} \Box_k \cdots \Box_k x_m^{(k)}\right) \longrightarrow I. \qquad \diamond$$

Proposition 5.4.14. *Let $m \in \mathbb{N}$ and $X = (x_1, \cdots, x_m) \in \mathcal{M}^m$. Then, the following diagram commutes:*

$$\begin{tikzcd}
(x_1 \cdots x_m)' x_1 \cdots x_m \arrow[r, "\epsilon_x"] \arrow[d, "C_m^1(X) 1_x"'] & I \arrow[d, equal] \\
x_m' \cdots x_1' x_1 \cdots x_m \arrow[r, "A_m(X)"'] & I
\end{tikzcd} \tag{5.4.15}$$

where $X = (x_1, \ldots, x_m)$ and $x = x_1 \cdots x_m$.

Proof. We will show this using induction on m.

Base case ($m = 0$): In this case, we get $x = I$ and $x' = I' = I$. Also, we get

$$\epsilon_I = 1_I \qquad (4.1.3)$$

$$C_0^1(X) = 1_I \qquad (5.2.22)$$

$$A_0(X) = 1_I. \qquad (5.4.6)$$

Thus, the diagram commutes.

Base case ($m = 1$): In this case, we get

$$C_1^1(x_1) = 1_{x'} \qquad (5.2.23)$$

$$A_1(x_1) = \epsilon_{x_1}. \qquad (5.4.8)$$

Thus, the diagram (5.4.15) becomes

$$\begin{tikzcd}
x_1' x_1 \arrow[r, "1_{x_1' x_1}"] \arrow[d, "\epsilon_{x_1}"'] & x_1' x_1 \arrow[d, "\epsilon_{x_1}"] \\
I \arrow[r, equal] & I.
\end{tikzcd}$$

This diagram commutes trivially.

Induction case ($m \geq 1$): Assume that the diagram (5.4.15) commutes for m. We wish to show that the diagram commutes for $m+1$. Let

$$X = (x_1, \ldots, x_{m+1}) \qquad \text{and} \qquad x = x_1 \cdots x_{m+1}$$

Also, let

$$Y = (x_1, \ldots, x_m) \qquad \text{and } y = x_1 \cdots x_m.$$

Consider the following diagram,

$$
\begin{tikzcd}
(x_1\cdots x_{m+1})'x_1\cdots x_{m+1} \arrow[rr, "\epsilon_x"] \arrow[d, "\delta_{y,x_{m+1}}1_x"'] & & I \\
x'_{m+1}(x_1\cdots x_m)'(x_1\cdots x_m)x_{m+1} \arrow[rr, "1_{x'_{m+1}}\epsilon_y 1_{x_{m+1}}"] \arrow[d, "1_{x'_{m+1}}C^1_m(Y)1_x"'] & & x'_{m+1}x_{m+1} \arrow[u, "\epsilon_{x_{m+1}}"'] \arrow[d, equal] \\
x'_{m+1}(x'_m\cdots x'_1)(x_1\cdots x_m)x_{m+1} \arrow[rr, "1_{x'_{m+1}}A_m(Y)1_{x_{m+1}}"'] & & x'_{m+1}x_{m+1}
\end{tikzcd}
$$

The bottom square commutes because of the induction hypothesis. Whereas, the top square commutes due to Proposition 5.4.4.

We have,

$$
\begin{aligned}
\left(1_{x'_{m+1}}C^1_m(Y)\right)\circ\delta_{y,x_{m+1}} &= \left(C^1_1(x_{m+1})C^1_m(Y)\right)\circ\delta_{y,x_{m+1}} && (5.2.23)\\
&= \left(C^1_1(x_{m+1})C^1_m(Y)\right)\circ C^1(y,x_{m+1}) && (5.2.11)\\
&= C^1_{m+1}(X). && (5.2.24)
\end{aligned}
$$

It follows that, the left vertical composite is equal to

$$C^1_{m+1}(X)1_x : (x_1\cdots x_{m+1})'x_1\cdots x_{m+1} \longrightarrow x'_{m+1}\cdots x'_1 x_1\cdots x_{m+1}.$$

Finally, the bottom-right composite is given by

$$\epsilon_{x_{m+1}} \circ \left(1_{x'_{m+1}} A_m(Y) 1_{x_{m+1}}\right) = A_{m+1}(X). \tag{5.4.11}$$

Since the above diagram commutes, we get

$$A_{m+1}(X) \circ \left(C^1_{m+1}(X) 1_x\right) = \epsilon_x$$

as required. $\square$

Proposition 5.4.16. *Let $m \in \mathbb{N}$ and $x_1, \cdots, x_m \in \mathcal{M}$. Then, the following diagram commutes.*

$$\begin{tikzcd}
(x_1 \cdots x_m)^{(k+1)} (x_1 \cdots x_m)^{(k)} \arrow[r, "\epsilon_{x^{(k)}}"] \arrow[d, "C^{k+1}_m(X) C^k_m(X)"'] & I \arrow[d, equal] \\
\left(x^{(k+1)}_m \square_k \cdots \square_k x^{(k+1)}_1\right) \left(x^{(k)}_1 \square_k \cdots \square_k x^{(k)}_m\right) \arrow[r, "A_m(X^{(k)})"] & I
\end{tikzcd} \tag{5.4.17}$$

where

$$X = (x_1, \ldots, x_m), \qquad x = x_1 \cdots x_m$$

and $X^{(k)}$ is as in Definition 5.4.12.

Proof. We will show this using induction on k.

Base case ($k = 0$): In this case, the diagram (5.4.17) reduces to the diagram (5.4.15). From Proposition 5.4.14 the diagram commutes.

Induction case ($k \geq 0$): Suppose, we have shown that the diagram Proposition 5.4.16 commutes for k, we wish to show that the diagram commutes for $k+1$.

Consider the following diagram

$$\begin{tikzcd}
(x_1 \cdots x_m)^{(k+2)}(x_1 \cdots x_m)^{(k+1)} \arrow[rr, "\epsilon_{x^{(k+1)}}"] \arrow[d, "\left(C_m^1(X)\right)^{(k+1)}\left(C_m^1(X)\right)^{(k)}"'] & & I \arrow[d, equal] \\
(x'_m \cdots x'_1)^{(k+1)}(x'_m \cdots x'_1)^{(k)} \arrow[rr, "\epsilon_{(x'_m \cdots x'_1)^{(k)}}"] \arrow[d, "C_m^{k+1}(X')C_m^k(X')"'] & & I \arrow[d, equal] \\
\left(x_m^{(k+2)} \square_{k+1} \cdots \square_{k+1} x_1^{(k+2)}\right)\left(x_1^{(k+1)} \square_{k+1} \cdots \square_{k+1} x_m^{(k+1)}\right) \arrow[rr, "A_m(X^{(k+1)})"'] & & I
\end{tikzcd}$$

Here, the top square commutes because of naturality of ϵ. The bottom square commutes because of induction hypothesis used on X'. Observe that

$$X^{(k+1)} = (X')^{(k)}.$$

The left vertical composite is given by

$$\begin{aligned}
&\left(C_m^{k+1}(X')C_m^k(X)\right) \circ \left(\left(C_m^1(X)\right)^{(k+1)}\left(C_m^1(X)\right)^{(k)}\right) \\
&\qquad = \left(C_m^{k+1}(X') \circ \left(C_m^1(X)\right)^{(k+1)}\right)\left(C_m^k(X') \circ \left(C_m^1(X)\right)^{(k)}\right) \\
&\qquad = C_m^{k+2}(X)C_m^{k+1}(X) && (5.2.36)
\end{aligned}$$

where the first equality follows from the functoriality of the multiplication in $\mathscr{M}$. Thus, we get

$$A_m(X^{(k+1)}) \circ \left(C_m^{k+2}(X)C_m^{k+1}(X)\right) = \epsilon_{x^{(k+1)}}$$

as required. $\square$

Proposition 5.4.18. *For $x \in \mathrm{Obj}(\mathscr{M})$ the following diagram commutes:*

$$\begin{tikzcd} x''x' \arrow[d, "B_x 1_{x'}"'] \arrow[dr, "\epsilon_{x'}"] & \\ xx' \arrow[r, "\eta_x^{-1}"'] & I \end{tikzcd} \tag{5.4.19}$$

where $B_x : x'' \longrightarrow x$ is as in Definition 5.3.4.

Proof. Consider the following diagram:

$$\begin{tikzcd} x''x' \arrow[d, "1_{x''}\epsilon_x^{-1}1_{x'}"'] \arrow[dr, equal] & \\ x''x'xx' \arrow[r, "1_{x''x'}\eta_x^{-1}"'] \arrow[d, "\epsilon_{x'}1_x1_{x'}"'] & x''x' \arrow[d, "\epsilon_{x'}"] \\ xx' \arrow[r, "\eta_x^{-1}"'] & I \end{tikzcd}$$

In this diagram, the top triangle commutes because of the cancellation identity. The bottom square commutes because of the functoriality of the tensor product. Thus, the above diagram commutes. Now, the left vertical composition is given by

$$(\epsilon_{x'}1_x1_{x'}) \circ \left(1_{x''}\epsilon_x^{-1}1_{x'}\right) = B_x 1_{x'}. \tag{5.3.5}$$

This proves the statement. □

Proposition 5.4.20. *For $x \in \mathrm{Obj}(\mathscr{M})$ the following diagram commutes:*

$$\begin{tikzcd} x'x'' \arrow[d, "1_{x'}B_x"'] \arrow[dr, "\eta_{x'}^{-1}"] & \\ x'x \arrow[r, "\epsilon_x"'] & I \end{tikzcd} \tag{5.4.21}$$

where $B_x : x'' \longrightarrow x$ is as in Definition 5.3.4.

Proof. Consider the following diagram:

$$
\begin{tikzcd}
x'x'' \arrow[r, "\eta_{x'}^{-1}"] \arrow[d, "1_{x'}1_{x''}\epsilon_x^{-1}"'] & I \\
x'x''x'x \arrow[r, "\eta_{x'}^{-1}1_{x'x}"] \arrow[d, "1_{x'}\epsilon_{x'}1_x"'] & x'x \arrow[u, "\epsilon_x"'] \\
x'x \arrow[ur, equal] &
\end{tikzcd}
$$

In this diagram, the bottom triangle commutes because of the cancellation identity. The top square commutes because of the functoriality of the tensor product. Thus, the above diagram commutes. Now, the left vertical composition is given by

$$(1_{x'}\epsilon_{x'}1_x)\circ\left(1_{x'}1_x\epsilon_x^{-1}\right) = 1_{x'}B_x. \tag{5.3.5}$$

This proves the statement. □

Lemma 5.4.22. *For $x \in \mathrm{Obj}(\mathscr{M})$ let ν_x refer to one of the following cancellation isomorphisms:*

$$\nu_x = \eta_x^{-1} : xx' \longrightarrow I \qquad \text{or} \qquad \nu_x = \epsilon_x : x'x \longrightarrow I.$$

Let $k \in \mathbb{N}$ and let $p = k\%2$ and $q = (k+1)\%2$. Then, the following diagrams commute

$$
\begin{tikzcd}
x^{(k+1)}x^{(k)} \arrow[d, "B_x^{k+1}B_x^k"'] \arrow[dr, "\epsilon_{x^{(k)}}"] & \\
x^{(q)}x^{(p)} \arrow[r, "\nu_x"'] & I
\end{tikzcd}
\tag{5.4.23}
$$

and

$$
\begin{tikzcd}
x^{(k)}x^{(k+1)} \arrow[d, "B_x^{k}B_x^{k+1}"'] \arrow[dr, "\eta_{x^{(k)}}^{-1}"] & \\
x^{(p)}x^{(q)} \arrow[r, "\nu_x"'] & I.
\end{tikzcd}
\tag{5.4.24}
$$

Proof. We will prove this using induction on k:

Base case ($k = 0$): In this case, both diagrams (5.4.23) and (5.4.24) commute trivially since from Definition 5.3.6 we get $B_x^0 = 1_x$ and $B_x^1 = 1_x$.

Induction case ($k \geq 1$): Consider the following diagram:

$$\begin{tikzcd}
x^{(k+2)}x^{(k+1)} \arrow[d, "B_{x^{(k)}}1_{x^{(k+1)}}"'] \arrow[dr, "\epsilon_{x^{(k+1)}}"] & \\
x^{(k)}x^{(k+1)} \arrow[r, "\eta^{-1}_{x^{(k)}}"'] \arrow[d, "B_x^k B_x^{k+1}"'] & I \arrow[d, equal] \\
x^{(p)}x^{(q)} \arrow[r, "\nu_x"'] & I.
\end{tikzcd}$$

In this diagram, the top triangle commutes due to Proposition 5.4.18 and the bottom square commutes due to induction hypothesis. The left vertical composition is given by

$$\left(B_x^k B_x^{k+1}\right) \circ \left(B_{x^{(k)}} 1_{x^{(k+1)}}\right) = B^{k+2} B^{k+1}. \tag{5.3.9}$$

Therefore, the diagram (5.4.19) commutes. Similarly, consider the diagram

$$\begin{tikzcd}
x^{(k+1)}x^{(k+2)} \arrow[d, "1_{x^{(k+1)}}B_{x^{(k)}}"'] \arrow[dr, "\eta^{-1}_{x^{(k+1)}}"] & \\
x^{(k+1)}x^{(k)} \arrow[r, "\epsilon_{x^{(k)}}"'] \arrow[d, "B_x^{k+1} B_x^k"'] & I \arrow[d, equal] \\
x^{(q)}x^{(p)} \arrow[r, "\nu_x"'] & I.
\end{tikzcd}$$

In this diagram, the top triangle commutes due to Proposition 5.4.20 and the bottom square commutes due to induction hypothesis. The left vertical composition is given by

$$\left(B_x^{k+1} B_x^k\right) \circ \left(1_{x^{(k+1)}} B_{x^{(k)}}\right) = B^{k+1} B^{k+2}. \tag{5.3.9}$$

Therefore, the diagram (5.4.21) commutes. □

Chapter 6: Topics in Monoid Theory

In this chapter, we introduce monoids and provide a construction of the free monoid generated by a set.

6.1 Free Monoids and Monoid Basis

This section defines a monoid and the universal property of a free monoid. We also provide a criterion for determining when a given monoid is free in Theorem 6.1.37.

Definition 6.1.1. A *monoid*

$$(M, I, \bullet)$$

has the following data:

- A set M, called the *underlying set*,
- an element, denoted $I \in M$, called the *unit element*, and
- a binary map

  $$- \bullet - : M \times M \longrightarrow M$$

  called the *multiplication*.

These satisfy the following conditions:

Unit conditions: For $x \in M$, the equlity

$$x \bullet I = x \qquad \text{and} \qquad I \bullet x = x \tag{6.1.2}$$

holds.

Associativity: For $x, y, z \in M$, the equlity

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z \tag{6.1.3}$$

holds. ◇

Notation 6.1.4. Let $(M, I, \bullet)$ be a monoid. Due to associativity, we omit parentheses when writing products of elements. For $n \in \mathbb{N}$ and $x_i \in M$ for $1 \leq i \leq n$, we write

$$x_1 \bullet \cdots \bullet x_n$$

to denote the product in that order. By convention, if $n = 0$, the product is defined to be the unit element I:

$$x_1 \bullet \cdots \bullet x_n = I \quad \text{when } n = 0.$$ ◇

Example 6.1.5. The set of natural numbers $\mathbb{N}$ forms a monoid under addition, with 0 as the unit element. When we refer to the natural numbers as a monoid, we mean this structure. We write

$$\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$$

for the set of nonzero natural numbers. ◇

Example 6.1.6. Monoids appear throughout mathematics. Monoids arising from category theory are of particular interest to us. Specifically, if $(\mathscr{M}, I, \bullet)$ is a strict monoidal category, then the set of objects $\mathrm{Obj}(\mathscr{M})$ forms a monoid with unit I and tensor as the multiplication. Moreover, if $\mathscr{M}$ is a small category, the set of morphisms

$$\mathrm{Mor}(\mathscr{M}) := \bigsqcup_{x,y\in \mathrm{Obj}(\mathscr{M})} \mathrm{Hom}_{\mathscr{M}}(x,y)$$

also forms a monoid, where the unit is 1_I and the multiplication is given by the tensor. $\diamond$

Definition 6.1.7. Let $M = (M, \bullet, I)$ and $N = (N, *, J)$ be monoids. A *monoid homomorphism* from M to N is a set map $f : M \longrightarrow N$ that satisfies following conditions:

Unit condition: The equality

$$f(I) = J \tag{6.1.8}$$

holds.

Multiplication condition: For $x, y \in M$, the equality

$$f(x \bullet y) = f(x) * f(y) \tag{6.1.9}$$

holds. $\diamond$

Remark 6.1.10. Monoids and monoid homomorphisms together form a category, denoted by $\mathcal{Mon}$. $\diamond$

Framework 6.1.11. Throughout this section, we fix a monoid $(M, \bullet, I)$ and a subset $X \subseteq M$. $\diamond$

Definition 6.1.12. The *subset generated by X under the multiplication*, denoted by $\langle X;\ \bullet\rangle \subseteq M$, is defined as

$$\langle X;\ \bullet\rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \in \mathbb{N},\ m \geq 1,\ x_i \in X \text{ for } 1 \leq i \leq m\}. \tag{6.1.13}$$

The *subset generated by X under non-trivial multiplication*, denoted by $\langle\langle X;\ \bullet\rangle\rangle \subseteq M$, is defined as

$$\langle\langle X;\ \bullet\rangle\rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \in \mathbb{N},\ m \geq 2,\ x_i \in X \text{ for } 1 \leq i \leq m\}. \tag{6.1.14}$$

◇

Warning 6.1.15. Observe that the subsets $\langle X;\ \bullet\rangle$ and $\langle\langle X;\ \bullet\rangle\rangle$ may not include the unit I, as the empty product is not considered. As a result, $\langle X;\ \bullet\rangle$ and $\langle\langle X;\ \bullet\rangle\rangle$ are generally not submonoids. ◇

Proposition 6.1.16. *The subset*

$$\langle X;\ \bullet\rangle \cup \{I\}$$

is a submonoid of M. We refer to this as the

emphsubmonoid generated by X.

Proof. First, by definition, we have

$$I \in \langle X;\ \bullet\rangle \cup \{I\}.$$

Now, let

$$x, y \in \langle X;\ \bullet\rangle \cup \{I\}.$$

We consider the following cases:

<u>Case I:</u> $x, y \in \langle X;\ \bullet\rangle$. Then there exist $m, n \geq 1$ and $x_i, y_j \in X$ such that

$$x = x_1 \bullet \cdots \bullet x_m, \qquad y = y_1 \bullet \cdots \bullet y_n.$$

Thus,

$$x \bullet y = x_1 \bullet \cdots \bullet x_m \bullet y_1 \bullet \cdots \bullet y_n \in \langle X;\ \bullet\rangle.$$

Case II: $(x = I,\ y \in \langle X;\ \bullet \rangle)$: Then

$$x \bullet y = I \bullet y = y \in \langle X;\ \bullet \rangle$$

Case III: $(x \in \langle X;\ \bullet \rangle,\ y = I)$: Then

$$x \bullet y = x \bullet I = x \in \langle X;\ \bullet \rangle.$$

Case IV: $(x = I = y)$: Then

$$x \bullet y = I \bullet I = I.$$

In all cases, the product $x \bullet y$ belongs to $\langle X;\ \bullet \rangle \cup \{I\}$. Therefore, $\langle X;\ \bullet \rangle \cup \{I\}$ is closed under the multiplication and contains the unit, so it is a submonoid of M. □

Notation 6.1.17. We denote the set of all monoid elements without the unit by M^+. That is,

$$M^+ := M \setminus \{I\}. \tag{6.1.18}$$

◇

Definition 6.1.19. We say $X \subseteq M$ is *independent with respect to the multiplication* if for every $n, m \in \mathbb{N}^+$, and $x_i, y_j \in X$ for $1 \le i \le n$ and $1 \le j \le m$, the following implication holds:

$$x_1 \bullet \cdots \bullet x_n = y_1 \bullet \cdots \bullet y_m \qquad \Longrightarrow \qquad n = m \text{ and } x_i = y_j. \tag{6.1.20}$$

◇

Proposition 6.1.21. *Suppose $X \subseteq M$ is independent with respect to the multiplication, then $I \notin X$.*

Proof. Suppose for contradiction $I \in X$. We have

$$I \bullet I = I.$$

This contradicts multiplicatively independent condition of X. □

Definition 6.1.22. We say $X \subseteq M$ is a *generating set with respect to the multiplication* if

$$M^+ \subseteq \langle X;\ \bullet \rangle. \tag{6.1.23}$$

That is, for every $x \in M^+$ there exits $m \in \mathbb{N}^+$ and $x_i \in X$ for $1 \le i \le m$ such that

$$x_1 \bullet \cdots \bullet x_m = x. \qquad \diamond$$

Definition 6.1.24. We say $X \subseteq M$ is a *multiplication basis* or a *monoid basis* if X is independent with respect to the multiplication and X is a generating set with respect to the multiplication. $\diamond$

Definition 6.1.25. Let $x \in M^+$. An *X-factorization* of x is a finite ordered collection $(x_1, \ldots, x_m)$ with $m \ge 1$ and $x_i \in X$ for $1 \le i \le m$ such that

$$x_1 \cdots x_n = x. \tag{6.1.26}$$

$\diamond$

Remark 6.1.27. Note that, the subset $X \subseteq M$ is independent with respect to the multiplication if and only if every X-factorization is unique. A subset $X \subseteq M$ is a generating set with respect to multiplication if and only if every $x \in M^+$ has an X-factorization. Thus, a subset $X \subseteq M$ is a multiplicative basis if and only if every $x \in M^+$ has a unique X-factorization. $\diamond$

Proposition 6.1.28. *Let M and N be monoids, and let*

$$f : M \longrightarrow N$$

be a monoid isomorphism. If $X \subseteq M$ is a multiplicative basis of M, then $f(X) \subseteq N$ is a multiplicative basis of N.

Proof. Suppose $X \subseteq M$ is a multiplicative basis of M. We need to show that $f(X)$ is both a multiplicatively independent subset and a generating subset.

First, let $n, m \in \mathbb{N}^+$, and $a_i, b_j \in f(X)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, such that

$$a_1 \bullet \cdots \bullet a_n = b_1 \bullet \cdots \bullet b_m.$$

Since f is a monoid isomorphism, we get $f^{-1}(a_i), f^{-1}(b_j) \in X$, and

$$f^{-1}(a_1) \bullet \cdots \bullet f^{-1}(a_n) = f^{-1}(b_1) \bullet \cdots \bullet f^{-1}(b_m).$$

Because X is a multiplicative basis, it follows that

$$n = m \qquad \text{and} \qquad f^{-1}(a_i) = f^{-1}(b_i).$$

Consequently, we get

$$n = m \qquad \text{and} \qquad a_i = b_i.$$

Next, let $a \in N^+$. Since f is a monoid homomorphism, $f^{-1}(a) \in M^+$. Since X is a multiplicative basis, there exists $n \in \mathbb{N}^+$, and $x_i \in X$ for $1 \leq i \leq n$ such that

$$f^{-1}(a) = x_1 \bullet \cdots \bullet x_n.$$

Applying f yields

$$a = f(x_1) \bullet \cdots \bullet f(x_n)$$

with each $f(x_i) \in f(X)$. This completes the proof. □

Proposition 6.1.29. *If a monoid M admits a multiplicative basis, then this basis is unique.*

Proof. Let $X, Y \subseteq M$ be multiplicative bases of M. If both are empty, the result is immediate. Suppose X is nonempty and let $x \in X$. By Proposition 6.1.21, $x \in M^+$. Since Y is a multiplicative basis, x has a unique Y-factorization:

$$x = y_1 \bullet \cdots \bullet y_m$$

where $m \geq 1$ and $y_j \in Y$ for $1 \leq j \leq m$. Each $y_j \in M^+$, so each has a unique X-factorization:

$$y_j = x_{j1} \bullet \cdots \bullet x_{ji_j}$$

with $i_j \geq 1$ and $x_{jk} \in X$ for $1 \leq k \leq i_j$. Substituting, we have

$$x = x_{11} \bullet \cdots \bullet x_{1i_1} \bullet \cdots \bullet x_{m1} \bullet \cdots \bullet x_{mi_m}.$$

By uniqueness of X-factorizations, we get

$$m = 1, \qquad i_1 = 1, \qquad \text{and} \qquad x_{11} = x.$$

Thus, $x = y_1$ and $x \in Y$. Therefore, $X \subseteq Y$. By symmetry, $Y \subseteq X$. Hence, $X = Y$. $\square$

Proposition 6.1.30. *Let M be a monoid with a monoid basis X. Then for any $x, y \in M$, if $x \neq I$ and $y \neq I$, it follows that $x \cdot y \neq I$.*

Proof. Suppose $X \subseteq M$ is a monoid basis of M. Let $x, y \in M$ be such that $x \neq I$ and $y \neq I$. Suppose for contradiction that $x \cdot y = I$. Since X is a monoid basis we get $n, m \in \mathbb{N}^+$ and $x_i, y_j \in X$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ such that

$$x = x_1 \cdots x_n \qquad \text{and} \qquad y = y_1 \cdots y_m.$$

Thus, we get

$$I = x \cdot y = x_1 \cdots x_n \cdot y_1 \cdots y_m.$$

Therefore, we have

$$\begin{aligned} x_1 &= x_1 \cdot I \\ &= x_1 \cdot x_1 \cdots x_n \cdot y_1 \cdots y_m. \end{aligned}$$

This contradicts the multiplicative independence of X. Thus, we get $x \cdot y \neq I$. □

Proposition 6.1.31. *Let M be a monoid with a monoid basis X. Then,*

$$M^+ = \langle X;\ \bullet \rangle. \tag{6.1.32}$$

Proof. By Definition 6.1.22, we have $M^+ \subseteq \langle X;\ \bullet \rangle$.

For the reverse direction, let $x \in \langle X;\ \bullet \rangle$. Then there exist $n \in \mathbb{N}^+$ and $x_i \in X$ for $1 \leq i \leq n$ such that

$$x = x_1 \bullet \cdots \bullet x_n.$$

Since X is a monoid basis, Proposition 6.1.21 gives $I \notin X$. Thus, every $x_i \neq I$. By repeated application of Proposition 6.1.30, the product x cannot be I. Thus, $x \in M^+$, as required. □

Definition 6.1.33. Let S be a set and let $\phi : S \longrightarrow M$ be a set map. We say that the pair

$$\big(M, \phi : S \longrightarrow M\big)$$

satisfies the *universal property of the free monoid generated by* S if the following conditions are satisfied:

Existence: Given a monoid N and a set map $u : S \longrightarrow N$ there exists a monoid homomorphism $f_u : M \longrightarrow N$ such that

$$f_u \circ \phi = u. \tag{6.1.34}$$

Uniqueness: Given a monoid N and a pair of monoid homomorphisms $f, g : M \longrightarrow N$, then the following implication holds:

$$f \circ \phi = g \circ \phi \qquad \Longrightarrow \qquad f = g. \tag{6.1.35}$$

◇

Remark 6.1.36. The above definition coincides with the definition of a free object in the category of monoids, as described in Definition 2.2.7. ◇

Theorem 6.1.37. *Suppose $X \subseteq M$ is a monoid basis of M. Then, the pair*

$$\big(M, \phi : X \hookrightarrow M\big),$$

where ϕ is the inclusion map, satisfies the universal property of the free monoid generated by X.

Proof. Assume $X \subseteq M$ is a monoid basis of M. Let $(N, \bullet, J)$ be a monoid and $u : X \longrightarrow N$ be a set map.

Define $f_u : M \longrightarrow N$ as follows: Let $x \in M$. If $x = I$ define

$$f_u(I) := J.$$

Now suppose we have $x \in M^+$. Since X is a basis we get a unique $n \in \mathbb{N}^+$ and unique $x_i \in X$ for $1 \le i \le n$ such that $x = x_1 \bullet \cdots \bullet x_n$. Define

$$f_u(x) := u(x_1) \bullet \cdots \bullet u(x_n).$$

Since the factorization $x = x_1 \bullet \cdots \bullet x_n$ is unique, f_u is a well-defined map.

Next, we will show that f_u is a monoid homomorphism. We already have the equality

$$f_u(I) = J.$$

Thus, the unit condition of a monoid homomorphism is satisfied.

Now suppose $x, y \in M$. We will show that

$$f_u(x \cdot y) = f_u(x) \bullet f_u(y).$$

We will consider the following cases:

<u>Case I</u> ($x, y \in M^+$): We get $x = x_1 \bullet \cdots \bullet x_n$ and $y = y_1 \bullet \cdots \bullet y_m$ where $n, m \in \mathbb{N}^+$ and $x_i, y_j \in X$. Then we have

$$x \bullet y = x_1 \bullet \cdots \bullet x_n \bullet y_1 \bullet \cdots \bullet y_m.$$

From Proposition 6.1.30 we get that $x \bullet y \neq I$. Thus, we get

$$f_u(x \bullet y) = u(x_1) \bullet \cdots \bullet u(x_n) \bullet u(y_1) \bullet \cdots \bullet u(y_m) = f_u(x) \bullet f_u(y).$$

<u>Case II</u> ($x = I,\ y \in M$): In this case, we get

$$\begin{aligned} f_u(I \bullet y) &= f_u(y) \\ &= J \bullet f_u(y) \\ &= f_u(I) \bullet f_u(y). \end{aligned}$$

Case III ($x \in M,\ y = I$): In this case, we get

$$\begin{aligned} f_u(x \bullet I) &= f_u(x) \\ &= f_u(x) \bullet J \\ &= f_u(y) \bullet f_u(I). \end{aligned}$$

This shows that the function $f_u : M \longrightarrow N$ satisfies the multiplicative condition of a monoid homomorphism. Thus, $f_u : M \longrightarrow N$ is a monoid homomorphism.

Observe that for $x \in X$, we get $f_u(x) = u(x)$. Thus, the equality

$$f_u \circ \Phi = u$$

holds. This completes the proof of the existence part of the universal property.

Next, we will show the uniqueness part of the universal property. Suppose

$$g, h : M \longrightarrow N$$

are monoid homomorphisms such that

$$f \circ \phi = g \circ \phi.$$

That is, we have $g(z) = h(z)$ for every $z \in X$. We will show that $g = h$.

Let $x \in M$. If $x \in M^+$ then we get unique $n \in \mathbb{N}^+$ and unique $x_i \in X$ such that

$$x = x_1 \bullet \cdots \bullet x_n.$$

We get

$$\begin{aligned} g(x) &= g(x_1 \bullet \cdots \bullet x_n) \\ &= g(x_1) \bullet \cdots \bullet g(x_n) \\ &= h(x_1) \bullet \cdots \bullet h(x_n) \\ &= h(x_1 \bullet \cdots \bullet x_n) \\ &= h(x). \end{aligned}$$

If $x = I$ then we get

$$g(I) = J = h(I).$$

This shows that $g = h$. Thus, the uniqueness part of the universal property is satisfied. This completes the proof. □

6.2 Words as the Free Monoid

In this section, we will describe a standard construction of the free monoid generated by a set S, namely the set of words formed from elements of S as letters. This construction is discussed on page 48 of [ML78]. For further information on free monoids and related topics, see [Lot97]. Our approach will emphasize a set-theoretic perspective in formalizing the construction of the free monoids.

Framework 6.2.1. Fix a set S. ◇

Notation 6.2.2. Here are some relevant notations that will prove useful throughout our discussion.

- The set of integers are denoted by

$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}.$$

- The set of natural numbers are denoted by

$$\mathbb{N} = \{0, 1, 2, \cdots\}.$$

- The set of positive integers are denoted by

$$\mathbb{N}^+ = \mathbb{Z}^+ = \{1, 2, \cdots\}.$$

- We subscribe to the standard inverval notation for integers. That is, for integers $n, m \in \mathbb{Z}$, we denote

$$(n, m) = \{i \in \mathbb{Z} \mid n < i < m\}, \qquad (n, m] = \{i \in \mathbb{Z} \mid n < i \leq m\},$$

$$[n, m) = \{i \in \mathbb{Z} \mid n \leq i < m\}, \qquad [n, m] = \{i \in \mathbb{Z} \mid n \leq i \leq m\}.$$

  In addition, we denote

$$[n] = [1, n].$$

  In this discussion, an interval will always refer to a finite interval.

- Let $A \subseteq \mathbb{Z}$ be a subset of integers. We use the following notations for the power set of A:

$$\mathcal{P}(A) = \{T \subseteq \mathbb{Z} \mid T \subseteq A\},$$

$$\mathcal{P}^+(A) = \{T \subseteq \mathbb{Z} \mid T \subseteq A,\ T \neq \varnothing\},$$

$$\mathcal{P}_I^+(A) = \{T \subseteq \mathbb{Z} \mid T \subset A,\ T \neq \varnothing,\ T \text{ is a finite interval.}\}$$

  We denote an element $T \in \mathcal{P}^+(A)$ by $T \subseteq^+ A$ and an element $T \in \mathcal{P}_I^+(A)$ by $T \subseteq_I^+ A$.

- For $n, m \in \mathbb{Z}$, we denote

$$\mathcal{P}(n,m) = \mathcal{P}([n,m]), \qquad \mathcal{P}(n) = \mathcal{P}([n]),$$

$$\mathcal{P}^+(n,m) = \mathcal{P}^+([n,m]), \qquad \mathcal{P}^+(n) = \mathcal{P}^+([n]),$$

$$\mathcal{P}_I^+(n,m) = \mathcal{P}_I^+([n,m]), \qquad \mathcal{P}_I^+(n) = \mathcal{P}_I^+([n]).$$

◇

Definition 6.2.3. Let $n \in \mathbb{N}$. An *n-letter word* is a function $u : [n] \longrightarrow S$. We denote the set of all n-letter words by $\mathcal{W}r(n)$. ◇

Definition 6.2.4. Note that the empty function is the only function with a empty domain. Thus, the only 0-letter word is the empty function, $\emptyset : \varnothing \longrightarrow S$. We call this the *unit word*, denoted by $J_{\mathcal{W}r}$. It follows that

$$\mathcal{W}r(0) = \{J_{\mathcal{W}r}\}.$$

◇

Definition 6.2.5. Let $n, m \in \mathbb{N}$. Let u and v be n and m-letter words respectively. Define the multiplication, $u \cdot v$, to be the following $n+m$-letter word.

$$u \cdot v\ (i) := \begin{cases} u(i) & \text{if } i \le n \\ v(i-n) & \text{if } i > n. \end{cases} \tag{6.2.6}$$

◇

Lemma 6.2.7. *Let $n, m, p \in \mathbb{N}$. Let u, v and w be n, m, and p-letter words respectively. Then,*

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w \tag{6.2.8}$$

as $n+m+p$-letter words.

Proof. Observe that for $i \in [n+m+p]$ we get

$$\begin{aligned}
u \cdot (v \cdot w)\ (i) &= \begin{cases} u(i) & \text{if } i \leq n \\ v \cdot w\ (i-n) & \text{if } i > n \end{cases} && (6.2.6) \\
&= \begin{cases} u(i) & \text{if } i \leq n \\ v(i-n) & \text{if } 0 < i-n \leq m \\ w(i-n-m) & \text{if } i-n > m \end{cases} && (6.2.6) \\
&= \begin{cases} u(i) & \text{if } i \leq n \\ v(i-n) & \text{if } n < i \leq n+m \\ w(i-n-m) & \text{if } i > n+m \end{cases} \\
&= \begin{cases} u \cdot v\ (i) & \text{if } i \leq n+m \\ w(i-n-m) & \text{if } i > n+m \end{cases} && (6.2.6) \\
&= (u \cdot v) \cdot w\ (i) && (6.2.6).
\end{aligned}$$

Since $i \in [n+m+p]$ is arbitrarily chosen, we get

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w$$

as required. □

Lemma 6.2.9. *Let $n \in \mathbb{N}$. Let u be an n-letter word and $J_{\mathcal{W}r} : [0] \longrightarrow S$ be the empty word. Then we have*

$$u \cdot J_{\mathcal{W}r} = u \qquad \text{and} \qquad J_{\mathcal{W}r} \cdot u = u \tag{6.2.10}$$

as n-letter words.

Proof. Observe that for $i \in [n]$ we get

$$\begin{aligned}
u \cdot J_{\mathcal{W}r}\ (i) &= \begin{cases} u(i) & \text{if } i \leq n \\ & \text{if } i > n \end{cases} && (6.2.6) \\
&= u(i) && \text{since } i \leq n.
\end{aligned}$$

Since $i \in [n]$ is arbitrarily chosen we get

$$u \cdot J_{\mathcal{W}r} = u.$$

Similarly, for $i \in [n]$ we get

$$J_{\mathcal{W}r} \cdot u\ (i) = \begin{cases} & \text{if } i \leq 0 \\ u(i) & \text{if } i > 0 \end{cases} \qquad (6.2.6)$$

$$= u(i) \qquad \text{since } i > 0.$$

Since $i \in [n]$ is arbitrarily chosen we get

$$J_{\mathcal{W}r} \cdot u = u$$

as required. □

Definition 6.2.11. Let $n \in \mathbb{N}$ and u be an n-letter word. Let $(p,q] \subseteq_I [n]$ with $0 \leq p \leq q \leq n$. Define a $q-p$-lenth word, $u_{(p,q]} : [q-p] \longrightarrow S$ as follows:

$$u_{(p,q]}\ (i) := u(p+i). \tag{6.2.12}$$

◇

Lemma 6.2.13. *Let $n \in \mathbb{N}$ and $u : [n] \longrightarrow S$ be an n-letter word. Then, we get*

$$u_{[n]} = u \tag{6.2.14}$$

as n-letter words.

Proof. Let $i \in [n]$. We have

$$u_{[n]}\ (i) = u(i). \qquad (6.2.12)$$

Since $i \in [n]$ is arbitrary, we get

$$u_{[n]} = u$$

as required. □

Lemma 6.2.15. *Let $n, m \in \mathbb{N}$, $u : [n] \longrightarrow S$ be an n-letter word and $v : [m] \longrightarrow S$ be an m-letter word. Let $(p, q] \subseteq_I [n]$ with $0 \le p \le q \le n$. Then, we get*

$$(u \cdot v)_{(p,q]} = u_{(p,q]} \tag{6.2.16}$$

as $q - p$-letter words.

Proof. Let $i \in [q-p]$. Therefore, we get $p < p + i \le q \le n$. Now observe

$$\begin{aligned} (u \cdot v)_{(p,q]}\,(i) &= u \cdot v\ (p+i) && (6.2.12) \\ &= u(p+i) && (6.2.6) \\ &= u_{(p,q]}\,(i). && (6.2.12) \end{aligned}$$

Since $i \in [n]$ is arbitrary, we get

$$(u \cdot v)_{(p,q]} = u_{(p,q]}$$

as required. □

Lemma 6.2.17. *Let $n, m \in \mathbb{N}$, $u : [n] \longrightarrow S$ be an n-letter word and $v : [m] \longrightarrow S$ be an m-letter word. Let $(p, q] \subseteq_I (n, n+m]$ with $n \le p \le q \le n+m$. Then, we get*

$$(u \cdot v)_{(p,q]} = v_{(p-n,q-n]} \tag{6.2.18}$$

as $q - p$-letter words.

Proof. Let $i \in [q-p]$. Therefore, we get $n \le p < p + i \le q \le n+m$. Now observe

$$\begin{aligned} (u \cdot v)_{(p,q]}\,(i) &= u \cdot v\ (p+i) && (6.2.12) \\ &= v(p+i-n) && (6.2.6) \\ &= u_{(p-n,q-n]}\,(i). && (6.2.12) \end{aligned}$$

Since $i \in [q-p]$ is arbitrary, we get

$$(u \cdot v)_{(p,q]} = u_{(p-n,q-n]}$$

as required. $\square$

Lemma 6.2.19. *Let $n, m \in \mathbb{N}$ and u and v be n and m-letter words respectively. Then, we get*

$$(u \cdot v)_{[n]} = u \tag{6.2.20}$$

as n-letter words.

Proof. Observe that

$$\begin{aligned} (u \cdot v)_{[n]} &= u_{[n]} && (6.2.16) \\ &= u. && (6.2.14) \end{aligned}$$

$\square$

Lemma 6.2.21. *Let $n, m \in \mathbb{N}$ and u and v be n and m-letter words respectively. Then, we get*

$$(u \cdot v)_{(n,n+m]} = v \tag{6.2.22}$$

as m-letter words.

Proof. Observe that

$$\begin{aligned} (u \cdot v)_{(n,n+m]} &= v_{(0,m]} && (6.2.18) \\ &= v. && (6.2.14) \end{aligned}$$

$\square$

Lemma 6.2.23. *Let $n \in \mathbb{N}$ be a number and $u : [n] \longrightarrow S$ be an n-letter word. Let $0 \le p \le n$. Then we have*

$$u = u_{[p]} \cdot u_{(p,n]} \tag{6.2.24}$$

as n-letter words.

Proof. Let $i \in [n]$. Then we have

$$u_{[p]} \cdot u_{(p,n]}\,(i) = \begin{cases} u_{[p]}(i) & \text{if } i \le p \\ u_{(p,n]}(i-p) & \text{if } i > p \end{cases} \tag{6.2.6}$$

$$= \begin{cases} u(i) & \text{if } i \le p \\ u(i-p+p) & \text{if } i > p \end{cases} \tag{6.2.12}$$

$$= u(p).$$

Since $i \in [n]$ is arbitrary, we get

$$u = u_{[p]} \cdot u_{(p,n]}$$

as required. □

Definition 6.2.25. A *word* with alphabet in S a tuple

$$(n, u)$$

where $n \in \mathbb{N}$ is the length and $u : [n] \longrightarrow S$ is the n-letter word. We will denote the collection of all words with alphabet in S by $\mathcal{Mon}\langle S \rangle$. ◇

Remark 6.2.26. Note that, $\mathcal{Mon}\langle S \rangle$ is the collection of all words,

$$\mathcal{Mon}\langle S \rangle = \bigsqcup_{n \in \mathbb{N}} \mathcal{Wr}(n).$$ ◇

Definition 6.2.27. The monoid of words with alphabet in S, denoted $\mathcal{Mon}\langle S \rangle$, is defined as follows:

- Let $\mathcal{Mon}\langle S \rangle$ be the underlying set.
- Let the empty word, $(0, J_{\mathcal{Wr}})$, be the unit word.
- For words (n, u) and (m, v), the multiplication is defined as

  $$(n,u) \cdot (m,v) := (n+m, u \cdot v) \tag{6.2.28}$$

  where $u \cdot v : [n+m] \longrightarrow S$ is as in Definition 6.2.5.

These data satisfies the monoid conditions as follows:

Associativity: Let $(n,u), (m,v)$ and (p,w) be words. Then we have

$$\begin{aligned}
(n,u) \cdot \big((m,v) \cdot (p,w)\big) &= (n,v) \cdot (m+p, v \cdot w) && (6.2.28) \\
&= (n+m+p, u \cdot (v \cdot w)) && (6.2.28) \\
&= (n+m+p, (u \cdot v) \cdot w) && (6.2.8) \\
&= (n+m, u \cdot v) \cdot (p,w) && (6.2.28) \\
&= \big((n,u) \cdot (m,v)\big) \cdot (p,w). && (6.2.28)
\end{aligned}$$

Unit condition: Let (n,u) be a word. Then we get

$$(n,u)\cdot(0,J_{\mathcal{W}r}) = (n,u\cdot J_{\mathcal{W}r}) \qquad (6.2.28)$$

$$= (n,u). \qquad (6.2.10)$$

Similarly,

$$(0,J_{\mathcal{W}r})\cdot(n,u) = (n,J_{\mathcal{W}r}\cdot u) \qquad (6.2.28)$$

$$= (n,u). \qquad (6.2.10) \qquad \diamond$$

Definition 6.2.29. Define a function

$$\hat{l} : \mathcal{M}on\,\langle S\rangle \longrightarrow \mathbb{N}$$

as

$$\hat{l}(n,u) = n. \tag{6.2.30}$$

From the monoid structure on $\mathcal{M}on\,\langle S\rangle$, the function $\hat{l} : \mathcal{M}on\,\langle S\rangle \longrightarrow \mathbb{N}$ is a monoid homomorphism. $\diamond$

Lemma 6.2.31. *The monoid of words with alphabet in S is a left cancellative monoid. That is, for $x,y,z \in \mathcal{M}on\,\langle S\rangle$,*

$$x\cdot y = x\cdot z \qquad \textit{implies} \qquad y = z. \tag{6.2.32}$$

Proof. Let $x=(n,u)$, $y=(m,v)$, and $z=(p,w)$ be words. Suppose we have

$$x\cdot y = x\cdot z.$$

From (6.2.28) we get

$$n+m = n+p \qquad \text{and} \qquad u\cdot v = u\cdot w.$$

Thus, we have $m = p$ and

$$\begin{aligned} y = (m,v) &= \big(m,(u\cdot v)_{(n,n+m]}\big) && (6.2.22) \\ &= \big(p,(u\cdot w)_{(n,n+p]}\big) && \text{from above} \\ &= (p,w) = z. && (6.2.22) \end{aligned}$$ □

Lemma 6.2.33. *The monoid of words with alphabet in S is a right cancellative monoid. That is, for $x,y,z \in \mathcal{M}on\langle S\rangle$ we get*

$$y\cdot x = z\cdot x \qquad \textit{implies} \qquad y = z. \tag{6.2.34}$$

Proof. Let $x = (n,u)$, $y = (m,v)$, and $z = (p,w)$ be words. Suppose we have

$$y\cdot x = z\cdot x.$$

From (6.2.28) we get

$$m+n = p+n \qquad \text{and} \qquad v\cdot u = w\cdot u.$$

Thus, we have $m = p$ and

$$\begin{aligned} y = (m,v) &= \big(m,(v\cdot u)_{[m]}\big) && (6.2.20) \\ &= \big(p,(w\cdot u)_{[p]}\big) && \text{from above} \\ &= (p,w) = z. && (6.2.20) \end{aligned}$$ □

Definition 6.2.35. Define a subset, $\mathsf{S} \subseteq \mathcal{M}on\langle S\rangle$ as follows:

$$\mathsf{S} := \{(n,u) \in \mathcal{M}on\langle S\rangle \mid n = 1\}. \tag{6.2.36}$$

◇

Proposition 6.2.37. *The subset* S *is in bijection with* S*.*

Proof. Define a function $f : S \longrightarrow \mathsf{S}$ as follows:

$$f(a) := (1, u_a) \in \mathsf{S}$$

where $u_a : [1] \longrightarrow S$ is given by $u(1) = a$. The function $f : S \longrightarrow \mathsf{S}$ is injective since for $a, b \in S$ with $f(a) = f(b)$ we get

$$(1, u_a) = (1, u_b).$$

Therefore, we get $u_a = u_b$. It follows that

$$a = u_a(1) = u_b(1) = b.$$

We will show that the function $f : S \longrightarrow \mathsf{S}$ is surjective. Let $(1, u) \in \mathsf{S}$. Let $a = u[1]$, then we get $u = u_a$ since $u_a[1] = a = u[a]$. Consequently, we get

$$f(a) = (1, u_a) = (1, u).$$

□

Theorem 6.2.38. *The subset* $\mathsf{S} \subseteq \mathcal{M}on\,\langle S \rangle$ *is a monoid basis.*

Proof. First, we will show that S is a generating set with respect to the multiplication. Let $x = (n, u) \in M^+$ be a non-unit word. We will show, using induction on n, that x has a S-factorization.

Base case ($n = 1$): In this case, we get $x \in \mathsf{S}$. Thus, it has the trivial S-factorization.

Induction case ($n > 1$): Let $x_1 = (1, u_{[1]})$ and $y = (n-1, u_{(1,n]})$. Observe that

$$\begin{aligned}
x_1 \cdot y &= (1, u_{[1]}) \cdot (n-1, u_{(1,n]}) \\
&= (n, u_{[1]} \cdot u_{(1,n]}) && (6.2.28) \\
&= (n, u) && (6.2.24) \\
&= x.
\end{aligned}$$

From the induction hypothesis, we get an $\mathtt{S}$-factorization of y. That is, we get $x_2, \ldots, x_m \in \mathtt{S}$ such that $y = x_2 \cdots x_m$. Since $x_1 = (1, u_{[1]})$ we have $x_1 \in \mathtt{S}$. Moreover, we get

$$x = x_1 \cdot y = x_1 \cdots x_m.$$

Thus, $(x_1, \ldots, x_m)$ forms an $\mathtt{S}$-factorization of x.

Next, we will show that $\mathtt{S}$ is independent with respect to the multiplication. Let $x = (n, u) \in \langle \mathtt{S}; \bullet \rangle$. We will show, using induction on n, that x has at most one $\mathtt{S}$-factorization. Note that, from the Definition 6.2.35 we conclude that $n \geq 1$.

<u>Base case</u> ($n = 1$): We get $x \in \mathtt{S}$. Thus, x has the trivial $\mathtt{S}$-factorization. Let $x = x_1 \cdots x_m$ be another $\mathtt{S}$-factorization of x. That is, we have $m \geq 1$ and $x_i \in \mathtt{S}$ for $1 \leq i \leq m$. Since $x_i \in X$, we get $\hat{l}(x_i) = 1$. Thus,

$$1 = \hat{l}(x) = \hat{l}(x_1 \cdots x_m) = \hat{l}(x_1) + \cdots + \hat{l}(x_m) = m.$$

Therefore, $x = x_1$ is the only $\mathtt{S}$-factorization of x.

<u>Induction case</u> ($n > 1$): Let

$$x = x_1 \cdots x_m = y_1 \cdots y_p$$

be two S-factorizations of x. That is, we have $x_i, y_j \in \mathtt{S}$ for $1 \leq i \leq m$ and $1 \leq j \leq p$. In particular, we have $x_1 = (1, u_1)$ and $y_1 = (1, v_1)$ for some 1-letter words u_1 and v_1. Let $x_R = x_2 \cdots x_m$ and $y_R = y_2 \cdots y_p$. Then, we get

$$x = x_1 \cdot x_R = y_1 \cdot y_R.$$

Observe that

$$\begin{aligned} x_1 &= (x_1 \cdot x_R)_{[1]} && (6.2.16) \\ &= x_{[1]} \\ &= (y_1 \cdot y_R)_{[1]} \\ &= y_1. && (6.2.16) \end{aligned}$$

Since $\mathcal{Mon}\langle S\rangle$ is a left cancellative monoid (Lemma 6.2.31), we get

$$x_R = x_2 \cdots x_m = y_2 \cdots y_p = y_R.$$

We have $\hat{l}(x_R) = n-1 < n$. Also, we get two $\mathtt{S}$-factorizations of $x_R = y_R$. From the induction hypothesis, we get $m = p$ and $x_i = y_i$ for $2 \le i \le m$. This, along with the fact that $x_1 = y_1$, we conclude that x has at most one $\mathtt{S}$-factorization. □

Theorem 6.2.39. *Let S be a set and let $\mathcal{Mon}\langle S\rangle$ be the monoid defined as in Definition 6.2.25. Let $i : S \longrightarrow \mathcal{Mon}\langle S\rangle$ be the inclusion from Proposition 6.2.37. Then, the pair*

$$(\mathcal{Mon}\langle S\rangle, i : S \longrightarrow \mathcal{Mon}\langle S\rangle)$$

satisfies the universal property of the free monoid.

Proof. By Theorem 6.2.38, $\mathtt{S}$ forms a monoid basis for $\mathcal{Mon}\langle S\rangle$. Therefore, by Theorem 6.1.37, the pair

$$(\mathcal{Mon}\langle S\rangle, i : S \longrightarrow \mathcal{Mon}\langle S\rangle)$$

satisfies the universal property of the free monoid. □

Theorem 6.2.40. *Let M be a monoid, and let $X \subseteq M$ with the inclusion given by*

$$\phi : X \hookrightarrow M.$$

Then, the pair

$$\left(M, \phi : X \hookrightarrow M\right)$$

satisfies the universal property of the free monoid generated by X if and only if X is a multiplicative basis of M.

Proof. We have already shown one direction at Theorem 6.1.37.

For the other direction, suppose the pair

$$\left(M, \phi : X \hookrightarrow M\right)$$

satisfies the universal property of the free monoid generated by X. Consider the construction of the monoid of words $\mathcal{M}on\langle X\rangle$ over the set X as in Definition 6.2.27. From Theorem 6.2.38, we know that $X \subseteq \mathcal{M}on\langle X\rangle$ is a multiplicative basis of $\mathcal{M}on\langle X\rangle$. From Theorem 6.2.39, we know that the pair

$$\left(\mathcal{M}on\langle X\rangle, i : X \longrightarrow \mathcal{M}on\langle X\rangle\right)$$

satisfies the universal property of the free monoid generated by X. It follows that, $\mathcal{M}on\langle X\rangle$ is isomorphic to M via a monoid homomorphism which maps $X \subseteq \mathcal{M}on\langle X\rangle$ to $X \subseteq M$. Since $X \subseteq \mathcal{M}on\langle X\rangle$ is a multiplicative basis of $\mathcal{M}on\langle X\rangle$, from Proposition 6.1.28 we conclude that $X \subseteq M$ is a multiplicative basis of M. $\square$

Chapter 7: Topics in Dashed Monoids

Our aim, as stated in the outline of Theorem 4.5.25, is to prove a coherence theorem for categorical groups by using semi-strict categorical groups as an intermediary. To do this, we first examine the structure of objects in a semi-strict categorical group. Let $(\mathscr{M}, I, (-)', \otimes)$ be a semi-strict categorical group with the set of objects denoted by M. Then, M forms a monoid where the multiplication is given by the tensor product and I is the unit. Additionally, there is a unary operation $(-)' : M \longrightarrow M$ on objects, which does not interact with the monoid structure except for the condition $I' = I$. This motivates the definition of a new algebraic structure, which we call a *dashed monoid*: a monoid equipped with a unary operation $(-)'$ satisfying $I' = I$.

In Definition 7.2.1, we state the universal property of the free dashed monoid. In Definition 7.2.5 and Theorem 7.2.43, we provide a criterion for when a dashed monoid is free. In Definition 7.6.2, we give an explicit construction of the free dashed monoid generated by a set, and in Theorem 7.7.56, we show that this construction satisfies the universal property.

7.1 Definition of a Dashed Monoid

In this section, we provide the formal definitions of dashed monoids and dashed monoid homomorphisms. We also introduce the notions of multiplicative basis and dash basis for dashed monoids.

Definition 7.1.1. A *dashed monoid*

$$\big(M, I, (-)', \bullet\big)$$

has the following data:

- A monoid,

  $$(M, \bullet, I),$$

  called the *underlying monoid*.

- A unary set map,

  $$(-)' : M \longrightarrow M,$$

  called the *dash map*.

These satisfy the following *unit condition*:

$$I' = I. \tag{7.1.2}$$

◇

Definition 7.1.3. A dashed monoid M is called:

- A *monoid with a distributive dash* if it satisfies the distributive property:

  $$(x \bullet y)' = y' \bullet x'$$

  for all $x, y \in M$.

- A *monoid with convolution* if it satisfies both the distributive property above and the convolution property:

$$x'' = x$$

for all $x \in M$.

- A *group* if it satisfies the full inverse property:

$$x' \bullet x = I = x \bullet x'$$

for all $x \in M$. ◇

Convention 7.1.4. Alternative notations for the dash of $x \in M$ include $x^\dagger$ and $x^{(1)}$. For $n \in \mathbb{N}$, we write $x^{(n)}$ to denote the n-fold application of the dash map to x, with the convention that $x^{(0)} = x$. ◇

Example 7.1.5. The set of natural numbers $\mathbb{N}$ forms a dashed monoid under addition, unit as 0, and the dash operation is the identity map. This will be considered the default dashed monoid structure on $\mathbb{N}$ unless specified otherwise. ◇

Example 7.1.6. Let $(\mathscr{M}, I, (-)', \otimes)$ be a semi-strict categorical group. As discussed earlier, the set of objects of $\mathscr{M}$ naturally forms a dashed monoid. Furthermore, if $\mathscr{M}$ is a small category, then the set of all morphisms

$$\mathrm{Mor}(\mathscr{M}) := \bigsqcup_{x,y \in \mathscr{M}} \mathrm{Hom}_{\mathscr{M}}(x, y)$$

also inherits a dashed monoid structure, where the unit is 1_I, the multiplication is given by the tensor product, and the dash operation is given by the dash functor. ◇

Example 7.1.7. Following are some examples of dashed monoids.

i. We can give dashed monoid structure to a monoid by simply setting the dash map to be the identity map. Let $(M,\cdot,I)$ be a monoid. Then $(M,\cdot,I,\mathrm{Id}_M)$ is a dashed monoid.

ii. A more natural approach is to consider a group as the dashed monoid with the inverse map as the dash map. Let $(G,\cdot,e,(-)^{-1})$ be a group then $(G,\cdot,e,(-)^{-1})$ is a dashed monoid.

iii. The integers form a dashed monoid with identity as a dashed map, $(\mathbb{Z},+,0,\mathrm{Id}_{\mathbb{Z}})$ is a dashed monoid. By convention, unless specified otherwise, the above structure will be the dashed monoid structure on the integers.

iv. Let $(M,\cdot,I,D_M)$ and $(N,*,J,D_N)$ be dashed monoids. Then

$$(M\times N,\ \cdot\times *,\ (I,J),\ D_M\times D_N)$$

is also a dashed monoid where the multiplication

$$\cdot\times * : (M\times N)\times(M\times N) \longrightarrow (M\times N)$$

is given by

$$\big((m_1,n_1),(m_2,n_2)\big) \longmapsto (m_1\cdot m_2,\ n_1 * n_2)$$

and $D_M\times D_N : M\times N \longrightarrow M\times N$ is given by

$$(m,n) \longmapsto (D_M(m),D_N(n)).$$

Since the Cartesian product of monoids is again a monoid,

$$(M\times N,\cdot\times *,(I,J))$$

is a monoid. It remains to check that $D_M\times D_N\ (I,J)=(I,J)$. Indeed, since

$$D_M\times D_N\ (I,J)=(D_M(I),D_N(J))=(I,J). \qquad \diamond$$

Definition 7.1.8. Let

$$M = (M, \cdot, I, (-)') \qquad \text{and} \qquad N = (N, *, J, (-)^{\dagger})$$

be two dashed monoids. A *morphism of dashed monoids*

$$f : M \longrightarrow N$$

has the following data:

- A monoid homomorphism,

  $$f : M \longrightarrow N,$$

  on the underlying monoidal categories

that satisfies the following dash condition: For every $x \in M$,

$$(f(x))^{\dagger} = f(x'). \tag{7.1.9}$$

$\diamond$

Definition 7.1.10. Let $M = (M, \cdot, I, (-)')$ be a dashed monoid. Define the identity dashed monoid homomorphism, $\mathrm{Id}_M : M \longrightarrow M$ as follows:

- Let $\mathrm{Id}_M : M \longrightarrow M$ be the underlying monoid homomorphism.

This satisfies the dash condition since the equality

$$\mathrm{Id}_M(x)' = x' = \mathrm{Id}_M(x')$$

holds for every $x \in M$. $\diamond$

Definition 7.1.11. Let

$$M = (M, \cdot, I, (-)'), \qquad N = (N, *, J, (-)^{\dagger}), \qquad \text{and} \qquad P = (P, \bullet, K, (-)^{(1)})$$

be dashed monoids, and

$$f : M \longrightarrow N, \qquad \text{and} \qquad g : N \longrightarrow P$$

be dashed monoid homomorphisms. Define the composite dashed monoid homomorphism $g \circ f : M \longrightarrow P$ as follows:

- Let $g \circ f$ be the underlying monoid homomorphism.

The above monoid homomorphism satisfies the dash condition since the chain of equalities

$$\begin{aligned}
\left(g \circ f\ (x)\right)^{(1)} &= \left(g(f(x))\right)^{(1)} \\
&= g\left((f(x))^{\dagger}\right) && (7.1.9) \\
&= g\left(f(x')\right) && (7.1.9) \\
&= g \circ f\ (x')
\end{aligned}$$

hold for every $x \in M$. ◇

Definition 7.1.12. The *category of dashed monoids*, denoted $\mathcal{DMon}$, is defined as follows:

- The dashed monoids as in Definition 7.1.1 form the class of objects.
- The dashed monoid homomorphisms as in Definition 7.1.8 form the morphisms.
- For a dashed monoid M, the identity homomorphism Id_M, is the identity dashed-monoid homomorphism as in Definition 7.1.10.
- The composition of dashed monoid homomorphism is as in Definition 7.1.11.

Since the composition in dashed monoids is induced from that in monoids, the associativity, the left identity and the right identity conditions are satisfied. ◇

Motivation 7.1.13. Recall the notion of a multiplicative basis for a monoid from Definition 6.1.24. Dashed monoids inherit this concept for their multiplication operation. We now introduce analogous definitions for the dash operation. ◇

Framework 7.1.14. For the rest of this section let $(M, I, (-)', \bullet)$ be a dashed monoid, and let $X \subseteq M$ be a subset. ◇

Definition 7.1.15. Define a subset $\langle X;\, (-)' \rangle \subseteq M$ as follows:

$$\langle X; (-)' \rangle := \{x^{(k)} \in M \mid x \in X \ \text{ and } \ k \in \mathbb{N}\}. \tag{7.1.16}$$

We call this the subset generated by X under the dash operation.

Define a subset $\langle\langle X;\, (-)' \rangle\rangle \subseteq M$ as follows:

$$\langle\langle X; (-)' \rangle\rangle := \{x^{(k)} \in M \mid x \in X \ \text{ and } \ k \in \mathbb{N}^{+}\}. \tag{7.1.17}$$

◇

Definition 7.1.18. We say that the subset $X \subseteq M$ is *independent with respect to the dash operation* if for $r, k \in \mathbb{N}$ and $x, y \in X$ the following implication holds:

$$x^{(r)} = y^{(k)} \qquad \Longrightarrow \qquad r = k \text{ and } x = y. \tag{7.1.19}$$

◇

Proposition 7.1.20. *Suppose $X \subseteq M$ is independent with respect to the dash operation, then $I \notin X$.*

Proof. Suppose for contradiction $I \in X$. We have

$$I' = I.$$

This contradicts dash independent condition of X. □

Definition 7.1.21. We say that the subset $X \subseteq M$ is a *generating set with respect to the dash operation* if

$$M^{+} \subseteq \langle X;\ (-)' \rangle. \tag{7.1.22}$$

In other words, X is a generating set with respect to the dash operation if for every $z \in M^{+}$ we get $r \in \mathbb{N}$ and $x \in X$ such that

$$x^{(r)} = z. \qquad \diamond$$

Definition 7.1.23. We say that the subset $X \subseteq M$ is a *dash basis* if X is independent with respect to the dash operation and X is a generating set with respect to the dash operation. $\diamond$

Proposition 7.1.24. *Let M and N be dashed monoids, and let*

$$f : M \longrightarrow N$$

be a dashed monoid isomorphism. If $X \subseteq M$ is a dash basis of M, then $f(X) \subseteq N$ is a dash basis of N.

Proof. Suppose $X \subseteq M$ is a dash basis of M. We need to show that $f(X)$ is both a dash independent subset and a dash generating subset.

First, let $r, k \in \mathbb{N}$, and $a, b \in f(X)$ such that

$$a^{(r)} = b^{(k)}.$$

Since f is a dashed monoid isomorphism, we get $f^{-1}(a), f^{-1}(b) \in X$, and

$$\left(f^{-1}(a)\right)^{(r)} = \left(f^{-1}(b)\right)^{(k)}.$$

Because X is a dash basis, it follows that

$$r = k \qquad \text{and} \qquad f^{-1}(a) = f^{-1}(b).$$

Applying f, we get

$$r = k \qquad \text{and} \qquad a = b.$$

Next, let $a \in N^+$. Since f is a dashed monoid homomorphism, $f^{-1}(a) \in M^+$. Since X is a dash basis, there exists $k \in \mathbb{N}$, and $x \in X$ such that

$$f^{-1}(a) = x^{(k)}.$$

Applying f yields

$$a = (f(x))^{(k)}$$

with $f(x) \in f(X)$. This completes the proof. □

Proposition 7.1.25. *If a dashed monoid M admits a dash basis, then this dash basis is unique.*

Proof. Let $X, Y \subseteq M$ be dash bases of M. If both are empty, the result is immediate. Suppose X is nonempty and let $x \in X$. By Proposition 7.1.20, $x \in M^+$. Since Y is a dash basis, we get unique $r \in \mathbb{N}$ and $y \in Y$ such that

$$x = y^{(r)}.$$

It follows that, $y \in M^+$. Since X is a dash basis, we get unique $k \in \mathbb{N}$ and $x \in X$ such that

$$y = x^{(k)}.$$

Substituting, we have

$$x = x^{(r+k)}.$$

By uniqueness property of dash basis X, we get $r+k=0$. Thus, $r=k=0$ and $x=y^{(0)}=y\in Y$. Therefore, we get $X\subseteq Y$. By symmetry, $Y\subseteq X$. Hence, $X=Y$. $\square$

Lemma 7.1.26. *Suppose $X\subseteq M$ is a multiplicatively independent subset. Then, we get*

$$\langle X;\ \bullet\rangle = X\sqcup\langle\langle X;\ \bullet\rangle\rangle. \tag{7.1.27}$$

Proof. Let $x\in M$. Suppose $x\in\langle X;\ \bullet\rangle$. Then, we get $m\in\mathbb{N}$ with $m\geq 1$ and $x_i\in X$ for $1\leq i\leq m$ such that

$$x=x_1\bullet\cdots\bullet x_m.$$

If $m=1$ then we get

$$x=x_1\in X.$$

Otherwise, we have $m\geq 2$ and

$$x=x_1\bullet x_2\bullet\cdots\bullet x_m\in\langle\langle X;\ \bullet\rangle\rangle.$$

On the other hand, suppose $x\in X\sqcup\langle\langle X;\ \bullet\rangle\rangle$. If $x\in X$, then we get

$$x\in\langle X;\ \bullet\rangle.$$

Otherwise, we have

$$x\in\langle\langle X;\ \bullet\rangle\rangle\subseteq\langle X;\ \bullet\rangle.$$

Now we will show that

$$X\cap\langle\langle X;\ \bullet\rangle\rangle=\varnothing.$$

Suppose $x\in X\cap\langle\langle X;\ \bullet\rangle\rangle$. Then we have $x\in X$ and $x\in\langle\langle X;\ \bullet\rangle\rangle$. Therefore, we get $m\in\mathbb{N}$ with $m\geq 2$ and $x_i\in X$ for $1\leq i\leq m$ such that

$$x=x_1\bullet x_2\bullet\cdots\bullet x_m.$$

This leads to a contradiction since X is independent with respect to the multiplication. Thus, we get

$$X \cap \langle\langle X;\ \bullet\rangle\rangle = \varnothing$$

as required. $\square$

Lemma 7.1.28. *Suppose $X \subseteq M$ is a dash independent subset. Then, we get*

$$\left\langle X;\ (-)' \right\rangle = X \sqcup \left\langle\left\langle X;\ (-)' \right\rangle\right\rangle. \tag{7.1.29}$$

Proof. Let $x \in M$. Suppose $x \in \left\langle X;\ (-)' \right\rangle$. Then, we get $k \in \mathbb{N}$ and $y \in X$ such that

$$x = y^{(k)}.$$

If $k = 0$ then we get

$$x = y^{(0)} = y \in X.$$

Otherwise, we get $k \geq 1$ and

$$x = y^{(k)} \in \left\langle\left\langle X;\ (-)' \right\rangle\right\rangle.$$

On the other hand, suppose $x \in X \sqcup \left\langle\left\langle X;\ (-)' \right\rangle\right\rangle$. If $x \in X$, then we get

$$x = x^{(0)} \in \left\langle X;\ (-)' \right\rangle.$$

Otherwise, we have

$$x \in \left\langle\left\langle X;\ (-)' \right\rangle\right\rangle \subseteq \left\langle X;\ (-)' \right\rangle.$$

Now we will show that

$$X \cap \left\langle\left\langle X;\ (-)' \right\rangle\right\rangle = \varnothing.$$

Suppose $x \in X \cap \langle\langle X;\, (-)' \rangle\rangle$. Then we have $x \in X$ and $x \in \langle\langle X;\, (-)' \rangle\rangle$. Therefore, we get $y \in X$ and $k \in \mathbb{N}$ with $k \geq 1$ such that $x = y^{(k)}$. Thus, we have

$$x^{(0)} = x = y^{(k)}.$$

This leads to a constradiction since X is independent with respect to the dash operation. Thus, we get

$$X \cap \langle\langle X;\, (-)' \rangle\rangle = \varnothing$$

as required. □

Lemma 7.1.30. *Let $A \subseteq X \subseteq M$ be subsets of M. If X is a multiplicatively independent set then so is A.*

Proof. Suppose X is a multiplicatively independent set. We wish to show that $A \subseteq X$ is a multiplicatively independent set. Let $n, m \geq 1$ and $x_i, y_j \in A$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ such that

$$x_1 \bullet \cdots \bullet x_n = y_1 \bullet \cdots \bullet y_m.$$

Since X is independent with respect to the multiplication and $x_i, y_j \in A \subseteq X$, we get $n = m$ and $x_i = y_j$. This shows that A is independent with respect to the multiplication. □

Definition 7.1.31. Let $A \subseteq X \subseteq M$ be subsets of M. Define a subset

$$e\,\langle X, A;\, \bullet \rangle \subseteq M$$

as the collection of elements of M that can be written as a product

$$x_1 \bullet \cdots \bullet x_m$$

with $m \geq 1$, each $x_i \in M$, and at least one $x_j \in A$. Specifically,

$$e\langle X, A;\ \bullet\rangle := \tag{7.1.32}$$
$$\{x_1 \bullet \cdots \bullet x_m \in M \mid m \geq 1,\ x_i \in X,\ \text{and}\ \ \exists\ 1 \leq j \leq m \text{ such that } x_j \in A\}.$$

Define a subset

$$e\langle\langle X, A;\ \bullet\rangle\rangle \subseteq M$$

as as the collection of elements of M that can be written as a product

$$x_1 \bullet \cdots \bullet x_m$$

with $m \geq 2$, each $x_i \in M$, and at least one $x_j \in A$. Specifically

$$e\langle\langle X, A;\ \bullet\rangle\rangle := \tag{7.1.33}$$
$$\{x_1 \bullet \cdots \bullet x_m \in M \mid m \geq 2,\ x_i \in X,\ \text{and}\ \ \exists\ 1 \leq j \leq m \text{ such that } x_j \in A\}. \qquad \diamond$$

Lemma 7.1.34. *Suppose $X \subseteq M$ is a multiplicatively independent subset. Let $A, B \subseteq X$ be disjoint subsets of X. Then, we get*

$$\langle A \sqcup B;\ \bullet\rangle = \langle A;\ \bullet\rangle \bigsqcup e\langle A \sqcup B,\ B;\ \bullet\rangle \tag{7.1.35}$$

and

$$\langle\langle A \sqcup B;\ \bullet\rangle\rangle = \langle\langle A;\ \bullet\rangle\rangle \bigsqcup e\langle\langle A \sqcup B,\ B;\ \bullet\rangle\rangle \tag{7.1.36}$$

Proof. Let $x \in M$. Suppose $x \in \langle A \sqcup B;\ \bullet\rangle$, then we get $m \geq 1$ and $x_i \in A \sqcup B$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

If for every $1 \leq i \leq m$, we have $x_i \in A$ then we get

$$x = x_1 \bullet \cdots \bullet x_m \in \langle A;\ \bullet\rangle.$$

Otherwise, we get at least one $1 \le j \le m$ such that $x_j \in B$. Thus, we have

$$x = x_1 \bullet \cdots \bullet x_m \in e\,\langle A \sqcup B, B;\ \bullet\rangle.$$

On the other hand, suppose $x \in \langle A;\ \bullet\rangle \sqcup e\,\langle A \sqcup B, B;\ \bullet\rangle$. If $x \in \langle A;\ \bullet\rangle$ then we get some $m \ge 1$ and $x_i \in A$ for $1 \le i \le m$ such that

$$x = x_1 \bullet \cdots x_m.$$

Therefore, we have

$$x = x_1 \bullet \cdots \bullet x_m \in \langle A \sqcup B;\ \bullet\rangle.$$

Otherwise, if $x \in e\left\langle A \sqcup B, B;\ (-)'\right\rangle$ then we get some $m \ge 1$ and $x_i \in A \cup B$ with at least one $x_j \in B$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

Therefore, we have

$$x = x_1 \bullet \cdots \bullet x_m \in \langle A \sqcup B;\ \bullet\rangle.$$

Now we will show that $\langle A;\ \bullet\rangle$ and $e\,\langle A \sqcup B, B;\ \bullet\rangle$ are disjoint. Suppose there is $x \in M$ such that

$$x \in \langle A;\ \bullet\rangle \cap e\,\langle A \sqcup B, B;\ \bullet\rangle.$$

We get $n, m \ge 1$, $x_i \in A$ for $1 \le i \le n$, and $y_j \in A \cup B$ for $1 \le j \le m$ with at least one $y_k \in B$ such that

$$x_1 \bullet \cdots \bullet x_n = x = y_1 \bullet \cdots \bullet x_m.$$

Becasue X is a dash independent set and $x_i, y_j \in A \sqcup B \subseteq X$, we get $n = m$ and $x_i = y_i$. In particular, we get $x_k = y_k \in B$. We already have $x_k \in A$. This is a contradiction since A and B are disjoint. Thus, we get

$$\langle A \sqcup B;\ \bullet\rangle = \langle A;\ \bullet\rangle \bigsqcup e\,\langle A \sqcup B, B;\ \bullet\rangle.$$

The proof of equation (7.1.36) is similar to the proof of equation (7.1.35). $\square$

Lemma 7.1.37. *Suppose $A \subseteq X \subseteq M$ are subsets of M. If x is a dash independent subset then so is A.*

Proof. Suppose X is a dash independent set. We wish to show that $A \subseteq X$ is a dash independent set. Let $n, m \in \mathbb{N}$ and $a, b \in A$ such that

$$a^{(n)} = b^{(m)}.$$

Since X is independent with respect to the dash-operation and $a, b \in X$, we get $n = m$ and $a = b$. This shows that A is independent with respect to the dash-operation as well. $\square$

Lemma 7.1.38. *Let $X \subseteq M$ be a dash independent subset. Let $A, B \subseteq X$ be disjoint subsets of X. Then, we get*

$$\langle A \sqcup B;\, (-)' \rangle = \langle A;\, (-)' \rangle \bigsqcup \langle B;\, (-)' \rangle \tag{7.1.39}$$

and

$$\langle\langle A \sqcup B;\, (-)' \rangle\rangle = \langle\langle A;\, (-)' \rangle\rangle \bigsqcup \langle\langle B;\, (-)' \rangle\rangle \tag{7.1.40}$$

Proof. Let $x \in M$. Suppose $x \in \langle A \sqcup B;\, (-)' \rangle$. Then, we get $r \in \mathbb{N}$ and $y \in A \sqcup B$ such that

$$x = y^{(r)}.$$

If $y \in A$ then we get

$$x = y^{(k)} \in \langle\langle A;\, (-)' \rangle\rangle$$

and if $y \in B$ we get

$$x = y^{(k)} \in \langle\langle B;\, (-)' \rangle\rangle.$$

On the other hand, suppose $x \in \langle A;\, (-)' \rangle \sqcup \langle B;\, (-)' \rangle$. If $x \in \langle A;\, (-)' \rangle$ then we get some $r \in \mathbb{N}$ and $y \in A$ such that

$$x = y^{(k)}.$$

Therefore, we have

$$x = y^{(k)} \in \langle A \sqcup B;\, (-)' \rangle.$$

If $x \in \langle B;\, (-)' \rangle$ then we get some $r \in \mathbb{N}$ and $y \in B$ such that

$$x = y^{(k)}.$$

Therefore, we have

$$x = y^{(k)} \in \langle A \sqcup B;\, (-)' \rangle.$$

Now we will show that $\langle\langle A;\, (-)' \rangle\rangle$ and $\langle\langle B;\, (-)' \rangle\rangle$ are disjoint. Suppose there is $x \in M$ such that

$$x \in \langle\langle A;\, (-)' \rangle\rangle \cap \langle\langle B;\, (-)' \rangle\rangle.$$

We get some $r, k \in \mathbb{N}$, $y \in A$, and $z \in B$ such that

$$y^{(k)} = x = z^{(r)}.$$

Since X is a dash independent set and $y, z \in A \sqcup B \subseteq X$ we get $k = r$ and $y = z$. This is a contradiction since A and B are disjoint. Thus, we get

$$\langle A \sqcup B;\, (-)' \rangle = \langle A;\, (-)' \rangle \bigsqcup \langle B;\, (-)' \rangle.$$

The proof of equation (7.1.40) is similar to the proof of equation (7.1.39). $\square$

7.2 Free Dashed Monoid and Dashed Monoid Basis

In this section, we will define the universal property of a free dashed monoid and provide a criterion for a dashed monoid to be a free dashed monoid.

Definition 7.2.1. Let S be a set, M be a dashed monoid, and $\phi : S \longrightarrow M$ be a function. We say that the pair

$$(M, \phi : S \longrightarrow M)$$

satisfies the *universal property of the free dashed monoid generated by S* if the following conditions are satisfied:

Existence: For a dashed monoid N and set map $u : S \longrightarrow N$ there exists a dashed monoid homomorphism $f_u : M \longrightarrow N$ such that

$$f_u \circ \phi = u. \tag{7.2.2}$$

Uniqueness: For a dashed monoid N and a pair of dashed monoid homomorphisms $f, g : M \longrightarrow N$ we get

$$f \circ \phi = g \circ \phi \qquad \text{implies} \qquad f = g. \tag{7.2.3}$$

◇

Remark 7.2.4. Note that, the universal property of a free dashed monoid is precisely the universal property of a free object (Definition 2.2.7) in the category of dashed monoids over the forgetful functor $U : \mathcal{DMon} \longrightarrow \mathcal{Set}$. ◇

Definition 7.2.5. Let M be a dashed monoid, and let $\mathtt{S} \subseteq M$ be a set. We say $\mathtt{S} \subseteq M$ is a *dashed monoid basis* if M

- admits a multiplicative basis $\mathtt{G}$,
- admits a dash basis $\mathtt{H}$, and
- has a dashed-monoid homomorphism $\hat{l} : M \longrightarrow \mathbb{N}$

such that the following conditions are satisfied:

Length-unit condition: For $x \in M$ we have

$$\hat{l}(x) = 0 \qquad \Longleftrightarrow \qquad x = I \tag{7.2.6}$$

and for $a \in \mathtt{S}$ we have

$$\hat{l}(a) = 1. \tag{7.2.7}$$

Interlocking condition: The equalities

$$\mathtt{G} = \mathtt{S} \sqcup \langle\langle \mathtt{H};\ (-)' \rangle\rangle \qquad \text{and} \tag{7.2.8}$$

$$\mathtt{H} = \mathtt{S} \sqcup \langle\langle \mathtt{G};\ \bullet \rangle\rangle \tag{7.2.9}$$

hold. $\diamond$

Proposition 7.2.10. *Let M and N be dashed monoids, and let*

$$f : M \longrightarrow N$$

be a dashed monoid isomorphism. If $\mathtt{S} \subseteq M$ is a dashed monoid basis of M, then $f(\mathtt{S}) \subseteq N$ is a dashed monoid basis of N.

Proof. Suppose $\mathtt{S} \subseteq M$ is a multiplicative basis of M. We need to show that $f(\mathtt{S})$ satisfies the conditions described in Definition 7.2.5.

Since M admits a dashed monoid basis S, it also admits a multiplicative basis $\mathtt{G}$, and a dash basis $\mathtt{H}$. From Propositions 6.1.28 and 7.1.24, $f(\mathtt{G})$ and $f(\mathtt{H})$ are multiplicative and dash basis of N respectively.

Let $\hat{l}_M : M \longrightarrow \mathbb{N}$ be the length function. We define

$$\hat{l}_N : N \longrightarrow \mathbb{N}$$

as

$$\hat{l}_N := \hat{l}_M \circ f^{-1}.$$

Since f is an isomorphism, from the length conditions (7.2.6) and (7.2.7) we get

$$\hat{l}_N(x) = 0 \qquad \Longleftrightarrow \qquad x = I$$

and for $a \in f(\mathtt{S})$

$$\hat{l}_N(a) = 1.$$

Finally, from the interlocking conditions (7.2.8) and (7.2.9), we have

$$\mathtt{G} = \mathtt{S} \sqcup \left\langle\left\langle \mathtt{H};\ (-)' \right\rangle\right\rangle \qquad \text{and}$$

$$\mathtt{H} = \mathtt{S} \sqcup \langle\langle \mathtt{G};\ \bullet \rangle\rangle\,.$$

Applying f we get

$$f(\mathtt{G}) = f(\mathtt{S}) \sqcup \left\langle\left\langle f(\mathtt{H});\ (-)' \right\rangle\right\rangle \qquad \text{and}$$

$$f(\mathtt{H}) = f(\mathtt{S}) \sqcup \langle\langle f(\mathtt{G});\ \bullet \rangle\rangle\,.$$

This, completes the proof. □

Proposition 7.2.11. *If a dashed monoid M admits a dashed monoid basis* $\mathtt{S}$*, then this basis is unique and given by*

$$\mathtt{S} = \mathtt{G} \cap \mathtt{H}. \tag{7.2.12}$$

Proof. Suppose $\mathtt{S} \subseteq M$ be dashed monoid bases of M. Then, there exists a multiplicative basis $\mathtt{G}$ and a dash basis $\mathtt{H}$ of M. From Propositions 6.1.29 and 7.1.25, these bases are unique.

From the interlocking conditions (7.2.8) and (7.2.9), we get

$$\mathtt{S} \subseteq \mathtt{G} \cap \mathtt{H}.$$

On the other hand, if $x \in \mathtt{G} \cap \mathtt{H}$ then we get

$$x \in S \qquad \text{or} \qquad x \in \left\langle\left\langle \mathtt{H};\ (-)' \right\rangle\right\rangle.$$

Since $x \in \mathtt{H}$, from Lemma 7.1.28, we conclude that $x \notin \left\langle\left\langle \mathtt{H};\ (-)' \right\rangle\right\rangle$. Thus, $x \in S$. This completes the proof. □

Framework 7.2.13. For the remainder of this section, let $(M, I, (-)', \bullet)$ be a dashed monoid that admits a dashed monoid basis $\mathtt{S}$, with multiplicative basis $\mathtt{G}$, dash basis $\mathtt{H}$, and length function

$$\hat{l} : M \longrightarrow \mathbb{N}.$$

◇

Definition 7.2.14. Define the *multiplicative composite subset*, denoted

$$\mathtt{R} \subseteq M$$

as follows:

$$\mathtt{R} := \langle\langle \mathtt{G};\ \bullet \rangle\rangle. \tag{7.2.15}$$

Note that, since $\mathtt{G}$ is a multiplicative basis, for every $x \in \mathtt{R}$ we get unique $m \geq 2$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

◇

Lemma 7.2.16. *Assume Framework 7.2.13. Then, we get*

$$\mathtt{G} = \left\langle \mathtt{S};\ (-)' \right\rangle \quad \bigsqcup \quad \left\langle\left\langle \mathtt{R};\ (-)' \right\rangle\right\rangle. \tag{7.2.17}$$

Proof. From Definition 7.2.5, we get that $\mathtt{H}$ is a dash basis, and consequently, $\mathtt{H}$ is a dash independent set. We have that $\mathtt{S}, \mathtt{R} \subseteq \mathtt{H}$ are disjoint subsets of $\mathtt{H}$. Thus, applying Lemma 7.1.38 we get

$$\left\langle\left\langle \mathtt{S} \sqcup \mathtt{R};\ (-)' \right\rangle\right\rangle = \left\langle\left\langle \mathtt{S};\ (-)' \right\rangle\right\rangle \sqcup \left\langle\left\langle \mathtt{R};\ (-)' \right\rangle\right\rangle.$$

We note that $\mathtt{S}$ is a dash-independent set because it is a subset of the dash-independent set $\mathtt{H}$, as established by Lemma 7.1.37. Applying Lemma 7.1.28, we get

$$\langle \mathtt{S};\, (-)' \rangle = \mathtt{S} \sqcup \langle\langle \mathtt{S};\, (-)' \rangle\rangle.$$

We will finish the proof with the following direct calculation:

$$\begin{aligned}
\mathtt{G} &= \mathtt{S} \sqcup \langle\langle \mathtt{H};\, (-)' \rangle\rangle && (7.2.8)\\
&= \mathtt{S} \sqcup \langle\langle \mathtt{S} \sqcup \mathtt{R};\, (-)' \rangle\rangle && (7.2.9) \text{ and } (7.2.15)\\
&= \mathtt{S} \sqcup \langle\langle \mathtt{S};\, (-)' \rangle\rangle \sqcup \langle\langle \mathtt{R};\, (-)' \rangle\rangle && (7.1.39)\\
&= \langle \mathtt{S};\, (-)' \rangle \sqcup \langle\langle \mathtt{R};\, (-)' \rangle\rangle. && (7.1.29)
\end{aligned}$$

□

Lemma 7.2.18. *Assume Framework 7.2.13. Let $x \in M$. Then, $x \in \mathtt{G}$ if and only if exactly one of the following holds:*

- *We have*

$$x = a^{(k)}$$

  for some unique $k \geq 0$ and $a \in \mathtt{S}$. In this case, we also get

$$\hat{l}\left(a^{(k)}\right) = 1. \tag{7.2.19}$$

- *We have*

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}$$

  for some unique $m \geq 2$, $k \geq 1$ and $x_i \in \mathtt{G}$. In this case, we also get

$$\hat{l}(x) \geq 2 \tag{7.2.20}$$

  and

$$\hat{l}(x_i) < \hat{l}(x). \tag{7.2.21}$$

Proof. This is a restatement of Lemma 7.2.16:

Let $x \in M$. From Lemma 7.2.16 we get that, $x \in \mathtt{G}$ if and only if either

$$x \in \left\langle \mathtt{S};\ (-)' \right\rangle \qquad \text{or} \qquad x \in \left\langle \left\langle \mathtt{R};\ (-)' \right\rangle \right\rangle.$$

If $x \in \left\langle \mathtt{S};\ (-)' \right\rangle$ then we get $a \in \mathtt{S}$ and $k \in \mathbb{N}$ such that

$$x = a^{(k)}.$$

From Definition 7.2.5 we get $\mathtt{S} \subseteq \mathtt{H}$. Since $\mathtt{H}$ is a dash-basis of M, from Lemma 7.1.37 we get that $\mathtt{S}$ is a dash independent subset of M. Thus, the choice of a and k is unique. Since $a \in \mathtt{S}$, we have $\hat{l}(a) = 1$. Thus, we get

$$\hat{l}(x) = \hat{l}\left(a^{(k)}\right) = (\hat{l}(a))^{(k)} = 1^{(k)} = 1.$$

If $x \in \left\langle \left\langle \mathtt{R};\ (-)' \right\rangle \right\rangle$ we get $k \geq 1$ and $z \in \mathtt{R}$ such that

$$x = z^{(k)}.$$

From the interlocking condition (7.2.9), we get that

$$\mathtt{R} = \langle\langle \mathtt{G};\ \bullet \rangle\rangle \subseteq \mathtt{H}.$$

From Lemma 7.1.37 we get that $\mathtt{R}$ is a dash independent set. Thus, the choice of k and z is unique. Since $\mathtt{G}$ is a multiplicative basis, we get unique $m \geq 2$ and $x_i \in \mathtt{G}$ such that

$$z = x_1 \bullet \cdots \bullet x_m.$$

Thus, we get unique $m \geq 2$, $k \geq 1$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}.$$

Since the length function $\hat{l} : M \longrightarrow \mathbb{N}$ is a dashed-monoid homomorphism, we get

$$\hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i).$$

Since $x_i \in \mathtt{G}$, from Proposition 6.1.21 we get $x_i \neq I$. Therefore, from (7.2.6) we get

$$\hat{l}(x_i) \geq 1.$$

We have

$$\hat{l}(x) = \hat{l}\left((x_1 \bullet x_2 \bullet \cdots \bullet x_m)^{(k)}\right) = \sum_{i=1}^{m} \hat{l}(x_i).$$

Since $m \geq 2$ and $\hat{l}(x_i) \geq 1$ for $1 \leq i \leq m$, we get $\hat{l}(x) \geq 2$ and $\hat{l}(x_i) < \hat{l}(x)$ as required. $\square$

Lemma 7.2.22. *Assume Framework 7.2.13. Then, the multiplicative basis $\mathtt{G} \subseteq M$ is closed under taking dash.*

Proof. Suppose $x \in \mathtt{G}$. We want to show that $x' \in \mathtt{G}$. From Lemma 7.2.16 we get

$$\mathtt{G} = \left\langle \mathtt{S};\ (-)' \right\rangle \sqcup \left\langle \left\langle \mathtt{R};\ (-)' \right\rangle \right\rangle.$$

If $x \in \left\langle \mathtt{S};\ (-)' \right\rangle$ then we get some $k \in \mathbb{N}$ and $y \in \mathtt{S}$ such that

$$x = y^{(k)}.$$

Thus, we get

$$x' = y^{(k+1)} \in \left\langle \mathtt{S};\ (-)' \right\rangle \subseteq \mathtt{G}.$$

Similarly, if $x \in \left\langle \left\langle \mathtt{R};\ (-)' \right\rangle \right\rangle$ then we get some $k \in \mathbb{N}$ with $k \geq 1$ and $y \in \mathtt{R}$ such that

$$x = y^{(k)}.$$

Thus, we get

$$x' = y^{(k+1)} \in \left\langle \left\langle \mathtt{R};\ (-)' \right\rangle \right\rangle \subseteq \mathtt{G}.$$

$\square$

Construction 7.2.23. Assume Framework 7.2.13. Let $u : S \longrightarrow M$ be a function. We will extend the function $u : \mathtt{S} \longrightarrow N$ to a function

$$u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$$

as follows:

We have a length homomorphism $\hat{l} : M \longrightarrow \mathbb{N}$. Suppose $x \in \mathtt{G}$. We will define $u^{\mathtt{G}}(x) \in N$ using induction on $\hat{l}(x)$. Since $x \in \mathtt{G}$, we have $\hat{l}(x) \geq 1$.

<u>Base case</u> $\big(\hat{l}(x) = 1\big)$: From Lemma 7.2.18 we get unique $k \in \mathbb{N}$ and unique $a \in \mathtt{S}$ such that

$$x = a^{(r)}.$$

We define

$$u^{\mathtt{G}}(x) := u(a)^{(k)}. \tag{7.2.24}$$

<u>Induction case</u> $\big(\hat{l}(x) \geq 2\big)$: From Lemma 7.2.18, unique $k \in \mathbb{N}$ with $k \geq 1$, $m \geq 2$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = (x_1 \bullet x_2 \bullet \cdots \bullet x_m)^{(k)}.$$

Furthermore, we have $\hat{l}(x_i) < \hat{l}(x)$. Using the induction hypothesis, we define

$$u^{\mathtt{G}}(x) := \big(u^{\mathtt{G}}(x_1) \bullet u^{\mathtt{G}}(x_2) \bullet \cdots \bullet u^{\mathtt{G}}(x_m)\big)^{(k)}. \tag{7.2.25}$$

◇

Lemma 7.2.26. *For $a \in \mathtt{S}$, the equality*

$$u^{\mathtt{G}}(a) = u(a). \tag{7.2.27}$$

Proof. Let $a \in \mathtt{S}$. Then, from the length condition (7.2.7), we get $\hat{l}(a) = 1$. It follows that,

$$\begin{aligned} u^{\mathtt{G}}(a) &= (u(a))^{(0)} && (7.2.24) \\ &= u(a). \end{aligned}$$

□

Lemma 7.2.28. *Let $x \in M$. Suppose $x \in \mathtt{G}$. Note that, from Lemma 7.2.22 we have $x' \in \mathtt{G}$. The equality*

$$\left(u^{\mathtt{G}}(x)\right)' = u^{\mathtt{G}}\left(x'\right) \tag{7.2.29}$$

holds.

Proof. Suppose $x \in \mathtt{G}$. We will consider the two cases as in Lemma 7.2.18.

Case I ($\hat{l}(x) = 1$): We get a unique $k \in \mathbb{N}$ and $a \in \mathtt{S}$ such that

$$x = a^{(k)}.$$

Therefore, we get

$$x' = a^{(k+1)}.$$

Observe that,

$$\begin{aligned}
\left(u^{\mathtt{G}}(x)\right)' &= \left(u(a)^{(k)}\right)' && (7.2.24)\\
&= u(a)^{(k+1)}\\
&= u^{\mathtt{G}}\left(x'\right). && (7.2.24)
\end{aligned}$$

Case II ($\hat{l}(x) \geq 2$): We get unique $k \geq 1$, $m \geq 2$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = (x_1 \bullet x_2 \bullet \cdots \bullet x_m)^{(k)}.$$

Therefore, we get

$$x' = (x_1 \bullet \cdots \bullet x_m)^{(k+1)}$$

Define $z \in N$ as follows:

$$z := u^{\mathtt{G}}(x_1) \bullet u^{\mathtt{G}}(x_2) \bullet \cdots \bullet u^{\mathtt{G}}(x_m).$$

Observe that

$$\begin{aligned}
\left(u^{\mathtt{G}}(x)\right)' &= \left(z^{(k)}\right)' && (7.2.24)\\
&= z^{(k+1)}\\
&= u^{\mathtt{G}}(x'). && (7.2.24)
\end{aligned}$$

This completes the proof. □

Definition 7.2.30. Let $u : \mathtt{S} \longrightarrow N$ be a function and let $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$ be the function as in Construction 7.2.23. Since $\mathtt{G}$ is a multiplicative basis of M, we can extend the function $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$ to a function

$$\overline{u} : M \longrightarrow N.$$

Explicitly, given $x \in M$, if $x = I$ then we define

$$\overline{u}(I) := J. \tag{7.2.31}$$

Otherwise, we have $x \in M^+$. Since $\mathtt{G}$ is a multiplicative basis of M, we get $m \in \mathbb{N}$ with $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

We define

$$\overline{u} := u^{\mathtt{G}}(x_1) \bullet \cdots \bullet u^{\mathtt{G}}(x_m). \tag{7.2.32}$$

◇

Lemma 7.2.33. *The function $\overline{u} : M \longrightarrow N$ as defined in Definition 7.2.30 is a monoid homomorphism.*

Proof. Note that $\mathtt{G}$ is a multiplicative basis of M. Since $\overline{u} : M \longrightarrow N$ defined by multiplicatively extending the function $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$, it follows that $\overline{u}$ is a monoid homomorphism. □

Lemma 7.2.34. *For $x \in \mathtt{G}$ we get*

$$\overline{u}(x) = u^{\mathtt{G}}(x). \tag{7.2.35}$$

Proof. Since $\overline{u} : M \longrightarrow N$ is defined by multiplicatively extending the function $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$, we get

$$\overline{u}(x) = u^{\mathtt{G}}(x). \qquad (7.2.32)$$ □

Lemma 7.2.36. *For $a \in \mathtt{S}$ we get*

$$\overline{u}(a) = u(a). \tag{7.2.37}$$

Proof. Let $a \in \mathtt{S}$. Then, we get $a \in \mathtt{G}$. Observe that

$$\begin{aligned} \overline{u}(a) &= u^{\mathtt{G}}(a) && (7.2.35) \\ &= u(a). && (7.2.27) \end{aligned}$$ □

Lemma 7.2.38. *The function $\overline{u} : M \longrightarrow N$ as defined in Definition 7.2.30 is a dashed monoid homomorphism.*

Proof. Lemma 7.2.33 shows that $\overline{u} : M \longrightarrow N$ is a monoid homomorphism. We will show that $\overline{u} : M \longrightarrow N$ satisfies the dash condition: $\overline{u}(x') = \overline{u}(x)'$ for every $x \in M$. Consider the following calculation:

$$\begin{aligned} M &= \{I\} \sqcup M^{+} && (6.1.18) \\ &= \{I\} \sqcup \langle \mathtt{G};\ \bullet \rangle && (6.1.32) \\ &= \{I\} \sqcup \mathtt{G} \sqcup \mathtt{R}. && (7.1.27) \text{ and } (7.2.15) \end{aligned}$$

Thus, we get either $x = I$, $x \in \mathtt{G}$, or $x \in \mathtt{R}$. We consider these three cases.

Case I ($x = I$): We get

$$\begin{aligned}
\overline{u}(I)' &= J' && (7.2.31)\\
&= J\\
&= \overline{u}(I) && (7.2.31)\\
&= \overline{u}(I').
\end{aligned}$$

Case II ($x \in \mathtt{G}$): From Lemma 7.2.22 we get $x' \in \mathtt{G}$. We have

$$\begin{aligned}
\overline{u}(x)' &= u^{\mathtt{G}}(x)' && (7.2.35)\\
&= u^{\mathtt{G}}(x') && (7.2.29)\\
&= \overline{u}(x'). && (7.2.35)
\end{aligned}$$

Case III: ($x \in \mathtt{R}$): We get $m \geq 2$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

Therefore, we have

$$x' = (x_1 \bullet \cdots \bullet x_m)'.$$

It follows that

$$x' \in \langle\langle \mathtt{R};\ (-)' \rangle\rangle \subseteq \mathtt{G}.$$

Observe,

$$\begin{aligned}
\overline{u}(x') &= u^{\mathtt{G}}(x') && (7.2.35)\\
&= \left(u^{\mathtt{G}}(x_1) \bullet \cdots \bullet u^{\mathtt{G}}(x_m)\right)' && (7.2.25)\\
&= (\overline{u}(x))'. && (7.2.32)
\end{aligned}$$

Since we have shown $\overline{u}(x') = \overline{u}(x)'$ in all three cases, we get that $\overline{u} : M \longrightarrow N$ is a dashed monoid homomorphism. □

Lemma 7.2.39. *Let $f : M \longrightarrow N$ be a dashed-monoid homomorphism. Define*

$$u := f\,|_{\mathtt{S}} : \mathtt{S} \longrightarrow N.$$

Then, we have

$$u^{\mathtt{G}} = f\,|_{\mathtt{G}} : \mathtt{G} \longrightarrow N \tag{7.2.40}$$

where $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$ is as in Construction 7.2.23.

Proof. Let $x \in M$. Suppose $x \in \mathtt{G}$. Using induction on $\hat{l}(x)$, we will show that

$$u^{\mathtt{G}}(x) = f(x).$$

Base case $\left(\hat{l}(x) = 1\right)$: From Lemma 7.2.18 we get unique $k \in \mathbb{N}$ and $a \in \mathtt{S}$ such that

$$x = a^{(k)}.$$

Observe that

$$\begin{aligned} u^{\mathtt{G}}(x) &= u(a)^{(r)} && (7.2.24) \\ &= f(a)^{(r)} && \text{(hypothesis)} \\ &= f\left(a^{(r)}\right) \\ &= f(x). \end{aligned}$$

Here, the third equality follows since f is a dashed monoid homomorphism.

Induction case $\left(\hat{l}(x) \geq 2\right)$: From Lemma 7.2.18 we get unique $k \geq 1$, $m \geq 2$, and $x_i \in \mathtt{G}$ such that

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}.$$

Since $x_i \in \mathtt{G}$, we have $\hat{l}(x_i) \geq 1$. Since $\hat{l} : M \longrightarrow \mathbb{N}$ is a dashed-monoid homomorphism, we get

$$\hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i).$$

Becasue $m \geq 2$, we get $\hat{l}(x_i) < \hat{l}(x)$. Using the induction hypothesis, we get

$$u^{\mathtt{G}}(x_i) = f(x_i).$$

Observe that

$$\begin{aligned} u^{\mathtt{G}}(x) &= \left(u^{\mathtt{G}}(x_1) \bullet \cdots \bullet u^{\mathtt{G}}(x_m)\right)^{(k)} && (7.2.25) \\ &= (f(x_1) \bullet \cdots \bullet f(x_m))^{(k)} && \text{(induction)} \\ &= f\left((x_1 \bullet \cdots \bullet x_m)^{(k)}\right) \\ &= f(x). \end{aligned}$$

Here, the third equality follows because f is a dashed monoid homomorphism.

Thus, by mathematical induction we get

$$u^{\mathtt{G}} = f\,|_{\mathtt{G}} : \mathtt{G} \longrightarrow N$$

as required. □

Lemma 7.2.41. *Suppose $f : M \longrightarrow N$ is a dashed monoid homomorphism. Let*

$$u := f\,|_{\mathtt{S}} : \mathtt{S} \longrightarrow N.$$

Then,

$$\overline{u} = f : M \longrightarrow N, \tag{7.2.42}$$

where $\overline{u} : M \longrightarrow N$ is as in Definition 7.2.30.

Proof. From Lemma 7.2.38, we get that $\overline{u} : M \longrightarrow N$ is a dashed monoid homomorphism. Given that $f : M \longrightarrow N$ is a dashed monoid homomorphism, it follows that $\overline{u}$ and f are both monoid homomorphisms. Since $\mathtt{G}$ is a multiplicative basis of M, it is sufficient to show that

$$\overline{u}\,|_{\mathtt{G}} = f\,|_{\mathtt{G}}\,.$$

Observe that

$$\begin{aligned} \overline{u}\,|_{\mathtt{G}} &= u^{\mathtt{G}} && (7.2.35) \\ &= f\,|_{\mathtt{G}}\,. && (7.2.40) \end{aligned}$$ □

This completes the proof.

Theorem 7.2.43. *Let $M = (M, \bullet, I, (-)')$ be a dashed monoid. Suppose M admits a dashed monoid basis $\mathtt{S}$ as in Definition 7.2.5. Then, the inclusion of sets $\mathtt{S} \subseteq M$ satisfies the universal property of dashed monoids*

Proof. We will show the existence condition (7.2.2) and the uniqueness condition (7.2.3) of the universal property of a free dashed monoid.

Existence: Let N be a dashed monoid and let $u : \mathtt{S} \longrightarrow N$ be a function. Consider the function

$$\overline{u} : M \longrightarrow N$$

as in Definition 7.2.30. From Lemma 7.2.38, we get that $\overline{u}$ is a dashed monoid homomorphism, and Lemma 7.2.36 shows that

$$\overline{u}\,|_{\mathtt{S}} = u.$$

Uniqueness: Let $f, g : M \longrightarrow N$ be dashed monoid homomorphisms such that

$$f\mid_{\mathsf{S}} = g\mid_{\mathsf{S}}.$$

Define

$$u := f\mid_{\mathsf{S}} = g\mid_{\mathsf{S}} : \mathsf{S} \longrightarrow N.$$

From Lemma 7.2.41, we have

$$f = \overline{u} = g.$$

This completes the proof. $\square$

7.3 Properties of Integer Sets

Before constructing the free dashed monoid generated by a set, we present some elementary yet nontrivial properties of subsets of integers. The propositions in this section are foundational and will be used repeatedly throughout this article. Given their basic nature, we will refer to them without explicit citation.

Definition 7.3.1. Let $n \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$. Then, $n + A \subseteq \mathbb{Z}$ is defined as follows:

$$n + A := \{n + a \mid a \in A\}. \tag{7.3.2}$$

◇

Notation 7.3.3. We will, at times, use the notation $A + n$ to refer to $n + A$. ◇

Proposition 7.3.4. *Let $x, n, m, p, q \in \mathbb{Z}$ and $A, B \subseteq \mathbb{Z}$. Then, following are true.*

$$x \in n + A \iff x - n \in A. \tag{7.3.5}$$

$$A \subseteq B \iff n + A \subseteq n + B. \tag{7.3.6}$$

$$A = B \iff n + A = n + B. \tag{7.3.7}$$

$$0 + A = A. \tag{7.3.8}$$

$$n + \varnothing = \varnothing. \tag{7.3.9}$$

$$n + (m + A) = (n + m) + A. \tag{7.3.10}$$

$$n + (A \cup B) = (n + A) \cup (n + B). \tag{7.3.11}$$

$$n + (A \cap B) = (n + A) \cap (n + B). \tag{7.3.12}$$

$$n + (p, q] = (n + p, n + q]. \tag{7.3.13}$$

Proof. (7.3.5): Suppose $x \in n + A$. Then, we get $x = n + a$ for some $a \in A$. It follows that $x - n = a \in A$. On the other hand, if $x - n \in A$ then $x = n + (x - n) \in n + A$.

(7.3.6): Suppose $A \subseteq B$. Let $x \in n + A$. From (7.3.5), we get $x - n \in A \subseteq B$. Thus, again from (7.3.5) we get $x \in n + B$. This shows

$$n + A \subseteq n + B.$$

For the reverse direction, assume $n + A \subseteq n + B$. Let $x \in A$. From (7.3.2) we get $n + x \in n + A \subseteq n + B$. Finally, from (7.3.5), we get $x = n + x - n \in B$. Thus, we have

$$A \subseteq B.$$

(7.3.7): We see this from the following chain of double implications.

$$\begin{aligned}
A = B &\iff (A \subseteq B) \text{ and } (B \subseteq A) \\
&\iff (n + A \subseteq n + B) \text{ and } (n + B \subseteq n + A) && (7.3.6) \\
&\iff n + A = n + B.
\end{aligned}$$

(7.3.8): We have

$$\begin{aligned}
0 + A &= \{0 + a \mid a \in A\} && (7.3.2) \\
&= \{a \mid a \in A\} \\
&= A.
\end{aligned}$$

(7.3.9): For the sake of contradiction assume $n + \varnothing \neq \varnothing$. Let $x \in n + \varnothing$ then, from (7.3.5), we get $x - n \in \varnothing$. This is a contradiction.

(7.3.10): We have

$$\begin{aligned}
n + (m + A) &= n + \{m + a \mid a \in A\} && (7.3.2) \\
&= \{n + m + a \mid a \in A\} && (7.3.2) \\
&= (n + m) + A. && (7.3.2)
\end{aligned}$$

(7.3.11): We get

$$
\begin{aligned}
n+(A\cup B) &= \{n+a \mid a\in A \ \text{ or } \ a\in B\} && (7.3.2)\\
&= \{n+a \mid a\in A\}\cup\{n+a \mid a\in B\}\\
&= (n+A)\cup(n+B). && (7.3.2)
\end{aligned}
$$

(7.3.12): We have

$$
\begin{aligned}
n+(A\cap B) &= \{n+a \mid a\in A \ \text{ and } \ a\in B\} && (7.3.2)\\
&= \{n+a \mid a\in A\}\cap\{n+a \mid a\in B\}\\
&= (n+A)\cap(n+B). && (7.3.2)
\end{aligned}
$$

(7.3.13): We see that

$$
\begin{aligned}
n+[p,q] &= \{n+x \mid p\le x\le q\} && (7.3.2)\\
&= \{y \mid p\le y-n \mid q\}\\
&= \{y \mid n+p\le y\le n+q\}\\
&= [n+p,n+q].
\end{aligned}
$$

□

Definition 7.3.14. Let $D\subseteq \mathcal{P}(\mathbb{Z})$. Then, $n+D\subseteq\mathcal{P}(\mathbb{Z})$ is defined as follows:

$$n+D := \{n+A \mid A\in D\}. \tag{7.3.15}$$

◇

Proposition 7.3.16. *Let $n,m \in \mathbb{Z}$, $A,B \subseteq \mathbb{Z}$, and $D,E \subseteq \mathcal{P}(\mathbb{Z})$. Then, following are true.*

$$A \in n+D \iff A-n \in D. \tag{7.3.17}$$

$$D \subseteq E \iff n+D \subseteq n+E. \tag{7.3.18}$$

$$D = E \iff n+D = n+E. \tag{7.3.19}$$

$$0+D = D. \tag{7.3.20}$$

$$n+\varnothing = \varnothing. \tag{7.3.21}$$

$$n+(m+D) = (n+m)+D. \tag{7.3.22}$$

$$n+(D\cup E) = (n+D)\cup(n+E). \tag{7.3.23}$$

$$n+(D\cap E) = (n+D)\cap(n+E). \tag{7.3.24}$$

Proof. (7.3.17): Suppose $X \in n+D$. Then, we get $X = n+A$ for some $A \in D$. It follows that $X-n = A \in D$. On the other hand, if $X-n \in D$ then $X = n+(X-n) \in n+D$.

(7.3.18): Suppose $D \subseteq E$. Let $X \in n+D$. From (7.3.17), we get $X-n \in D \subseteq E$. Thus, again from (7.3.17) we get $X \in n+E$. This shows

$$n+D \subseteq n+E.$$

For the reverse direction, assume $n+D \subseteq n+E$. Let $X \in D$. From (7.3.15) we get $n+X \in n+D \subseteq n+E$. Finally, from (7.3.17), we get $X = n+X-n \in B$. Thus, we have

$$D \subseteq E.$$

(7.3.19): We see this from the following chain of double implications.

$$\begin{aligned}
D = E &\iff (D \subseteq E) \text{ and } (E \subseteq D) \\
&\iff (n + D \subseteq n + E) \text{ and } (n + E \subseteq n + D) && (7.3.18) \\
&\iff n + D = n + E.
\end{aligned}$$

(7.3.20): We have

$$\begin{aligned}
0 + D &= \{0 + A \mid A \in D\} && (7.3.15) \\
&= \{A \mid A \in D\} \\
&= D.
\end{aligned}$$

(7.3.21): For the sake of contradiction assume $n + \varnothing \neq \varnothing$. Let $X \in n + \varnothing$ then, from (7.3.17), we get $X - n \in \varnothing$. This is a contradiction.

(7.3.22): We have

$$\begin{aligned}
n + (m + D) &= n + \{m + A \mid A \in D\} && (7.3.15) \\
&= \{n + m + A \mid A \in D\} && (7.3.15) \\
&= (n + m) + D. && (7.3.15)
\end{aligned}$$

(7.3.23): We get

$$\begin{aligned}
n + (D \cup E) &= \{n + A \mid A \in D \text{ or } A \in E\} && (7.3.15) \\
&= \{n + A \mid A \in D\} \cup \{n + A \mid A \in E\} \\
&= (n + D) \cup (n + E). && (7.3.15)
\end{aligned}$$

(7.3.24): We have

$$\begin{aligned} n+(D\cap E) &= \{n+A \mid A\in D \text{ and } A\in E\} && (7.3.15)\\ &= \{n+A \mid A\in D\}\cap\{n+A \mid A\in E\}\\ &= (n+D)\cap(n+E). && (7.3.15) \end{aligned}$$

□

Proposition 7.3.25. *Let $n\in\mathbb{Z}$, $B\subseteq\mathbb{Z}$, and $D\subseteq\mathcal{P}(B)$. Then, $n+D\subseteq\mathcal{P}(n+B)$. Moreover, if $D\subseteq\mathcal{P}_I^+(B)$ then $n+D\subseteq\mathcal{P}_I^+(n+B)$.*

Proof. Let $A\in n+D$. We get $-n+A\in D$. Since $D\subseteq\mathcal{P}(B)$ we get $-n+A\subseteq B$. This implies $A\subseteq n+B$. This shows that $n+D\subseteq\mathcal{P}(n+B)$.

Now assume $D\subseteq\mathcal{P}_I^+(B)$. Let $n+A\in n+D$ where $A\in D$. We have that $A\in D$ is a non-empty interval. Therefore, we get that $n+A$ is a non-empty interval. This shows that $n+D\subseteq\mathcal{P}_I^+(n+B)$. □

7.4 Bracketing on a Word

Definition 6.2.25 gives a formal definition of words with letters from a set S. In Theorem 6.2.39, we saw that the set of all such words forms the free monoid generated by S. We now want to extend this construction to include a formal dash operation on words. This dash operation is neither involutive nor does it distribute over multiplication, so we must keep track of which groupings of letters within a word have dashes applied to them.

For example, let $S=\{a,b,c\}$. The word

$$(a''(bc)''')'b$$

should be treated as a distinct element in the free dashed-monoid.

To handle this, we introduce the concept of *bracketings*. A bracketing serves two main purposes: (1) it records the locations of groups of letters that are dashed together, and (2) it keeps track of nested groupings.

In the example above, if we focus only on the locations of grouped letters, we get the following schematic:

$$((*)(**))*$$

Here, each $*$ represents a letter from S, and each bracketed group has at least one dash. To reconstruct the actual word from this schematic, we need to specify which letter each $*$ stands for and how many dashes are on each bracket. The following formal definitions make this precise.

Definition 7.4.1. Let $n \in \mathbb{N}$. A *bracketing* on n letters is a collection D of non-empty sub-intervals of $[n]$, that is,

$$D \subseteq \mathcal{P}_I^+(n),$$

such that for all $A, B \in D$, the following *bracketing condition* holds:

$$A \cap B = \emptyset, \qquad \text{or} \quad A \subseteq B, \quad \text{or} \quad B \subseteq A. \tag{7.4.2}$$

The elements of D are called *brackets*. We denote the set of all bracketings on n letters by $\mathcal{B}r(n)$. ◇

Convention 7.4.3. Define

$$\mathcal{B}r := \bigsqcup_{n \in \mathbb{N}} \mathcal{B}r(n)$$

as the set of all bracketings, and

$$\mathcal{B}r^+ := \bigsqcup_{n \in \mathbb{N}^+} \mathcal{B}r(n)$$

as the set of all non-empty bracketings. ◇

Definition 7.4.4. Let $n \in \mathbb{N}$. Then, the empty collection $\varnothing \subseteq \mathcal{P}_I^+(n)$ vacuously satisfies the bracketing condition. We call this the *empty bracketing* on n letters.

We define the *unit bracketing*, denoted $J_{\mathcal{B}r}$, to be the empty bracketing, $\varnothing \subseteq \mathcal{P}_I^+(0)$, on 0 letters. That is,

$$J_{\mathcal{B}r} := \varnothing \in \mathcal{B}r(0). \tag{7.4.5}$$

Note that, the unit bracketing is the only possible bracketing on 0 letters. ◇

Example 7.4.6. Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$, then every $A \in D$ corresponds to a bracket. We require that A is an interval since a bracket can not have gaps within them. Conventionally, we assume that A is non-empty since one can simply omit empty bracket. The bracketing condition ensures that each bracket contains only the closed brackets.

Here are some examples of bracketings on a word with five letters.

Bracketing (D)	Schematic
$\varnothing$	$* * * * *$
$\{\{1,2,3,4,5\}\}$	$(* * * * *)$
$\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$	$(*)\,(*)\,(*)\,(*)\,(*)$
$\{\{1,2\},\{1,2,3\},\{4,5\}\}$	$((**)\,*)\,(**)$
$\{\{1,2\},\{4,5\},\{1,2,3,4,5\}\}$	$((**)\ *\ (**))$

◇

Proposition 7.4.7. *Let $n,m \in \mathbb{N}$. Let $D \in \mathcal{B}r(n)$ and $E \in \mathcal{B}r(m)$ be bracketings on n and m letters respectively. Then,*

$$D \cap (n+E) = \varnothing$$

and

$$D \sqcup (n+E) \in \mathcal{B}r(n+m).$$

Proof. First, we will show $D \cap (n+E) = \varnothing$. Let $A \in D \cap (n+E)$, that is, $A \in D$ and $A = n+B$ for some $B \in E$. Then, we get $A \subseteq (0,n]$ and $A \subseteq n+(0,m] = (n, n+m]$. Thus, we get $A = \varnothing$. This is a contradiction since $A \in D$ and D is a bracketing implies $A \neq \varnothing$. Hence, we get $D \cap (n+E) = \varnothing$.

To see that $D \cup (n+E)$ is a bracketing on $n+m$ letters, we need to check following two things:

i. $D \cup (n+E) \subseteq \mathcal{P}_I^+(n+m)$, and

ii. for every $A, B \in D \cup (n+E)$ either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$.

Since D is a bracketing on n letters, we know that $D \subseteq \mathcal{P}_I^+(n) \subseteq \mathcal{P}_I^+(n+m)$. Since E is a bracketing on m letters, we get that $E \subseteq \mathcal{P}_I^+(m)$. We have

$$n+E \subseteq \mathcal{P}_I^+(n+[m]) = \mathcal{P}_I^+([n+1, n+m]) \subseteq \mathcal{P}_I^+(n+m).$$

Thus, we get

$$D \cup (n+E) \subseteq \mathcal{P}_I^+(n+m).$$

Now for the bracketing condition, let $A, B \in D \cup (n+E)$. We will consider the following cases:

<u>Case 1</u> ($A, B \in D$): Since D is a bracketing and $A, B \in D$ we get either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$.

<u>Case 2</u> ($A, B \in n+E$): That is, we have $X, Y \in E$ such that $A = n+X$ and $B = n+Y$. Since E is a bracketing, from the bracketing condition, we get either $X \cap Y = \varnothing$,

$X \subseteq Y$, or $Y \subseteq X$. If $X \cap Y = \varnothing$, then we get

$$\begin{aligned} A \cap B &= (n + X) \cap (n + Y) \\ &= n + (X \cap Y) \\ &= n + \varnothing \\ &= \varnothing. \end{aligned}$$

If $X \subseteq Y$, then we get

$$\begin{aligned} A &= n + X \\ &\subseteq n + Y \\ &= B. \end{aligned}$$

Similarly, we get $B \subseteq A$ if $Y \subseteq X$.

Case 3 ($A \in D, B \in n + E$): That is, $A \in D$ and we have $Y \in E$ such that $B = n + Y$. Since D is a bracketing on n letters, we get $A \subseteq (0, n]$. Similarly, we get $Y \subseteq (0, m]$. This implies $B = n + Y \subseteq (n, m]$. Thus, we get $A \cap B = \varnothing$.

Since in each of the cases we get either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$, we conclude that $D \cup (n + E)$ is a bracketing on $n + m$ letters. □

Definition 7.4.8. Let $n, m \in \mathbb{N}$ and $D \in \mathcal{Br}(n), E \in \mathcal{Br}(m)$ be bracketings. Define the multiplication of the bracketings, $D \bullet E \in \mathcal{Br}(n + m)$, as

$$D \bullet E := D \sqcup (n + E). \tag{7.4.9}$$

From Proposition 7.4.7 we know that $D \bullet E$ is a bracketing on $n + m$ letters. ◇

Example 7.4.10. Let

$$D = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\big\} \in \mathcal{Br}(5)$$

and

$$E = \{1\} \in \mathcal{B}r(3).$$

Then, we have

$$D \bullet E = \{\{1,2\},\ \{1,2,3\},\ \{4,5\},\{6\}\} \in \mathcal{B}r(8).$$

Using the visual representation, we get

$$D = ((**)\ *)\ (**) \qquad \text{and} \qquad E = (*)\ **.$$

Then, the product is given by the concatenation,

$$D \bullet E = ((**)\ *)\ (**)\ (*)\ **.$$

◇

Lemma 7.4.11. *Let $n, m, p \in \mathbb{N}$ and $D \in \mathcal{B}r(n), E \in \mathcal{B}r(m), F \in \mathcal{B}r(p)$ be bracketings. Then we have*

$$D \bullet (E \bullet F) = (D \bullet E) \bullet F \tag{7.4.12}$$

as bracketings on $n+m+p$ letters.

Proof. Observe that,

$$\begin{aligned}
D \bullet (E \cdot F) &= D \cup (n + (E \bullet F)) && (7.4.9)\\
&= D \cup (n + (E \cup (m + F))) && (7.4.9)\\
&= D \cup (n + E) \cup (n + (m + F)) && (7.3.23)\\
&= D \cup (n + E) \cup ((n + m) + F) && (7.3.22)\\
&= (D \bullet E) \cup ((n + m) + F) && (7.4.9)\\
&= (D \bullet E) \bullet F. && (7.4.9)
\end{aligned}$$

□

Remark 7.4.13. Since the bracketing multiplication is associative, we will omit the use of brackets to show the order of multiplication. ⋄

Lemma 7.4.14. *Let $n \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$. Recall, $J_{\mathcal{Br}} \in \mathcal{Br}(0)$ is the unit bracketing as in Definition 7.4.4. We have*

$$D \bullet J_{\mathcal{Br}} = D \qquad \text{and} \qquad J_{\mathcal{Br}} \bullet D = D \tag{7.4.15}$$

as bracketings on n letters.

Proof. We get

$$\begin{aligned} D \bullet I &= D \bullet \varnothing && (7.4.5) \\ &= D \cup (n + \varnothing) && (7.4.9) \\ &= D \cup \varnothing && (7.3.21) \\ &= D. \end{aligned}$$

Next, we get

$$\begin{aligned} I \bullet D &= \varnothing \bullet D && (7.4.5) \\ &= \varnothing \cup (0 + D) && (7.4.9) \\ &= \varnothing \cup D && (7.3.20) \\ &= D. \end{aligned}$$

□

Proposition 7.4.16. *Let $n \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$ be a bracketing. Then,*

$$E := \{A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ or } \ A = [n]\}$$

is a bracketing on n letters.

Proof. To see that E is a bracketing on n letters, we need to check following two things:

i. $E \subseteq \mathcal{P}_I^+(n)$, and

ii. for every $A, B \in E$ either $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$.

By the construction, we have $E \subseteq \mathcal{P}_I^+(n)$. Now, let $A, B \in E$. We consider following cases:

Case 1 ($A, B \in D$): In this case, the bracketing condition is satisfied since D itself is a bracketing.

Case 2 ($B = [n]$): In this case, since $E \subseteq \mathcal{P}_I^+(n)$, we have $A \subseteq [n] = B$.

This completes the proof. □

Definition 7.4.17. Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Define the dash of the bracketing, $D' \in \mathcal{B}r(n)$, as follows:

$$D' := \{A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ or } \ A = [n]\}. \tag{7.4.18}$$

From Proposition 7.4.16 we know that D' is a bracketing on n letters. ⋄

Proposition 7.4.19. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then, we get*

$$D' = \begin{cases} D & \text{if } \ n = 0 \\ D \cup \{[n]\} & \text{if } \ n \geq 1. \end{cases} \tag{7.4.20}$$

Proof. We will consider following cases:

Case 1($n = 0$): In this case, we have both $D, D' \in \mathcal{B}r(0)$. Since $\mathcal{B}r(0)$ only consists of $J_{\mathcal{B}r}$, we get $D = D'$.

Case 2($n \geq 1$): In this case, we have $[n] \in \mathcal{P}_I^+(n)$. Observe that,

$$D' = \{A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ or } \ A = [n]\} \tag{7.4.18}$$

$$= \{A \in zP_I^+(n) \mid A \in D\} \cup \{A \in \mathcal{P}_I^+(n) \mid A = [n]\}$$

$$= D \cup \{[n]\}.$$

□

Example 7.4.21. Let

$$D = \{\{1,2\},\ \{1,2,3\},\ \{4,5\}\} \in \mathcal{B}r(5).$$

Then, we have

$$D' = \{\{1,2\},\ \{1,2,3\},\ \{4,5\},\ \{1,2,3,4,5\}\} \in \mathcal{B}r(5).$$

Using the visual representation, we get

$$D = ((**)\ *)\ (**).$$

Then, the dash of D is given by adding the outermost bracket if there is no such bracket,

$$D' = (((**)\ *)\ (**)\ (*)).$$

⋄

Lemma 7.4.22. *Consider the unit bracketing, $J_{\mathcal{B}r} \in \mathcal{B}r(0)$. We have*

$$J'_{\mathcal{B}r} = J_{\mathcal{B}r} \tag{7.4.23}$$

as bracketing on 0 *letters.*

Proof. Follows immediately from the case $n = 0$ of Proposition 7.4.19. □

Proposition 7.4.24. *Let $n \geq 1$ and $D \in \mathcal{Br}(n)$ be a bracketing. For $k \geq 1$ let $D^{(k)}$ denote the bracketing on n letters that we get after taking k dashes of D. Then we get*

$$D^{(k)} = D \cup \{[n]\} \tag{7.4.25}$$

as bracketings on n letters.

Proof. We will use induction on k.

Base case ($k = 1$): We get

$$D^{(1)} = D' = D \cup \{[n]\}. \qquad (7.4.20)$$

Induction case ($k \geq 1$): Assume that $D^{(k)} = D \cup \{[n]\}$. We get

$$\begin{aligned} D^{(k+1)} &= \left(D^{(k)}\right)' \\ &= (D \cup \{[n]\})' && \text{Induction} \\ &= D \cup \{[n]\} \cup \{[n]\} && \text{Base case} \\ &= D \cup \{[n]\}. \end{aligned}$$

□

Proposition 7.4.26. *Let $n \in \mathbb{N}$, $D \in \mathcal{Br}(n)$ be a bracketing, and $C \subseteq [n]$. Then,*

$$E := \left\{A \in \mathcal{P}_I^+(n) \mid A \in D \text{ and } A \neq C\right\}$$

is a bracketing on n letters.

Proof. To see that E is a bracketing on n letters, we need to check following two things:

i. $E \subseteq \mathcal{P}_I^+(n)$, and

ii. for every $A, B \in E$ either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$.

By the construction, E is a subset of $\mathcal{P}_I^+(n)$. Now let $A, B \in E$. Then, we get $A, B \in D$. The bracketing condition is satisfied since D itself is a bracketing. This shows that E is a bracketing on n letters. $\square$

Definition 7.4.27. Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Define the *floor* of the bracketing D, denoted $\lfloor D \rfloor \in \mathcal{B}r(n)$, as follows:

$$\lfloor D \rfloor := \{A \in \mathcal{P}_I^+(n) \mid A \in D \text{ and } A \neq [n]\}. \tag{7.4.28}$$

From Proposition 7.4.26 we know that $\lfloor D \rfloor$ is a bracketing on n letters. $\diamond$

Proposition 7.4.29. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then, we get*

$$\lfloor D \rfloor = \begin{cases} D & \text{if } n = 0 \\ D \setminus \{[n]\} & \text{if } n \geq 1. \end{cases} \tag{7.4.30}$$

Proof. We will consider following cases:

Case 1($n = 0$): In this case, we have both $D, \lfloor D \rfloor \in \mathcal{B}r(0)$. Since $\mathcal{B}r(0)$ only consists of $J_{\mathcal{B}r}$, we get $D = \lfloor D \rfloor$.

Case 2($n \geq 1$): In this case, we have $[n] \in \mathcal{P}_I^+(n)$. Observe that,

$$\begin{aligned} \lfloor D \rfloor &= \{A \in \mathcal{P}_I^+(n) \mid A \in D \text{ and } A \neq [n]\} && (7.4.28) \\ &= \{A \in zP_I^+(n) \mid A \in D\} \cap \{A \in \mathcal{P}_I^+(n) \mid A \neq [n]\} \\ &= D \setminus \{[n]\}. \end{aligned}$$

$\square$

Example 7.4.31. Let

$$D = \{\{1,2\},\ \{1,2,3\},\ \{4,5\},\ \{1,2,3,4,5\}\} \in \mathcal{B}r(5).$$

and

$$E = \{1\} \in \mathcal{B}r(3).$$

Then, we have

$$\lfloor D \rfloor = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\big\} \in \mathcal{Br}(5).$$

Using the visual representation, we get

$$D = (((**)\ *)\ (**)).$$

Then, the floor of D is given by removing the outermost bracket if there is one,

$$\lfloor D \rfloor = ((**)\ *)\ (**). \qquad \diamond$$

Lemma 7.4.32. *Consider the unit bracketing, $J_{\mathcal{Br}} \in \mathcal{Br}(0)$. We have*

$$\lfloor J_{\mathcal{Br}} \rfloor = J_{\mathcal{Br}} \tag{7.4.33}$$

as bracketing on 0 letters.

Proof. Follows immediately from the case $n = 0$ of Proposition 7.4.29. □

Proposition 7.4.34. *Let $n \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$ be a bracketing. We get*

$$\lfloor \lfloor D \rfloor \rfloor := \Big\lfloor \big(\lfloor D \rfloor\big) \Big\rfloor = \lfloor D \rfloor \tag{7.4.35}$$

as bracketings on n letters.

Proof. If $n = 0$ then, from Proposition 7.4.29, we get $\lfloor D \rfloor = D$. As a result, we get

$$\lfloor \lfloor D \rfloor \rfloor = \lfloor D \rfloor = D.$$

If $n \geq 1$ then, from Proposition 7.4.29, we get $\lfloor D \rfloor = D \setminus \big\{[n]\big\}$. Thus, we get
Base case ($k = 1$): We get

$$\begin{aligned} \lfloor \lfloor D \rfloor \rfloor &= \lfloor D \setminus \{[n]\}] && (7.4.30) \\ &= (D \setminus \{[n]\}) \setminus \{[n]\} && (7.4.30) \\ &= D \setminus \{[n]\} \\ &= \lfloor D \rfloor . && (7.4.30) \end{aligned}$$

□

Proposition 7.4.36. *Let $n \in \mathbb{N}^+$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then, for $k \in \mathbb{N}$ we have*

$$\left\lfloor D^{(k)} \right\rfloor = \lfloor D \rfloor . \tag{7.4.37}$$

Proof. For the case $k = 0$, we get $D^{(0)} = D$. Therefore, the proposition is true for $k = 0$. Now assume $k \geq 1$. We have

$$\begin{aligned}
\left\lfloor D^{(k)} \right\rfloor &= \lfloor D \cup \{[n]\} \rfloor && (7.4.25) \\
&= (D \cup \{[n]\}) \setminus \{[n]\} && (7.4.30) \\
&= D \setminus \{[n]\} \\
&= \lfloor D \rfloor . && (7.4.30)
\end{aligned}$$

□

Proposition 7.4.38. *Let $n \in \mathbb{N}^+$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then, $\lfloor D \rfloor = D$ if and only if $[n] \notin D$.*

Proof. Suppose $\lfloor D \rfloor = D$. From Proposition 7.4.29, we get $\lfloor D \rfloor = D \setminus \{[n]\}$. Therefore, we have $D = D \setminus \{[n]\}$ implying $[n] \notin D$.

On the other hand, suppose $[n] \notin D$. Then, we get

$$\begin{aligned}
\lfloor D \rfloor &= D \setminus \{[n]\} && (7.4.30) \\
&= D .
\end{aligned}$$

□

Proposition 7.4.39. *Let $n \in \mathbb{N}^+$ and $D \in \mathcal{B}r(n)$ be a bracketing. Suppose $[n] \in D$. Then, for $k \geq 1$ we have*

$$(\lfloor D \rfloor)^{(k)} = D . \tag{7.4.40}$$

Proof. Observe that

$$\begin{aligned}(\lfloor D \rfloor)^{(k)} &= (D \setminus \{[n]\})^{(k)} && (7.4.30)\\ &= (D \setminus \{[n]\}) \cup \{[n]\} && (7.4.25)\\ &= D.\end{aligned}$$

The last equality follows from the assumption that $[n] \in D$. □

Proposition 7.4.41. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Let $(p,q] \subseteq_I (0,n]$ with $0 \le p \le q \le n$. Then,*

$$E := -p + \{B \in D \mid B \subseteq (p,q]\}$$

is a bracketing on $p-q$ letters.

Proof. To see that E is a bracketing on $p-q$ letters, we need to check following two things:

i. $E \subseteq \mathcal{P}_I^+(p-q)$, and

ii. for every $A,B \in E$ either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$.

First, we will show that every element of E is a non-empty sub-interval of $(0,q-p]$. Let $A \in E$, then we get $X \in D$ with $X \subseteq (p,q]$ such that $A = -p + X$. Since $X \in D$ we get that X is a non-empty interval and hence $A = -p + X$ is a non-empty interval. Since $X \subseteq (p,q]$ we get

$$A = -p + X \subseteq -p + (p,q] = (0,q-p].$$

This shows that $E \subseteq \mathcal{P}_I^+(q-p)$.

It remains to check that E satisfies the bracketing condition. Let $A,B \in E$. Then we get $X,Y \in D$ with $X,Y \subseteq (p,q]$ such that $A = -p + X$ and $C = -p + Y$. Since D

is a bracketing, we get either $X \cap Y = \varnothing$, $X \subseteq Y$, or $Y \subseteq Z$. This implies that either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$. Thus, E is a bracketing on $q-p$ letters. $\square$

Definition 7.4.42. Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Let $(p,q] \subseteq_I [n]$ with $0 \leq p \leq q \leq n$. Define a bracketing, $D_{(p,q]}$, on $q-p$ letters as follows:

$$D_{(p,q]} := -p + \{B \in D \mid B \subseteq (p,q]\}. \tag{7.4.43}$$

Proposition 7.4.41 shows that $D_{(p,q]}$ is a bracketing on $q-p$ letters. $\diamond$

Example 7.4.44. Let

$$D = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\big\} \in \mathcal{B}r(5).$$

Then, we get

$$\begin{aligned}
D_{(0,3]} &= \big\{\{1,2\},\ \{1,2,3\}\big\} && \in \mathcal{B}r(3), \\
D_{(0,2]} &= \big\{\{1,2\}\big\} && \in \mathcal{B}r(2), \\
D_{(3,5]} &= \big\{\{1,2\}\big\} && \in \mathcal{B}r(2), \\
D_{(2,5]} &= \big\{\{2,3\}\big\} && \in \mathcal{B}r(3), \text{ and} \\
D_{(1,4]} &= \big\{\big\} && \in \mathcal{B}r(3).
\end{aligned}$$

Using the visual representation, we get

$$D = ((**)\ *)\ (**).$$

Then, the restrictions are given by the,

$$
\begin{aligned}
&D_{(0,3]} = ((**)\,*),\\
&D_{(0,2]} = (**),\\
&D_{(3,5]} = (**),\\
&D_{(2,5]} = *\,(**), \quad \text{and}\\
&D_{(1,4]} = ***.
\end{aligned}
$$

◇

Proposition 7.4.45. *Let $n \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$ be a bracketing. Then, we get*

$$D_{[n]} = D. \tag{7.4.46}$$

Proof. We get this from the following:

$$
\begin{aligned}
D_{[n]} = D_{(0,n]} &= -0 + \{B \in D \mid B \subseteq (0,n]\} && (7.4.43)\\
&= \{B \in D\}\\
&= D.
\end{aligned}
$$

□

Proposition 7.4.47. *Let $n, m \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$, $E \in \mathcal{Br}(m)$ be bracketings. Let $(p,q] \subseteq_I [n]$ with $0 \leq p \leq q \leq n$. Then, we get*

$$\left(D \cdot E\right)_{(p,q]} = D_{(p,q]} \tag{7.4.48}$$

as bracketings on $q - p$ letters.

Proof. We will show $\left(D \cdot E\right)_{(p,q]} = D_{(p,q]}$ by showing that

$$\left(D \cdot E\right)_{(p,q]} \subseteq D_{(p,q]} \qquad \text{and} \qquad D_{(p,q]} \subseteq \left(D \cdot E\right)_{(p,q]}.$$

Let $A \in \mathcal{P}_I^+(q-p)$. Suppose $A \in \left(D \cdot E\right)_{(p,q]}$. We get some $X \in D \cdot E$ with $X \subseteq (p,q]$ such that $-p + X = A$. We have

$$X \in D \cdot E = D \sqcup (n + E).$$

If $X \in n + E$, then we must have $X \subseteq (n,m]$. Since $X \subseteq (p,q] \subseteq [n]$, this case is not possible. Thus, we conclude that $X \in D$. Since $X \in D$ and $X \subseteq (p,q]$, we get that

$$A = -p + X \in D_{(p,q]}.$$

This shows that $\left(D \cdot E\right)_{(p,q]} \subseteq D_{(p,q]}$.

On the other hand, suppose $A \in D_{(p,q]}$. We get some $X \in D$ with $X \subseteq (p,q]$ such that $-p + X = A$. We get $X \in D \sqcup (n + E) = D \cdot E$ and $X \subseteq (p,q]$. Thus,

$$A = p + X \in \left(D \cdot E\right)_{(p,q]}.$$

This shows that $D_{(p,q]} \subseteq \left(D \cdot E\right)_{(p,q]}$.

Therefore, we conclude

$$\left(D \cdot E\right)_{(p,q]} = D_{(p,q]}.$$

Proposition 7.4.49. *Let $n, m \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings. Let $(p,q] \subseteq_I (n, n+m]$ with $n \le p \le q \le n+m$. Then, we get*

$$\left(D \cdot E\right)_{(p,q]} = E_{(p-n,q-n]} \tag{7.4.50}$$

as bracketings on $q-p$ letters.

Proof. We will show $\left(D \cdot E\right)_{(p,q]} = E_{(p-n,q-n]}$ by showing that

$$\left(D \cdot E\right)_{(p,q]} \subseteq E_{(p-n,q-n]} \qquad \text{and} \qquad E_{(p-n,q-n]} \subseteq \left(D \cdot E\right)_{(p,q]}.$$

Let $A \in \mathcal{P}_I^+(q-p)$. Suppose $A \in \left(D \cdot E\right)_{(p,q]}$. We get some $X \in D \cdot E$ with $X \subseteq (p,q]$ such that $-p+X = A$. We have

$$X \in D \cdot E = D \sqcup (n+E).$$

If $X \in D$, then we must have $X \subseteq [n]$. Since $X \subseteq (p,q] \subseteq (n, n+m]$, this case is not possible. Thus, we conclude that $X \in n+E$. That is, $-n+X \in E$. Moreover, since $X \subseteq (p,q]$ we get $-n+X \subseteq (p-n, q-n]$. This shows that

$$A = -(p-n) + (-n+X) \in E_{(p-n,q-n]}.$$

Thus, we get

$$\left(D \cdot E\right)_{(p,q]} \subseteq E_{(p-n,q-n]}.$$

On the other hand, suppose $A \in E_{(p-n,q-n]}$. We get some $X \in E$ with $X \subseteq (p-n, q-n]$ such that $-(p-n)+X = A$. We get

$$n+X \in D \sqcup (n+E) = D \cdot E$$

and $n+X \subseteq (p,q]$. Thus,

$$A = -p+n+X \in \left(D \cdot E\right)_{(p,q]}.$$

Consequently, we get

$$E_{(p-n,q-n]} \subseteq \left(D \cdot E\right)_{(p,q]}.$$

Therefore, we conclude

$$\left(D \cdot E\right)_{(p,q]} = E_{(p-n,q-n]}.$$

Lemma 7.4.51. *Let $n, m \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$, $E \in \mathcal{Br}(m)$ be bracketings. Then, we get*

$$\big(D \cdot E\big)_{[n]} = D \tag{7.4.52}$$

as bracketings on n letters.

Proof. We see that

$$\begin{aligned} \big(D \cdot E\big)_{[n]} &= D_{[n]} && (7.4.48) \\ &= D. && (7.4.46) \end{aligned}$$

□

Lemma 7.4.53. *Let $n, m \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$, $E \in \mathcal{Br}(m)$ be bracketings. Then, we get*

$$\big(D \cdot E\big)_{(n,n+m]} = E \tag{7.4.54}$$

as bracketings on m letters.

Proof. We see that

$$\begin{aligned} \big(D \cdot E\big)_{(n,n+m]} &= E_{(0,m]} && (7.4.50) \\ &= E. && (7.4.46) \end{aligned}$$

□

7.5 Dash Assignments on a Bracketing

The bracketing introduced in the previous section gives the structural outline for a dashed word. In this section, we refine that outline by specifying additional information. For instance, let $S = \{a, b, c\}$ and consider the word

$$(a''(bc)''')'b$$

as mentioned at the start of Section 7.4. The corresponding bracketing schematic is

$$((*)(**))*\,.$$

Here, each bracket represents a group of letters that are dashed together. To indicate how many dashes are applied to each group, we define a function from the set of brackets to the positive integers. This function assigns to each bracket the number of times the dash operation is applied to that group. We call such a function a *dash assignment*. For the example above, the dash assignment schematic is

$$((*)^{(2)}\,(**)^{(3)})^{(1)}\;\;*\,.$$

Definition 7.5.1. Let $n \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$ be a bracketing on n letters. A *dash-assignment* on the bracketing D is a function $d : D \longrightarrow \mathbb{N}^+$. We denote the set of all dash-assignments on D by $\mathcal{Ds}(D)$. ◇

Definition 7.5.2. Let $n \in \mathbb{N}$ and $\varnothing \in \mathcal{Br}(n)$ be the empty bracketing on n-letters. Then, the empty map $\emptyset : \varnothing \longrightarrow \mathbb{N}^+$ is the only dash-assignment. We call this the *empty dash-assignment* on the empty bracketing on n letters. The *unit dash-assignment*, denoted $J_{\mathcal{Ds}}$, is the empty dash-assignment on the unit bracketing. That is,

$$J_{\mathcal{Ds}} := \emptyset \in \mathcal{Ds}(J_{\mathcal{Br}})\,. \tag{7.5.3}$$

◇

Example 7.5.4. Let $n \in \mathbb{N}$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$. Then, every $A \in D$ corresponds to a bracket and the positive integer, $d(A)$, corresponds to the number of dashes on that bracket.

For example, let

$$D = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\big\} \in \mathcal{Br}(6)$$

be a bracketing on 6 letters. Then, an example of dash-assignment $d \in \mathcal{Ds}(D)$ is given by

$$\{1,2\} \longmapsto 2,$$
$$\{1,2,3\} \longmapsto 1,$$
$$\{4,5\} \longmapsto 3.$$

A visual representation of above dash-assignment is given by

$$d = ((**)'' \; *)' \; (**)''' \; * \,.$$

Let

$$E = \big\{\{2\},\ \{2,3,4\},\ \{1,2,3,4,5\}\big\} \in \mathcal{Br}(5)$$

be a bracketing on 5 letters. Then, an example of dash-assignment $e \in \mathcal{Ds}(D)$ is given by

$$\{2\} \longmapsto 7,$$
$$\{2,3,4\} \longmapsto 18,$$
$$\{1,2,3,4,5\} \longmapsto 11.$$

A visual representation of above dash-assignment is given by

$$e = (* \; ((*)^{(7)} \; **)^{(18)} \; *)^{(11)}.$$ ◇

Definition 7.5.5. Let $n,m \in \mathbb{N}$, $D \in \mathcal{Br}(n)$, and $E \in \mathcal{Br}(m)$. Let $d \in \mathcal{Ds}(D)$ and $e \in \mathcal{Ds}(E)$ be dash-assignments on D and E respectively. Define the multiplication dash-assignments, $d \bullet e \in \mathcal{Ds}(D \bullet E)$, as follows:

$$d \bullet e\ (A) = \begin{cases} d(A) & \text{if } A \in D \\ e(-n+A) & \text{if } A \in n+E. \end{cases} \tag{7.5.6}$$

◇

Example 7.5.7. Let

$$d = ((**)'' *)' (**)''' * \in \mathcal{D}s(D) \quad \text{and} \quad e = (* ((*)^{(7)} **)^{(18)} *)^{(11)} \in \mathcal{D}s(E)$$

be dash-assignments as in Example 7.5.4. Then, the multiplication of the dash-assignments $d \bullet e \in \mathcal{D}s(D \bullet E)$ is given by

$$\begin{aligned} \{1,2\} &\longmapsto 2, \\ \{1,2,3\} &\longmapsto 1, \\ \{4,5\} &\longmapsto 3, \\ \{8\} &\longmapsto 7, \\ \{8,9,10\} &\longmapsto 18, \\ \{7,8,9,10,11\} &\longmapsto 11. \end{aligned}$$

A visual representation of above dash-assignment is given by the concatenation.

$$d \bullet e = ((**)'' *)' (**)''' * (* ((*)^{(7)} **)^{(18)} *)^{(11)}. \qquad \diamond$$

Lemma 7.5.8. *Let $n, m, p \in \mathbb{N}$ and $D \in \mathcal{B}r(n), E \in \mathcal{B}r(m), F \in \mathcal{B}r(p)$ be bracketings. Let $d \in \mathcal{D}s(D)$, $e \in \mathcal{D}s(E)$, and $f \in \mathcal{D}s(F)$ be dash-assignments. Then we have*

$$d \bullet (e \bullet f) = (d \bullet e) \bullet f \tag{7.5.9}$$

as dash-assignments on $D \bullet E \bullet F$.

Proof. Let $A \in D \bullet E \bullet F$. Observe that,

$$
\begin{aligned}
d \bullet (e \bullet f)\,(A) &= \begin{cases} d(A) & \text{if } A \in D \\ e \bullet f\,(-n+A) & \text{if } A \in n + E \cdot F \end{cases} && (7.5.6) \\
&= \begin{cases} d(A) & \text{if } A \in D \\ e(-n+A) & \text{if } A \in n + E \\ f(-n-m+A) & \text{if } A \in n+m+E \end{cases} && (7.5.6) \\
&= \begin{cases} d \bullet e\,(A) & \text{if } A \in D \bullet E \\ f(-(n+m)+A) & \text{if } A \in (n+m)+E \end{cases} && (7.5.6) \\
&= (d \bullet e) \bullet f\,(A). && (7.5.6)
\end{aligned}
$$

Since $A \in D \bullet E \bullet F$ is arbitrary, we get

$$d \bullet (e \bullet f) = (d \bullet e) \bullet f$$

as required. □

Lemma 7.5.10. *Let $n \in \mathbb{N}$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Then,*

$$d \bullet J_{\mathcal{Ds}} = d \qquad \text{and} \qquad J_{\mathcal{Ds}} \bullet d = d \tag{7.5.11}$$

as dash-assignments on D.

Proof. Let $A \in D$. We get

$$
\begin{aligned}
d \bullet J_{\mathcal{Ds}}\,(A) &= \begin{cases} d(A) & \text{if } A \in D \\ & \text{if } A \in J_{\mathcal{Br}} \end{cases} && (7.5.6) \\
&= \begin{cases} d(A) & \text{if } A \in D \\ & \text{if } A \in \varnothing \end{cases} && (7.4.5) \\
&= d.
\end{aligned}
$$

Since $A \in D$ is arbitrary, we get

$$d \bullet J_{\mathcal{Ds}} = d.$$

Next, we get

$$J_{\mathcal{D}s} \bullet d\ (A) = \begin{cases} & \text{if } A \in J_{\mathcal{B}r} \\ d(A) & \text{if } A \in D \end{cases} \qquad (7.5.6)$$

$$= \begin{cases} & \text{if } A \in \varnothing \\ d(A) & \text{if } A \in D \end{cases} \qquad (7.4.5)$$

$$= d(A).$$

Since $A \in D$ is arbitrary, we get

$$J_{\mathcal{D}s} \bullet d = d.$$

□

Definition 7.5.12. Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing. Let $d \in \mathcal{D}s(D)$ be a dash-assignment. Define the dash of the dash-assignment, $d' \in \mathcal{D}s\left(D'\right)$, as follows:

$$d'\ (A) := \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+1 & \text{if } A = [n] \text{ and } A \in D \\ 1 & \text{if } A = [n] \text{ and } A \notin D. \end{cases} \qquad (7.5.13)$$

◇

Lemma 7.5.14. *Consider the unit dash-assignment, $J_{\mathcal{D}s} \in \mathcal{D}s(J_{\mathcal{B}r})$. We have*

$$J'_{\mathcal{D}s} = J_{\mathcal{D}s} \qquad (7.5.15)$$

as dash-assignments on the unit bracketing.

Proof. Since both $J'_{\mathcal{D}s}$ and $J_{\mathcal{D}s}$ are functions with the domain $J_{\mathcal{B}r} = \varnothing$, they are empty functions and thus trivially equal. For the sake of argument, let $A \in J_{\mathcal{B}r}$. Since $J_{\mathcal{B}r}$ is a bracketing, we have $A \neq \varnothing = [0]$. Therefore, from Definition 7.5.12, we get $d'(A) = d(A)$. □

Proposition 7.5.16. *Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing. Let $d \in \mathcal{D}s(D)$ be a dash-assignment. For $k \in \mathbb{N}^+$, let $d^{(k)} \in \mathcal{D}s\left(D^{(k)}\right)$ denote the dash-assignment that we get after taking k dashes. Then, we have*

$$d^{(k)}\ (A) = \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+k & \text{if } A = [n] \text{ and } A \in D \\ k & \text{if } A = [n] \text{ and } A \notin D. \end{cases} \qquad (7.5.17)$$

Proof. We will use induction on k.

Base case ($k = 1$): Observe

$$d^{(1)}(A) = d'(A) = \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+1 & \text{if } A = [n] \text{ and } A \in D \\ 1 & \text{if } A = [n] \text{ and } A \notin D. \end{cases}$$

Induction case ($k \geq 1$): Assume that

$$d^{(k)}(A) = \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+k & \text{if } A = [n] \text{ and } A \in D \\ k & \text{if } A = [n] \text{ and } A \notin D. \end{cases}$$

We get

$$\begin{aligned} d^{(k+1)}(A) &= \left(d^{(k)}\right)'(A) \\ &= \begin{cases} d^{(k)}(A) & \text{if } A \neq [n] \\ d^{(k)}(A)+1 & \text{if } A = [n] \text{ and } A \in D^{(k)} \\ 1 & \text{if } A = [n] \text{ and } A \notin D^{(k)}. \end{cases} && (7.5.13) \\ &= \begin{cases} d^{(k)}(A) & \text{if } A \neq [n] \\ d^{(k)}(A)+1 & \text{if } A = [n] \text{ and } A \in D \cup \{[n]\} \\ 1 & \text{if } A = [n] \text{ and } A \notin D \cup \{[n]\}. \end{cases} && (7.4.25) \\ &= \begin{cases} d^{(k)}(A) & \text{if } A \neq [n] \\ d^{(k)}(A)+1 & \text{if } A = [n] \end{cases} \\ &= \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+k+1 & \text{if } A = [n] \text{ and } A \in D \\ k+1 & \text{if } A = [n] \text{ and } A \notin D. \end{cases} && \text{Induction} \end{aligned}$$

This completes the proof. □

Example 7.5.18. Consider the dash-assignments

$$d = ((**)''\ *)'\ (**)'''\ * \in \mathcal{D}s(D) \quad \text{and} \quad e = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(11)} \in \mathcal{D}s(E)$$

as described in Example 7.5.4. Then, the dash-assignment $d' \in \mathcal{D}s\left(D'\right)$ is given by:

$$\{1,2\} \longmapsto 2,$$

$$\{1,2,3\} \longmapsto 1,$$

$$\{4,5\} \longmapsto 3,$$

$$\{1,2,3,4,5,6\} \longmapsto 1.$$

A visual representation of this dash-assignment is:

$$d' = (((**)'' *)' (**)''' *)'.$$

The dash-assignment $e^{(5)} \in \mathcal{D}s\left(E^{(5)}\right)$ is given by:

$$\{2\} \longmapsto 7,$$

$$\{2,3,4\} \longmapsto 18,$$

$$\{1,2,3,4,5\} \longmapsto 16.$$

A visual representation of this dash-assignment is:

$$e^{(5)} = (* ((*)^{(7)} \ * *)^{(18)} \ *)^{(16)}.$$

In general, applying dashes to a dash-assignment involves adding an outermost bracket if there isn't one already and increasing the number on the outermost bracket by the appropriate dash value. ◇

Definition 7.5.19. Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing, and $d \in \mathcal{D}s(D)$ be a dash-assignment. Define the *floor of the dash-assignment*, denoted $\lfloor d \rfloor \in \mathcal{D}s(\lfloor D)\rfloor$, as follows: Let $A \in \lfloor D \rfloor$ then we have $A \in D$. Define

$$\lfloor d \rfloor (A) := d(A). \tag{7.5.20}$$

◇

Example 7.5.21. Consider the dash-assignments

$$d = ((**)'' \; *)' \; (**)''' \; * \in \mathcal{D}s(D) \quad \text{and} \quad e = (* \; ((*)^{(7)} \; **)^{(18)} \; *)^{(11)} \in \mathcal{D}s(E)$$

as described in Example 7.5.4. Then, we have

$$\lfloor d \rfloor = d.$$

The dash-assignment $\lfloor e \rfloor \in \mathcal{D}s(\lfloor E \rfloor)$ is given by

$$\{2\} \longmapsto 7,$$

$$\{2,3,4\} \longmapsto 18.$$

A visual representation of this dash-assignment is:

$$\lfloor e \rfloor = * \; ((*)^{(7)} \; **)^{(18)} \; *.$$

The floor of a dash-assignment is obtained by removing the outermost bracket and any associated dashes, if present. ⋄

Lemma 7.5.22. *Consider the empty dash-assignment, $J_{\mathcal{D}s} \in \mathcal{D}s(J_{\mathcal{B}r})$. We have*

$$\lfloor J_{\mathcal{D}s} \rfloor = J_{\mathcal{D}s} \tag{7.5.23}$$

as dash-assignments on the unit bracketing.

Proof. From Lemma 7.4.32 we get $\lfloor J_{\mathcal{B}r} \rfloor = J_{\mathcal{B}r}$. Therefore, both $\lfloor J_{\mathcal{D}s} \rfloor$ and $J_{\mathcal{D}s}$ are functions with domain $J_{\mathcal{B}r} = \varnothing$. Consequently, both are empty functions and hence vacuously equal. □

Proposition 7.5.24. *Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing, and $d \in \mathcal{D}s(D)$ be a dash-assignment. Then, we have*

$$\lfloor \lfloor d \rfloor \rfloor = \lfloor d \rfloor \tag{7.5.25}$$

as dash-assignments on $\lfloor D \rfloor$.

Proof. From Proposition 7.4.34 we gave $\lfloor \lfloor D \rfloor \rfloor = \lfloor D \rfloor$. Therefore, it is enough to show that $\lfloor \lfloor d \rfloor \rfloor\ (A) = \lfloor d \rfloor\ (A)$ for $A \in \lfloor D \rfloor$.

Let $A \in \lfloor D \rfloor$. We get

$$\lfloor \lfloor d \rfloor \rfloor\ (A) = \lfloor d \rfloor\ (A) \qquad (7.5.20).$$

Thus, we get

$$\lfloor \lfloor d \rfloor \rfloor = \lfloor d \rfloor. \qquad \square$$

Proposition 7.5.26. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{Br}(n)$ be a bracketing, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Then, for $k \in \mathbb{N}$ the equality*

$$\left\lfloor d^{(k)} \right\rfloor = \lfloor d \rfloor \tag{7.5.27}$$

of dash-assignments on $\lfloor D \rfloor$ holds.

Proof. For the case $k = 0$, we get $d^{(0)} = d$. Therefore, the proposition is true for $k = 0$. Now assume $k \geq 1$. From Proposition 7.4.36 we get

$$\left\lfloor D^{(k)} \right\rfloor = \lfloor D \rfloor.$$

Therefore, it is enoguh to show that $\left\lfloor d^{(k)} \right\rfloor\ (A) = \lfloor d \rfloor\ (A)$ for $A \in \lfloor D \rfloor$. Let $A \in \lfloor D \rfloor$. Then, we have $A \neq [n]$. Observe that

$$\begin{aligned}
\left\lfloor d^{(k)} \right\rfloor\ (A) &= d^{(k)}\ (A) && (7.5.20) \\
&= d(A) && (7.5.17) \\
&= \lfloor d \rfloor\ (A). && (7.5.20)
\end{aligned} \qquad \square$$

Proposition 7.5.28. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Then, $\lfloor d \rfloor = d$ if and only is $[n] \notin D$.*

Proof. Suppose $\lfloor d \rfloor = d$. Since we have $\lfloor d \rfloor : \lfloor D \rfloor \longrightarrow \mathbb{N}^+$ and $d : D \longrightarrow \mathbb{N}^+$, we get

$$\lfloor D \rfloor = D.$$

From Proposition 7.4.38 we conclude $[n] \notin D$.

On the other hand, suppose $[n] \notin D$. Again from Proposition 7.4.38 we get $\lfloor D \rfloor = D$. Let $A \in \lfloor D \rfloor = D$. From Definition 7.5.19 we get $\lfloor d \rfloor\ (A) = d(A)$. Since $A \in \lfloor D \rfloor = D$ is arbitrarily chosen, we conclude $\lfloor d \rfloor = d$. □

Proposition 7.5.29. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{B}r(n)$, and $d \in \mathcal{D}s(D)$ be a dash-assignment. Assign*

$$k = \begin{cases} 0 & \text{if } [n] \notin D \\ d([n]) & \text{if } [n] \in D. \end{cases}$$

Then, the equality

$$(\lfloor d \rfloor)^{(k)} = d \tag{7.5.30}$$

of dash assignments on D holds.

Proof. We will consider the following two cases.

Case 1($[n] \notin D$): In this case we have $k = 0$. Therefore, we get

$$\begin{aligned} (\lfloor d \rfloor)^{(0)} &= \lfloor d \rfloor \\ &= d. && \text{Proposition 7.5.28} \end{aligned}$$

Case 2($[n] \in D$): In this case we have $k = d([n])$. Since $d : D \longrightarrow \mathbb{N}^+$ is a function with codomain $\mathbb{N}^+$ we get $k \geq 1$. From Proposition 7.4.39 we get

$$(\lfloor D \rfloor)^{(k)} = D.$$

Therefore, it is enough to show that $(\lfloor d \rfloor)^{(k)}\ (A) = d(A)$ for $A \in D$. Observe that

$$
\begin{aligned}
(\lfloor d \rfloor)^{(k)}\ (A) &= \begin{cases} \lfloor d \rfloor\ (A) & \text{if } A \neq [n] \\ \lfloor d \rfloor\ (A) + k & \text{if } A = [n] \text{ and } A \in \lfloor D \rfloor \\ k & \text{if } A = [n] \text{ and } A \notin \lfloor D \rfloor \end{cases} && (7.5.17) \\
&= \begin{cases} \lfloor d \rfloor\ (A) & \text{if } A \neq [n] \\ k & \text{if } A = [n] \end{cases} \\
&= \begin{cases} d(A) & \text{if } A \neq [n] \\ k & \text{if } A = [n] \end{cases} && (7.5.20) \\
&= \begin{cases} d(A) & \text{if } A \neq [n] \\ d([n]) & \text{if } A = [n] \end{cases} && (7.5.20) \\
&= d(A).
\end{aligned}
$$

Here, the second equality follows from the fact that $[n] \notin \lfloor D \rfloor$. □

Example 7.5.31. Let

$$d = ((**)''\ *)'\ (**)'''\ * \in \mathcal{D}s(D) \quad \text{and} \quad e = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(11)} \in \mathcal{D}s(E)$$

be dash-assignments as in Example 7.5.4. Since $[6] \notin D$ we get

$$\lfloor d \rfloor = d.$$

Since $[5] \in E$, we get $k = e([6]) = 11$. We have

$$(\lfloor e \rfloor)^{(11)} = e.$$ ◇

Definition 7.5.32. Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing, and $d \in \mathcal{D}s(D)$ be a dash-assignment. Let $(p,q] \subseteq_I [n]$ for $0 \leq p \leq q \leq n$. Define a dash-assignment, $d_{(p,q]}$ on $D_{(p,q]}$ as follows:

$$d_{(p,q]}\ (A) := d(p + A) \tag{7.5.33}$$

for $A \in D_{(p,q]}$. This assignment is well-defined since for $A \in D_{(p,q]}$, we get $p + P \in D$. Therefore, $d(p + A) \in \mathbb{N}^+$. ◇

Example 7.5.34. Let

$$d = ((**)'' \ *)' \ (**)''' \ * \in \mathcal{Ds}(D) \quad \text{and} \quad e = (* \ ((*)^{(7)} \ **)^{(18)} \ *)^{(11)} \in \mathcal{Ds}(E)$$

be dash-assignments as described in Example 7.5.4. Then, the dash-assignment $d_{[3]} \in \mathcal{Ds}\left(D_{[3]}\right)$ is given by:

$$\{1,2\} \longmapsto 2,$$
$$\{1,2,3\} \longmapsto 1.$$

A visual representation of this dash-assignment is:

$$d_{[3]} = ((**)'' \ *)'.$$

The dash-assignment $d_{(3,6]} \in \mathcal{Ds}\left(D_{(3,6]}\right)$ is given by:

$$\{1,2\} \longmapsto 3.$$

A visual representation of this dash-assignment is:

$$d_{(3,6]} = (**)''' \ * .$$

The dash-assignment $e_{(2,4]} \in \mathcal{Ds}\left(D_{(2,4]}\right)$ is given by the empty function since $D_{(2,4]} = \varnothing$. A visual representation of this dash-assignment is:

$$e_{(2,4]} = ** . \qquad \diamond$$

Proposition 7.5.35. *Let $n \in \mathbb{N}$, $D \in \mathcal{Br}(n)$ be a bracketing, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Then, we get*

$$d_{[n]} = d \tag{7.5.36}$$

as dash-assignments on D.

Proof. From Proposition 7.4.45 we get $D_{[n]} = D$. Therefore, it is enoguh to show that $d_{[n]}(A) = d(A)$ for $A \in D$. Consider the following calculation for $A \in D$:

$$
\begin{aligned}
d_{[n]}(A) = d_{(0,n]}(A) &= d(0 + A) && (7.5.33) \\
&= d(A).
\end{aligned}
$$

Thus, we conclude $d_{[n]} = d$. ☐

Proposition 7.5.37. *Let $n, m \in \mathbb{N}$, $D \in \mathcal{Br}(n)$, $E \in \mathcal{Br}(m)$ be bracketings, and $d \in \mathcal{Ds}(D)$, $e \in \mathcal{Ds}(E)$ be dash assignments. Let $(p,q] \subseteq_I [n]$ with $0 \le p \le q \le n$. Then, we get*

$$
(d \bullet e)_{(p,q]} = d_{(p,q]} \tag{7.5.38}
$$

as dash assignments on $D_{(p,q]}$.

Proof. From Proposition 7.4.47 we get $(D \bullet E)_{(p,q]} = D_{(p,q]}$. Therefore, it is enoguh to show that $(d \bullet e)_{(p,q]}(A) = d_{(p,q]}(A)$ for $A \in D_{(p,q]}$.

Let $A \in D_{(p,q]}$. We get some $X \in D$ with $X \subseteq (p,q]$ such that $-p + X = A$. It follows that

$$
p + A = X \subseteq (p,q] \subseteq [n].
$$

Consequently, we get

$$
\begin{aligned}
(d \cdot e)_{(p,q]}(A) &= d \cdot e\,(p + A) && (7.5.33) \\
&= d(p + A) && (7.5.6) \\
&= d_{(p,q]}(A). && (7.5.33)
\end{aligned}
$$

Since $A \in D_{(p,q]}$ is arbitrary, we get

$$
(d \cdot e)_{(p,q]} = d_{(p,q]}.
$$

☐

Proposition 7.5.39. *Let $n, m \in \mathbb{N}$, $D \in \mathcal{Br}(n)$, $E \in \mathcal{Br}(m)$ be bracketings, and $d \in \mathcal{Ds}(D)$, $e \in \mathcal{Ds}(E)$ be dash assignments. Let $(p,q] \subseteq_I (n, n+m]$ with $n \le p \le q \le n+m$. Then, we get*

$$\left(d \cdot e\right)_{(p,q]} = e_{(p-n,q-n]} \tag{7.5.40}$$

as dash assignments on $E_{(p-n,q-n]}$.

Proof. From Proposition 7.4.49 we get $(D \bullet E)_{(p,q]} = E_{(p-n,q-n]}$. Therefore, it is enoguh to show that $(d \bullet e)_{(p,q]}\,(A) = e_{(p-n,q-n]}(A)$ for $A \in D_{(p-n,q-n]}$.

Let $A \in E_{(p-n,q-n]}$. We get some $X \in E$ with $X \subseteq (p-n, q-n]$ such that $-(p-n)+X = A$. We have $p + A = n + X \subseteq (p,q] \subseteq (n, n+m]$. Observe that,

$$\begin{aligned}
\left(d \cdot e\right)_{(p,q]}(A) &= d \cdot e\ (p+A) && (7.5.33)\\
&= e(-n+p+A) && (7.5.6)\\
&= e_{(p-n,q-n]}\,(A). && (7.5.33)
\end{aligned}$$

Since $A \in E_{(p-n,q-n]}$ is arbitrary, we get

$$\left(d \cdot e\right)_{(p,q]} = e_{(p-n,q-n]}.$$

□

Lemma 7.5.41. *Let $n, m \in \mathbb{N}$, $D \in \mathcal{Br}(n)$, $E \in \mathcal{Br}(m)$ be bracketings. Let $d \in \mathcal{Ds}(D)$ and $e \in \mathcal{Ds}(E)$ be dash-assignments. Then, we get*

$$\left(d \cdot e\right)_{[n]} = d \tag{7.5.42}$$

as dash-assignments on D.

Proof. We see that

$$\begin{aligned}
\left(d \cdot e\right)_{[n]} &= e_{[n]} && (7.5.38)\\
&= d. && (7.5.36)
\end{aligned}$$

□

Lemma 7.5.43. *Let $n, m \in \mathbb{N}$, $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings. Let $d \in \mathcal{D}s(D)$ and $e \in \mathcal{D}s(E)$ be dash-assignments. Then, we get*

$$\left(d \cdot e\right)_{(n,n+m]} = e \tag{7.5.44}$$

as dash-assignments on E.

Proof. We see that

$$\begin{aligned}
\left(d \cdot e\right)_{(n,n+m]} &= e_{(0,m]} && (7.5.40) \\
&= e. && (7.5.36)
\end{aligned}$$

□

7.6 Dashed Words

In Sections 7.4 and 7.5, we introduced the concepts of bracketing and dash assignment for words of length n. These serve as the foundational structures for words in the free dashed monoid. In this section, we complete the construction by explicitly defining the free dashed monoid generated by a set S, denoted $\mathcal{DMon}\langle S\rangle$, utilizing bracketing and dash-assignment.

Building on the groundwork from previous sections, we will demonstrate that this construction indeed yields the free dashed monoid. Specifically, we will show that $\mathcal{DMon}\langle S\rangle$ possesses the structure of a free dashed monoid as described in Definition 7.2.5.

Framework 7.6.1. Throughout this section let S be a set. $\diamond$

Definition 7.6.2. A *dashed-word* over the set S is a dependent quadruple

$$(n, u, D, d)$$

where:

- n is a natural number representing the length of the dashed-word,
- $u : [n] \longrightarrow S$ (that is, $u \in \mathcal{Wr}(n)$) assigns to each position a letter from S,
- $D \in \mathcal{Br}(n)$ is a bracketing on n letters,
- $d : D \longrightarrow \mathbb{N}^+$ (that is, $d \in \mathcal{Ds}(D)$) is a dash-assignment on the bracketing D.

We denote the set of all dashed words over the set S by $\mathcal{DMon}\langle S\rangle$. $\diamond$

Remark 7.6.3. We emphasize that the construction in Construction 4.5.2 is also denoted by $\mathcal{DMon}\langle S\rangle$. In Section 7.8, we will demonstrate that both constructions satisfy the universal property of the free dashed monoid generated by S, justifying the use of the same notation. While Construction 4.5.2 offers a more straightforward approach, Definition 7.6.2 provides the detailed structure necessary for the results in Section 7.8. For the remainder of this chapter, $\mathcal{DMon}\langle S\rangle$ will refer to the construction given in Definition 7.6.2. ◇

Definition 7.6.4. We define the monoid of dashed-words over the set S as follows:

- The underlying set is $\mathcal{DMon}\langle S\rangle$ as in Definition 7.6.2.
- The unit dashed-word, denoted $J \in \mathcal{DMon}\langle S\rangle$, is given by

$$J := (0, J_{\mathcal{Wr}}, J_{\mathcal{Br}}, J_{\mathcal{Ds}}) \tag{7.6.5}$$

  where $J_{\mathcal{Wr}}$ is the unit word as described in Definition 6.2.4, $J_{\mathcal{Br}}$ is the unit bracketing as in Definition 7.4.4, and $J_{\mathcal{Ds}}$ is the unit dash-assignment as in Definition 7.5.2.

- For dashed words (n,u,D,d) and (m,v,E,e), the multiplication is given by

$$(n,u,D,d) \bullet (m,v,E,e) := (n+m, u \bullet v, D \bullet E, d \bullet e) \tag{7.6.6}$$

  where $u \bullet v \in \mathcal{Wr}(n+m)$ is as in Definition 6.2.5, $D \bullet E \in \mathcal{Br}(n+m)$ is as in Definition 7.4.8, and $d \bullet e \in \mathcal{Ds}(D \bullet E)$ is as in Definition 7.5.5.

The associativity for dashed-words follows from equations Lemmas 6.2.7, 7.4.11, and 7.5.8. The unit conditions for dashed-words follow from equations Lemmas 6.2.9, 7.4.14, and 7.5.10. ◇

Definition 7.6.7. We define the dashed-monoid of dashed-words over the set S as follows:

- The underlying monoid is $\mathcal{DMon}\langle S\rangle$ as in Definition 7.6.4.
- The dash map $(-)' : \mathcal{DMon}\langle S\rangle \longrightarrow \mathcal{DMon}\langle S\rangle$ is given by

$$(n,u,D,d)' := (n,u,D',d') \tag{7.6.8}$$

where $D' \in \mathcal{Br}(n)$ is as in Definition 7.4.17 and $d' \in \mathcal{Ds}\left(D'\right)$ is as in Definition 7.5.12.

The unit condition for dash follows from Lemmas 7.4.22 and 7.5.14. ⋄

Example 7.6.9. Let $S = \{a,b,c\}$. Consider the dashed-word

$$x = (6,u,D,e) \in \mathcal{DMon}\langle S\rangle$$

described as follows:

$$\begin{aligned} u = \{1 &\longmapsto a, \\ 2 &\longmapsto b, \\ 3 &\longmapsto c, \\ 4 &\longmapsto a, \\ 5 &\longmapsto b, \\ 6 &\longmapsto a\}. \end{aligned}$$

The bracketing $D \in \mathcal{Br}(6)$ and the dash-assignment $d \in \mathcal{Ds}(D)$ are same as in Example 7.5.4. A visual representation of the dashed-word is as follows:

$$x = ((ab)''\ c)'\ (ab)''\ a.$$

The dashed-word

$$y = (5, v, E, e) \in \mathcal{DMon}\langle S\rangle$$

is described as follows:

$$\begin{aligned} u = \{1 &\longmapsto b, \\ 2 &\longmapsto b, \\ 3 &\longmapsto c, \\ 4 &\longmapsto a, \\ 5 &\longmapsto a\}. \end{aligned}$$

The bracketing $E \in \mathcal{Br}(5)$ and the dash-assignment $e \in \mathcal{Ds}(E)$ are same as in Example 7.5.4. A visual representation of the dashed-word is as follows:

$$y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

The multiplication of these dashed-words is given by the concatenation:

$$x \bullet y = ((ab)''\ c)'\ (ab)''\ a(b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

The dash of the dashed-words is given by

$$\begin{aligned} x' &= (((ab)''\ c)'\ (ab)''\ a)' \\ y^{(5)} &= (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(16)}. \end{aligned}$$

◇

Definition 7.6.10. Define a *length* function $\hat{l} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N}$ as follows:

$$\hat{l}((n, u, D, d)) = n. \tag{7.6.11}$$

◇

Proposition 7.6.12. *Consider the set of natural numbers* $\mathbb{N}$ *with the dashed-monoid structure as in Example 7.1.5. Then, the length function* $\hat{l} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N}$ *is a dashed-monoid homomorphism. That is, for all* $x, y \in \mathcal{DMon}\langle S\rangle$ *we get*

$$\hat{l}(J) = 0, \tag{7.6.13}$$

$$\hat{l}(x \bullet y) = \hat{l}(x) + \hat{l}(y), \quad \text{and} \tag{7.6.14}$$

$$\hat{l}(x') = \hat{l}(x). \tag{7.6.15}$$

Proof. Let $J = (0, J_{\mathcal{Wr}}, J_{\mathcal{Br}}, J_{\mathcal{Ds}})$ be the unit dashed-word. We get

$$\hat{l}(J) = 0.$$

Now let $x = (n, u, D, d)$ and $y = (m, v, E, e)$ be two dashed-words. From the equation (7.6.6) we get $x \bullet y = (n+m, u \bullet v, D \bullet E, d \bullet e)$. Therefore, we get

$$\hat{l}(x \bullet y) = n + m = \hat{l}(x) + \hat{l}(y).$$

Finally, let $x = (n, u, D, d)$ be a dashed-word. From the equation (7.6.8) we get $x' = (n, u, D', d')$. Therefore, we get

$$\hat{l}(x') = n = \hat{l}(x) = (\hat{l}(x))'. \qquad \square$$

Proposition 7.6.16. *Let* $x \in \mathcal{DMon}\langle S\rangle$*. Then,* $\hat{l}(x) = 0$ *if and only if* $x = J$*.*

Proof. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle$. Suppose $\hat{l}(x) = 0$, that is, $n = 0$. It follows that, $u : [0] \longrightarrow S$ is the empty function $J_{\mathcal{Wr}}$. Also, we have $D \subseteq \mathcal{P}_J^+(0) = \varnothing$. Thus, we get $D = J_{\mathcal{Br}}$. It follows that, $d : J_{\mathcal{Br}} \longrightarrow \mathbb{N}$ is the empty function. Therefore, we get $d = J_{\mathcal{Ds}}$. Thus, we have

$$x = (0, J_{\mathcal{Wr}}, J_{\mathcal{Br}}, J_{\mathcal{Ds}}) = J.$$

On the other hand, if $x = J$ then from equation (7.6.13) get $\hat{l}(x) = 0$. $\square$

Notation 7.6.17. Let $\mathcal{DMon}\langle S\rangle^+$ denote the collection of all non-zero length dashed-words. That is,

$$\begin{aligned}\mathcal{DMon}\langle S\rangle^+ &:= \{x \in \mathcal{DMon}\langle S\rangle \mid x \neq J\} \\ &= \{x \in \mathcal{DMon}\langle S\rangle \mid \hat{l}(x) > 0\} \qquad \textit{Proposition } 7.6.16.\end{aligned}$$ ◇

Notation 7.6.18. For $x \in \mathcal{DMon}\langle S\rangle$ and $k \in \mathbb{N}$ let $x^{(k)}$ denote the dashed-word obtained by applying the dash operation k-times. In particular, we have

$$x^{(0)} = x \qquad \text{and} \qquad x^{(1)} = x'.$$ ◇

Proposition 7.6.19. *Let $k \in \mathbb{N}$ and $x \in \mathcal{DMon}\langle S\rangle$. Then we get*

$$\hat{l}\left(x^{(k)}\right) = \hat{l}(x). \tag{7.6.20}$$

Proof. We will consider the cases $k = 0$ and $k \geq 1$. If $k = 0$, then we have $x^{(0)} = x$. Thus, we get

$$\hat{l}\left(x^{(0)}\right) = \hat{l}(x).$$

Now, let $k \geq 1$. From Proposition 7.6.12, we have $\hat{l}(x') = \hat{l}(x)$. Therefore, applying this fact repeatedly, we get

$$\hat{l}\left(x^{(k)}\right) = \hat{l}(x).$$ □

Definition 7.6.21. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle$. Define *floor* of x, denoted $\lfloor x \rfloor$, as follows:

$$\lfloor x \rfloor := (n, u, \lfloor D \rfloor, \lfloor d \rfloor) \tag{7.6.22}$$

where $\lfloor D \rfloor \in \mathcal{Br}(n)$ is as in Definition 7.4.27 and $\lfloor d \rfloor \in \mathcal{Ds}(\lfloor D)\rfloor$ is as in Definition 7.5.19. ◇

Proposition 7.6.23. *We have*

$$\lfloor J \rfloor = J. \tag{7.6.24}$$

Proof. Observe that

$$\begin{aligned}
\lfloor J \rfloor &= \lfloor (0, J_{\mathcal{W}r}, J_{\mathcal{B}r}, J_{\mathcal{D}s}) \rfloor && (7.6.5)\\
&= (0, J_{\mathcal{W}r}, \lfloor J_{\mathcal{B}r} \rfloor, \lfloor J_{\mathcal{D}s} \rfloor) && (7.6.22)\\
&= (0, J_{\mathcal{W}r}, J_{\mathcal{B}r}, J_{\mathcal{D}s}) && (7.4.33) \text{ and } (7.5.23)\\
&= J. && (7.6.5)
\end{aligned}$$

□

Example 7.6.25. Let $S = \{a, b, c\}$. Consider the dashed-words

$$xx = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}$$

as in Example 7.6.9. The floor of the dashed-word x is given by

$$\lfloor x \rfloor = x = ((ab)''\ c)'\ (ab)''\ a$$

and the floor of the dashed-word y is given by

$$\lfloor y \rfloor = b\ ((b)^{(7)}\ ca)^{(18)}\ a.$$

◇

Proposition 7.6.26. *Let $x \in \mathcal{DMon}\langle S \rangle$ be a dashed-word. Then, we get*

$$\hat{l}(\lfloor x)\rfloor = \hat{l}(x). \tag{7.6.27}$$

Proof. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S \rangle$ where $n \in \mathbb{N}$, $u : [n] \longrightarrow S$, $D \in \mathcal{B}r(n)$, and $d \in \mathcal{D}s(D)$. Observe that

$$\begin{aligned}
\hat{l}(\lfloor x)\rfloor &= \hat{l}((n, u, \lfloor D), \lfloor d \rfloor)\rfloor && (7.6.22)\\
&= n && (7.6.11)\\
&= \hat{l}(x). && (7.6.11)
\end{aligned}$$

□

Proposition 7.6.28. *Let $x \in \mathcal{DMon}\langle S\rangle$ be a dashed-word. Then, we get*

$$\lfloor\lfloor x\rfloor\rfloor = \lfloor x\rfloor\,. \tag{7.6.29}$$

Proof. Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$ where $n \in \mathbb{N}$, $u : [n] \longrightarrow S$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$. Observe that

$$\begin{aligned}
\lfloor\lfloor x\rfloor\rfloor &= \lfloor(n,u,\lfloor D\rfloor,\lfloor d\rfloor)\rfloor && (7.6.22)\\
&= (n,u,\lfloor\lfloor D\rfloor\rfloor,\lfloor\lfloor d\rfloor\rfloor) && (7.6.22)\\
&= (n,u,\lfloor D\rfloor,\lfloor d\rfloor) && (7.4.35) \text{ and } (7.5.27)\\
&= \lfloor x\rfloor\,. && (7.6.22)
\end{aligned}$$

□

Proposition 7.6.30. *Let $x \in \mathcal{DMon}\langle S\rangle$ be a dashed-word and $k \in \mathbb{N}^+$. Then, we get*

$$\left\lfloor x^{(k)}\right\rfloor = \lfloor x\rfloor\,. \tag{7.6.31}$$

Proof. Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$ where $n \in \mathbb{N}$, $u : [n] \longrightarrow S$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$. Observe that

$$\begin{aligned}
\left\lfloor x^{(k)}\right\rfloor &= \left\lfloor(n,u,D^{(k)},d^{(k)})\right\rfloor && (7.6.8)\\
&= (n,u,\left\lfloor D^{(k)}\right\rfloor,\left\lfloor d^{(k)})\right\rfloor && (7.6.22)\\
&= (n,u,\lfloor D\rfloor,\lfloor d\rfloor) && (7.4.37) \text{ and } (7.5.27)\\
&= \lfloor x\rfloor\,. && (7.6.22)
\end{aligned}$$

□

Proposition 7.6.32. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$ be a dashed-word with $n \geq 1$. The, $\lfloor x\rfloor = x$ if and only if $[n] \notin D$.*

Proof. Observe that

$$\lfloor x \rfloor = (n, u, \lfloor D \rfloor, \lfloor d \rfloor).$$

Therefore, from Propositions 7.4.38 and 7.5.28 we get that $\lfloor x \rfloor = x$ if and only if $[n] \notin D$. $\square$

Proposition 7.6.33. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S \rangle$ be a dashed-word with $n \geq 1$. Assign*

$$k = \begin{cases} 0 & \text{if } [n] \notin D \\ d([n]) & \text{if } [n] \in D. \end{cases}$$

Then, the equality

$$(\lfloor x \rfloor)^{(k)} = x \tag{7.6.34}$$

of dashed-words hold.

Proof. Note that $D \in \mathcal{Br}(n)$ and $d \in \mathcal{Ds}(D)$ satisfy the conditions of Propositions 7.4.39 and 7.5.29 respectively. Therefore, we get

$$\begin{aligned} (\lfloor x \rfloor)^{(k)} &= \left(n, u, \lfloor D \rfloor, \lfloor d \rfloor\right)^{(k)} && (7.6.22) \\ &= \left(n, u, \lfloor D \rfloor^{(k)}, \lfloor d \rfloor^{(k)}\right) && (7.6.8) \\ &= (n,u,D,d) && (7.4.40) \text{ and } (7.5.30) \\ &= x. \end{aligned}$$

$\square$

Definition 7.6.35. Let (n,u,D,d) be a dashed word. Let $(p,q] \subseteq_I [n]$ with $0 \leq p \leq q \leq n$. Define a dashed word $(n,u,D,d)_{(p,q]}$ as follows:

$$(n,u,d,D)_{(p,q]} := (q-p, u_{(p,q]}, D_{(p,q]}, d_{(p,q]}) \tag{7.6.36}$$

where $u_{(q,p]} \in \mathcal{Wr}(q-p)$ is as in Definition 6.2.11. The bracketing, $D_{(p,q]} \in \mathcal{Br}(q-p)$, is as in Definition 7.4.42. The dash-assignment, $d_{(p,q]} \in \mathcal{Ds}\left(D_{(p,q]}\right)$ is as in Definition 7.5.32. $\diamond$

Example 7.6.37. Let $S = \{a, b, c\}$. Consider the dashed-words

$$x = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}$$

as in Example 7.6.9. Then, we have

$$\begin{aligned} x_{[3]} &= ((ab)''\ c)' \\ x_{(3,6]} &= (ab)''\ a \\ y_{(2,5]} &= caa. \end{aligned}$$

◇

Proposition 7.6.38. *Let $x \in \mathcal{DMon}\langle S\rangle$ and $n = \hat{l}(x)$. Let $L \subseteq_I [n]$ be a sub-interval. Then we get*

$$\hat{l}(x_L) = |L|. \tag{7.6.39}$$

Proof. Let $L = (p, q]$ with $0 \le p \le q \le n$. Then from Definition 7.6.35 we get

$$\hat{l}(x_L) = q - p = |L|.$$

□

Proposition 7.6.40. *Let $x = (n, u, D, d)$ be a dashed word. Then, we have*

$$x_{[n]} = x. \tag{7.6.41}$$

Proof. Observe that

$$\begin{aligned} x_{[n]} &= (n, u_{[n]}, D_{[n]}, d_{[n]}) && (7.6.36) \\ &= (n, u, D, d) && (6.2.14), (7.4.46), \text{and } (7.5.36) \\ &= x. \end{aligned}$$

□

Proposition 7.6.42. *Let* $x = (n,u,D,d)$ *and* $y = (m,v,E,e)$ *be dashed-words. Let* $(p,q] \subseteq_I [n]$ *with* $0 \le p \le q \le n$*. Then, we get*

$$(x \bullet y)_{(p,q]} = x_{(p,q]}. \tag{7.6.43}$$

Proof. Observe that

$$\begin{aligned}
&(x \bullet y)_{(p,q]} \\
&= (n+m, u \bullet v, D \bullet E, d \bullet e)_{(p,q]} && (7.6.6) \\
&= \big(q-p, (u \bullet v)_{(p,q]}, (D \bullet E)_{(p,q]}, (d \bullet e)_{(p,q]}\big) && (7.6.36) \\
&= \big(q-p, u_{(p,q]}, D_{(p,q]}, d_{(p,q]}\big) && (6.2.16), (7.4.48), \text{ and } (7.5.38) \\
&= x_{(p,q]}. && (7.6.36)
\end{aligned}$$

□

Proposition 7.6.44. *Let* $x = (n,u,D,d)$ *and* $y = (m,v,E,e)$ *be dashed-words. Let* $(p,q] \subseteq_I [n]$ *with* $n \le p \le q \le n+m$*. Then, we get*

$$(x \bullet y)_{(p,q]} = y_{(p-n,q-n]}. \tag{7.6.45}$$

Proof. Observe that

$$\begin{aligned}
&(x \bullet y)_{(p,q]} \\
&= (n+m, u \bullet v, D \bullet E, d \bullet e)_{(p,q]} && (7.6.6) \\
&= \big(q-p, (u \bullet v)_{(p,q]}, (D \bullet E)_{(p,q]}, (d \bullet e)_{(p,q]}\big) && (7.6.36) \\
&= \big(q-p, v_{(p-n,q-n]}, E_{(p-n,q-n]}, e_{(p-n,q-n]}\big) && (6.2.18), (7.4.50), \text{ and } (7.5.40) \\
&= y_{(p-n,q-n]}. && (7.6.36)
\end{aligned}$$

□

Proposition 7.6.46. *Let* $x = (n, u, D, d)$ *and* $y = (m, v, E, e)$ *be two dashed words. Then, we get*

$$(x \bullet y)_{[n]} = x. \tag{7.6.47}$$

Proof. Observe that,

$$\begin{aligned} (x \bullet y)_{[n]} &= x_{[n]} && (7.6.43) \\ &= x. && (7.6.41) \end{aligned}$$

□

Proposition 7.6.48. *Let* $x = (n, u, D, d)$ *and* $y = (m, v, E, e)$ *be two dashed words. Then, we get*

$$(x \bullet y)_{(n, n+m]} = y. \tag{7.6.49}$$

Proof. Observe that,

$$\begin{aligned} (x \bullet y)_{(n, n+m]} &= y_{[m]} && (7.6.45) \\ &= y. && (7.6.41) \end{aligned}$$

□

Lemma 7.6.50. *The dashed monoid of dashed-words over the set* S *is a left cancellative monoid. That is, for* $x, y, z \in \mathcal{DMon}\langle S \rangle$,

$$x \bullet y = x \bullet z \qquad \textit{implies} \qquad y = z. \tag{7.6.51}$$

Proof. Let $x, y, z \in \mathcal{DMon}\langle S \rangle$ with $\hat{l}(x) = n$, $\hat{l}(y) = m$, and $\hat{l}(z) = p$. Suppose we have

$$x \bullet y = x \bullet z.$$

We get

$$n + m = \hat{l}(x \bullet y) = \hat{l}(x \bullet z) = n + p.$$

Thus, we have $m = p$. Observe that

$$
\begin{aligned}
y &= (x \bullet y)_{(n,n+m]} && (7.6.49) \\
&= (x \bullet z)_{(n,n+p]} \\
&= z. && (7.6.49)
\end{aligned}
$$
□

Lemma 7.6.52. *The dashed monoid of dashed-words over the set S is a right cancellative monoid. That is, for $x, y, z \in \mathcal{DMon}\langle S \rangle$,*

$$y \bullet x = z \bullet x \qquad \textit{implies} \qquad y = z. \tag{7.6.53}$$

Proof. Let $x, y, z \in \mathcal{DMon}\langle S \rangle$ with $\hat{l}(x) = n$, $\hat{l}(y) = m$, and $\hat{l}(z) = p$. Suppose we have

$$y \bullet x = z \bullet x.$$

We get

$$m + n = \hat{l}(y \bullet x) = \hat{l}(z \bullet x) = p + n.$$

Thus, we have $m = p$. Observe that

$$
\begin{aligned}
y &= (y \bullet x)_{[m]} && (7.6.47) \\
&= (z \bullet x)_{[p]} \\
&= z. && (7.6.47)
\end{aligned}
$$
□

7.7 Dashed words as the Free Dashed Monoid

In this section, we will show that the dashed monoid $\mathcal{DMon}\langle S \rangle$ has a free dashed monoid like structure (Definition 7.2.5). To achieve this, we will use $\hat{l} : \mathcal{DMon}\langle S \rangle \longrightarrow \mathbb{N}$ as in Definition 7.6.10 as the length function. We need to provide multiplicative basis $\mathtt{G}$ and dash basis $\mathtt{H}$ of $\mathcal{DMon}\langle S\rangle$. Finally, we will show that these satisfy the interlocking conditions as in (7.2.8) and (7.2.9).

We will start by constructing a multiplicative basis $\mathtt{G}$ for $\mathcal{DMon}\langle S\rangle$.

Definition 7.7.1. Define a subset $\mathtt{G} \subseteq \mathcal{DMon}\langle S\rangle$ as follows:

$$\mathtt{G} := \left\{(n,u,D,d) \mid n = 1 \text{ or } \left(n \geq 2 \text{ and } [n] \in D\right)\right\}. \tag{7.7.2}$$

◇

Remark 7.7.3. Observe that $J \notin \mathtt{G}$ since $\hat{l}(J) = 0$. Therefore, we have

$$\mathtt{G} \subseteq \mathcal{DMon}\langle S\rangle^{+}.$$

◇

Example 7.7.4. Let $S = \{a,b,c\}$. Consider the dashed-words

$$x = (6,u,D,d) \qquad \text{and} \qquad y = (5,u,D,d)$$

as described in Example 7.6.9. These dashed-words are described as follows:

$$x = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

Then, $x \notin \mathtt{G}$ since $6 \geq 2$ and $[6] \notin D$. On the other hand, $y \in \mathtt{G}$ since $5 \geq 2$ and $[5] \in E$. Moreover, $a \in \mathtt{G}$ and $b^{(7)} \in \mathtt{G}$ since we have $\hat{l}(a) = \hat{l}\left(b^{(7)}\right) = 1$. ◇

Definition 7.7.5. Let $n \in \mathbb{N}^{+}$ and $D \in \mathcal{Br}(n)$ be a bracketing. Define the *leading left interval*, $L \subseteq [n]$, as

$$L := \{1\} \cup \left(\bigcup_{A \in D;\ 1 \in A} A\right). \tag{7.7.6}$$

The set L as defined above is an interval since it is a union of intervals containing 1. Suppose $L = [p]$ for some $1 \leq p \leq n$. Define the *trailing right interval*, $R \subseteq [n]$, as the complement of L:

$$R := (p,n]. \tag{7.7.7}$$

◇

Remark 7.7.8. Note that the collection

$$\mathfrak{L} := \{A \in D \mid 1 \in A\}$$

is a finite collection with a linear order with respect to set inclusion. The leading left interval L is the largest interval in the above collection, provided that the collection is non-empty; otherwise, L is equal to $[1]$. ◇

Proposition 7.7.9. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{Br}(n)$ be a bracketing, and $L = [p]$ for some $1 \leq p \leq n$ be the leading left interval as in Definition 7.7.5. Then, $L \in D$ if and only if there exists $X \in D$ such that $1 \in X$.*

Proof. Suppose $L \in D$. Then, by Definition 7.7.5 we have $1 \in L$. Thus, we can take $X = L$.

On the other hand, suppose there exists $X \in D$ such that $1 \in X$. Since L is the largest such interval in D, we have $L \in D$.

Proposition 7.7.10. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{Br}(n)$ be a bracketing, and L and R be the leading left interval and the trailing right interval as in Definition 7.7.5. Then, for $A \in D$ we get either $A \subseteq L$ or $A \subseteq R$.*

Proof. We will prove this by considering following cases.

Case 1 ($L \in D$): In this case, since D is a bracketing and $A, L \in D$ we get either $A \cap L = \varnothing$, $A \subseteq L$, or $L \subseteq A$. In this first subcase, we get

$$A \subseteq [n] \setminus L = R.$$

In the second subcase, we get $A \subseteq L$ as required. For the third subcase, we get $1 \subseteq A$ since $1 \in L$ and $L \subseteq A$. Since L is the largest interval in D such that $1 \in L$ we get $A \subseteq L$.

Case 2 ($L \notin D$): In this case, from Proposition 7.7.9, we get that for every $X \in D$, it holds that $1 \notin X$. Therefore, we have

$$L = \{1\}. \qquad \square$$

Taking $X = A$, we get that $1 \notin A$. Thus, we have

$$A \subseteq [n] \setminus \{1\} = [n] \setminus L = R.$$

Definition 7.7.11. Define functions

$$\texttt{Head} : \mathcal{DMon}\langle S\rangle^{+} \longrightarrow \mathcal{DMon}\langle S\rangle^{+}$$

and

$$\texttt{Tail} : \mathcal{DMon}\langle S\rangle^{+} \longrightarrow \mathcal{DMon}\langle S\rangle$$

as follows: Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle^{+}$ be a non-unit dashed-word. Then, we have $n \geq 1$. Let $L, R \subseteq_I [n]$ be the leading left intervals and the trailing right interval as in Definition 7.7.5. Since L is non-empty, we get $x_L \in \mathcal{DMon}\langle S\rangle^{+}$. Define

$$\texttt{Head}(x) := x_L, \qquad \text{and} \tag{7.7.12}$$

$$\texttt{Tail}(x) := x_R. \tag{7.7.13}$$

$\diamond$

Example 7.7.14. Let $S = \{a, b, c\}$. Consider the dashed-words

$$x = (6, u, D, d) \qquad \text{and} \qquad y = (5, u, D, d)$$

as described in Example 7.6.9. These dashed-words are described as follows:

$$x = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

Then, we have

$$\mathtt{Head}(x) = ((ab)''\ c)'$$

$$\mathtt{Tail}(x) = (ab)''\ a.$$

Similarly, we get

$$\mathtt{Head}(y) = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}$$

$$\mathtt{Tail}(y) = J.$$

Moreover, we have

$$\begin{aligned} \mathtt{Head}(abc) &= a && \text{and} && \mathtt{Tail}(abc) = bc, \\ \mathtt{Head}\left(b^{(7)}\right) &= b^{(7)} && \text{and} && \mathtt{Tail}\left(b^{(7)}\right) = J, \\ \mathtt{Head}\left(a'a\right) &= a' && \text{and} && \mathtt{Tail}\left(a'a\right) = a. \end{aligned} \qquad \diamond$$

Proposition 7.7.15. *Let $x \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, we get*

$$\hat{l}(\mathtt{Head}(x)) \leq \hat{l}(x) \tag{7.7.16}$$

and

$$\hat{l}(\mathtt{Tail}(x)) < \hat{l}(x). \tag{7.7.17}$$

Proof. Let $n = \hat{l}(x)$ and $L = [p], R = (p,n] \subseteq_I [n]$ be the leading left inverval and the trailing right interval where $1 \leq p \leq n$. We see that

$$\begin{aligned} \hat{l}(\mathtt{Head}(x)) &= \hat{l}\left(x_{[p]}\right) && (7.7.12) \\ &= p \\ &\leq n \\ &= \hat{l}(x) \end{aligned}$$

and

$$\begin{aligned}\hat{l}(\texttt{Tail}(x)) &= \hat{l}\left(x_{(p,n]}\right) && (7.7.13)\\ &= n-p\\ &< n\\ &= \hat{l}(x).\end{aligned}$$

□

Proposition 7.7.18. *Let $x \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, we get*

$$\texttt{Head}(\texttt{Head}(x)) = \texttt{Head}(x). \tag{7.7.19}$$

Proof. Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ and $L = [p]$ be the leading left interval of x, where $1 \le p \le n$. Thus, we get

$$y := \texttt{Head}(x) = x_{[p]} = (p, u_{[p]}, D_{[p]}, d_{[p]}).$$

Let $M = [q]$ be the leading left interval for y, where $1 \le q \le p$. Thus, we get

$$\texttt{Head}(\texttt{Head}(x)) = y_{[q]} = \left(x_{[p]}\right)_{[q]}.$$

We claim that $p = q$. If $p = 1$ then we get $q = 1$, and we are done. Assume $2 \le p$. Therefore, we get $L = [p] \neq \{1\}$. Recall that,

$$L = \{1\} \cup \left(\bigcup_{A \in D; 1 \in A} A\right). \qquad (7.7.6)$$

Since $L \neq \{1\}$, there exists $A \in D$ with $1 \in A$. From Proposition 7.7.9 we get $L = [p] \in D$. From Definition 7.4.42, we conclude $[p] \in D_{[p]}$.

We have

$$M = \{1\} \cup \left(\bigcup_{A \in D_{[p]}; 1 \in A} A\right). \qquad (7.7.6)$$

Since $[p] \in D_{[p]}$ and $1 \in [p]$, we get $[p] \subseteq [q] = M$ implying $p \leq q$. Therefore, we conclude $p = q$.

Finally, we get

$$\begin{aligned}
\mathtt{Head}(\mathtt{Head}(x)) &= \left(x_{[p]}\right)_{[q]} \\
&= \left(x_{[p]}\right)_{[p]} \\
&= x_{[p]} && (7.6.41) \\
&= \mathtt{Head}(x).
\end{aligned}$$

□

Proposition 7.7.20. *Let $x \in \mathcal{DMon}\langle S\rangle^{+}$ be non-unit dashed-word and $y \in \mathcal{DMon}\langle S\rangle$. Then,*

$$\mathtt{Head}(x \bullet y) = \mathtt{Head}(x). \tag{7.7.21}$$

Proof. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle^{+}$ and $y = (m, v, E, e) \in \mathcal{DMon}\langle S\rangle$. Let L be the leading left interval of $x \bullet y$. Consider the family, $\mathfrak{L}$, of sub-intervals of $[n+m]$ given by

$$\mathfrak{L} = \{A \in D \bullet E \mid 1 \in A\}$$

Then, we get

$$L = \{1\} \cup \left(\bigcup_{A \in \mathfrak{L}} A \right). \qquad (7.7.6)$$

Let M be the leading left interval of x. Consider the family, $\mathfrak{M}$, of sub-intervals of $[n]$ given by

$$\mathfrak{M} = \{A \in D \mid 1 \in A\}$$

Then, we get

$$M = \{1\} \cup \left(\bigcup_{A \in \mathfrak{M}} A \right). \qquad (7.7.6)$$

From Definition 7.7.11 we get

$$\mathtt{Head}(x \bullet y) = (x \bullet y)_L \qquad \text{and} \qquad \mathtt{Head}(x) = x_M.$$

We claim that, $\mathfrak{L} = \mathfrak{M}$. To see this, suppose $A \in \mathfrak{L}$. That means

$$A \in D \bullet E = D \sqcup (n+E) \qquad \text{and} \qquad 1 \in A.$$

If $A \in n+E$ then we have $A \subseteq (n, n+m]$. Since $x \in \mathcal{DMon}\langle S\rangle^+$, we get $n \geq 1$. This implies, $1 \notin A$ which is a contradiction. Thus, we get $A \in D$. We already have $1 \in A$, thus we get $A \in \mathfrak{M}$. This shows

$$\mathfrak{L} \subseteq \mathfrak{M}.$$

On the other hand, suppose $A \in \mathfrak{M}$. Then we get

$$A \in D \qquad \text{and} \qquad 1 \in A.$$

Therefore,

$$A \in D \subseteq D \sqcup (n+E) = D \bullet E.$$

That is, $A \in \mathfrak{L}$. This shows

$$\mathfrak{L} \supseteq \mathfrak{M}.$$

As a result, we get $\mathfrak{L} = \mathfrak{M}$ implying $L = M$.

Noting $M \subseteq [n]$, we conclude

$$\begin{aligned}
\mathtt{Head}(x \bullet y) &= (x \bullet y)_L \\
&= (x \bullet y)_M \\
&= x_M && (7.6.43) \\
&= \mathtt{Head}(x).
\end{aligned}$$

□

Proposition 7.7.22. *Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Let $L, R \subseteq [n]$ be the left leading interval and the right leading inverval as in Definition 7.7.5. Then, we get*

$$D = D_L \bullet D_R \tag{7.7.23}$$

as bracketings on n letters.

Proof. Let $L = [p]$ and $R = (p, n]$ for where $1 \leq p \leq n$. Thus, we wish to show that

$$D = D_{[p]} \bullet D_{(p,n]}.$$

Suppose $A \in D$. Then, from Proposition 7.7.10, we get $A \subseteq [p]$ or $A \subseteq (p, n]$. We will consider these two cases.

Case 1 ($A \subseteq [p]$): We get

$$\begin{aligned} A &\in D_{[p]} && (7.4.43) \\ &\subseteq D_{[p]} \sqcup p + D_{(p,n]} \\ &= D_{[p]} \bullet D_{(p,n]}. && (7.4.9) \end{aligned}$$

Case 2 ($A \subseteq (p, n]$): Since $A \in D$ and $A \subseteq (p, n]$, we get $-p + A \in D_{(p,n]}$. Thus, we get $A \in p + D_{(p,n]}$. As a consequence we get

$$A \in D_{[p]} \sqcup (p + D_{(p,n]}) = D_{[p]} \bullet D_{(p,n]}.$$

Since we have shown $A \in D_{[p]} \bullet D_{(p,n]}$ in both the cases, we conclude

$$D \subseteq D_{[p]} \bullet D_{(p,n]}.$$

On the other hand, suppose

$$A \in D_{[p]} \bullet D_{(p,n]} = D_{[p]} \sqcup p + D_{(p,n]}$$

. We will consider these two cases.

Case 1 ($A \in D_{[p]}$): We get that $A \in D$ and $A \subseteq [p]$. In particular, we get $A \in D$.

Case 2 ($A \in p + D_{(p,n]}$): We get $X \in D_{(p,n]}$ such that $A = p + X$. Since $X \in D_{(p,n]}$, we get $Y \in D$ with $Y \subseteq (p,n]$ such that $X = -p + Y$. Observe that

$$A = p + X = p + (-p + Y) = Y.$$

Since $Y \in D$, we conclude $A \in D$.

We have shown $A \in D$ in both the cases, therefore we get

$$D \supseteq D_{[p]} \bullet D_{(p,n]}.$$

Thus, we conclude

$$D = D_{[p]} \bullet D_{(p,n]} = D_L \bullet D_R. \qquad \square$$

Proposition 7.7.24. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Let $L, R \subseteq [n]$ be the leading left interval and the leading right interval as in Definition 7.7.5. Then, we get*

$$d = d_L \bullet d_R \tag{7.7.25}$$

as dash-assignments on D.

Proof. Let $L = [p]$ and $R = (p,n]$ where $1 \le p \le n$. We will show that

$$d = d_{[p]} \bullet d_{(p,n]}.$$

From Proposition 7.7.22, we have

$$D = D_{[p]} \bullet D_{(p,n]} = D_{[p]} \sqcup (p + D_{(p,n]}).$$

Therefore, it is enough to show

$$d(A) = d_{[p]} \bullet d_{(p,n]}\,(A)$$

for $A \in D_{[p]} \sqcup (p + D_{(p,n]})$.

We will consider the following cases.

<u>Case 1</u> ($A \subseteq D_{[p]}$): Observe that

$$\begin{aligned} d_{[p]} \bullet d_{(p,n]}\,(A) &= d_{(0,p]}\,(A) && (7.5.6) \\ &= d(A). && (7.5.33) \end{aligned}$$

<u>Case 2</u> ($A \subseteq p + D_{(p,n]}$): Observe that

$$\begin{aligned} d_{(0,p]} \bullet d_{(p,n]}\,(A) &= d_{(p,n]}\,(-p + A) && (7.5.6) \\ &= d(p + (-p + A)) && (7.5.33) \\ &= d(A). \end{aligned}$$

We have shown

$$d_{(0,p]} \bullet d_{(p,n]}\,(A) = d(A)$$

in both the cases, we conclude that

$$d_{(0,p]} \bullet d_{(p,n]} = d. \qquad \square$$

Lemma 7.7.26. *Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S \rangle^{+}$ be a non-unit dashed-word. Then,*

$$x = \mathtt{Head}(x) \bullet \mathtt{Tail}(x). \tag{7.7.27}$$

Proof. Let $L = [p], R = (p,n] \subseteq_I [n]$ be the leading left interval and the right left interval as in Definition 7.7.5 where $1 \le p \le n$. Then, we have

$$\mathtt{Head}(x) = x_L \qquad \text{and} \qquad \mathtt{Tail}(x) = x_R.$$

Observe that

$$
\begin{aligned}
x &= (n,u,D,d) \\
&= (p+(n-p),\ u_L \bullet u_R,\ D_L \bullet D_R,\ d_L \bullet d_R) && \text{(6.2.24), (7.7.23), and (7.7.25)} \\
&= (p,u_L,D_L,d_L) \bullet (n-p,u_R,D_R,d_R) && \text{(7.6.6)} \\
&= x_L \bullet x_R && \text{(7.6.36)} \\
&= \mathtt{Head}(x) \bullet \mathtt{Tail}(x). && \text{(7.7.12) and (7.7.13)} \quad \square
\end{aligned}
$$

Proposition 7.7.28. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, $x \in \mathtt{G}$ if and only if $\mathtt{Head}(x) = x$.*

Proof. Let $L \subseteq [n]$ be the leading left interval as in Definition 7.7.5, that is

$$L = \{1\} \cup \left(\bigcup_{A \in D; 1 \in A} A \right).$$

We get

$$\mathtt{Head}(x) = x_L. \qquad \text{(7.7.12)}$$

Suppose $x \in \mathtt{G}$, then we get $n = 1$ or $[n] \in D$. We will consider these two cases:

<u>Case 1</u> ($n = 1$): We get $L = \{1\} = [n]$. Thus, we have

$$
\begin{aligned}
\mathtt{Head}(x) &= x_L \\
&= x_{[n]} \\
&= x. && \text{(7.6.41)}
\end{aligned}
$$

Case 2 ($[n] \in D$): For every $A \in D$ we have $A \subseteq [n]$. Since $[n] \in D$ and $1 \in [n]$, we get $L = [n]$. Observe that

$$\begin{aligned} \texttt{Head}(x) &= x_L \\ &= x_{[n]} \\ &= x. \end{aligned} \tag{7.6.41}$$

On the other hand, suppose $\texttt{Head}(x) = x_L = x$. If $n = 1$ then we get $x \in \texttt{G}$, and we are done. Assume $n \geq 2$. We get

$$|L| = \hat{l}(x_L) = \hat{l}(x) = n.$$

Thus, L is a length n sub-interval of $[n]$. Therefor, we conclude $L = [n]$. Since $n \geq 1$, we get $[n] = L$ is the largest interval in D such that $1 \in L$. In particular, we get $[n] = L \in D$. This shows that $x \in \texttt{G}$. □

Lemma 7.7.29. *Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then we get* $\texttt{Head}(x) \in \texttt{G}$.

Proof. From Proposition 7.7.18 we get

$$\texttt{Head}(\texttt{Head}(x)) = \texttt{Head}(x).$$

From Lemma 7.7.29 we get $\texttt{Head}(x) \in \texttt{G}$. □

Theorem 7.7.30. *The subset $\texttt{G} \subseteq \mathcal{DMon}\langle S\rangle$ is a multiplicative basis of the dashed monoid $\mathcal{DMon}\langle S\rangle$.*

Proof. We will show that $\texttt{G}$ is a generating set with respect to the multiplication and an independent set with respect to the multiplication.

First, we will show $\mathtt{G}$ is a generating set with respect to the multiplication. Let $x \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. We will show, using induction on $\hat{l}(x)$, that there exists $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

Base case ($\hat{l}(x) = 1$): From Definition 7.7.1 we get $x \in \mathtt{G}$. We take $m = 1$ and $x_1 = x$.

Induction case ($\hat{l}(x) \geq 2$): From Lemma 7.7.26 we get

$$x = \mathtt{Head}(x) \bullet \mathtt{Tail}(x).$$

If $\mathtt{Tail}(x) = J$, then we get $\mathtt{Head}(x) = x$. From Proposition 7.7.28, we get $x \in \mathtt{G}$. Now assume $\mathtt{Tail}(x)$ is a non-unit dashed word. From Proposition 7.7.15 we get

$$\hat{l}(\mathtt{Tail}(x)) < \hat{l}(x).$$

Thus, from the induction hypothesis we get $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$\mathtt{Tail}(x) = x_1 \bullet \cdots \bullet x_m.$$

From Lemma 7.7.29 we have $\mathtt{Head}(x) \in \mathtt{G}$. Therefore, we get

$$x = \mathtt{Head}(x) \bullet \mathtt{Tail}(x) = \mathtt{Head}(x) \bullet x_1 \cdots x_m$$

where $\mathtt{Head}(x), x_i \in \mathtt{G}$ for $1 \leq i \leq m$.

From the mathematical induction, we conclude that $\mathtt{G}$ is a generating set with respect to the multiplication.

Next, we will show that $\mathtt{G}$ is an independent set with respect to the multiplication. Let $x \in \mathcal{DMon}\langle S\rangle^+$. Using induction on $\hat{l}(x)$, we will show that given $m, l \geq 1$ and $x_i, y_j \in \mathtt{G}$ for $1 \leq i \leq m$ and $1 \leq j \leq l$ such that

$$x = x_1 \bullet \cdots \bullet x_m = y_1 \bullet \cdots \bullet y_l,$$

we get $m = l$ and $x_i = y_i$ for $1 \le i \le m$.

Base case ($\hat{l}(x) = 1$): Suppose we have $m, l \ge 1$ and $x_i, y_j \in \mathtt{G}$ for $1 \le i \le m$ and $1 \le j \le l$ such that

$$x = x_1 \bullet \cdots \bullet x_m = y_1 \bullet \cdots \bullet y_l.$$

We get

$$1 = \hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i). \tag{7.6.14}$$

Since $x_i \in \mathtt{G}$, we have $\hat{l}(x_i) \ge 1$. It follows that $m = 1$ and $x_1 = x$. Similarly, we get $l = 1$ and $y_1 = x$. This completes the base case.

Induction case ($\hat{l}(x) \ge 2$): Suppose we have $m, l \ge 1$ and $x_i, y_j \in \mathtt{G}$ for $1 \le i \le m$ and $1 \le j \le l$ such that

$$x = x_1 \bullet \cdots \bullet x_m = y_1 \bullet \cdots \bullet y_l.$$

In particular, we have $x_1, y_1 \in \mathtt{G}$. Let $z = x_2 \cdots x_m$ and $w = y_2 \cdots y_l$. Then, we get

$$x = x_1 \bullet z = y_1 \bullet w.$$

Observe that

$$\begin{aligned}
x_1 &= \mathtt{Head}(x_1) && \text{Proposition 7.7.28} \\
&= \mathtt{Head}(x_1 \bullet z) && (7.7.21) \\
&= \mathtt{Head}(y_1 \bullet w) && \\
&= \mathtt{Head}(y_1) && (7.7.21) \\
&= y_1. && \text{Proposition 7.7.28}
\end{aligned}$$

Since $\mathcal{DMon}\langle S \rangle$ is a left cancellative monoid (Lemma 7.6.50), we get

$$x_2 \cdots x_m = y = z = y_2 \cdots y_l.$$

If $y = J$ then we get $m = 1$ and $x = x_1$. Since $z = y$, we get $z = J$. Consequently, we have $l = 1$ and $x = y_1$. This gives the required condition in $y = J$ case. Now assume $y \in \mathcal{DMon}\langle S\rangle^+$. Since $x_1 \in \mathtt{G}$ we get $\hat{l}(x_1) \geq 1$. It follows that

$$\hat{l}(y) < \hat{l}(x_1) + \hat{l}(y) = \hat{l}(x).$$

From the induction hypothesis we get $m = l$ and $x_i = y_i$ for $2 \leq i \leq m$. Above, along with the fact that $x_1 = y_1$, gives us $m = l$ and $x_i = y_i$ for $1 \leq i \leq m$.

From the mathematical induction, we conclude that $\mathtt{G}$ is an independent set with respect to the multiplication. Consequently, $\mathtt{G}$ is a multiplicative basis of $\mathcal{DMon}\langle S\rangle$.$\square$

Definition 7.7.31. Define a subset $\mathtt{S} \subseteq \mathcal{DMon}\langle S\rangle$ as follows:

$$\mathtt{S} := \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n = 1 \text{ and } D = \varnothing\}. \tag{7.7.32}$$

Define a function $i_S : S \longrightarrow \mathcal{DMon}\langle S\rangle$ as follows:

$$i_S(a) := (1,\ \overline{a},\ \varnothing,\ \emptyset) \tag{7.7.33}$$

where $\overline{a} : [1] \longrightarrow S$ is given by

$$\overline{a}(1) = a. \qquad \diamond$$

Proposition 7.7.34. *The set S is in bijection with the subset $\mathtt{S} \subseteq \mathcal{DMon}\langle S\rangle$ via the function i_S.*

Proof. Let $a, b \in S$ and suppose

$$i_S(a) = i_S(b).$$

Then, we get $\overline{a} = \overline{b}$. Thus,

$$a = \overline{a}(1) = \overline{b}(1) = b.$$

This shows that $i_S : S \longrightarrow \mathcal{DMon}\langle S\rangle$ is injective.

Now, let $x = (1, u, \varnothing, \emptyset) \in \mathtt{S}$. Let $a = u(1) \in S$. Observe that

$$i_S(a) = (1, \overline{a}, \varnothing, \emptyset)$$

where

$$\overline{a}(1) = a = u(1).$$

Thus, we get $\overline{a} = u$ therefore

$$i_S(a) = x.$$

This shows that $\mathtt{S} \subseteq i_S(S)$.

Finally, let $a \in S$. Then, we have

$$i_S(a) = (1, \overline{a}, \varnothing, \emptyset) \in \mathtt{S}.$$

Thus, we have $i_S(S) \subseteq \mathtt{S}$. Therefore, we conclude that i_S is a bijection between S and $\mathtt{S}$. □

Definition 7.7.35. Define a subset $\mathtt{R} \subseteq \mathcal{DMon}\langle S\rangle$ as follows:

$$\begin{aligned} \mathtt{R} &:= \langle\langle \mathtt{G};\ \bullet\rangle\rangle && (7.7.36) \\ &= \{x_1 \bullet \cdots \bullet x_m \mid m \geq 2,\ x_i \in \mathtt{G} \text{ for } 1 \leq i \leq m\}. && (6.1.14) \end{aligned}$$ ◇

Proposition 7.7.37. *Let $\mathtt{R} \subseteq \mathcal{DMon}\langle S\rangle$ be as in Definition 7.7.35. Then,*

$$\mathtt{R} = \{(n, u, D, d) \in \mathcal{DMon}\langle S\rangle \mid n \geq 2 \text{ and } [n] \notin D\}. \tag{7.7.38}$$

Proof. Let $x = (n, u, D, d) \in \mathtt{R}$. Then, we get $m \geq 2$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet x_2 \bullet \cdots \bullet x_m.$$

Since $x_i \in \mathtt{G}$, we get $\hat{l}(x_i) \geq 1$. Moreover, since $m \geq 2$ we get

$$n = \hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i) \geq 2.$$

For the sake of contradiction, assume $[n] \in D$. Then, from Definition 7.7.1 we get $x \in \mathtt{G}$. Since $\mathtt{G}$ is a multiplicative basis, it is a multiplicatively independent set. Lemma 7.1.26 asserts that

$$\mathtt{G} \cap \mathtt{R} = \varnothing.$$

This leads to a contradiction since we have $x \in \mathtt{R}$ and $x \in \mathtt{G}$.

On the other hand, suppose $x = (n, u, D, d) \in \mathcal{DMon}\langle S \rangle$ such that $n \geq 2$ and $[n] \notin D$. Since $n \geq 2$ we get $x \neq J$. Since $\mathtt{G}$ is a multiplicative basis, we get $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

To show that $x \in \mathtt{R}$, we need to show that $m \geq 2$. If $m = 1$ then we have $x = x_1 \in \mathtt{G}$. Since $n \geq 2$ and $[n] \notin D$, we have $x \notin \mathtt{G}$, proving that $m \neq 1$. Therefore, we must have $m \geq 2$. It follows that, $x \in \mathtt{R}$. □

Proposition 7.7.39. *Let* $\mathtt{S} \subseteq \mathcal{DMon}\langle S \rangle$ *be as in Definition 7.7.31 and* $\mathtt{R} \subseteq \mathcal{DMon}\langle S \rangle$ *be as in Definition 7.7.35. Then, we have*

$$\mathtt{S} \cap \mathtt{R} = \varnothing. \tag{7.7.40}$$

Proof. Suppose $x = (n, u, D, d) \in \mathtt{S} \cap \mathtt{R}$. Since $x \in \mathtt{S}$, we get $n = 1$. Since $x \in \mathtt{R}$, we get $n \geq 2$. This is a contradiction. Therefore, we have

$$\mathtt{S} \cap \mathtt{R} = \varnothing.$$ □

Definition 7.7.41. Define a subset $\mathtt{H} \subseteq \mathcal{DMon}\langle S\rangle$ as follows:

$$\mathtt{H} := \mathtt{S} \sqcup \mathtt{R}. \tag{7.7.42}$$

◇

Proposition 7.7.43. *We have*

$$\mathtt{H} = \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n \geq 1 \ \text{ and } \ [n] \notin D\}. \tag{7.7.44}$$

Proof. Observe that

$$\begin{aligned}
\mathtt{H} &= \mathtt{S} \sqcup \mathtt{R} && (7.7.42)\\
&= \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n = 1 \ \text{ and } \ [n] \notin D\}\\
&\qquad \sqcup \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n \geq 2 \ \text{ and } [n] \notin D\} && (7.7.32) \text{ and } (7.7.42)\\
&= \{(n,u,D,d) \mid n \geq 1 \ \text{ and } \ [n] \notin D\}.
\end{aligned}$$

□

Lemma 7.7.45. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, $x \in \mathtt{H}$ if and only if $\lfloor x \rfloor = x$.*

Proof. Since x is a non-unit dashed-word, we get $n \geq 1$. From Proposition 7.6.32 we get $\lfloor x \rfloor = x$ if and only if $[n] \notin D$. From Proposition 7.7.43, this is equivalent to $x \in \mathtt{H}$. □

Lemma 7.7.46. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, $\lfloor x \rfloor \in \mathtt{H}$.*

Proof. Since x is a non-unit dashed-word, we get $\hat{l}(x) = n \geq 1$. We have

$$\begin{aligned}
\hat{l}(\lfloor x)\rfloor &= \hat{l}(x) && (7.6.27)\\
&= n \geq 1.
\end{aligned}$$

From Proposition 7.6.28 we get

$$\lfloor \lfloor x \rfloor \rfloor = \lfloor x \rfloor .$$

Using Lemma 7.7.45 we get $\lfloor x \rfloor \in \mathtt{H}$. □

Lemma 7.7.47. *Let $x = (n,u,D,d) \in \mathtt{H}$. Suppose*

$$x^{(k)} = x^{(l)}$$

for some $k,l \in \mathbb{N}$. Then, we get $k = l$.

Proof. From Proposition 7.7.43 we get $n \geq 1$ and $[n] \notin D$. Without the loss of generality, we will consider the following cases:

Case 1 ($k = 0$): Suppose $l \geq 1$. Then, we get $x = x^{(l)}$. In particular, we get

$$D = D^{(l)}.$$

From Proposition 7.4.19 we get

$$D(l) = D \cup \{[n]\}.$$

Thus, we get $[n] \in D$. This is a contradiction. Therefore, we conclude $l = 0$.

Case 2 ($k,l \geq 1$): Since $[n] \notin D$, from Proposition 7.5.16 we get

$$d^{(k)}\left([n]\right) = k \qquad \text{and} \qquad d^{(l)}\left([n]\right) = l.$$

From the equality $x^{(k)} = x^{(l)}$ we get

$$d^{(k)} = d^{(l)}.$$

It follows that $k = l$ as required. □

Theorem 7.7.48. *The subset $\mathtt{H} \subseteq \mathcal{DMon}\langle S \rangle$ is a dash basis of $\mathcal{DMon}\langle S \rangle$.*

Proof. We will show that $\mathtt{H}$ is a generating set and an independent set with respect to the dash operation.

First, we will show that $\mathtt{H}$ is a generating set with respect to the dash operation. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. From Lemma 7.7.46, we get that $\lfloor x \rfloor \in \mathtt{H}$. Assign

$$k = \begin{cases} 0 & \text{if } [n] \notin D \\ d([n]) & \text{if } [n] \in D. \end{cases}$$

Then, from Proposition 7.6.33 we get

$$(\lfloor x \rfloor)^{(k)} = x.$$

This shows that $\mathtt{H}$ is a generating set with respect to the dash operation.

Next, we will show that $\mathtt{H}$ is independent with respect to the dash operation. Let $x, y \in \mathtt{H}$. Suppose we have

$$x^{(k)} = y^{(l)}$$

for some $k, l \in \mathbb{N}$. Since $x, y \in \mathtt{H}$, from Lemma 7.7.45 we get

$$x = \lfloor x \rfloor \qquad \text{and} \qquad y = \lfloor y \rfloor .$$

Observe that

$$\begin{aligned} x &= \lfloor x \rfloor \\ &= \left\lfloor x^{(k)} \right\rfloor && (7.6.31) \\ &= \left\lfloor y^{(k)} \right\rfloor && \text{assumption} \\ &= \lfloor y \rfloor && (7.6.31) \\ &= y. \end{aligned}$$

From Lemma 7.7.47 we get $k = l$. This shows that $\mathtt{H}$ is independent with respect to the dash operation. □

Proposition 7.7.49. *The equality*

$$\langle\langle \mathtt{H};\ (-)' \rangle\rangle = \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n \geq 1 \text{ and } [n] \in D\} \tag{7.7.50}$$

holds.

Proof. Let $x = (n,u,D,d) \in \langle\langle \mathtt{H};\ (-)' \rangle\rangle$. Then, we get $y = (m,v,E,e) \in \mathtt{H}$ and $k \geq 1$ such that

$$x = y^{(k)}.$$

That is,

$$\begin{aligned} (n,u,D,d) &= (m,v,E,e)^{(k)} \\ &= (m,v,E^{(k)},e^{(k)}) && (7.6.8). \end{aligned}$$

Therefore, we have

$$n = m, \qquad u = v, \qquad D = E^{(k)}, \qquad \text{and} \qquad d = e^{(k)}.$$

Since $y \in \mathtt{H}$, we get $m \geq 1$ and $[m] \notin E$. This implies $n \geq 1$. Since $m \geq 1$ and $k \geq 1$, using Proposition 7.4.24, we get

$$\begin{aligned} D &= E^{(k)} \\ &= E \cup \{[m]\} && (7.4.25) \\ &= E \cup \{[n]\}. \end{aligned}$$

Therefore, we get $[n] \in D$.

On the other hand, suppose $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$ such that $n \geq 1$ and $[n] \in D$. Let $k = d([n])$. Note that $k \geq 1$ since $d : D \longrightarrow \mathbb{N}^+$. From Lemma 7.7.46 we get $\lfloor x \rfloor \in \mathtt{H}$. From Proposition 7.6.33 we get

$$(\lfloor x \rfloor)^{(k)} = x.$$

This shows that $x \in \langle\langle \mathtt{H};\ (-)' \rangle\rangle$. $\square$

Proposition 7.7.51. *We have*

$$\mathtt{S} \cap \langle\langle \mathtt{H};\ (-)' \rangle\rangle = \varnothing. \tag{7.7.52}$$

Proof. Suppose $x = (n, u, D, d) \in \mathtt{S} \cap \langle\langle DBasis;\ (-)' \rangle\rangle$. From Definition 7.7.31 we get $n = 1$ and $D = \varnothing$. From Proposition 7.7.49 we get $n \geq 1$ and $[n] \in D$. This leads to a contradiction. Therefore,

$$\mathtt{S} \cap \langle\langle \mathtt{H};\ (-)' \rangle\rangle = \varnothing.$$

$\square$

Proposition 7.7.53. *The equality*

$$\mathtt{S} \sqcup \langle\langle \mathtt{H};\ (-)' \rangle\rangle = \mathtt{G} \tag{7.7.54}$$

holds.

Proof. We see that

$$\begin{aligned}
&\mathtt{S} \sqcup \langle\langle \mathtt{H};\ (-)' \rangle\rangle \\
&= \{(n, u, D, d) \mid (n = 1 \ \text{ and } D = \varnothing) \ \text{ or} \\
&\qquad\qquad (n \geq 1 \ \text{ and } \ [n] \in D)\} && \text{(7.7.32) and (7.7.50)} \\
&= \{(n, u, D, d) \mid n = 1 \ \text{ or } \ (n \geq 2 \ \text{ and } \ [n] \in D)\} \\
&= \mathtt{G}. && \text{(7.7.2)}
\end{aligned}$$

Here, the third equality follows since for $n = 1$ we have either $D = \{\}$ or $D = \{[1]\}$. $\square$

Theorem 7.7.55. *The subset $\mathtt{S} \subseteq \mathcal{DMon}\,\langle S \rangle$ is a dashed monoid basis.*

Proof. Let $\hat{l} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N}$ as in Definition 7.6.10 be the length function. Proposition 7.6.16 gives the required property for the length function.

Let $\mathtt{G}$ be the multiplicative basis as shown in Theorem 7.7.30. Let $\mathtt{H}$ be the dash basis as in Definition 7.7.41. Proposition 7.7.53 and Definition 7.7.41 give the required interlocking conditions. □

Theorem 7.7.56. *The inclusion of sets* $\mathtt{S} \subseteq \mathcal{DMon}\langle S\rangle$ *satisfies the universal property of dashed monoid.*

Proof. From Theorem 7.7.55 we know that $\mathtt{S} \subseteq \mathcal{DMon}\langle S\rangle$ is a dashed monoid basis. The theorem follows immediately from Theorem 7.2.43 . □

7.8 Key Results

We will conclude this chapter with the following key results.

Theorem 7.8.1. *Let M be a dashed monoid, and let $S \subseteq M$ with the inclusion given by*

$$\phi : S \hookrightarrow M.$$

Then, the pair

$$\left(M, \phi : S \hookrightarrow M\right)$$

satisfies the universal property of the free dashed monoid generated by S if and only if S is a dashed monoid basis of M.

Proof. We have already shown one direction at Theorem 7.2.43.

For the other direction, suppose the pair

$$\left(M, \phi : S \hookrightarrow M\right)$$

satisfies the universal property of the free dashed monoid generated by S. Consider the construction of dashed monoid of dashed words $\mathcal{DMon}\langle S\rangle$ over the set S as in Definition 7.6.7. From Theorem 7.7.55, we know that $S \subseteq \mathcal{DMon}\langle S\rangle$ is a dashed monoid basis of $\mathcal{DMon}\langle S\rangle$. From Theorem 7.7.56, we know that the pair

$$\left(\mathcal{Mon}\langle S\rangle, i : S \longrightarrow \mathcal{Mon}\langle S\rangle\right)$$

satisfies the universal property of the free dashed monoid generated by S. It follows that, $\mathcal{DMon}\langle S\rangle$ is isomorphic to M via a dashed monoid homomorphism which maps $\mathtt{S} \subseteq \mathcal{DMon}\langle S\rangle$ to $S \subseteq M$. Since $\mathtt{S} \subseteq \mathcal{DMon}\langle X\rangle$ is a dashed monoid basis of $\mathcal{DMon}\langle S\rangle$, from Proposition 7.2.10 we conclude that $S \subseteq M$ is a dashed monoid basis of M. $\square$

Theorem 7.8.2. *Let S be a set. The dashed monoid $\mathcal{DMon}\langle S\rangle$ as in Construction 4.5.2 satisfies the universal property of the free dashed monoid generated by S.*

Proof. Let $(M, I, (-)', \cdot)$ be a dashed monoid and let $u : S \longrightarrow M$ be a function. We will construct the induced dashed homomorphism as follows:

Definition 7.8.3. Define

$$F : \mathcal{DMon}\langle S\rangle \longrightarrow M$$

as follows:

$$F(J) = I \tag{7.8.4}$$

$$F(a) = u(a) \qquad \text{for } a \in S \tag{7.8.5}$$

$$F(x \bullet y) = F(x) \cdot F(y) \qquad \text{for } x, y \in \mathcal{DMon}\langle S\rangle, \ \text{ and} \tag{7.8.6}$$

$$F(x') = F(x)' \qquad \text{for } x \in \mathcal{DMon}\langle S\rangle. \tag{7.8.7}$$

This function is well-defined since M is a dashed monoid. From the definition, F is a dashed monoid homomorphism and and satisfies the inclusion condition (7.2.2). $\diamond$

Now, suppose $(M, I, (-)', \cdot)$ be a dashed monoid and let

$$F, G : \mathcal{DMon}\langle S\rangle \longrightarrow M$$

be dashed monoid homomorphisms such that

$$F(a) = G(a)$$

for $a \in S$. Since F, G are both dashed monoid homomorphisms and agree on $S \subseteq \mathcal{DMon}\langle S\rangle$, from Construction 4.5.2 we conclude that

$$F = G. \qquad \square$$

Theorem 7.8.8. *Let M be a dashed monoid, $S \subseteq M$ be a subset and $\phi : S \hookrightarrow M$ be the inclusion function. Suppose the pair*

$$(M, \phi : S \hookrightarrow M)$$

satisfies the universal property of the free monoid generated by S. Then, M is a free monoid with monoid basis $\mathtt{G} \subseteq M$.

Furthermore, the monoid basis $\mathtt{G}$ is characterized as follows: Let

$$\hat{l} : M \longrightarrow \mathbb{N}$$

be the induced dashed monoin homomorphism defined by setting

$$\hat{l}(a) = 1 \qquad \text{for every } a \in S.$$

Then, $x \in \mathtt{G}$ if and only if exactly one of the following holds:

- *We have*

$$x = a^{(k)}$$

for some unique $k \geq 0$ *and* $a \in \mathtt{S}$*. In this case, we also get*

$$\hat{l}\left(a^{(k)}\right) = 1. \tag{7.8.9}$$

- *We have*

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}$$

for some unique $m \geq 2$*,* $k \geq 1$ *and* $x_i \in \mathtt{G}$*. In this case, we also get*

$$\hat{l}(x) \geq 2 \tag{7.8.10}$$

and

$$\hat{l}(x_i) < \hat{l}(x). \tag{7.8.11}$$

Proof. From Theorem 7.8.1 we get that $S \subseteq M$ is a dashed monoid basis of M. The theorem follows from Lemma 7.2.18. □

Chapter 8: Construction of the Free Semi-Strict Categorical Group

In this chapter, we construct the free semi-strict categorical group on a set S, denoted $\mathcal{SSCatGrp}\langle S\rangle$. Our approach is inspired by Mac Lane's construction of the free monoidal category, as described in the Coherence section of the Monoids chapter (see [ML78, Section VII.2, p. 165]), and its generalization to arbitrary sets (see [ML78, Exercise 3, p. 170]).

The objects of $\mathcal{SSCatGrp}\langle S\rangle$ are given by the free dashed monoid $\mathcal{DMon}\langle S\rangle$ generated by S. Let

$$\boldsymbol{Q} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathcal{Grp}\langle S\rangle$$

be the dashed monoid homomorphism induced by viewing the free group $\mathcal{Grp}\langle S\rangle$ as a dashed monoid. For $x, y \in \mathcal{DMon}\langle S\rangle$, there is a unique morphism $\star_{x,y} : x \longrightarrow y$ in $\mathcal{SSCatGrp}\langle S\rangle$ if and only if $\boldsymbol{Q}(x) = \boldsymbol{Q}(y)$. The detailed construction is given in Definition 8.1.2. In Theorem 9.3.1, we will show that $\mathcal{SSCatGrp}\langle S\rangle$ satisfies the universal property of the free semi-strict categorical group.

8.1 The Free Semi-Strict Categorical Group

Framework 8.1.1. Throughout this chapter, let S be a set. The following constructions and results will be used:

- Let $\mathcal{DMon}\langle S\rangle$ be the free dashed-monoid generated by S, as in Construction 4.5.2. The universal property is established in Theorem 7.8.2.

- Let $\mathcal{Grp}\langle S\rangle$ be the free group generated by S, as in Construction 4.5.4.

- Let $Q_{\mathcal{DMon}} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathcal{Grp}\langle S\rangle$ be the dashed-monoid homomorphism, as in Construction 4.5.5; we will abbreviate this as Q.

- By Theorem 7.8.1, $S \subseteq \mathcal{DMon}\langle S\rangle$ forms a dashed monoid basis.

- Theorem 7.8.8 asserts that $\mathcal{DMon}\langle S\rangle$ is also a free monoid and characterizes its monoid basis $\mathtt{G}$. $\diamond$

Definition 8.1.2. Define a category, denoted $\mathcal{SSCatGrp}\langle S\rangle$, as follows:

- The set of objects is given by

$$\mathrm{Obj}(\mathcal{SSCatGrp}\langle S\rangle) := \mathcal{DMon}\langle S\rangle\,. \tag{8.1.3}$$

- For $x, y \in \mathcal{DMon}\langle S\rangle$, the set of morphisms $\mathrm{Hom}_{\mathcal{SSCatGrp}\langle S\rangle}(x, y)$ is defined as

$$\mathrm{Hom}_{\mathcal{SSCatGrp}\langle S\rangle}(x, y) := \begin{cases} \{\star_{x,y}\} & \text{if } Q(x) = Q(y) \\ \varnothing & \text{if } Q(x) \neq Q(y)\,. \end{cases} \tag{8.1.4}$$

  This means that there exists a unique morphism from x to y if and only if $Q(x) = Q(y)$. We will refer to morphisms in $\mathcal{SSCatGrp}\langle S\rangle$ as arrows. Note that for $x, y \in \mathcal{DMon}\langle S\rangle$, the set of arrowss

$$\mathrm{Hom}_{\mathcal{SSCatGrp}\langle S\rangle}(x, y)$$

  is either empty if $Q(x) \neq Q(y)$ or a singleton, making $\mathcal{SSCatGrp}\langle S\rangle$ a thin category. It follows that, any arrow in $f : x \longrightarrow y$ in $\mathcal{SSCatGrp}\langle S\rangle$ is uniquely determined by the source and target objects. We will use the notation $f : x \multimap\!\!\!\rightarrow y$ to denote the unique arrow $\star_{x,y}$ from x to y.

- For $x \in \mathcal{DMon}\langle S\rangle$ we have the tautology $Q(x) = Q(x)$. Define the identity morphism as

$$1_x := \star_{x,x}. \tag{8.1.5}$$

- Let $x, y, z \in \mathcal{DMon}\langle S\rangle$, $\star_{x,y} : x \multimap y$, and $\star_{y,z} : y \multimap z$ be composible arrows. The law of composition is given by

$$\star_{y,z} \circ \star_{x,y} := \star_{x,z}. \tag{8.1.6}$$

  This assigenement is well-defined since we have $Q(x) = Q(y)$ and $Q(y) = Q(z)$, therefore we get $Q(x) = Q(z)$.

We will show that the above assignment of composition is associative and satisfies the unit law. Let $x, y, z, w \in \mathcal{DMon}\langle S\rangle$ and $\star_{x,y} : x \multimap y$, $\star_{y,z} : y \multimap z$, and $\star_{z,w} : z \multimap w$ be composible arrows. Then from the definition of composition it follows that,

$$\begin{aligned}
(\star_{z,w} \circ \star_{y,z}) \circ \star_{x,y} &= \star_{y,w} \circ \star_{x,y} \\
&= \star_{x,w} \\
&= \star_{z,w} \circ \star_{x,z} \\
&= \star_{z,w} \circ (\star_{y,z} \circ \star_{x,y}).
\end{aligned}$$

Let $x, y \in \mathcal{DMon}\langle S\rangle$ and $\star_{x,y} : x \multimap y$ be a morphism. Then we get

$$1_y \circ \star_{x,y} = \star_{y,y} \circ \star_{x,y} = \star_{x,y},$$

and

$$\star_{x,y} \circ 1_x = \star_{x,y} \circ \star_{x,x} = \star_{x,y}.$$

Thus, the composition law is associative and satisfies the unit law. Therefore, $\mathcal{SSCatGrp}\langle S\rangle$ is a category. $\diamond$

Definition 8.1.7. Define a functor

$$-\bullet- : \mathcal{SSCatGrp}\langle S\rangle \times \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{SSCatGrp}\langle S\rangle$$

as follows:

- For objects $x, y \in \mathcal{DMon}\langle S\rangle$, we define

$$(x,y) \longmapsto x \bullet y. \tag{8.1.8}$$

- For arrows $\star_{x_1,y_1} : x_1 \multimap y_1$ and $\star_{x_2,y_2} : x_2 \multimap y_2$ in $\mathcal{SSCatGrp}\langle S\rangle$, we define

$$\star_{x_1,y_1} \bullet \star_{x_2,y_2} : x_1 \bullet x_2 \multimap y_1 \bullet y_2$$

  as

$$\star_{x_1,y_1} \bullet \star_{x_2,y_2} := \star_{x_1\bullet x_2,\ y_1\bullet y_2}. \tag{8.1.9}$$

  This assignment is well-defined since $\star_{x_1,y_1} : x_1 \multimap y_1$ and $\star_{x_2,y_2} : x_2 \multimap y_2$ being arrows in $\mathcal{SSCatGrp}\langle S\rangle$ implies that

$$Q(x_1) = Q(y_1) \qquad \text{and} \qquad Q(x_2) = Q(y_2).$$

  Consequently, we have

$$\begin{aligned} Q(x_1 \bullet x_2) &= Q(x_1)\,Q(x_2) && (4.5.12) \\ &= Q(y_1)\,Q(y_2) \\ &= Q(y_1 \bullet y_2). && (4.5.12) \end{aligned}$$

We will now verify that the above assignment satisfies the composition and identity laws of a functor. Let $\star_{x_i,y_i} : x_i \multimap y_i$ and $\star_{y_i,z_i} : y_i \multimap z_i$ for $i \in \{1,2\}$ be arrows in $\mathcal{SSCatGrp}\langle S\rangle$. We have

$$
\begin{aligned}
&\left(\star_{y_1,z_1} \circ \star_{x_1,y_1}\right) \bullet \left(\star_{y_2,z_2} \circ \star_{x_2,y_2}\right) \\
&= \star_{x_1,z_1} \bullet \star_{x_2,z_2} && (8.1.6) \\
&= \star_{x_1 \bullet x_2, z_1 \bullet z_2} && (8.1.9) \\
&= \star_{y_1 \bullet y_2, z_1 \bullet z_2} \circ \star_{x_1 \bullet x_2, y_1 \bullet y_2} && (8.1.6) \\
&= \left(\star_{y_1,z_1} \bullet \star_{y_2,z_2}\right) \circ \left(\star_{x_1,y_1} \bullet \star_{x_2,y_2}\right) && (8.1.9).
\end{aligned}
$$

This demonstrates that the composition law is satisfied. For the identity law, let $x, y \in \mathcal{DMon}\langle S\rangle$. We have

$$
\begin{aligned}
1_x \bullet 1_y &= \star_{x,x} \bullet \star_{y,y} \\
&= \star_{x \bullet y, x \bullet y} \\
&= 1_{x \bullet y}.
\end{aligned}
$$

Thus, the identity law is satisfied. Therefore, the above assignment defines a functor. ◇

Definition 8.1.10. Define a monoidal category, denoted $\mathcal{SSCatGrp}\langle S\rangle$, as follows:

- The category $\mathcal{SSCatGrp}\langle S\rangle$ as in Definition 8.1.2 serves as the underlying category.

- The functor

$$
- \bullet - : \mathcal{SSCatGrp}\langle S\rangle \times \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{SSCatGrp}\langle S\rangle
$$

  as in Definition 8.1.7 serves as the tensor functor.

- Let $J \in \mathcal{DMon}\langle S\rangle$ be the unit object.

- Let $x, y, z \in \mathcal{DMon}\langle S\rangle$. Define the associator $\alpha_{x,y,z} : x \bullet (y \bullet z) \longrightarrow (x \bullet y) \bullet z$ as

$$\alpha_{x,y,z} := 1_{x \bullet y \bullet z}. \tag{8.1.11}$$

  Above assigenment is well-defined $\mathcal{DMon}\langle S\rangle$ is a monoid and we have

$$(x \bullet y) \bullet z = x \bullet (y \bullet z) = x \bullet y \bullet z.$$

- Let $x \in \mathcal{DMon}\langle S\rangle$. Define the left unitor $\lambda_x : J \bullet x \longrightarrow x$ as

$$\lambda_x := 1_x. \tag{8.1.12}$$

  Define the right unitor $\rho_x : x \bullet J \longrightarrow x$ as

$$\rho_x := 1_x. \tag{8.1.13}$$

  Above assignments are well-defined since $\mathcal{DMon}\langle S\rangle$ is a monoid and we have

$$J \bullet x = x \qquad \text{and} \qquad x \bullet J = x.$$

We will show that the above assignments satisfy the coherence axioms of a monoidal category. For any $x, y, z \in \mathcal{DMon}\langle S\rangle$ we have

$$\alpha_{x,y,z} = 1_{x \bullet y \bullet z}.$$

Thus, the pentagon condition is trivially satisfied. For any $x \in \mathcal{DMon}\langle S\rangle$ we have $\lambda_x = 1_x$ and $\rho_x = 1_x$. Thus, the triangle conditions are trivially satisfied.

Therefore, the above assignments satisfy the coherence axioms of a monoidal category. Thus, $\mathcal{SSCatGrp}\langle S\rangle$ is a monoidal category. $\diamond$

Definition 8.1.14. Define a functor

$$(-)' : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{SSCatGrp}\langle S\rangle$$

as follows:

- For an object $x \in \mathcal{DMon}\langle S\rangle$ we define

$$x \longmapsto x'. \tag{8.1.15}$$

- Let $\star_{x,y} : x \multimap y$ be an arrow in $\mathcal{SSCatGrp}\langle S\rangle$. Define

$$(\star_{x,y})' : x' \longrightarrow y'$$

  as

$$(\star_{x,y})' := \star_{x',y'}. \tag{8.1.16}$$

  This assignment is well-defined since $\star_{x,y} : x \multimap y$ being an arrow in $\mathcal{SSCatGrp}\langle S\rangle$ implies that

$$Q(x) = Q(y) \qquad (8.1.4).$$

  Thus, it follows that

$$\begin{aligned} Q(x') &= Q(x)^{-1} && (4.5.13) \\ &= Q(y)^{-1} \\ &= Q(y') && (4.5.13). \end{aligned}$$

We will now verify that the above assignment satisfies the composition condition and the identity condition of a functor. Let $x, y, z \in \mathcal{DMon}\langle S\rangle$ and $\star_{x,y} : x \multimap y, \star_{y,z} :$

$y \multimap z$ be arrows in $\mathcal{SSCatGrp}\langle S \rangle$. We have

$$
\begin{aligned}
(\star_{y,z} \circ \star_{x,y})' &= (\star_{x,z})' && (8.1.6)\\
&= \star_{x',z'} && (8.1.16)\\
&= \star_{y',z'} \circ \star_{x',y'} && (8.1.6)\\
&= (\star_{y,z})' \circ (\star_{x,y})'. && (8.1.16)
\end{aligned}
$$

This shows that the composition condition is satisfied. Now, let $x \in \mathcal{DMon}\langle S \rangle$. We have

$$
\begin{aligned}
1'_x &= (\star_{x,x})' && (8.1.5)\\
&= \star_{x',x'} && (8.1.16)\\
&= 1_{x'}. && (8.1.5)
\end{aligned}
$$

This shows that the identity condition is satisfied. Therefore, the above assignment is a functor. $\diamond$

Definition 8.1.17. Define a categorical group, denoted $\mathcal{SSCatGrp}\langle S \rangle$, as follows:

- The monoidal category $\mathcal{SSCatGrp}\langle S \rangle$ as described in Definition 8.1.10 serves as the underlying monoidal category. Given that $\mathcal{SSCatGrp}\langle S \rangle$ is a thin category, it is inherently a monoidal groupoid.
- The functor $(-)' : \mathcal{SSCatGrp}\langle S \rangle \longrightarrow \mathcal{SSCatGrp}\langle S \rangle$ as defined in Definition 8.1.14 serves as the negator functor.
- For each $x \in \mathcal{DMon}\langle S \rangle$, we define the left cancellation isomorphism $\epsilon_x : x' \bullet x \multimap J$ by

$$\epsilon_x := \star_{x' \bullet x, J}. \tag{8.1.18}$$

This definition is well-defined since $\mathcal{DMon}\langle S\rangle$ is a dashed-monoid and satisfies

$$\mathcal{Q}\left(x' \bullet x\right) = \mathcal{Q}(x)^{-1}\, \mathcal{Q}(x) = e = \mathcal{Q}(J).$$

- For each $x \in \mathcal{DMon}\langle S\rangle$, we define the right cancellation isomorphism $\lambda_x : J \longrightarrow x \bullet x'$ by

$$\lambda_x := \star_{J, x \bullet x'}. \tag{8.1.19}$$

This definition is well-defined since $\mathcal{DMon}\langle S\rangle$ is a dashed-monoid and satisfies

$$\mathcal{Q}(J) = e = \mathcal{Q}(x)\, \mathcal{Q}(x)^{-1} = \mathcal{Q}\left(x \bullet x'\right).$$

Since $\mathcal{SSCatGrp}\langle S\rangle$ is a thin category, all diagrams within it commute, ensuring that the cancellation identities hold. Consequently, $\mathcal{SSCatGrp}\langle S\rangle$ is a categorical group. $\diamond$

8.2 Three Subsets of Dashed Words

In this section, we define three nested subsets:

$$\mathtt{A} \subseteq \mathtt{B} \subseteq \mathtt{C} \subseteq \mathcal{DMon}\langle S \rangle,$$

where $\mathcal{DMon}\langle S \rangle$ is the free dashed-monoid generated by S. These are

$$\mathtt{C} := \left\{ a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \mid m \in \mathbb{N},\ a_i \in S,\ k_i \in \mathbb{N} \right\}$$

$$\mathtt{B} := \left\{ a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \mid m \in \mathbb{N},\ a_i \in S,\ k_i \in \{0,1\} \right\}$$

$$\mathtt{A} := \left\{ a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \mid m \in \mathbb{N},\ a_i \in S,\ k_i \in \{0,1\},\ a_i^{(0)} \text{ and } a_i^{(1)} \text{ are not adjacent} \right\}.$$

We will precisely define these subsets in Definitions 8.2.1, 8.2.16, and 8.2.20, respectively. Furthermore, we will define two functions

$$\hat{C} : \mathcal{DMon}\langle S \rangle \longrightarrow \mathbb{N}, \qquad \text{and} \qquad \hat{A} : \mathtt{B} \longrightarrow \mathbb{N}$$

, called the counting functions, that assign a natural number to dashed words at Definitions 8.2.10 and 8.2.23, respectively. These counting functions will be used to characterize the subsets $\mathtt{C}$, and $\mathtt{A}$ in Propositions 8.2.15 and 8.2.25. Note that, due to similarities between the subsets $\mathtt{C}$ and $\mathtt{B}$, we do not require a counting function for $\mathtt{B}$. In section Section 9.1, we will use the subsets $\mathtt{C}$, $\mathtt{B}$, and $\mathtt{A}$ to characterize the morphisms in the free semi-strict categorical group $\mathcal{SSCatGrp}\langle S \rangle$.

Before we proceed, we would like draw parallels between the subsets $\mathtt{C}$, $\mathtt{B}$, and $\mathtt{A}$ and the different algebraic structures on dashed monoids. Recall Definition 7.1.3.

- A *monoid with distributive dash* is a dashed-monoid that satisfies the distributive property: $(x \bullet y)' = y' \bullet x'$ for all x, y.

- A *monoid with convolution* is a monoid with distributive dash that additionally satisfies the property: $x'' = x$ for all x.

- A *group* is a dashed-monoid that satisfies the full inverse relations: $x' \bullet x = I = x \bullet x'$ for all x.

Observe that subset $\mathtt{C}$ is in bijection with the free monoid with distributive dash generated by S, the subset $\mathtt{B}$ is in bijection with the free monoid with convolution generated by S, and the subset $\mathtt{A}$ is in bijection with the free group generated by S.

Definition 8.2.1. Define a subset $\mathtt{C} \subseteq \mathcal{DMon}\langle S\rangle$, referred to as the *type-C set*, as follows: Let

$$\begin{aligned} T &:= \left\langle \mathrm{S};\ (-)' \right\rangle \\ &= \{a^{(k)} \mid a \in \mathrm{S},\ k \in \mathbb{N}\}. \end{aligned}$$

We define

$$\begin{aligned} \mathtt{C} &:= \{J\} \cup \langle T;\ \bullet\rangle \\ &= \left\{a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \mid m \in \mathbb{N},\ a_i \in S,\ k_i \in \mathbb{N}\right\}. \end{aligned} \tag{8.2.2}$$

◇

Remark 8.2.3. Observe that $\mathtt{C}$ forms a submonoid of $\mathcal{DMon}\langle S\rangle$, but it is not closed under the dash operation. Specifically, while $a \in S$ implies $a \bullet a \in \mathtt{C}$, it does not imply $(a \bullet a)' \in \mathtt{C}$. Consequently, $\mathtt{C}$ is not a sub-dashed monoid. ◇

Example 8.2.4. Consider the following words in $\mathcal{DMon}\langle a,b,c\rangle$:

$$x = (a \bullet b)' \bullet c'' \bullet c',$$

$$y = b' \bullet a' \bullet c'' \bullet c',$$

$$z = b' \bullet a' \bullet c \bullet c', \text{ and}$$

$$w = b' \bullet a'.$$

Observe that $x \notin \mathtt{C}$, while $y,z,w \in \mathtt{C}$. ◇

Definition 8.2.5. Recall Theorem 7.8.8. We define a function

$$\widetilde{C} : \mathtt{G} \longrightarrow \mathbb{N}$$

with induction on the length of the word as follows:

Let $x \in \mathtt{G}$. If $\hat{l}(x) = 1$ then we get

$$x = a^{(k)}$$

for some unique $k \in \mathbb{N}$ and $a \in S$. In this case, we define

$$\widetilde{C}\left(a^{(k)}\right) = 0. \tag{8.2.6}$$

If $\hat{l}(x) \geq 2$ then we get

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}$$

for some unique $m \geq 2$, $k \geq 1$, and $x_i \in \mathtt{G}$. We define

$$\widetilde{C}(x) = \left(\sum_{i=1}^{m} \widetilde{C}(x_i)\right) + k\left(\hat{l}(x) - 1\right). \tag{8.2.7}$$

In this case, since $k \geq 1$ and $\hat{l}(x) \geq 2$, we get

$$\widetilde{C}(x) \geq 1. \tag{8.2.8}$$

As a result, for $x \in \mathtt{G}$

$$\widetilde{C}(x) = 0 \qquad \Longleftrightarrow \qquad x = a^{(k)} \text{ for some } k \in \mathbb{N} \text{ and } a \in S. \tag{8.2.9}$$

$\diamond$

Definition 8.2.10. Define the *type-C counting function*, denoted by

$$\hat{C} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N},$$

as follows: Recall from Definition 8.2.5 the function

$$\widetilde{C} : \mathtt{G} \longrightarrow \mathbb{N}$$

where $\mathtt{G}$ is the multiplicative basis of $\mathcal{DMon}\langle S\rangle$. Extend $\widetilde{C}$ multiplicatively to obtain

$$\hat{C} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N}.$$

That is,

$$\hat{C}(x_1 \bullet \cdots \bullet x_m) = \widetilde{C}(x_1) + \cdots + \widetilde{C}(x_m). \tag{8.2.11}$$

for $m \in \mathbb{N}$ and $x_i \in \mathtt{G}$. $\diamond$

Proposition 8.2.12. *Let $x \in \mathcal{DMon}\langle S\rangle$ with $x \neq J$. Then, the following holds:*

$$\hat{C}\left(x^{(k)}\right) = \hat{C}(x) + k \cdot (\hat{l}(x) - 1). \tag{8.2.13}$$

Proof. We will consider the following two cases:

Case I ($x \in \mathtt{G}$): In this case, there are two subcases. First,

$$x = a^{(r)}$$

for some unique $r \in \mathbb{N}$ and $a \in S$. Then,

$$\begin{aligned}
\hat{C}\left(x^{(k)}\right) &= \hat{C}\left(a^{(r+k)}\right) \\
&= 0 && (8.2.6) \text{ and } (8.2.11) \\
&= \hat{C}\left(a^{(k)}\right) && (8.2.6) \text{ and } (8.2.11) \\
&= \hat{C}(x) \\
&= \hat{C}(x) + k(\hat{l}(x) - 1).
\end{aligned}$$

Note that here $\hat{l}(x) = 1$.

Second,

$$x = (x_1 \bullet \cdots \bullet x_m)^{(r)}$$

for some unique $m \geq 1$, $r \geq 1$, and $x_i \in \mathtt{G}$. Then,

$$\begin{aligned}
\hat{C}\left(x^{(k)}\right) &= \hat{C}\left((x_1 \bullet \cdots \bullet x_m)^{(r+k)}\right) \\
&= \left(\sum_{i=1}^{m} \widetilde{C}(x_i)\right) + (r+k)\left(\hat{l}\left(x^{(k)}\right) - 1\right) && (8.2.7) \text{ and } (8.2.11) \\
&= \left(\sum_{i=1}^{m} \widetilde{C}(x_i)\right) + r\left(\hat{l}(x) - 1\right) + k\left(\hat{l}(x) - 1\right) \\
&= \hat{C}(x) + k\left(\hat{l}(x) - 1\right). && (8.2.7) \text{ and } (8.2.11)
\end{aligned}$$

Note that, since dash in natural numbers is identity, we get $\hat{l}\left(x^k\right) = \hat{l}(x)$.

<u>Case II</u> ($x \notin \mathtt{G}$): In this case, we get

$$x = x_1 \bullet \cdots \bullet x_m$$

for some unique $m \in \mathbb{N}$ and $x_i \in \mathtt{G}$. If $k = 0$ then

$$\begin{aligned}
\hat{C}\left(x^{(k)}\right) &= \hat{C}\left(x^{(0)}\right) \\
&= \hat{C}(x) \\
&= \hat{C}(x) + k(\hat{l}(x) - 1).
\end{aligned}$$

If $k \geq 1$ then

$$\begin{aligned}
\hat{C}\left(x^{(k)}\right) &= \left(\sum_{i=1}^{m} \widetilde{C}(x_i)\right) + k\left(\hat{l}\left(x^{(k)}\right) - 1\right) && \text{(8.2.7) and (8.2.11)} \\
&= \hat{C}(x) + k\left(\hat{l}(x) - 1\right). && \text{(8.2.11)}
\end{aligned}$$

This completes the proof. □

Example 8.2.14. Let $a, b, c, d, e \in \mathrm{S}$. Let

$$x = a \bullet (b \bullet (c \bullet d)' \bullet e)'.$$

Then, we get

$$\begin{aligned}
\hat{C}(x) &= \hat{C}(a) + \hat{C}\left((b \bullet (c \bullet d)' \bullet e)'\right) && \text{(8.2.11)} \\
&= \hat{C}\left(b \bullet (c \bullet d)' \bullet e\right) + 1 \times (\hat{l}\left(b \bullet (c \bullet d)' \bullet e\right) - 1) && \text{(8.2.6) and (8.2.13)} \\
&= \hat{C}(b) + \hat{C}\left((c \bullet d)'\right) + \hat{C}(e) + (4 - 1) && \text{(8.2.11)} \\
&= \hat{C}(c \bullet d) + 1 \times (\hat{l}(c \bullet d) - 1) + 3 && \text{(8.2.6) and (8.2.13)} \\
&= \hat{C}(c) + \hat{C}(d) + (2 - 1) + 3 && \text{(8.2.11)} \\
&= 0 + 1 + 3 && \text{(8.2.6)} \\
&= 4.
\end{aligned}$$

Similarly, for $y = a \bullet e' \bullet (c \bullet d)'' \bullet b'$, we have

$$\hat{C}(y) = 2$$

and for $z = a \bullet e' \bullet c'' \bullet d'' \bullet b'$, we get

$$\hat{C}(z) = 0.$$

◇

Proposition 8.2.15. *Let $x \in \mathcal{DMon}\langle S \rangle$. Then, $x \in \mathrm{C}$ if and only if $\hat{C}(x) = 0$.*

Proof. Suppose $x \in \mathtt{C}$. Then, we can write

$$x = a_1^{(r_1)} \bullet \cdots \bullet a_m^{(r_m)} \qquad (8.2.2)$$

where $m \in \mathbb{N}$, $a_i \in S$, and $r_i \in \mathbb{N}$ for $1 \le i \le m$. Then,

$$\begin{aligned} \mathtt{C} &= \sum_{i=1}^{m} \widetilde{C}\left(a_i^{(r_i)}\right) && (8.2.11) \\ &= 0. && (8.2.6) \end{aligned}$$

Conversely, suppose $x \in \mathcal{DMon}\langle S \rangle$ such that $\hat{C}(x) = 0$. We get

$$x = x_1 \bullet \cdots \bullet x_m$$

for some unique $m \in \mathbb{N}$ and $x_i \in \mathtt{G}$. We have

$$\hat{C}(x) = \sum_{i=1}^{m} \widetilde{C}(x_i). \qquad (8.2.11)$$

Since $\hat{C}(x) = 0$, we conclude

$$\widetilde{x_i} = 0$$

for every $1 \le i \le m$. From the implication (8.2.9) we get

$$x_i = a_i^{(r_i)}$$

for some unique $r_i \ge 0$ and $a_i \in S$. Thus,

$$x = a_1^{(r_1)} \bullet \cdots \bullet a_m^{(r_m)} \in \mathtt{C}.$$

□

Definition 8.2.16. Define a subset $\mathtt{B} \subseteq \mathtt{C}$, referred to as the *type-B set*, as follows:

$$\mathtt{B} := \left\{a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \mid m \in \mathbb{N},\ a_i \in \mathtt{S},\ k_i \in \{0,1\}\right\}. \qquad (8.2.17)$$

◇

Remark 8.2.18. Observe that $\mathtt{B}$ forms a submonoid of $\mathcal{DMon}\langle S\rangle$ since $\mathtt{B}$ is closed under multiplication, but it is not closed under the dash operation. For example, while $a \in S$ implies $a' \in \mathtt{B}$, it does not imply $a'' \in \mathtt{B}$. Consequently, $\mathtt{B}$ is not a sub-dashed monoid. ◇

Example 8.2.19. Consider the following words in $\mathcal{DMon}\langle a,b,c\rangle$:

$$x = (a \bullet b)' \bullet c'' \bullet c',$$

$$y = b' \bullet a' \bullet c'' \bullet c',$$

$$z = b' \bullet a' \bullet c \bullet c', \text{ and}$$

$$w = b' \bullet a'.$$

Observe that $x, y \notin \mathtt{C}$, while $z, w \in \mathtt{C}$. ◇

Definition 8.2.20. Define a subset $\mathtt{A} \subseteq \mathtt{B}$, referred to as the **type-A set**, as follows:

$$\mathtt{A} := \{a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \in \mathtt{B} \mid a_i = a_{i+1} \text{ implies } k_i = k_{i+1} \text{ for } 1 \leq i < m\}. \tag{8.2.21}$$

◇

Proposition 8.2.22. *Consider the induced dashed-monoid homomorphism*

$$\mathcal{Q} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathcal{Grp}\langle S\rangle$$

as defined in Construction 4.5.5. Then, the restriction

$$\mathcal{Q}\,|_{\mathtt{A}} : \mathtt{A} \longrightarrow \mathcal{Grp}\langle S\rangle$$

is a bijection.

Proof. We will show that $\mathcal{Q}\,|_{\mathtt{A}}$ is both injective and surjective. Let $x, y \in \mathtt{A}$ such that $\mathcal{Q}(x) = \mathcal{Q}(y)$. Then we have

$$x = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \qquad \text{and} \qquad y = b_1^{(l_1)} \bullet \cdots \bullet b_p^{(l_p)}$$

where $m, p \in \mathbb{N}$, $a_i, b_j \in S$, and $k_i, l_j \in \{0,1\}$ for $1 \le i \le m$ and $1 \le j \le p$. We get

$$\mathcal{Q}(x) = a_1^{(k_1\%2)} \cdots a_m^{(k_m\%2)}, \qquad \text{(4.5.11) through (4.5.13)}$$

where $a_i^{(k_i\%2)} = a_i$ if $k_i = 0$ and $a_i^{(k_i\%2)} = a_i^{-1}$ if $k_i = 1$. Furthermore, there is no cancellation pair aa^{-1} or $a^{-1}a$ present in the above representation of $\mathcal{Q}(x)$ since $a_i = a_{i+1}$ implies $k_i = k_{i+1}$. Thus, the representation

$$a_1^{(k_1\%2)} \cdots a_m^{(k_m\%2)}$$

is the reduced representation of $\mathcal{Q}(x)$ in $\mathcal{Grp}\langle S \rangle$. Similarly, we get the reduced representation

$$\mathcal{Q}(y) = b_1^{(l_1\%2)} \cdots b_p^{(l_p\%2)}.$$

Since $\mathcal{Q}(x) = \mathcal{Q}(y)$, we conclude that $m = p$, $a_i = b_i$, and $k_i = l_i$ for $1 \le i \le m$. Thus, $x = y$. This shows that $\mathcal{Q}|_{\mathtt{A}}$ is injective.

Now let $z \in \mathcal{Grp}\langle S \rangle$. Let

$$z = a_1^{(k_1)} \cdots a_m^{(k_m)}$$

be the reduced representation of z in $\mathcal{Grp}\langle S \rangle$. Therefore, we have $a_i \in S$ and $k_i \in \{0,1\}$. In particular, we have $a_i^{(k_i)} = a_i$ if $k_i = 0$, and $a_i^{(k_i)} = a_i^{-1}$ if $k_i = 1$. Additionally, there is no cancellation pair aa^{-1} or $a^{-1}a$ present in the representation of z. Consequently, if $a_i = a_{i+1}$, we must have $k_i = k_{i+1}$. Consider

$$x = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \in \mathtt{A}.$$

From Construction 4.5.5, we get

$$\mathcal{Q}(x) = z.$$

Therefore, $\mathcal{Q}|_{\mathtt{A}}$ is surjective. □

Definition 8.2.23. Define a counting function, referred to as the *type-A counting function*, and denoted by

$$\hat{A} : \mathtt{B} \longrightarrow \mathbb{N},$$

as follows: Let

$$x = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \in \mathtt{B},$$

where $m \in \mathbb{N}$, $a_i \in \mathtt{S}$, and $k_i \in \{0,1\}$ for $1 \leq i \leq m$. Define $\hat{A}(x)$ as the number of simple cancellation pairs, $a \bullet a'$ or $a' \bullet a$, present in x. Specifically,

$$\hat{A}(x) := |\{1 \leq i < m \mid a_i = a_{i+1} \text{ and } k_i \neq k_{i+1}\}|. \tag{8.2.24}$$

◇

Proposition 8.2.25. *Let $x \in \mathtt{B}$. Then, $x \in \mathtt{A}$ if and only if $\hat{A}(x) = 0$.*

Proof. Let

$$x = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \in \mathtt{B},$$

where $m \in \mathbb{N}$, $a_i \in \mathtt{S}$, and $k_i \in \{0,1\}$ for $1 \leq i \leq m$.

Suppose $x \in \mathtt{A}$. Then, for every $1 \leq i < m$, we have $a_i = a_{i+1} \implies k_i = k_{i+1}$. Consequently, $\hat{A}(x) = 0$.

Conversely, suppose $\hat{A}(x) = 0$. This implies there is no $1 \leq i < m$ such that $a_i = a_{i+1}$ and $k_i \neq k_{i+1}$. In other words, for every $1 \leq i < m$, $a_i = a_{i+1}$ implies $k_i = k_{i+1}$. Therefore, $x \in \mathtt{A}$. □

Lemma 8.2.26. *We have*

$$\hat{A}(J) = 0. \tag{8.2.27}$$

Proof. Since J does not contain any simple cancellation pairs, it follows that $\hat{C}(J) = 0$. □

8.3 Type-C Morphism

Recall from Definition 5.2.21 the definition of distribution morphisms in a semi-strict categorical group. In this section, we construct the type-C forward arrows, which are the expanded instances of these distribution arrows in $\mathcal{SSCatGrp}\langle S\rangle$. We will show that for every $x \in \mathcal{DMon}\langle S\rangle$, there exists a unique dashed word $x_{\mathtt{C}} \in \mathtt{C}$ and a chain of type-C forward arrows connecting x to $x_{\mathtt{C}}$ (see Definition 8.3.29). Furthermore, given a semi-strict categorical group $\mathcal{M}$, a function $u : S \longrightarrow \mathrm{Obj}(\mathcal{M})$, and the corresponding induced dashed monoid homomorphism

$$F : \mathcal{DMon}\langle S\rangle \longrightarrow \mathrm{Obj}(\mathcal{M}),$$

we will construct a morphism

$$\gamma_x : F(x) \longrightarrow F(x_{\mathtt{C}}) \in \mathcal{M}$$

called the *type-C morphism*.

Framework 8.3.1. Amend Framework 8.1.1 as follows: Let $\mathcal{SSCatGrp}\langle S\rangle$ be the free semi-strict categorical group generated by S as defined in Definition 8.1.2.

Furthermore, let $\mathcal{M}$ be a semi-strict categorical group with $M := \mathrm{Obj}(\mathcal{M})$. Recall from Example 7.1.6 that M is a dashed monoid. Let $u : S \longrightarrow M$ be a function, and let

$$F : \mathcal{DMon}\langle S\rangle \longrightarrow M$$

be the unique induced dashed monoid homomorphism as in Definition 7.8.3. $\diamond$

Definition 8.3.2. Let $m, k \in \mathbb{N}$ and M be a dashed monoid. Define a function

$$\Box_k : M^m \longrightarrow M,$$

called the *parity function*, as follows:

$$\Box_k(x_1,\dots,x_m) = \begin{cases} x_1 \bullet \cdots \bullet x_m & \text{if } k \text{ is even} \\ x_m \bullet \cdots \bullet x_1 & \text{if } k \text{ is odd.} \end{cases} \tag{8.3.3}$$

◇

Notation 8.3.4. For convenience, we will use the following shorthand notation:

$$x_1 \Box_k \cdots \Box_k x_m := \Box_k(x_1,\dots,x_m).$$

When the value of k is clear from the context, we may omit the subscript k. ◇

Proposition 8.3.5. *Let $m,k \in \mathbb{N}$, and let $x_i \in \mathcal{DMon}\langle S\rangle$ for $1 \le i \le m$. Let $\Box := \Box_k$. Then, the equality*

$$Q\left((x_1 \bullet \cdots \bullet x_m)^{(k)}\right) = Q\left(x_1^{(k)} \Box \cdots \Box x_m^{(k)}\right) \tag{8.3.6}$$

holds.

Proof. Observe that

$$\begin{aligned}
Q\left((x_1 \bullet \cdots \bullet x_m)^{(k)}\right) &= \begin{cases} Q(x_1 \bullet \cdots \bullet x_m) & \text{if } k \text{ is even} \\ Q(x_1 \bullet \cdots \bullet x_m)^{-1} & \text{if } k \text{ is odd.} \end{cases} && (4.5.13) \\
&= \begin{cases} Q(x_1) \cdots Q(x_m) & \text{if } k \text{ is even} \\ (Q(x_1) \cdots Q(x_m))^{-1} & \text{if } k \text{ is odd.} \end{cases} && (4.5.12) \\
&= \begin{cases} Q(x_1) \cdots Q(x_m) & \text{if } k \text{ is even} \\ Q(x_m)^{-1} \cdots Q(x_1)^{-1} & \text{if } k \text{ is odd.} \end{cases} && \text{Construction 4.5.4} \\
&= \begin{cases} Q\left(x_1^{(k)}\right) \cdots Q\left(x_m^{(k)}\right) & \text{if } k \text{ is even} \\ Q\left(x_m^{(k)}\right) \cdots Q\left(x_1^{(k)}\right) & \text{if } k \text{ is odd.} \end{cases} && (4.5.13) \\
&= \begin{cases} Q\left(x_1^{(k)} \bullet \cdots \bullet x_m^{(k)}\right) & \text{if } k \text{ is even} \\ Q\left(x_m^{(k)} \bullet \cdots \bullet x_1^{(k)}\right) & \text{if } k \text{ is odd.} \end{cases} && (4.5.12) \\
&= Q\left(x_1^{(k)} \Box \cdots \Box x_m^{(k)}\right). && (8.3.3)
\end{aligned}$$

This completes the proof. □

Definition 8.3.7. Define a subset $\mathtt{Y} \subseteq \mathtt{G}$ as follows:

$$\mathtt{Y} := \left\{ (x_1 \bullet \cdots \bullet x_m)^{(k)} \mid m \geq 2,\ k \geq 1, \text{ and } x_i \in \mathtt{G} \right\}. \tag{8.3.8}$$

From Theorem 7.8.8, $\mathtt{Y}$ is indeed a subset of $\mathtt{G}$.

Let

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)} \in \mathtt{Y}.$$

Define $y \in \mathcal{DMon}\langle S \rangle$ as

$$y := x_1^{(k)} \square \cdots \square x_m^{(k)}$$

where $\square = \square_k$. Then, from Proposition 8.3.5, we have

$$\mathcal{Q}(x) = \mathcal{Q}(y).$$

Thus, there exists an arrow

$$\tilde{\gamma}_X : x \multimap y$$

in $\mathcal{CatGrp}\langle S \rangle$, where $X = (x_1, \cdots, x_m)$. We refer to arrows constructed in this manner as *distribution arrows*.

For $x \in \mathcal{DMon}\langle S \rangle$, an arrow $f : x \multimap y$ is called a *type-C forward arrow* if f is an expanded instance of a distribution arrow. ⋄

Remark 8.3.9. Observe that a distribution arrow is necessarily a non-identity arrow. Specifically, if $f : x \multimap y$ is a distribution arrow, then it must satisfy $x \neq y$. It follows that a type-C forward arrow is non-identity if it is a non-trivial expanded instance of a distribution arrow. ⋄

Remark 8.3.10. We recall Definition 7.1.31: let $A \subseteq X \subseteq \mathcal{DMon}\langle S \rangle$ be subsets, then

$$e\langle X, A;\ \bullet \rangle :=$$
$$\{x_1 \bullet \cdots \bullet x_m \in M \mid m \geq 1,\ x_i \in X, \text{ and } \exists\ 1 \leq j \leq m \text{ such that } x_j \in A\}.$$ ⋄

Proposition 8.3.11. *Let $f : x \multimap\!\!\rightarrow y$ be a non-identity type-C forward arrow. Then,*

$$x \in e\,\langle \mathtt{G}, \mathtt{Y};\ \bullet \rangle\,. \tag{8.3.12}$$

Proof. Since $\mathtt{G}$ is a multiplicative basis of $\mathcal{DMon}\,\langle S \rangle$, we get

$$x = x_1 \bullet \cdots \bullet x_m$$

where $m \in \mathbb{N}$, $x_i \in \mathtt{G}$. Since f is non-identity type-C forward arrow, it is a non-trivial expanded instance of a distribution arrow. Thus, there exists $1 \le i \le m$ such that $x_i \in \mathtt{Y}$. This completes the proof. $\square$

Example 8.3.13. Let $a, b, c, d, e \in \mathtt{S}$. We have $(b \bullet (c \bullet d)' \bullet e)' \in \big\langle \langle \mathtt{R};\ (-)' \rangle \big\rangle$. The arrow

$$h_1 : \big((b \bullet (c \bullet d)' \bullet e)' \multimap\!\!\rightarrow e' \bullet (c \bullet d)'' \bullet b' \big)$$

is a distribution arrow and therefore a type-C forward arrow. The arrow

$$f := 1_a \bullet h_1 : \big(a \bullet (b \bullet (c \bullet d)' \bullet e)' \multimap\!\!\rightarrow a \bullet e' \bullet (c \bullet d)'' \bullet b' \big)$$

is a type-C forward arrow. Similarly, the arrow

$$h_2 : \big((c \bullet d)'' \multimap\!\!\rightarrow c'' \bullet d'' \big)$$

is a distribution arrow. Therefore,

$$g := 1_{a \bullet e'} \bullet h_2 \bullet 1'_b : \big(a \bullet e' \bullet (c \bullet d)'' \bullet b' \multimap\!\!\rightarrow e' \bullet c'' \bullet d'' \bullet b' \big)$$

is a type-C forward arrow. $\diamond$

Definition 8.3.14. Let $x \in \mathcal{DMon}\,\langle S \rangle$. We define x to be *irreducible with respect to type-C forward arrows* if every type-C forward arrow $f : x \multimap\!\!\rightarrow y$ with x as the source object satisfies $f = 1_x$. Conversely, x is said to be *reducible* with respect to type-C forward arrows if there exists a non-identity type-C forward arrow $f : x \longrightarrow y$ with source object equal to x. $\diamond$

Proposition 8.3.15. *Let $X \subseteq \mathcal{DMon}\langle S\rangle$ to be the subset consisting of elements that are reducible with respect to type-C forward arrows. Then, the following equality holds:*

$$X = e\langle \mathtt{G},\ \mathtt{Y};\ \bullet\rangle. \tag{8.3.16}$$

Furthermore, the subset of elements that are irreducible with respect to type-C forward arrows is precisely the type-C set:

$$\mathcal{DMon}\langle S\rangle \setminus X = \mathtt{C}. \tag{8.3.17}$$

Proof. Let $x \in \mathcal{DMon}\langle S\rangle$. Suppose x is reducible with respect to type-C forward arrows. Then, there exists a non-identity type-C forward arrow $f : x \multimap\!\!\!\rightarrow y$. From Proposition 8.3.11, we conclude

$$x \in e\langle \mathtt{G},\ \mathtt{Y};\ \bullet\rangle.$$

On the other hand, suppose $x \in e\langle \mathtt{G},\ \mathtt{Y};\ \bullet\rangle$. Then, there exists $m \geq 1$, $x_i \in \mathtt{G}$, and there exists $1 \leq i \leq m$ such that $x_i \in Y$. From Definition 8.3.7, we get a distribution arrow $g : x_i \multimap\!\!\!\rightarrow y_i$. Consider a type-C forward arrow

$$f := 1_{x_1 \bullet \cdots \bullet x_{i-1}} \bullet g \bullet 1_{x_{i+1} \bullet \cdots \bullet x_m}$$

The source object of f is x. Therefore, x is reducible with respect to type-C forward arrows. This completes the proof of equation (8.3.16).

Now, from Theorem 7.8.8, we get that

$$\mathtt{G} = \left\langle \mathtt{S};\ (-)' \right\rangle \sqcup \mathtt{Y}.$$

Consider the following calculation:

$$
\begin{aligned}
\mathcal{DMon}\langle S\rangle &= \{J\} \sqcup \mathcal{DMon}\langle S\rangle^{+} && (6.1.18)\\
&= \{J\} \sqcup \langle \mathtt{G};\ \bullet\rangle && (6.1.32)\\
&= \{J\} \sqcup \left\langle \left\langle \mathtt{S};\ (-)'\right\rangle \sqcup Y;\ \bullet\right\rangle && \text{from above}\\
&= \{J\} \sqcup \left\langle \left\langle \mathtt{S};\ (-)'\right\rangle;\ \bullet\right\rangle \sqcup e\,\langle \mathtt{G}, Y; \bullet\rangle && (7.1.35)\\
&= \mathtt{C} \sqcup X. && (8.2.2) \text{ and } (8.3.16)
\end{aligned}
$$

Here, the third equality follows because both $\left\langle \mathtt{S};\ (-)'\right\rangle$ and Y are subsets of $\mathtt{G}$ and $\mathtt{G}$ is a multiplicatively independent set. It follows that

$$\mathtt{C} = \mathcal{DMon}\langle S\rangle \setminus X$$

as required. □

Example 8.3.18. Let $a, b, c, d, e \in \mathtt{S}$,

$$y = a \bullet e' \bullet (c \bullet d)''' \bullet b' \qquad \text{and} \qquad z = e' \bullet c'' \bullet d'' \bullet b'.$$

Then, y is reducible with respect to type-C forward arrows since there is type-C forward arrow with source y, namely,

$$g : a \bullet e' \bullet (c \bullet d)'' \bullet b' \multimap\!\!\!\rightarrow a \bullet e' \bullet c'' \bullet d'' \bullet b'$$

as in Example 8.3.13. Whereas, z is irreducible with respect to type-C forward arrows since there are no more distributions that can take place. Therefore, $z \in \mathtt{C}$. ◇

Proposition 8.3.19. *Let $f : x \multimap\!\!\!\rightarrow y$ be a non-identity type-C forward arrow. Then,*

$$\hat{C}(y) < \hat{C}(x). \tag{8.3.20}$$

Proof. We will first show this proposition for distribution arrows. Suppose $f : x \multimap\!\!\to y$ is a distribution arrow. Then, we get

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}$$

for some $m \geq 2$, $k \geq 1$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ and

$$y = x_1^{(k)} \Box \cdots \Box x_m^{(k)}$$

where $\Box = \Box_k$. Consider the following calculation:

$$\begin{aligned}
\hat{C}(y) &= \sum_{i=1}^{m} \hat{C}\left(x_i^{(k)}\right) && (8.2.11)\\
&= \sum_{i=1}^{m} \left(\hat{C}(x_i) + k(\hat{l}(x_i) - 1)\right) && (8.2.13)\\
&= \sum_{i=1}^{m} \hat{C}(x_i) + k \sum_{i=1}^{m} \hat{l}(x_i) \ - km &&\\
&= \hat{C}(x_1 \bullet \cdots \bullet x_m) + k(\hat{l}(x_1 \bullet \cdots \bullet x_m) - 1) + k - km && (8.2.11)\\
&< \hat{C}(x_1 \bullet \cdots \bullet x_m) + k(\hat{l}(x_1 \bullet \cdots \bullet x_m) - 1) && \text{since } k \geq 1 \text{ and } m \geq 2\\
&= \hat{C}\left((x_1 \bullet \cdots \bullet x_m)^{(n)}\right) && (8.2.13)\\
&= \hat{C}(x).
\end{aligned}$$

Thus, the proposition holds for distribution arrows.

Now, assume $f : x \multimap\!\!\to y$ is a non-unit type-C forward arrow. Therefore, f is a non-trival expanded instance of a distribution arrow. Thus, we get $z, a, b, w \in \mathcal{DMon}\langle S \rangle$ such that $\widehat{f} : a \multimap\!\!\to b$ is a distribution arrow and

$$f = 1_z \bullet \widehat{f} \bullet 1_w : z \bullet a \bullet w \multimap\!\!\to z \bullet b \bullet w.$$

Therefore, we get

$$
\begin{aligned}
\hat{C}(y) &= \hat{C}(z \bullet b \bullet w) && (8.2.11)\\
&= \hat{C}(z) + \hat{C}(b) + \hat{C}(w) && (8.2.11)\\
&< \hat{C}(z) + \hat{C}(a) + \hat{C}(w) && \text{by the above calculation}\\
&< \hat{C}(z \bullet a \bullet w) && (8.2.11)\\
&= \hat{C}(x).
\end{aligned}
$$

□

Definition 8.3.21. Let $x \in \mathcal{DMon}\langle S\rangle$. A *chain anchored at x* is a sequence

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots$$

of composable arrows starting at x. A *chain of type-C forward arrows anchored at x* is a chain anchored at x such that for all $i \geq 1$, f_i is a type-C forward arrow. We say that a chain *terminates* at $y \in \mathcal{DMon}\langle S\rangle$ if there is $N \in \mathbb{N}$ such that for all $n > N$, we have $f_n = 1_y$. We will denote such a terminating chain by

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_N} y.$$

◇

Lemma 8.3.22. *Let $x \in \mathcal{DMon}\langle S\rangle$. Then, there exists a terminating chain*

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_N} y.$$

of type-C forward arrows anchored at x such that $y \in \mathtt{C}$.

Proof. We will prove this using induction on $\hat{C}(x)$.

Base case $\left(\hat{C}(x) = 0\right)$: From Proposition 8.2.15 we get $x \in \mathtt{C}$. Thus, we can take $f_i = 1_x$ for every $i \geq 1$.

Induction case $\left(\hat{C}(x) \geq 1\right)$: From Proposition 8.2.15 we get $x \notin \mathtt{C}$. From Proposition 8.3.15 we get that x is reducible with respect to type-C forward arrow. Thus, we get $x_1 \in \mathcal{DMon}\langle S\rangle$ and $f_1 : x \longrightarrow x_1$ such that f_1 is a non-identity type-C forward arrow. From Proposition 8.3.19 we get that $\hat{C}(x_1) < \hat{C}(x)$. Thus, from the induction hypothesis we get a terminating chain

$$x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_N} y.$$

of type-C forward arrows anchored at x_1 such that $y \in \mathtt{C}$. Pre-composing with $f_1 : x \longrightarrow x_1$, we get the required terminating chain of type-C forward arrows. □

Example 8.3.23. Let $a, b, c, d, e \in \mathtt{S}$. Let $x = a \bullet (b \bullet (c \bullet d)' \bullet e)'$. Then,

$$\begin{tikzcd} a \bullet (b \bullet (c \bullet d)' \bullet e)' \arrow[d, "f"] \\ a \bullet e' \bullet (c \bullet d)' \bullet a' \arrow[d, "g"] \\ a \bullet e' \bullet c'' \bullet d'' \bullet a' \end{tikzcd}$$

is a terminating chain of type-C forward arrows that ends in $\mathtt{C}$. Here f, g are as in Example 8.3.13. ◇

Definition 8.3.24. Assume Framework 8.3.1. Let

$$\mathcal{FA}_{\mathtt{C}} := \left\{ f \in \mathrm{Mor}\left(\mathcal{SSCatGrp}\langle S\rangle\right) \mid f \text{ is tpye-C forward arrow} \right\}$$

be the set of type-C forward arrows. Define a mapping

$$F_{\mathtt{C}} : \mathcal{FA}_{\mathtt{C}} \longrightarrow \mathrm{Mor}(\mathscr{M})$$

as follows:

Let $f : x \multimap y$ be a type-C arrow. Recall that type-C forward arrows are expanded instances of distribution arrows. Thus, we will first define $F_{\mathtt{C}}$ for distribution arrows and then extend it to type-C forward arrows. Now suppose $f : x \multimap y$ is a distribution arrow. Then, we get

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)} \in \mathtt{Y}$$

where $k \geq 1$, $m \geq 2$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$. Moreover, we get

$$f = \tilde{\gamma}_X : (x_1 \bullet \cdots \bullet x_m)^{(k)} \multimap x_1^{(k)} \Box \cdots \Box x_m^{(k)}$$

is a distribution arrow as in Definition 8.3.7, where $X = (x_1, \ldots, x_m)$. Define

$$F_{\mathtt{C}}(\tilde{\gamma}_X) : (F(x_1) \cdots F(x_m))^{(k)} \longrightarrow (Fx_1)^{(k)} \Box \cdots \Box (Fx_m)^{(k)}$$

as

$$F_{\mathtt{C}}(\widehat{\gamma}_X) := C_m^k(F(X)) \tag{8.3.25}$$

where $C_m^k \in \mathcal{D}ist_m$ is the distribution natural transformation as in Theorem 5.2.26 and $F(X) = (F(x_1),\ \ldots,\ F(x_m))$.

Since type-C forward arrows are expanded instances distribution arrows, we define

$$F_{\mathtt{C}}(1_a \bullet \widetilde{\gamma}_X \bullet 1_b) := 1_{F(a)} \gamma_{F(X)} 1_{F(b)}$$

where $\widetilde{\gamma}_X$ is a distribution arrow. ◇

Theorem 8.3.26. *Let $x \in \mathcal{DM}on\,\langle S \rangle$. Then, any two terminating chains,*

$$x \xrightarrow{f_1} y_1 \xrightarrow{f_2} y_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} y_n = a$$

and

$$x \xrightarrow{g_1} z_1 \xrightarrow{g_2} z_2 \xrightarrow{g_3} \cdots \xrightarrow{g_r} z_r = b,$$

of type-C forward arrows anchored at x such that $a, b \in \mathtt{C}$, then we get

$$a = b. \tag{8.3.27}$$

Moreover, we get

$$F_{\mathtt{C}}(f_n) \circ \cdots \circ F_{\mathtt{C}}(f_1) = F_{\mathtt{C}}(g_r) \circ \cdots \circ F_{\mathtt{C}}(g_1). \tag{8.3.28}$$

Proof. We will show this using induction on $\hat{C}(x)$.

Base case $\left(\hat{C}(x) = 0\right)$: From Proposition 8.2.15 we get $x \in \mathtt{C}$. From Proposition 8.3.15 we get that x is irreducible with respect to type-C forward arrows. Thus, we get $a = x = b$, $f_i = g_i = 1_x$, and

$$F_{\mathtt{C}}(f_i) = F_{\mathtt{C}}(g_i)$$

for $i \geq 1$.

Induction case $\left(\hat{C}(x) \geq 1\right)$: Since $\mathtt{G}$ is a multiplicative basis of $\mathcal{DMon}\langle S \rangle$, we get

$$x = x_1 \bullet \cdots \bullet x_m$$

where $x_i \in \mathtt{G}$. Since $\hat{C}(J) = 0$ and $\hat{C}(x) \geq 1$, we get $x \neq J$. Consequently, we have $m \geq 1$. Since $\hat{C}(x) \geq 1$, we get that $x \notin \mathtt{C}$. It follows that at least one f_i for $1 \leq i \leq n$ and g_j for $1 \leq j \leq r$ must be non-identity. Rewrite the chains f_i and g_j such that all composition arrows are non identity until chains reach a and b respectively.

Now, consider $f_1 : x \multimap y_1$ and $g_1 : x \multimap z_1$. Since a non-identity type-C forward arrow is a non-trivial expanded instance of a distribution arrow, we get a distribution arrow

$$\tilde{\gamma}_p : x_p \multimap u$$

for some $1 \leq p \leq m$ such that

$$f_1 = \langle \tilde{\gamma}_p \rangle : x_1 \bullet \cdots \bullet x_p \bullet \cdots \bullet x_m \multimap x_1 \bullet \cdots \bullet u \bullet \cdots \bullet x_m.$$

Similarly, we get a distribution arrow

$$\tilde{\gamma}_q : x_q \multimap v$$

for some $1 \leq q \leq m$ such that

$$g_1 = \langle \tilde{\gamma}_q \rangle : x_1 \bullet \cdots \bullet x_q \bullet \cdots \bullet x_m \multimap x_1 \bullet \cdots \bullet v \bullet \cdots \bullet x_m.$$

Since the distribution arrows are completely determined by the source object, if we have $p = q$ then we get $\gamma_p = \gamma_q$. In turn, we get $f_1 = g_1$. In this case, we proceed by induction by since we have $\hat{C}(x_1) < \hat{C}(x)$ from Proposition 8.3.19.

Now, suppose without the loss of generality $p < q$. That is, f_1 and g_1 act on separate factors. Consider the following diagram:

$$\begin{tikzcd}
 & x_1 \bullet \cdots \bullet x_p \bullet \cdots \bullet x_q \bullet \cdots \bullet x_m \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\
x_1 \bullet \cdots \bullet u \bullet \cdots \bullet x_q \bullet \cdots \bullet x_m \arrow[dr, "\overline{f_1}"] \arrow[dd, "f_2"'] & & x_1 \bullet \cdots \bullet x_p \bullet \cdots \bullet v \bullet \cdots \bullet x_m \arrow[dl, "\overline{g_1}"'] \arrow[dd, "g_2"] \\
 & x_1 \bullet \cdots \bullet u \bullet \cdots \bullet v \bullet \cdots \bullet x_m \arrow[d, "h_2"] & \\
\vdots \arrow[d, "f_n"'] & \vdots \arrow[d, "h_t"] & \vdots \arrow[d, "g_r"] \\
a \arrow[r, equal] & c \arrow[r, equal] & b
\end{tikzcd}$$

where

$$\overline{f}_1 := \langle \tilde{\gamma_q} \rangle \qquad \text{and} \qquad \overline{g}_1 := \langle \tilde{\gamma_p} \rangle,$$

are type-C forward arrows and

$$h_2, \ldots, h_t$$

is the terminating chain of type-C forward arrows that we get from Lemma 8.3.22.

Observe that each component arrow of the above diagram is a type-C forward arrow. We will show that the image of the diagram under $F_{\mathtt{C}}$ commutes. Consider the following calculation:

$$\begin{aligned} F_{\mathtt{C}}(\overline{f}_1) \circ F_{\mathtt{C}}(f_1) &= 1_{Fx_1} \cdots 1_{Fu} \cdots \gamma_q \cdots 1_{Fx_m} \ \circ \ 1_{Fx_1} \cdots \gamma_p \cdots 1_{Fx_q} \cdots 1_{Fx_m} && (8.3.25) \\ &= 1_{Fx_1} \cdots \gamma_p \cdots 1_{Fv} \cdots 1_{Fx_m} \ \circ \ 1_{Fx_1} \cdots 1_{Fx_p} \cdots \gamma_q \cdots 1_{Fx_m} \\ &= F_{\mathtt{C}}(\overline{g}_1) \circ F_{\mathtt{C}}(g_1) && (8.3.25) \end{aligned}$$

This shows that the image of the top diamond commutes.

Next, we will show that the image of the left and right rectangle commutes. Let

$$w_p = x_1 \bullet \cdots \bullet u \bullet \cdots \bullet x_q \bullet \cdots \bullet x_m.$$

Since $f_1 : x \multimap\!\!\!\to w_p$ is a non-identity type-C forward arrow, from Proposition 8.3.19 we get $\hat{C}\left(w_p\right) < \hat{C}(x)$. We have two terminating chains of type-C forward arrows anchored at w_p that end in $\mathtt{C}$, namely

$$f_2, \ldots, f_n \qquad \text{and} \qquad \overline{f}_1, h_2, \ldots, h_t.$$

Using the induction hypothesis, we get

$$a = c$$

and

$$F_{\mathtt{C}}(f_n) \circ \cdots \circ F_{\mathtt{C}}(f_2) = F_{\mathtt{C}}(h_t) \circ \cdots \circ F_{\mathtt{C}}(h_2) \circ F_{\mathtt{C}}(\overline{f}_1).$$

Using the same argument, we get

$$b = c$$

and

$$F_{\mathtt{C}}(g_r)\circ\cdots\circ F_{\mathtt{C}}(g_2)=F_{\mathtt{C}}(h_t)\circ\cdots\circ F_{\mathtt{C}}(h_2)\circ F_{\mathtt{C}}(\overline{g}_1).$$

Thus, we get

$$a=b$$

and

$$\begin{aligned}F_{\mathtt{C}}(f_n)\circ\cdots\circ F_{\mathtt{C}}(f_1)&=F_{\mathtt{C}}(h_t)\circ\cdots\circ F_{\mathtt{C}}(h_2)\circ F_{\mathtt{C}}(\overline{f}_1)\circ F_{\mathtt{C}}(f_1)\\&=F_{\mathtt{C}}(h_t)\circ\cdots\circ F_{\mathtt{C}}(h_2)\circ F_{\mathtt{C}}(\overline{g}_1)\circ F_{\mathtt{C}}(g_1)\\&=F_{\mathtt{C}}(g_r)\circ\cdots\circ F_{\mathtt{C}}(g_1)\end{aligned}$$

as required. □

Definition 8.3.29. Let $x\in\mathcal{DMon}\langle S\rangle$. From Lemma 8.3.22 we get a terminating chain of type-C forward arrows

$$x\xrightarrow{f_1}x_1\xrightarrow{f_2}x_2\xrightarrow{f_3}\cdots\xrightarrow{f_n}x_n=y$$

such that $y\in\mathtt{C}$. From Theorem 8.3.26, we get that $y\in\mathtt{C}$ is uniquely determined by $x\in\mathcal{DMon}\langle S\rangle$. We call $y\in\mathtt{C}$ defined in this way the *totally distributed word*, denoted by

$$x_{\mathtt{C}}:=y. \tag{8.3.30}$$

We denote the composite arrow $\star_{x,x_{\mathtt{C}}}:x\longrightarrow x_{\mathtt{C}}$ by

$$\widehat{\gamma}_x:=\star_{x,x_{\mathtt{C}}}:x\longrightarrow x_{\mathtt{C}}. \tag{8.3.31}$$

It follows that

$$Q(x_{\mathtt{C}})=Q(x). \tag{8.3.32}$$

Moreover, we define a morphism $\gamma_x : F(x) \longrightarrow F(x_{\mathtt{C}})$, called the type-C morphism in $\mathscr{M}$ as follows:

$$\gamma_x := F_{\mathtt{C}}(f_n) \circ \cdots \circ F_{\mathtt{C}}(f_1). \tag{8.3.33}$$

From Theorem 8.3.26, the morphism γ_x is uniquely determined. $\diamond$

Example 8.3.34. Let $a, b, c, d, e \in \mathtt{S}$. Then, for $x = a \bullet (b \bullet (c \bullet d)' \bullet e)'$ we get

$$\begin{tikzcd}
a \bullet (b \bullet (c \bullet d)' \bullet e)' \arrow[d, "f_1"] \\
a \bullet e' \bullet (c \bullet d)'' \bullet b' \arrow[d, "f_2"] \\
a \bullet e' \bullet c'' \bullet d'' \bullet b'
\end{tikzcd}$$

where f_1 and f_2 are type-C forward arrows. Since the chain terminates at $a \bullet e' \bullet c'' \bullet d'' \bullet b' \in \mathtt{C}$, we get

$$x_{\mathtt{C}} = a \bullet e' \bullet c'' \bullet d'' \bullet b'.$$ $\diamond$

Example 8.3.35. Let $a, b, c, d, e \in \mathtt{S}$ and $x = ((a \bullet b)'' \bullet c)' \bullet (d \bullet e)'$. Consider following two chains of type-C forward arrows starting at x:

$$\begin{tikzcd}
((a \bullet b)'' \bullet c)' \bullet (d \bullet e)' \arrow[d, "f_1"] & ((a \bullet b)'' \bullet c)' \bullet (d \bullet e)' \arrow[d, "g_1"] \\
c' \bullet (a \bullet b)''' \bullet (d \bullet e)' \arrow[d, "f_2"] & ((a \bullet b)'' \bullet c)' \bullet e' \bullet d' \arrow[d, "g_2"] \\
c' \bullet b''' \bullet a''' \bullet (d \bullet e)' \arrow[d, "f_3"] & c' \bullet (a \bullet b)''' \bullet e' \bullet d' \arrow[d, "g_3"] \\
c' \bullet b''' \bullet a''' \bullet e' \bullet d' & c' \bullet b''' \bullet a''' \bullet e' \bullet d'
\end{tikzcd}$$

where each f_i and g_j for $i, j \in \{1,2,3\}$ is a type-C forward arrow. We get

$$x_{\mathtt{C}} = c' \bullet b''' \bullet a''' \bullet e' \bullet d'$$

via both chains. Moreover, Theorem 8.3.26 guaranties that

$$F_{\mathtt{C}}(f_3) \circ F_{\mathtt{C}}(f_2) \circ F_{\mathtt{C}}(f_1) = F_{\mathtt{C}}(g_3) \circ F_{\mathtt{C}}(g_2) \circ F_{\mathtt{C}}(g_1) \circ \qquad \diamond$$

Proposition 8.3.36. *Let $x \in \mathtt{C}$. Then, the equality*

$$x_{\mathtt{C}} = x \tag{8.3.37}$$

holds.

Proof. Since $x \in \mathtt{C}$, we can consider

$$1_x : x \multimap\!\!\rightarrow x$$

as a chain of type-C forward arrows terminating in $\mathtt{C}$. Thus, from Definition 8.3.29 we get

$$x_{\mathtt{C}} = x$$

as required. $\square$

Proposition 8.3.38. *Let $x, y \in \mathcal{DMon}\langle S \rangle$. Then, the equalities*

$$(x \bullet y)_{\mathtt{C}} = x_{\mathtt{C}} \bullet y_{\mathtt{C}} \tag{8.3.39}$$

and

$$\gamma_{x \bullet y} = \gamma_x \cdot \gamma_y \tag{8.3.40}$$

holds.

Proof. Let $x, y \in \mathcal{DMon}\langle S\rangle$. From Definition 8.3.29 we get terminating chains

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} x_n = x_{\mathtt{C}}$$

and

$$y \xrightarrow{g_1} y_1 \xrightarrow{g_2} y_2 \xrightarrow{g_3} \cdots \xrightarrow{g_m} y_m = y_{\mathtt{C}}$$

of type-C forward arrows. Consider the following chain:

$$x \bullet y \xrightarrow{f_1 \bullet 1_y} x_1 \bullet y \cdots \xrightarrow{f_n \bullet 1_y} x_{\mathtt{C}} \bullet y \xrightarrow{1_{x_{\mathtt{C}}} \bullet g_1} x_{\mathtt{C}} \bullet y_1 \cdots \xrightarrow{1_{x_{\mathtt{C}}} \bullet g_m} x_{\mathtt{C}} \bullet y_{\mathtt{C}}$$

Observe that each component arrow of the above diagram is a type-C forward arrow. Since $\mathtt{C}$ is a sub-monoid of $\mathcal{DMon}\langle S\rangle$, we get that the chain terminates at $x_{\mathtt{C}} \bullet y_{\mathtt{C}} \in \mathtt{C}$. Thus, from Definition 8.3.29 we get

$$(x \bullet y)_{\mathtt{C}} = x_{\mathtt{C}} \bullet y_{\mathtt{C}}$$

and

$$\begin{aligned}
\gamma_{x \bullet y} &= 1_{F(x_{\mathtt{C}})} F(g_m) \circ \cdots \circ 1_{F(x_{\mathtt{C}})} F(g_1) \circ 1_{f_n} 1_{F(y)} \circ \cdots \circ F(f_1) 1_{F(y)} && (8.3.33) \\
&= 1_{F(x_{\mathtt{C}})} \gamma_y \circ \gamma_x 1_{F(y)} && (8.3.33) \\
&= \gamma_x \cdot \gamma_y.
\end{aligned}$$

□

Proposition 8.3.41. *Let $x \in \mathcal{DMon}\langle S\rangle$ and $f : x \longrightarrow y$ be a type-C forward arrow. Then, the equality*

$$x_{\mathtt{C}} = y_{\mathtt{C}} \tag{8.3.42}$$

holds. Moreover, we get the equality

$$\gamma_x = \gamma_y \circ F_{\mathtt{C}}(f) \tag{8.3.43}$$

holds.

Proof. From Lemma 8.3.22, we get a terminating chain of type-C forward arrows:

$$y \xrightarrow{f_1} y_1 \xrightarrow{f_2} y_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} y_n = y_{\mathsf{C}}.$$

Pre-composing the above chain with the type-C forward arrow $f : x \longrightarrow y$ gives us a chain of type-C forward arrows anchored at x and terminating at y_{C}. From Definition 8.3.29, we get that

$$x_{\mathsf{C}} = y_{\mathsf{C}}$$

and

$$\begin{aligned} \gamma_x &= F_{\mathsf{C}}(f_n) \circ \cdots \circ F_{\mathsf{C}}(f_1) \circ F_{\mathsf{C}}(f). && (8.3.33) \\ &= \gamma_x \circ F_{\mathsf{C}}(f). && (8.3.33) \end{aligned}$$

□

8.4 Type-B Morphism

Building on Framework 8.3.1 from the previous section, we now define, for each $x \in \mathcal{DMon}\langle S \rangle$, an object $x_{\mathsf{B}} \in \mathsf{B}$. Moreover, we introduce the type-B morphism

$$\beta_x : F(x_{\mathsf{C}}) \longrightarrow F(x_{\mathsf{B}})$$

in $\mathcal{M}$.

Definition 8.4.1. For a natural number $k \in \mathbb{N}$ define

$$k\%2 := \begin{cases} 0 & \text{if } k \text{ is even} \\ 1 & \text{if } k \text{ is odd.} \end{cases} \tag{8.4.2}$$

◇

Remark 8.4.3. Let G be a group. For an element $a \in G$ and $k \in \mathbb{N}$, since $(a^{-1})^{-1} = a$, the equality

$$a^{(k)} = a^{(k\%2)}$$

holds. ◇

Definition 8.4.4. Let $x \in \mathcal{DMon}\langle S\rangle$. We have $x_{\mathtt{C}} \in \mathtt{C}$, and from Definition 8.2.1 we get $m \in \mathbb{N}$, $a_i \in \mathtt{S}$, and $k_i \in \mathbb{N}$ for $1 \le i \le m$ such that

$$x_{\mathtt{C}} = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)}.$$

Define $x_{\mathtt{B}} \in \mathtt{B}$ as

$$x_{\mathtt{B}} := a_1^{(k_1\%2)} \bullet \cdots \bullet a_m^{(k_m\%2)}. \tag{8.4.5}$$

Observe that

$$\begin{aligned} \mathcal{Q}(x_{\mathtt{C}}) &= \mathcal{Q}\left(a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)}\right) \\ &= a_1^{(k_1)} \cdots a_m^{(k_m)} \\ &= a_1^{(k_1\%2)} \cdots a_m^{(k_m\%2)} \\ &= \mathcal{Q}\left(a_1^{(k_1\%2)} \bullet \cdots \bullet a_m^{(k_m\%2)}\right) \\ &= \mathcal{Q}(x_{\mathtt{B}}). \end{aligned} \tag{8.4.6}$$

We denote the unique arrow from $x_{\mathtt{C}}$ to $x_{\mathtt{B}}$ as

$$\tilde{\beta}_x := \star_{x_{\mathtt{C}},x_{\mathtt{B}}} : x_{\mathtt{C}} \longrightarrow x_{\mathtt{B}}. \tag{8.4.7}$$

$\diamond$

Proposition 8.4.8. *Let $x \in \mathtt{B}$. Then, we the equality*

$$x_{\mathtt{B}} = x \tag{8.4.9}$$

holds.

Proof. Since $x \in \mathtt{B}$ we get

$$x = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)}$$

where $m \in \mathbb{N}$, $a_i \in \mathtt{S}$, and $k_i \in \{0,1\}$ for $1 \le i \le m$. We have $\mathtt{B} \subseteq \mathtt{C}$. Therefore, from Proposition 8.3.36 we get

$$x_{\mathtt{C}} = x = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)}.$$

Since $k_i \in \{0,1\}$ we get $k_i\%2 = k_i$. It follows that

$$x_{\mathtt{B}} = a_1^{(k_1\%2)} \bullet \cdots \bullet a_m^{(k_m\%2)} = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} = x. \qquad \square$$

Definition 8.4.10. Assume Framework 8.3.1. For an element $x \in \mathcal{DMon}\langle S\rangle$ define a morphism

$$\beta_x : F(x_{\mathtt{C}}) \longrightarrow F(x_{\mathtt{B}})$$

in $\mathcal{M}$ as follows:

Let $x \in \mathcal{DMon}\langle S\rangle$. Then, we get

$$x_{\mathtt{C}} = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)}$$

and

$$x_{\mathtt{B}} = a_1^{(k_1\%2)} \bullet \cdots \bullet a_m^{(k_m\%2)}$$

where $m \in \mathbb{N}$, $a_i \in \mathtt{S}$, and $k_i \in \mathbb{N}$ for $1 \le i \le m$. It follows that

$$F(x_{\mathtt{C}}) = F(a_1)^{(k_1)} \bullet \cdots \bullet F(a_m)^{(k_m)}$$

and

$$F(x_{\mathtt{B}}) = F(a_1)^{(k_1\%2)} \bullet \cdots \bullet F(a_m)^{(k_m\%2)}$$

Define

$$\beta_x := B_m^L(X) \tag{8.4.11}$$

where

$$L = (k_1, \ldots, k_m) \in \mathbb{N}^m, \qquad X = (F(a_1), \ldots, F(a_m)) \in M^m,$$

and B^L_m is a natural transformation as in Definition 5.3.10. ◇

Proposition 8.4.12. *Let $x, y \in \mathcal{DMon}\langle S\rangle$. Then, we have the equalities:*

$$(x \bullet y)_{\mathtt{B}} = x_{\mathtt{B}} \bullet y_{\mathtt{B}} \tag{8.4.13}$$

and

$$\beta_{x \bullet y} = \beta_x \cdot \beta_y. \tag{8.4.14}$$

Proof. Let $x, y \in \mathcal{DMon}\langle S\rangle$. Then we get

$$x_{\mathtt{C}} = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)}$$

and

$$y_{\mathtt{C}} = b_1^{(l_1)} \bullet \cdots \bullet b_n^{(l_n)}$$

where $m, n \in \mathbb{N}$, $a_i, b_j \in \mathtt{S}$, and $k_i, l_j \in \{0, 1\}$ for $1 \leq i \leq m$ and $1 \leq j \leq m$. From Proposition 8.3.38 we get

$$(x \bullet y)_{\mathtt{C}} = x_{\mathtt{C}} \bullet y_{\mathtt{C}} = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \bullet b_1^{(l_1)} \bullet \cdots \bullet b_n^{(l_n)}.$$

From Definition 8.4.4 we get

$$(x \bullet y)_{\mathtt{B}} = a_1^{(k_1 \% 2)} \bullet \cdots \bullet a_m^{(k_m \% 2)} \bullet b_1^{(l_1 \% 2)} \bullet \cdots \bullet b_n^{(l_n \% 2)} = x_{\mathtt{B}} \bullet y_{\mathtt{B}}.$$

Let

$$K = (k_1, \ldots, k_m), \qquad L = (l_1, \ldots, l_n), \quad \text{and} \quad K \bullet L = (k_1, \ldots, k_m, l_1, \ldots, l_n).$$

Also, let

$$X = (x_1, \ldots, x_m), \qquad Y = (y_1, \ldots, y_n), \quad \text{and} \quad X \bullet Y = (x_1, \ldots, x_m, y_1, \ldots, y_n).$$

Consider the following calculation:

$$
\begin{aligned}
\beta_{x\bullet y} &= B^{K\bullet L}_{m+n}(X \bullet Y) && (8.4.11)\\
&= B^{k_1}(x_1)\bullet\cdots\bullet B^{k_m}(x_m)\cdot B^{l_1}(y_1)\cdots B^{l_n}(y_n) && (5.3.11)\\
&= B^K_m(X)\cdot B^L_n(Y) && (5.3.11)\\
&= \beta_x\cdot\beta_y. && (8.4.11)
\end{aligned}
$$

□

Proposition 8.4.15. *Let $x, y \in \mathcal{DMon}\langle S\rangle$ such that $x_{\mathtt{C}} = y_{\mathtt{C}}$. Then, the equality*

$$x_{\mathtt{B}} = y_{\mathtt{B}} \tag{8.4.16}$$

holds. Moreover, we have the equality

$$\beta_x = \beta_y. \tag{8.4.17}$$

Proof. Since $x_{\mathtt{C}} \in \mathtt{C}$, we get

$$x_{\mathtt{C}} = a_1^{(k_1)} \bullet\cdots\bullet a_m^{(k_m)}$$

where $m \in \mathbb{N}$, $a_i \in \mathtt{S}$, and $k_i \in \mathbb{N}$ for $1 \le i \le m$. Since we have assumed that $x_{\mathtt{C}} = y_{\mathtt{C}}$, we get

$$y_{\mathtt{C}} = a_1^{(k_1)} \bullet\cdots\bullet a_m^{(k_m)}.$$

From Definition 8.4.10 we get

$$x_{\mathtt{B}} = y_{\mathtt{B}} = a_1^{(k_1\%2)} \bullet\cdots\bullet a_m^{(k_m\%2)}.$$

Moreover, we get

$$\beta_x = \beta_y = B^L_m(X)$$

where

$$L = (k_1, \ldots, k_m) \in \mathbb{N}^m, \qquad X = (F(a_1), \ldots, F(a_m)) \in M^m.$$

This completes the proof. □

Example 8.4.18. Let $a, b, c, d, e \in \mathtt{S}$. Then, for $x = a \bullet (b \bullet (c \bullet d)' \bullet e)'$ we get

$$x_{\mathtt{A}} = a \bullet e' \bullet c'' \bullet d'' \bullet b'$$

and

$$x_{\mathtt{B}} = a \bullet e' \bullet c \bullet d \bullet b'.$$ ◇

8.5 Type-A Morphism

Continuing from Section 8.4, we define, for each $x \in \mathcal{DMon}\langle S \rangle$, an element $x_{\mathtt{A}} \in \mathtt{A}$. We continue to assume Framework 8.3.1. We will construct the type-A morphism

$$\alpha_x : F(x_{\mathtt{B}}) \longrightarrow F(x_{\mathtt{A}})$$

in $\mathscr{M}$ as described in Definition 8.5.17. In Proposition 8.5.28, we will establish the key property that there exists an arrow $f : x \multimap y$ in $\mathcal{SSCatGrp}\langle S \rangle$ if and only if $x_{\mathtt{A}} = y_{\mathtt{A}}$. This result will be used in Section 9.1 to define a functor $F : \mathcal{SSCatGrp}\langle S \rangle \longrightarrow \mathscr{M}$.

Definition 8.5.1. The *simple cancellation pair* is an object of type $a \bullet a'$ or $a' \bullet a$ for some $a \in \mathtt{S}$. Let $a \in \mathtt{S}$. Then, we have

$$\begin{aligned} Q\left(a \bullet a'\right) &= aa^{-1} \\ &= e \\ &= Q(J) \end{aligned}$$

and

$$\begin{aligned} Q\left(a' \bullet a\right) &= a^{-1}a \\ &= e \\ &= Q(J). \end{aligned}$$

The *simple cancellation arrows* are the unique arrows $\star_{a\bullet a',J} : a \bullet a' \multimap J$ and $\star_{a'\bullet a,J} : a' \multimap a \longrightarrow J$. For $x \in \mathtt{B}$, an arrow $f : x \multimap y$ is called a *type-A forward arrow* if f is an expanded instance of a simple cancellation arrow. $\diamond$

Definition 8.5.2. Let $x \in \mathtt{B}$. We say that x is *irreducible with respect to type-A forward arrows* if for any type-A forward arrow $f : x \multimap y$ with x as the source object satisfies $f = 1_x$. Conversely, x is said to be *reducible* with respect to type-A forward arrows if there exist a non-identity type-A forward arrow $f : x \multimap y$ with source object equal to a. $\diamond$

Example 8.5.3. Let $a, b, c \in \mathtt{S}$ and $x = a \bullet a' \bullet c \bullet b' \bullet b \in \mathtt{B}$. Then,

$$f := \widetilde{\eta}_a^{-1} \bullet 1_{c\bullet b'\bullet b\bullet} : a \bullet a' \bullet c \bullet b' \bullet b \multimap c \bullet b' \bullet b$$

is a type-A forward arrow. Similarly,

$$g := 1_c \bullet \widetilde{\epsilon}_b : c \bullet b' \bullet b \multimap c$$

is a type-A forward arrow. $\diamond$

Proposition 8.5.4. *Let $x \in \mathtt{B}$. Then, x is irreducible with respect to the type-A forward arrows if and only if $x \in \mathtt{A}$.*

Proof. Let

$$x = a_1^{(k_1)} \bullet \cdots \bullet x_m^{(k_m)}$$

where $m \in \mathbb{N}$, $a_i \in \mathtt{S}$, $k_i \in 0,1$ for $1 \le i \le m$. Suppose, x is irreducible with respectively to the type-A forward arrows. Since, non-identity type-A forward arrows are the expanded instances of simple-cancellation arrow, there must not exist a simple cancellation pair, $a \bullet a'$ or $a' \bullet a$, as a part of x. Thus, we get

$$a_i = a_{i+1} \quad \Longrightarrow \quad k_i = k_{i+1}.$$

That is, $x \in \mathtt{A}$.

On the other hand, suppose $x \in \mathtt{A}$. That is, we have

$$a_i = a_{i+1} \quad \Longrightarrow \quad k_i = k_{i+1}.$$

Then, there is no simple cancellation pair, $a \bullet a'$ or $a' \bullet a$, as a part of x. This implies that any type-A forward arrow with source x, must be 1_x. Thus, we get that x is irreducible with respect to the type-A forward arrows. $\square$

Proposition 8.5.5. *Let $f : x \multimap\!\!\rightarrow y$ be a non-identity type-A forward arrow. Then,*

$$\hat{A}(y) < \hat{A}(x). \tag{8.5.6}$$

Proof. Let

$$x = a_1^{(k_1)} \bullet \cdots \bullet x_m^{(k_m)} \in \mathtt{B}$$

where $m \in \mathbb{N}$, $a_i \in \mathtt{S}$, $k_i \in 0,1$ for $1 \le i \le m$.

Since the non-identity type-A forward arrows are expanded instances of simple cancellation arrows, x must contain a simple cancellation pair, $a \bullet a'$ or $a' \bullet a$. Suppose $a_j^{(k_j)} \bullet a_{j+1}^{(k_{j+1})}$ is that pair. Thus, we get

$$y = a_1^{(k_1)} \bullet \cdots \bullet a_{j-1}^{(k_{j-1})} \bullet a_{j+2}^{(k_{j+2})} \bullet \cdots \bullet x_m^{(k_m)}.$$

Therefore, f reduces one simple cancellation pair. It follows that

$$\hat{A}(y) < \hat{A}(x).$$ $\square$

Example 8.5.7. Let $a, b, c \in \mathtt{S}$ be distinct objects and $x = a \bullet a' \bullet c \bullet b' \bullet b \in \mathtt{B}$. Then,

$$\hat{A}(x) = 2$$

since there are two cancellation pairs, $a \bullet a'$ and $b' \bullet b$. Similarly, for $y = a \bullet a' \bullet a \bullet b'$ we get

$$\hat{A}(y) = 2$$

since there are two cancellation pairs, $a \bullet a'$ and $a' \bullet a$. Even though the element a' is common in both pairs, we will count these are separate pairs. ◇

Definition 8.5.8. Let $x \in \mathcal{DMon}\langle S \rangle$. A *chain of type-C forward arrows anchored at x* is a chain

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots$$

anchored at x such that for all $i \geq 1$, f_i is a type-C forward arrow. We say that a chain *terminates* at $y \in \mathcal{DMon}\langle S \rangle$ if there is $N \in \mathbb{N}$ such that for all $n > N$, we have $f_n = 1_y$. We will denote such a terminating chain by

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_N} y.$$ ◇

Lemma 8.5.9. *Let $x \in \mathcal{DMon}\langle S \rangle$. Then, there exists a terminating chain*

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_N} y.$$

of type-A forward arrows anchored at x such that $y \in \mathtt{A}$.

Proof. We will prove this using induction on $\hat{A}(x)$.

Base case $(\hat{A}(x) = 0)$: From Proposition 8.2.25 we get $x \in \mathtt{A}$. Thus, we can take $f_i = 1_x$ for every $i \geq 1$.

Induction case $(\hat{A}(x) \geq 1)$: From Proposition 8.2.25 we get $x \notin \mathtt{A}$. From Proposition 8.5.4 we get that x is reducible with respect to type-A forward arrow. Thus, we get $y \in \mathtt{B}$ and $f_1 : x \longrightarrow y$ such that f_1 is a non-identity type-A forward arrow. From

Proposition 8.5.5 we get that $\hat{A}(y) < \hat{A}(x)$. Thus, we get a chain of type-A forward arrows

$$x_1 \xrightarrow[\circ]{f_2} x_2 \xrightarrow[\circ]{f_3} \cdots$$

that terminates at $y \in \mathtt{A}$. Pre-composing with $f_1 : x \multimap x_1$, we get our required chain. $\square$

Definition 8.5.10. Assume Framework 8.3.1. Let

$$\mathcal{FA}_{\mathtt{A}} := \left\{ f \in \mathrm{Mor}\left(\mathcal{SSCatGrp}\langle S\rangle\right) \mid f \text{ is tpye-A forward arrow} \right\}$$

be the set of type-A forward arrows. Define a mapping

$$F_{\mathtt{A}} : \mathcal{FA}_{\mathtt{A}} \longrightarrow \mathrm{Mor}(\mathscr{M})$$

as follows:

The identity arrows are mapped to the corresponding identity morphisms in $\mathscr{M}$. That is,

$$F_{\mathtt{A}}(1_x) = 1_{(Fx)}. \tag{8.5.11}$$

For $a \in \mathtt{S}$, let $\widetilde{\eta}_a^{-1} : a \bullet a' \longrightarrow J$ and $\widetilde{\epsilon}_a a' \bullet a \longrightarrow J$ be simple cancellation arrows. We define

$$F_{\mathtt{A}}(\widetilde{\epsilon}_a) := \epsilon_{F(a)} \tag{8.5.12}$$

$$F_{\mathtt{A}}(\widetilde{\eta}_a) := \eta_{F(a)}. \tag{8.5.13}$$

Since type-A forward arrows are expanded instances of simple cancellation arrows, $F_{\mathtt{A}}$ maps type-A forward arrows to the corresponding expanded instances of cancellation morphisms in $\mathscr{M}$. That is,

$$F_{\mathtt{C}}(1_x \bullet \widetilde{\alpha}_a \bullet 1_y) = 1_{F(x)}\alpha_{F(a)}1_{F(c)}$$

where $\widetilde{\alpha}_a$ is a simple cancellation arrow. $\diamond$

Theorem 8.5.14. *Let* $x \in \mathtt{B}$*. Then, any two terminating chains,*

$$x \xrightarrow{f_1} y_1 \xrightarrow{f_2} y_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} y_n = y$$

and

$$x \xrightarrow{g_1} z_1 \xrightarrow{g_2} z_2 \xrightarrow{g_3} \cdots \xrightarrow{g_r} z_r = z$$

of type-A forward arrows anchored at x *such that* $y, z \in \mathtt{A}$*, we get*

$$y = z. \tag{8.5.15}$$

Moreover, we get

$$F_{\mathtt{A}}(f_n) \circ \cdots \circ F_{\mathtt{A}}(f_1) = F_{\mathtt{A}}(g_r) \circ \cdots \circ F_{\mathtt{A}}(g_1). \tag{8.5.16}$$

Proof. We will show this using induction on $\hat{A}(x)$.

Base case $\big(\hat{A}(x) = 0\big)$: From Proposition 8.2.25 we get $x \in \mathtt{A}$. From Proposition 8.5.4 we get that x is irreducible with respect to type-A forward arrows. Thus, we get $y = x = z$ and $f_i = g_i = 1_x$ for $i \geq 1$.

Induction case $\big(\hat{A}(x) \geq 1\big)$: Since $\hat{C}(x) \geq 1$, we get that $x \notin \mathtt{A}$. It follows that at least one f_i for $1 \leq i \leq n$ and g_j for $1 \leq j \leq r$ must be non-identity. Now, rewrite the chains f_i and g_j such that all composition arrows are non identity.

Let

$$x = a_1^{(k_1)} \bullet \cdots \bullet x_m^{(k_m)} \in \mathtt{B}$$

where $m \in \mathbb{N}$, $a_i \in \mathtt{S}$, $k_i \in 0,1$ for $1 \leq i \leq m$. Consider $f_1 : x \longrightarrow y_1$ and $g_1 : x \longrightarrow z_1$. Since f_1 and g_1 non-identity arrows, there exists a simple cancellation pair

$$x_p := a_p^{(k_p)} \bullet a_{p+1}^{(k_{p+1})}$$

for some $1 \le p \le m$ and a simple cancellation arrow

$$\widehat{\gamma}_p : x_p \longrightarrow J$$

such that

$$f_1 = \langle \widehat{\gamma}_{a_p} \rangle : a_1^{(k_1)} \bullet \cdots \bullet x_p \bullet \cdots \bullet a_m^{(k_m)} \multimap a_1^{(k_1)} \bullet \cdots \bullet J \bullet \cdots \bullet a_m^{(k_m)}.$$

Similarly, there exists a simple cancellation pair

$$x_q := a_q^{(k_q)} \bullet a_{q+1}^{(k_{q+1})}$$

for some $1 \le q \le m$ and a simple cancellation arrow

$$\widehat{\gamma}_q : x_q \longrightarrow J$$

such that

$$f_1 = \langle \widehat{\gamma}_{a_q} \rangle : a_1^{(k_1)} \bullet \cdots \bullet x_q \bullet \cdots \bullet a_m^{(k_m)} \multimap a_1^{(k_1)} \bullet \cdots \bullet J \bullet \cdots \bullet a_m^{(k_m)}.$$

If we have $p = q$ then we get $\gamma_p = \gamma_q$. Consequently, we get $f_1 = g_1$ and $x_1 = y_1$. In this case, we considering the chains starting at $x_1 = y_1$. Since f_1 is a non-identity type-A forward arrow, from Proposition 8.5.5 we get $\hat{A}(x_1) < \hat{A}(x)$. We conclude the theorem by induction hypothesis.

Now, suppose $p < q$. We will consider following two cases:

<u>Case I</u> ($q = p + 1$): From the cancellation pair

$$a_p^{(k_p)} \bullet a_{p+1}^{(k_{p+1})}$$

we get $a_p = a_{p+1}$ and $k_p \neq k_{p+1}$. Similarly, from the cancellation pair

$$a_{p+1}^{(k_{p+1})} \bullet a_{p+2}^{(k_{p+2})}$$

we get $a_{p+1} = a_{p+2}$ and $k_{p+1} \neq k_{p+2}$. Let's denote

$$a := a_p = a_{p+1} = a_{p+2}.$$

Suppose $k_p = 0$. Then, we have $k_{p+1} = 1$ and $k_{p+2} = 0$. In this case, we get

$$x = x_1^{(k_1)} \bullet \cdots \bullet a_{p-1}^{(k_{p-1})} \bullet \quad a \bullet a' \bullet a \quad \bullet a_{p+3}^{(k_{p+3})} \bullet \cdots \bullet a_m^{(k_m)}$$

and

$$\begin{aligned} y_1 &= a_1^{(k_1)} \bullet \cdots \bullet a_{p-1}^{(k_{p-1})} \bullet a \bullet a_{p+3}^{(k_{p+3})} \bullet \cdots \bullet a_m^{(k_m)} \\ &= z_1. \end{aligned}$$

Since $f_1 : x \longrightarrow y_1$ is a non-identity type-A forward arrow, we get $\hat{A}(y_1) < \hat{A}(x)$. Thus, from the induction hypothesis, we get $y = z$.

Now we will show that $F_{\mathtt{A}}(f_1) = F_{\mathtt{A}}(g_1)$. The arrows f_1 and g_1 are expanded instances of $\widehat{\gamma}_p$ and $\widehat{\gamma}_{p+1}$. Since $F_{\mathtt{A}}$ maps expanded instances to the corresponding expanded instances, it is enough to show

$$F_{\mathtt{A}}(\widehat{\gamma}_p)1_{F(a)} = 1_{F(a)}F_{\mathtt{A}}(\widehat{\gamma}_{p+1}) : F(a)F(a)'F(a) \longrightarrow F(a).$$

Observe that

$$\begin{aligned} \widehat{\gamma}_p &= \widehat{\eta}_a^{-1} : a \bullet a' \longrightarrow J \\ \widehat{\gamma}_{p+1} &= \widehat{\epsilon}_a : a' \bullet a \longrightarrow J. \end{aligned}$$

Therefore, we have

$$\begin{aligned} F_{\mathtt{A}}(\widehat{\eta}_a^{-1})1_{F(a)} &= \eta_{F(a)}^{-1}1_{F(a)} && (8.5.13) \\ &= 1_{F(a)}\epsilon_{F(a)} && (4.1.4) \\ &= 1_{F(a)}F_{\mathtt{A}}(\widehat{\epsilon}_a) && (8.5.12) \end{aligned}$$

as required. The case when $k_1 = 1$ is similar: In this case, we get

$$x = x_1^{(k_1)} \bullet \cdots \bullet a_{p-1}^{(k_{p-1})} \bullet \quad a' \bullet a \bullet a' \quad \bullet a_{p+3}^{(k_{p+3})} \bullet \cdots \bullet a_m^{(k_m)}$$

and

$$\begin{aligned} y_1 &= a_1^{(k_1)} \bullet \cdots \bullet a_{p-1}^{(k_{p-1})} \bullet a \bullet a_{p+3}^{(k_{p+3})} \bullet \cdots \bullet a_m^{(k_m)} \\ &= z_1. \end{aligned}$$

Moreover, we get

$$\begin{aligned} F_{\mathtt{A}}(\widehat{\gamma}_p) 1_{F(a)'} &= F_{\mathtt{A}}(\widehat{\epsilon}_a) 1_{F(a)'} \\ &= \epsilon_{F(a)} 1_{F(a)'} && (8.5.12) \\ &= 1_{F(a)'} \eta_{F(a)}^{-1} && (4.1.5) \\ &= 1_{F(a)'} F_{\mathtt{A}}(\widehat{\eta}_a^{-1}) && (8.5.12) \\ &= 1_{F(a)'} F_{\mathtt{A}}(\widehat{\gamma}_{p+1}) \end{aligned}$$

as required.

Case II $(p < q-1)$: That is, f_1 and g_1 act on separate parts of x. Consider the following diagram:

$$\begin{tikzcd}
 & a_1^{(k_1)} \bullet \cdots \bullet x_p \bullet \cdots \bullet x_q \bullet \cdots \bullet a_m^{(k_m)} \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\
a_1^{(k_1)} \bullet \cdots \bullet J \bullet \cdots \bullet x_q \bullet \cdots \bullet a_m^{(k_m)} \arrow[dr, "\overline{f_1}"] \arrow[dd, "f_2"'] & & a_1^{(k_1)} \bullet \cdots \bullet x_p \bullet \cdots \bullet J \bullet \cdots \bullet a_m^{(k_m)} \arrow[dl, "\overline{g_1}"'] \arrow[dd, "g_2"] \\
 & a_1^{(k_1)} \bullet \cdots \bullet J \bullet \cdots \bullet J \bullet \cdots \bullet a_m^{(k_m)} \arrow[d, "h_2"] & \\
\vdots \arrow[d, "f_n"'] & \vdots \arrow[d, "h_t"] & \vdots \arrow[d, "g_r"] \\
y \arrow[r, equal] & w \arrow[r, equal] & z
\end{tikzcd}$$

where

$$x_p = a_p^{(k_p)} \bullet a_{p+1}^{(k_{p+1})}, \qquad x_q = a_q^{(k_q)} \bullet a_{q+1}^{(k_{q+1})}$$

and

$$\overline{f}_1 := \langle \gamma_q \rangle \qquad \text{and} \qquad \overline{g}_1 := \langle \gamma_p \rangle$$

are type-A forward arrows. The element

$$a_1^{(k_1)} \bullet \cdots \bullet J \bullet \cdots \bullet J \bullet \cdots \bullet a_m^{(k_m)}$$

is a member of $\mathtt{B}$, therefore from Lemma 8.5.9 we get a chain $h_2, \ldots, h_t$ of type-A forward arrows terminating in $w \in \mathtt{A}$.

Consider

$$u_p := a_1^{(k_1)} \bullet \cdots \bullet J \bullet \cdots \bullet x_q \bullet \cdots \bullet a_m^{(k_m)}.$$

Since $f_1 : x \longrightarrow u_p$ is a non-identity type-A forward arrow, from Proposition 8.3.19 we get $\hat{A}\left(u_p\right) < \hat{A}(x)$. We have two chains of type-A forward arrows, namely $f_2, \ldots, f_n$

and $\overline{f}_1, h_2, \ldots, h_t$, anchored at u_p and terminating in $y, w \in \mathtt{A}$ respectively. Using induction hypothesis, we get

$$y = w$$

and

$$F_{\mathtt{A}}(f_n) \circ \cdots \circ F_{\mathtt{A}}(f_2) = F_{\mathtt{A}}(h_t) \circ \cdots \circ F_{\mathtt{A}}(h_2) \circ F_{\mathtt{A}}(\overline{f}_1).$$

Using a similar argument, we get

$$w = z$$

and

$$F_{\mathtt{A}}(g_r) \circ \cdots \circ F_{\mathtt{A}}(g_2) = F_{\mathtt{A}}(h_t) \circ \cdots \circ F_{\mathtt{A}}(h_2) \circ F_{\mathtt{A}}(\overline{g}_1).$$

Thus, we get

$$y = z$$

and

$$\begin{aligned} F_{\mathtt{A}}(f_n) \circ \cdots \circ F_{\mathtt{A}}(f_1) &= F_{\mathtt{A}}(h_t) \circ \cdots \circ F_{\mathtt{A}}(h_2) \circ F_{\mathtt{A}}(\overline{f}_1) \circ F_{\mathtt{A}}(f_1) \\ &= F_{\mathtt{A}}(h_t) \circ \cdots \circ F_{\mathtt{A}}(h_2) \circ F_{\mathtt{A}}(\overline{g}_1) \circ F_{\mathtt{A}}(g_1) \\ &= F_{\mathtt{A}}(g_r) \circ \cdots \circ F_{\mathtt{A}}(g_1) \end{aligned}$$

as required. □

Definition 8.5.17. Let $x \in \mathcal{DMon}\langle S \rangle$, from Definition 8.4.4 we get $x_{\mathtt{B}} \in \mathtt{B}$. From Lemma 8.5.9 we get a terminating chain of type-A forward arrows,

$$x_{\mathtt{B}} \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} x_n = y$$

such that $y \in \mathtt{A}$. From Theorem 8.5.14, we get that $y \in \mathtt{A}$ is uniquely determined by $x_{\mathtt{B}} \in \mathtt{B}$ and hence y is uniquely determined by $x \in \mathcal{DMon}\langle S \rangle$. We call $y \in \mathtt{A}$ defined in this way the *totally simplified word*, denoted by

$$x_{\mathtt{A}} := y. \tag{8.5.18}$$

We denote the composite arrow $\star_{x_{\mathtt{B}},x_{\mathtt{A}}} : x_{\mathtt{B}} \multimap x_{\mathtt{A}}$ by

$$\widehat{\alpha}_x := \star_{x_{\mathtt{B}},x_{\mathtt{A}}} : x_{\mathtt{B}} \longrightarrow x_{\mathtt{A}}. \tag{8.5.19}$$

It follows that

$$\mathcal{Q}(x_{\mathtt{A}}) = \mathcal{Q}(x_{\mathtt{B}}). \tag{8.5.20}$$

Moreover, we define a morphism $\alpha_x : F(x_{\mathtt{B}}) \longrightarrow F(x_{\mathtt{A}})$ in $\mathscr{M}$ as follows:

$$\alpha_x := F_{\mathtt{A}}(f_n) \circ \cdots \circ F_{\mathtt{A}}(f_1). \tag{8.5.21}$$

From Theorem 8.5.14, the morphism α_x is uniquely determined by $x \in \mathcal{DMon}\langle S \rangle$. ⋄

Proposition 8.5.22. *Let $x \in \mathcal{DMon}\langle S \rangle$. Then, the equality*

$$\mathcal{Q}(x_{\mathtt{A}}) = \mathcal{Q}(x) \tag{8.5.23}$$

holds.

Proof. Observe that

$$\begin{aligned} \mathcal{Q}(x_{\mathtt{A}}) &= \mathcal{Q}(x_{\mathtt{B}}) && (8.5.20) \\ &= \mathcal{Q}(x_{\mathtt{C}}) && (8.4.6) \\ &= \mathcal{Q}(x). && (8.3.32) \end{aligned}$$

□

Proposition 8.5.24. *Let $x, y \in \mathcal{DMon}\langle S \rangle$. Then, $\mathcal{Q}(x) = \mathcal{Q}(y)$ if and only if $x_{\mathtt{A}} = y_{\mathtt{A}}$.*

Proof. Suppose $\mathcal{Q}(x) = \mathcal{Q}(y)$. We get

$$\begin{aligned} \mathcal{Q}(x_{\mathtt{A}}) &= \mathcal{Q}(x) && (8.5.23) \\ &= \mathcal{Q}(y) && \text{assumption} \\ &= \mathcal{Q}(y_{\mathtt{A}}). && (8.5.23) \end{aligned}$$

From Proposition 8.2.22 we get that $\mathcal{Q}$ is a bijection between $\mathtt{A}$ and $\mathcal{Grp}\langle S\rangle$. Since $x_{\mathtt{A}}, y_{\mathtt{A}} \in \mathtt{A}$, we conclude that $x_{\mathtt{A}} = y_{\mathtt{A}}$.

Conversely, suppose $x_{\mathtt{A}} = y_{\mathtt{A}}$. We get

$$\begin{aligned} \mathcal{Q}(x) &= \mathcal{Q}(x_{\mathtt{A}}) && (8.5.23) \\ &= \mathcal{Q}(y_{\mathtt{A}}) && \text{assumption} \\ &= \mathcal{Q}(y). && (8.5.23) \end{aligned}$$

□

Proposition 8.5.25. *Let $x, y \in \mathcal{DMon}\langle S\rangle$. Then, the equality*

$$(x \bullet y)_{\mathtt{A}} = (x_{\mathtt{A}} \bullet y_{\mathtt{A}})_{\mathtt{A}}. \tag{8.5.26}$$

Moreover, the morphism $\alpha_{x\bullet y} : (x \bullet y)_{\mathtt{B}} \longrightarrow (x \bullet y)_{\mathtt{A}}$ is given by the following composition:

$$\alpha_{x\bullet y} = \alpha_{x_{\mathtt{A}}\bullet y_{\mathtt{A}}} \circ \left(\alpha_x \cdot \alpha_y\right). \tag{8.5.27}$$

Note that, in general we get

$$(x \bullet y)_{\mathtt{A}} \neq x_{\mathtt{A}} \bullet y_{\mathtt{A}}.$$

Proof. Let $x, y \in \mathcal{DMon}\langle S\rangle$. Consider the following calculation:

$$\begin{aligned}
\mathcal{Q}((x \bullet y)_{\mathtt{A}}) &= \mathcal{Q}(x \bullet y) && (8.5.23)\\
&= \mathcal{Q}(x)\,\mathcal{Q}(y) && (4.5.12)\\
&= \mathcal{Q}(x_{\mathtt{A}})\,\mathcal{Q}(y_{\mathtt{A}}) && (8.5.23)\\
&= \mathcal{Q}(x_{\mathtt{A}} \bullet y_{\mathtt{A}}) && (4.5.12)\\
&= \mathcal{Q}\big((x_{\mathtt{A}} \bullet y_{\mathtt{A}})_{\mathtt{A}}\big). && (8.5.23)
\end{aligned}$$

Since both $(x \bullet y)_{\mathtt{A}}$ and $(x_{\mathtt{A}} \bullet y_{\mathtt{A}})_{\mathtt{A}}$ are in $\mathtt{A}$ and $\mathcal{Q}$ is a bijection between $\mathtt{A}$ and $\mathcal{Grp}\langle S\rangle$, we get

$$(x \bullet y)_{\mathtt{A}} = (x_{\mathtt{A}} \bullet y_{\mathtt{A}})_{\mathtt{A}}.$$

This shows that the equality (8.5.26) holds.

For the second part, from Definition 8.5.17 we get terminating chains

$$x_{\mathtt{B}} \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} x_n = x_{\mathtt{A}}$$

and

$$y_{\mathtt{B}} \xrightarrow{g_1} y_1 \xrightarrow{g_2} y_2 \xrightarrow{g_3} \cdots \xrightarrow{g_m} y_m = y_{\mathtt{A}}$$

of type-A forward arrows. Consider the following chain:

$$x_{\mathtt{B}} \bullet y_{\mathtt{B}} \xrightarrow{f_1 \bullet 1_{y_{\mathtt{B}}}} x_1 \bullet y_{\mathtt{B}} \cdots \xrightarrow{f_n \bullet 1_{y_{\mathtt{B}}}} x_{\mathtt{A}} \bullet y_{\mathtt{B}} \xrightarrow{1_{x_{\mathtt{A}}} \bullet g_1} x_{\mathtt{A}} \bullet y_1 \cdots \xrightarrow{1_{x_{\mathtt{A}}} \bullet g_m} x_{\mathtt{A}} \bullet y_{\mathtt{A}}$$

From Proposition 8.4.12 we have $x_{\mathtt{B}} \bullet y_{\mathtt{B}} = (x \bullet y)_{\mathtt{B}}$. Recall that type-A set is not closed under multiplication. Therefore the above chain need not terminate in $\mathtt{A}$. Since $\mathtt{A} \subseteq \mathtt{B}$ and $\mathtt{B}$ is closed under multiplication, we have

$$x_{\mathtt{A}} \bullet y_{\mathtt{A}} \in \mathtt{B}.$$

From Proposition 8.4.8 we get

$$(x_{\mathtt{A}} \bullet y_{\mathtt{A}})_{\mathrm{B}} = x_{\mathtt{A}} \bullet y_{\mathtt{A}}.$$

Now from *Definition* 8.5.17, we get a terminating chain

$$x_{\mathtt{A}} \bullet y_{\mathtt{A}} \xrightarrow{h_1} z_1 \xrightarrow{h_2} z_2 \xrightarrow{h_3} \cdots \xrightarrow{h_p} z_p = (x_{\mathtt{A}} \bullet y_{\mathtt{A}})_{\mathtt{A}}$$

of type-A forward arrows anchored at

$$(x_{\mathtt{A}} \bullet y_{\mathtt{A}})_{\mathrm{B}} = x_{\mathtt{A}} \bullet y_{\mathtt{A}}$$

and terminating in

$$(x_{\mathtt{A}} \bullet y_{\mathtt{A}})_{\mathtt{A}} = (x \bullet y)_{\mathtt{A}} \in \mathtt{A}.$$

Composing the above two chains, we get a chain of type-A forward arrows anchored at $(x \bullet y)_{\mathrm{B}}$ and terminating at $(x \bullet y)_{\mathtt{A}} \in \mathtt{A}$. Therefore, we get

$$\begin{aligned}
\alpha_{x \bullet y} = F_{\mathtt{A}}(h_p) \circ \cdots \circ F_{\mathtt{A}}(h_1) \circ 1_{F(x_{\mathtt{A}})} F_{\mathtt{A}}(g_m) \circ \cdots \circ 1_{F(x_{\mathtt{A}})} F_{\mathtt{A}}(g_1) & \\
\circ F_{\mathtt{A}}(f_n) 1_{F(y_{\mathrm{B}})} \circ \cdots \circ F_{\mathtt{A}}(f_1) 1_{F(y_{\mathrm{B}})} & \qquad (8.5.21) \\
= \alpha_{x_{\mathtt{A}} \bullet y_{\mathtt{A}}} \circ 1_{F(x_{\mathtt{A}})} \alpha_y \circ \alpha_x 1_{F(y_{\mathrm{B}})} & \qquad (8.5.27) \\
= \alpha_{x_{\mathtt{A}} \bullet y_{\mathtt{A}}} \circ (\alpha_x \cdot \alpha_y). & \qquad \square
\end{aligned}$$

Proposition 8.5.28. *Let $x, y \in \mathcal{DM}on\langle S \rangle$ such that $x_{\mathrm{B}} = y_{\mathrm{B}}$. Then, we get*

$$x_{\mathtt{A}} = y_{\mathtt{A}}. \qquad (8.5.29)$$

Moreover, the equality

$$\alpha_x = \alpha_y \qquad (8.5.30)$$

holds.

Proof. Since $x_{\mathtt{B}} \in \mathtt{B}$, we get

$$x_{\mathtt{B}} = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)}$$

where $m \in \mathbb{N}$, $a_i \in \mathtt{S}$, and $k_i \in \{0,1\}$ for $1 \le i \le m$. From Lemma 8.5.9 we get a terminating chain of type-A forward arrows

$$x_{\mathtt{B}} \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} x_n = x_{\mathtt{A}}$$

Since we have assumed that $x_{\mathtt{B}} = y_{\mathtt{B}}$, the terminating chain for x also works for y. Therefore, from Definition 8.5.17 we get

$$x_{\mathtt{A}} = y_{\mathtt{A}}$$

and

$$\alpha_x = \alpha_y. \qquad \square$$

Chapter 9: Coherence Theorems for Categorical Groups: Proofs

In this chapter, we establish that the construction of the semi-strict categorical group $\mathcal{SSCatGrp}\langle S\rangle$ given in Definition 8.1.2 satisfies the universal property of the free semi-strict categorical group (see Definition 4.4.7). This result enables us to formally prove the coherence theorems stated in Section 4.5.

9.1 Construction of the Induced Categorical Group Functor

In this section, we focus on constructing the induced categorical group functor. The following framework builds upon Framework 8.3.1, which we restate here for clarity.

Framework 9.1.1. Throughout this section, let S be a set. The following constructions and results will be used:

- Let $\mathcal{DMon}\langle S\rangle$ be the free dashed-monoid generated by S, as in Construction 4.5.2. The universal property is established in Theorem 7.8.2.
- Let $\mathcal{Grp}\langle S\rangle$ be the free group generated by S, as in Construction 4.5.4.
- Let $Q_{\mathcal{DMon}} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathcal{Grp}\langle S\rangle$ be the dashed-monoid homomorphism, as in Construction 4.5.5; we will abbreviate this as Q.

- By Theorem 7.8.1, $S \subseteq \mathcal{DMon}\langle S\rangle$ forms a dashed monoid basis.

- Theorem 7.8.8 asserts that $\mathcal{DMon}\langle S\rangle$ is also a free monoid and characterizes its monoid basis $\mathtt{G}$.

Let $\mathcal{SSCatGrp}\langle S\rangle$ be the free semi-strict categorical group generated by S as defined in Definition 8.1.2. Let $\mathscr{M}$ be a semi-strict categorical group with $M := \mathrm{Obj}(\mathscr{M})$. Recall from Example 7.1.6 that M is a dashed monoid. Let $u : S \longrightarrow M$ be a function, and let

$$F : \mathcal{DMon}\langle S\rangle \longrightarrow M$$

be the unique induced dashed monoid homomorphism as in Definition 7.8.3.

Furthermore, for each $x \in \mathcal{DMon}\langle S\rangle$, the morphisms

$$F(x) \xrightarrow{\gamma_x} F(x_{\mathtt{C}}) \xrightarrow{\beta_x} F(x_{\mathtt{B}}) \xrightarrow{\alpha_x} F(x_{\mathtt{A}})$$

in $\mathscr{M}$ be as in Definitions 8.3.29, 8.4.10, and 8.5.17, respectively. $\diamond$

Definition 9.1.2. Define a functor

$$F : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathscr{M}$$

as follows:

- From Definition 8.1.2 we have that

$$\mathrm{Obj}\left(\mathcal{SSCatGrp}\langle S\rangle\right) = \mathcal{DMon}\langle S\rangle\,.$$

  We define the functor $F : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathscr{M}$ on objects to be the induced dashed monoid homomorphism

$$F : \mathcal{DMon}\langle S\rangle \longrightarrow M$$

  as in Framework 9.1.1.

- Let $f : x \multimap y$ be an arrow in $\mathcal{SSCatGrp}\langle S\rangle$. We get $\mathcal{Q}(x) = \mathcal{Q}(y)$. From Proposition 8.5.24, we get $x_{\mathtt{A}} = y_{\mathtt{A}}$. We define the morphism

$$F(f) : F(x) \longrightarrow F(y)$$

as the following composition of morphisms in $\mathcal{M}$:

$$F(f) := \gamma_y^{-1} \circ \beta_y^{-1} \circ \alpha_y^{-1} \circ \alpha_x \circ \beta_x \circ \gamma_x. \tag{9.1.3}$$

This is described in the following diagram:

$$\begin{tikzcd}
F(x) \arrow[r, "F(f)"] \arrow[d, "\gamma_x"'] & F(y) \arrow[d, "\gamma_y"] \\
F(x_{\mathtt{C}}) \arrow[d, "\beta_x"'] & F(y_{\mathtt{C}}) \arrow[d, "\beta_y"] \\
F(x_{\mathtt{B}}) \arrow[d, "\alpha_x"'] & F(y_{\mathtt{B}}) \arrow[d, "\alpha_y"] \\
F(x_{\mathtt{A}}) \arrow[r, equal] & F(y_{\mathtt{A}})
\end{tikzcd}$$

Now, we will show that the above assignement satisfies the identity condition and the composition condition: Observe that for $x \in \mathcal{DMon}\langle S\rangle$ we have

$$\begin{aligned}
F(1_x) &= \gamma_x^{-1} \circ \beta_x^{-1} \circ \alpha_x^{-1} \circ \alpha_x \circ \beta_x \circ \gamma_x && (9.1.3) \\
&= 1_{F(x)}.
\end{aligned}$$

Let $f : x \longrightarrow y$ and $g : y \longrightarrow z$ be composable arrows in $\mathcal{SSCatGrp}\langle S\rangle$. We get

$$\begin{aligned}
F(g \circ f) &= \gamma_z^{-1} \circ \beta_z^{-1} \circ \alpha_z^{-1} \circ \alpha_x \circ \beta_x \circ \gamma_x && (9.1.3) \\
&= \gamma_z^{-1} \circ \beta_z^{-1} \circ \alpha_z^{-1} \circ \alpha_y \circ \beta_y \circ \gamma_y \circ \gamma_y^{-1} \circ \beta_y^{-1} \circ \alpha_y^{-1} \circ \alpha_x \circ \beta_x \circ \gamma_x \\
&= F(g) \circ F(f). && (9.1.3)
\end{aligned}$$

$\diamond$

Proposition 9.1.4. *Consider the functors*

$$L, R : \mathcal{SSCatGrp}\langle S\rangle \times \mathcal{SSCatGrp}\langle S\rangle \longrightarrow M$$

given by

$$L(x,y) = F(x)\cdot F(y) \qquad \textit{on objects}$$

$$L(f,g) = F(f)\cdot F(g) \qquad \textit{on morphisms}$$

and

$$R(x,y) = F(x \bullet y) \qquad \textit{on objects}$$

$$R(f,g) = F(f \bullet g) \qquad \textit{on morphisms.}$$

Then, the equality $L = R$ of functors holds.

Proof. We have

$$F(x)\cdot F(y) = F(x \bullet y). \qquad (7.8.6)$$

Therefore, we conclude

$$L(x,y) = R(x,y)$$

for objects $x, y \in \mathcal{DMon}\langle S\rangle$.

Now, let $f : x \longrightarrow z$ and $g : y \longrightarrow w$ be arrows in $\mathcal{SSCatGrp}\langle S\rangle$. Therefore, we have

$$\mathcal{Q}(x) = \mathcal{Q}(z) \qquad \text{and} \qquad \mathcal{Q}(y) = \mathcal{Q}(w).$$

From Proposition 8.5.24, we get

$$x_{\mathtt{A}} = z_{\mathtt{A}} \qquad \text{and} \qquad y_{\mathtt{A}} = w_{\mathtt{A}}.$$

It follows that

$$\alpha_{x_{\mathtt{A}} \bullet y_{\mathtt{A}}} = \alpha_{z_{\mathtt{A}} \bullet w_{\mathtt{A}}}.$$

Observe that

$$\begin{aligned}
&F(f \bullet g) \\
&= \gamma_{z \bullet w}^{-1} \circ \beta_{z \bullet w}^{-1} \circ \alpha_{z \bullet w}^{-1} \circ \alpha_{x \bullet y} \circ \beta_{x \bullet y} \circ \gamma_{x \bullet y} && (9.1.3) \\
&= (\gamma_z^{-1} \gamma_w^{-1}) \circ (\beta_z^{-1} \beta_w^{-1}) \circ (\alpha_z^{-1} \alpha_w^{-1}) \circ \alpha_{(z \bullet w)_{\mathtt{A}}}^{-1} \\
&\qquad \circ \alpha_{(x \bullet y)_{\mathtt{A}}} \circ (\alpha_x \alpha_y) \circ (\beta_x \beta_y) \circ (\gamma_x \gamma_y) && (8.3.40), (8.4.14), \text{ and } (8.5.27) \\
&= (\gamma_z^{-1} \gamma_w^{-1}) \circ (\beta_z^{-1} \beta_w^{-1}) \circ (\alpha_z^{-1} \alpha_w^{-1}) \\
&\qquad\quad \circ (\alpha_x \alpha_y) \circ (\beta_x \beta_y) \circ (\gamma_x \gamma_y) && \text{from above} \\
&= \left(\gamma_z^{-1} \circ \beta_z^{-1} \circ \alpha_z^{-1} \circ \alpha_x \circ \beta_x \circ \gamma_x\right) \\
&\qquad\quad \left(\gamma_w^{-1} \circ \beta_w^{-1} \circ \alpha_w^{-1} \circ \alpha_y \circ \beta_y \circ \gamma_y\right) && \text{functoriality} \\
&= F(f)F(g). && (9.1.3)
\end{aligned}$$

This completes the proposition. □

Definition 9.1.5. Define a monoidal functor

$$(F, F_0, F_2) : \mathcal{SSCatGrp} \longrightarrow \mathcal{M}$$

as follows:

Unit morphism: Define a morphism $F_0 : I \longrightarrow F(J)$ in $\mathcal{M}$ as follows: We have

$$F(J) = I. \qquad (7.8.4)$$

Define

$$F_0 := 1_I : I \longrightarrow F(J). \tag{9.1.6}$$

Tensor natural transformation: Consider the functors $L, R : \mathcal{SSCatGrp}\langle S\rangle \times \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{M}$ as in Proposition 9.1.4. We have $L = R$ as functors. Define the natural transformation

$$F_2 : L \Rightarrow R$$

to be the identity natural transformation:

$$F_2 := \mathrm{Id}_L. \tag{9.1.7}$$

Thas is, for $x, y \in \mathcal{DMon}\langle S\rangle$ we define

$$F_2(x, y) := 1_{F(x)\cdot F(y)} : F(x)\cdot F(y) \longrightarrow F(x \bullet y). \tag{9.1.8}$$

◇

Proposition 9.1.9. *Let*

$$(F, F_0, F_2) : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{M}$$

be the monoidal functor as in Definition 9.1.2 and $x \in \mathcal{DMon}\langle S\rangle$. Let $f : x \multimap\!\!\to y$ be a type-C forward arrow. Then, the equality

$$F(f) = F_{\mathsf{C}}(f) \tag{9.1.10}$$

holds where F_{C} is as in Definition 8.3.24.

Proof. Let $f : x \multimap\!\!\to y$ be a type-A forward arrow. Then, from Proposition 8.3.41 we get

$$x_{\mathsf{C}} = y_{\mathsf{C}} \qquad \text{and} \qquad \gamma_x = \gamma_y \circ F_{\mathsf{C}}(f).$$

Consequently, from Propositions 8.4.15 and 8.5.28 we get

$$\beta_x = \beta_y \qquad \text{and} \qquad \alpha_x = \alpha_y.$$

Thus, we have

$$\begin{aligned} F(f) &= \gamma_y^{-1} \circ \beta_y^{-1} \circ \alpha_y^{-1} \circ \alpha_x \circ \beta_x \circ \gamma_x && (9.1.3) \\ &= \gamma_y^{-1} \circ \gamma_y \circ F_{\mathtt{C}}(f) && \text{from above} \\ &= F_{\mathtt{C}}(f) \end{aligned}$$

as required. □

Lemma 9.1.11. *Let*

$$(F, F_0, F_2) : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{M}$$

be the monoidal functor as in Definition 9.1.2 and $a \in \mathtt{S}$. Then, for a natural number $k \in \mathbb{N}$, the equality

$$F(\tilde{\epsilon}_{a^{(k)}}) = \epsilon_{F(a^{(k)})}$$

holds.

Proof. Let $k \in \mathbb{N}$ and let

$$x := a^{(k+1)} \bullet a^{(k)}.$$

Then, x is the domain of $\widetilde{\epsilon}_{a^{(k)}}$. Observe that $x \in \mathtt{C}$ and

$$F(x) = F(a)^{(k+1)} F(a)^{(k)}.$$

Let

$$p = k\%2 \qquad \text{and} \qquad q = (k+1)\%2.$$

Then, we get

$$x_{\mathtt{B}} = a^{(q)} \bullet a^{(p)}. \qquad \text{and} \qquad F(x_{\mathtt{B}}) = F(a)^{(q)} F(a)^{(p)}.$$

Observe that $x_{\mathtt{B}}$ is a simple cancellation pair. That is,

$$x_{\mathtt{B}} = a' \bullet a \qquad \text{or} \qquad x_{\mathtt{B}} = a \bullet a'.$$

We have

$$\begin{aligned}
Q(x_{\mathtt{A}}) &= Q(x) && (8.5.23)\\
&= Q\left(a^{(k+1)} \bullet a^{(k)}\right)\\
&= e && (4.5.11) \text{ through } (4.5.13)\\
&= Q(J). && (4.5.10)
\end{aligned}$$

Therefore, from Proposition 8.2.22, we get that

$$x_{\mathtt{A}} = J.$$

It follows that

$$F(x_{\mathtt{A}}) = I. \qquad (7.8.4)$$

Thus, the arrow

$$\star_{x_{\mathrm{B}}, x_{\mathtt{A}}} : x_{\mathrm{B}} \multimap J$$

is a simple cancellation arrow. Moreover, this arrow forms a terminating chain of type-A forward arrows. Define the morphism $\nu_a : F(x_{\mathrm{B}}) \longrightarrow F(x_{\mathtt{A}})$ as follows:

$$\nu_{F(a)} := \begin{cases} \epsilon_{F(a)} & \text{if } x_{\mathrm{B}} = a' \bullet a \\ \eta_{F(a)}^{-1} & \text{if } x_{\mathrm{B}} = a \bullet a'. \end{cases}$$

We get

$$\begin{aligned}
\alpha_x &= F_{\mathtt{A}}(\star_{x_{\mathrm{B}}, x_{\mathtt{A}}}) && (8.5.21)\\
&= \nu_{F(a)}. && (8.5.12) \text{ and } (8.5.13)
\end{aligned}$$

Next, we have

$$\begin{aligned}
\beta_x &= B_2^{(k+1,k)}(F(a), F(a)) && (8.4.11)\\
&= B^{k+1}(F(a)) B^k(F(a)). && (5.3.11)
\end{aligned}$$

Since $x \in \mathtt{C}$, we get

$$\gamma_x = 1_{F(x)}.$$

Similarly, since $J \in \mathtt{A} \subseteq \mathtt{B} \subseteq \mathtt{C}$, we have

$$\gamma_J = \beta_J = \alpha_J = 1_{F(J)}.$$

Thus, we get

$$\begin{aligned} F(\tilde{\epsilon}_{a^{(k)}}) &= \gamma_J^{-1} \circ \beta_J^{-1} \circ \alpha_J^{-1} \circ \alpha_x \circ \beta_x \circ \gamma_x && (9.1.3) \\ &= \nu_{F(a)} \circ B^{k+1}(F(a))B^k(F(a)) \\ &= \epsilon_{F(a)^{(k)}}. && (5.4.23) \end{aligned}$$

□

Lemma 9.1.12. *Let*

$$(F, F_0, F_2) : \mathcal{SSCatGrp}\langle S \rangle \longrightarrow \mathcal{M}$$

be the monoidal functor as in Definition 9.1.2. Let $m \in \mathbb{N}$ and $x_i \in \mathcal{DMon}\langle S \rangle$ for $1 \le i \le m$ such that

$$F(\tilde{\epsilon}_{x_i}) = \epsilon_{F(x_i)}.$$

Let $X = (x_1, \ldots, x_m)$ and $F(X) = (F(x_1), \ldots, F(x_m))$. Then, the equality

$$F(\tilde{A}_m(X)) = A_m(F(X)) \tag{9.1.13}$$

holds where $\tilde{A}$ is a natural transformation as in Definition 5.4.5 for the categorical group $\mathcal{SSCatGrp}\langle S \rangle$ and A is a natural transformation as in Definition 5.4.5 for the categorical group $\mathcal{M}$.

Proof. Since F is a strict monoidal functor and we have

$$F(\tilde{\epsilon}_{x_i}) = \epsilon_{F(x_i)}.$$

Taking expanded instances, we get

$$F\left(\left\langle \tilde{\epsilon}_{x_i} \right\rangle\right) = \left\langle \epsilon_{F(x_i)} \right\rangle$$

for $1 \le i \le m$. Observe that

$$\begin{aligned} F(\tilde{A}_m(X)) &= F\left(\left\langle \tilde{\epsilon}_{x_m} \right\rangle \circ \cdots \circ \left\langle \tilde{\epsilon}_{x_1} \right\rangle\right) && (5.4.8) \\ &= \left\langle \epsilon_{F(x_m)} \right\rangle \circ \cdots \circ \left\langle \epsilon_{F(x_1)} \right\rangle && \text{from above} \\ &= A_m(F(X)). && (5.4.8) \end{aligned}$$

□

Lemma 9.1.14. *Let*

$$(F, F_0, F_2) : \mathcal{SSCatGrp}\langle S \rangle \longrightarrow \mathcal{M}$$

be the monoidal functor as in Definition 9.1.2. Let $k, m \in \mathbb{N}$ and $x_i \in \mathcal{DMon}\langle S \rangle$ for $1 \le i \le m$ such that

$$F(\tilde{\epsilon}_{x_i^{(k)}}) = \epsilon_{F(x_i)^{(k)}}.$$

Then, the equality

$$F(\tilde{\epsilon}_x) = \epsilon_{F(x)} \tag{9.1.15}$$

holds where $x = (x_1 \bullet \cdots \bullet x_m)^{(k)}$.

Proof. For the scope of this proof we will use the following notation:

$$\begin{aligned} X &= (x_1, \ldots, x_m) \\ F(X) &= (F(x_1), \ldots, F(x_m)) \\ X^{(k)} &= \begin{cases} (x_1^{(k)}, \ldots, x_m^{(k)}) & \text{if } k \text{ is even} \\ (x_m^{(k)}, \ldots, x_1^{(k)}) & \text{if } k \text{ is odd} \end{cases} && \text{as in Definition 5.4.12} \\ F(X^{(k)}) = F(X)^{(k)} &= \begin{cases} (F(x_1)^{(k)}, \ldots, F(x_m)^{(k)}) & \text{if } k \text{ is even} \\ (F(x_m)^{(k)}, \ldots, F(x_1)^{(k)}) & \text{if } k \text{ is odd} \end{cases} && \text{as in Definition 5.4.12} \end{aligned}$$

Consider the distribution arrows

$$f : (x_1 \bullet \cdots \bullet x_m)^{(k)} \multimapdotbothvert\!\!\longrightarrow x_1^{(k)} \square \cdots \square x_m^{(k)}$$

and

$$g : (x_1 \bullet \cdots \bullet x_m)^{(k+1)} \longrightarrow x_m^{(k+1)} \square \cdots \square x_1^{(k+1)}$$

where $\square = \square_k$ Since f and g are distribution arrows, they are type-C forward arrows. Now consider the arrow

$$g \bullet f : (x_1 \bullet \cdots \bullet x_m)^{(k+1)} \bullet (x_1 \bullet \cdots \bullet x_m)^{(k)} \longrightarrow x_m^{(k+1)} \square \cdots \square x_1^{(k+1)} \bullet x_1^{(k)} \square \cdots \square x_m^{(k)}$$

We get

$$\begin{aligned} F(g \bullet f) &= F(g)F(f) \\ &= F_{\mathsf{C}}(g)F_{\mathsf{C}}(f) && (9.1.10) \\ &= C_m^{k+1}(F(X))C_m^k(F(X)). && (8.3.25) \end{aligned}$$

Let $\tilde{A}$ be the natural transformation as in Definition 5.4.5 for the categorical group $\mathcal{SSCatGrp}\langle S \rangle$ and A be the natural transformation as in Definition 5.4.5 for the categorical group $\mathcal{M}$. We get the arrow

$$\tilde{A}_m(X^{(k)}) : x_m^{(k+1)} \square \cdots \square x_1^{(k+1)} \bullet x_1^{(k)} \square \cdots \square x_m^{(k)} \longrightarrow J.$$

Since $\mathcal{SSCatGrp}\langle S \rangle$ is a thin categorical, the composition arrow

$$\tilde{A}_m(X^{(k)}) \circ (g \bullet f) : (x_1 \bullet \cdots \bullet x_m)^{(k+1)} \bullet (x_1 \bullet \cdots \bullet x_m)^{(k)} \longrightarrow J$$

is equal to the cancellation arrow $\tilde{\epsilon}_x : x' \bullet x \longrightarrow J$.

From the assumption we have

$$F(\tilde{\epsilon}_{x_i^{(k)}}) = \epsilon_{F(x_i)^{(k)}}.$$

From Lemma 9.1.12, we get

$$F(\tilde{A}_m(X^{(k)})) = A_m(F(X^{(k)})) = A_m(F(X)^{(k)}).$$

We finish the proof by applying Proposition 5.4.16 as follows:

$$\begin{aligned} F(\tilde{\epsilon}_x) &= F\left(\tilde{A}_m(X^{(k)})\right) \circ F(g \bullet f) \\ &= A_m(F(X)^{(k)}) \circ C_m^{k+1}(F(X)) C_m^k(F(X)) \\ &= \epsilon_{F(x)}. \end{aligned}$$

□

Lemma 9.1.16. *Let*

$$(F, F_0, F_2) : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{M}$$

be the monoidal functor as in Definition 9.1.2 and $x \in \mathtt{G}$. *Then, the equality*

$$F(\tilde{\epsilon}_x) = \epsilon_{F(x)}$$

holds.

Proof. We will prove this lemma using induction on $\hat{l}(x)$.

Base case ($\hat{l}(x) = 1$): In this case, from Theorem 7.8.8, we have $x = a^{(k)}$ for some $a \in \mathtt{S}$ and $k \in \mathbb{N}$. From Lemma 9.1.16, we get

$$F(\tilde{\epsilon}_x) = \epsilon_{F(x)}$$

as required.

Induction case ($\hat{l}(x) \geq 2$): Assume that the lemma holds for all $w \in \mathtt{G}$ with $\hat{l}(w) < \hat{l}(x)$. Since $\hat{l}(x) \geq 2$, from Theorem 7.8.8 we get

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}$$

where $k \geq 1$, $m \geq 2$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$. Moreover, we have

$$\hat{l}\,(x_i) < \hat{l}\,(x).$$

Therefore, we get

$$\hat{l}\left(x_i^{(k)}\right) = \hat{l}\,(x_i) < \hat{l}\,(x).$$

From the induction hypothesis we get

$$F\left(\tilde{\epsilon}_{x_i^{(k)}}\right) = \epsilon_{F(x_i^{(k)})} = \epsilon_{F(x_i)^{(k)}}.$$

for $1 \leq i \leq m$. Invoking Lemma 9.1.14, we get

$$F(\tilde{\epsilon}_x) = \epsilon_{F(x)}$$

as required. ☐

Theorem 9.1.17. *Let*

$$(F, F_0, F_2) : \mathcal{SSCatGrp}\,\langle S\rangle \longrightarrow \mathscr{M}$$

be the monoidal functor as in Definition 9.1.2. Then, the equality

$$F(\tilde{\epsilon}_x) = \epsilon_{F(x)}$$

holds for every $x \in \mathcal{DMon}\,\langle S\rangle$.

Proof. Let $x \in \mathcal{DMon}\,\langle S\rangle$. Since $\mathtt{G}$ is a multiplicative basis of $\mathcal{DMon}\,\langle S\rangle$ we get

$$x = x_1 \bullet \cdots \bullet x_m$$

where $m \geq 0$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$. From Lemma 9.1.16, we get

$$F(\tilde{\epsilon}_{x_i}) = \epsilon_{F(x_i)}.$$

From Lemma 9.1.14, we get

$$F(\tilde{\epsilon}_x) = \epsilon_{F(x)}$$

as required. $\square$

Definition 9.1.18. Let

$$(F, F_0, F_2) : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{M}$$

be the induced monoidal functor as in Definition 9.1.5. From Theorem 4.2.16 we get a categorical group functor

$$(F, F_0, F_1, F_2) : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{M}.$$

We define the above categorical group functor to be the *induced categorical group functor*. $\diamond$

Theorem 9.1.19. *Let*

$$(F, F_0, F_1, F_2) : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{M}$$

be the induced categorical group functor as in Definition 9.1.18. Then, the functor F is a strict categorical group functor.

Proof. From equations (9.1.6) and (9.1.7) of Definition 9.1.5 we get that the underlying monoidal functor is strict. From equation (7.8.7) of Definition 9.1.2 we get that

$$F(x)' = F(x')$$

for every $x \in \mathcal{DMon}\langle S\rangle$. Moreover, from Theorem 9.1.17 we get that the equality

$$F(\tilde{\epsilon}_x) = \epsilon_{F(x)}$$

holds for every $x \in \mathcal{DMon}\langle S\rangle$. Therefore, from Theorem 4.2.19 we conclude that the categorical group functor F is strict. $\square$

9.2 Induced Natural Transformations

In this section, we establish the existence and uniqueness of a categorical group natural transformation, as required by the universal property of the free semi-strict categorical group in Definition 4.4.7.

Framework 9.2.1. Throughout this section, let S be a set. Let $\mathcal{DMon}\langle S\rangle$ denote the free dashed-monoid generated by S, as in Construction 4.5.2, with its universal property established in Theorem 7.8.2.

Let $\mathcal{SSCatGrp}\langle S\rangle$ be the free semi-strict categorical group generated by S, as in Definition 8.1.2. Let $\mathcal{M}$ be a semi-strict categorical group with $M := \mathrm{Obj}(\mathcal{M})$, so that M is a dashed monoid by Example 7.1.6.

Consider two categorical group functors

$$F, G : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{M}.$$

◇

Definition 9.2.2. Define a set of morphisms, $N \subseteq \mathrm{Mor}(\mathcal{M})$ as follows:

$$N := \{(x, f : F(x) \longrightarrow G(x)) \mid x \in \mathcal{DMon}\langle S\rangle\}. \tag{9.2.3}$$

◇

Motivation 9.2.4. While $\mathrm{Mor}(\mathcal{M})$ is a class in the Grothendieck universe $\mathcal{U}$, the subset N defined above is in fact a set. This is because S is a set, so $\mathcal{DMon}\langle S\rangle$ is also a set, and for each $x \in \mathcal{DMon}\langle S\rangle$, the collection of morphisms $F(x) \longrightarrow G(x)$ in $\mathcal{M}$ forms a set. Thus, the set of all such pairs (x, f) is indeed a set.

We will later show in Definition 9.2.21 that N is a dashed monoid. Consequently, any function

$$\widehat{\phi} : S \longrightarrow N$$

induces a dashed monoid homomorphism

$$\phi : \mathcal{DMon}\langle S\rangle \longrightarrow N.$$

This induced dashed monoid homomorphism ϕ will be used to construct a unique categorical group natural transformation (see Theorems 9.2.40 and 9.2.41). $\diamond$

Definition 9.2.5. Consider the morphism

$$e : F(J) \longrightarrow G(J)$$

given by

$$e := G_0 \circ F_0^{-1}. \tag{9.2.6}$$

Define

$$(J, e) \in M$$

to be the unit element of N. $\diamond$

Definition 9.2.7. Let $x, y \in \mathcal{DMon}\langle S\rangle$ and

$$f : F(x) \longrightarrow G(x) \quad \text{and} \quad g : F(y) \longrightarrow G(y)$$

be morphisms in $\mathcal{M}$. Define a morphism

$$f * g : F(x \bullet y) \longrightarrow G(x \bullet y)$$

as

$$f * g := G_2(x, y) \circ (f g) \circ F_2(x, y). \tag{9.2.8}$$

Define the multiplication on N as

$$(x, f) * (y, g) = (x \bullet y, f * g). \qquad \diamond$$

Proposition 9.2.9. *Let* $x, y, z \in \mathcal{DMon}\langle S\rangle$ *and*

$$f : F(x) \longrightarrow G(x), \qquad g : F(y) \longrightarrow G(y) \qquad \text{and} \qquad h : F(z) \longrightarrow G(z)$$

be morphisms in $\mathscr{M}$. *Then, the equality*

$$(x, f) * ((y, g) * (z, h)) = ((x, f) * (y, g)) * (z, h) \tag{9.2.10}$$

holds in N.

Proof. We have

$$(x, f) * ((y, g) * (z, h)) = (x \bullet y \bullet z, f * (g * h))$$

and

$$((x, f) * (y, g)) * (z, h) = (x \bullet y \bullet z, (f * g) * h).$$

Therefore, we need to show $f*(g*h)=(f*g)*h$. Consider the diagram

$$\begin{tikzcd}
F(x\bullet y\bullet z) \arrow[r, equal] \arrow[d, "F_2(x,y\bullet z)^{-1}"'] & F(x\bullet y\bullet z) \arrow[d, "F_2(x\bullet y,z)^{-1}"] \\
F(x)F(y\bullet z) \arrow[d, "1_{F(x)}F_2(y,z)^{-1}"'] & F(x\bullet y)\bullet F(z) \arrow[d, "F_2(x,y)^{-1}1_{F(z)}"] \\
F(x)F(y)F(z) \arrow[r, equal] \arrow[d, "fgh"'] & F(x)F(y)F(z) \arrow[d, "fgh"] \\
G(x)G(y)G(z) \arrow[r, equal] \arrow[d, "1_{G(x)}G_2(y,z)"'] & G(x)G(y)G(z) \arrow[d, "G_2(x,y)1_{G(z)}"] \\
G(x)G(y\bullet z) \arrow[d, "G_2(x,y\bullet z)"'] & G(x\bullet y)\bullet G(z) \arrow[d, "G_2(x\bullet y,z)"] \\
G(x\bullet y\bullet z) \arrow[r, equal] & G(x\bullet y\bullet z).
\end{tikzcd}$$

Here, the top and bottom rectangle commute because of the associator condition (3.1.10) of the monoidal functors F and G respectively. The middle square commutes trivially. The left vertical composite is $f*(g*h)$ and the right composite is $(f*g)*h$. Since the above diagram commutes, we get

$$f*(g*h)=(f*g)*h$$

as required. $\square$

Proposition 9.2.11. *Let $x \in \mathcal{DMon}\langle S\rangle$ and $f : F(x) \longrightarrow G(x)$. Then, the equality*

$$(J,e)*(x,f) = (x,f) \tag{9.2.12}$$

holds in N where e is the morphism as in Definition 9.2.5.

Proof. We have

$$(J,e)*(x,f) = (x, e*f).$$

Thus, we want to show $e*f = f$. Consider the diagram

$$\begin{tikzcd}
F(x) \arrow[rr, equal] \arrow[d, "F_2(J,x)^{-1}"'] & & F(x) \arrow[dd, equal] \\
F(J)F(x) \arrow[d, "F_0^{-1}1_{F(x)}"'] & & \\
IF(x) \arrow[rr, equal] \arrow[d, "1_I f"'] & & F(x) \arrow[d, "f"] \\
IG(x) \arrow[rr, equal] \arrow[d, "F_0 1_{G(x)}"'] & & G(x) \arrow[dd, equal] \\
G(J)G(x) \arrow[d, "F_2(J,x)"'] & & \\
G(x) \arrow[rr, equal] & & G(x).
\end{tikzcd}$$

Here, the top and bottom rectangles commute because of the left unitor condition (3.1.11) of the monoidal functors F and G respectively. The middle square commutes from the naturality of the left unitor. The left vertical composite is given by $e*f : F(x) \longrightarrow G(x)$. Since the above diagram commutes we get

$$e*f = f$$

as required. $\square$

Proposition 9.2.13. *Let $x \in \mathcal{DMon}\langle S\rangle$ and $f : F(x) \longrightarrow G(x)$. Then, the equality*

$$(x, f) * (J, e) = (x, f) \tag{9.2.14}$$

holds in N where e is the morphism as in Definition 9.2.5.

Proof. We have

$$(x, f) * (J, e) = (x, f * e).$$

Thus, we want to show $f * e = f$. Consider the diagram

$$\begin{tikzcd}
F(x) \arrow[rr, equal] \arrow[d, "F_2(x,J)^{-1}"'] & & F(x) \arrow[dd, equal] \\
F(x)F(J) \arrow[d, "1_{F(x)}F_0^{-1}"'] & & \\
F(x)I \arrow[rr, equal] \arrow[d, "f1_I"'] & & F(x) \arrow[d, "f"] \\
G(x)I \arrow[rr, equal] \arrow[d, "1_{G(x)}F_0"'] & & G(x) \arrow[dd, equal] \\
G(x)G(J) \arrow[d, "F_2(x,J)"'] & & \\
G(x) \arrow[rr, equal] & & G(x).
\end{tikzcd}$$

Here, the top and bottom rectangles commute because of the left unitor condition (3.1.12) of the monoidal functors F and G respectively. The middle square commutes from the naturality of the left unitor. The left vertical composite is given by $f * e$:

$F(x) \longrightarrow G(x)$. Since the above diagram commutes we get

$$f * e = f$$

as required. □

Definition 9.2.15. Let $x \in \mathcal{DMon}\langle S\rangle$. Let $f : F(x) \longrightarrow G(x)$ be a morphism in $\mathcal{M}$. Define a morphism

$$D(f): F(x') \longrightarrow G(x')$$

as

$$D(f) := G_1(x) \circ f' \circ F_1(x)^{-1}. \tag{9.2.16}$$

Define dash operation on N as

$$D(x, f) = (x', D(f)).$$

◇

Proposition 9.2.17. *The following diagram commutes*

$$\begin{tikzcd}
I' \arrow[r, equal] \arrow[d, "F_0'"'] & I \arrow[dd, "F_0"] \\
F(J)' \arrow[d, "F_1(J)"'] & \\
F(J') \arrow[r, equal] & F(J).
\end{tikzcd} \tag{9.2.18}$$

Proof. Consider the diagram

$$\begin{tikzcd}
I' = I'I \arrow[r, "\epsilon_I"] \arrow[d, "F_0'"'] & I \arrow[r, equal] \arrow[d, "\epsilon^{-1}_{F(J)}"] & I. \arrow[dd, "F_0"] \\
F(J)' = F(J)'I \arrow[r, "1_{F(J)'}F_0"] \arrow[d, "F_1(J)"'] & F(J)'F(J) \arrow[d, "F_1(J)1_{F(J)}"] & \\
F(J') = F(J')I \arrow[r, "1_{F(J')}F_0"'] & F(J')F(J) \arrow[r, "F_2(J',J)"'] & F(J' \bullet J) = F(J).
\end{tikzcd}$$

Here, the top left square commutes because of the naturality of ϵ. From the functoriality of tensor, the bottom right square commutes. The right rectangle commutes from the left cancellation condition (4.1.23) of the categorical group functor F.

From the unit condition (4.1.3), we have that the top horizontal composite is equal to

$$1_I : I \longrightarrow I.$$

From the right unitor condition (3.1.12), we have that the bottom horizontal composite is equal to

$$F(\rho_{J'}) : F(J') \longrightarrow F(J' \bullet J).$$

Since $\rho_{J'} = 1_{J'}$, we get that the bottom composite is equal to

$$1_{F(J)} : F(J') \longrightarrow F(J).$$

Since the above diagram commutes, we get that

$$F_1(J) \circ F_0' = F_0$$

as required. $\square$

Proposition 9.2.19. *The equality*

$$D(J,e) = (J,e). \tag{9.2.20}$$

holds in N where e is the morphism as in Definition 9.2.5.

Proof. We know that

$$D(J,e) = (J',D(e)) = (J,D(e)).$$

We need to show that $D(e) = e : F(J) \longrightarrow G(J)$. Observe that

$$\begin{aligned}
D(e) &= G_1(J) \circ e \circ F_1(J)^{-1} && (9.2.16)\\
&= G_1(J) \circ G_0 \circ F_0^{-1} \circ F_1(J)^{-1} && (9.2.6)\\
&= G_0 \circ F_0^{-1} && (9.2.18)\\
&= e. && (9.2.6)
\end{aligned}$$

□

Definition 9.2.21. Define a dashed monoid, denoted N, as follows:

Set: Let N as in Definition 9.2.2 be the underlying set.

Unit: Let $(J,e) \in N$ as in Definition 9.2.5 be the unit.

Multiplication: Definition 9.2.7 defines the multiplication.

Dash: Definition 9.2.15 defines the dash operation.

Associativity: Proposition 9.2.9 gives the associativity of the multiplication.

Left and right unity: Propositions 9.2.11 and 9.2.13 gives the unity equations.

Unit dash: Proposition 9.2.19 gives the unit dash condition. ◇

Definition 9.2.22. Let $\widehat{\phi} : S \longrightarrow N$ be a function that assigns to each $a \in S$ a morphism

$$\widehat{\phi}_a : F(a) \longrightarrow G(a).$$

Define a dashed monoid homomorphism

$$\phi : \mathcal{DMon}\langle S\rangle \longrightarrow N$$

to be the unique dashed monoid homomorphism that we get from the universal property of the free dashed monoid (see Definition 7.2.1). Above dashed monoid homomorphism ϕ is completely determined by the following equations:

$$\phi(a) = \widehat{\phi}(a) \qquad \text{for every } a \in S, \tag{9.2.23}$$

$$\phi(J) = I \tag{9.2.24}$$

$$\phi(x \bullet y) = \phi(x) \cdot \phi(y) \qquad \text{for every } x, y \in \mathcal{DMon}\langle S\rangle \tag{9.2.25}$$

$$\phi(x') = \phi(x)' \qquad \text{for every } x \in \mathcal{DMon}\langle S\rangle\,. \tag{9.2.26}$$

◇

Definition 9.2.27. Let $f : x \longrightarrow y$ be an arrow in $\mathcal{SSCatGrp}\langle S\rangle$. We say that ϕ is natural with respect to f if the following diagram commutes:

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\phi(x)"'] & F(y) \arrow[d, "\phi(y)"] \\ G(x) \arrow[r, "G(f)"'] & G(y). \end{tikzcd} \tag{9.2.28}$$

It follows that if ϕ is natural with respect to every arrow in $\mathcal{SSCatGrp}\langle S\rangle$ then ϕ is a natural transformation. ◇

Remark 9.2.29. To show that ϕ is natural with respect to every arrow, we will follow the following principle. Every arrow in $\mathcal{SSCatGrp}\langle S\rangle$ is a composite of expanded instances of cancellation arrows. Thus, we will show that ϕ is natural with respect to cancellation arrows and the naturality respects composition and taking expanded instance. ◇

Proposition 9.2.30. *Let*

$$f : x' \bullet x \multimap J$$

be an arrow in $\mathcal{SSCatGrp}\langle S\rangle$. Then, the following diagram commutes

$$\begin{tikzcd}
F(x' \bullet x) \arrow[r, "F(f)"] \arrow[d, "\phi(x' \bullet x)"'] & F(J) \arrow[d, "\phi(J)"] \\
G(x' \bullet x) \arrow[r, "G(f)"'] & G(J).
\end{tikzcd} \tag{9.2.31}$$

Proof. Consider the diagram

$$\begin{tikzcd}
F(x' \bullet x) \arrow[r, "F(f)"] \arrow[d, "\left(F_1(x)^{-1}1_{F(x)}\right) \circ F_2(x',x)^{-1}"'] & F(J) \arrow[d, "F_0^{-1}"] \\
F(x)^{\dagger}F(x) \arrow[r, "\epsilon_{F(x)}"] \arrow[d, "\phi(x)^{\dagger}\phi(x)"'] & I \arrow[d, equal] \\
G(x)^{\dagger}G(x) \arrow[r, "\epsilon_{G(x)}"] \arrow[d, "\left(G_1(x)^{-1}1_{G(x)}\right) \circ G_2(x',x)^{-1}"'] & I \arrow[d, "F_0"] \\
G(x' \bullet x) \arrow[r, "G(f)"'] & G(J).
\end{tikzcd}$$

Here, from the cancellation condition (4.1.23), we get that the top and bottom squares commute. The middle square commute because of the naturality of ϵ. From Definition 9.2.7 of multiplication in N and Definition 9.2.15 of dash operation in N, we get that the left vertical composite is equal to

$$D(\phi(x)) * \phi(x) : F(x' \bullet x) \longrightarrow G(x' \bullet x).$$

From Definition 9.2.5, the right vertical composite is equal to

$$e : F(J) \longrightarrow G(J).$$

Now, since ϕ is a dashed monoid homomorphism, we get that left vertical composite is equal to

$$D(\phi(x)) * \phi(x) = \phi(x' \bullet x)$$

and the right vertical composite is equal to

$$e = \phi(J).$$

This completes the proof. $\square$

Proposition 9.2.32. *Let*

$$f : x \bullet x' \longrightarrow J$$

be an arrow in $\mathcal{SSCatGrp}\langle S\rangle$. Then, the following diagram commutes

$$\begin{tikzcd}
F(x \bullet x') \arrow[r, "F(f)"] \arrow[d, "\phi(x \bullet x')"'] & F(J) \arrow[d, "\phi(J)"] \\
G(x \bullet x') \arrow[r, "G(f)"'] & G(J).
\end{tikzcd} \tag{9.2.33}$$

Proof. Consider the diagram

$$\begin{tikzcd}
F(x \bullet x') \arrow[r, "F(f)"] \arrow[d, "\left(1_{F(x)} F_1(x)^{-1}\right) \circ F_2(x,x')^{-1}"'] & F(J) \arrow[d, "F_0^{-1}"] \\
F(x)F(x)^\dagger \arrow[r, "\eta_{F(x)}^{-1}"] \arrow[d, "\phi(x)\phi(x)^\dagger"'] & I \arrow[d, equal] \\
G(x)G(x)^\dagger \arrow[r, "\eta_{G(x)}^{-1}"] \arrow[d, "\left(1_{G(x)} G_1(x)^{-1}\right) \circ G_2(x,x')^{-1}"'] & I \arrow[d, "F_0"] \\
G(x \bullet x') \arrow[r, "G(f)"'] & G(J).
\end{tikzcd}$$

Here, from the cancellation condition (4.1.24), we get that the top and bottom squares commute. The middle square commute because of the naturality of ϵ. From Definition 9.2.7 of multiplication in N and Definition 9.2.15 of dash operation in N, we get that the left vertical composite is equal to

$$\phi(x) * D(\phi(x)) : F(x' \bullet x) \longrightarrow G(x' \bullet x).$$

From Definition 9.2.5, the right vertical composite is equal to

$$e : F(J) \longrightarrow G(J).$$

Now, since ϕ is a dashed monoid homomorphism, we get that left vertical composite is equal to

$$\phi(x) * D(\phi(x)) = \phi(x \bullet x')$$

and the right vertical composite is equal to

$$e = \phi(J).$$

This completes the proof. □

Proposition 9.2.34. *Let*

$$f : x \multimap y \qquad \text{and} \qquad g : y \multimap z$$

be arrows in $\mathcal{SSCatGrp}\langle S\rangle$. *Suppose* ϕ *is natural with respect to* f *and* g *then* ϕ *is natural with respect to* $g \circ f$.

Proof. Consider the diagram

$$\begin{tikzcd}
F(x) \arrow[r, "F(f)"] \arrow[d, "\phi(x)"'] & F(y) \arrow[r, "F(g)"] \arrow[d, "\phi(y)"] & F(z) \arrow[d, "\phi(z)"] \\
G(x) \arrow[r, "G(f)"'] & G(y) \arrow[r, "G(g)"'] & G(z)
\end{tikzcd}$$

Here, the left and right squares commute because ϕ is natural with respect to both f and g. The top horizontal composite is equal to $F(g \circ f)$ and the bottom horizontal composite is equal to $G(g \circ f)$. Thus, ϕ is natural with respect to $g \circ f$. $\square$

Proposition 9.2.35. *Let*

$$f : x \multimap z \qquad \text{and} \qquad g : y \multimap w$$

be an arrows in $\mathcal{SSCatGrp}\langle S \rangle$. Suppose ϕ is natural with respect to f and g, then ϕ is natural with respect to $f \bullet g$.

Proof. Consider the diagram

$$\begin{tikzcd}
F(x \bullet y) \arrow[r, "F(f \bullet g)"] \arrow[d, "F_2(x,y)^{-1}"'] & F(z \bullet y) \arrow[d, "F_2(z,w)^{-1}"] \\
F(x)F(y) \arrow[r, "F(f)F(g)"] \arrow[d, "\phi(x)\phi(y)"'] & F(z)F(w) \arrow[d, "\phi(z)\phi(w)"] \\
G(x)G(y) \arrow[r, "G(f)G(g)"] \arrow[d, "G_2(x,y)"'] & G(z)G(w) \arrow[d, "G_2(z,w)"] \\
G(x \bullet y) \arrow[r, "G(f \bullet g)"'] & G(z \bullet w).
\end{tikzcd}$$

Here, the top and bottom squares commute because of the naturality of F_2 and G_2 respectively. Since ϕ is natural with respect to f and g, the middle square commutes. From Definition 9.2.7 of multiplication in N, we get that the left vertical composite is equal to

$$\phi(x) * \phi(y) : F(x \bullet y) \longrightarrow G(x \bullet y)$$

and the right vertical composite is equal to

$$\phi(z) * \phi(w) : F(z \bullet w) \longrightarrow G(z \bullet w).$$

Now, since ϕ is a dashed monoid homomorphism, we get that left vertical composite is equal to

$$\phi(x) * \phi(y) = \phi(x \bullet y) \tag{9.2.25}$$

and the right vertical composite is equal to

$$\phi(z) * \phi(w) = \phi(z \bullet w). \tag{9.2.25}$$

Thus, we get that ϕ is natural with respect to $f \bullet g$. $\square$

Lemma 9.2.36. *The dashed monoid homomorphism*

$$\phi : \mathcal{DMon}\langle S\rangle \longrightarrow M$$

is a natural transformation from F to G.

Proof. It is enough to show that ϕ is natural with respect to every arrow in $\mathcal{SSCatGrp}\langle S\rangle$. From Propositions 9.2.30 and 9.2.32, we get that ϕ is natural with respect to cancellation morphisms. From Propositions 9.2.34 and 9.2.35, we know that naturality respects composition and taking expanded instance. Since every arrow in $\mathcal{SSCatGrp}\langle S\rangle$ is a composite of expanded instances of cancellation arrows, ϕ is a natural transformation. $\square$

Lemma 9.2.37. *The natural transformation*

$$\phi : F \Rightarrow G$$

satisfies the unit condition (3.1.26).

Proof. Since ϕ is a dashed monoid homomorphism, we get

$$\begin{aligned} \phi(J) &= e && (9.2.24) \\ &= G_0 \circ F_0^{-1}. && (9.2.6) \end{aligned}$$ $\square$

Lemma 9.2.38. *The natural transformation*

$$\phi : F \Rightarrow G$$

satisfies the tensor condition (3.1.27).

Proof. Since ϕ is a dashed monoid homomorphism, we get

$$\begin{aligned}
\phi(x \bullet y) &= \phi(x) * \phi(y) && (9.2.25)\\
&= G_2(x,y) \circ (\phi(x)\phi(y)) \circ F_2(x,y)^{-1}. && (9.2.6)
\end{aligned}$$

□

Lemma 9.2.39. *The natural transformation*

$$\phi : F \Rightarrow G$$

satisfies the negator condition (4.1.36).

Proof. Since ϕ is a dashed monoid homomorphism, we get

$$\begin{aligned}
\phi(x') &= \phi(x)^\dagger && (9.2.26)\\
&= G_1(x) \circ \phi(x)^\dagger \circ F_1(x)^{-1}. && (9.2.6)
\end{aligned}$$

□

Theorem 9.2.40. *Let S be a set, $\mathscr{M}$ be a semi-strict categorical group. Let*

$$F, G : \mathcal{SSCatGrp}\langle S \rangle \longrightarrow \mathscr{M}$$

be categorical group functors, and a collection of isomorphisms

$$\widehat{\phi} = \left(\widehat{\phi}_a : F(a) \longrightarrow G(a)\right)_{a \in S}$$

packaged as a function

$$\widehat{\phi} : S \longrightarrow N.$$

Then, there is a categorical group natural transformation $\phi : F \Rightarrow G$ *such that*

$$\phi(a) = \widehat{\phi}_a$$

for every $a \in S$.

Proof. We will define the categorical group natural transformation

$$\phi : F \Rightarrow G$$

as follows:

Objects: Definition 9.2.22 gives the value of ϕ on objects.

Naturality: Lemma 9.2.36 gives the naturality condition.

Unit condition: Lemma 9.2.37 gives the unit condition.

Tensor condition: Lemma 9.2.38 gives the tensor condition.

Negator condition: Lemma 9.2.39 gives the negator condition.

Above defines the required categorical group natural transformation. Finally, from Definition 9.2.22, we get that

$$\phi(a) = \widehat{\phi}_a$$

for $a \in S$. $\square$

Theorem 9.2.41. *Let* S *be a set,* $\mathscr{M}$ *be a semi-strict categorical group, and*

$$F, G : \mathcal{SSCatGrp}\langle S \rangle \longrightarrow \mathscr{M}$$

be categorical group functors. Let

$$\phi, \psi : F \Rightarrow G$$

be categorical group natural transformations such that

$$\phi(a) = \psi(a)$$

for $a \in \mathrm{S}$*. Then, we have*

$$\phi = \psi.$$

Proof. Let N be the dashed monoid as in Definition 9.2.21. Consider the natural transformations ϕ, ψ as functions from $\mathcal{DMon}\langle S\rangle$ to N. Since ϕ and ψ are categorical group natural transformations, it follows from the unit, tensor and negator conditions (3.1.26), (3.1.27), and (4.1.36), that $\phi, \psi : \mathcal{DMon}\langle S\rangle \longrightarrow N$ are dashed monoid homomorphisms. Since we have

$$\phi(a) = \psi(a)$$

for $a \in S$, from the uniqueness of universal property of free dashed monoids, we get $\phi = \psi$. $\square$

9.3 Main Results

In this section, we present the main coherence results for categorical groups.

Theorem 9.3.1. *Let* S *be a set, and let* $\mathcal{SSCatGrp}\langle S\rangle$ *denote the semi-strict categorical group as in Definition 8.1.17. Recall that*

$$\mathrm{Obj}\left(\mathcal{SSCatGrp}\langle S\rangle\right) = \mathcal{DMon}\langle S\rangle \tag{8.1.3}$$

where $\mathcal{DMon}\langle S\rangle$ *is the free dashed monoid generated by* S *as in Construction 4.5.2. Then, the pair*

$$(\mathcal{SSCatGrp}\langle S\rangle\,,\; i_S : S \hookrightarrow \mathcal{DMon}\langle S\rangle)$$

satisfies the universal property of the free semi-strict categorical group (see Definition 4.4.7), where $i_S : S \hookrightarrow \mathcal{DMon}\langle S\rangle$ *is the canonical inclusion.*

Proof. Let $\mathscr{M}$ be a semi-strict categorical group, and let

$$f : S \longrightarrow \mathrm{Obj}(\mathscr{M})$$

be a function. By Definition 9.1.2, there exists a functor

$$F : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathscr{M}$$

induced by f. From Theorem 9.1.19, F is a strict categorical group functor. Furthermore, by equation (7.8.5), we have

$$\mathrm{Obj}(F) \circ i_S = f.$$

This proves the existence of a strict categorical group functor as required by the universal property.

Next, let $\mathscr{M}$ be a semi-strict categorical group, and let

$$F, G : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathscr{M}$$

be categorical group functors. Given a collection of isomorphisms

$$\widehat{\phi} = \big(\widehat{\phi}_a : F(a) \longrightarrow G(a)\big)_{a \in S}$$

in $\mathscr{M}$, Theorem 9.2.40 provides a categorical group natural transformation

$$\phi : F \Rightarrow G$$

such that $\phi_a = \widehat{\phi}_a$ for all $a \in S$.

Finally, suppose

$$F, G : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathscr{M}$$

are categorical group functors, and

$$\phi, \psi : F \Rightarrow G$$

are categorical group natural transformations with

$$\phi_a = \psi_a$$

for all $a \in S$. By Theorem 9.2.41, it follows that $\phi = \psi$.

Therefore, the pair

$$(\mathcal{SSCatGrp}\langle S\rangle,\ i_S : S \longrightarrow \mathcal{DMon}\langle S\rangle)$$

satisfies the universal property of the free semi-strict categorical group. □

Theorem 9.3.2. *Let S be a set. The free semi-strict categorical group*

$$\mathcal{SSCatGrp}\langle S\rangle$$

as constructed in Definition 8.1.17 is a thin category.

Proof. By the definition of morphisms in $\mathcal{SSCatGrp}\langle S\rangle$, we have

$$\mathrm{Hom}_{\mathcal{SSCatGrp}\langle S\rangle}(x,y) = \begin{cases} \{\star_{x,y}\} & \text{if } Q(x) = Q(y), \\ \varnothing & \text{if } Q(x) \neq Q(y). \end{cases} \tag{8.1.4}$$

Therefore, for any pair of objects x, y in $\mathcal{SSCatGrp}\langle S\rangle$, there is at most one morphism from x to y. This shows that $\mathcal{SSCatGrp}\langle S\rangle$ is a thin category. □

Theorem 9.3.3. *Let S be a set, and let $\mathcal{SSCatGrp}\langle S\rangle$ denote the free semi-strict categorical group as constructed in Definition 8.1.17. Consider the induced strict categorical group functor*

$$Q_{\mathcal{SSCatGrp}} : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{Grp}\langle S\rangle$$

where the free group $\mathcal{Grp}\langle S\rangle$ is regarded as a discrete strict categorical group. Note that, the action of above functor coincides with

$$Q = Q_{\mathcal{DMon}} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathcal{Grp}\langle S\rangle$$

as defined in Construction 4.5.5.

Then, the functor $Q_{\mathcal{SSCatGrp}}$ is full, faithful, and essentially surjective. Therefore, $Q_{\mathcal{SSCatGrp}}$ is a categorical equivalence. Therefore, $Q_{\mathcal{SSCatGrp}}$ is a categorical equivalence.

Proof. Let $z \in \mathcal{Grp}\langle S\rangle$. Then, $z \in \mathcal{Grp}\langle S\rangle$ can be uniquely written in reduced form as

$$z = a_1^{(k_1)} \cdots a_m^{(k_m)}$$

where $a_i \in S$, $k_i \in \{0,1\}$, with $a_i^{(0)} = a_i$ and $a_i^{(1)} = a_i^{-1}$, and no cancellation pairs $a_i a_i^{-1}$ or $a_i^{-1} a_i$ appear in the product. Consider

$$x = a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \in \mathcal{DMon}\langle S\rangle\,.$$

Then, we get

$$Q_{\mathcal{DMon}}(x) = z,$$

and hence

$$Q_{\mathcal{SSCatGrp}}(1_x) = 1_z.$$

This shows that $Q_{\mathcal{SSCatGrp}}$ is essentially surjective.

Now, let $x, y \in \mathcal{DMon}\langle S\rangle$. We have

$$\mathrm{Hom}_{\mathcal{SSCatGrp}\langle S\rangle}(x,y) = \begin{cases} \{\star_{x,y}\} & \text{if } Q(x) = Q(y), \\ \varnothing & \text{if } Q(x) \neq Q(y). \end{cases} \tag{8.1.4}$$

Since $\mathcal{Grp}\langle S\rangle$ is a discrete category, the function

$$Q_{\mathcal{SSCatGrp}} : \mathrm{Hom}_{\mathcal{SSCatGrp}\langle S\rangle}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{Grp}\langle S\rangle}(x,y) \qquad \square$$

is an isomorphism. Consequently, $Q_{\mathcal{SSCatGrp}}$ is full and faithful.

Theorem 9.3.4. *Let S be a set, and let $\mathcal{CatGrp}\langle S\rangle$ be the free categorical group $\mathcal{CatGrp}\langle S\rangle$ as in Construction 4.5.19. Then, $\mathcal{CatGrp}\langle S\rangle$ is a thin category.*

Proof. Let $\mathcal{SSCatGrp}\langle S\rangle$ be the free semi-strict categorical group generated by S as in Definition 8.1.17. By Theorem 9.3.1, $\mathcal{SSCatGrp}\langle S\rangle$ satisfies the universal property of the free semi-strict categorical group generated by S. From Theorem 4.4.13, $\mathcal{CatGrp}\langle S\rangle$ is categorically equivalent $\mathcal{SSCatGrp}\langle S\rangle$ via a categorical group equivalence. Since Theorem 9.3.2 establishes that $\mathcal{SSCatGrp}\langle S\rangle$ is a thin category, it follows that $\mathcal{CatGrp}\langle S\rangle$ is also thin. □

Theorem 9.3.5. *Let S be a set, and $\mathcal{CatGrp}\langle S\rangle$ be the free categorical group generated by S as in Construction 4.5.19. Consider the induced strict categorical group functor*

$$Q_{\mathcal{CatGrp}} : \mathcal{CatGrp}\langle S\rangle \longrightarrow \mathcal{Grp}\langle S\rangle$$

as in Definition 4.5.23, where the free group $\mathcal{Grp}\langle S\rangle$ is regarded as a discrete strict categorical group.

Then, the functor $Q_{\mathcal{CatGrp}}$ is full, faithful, and essentially surjective. Therefore, $Q_{\mathcal{CatGrp}}$ is a categorical equivalence.

As a consequence, there is a morphisms $f : x \longrightarrow y$ in $\mathcal{CatGrp}\langle S\rangle$ if and only if $Q_{\mathcal{CatGrp}}(x) = Q_{\mathcal{CatGrp}}(y)$.

Proof. Let $\mathcal{SSCatGrp}\langle S\rangle$ be the free semi-strict categorical group generated by S as in Definition 8.1.17. By Theorem 9.3.1, $\mathcal{SSCatGrp}\langle S\rangle$ satisfies the universal property of the free semi-strict categorical group generated by S. From Theorem 4.4.13, $\mathcal{CatGrp}\langle S\rangle$ is categorically equivalent $\mathcal{SSCatGrp}\langle S\rangle$ via a categorical group equivalence

$$P : \mathcal{CatGrp}\langle S\rangle \longrightarrow \mathcal{SSCatGrp}\langle S\rangle .$$

Observe that P is the induced strict categorical group functor. Thus, we get

$$Q_{\mathcal{CatGrp}} = Q_{\mathcal{SSCatGrp}} \circ P .$$

Since Theorem 9.3.3 establishes that the functor $Q_{\mathcal{SSCatGrp}}$ is full, faithful, and essentially surjective, it follows that $Q_{\mathcal{CatGrp}}$ is also full, faithful, and essentially surjective. □

Theorem 9.3.6. *Let $\mathcal{M}$ be a categorical group. Then, every formal diagram in $\mathcal{M}$ commutes.*

Proof. Let $M := \mathrm{Obj}(\mathcal{M})$ and let

$$\mathcal{EV} : \mathcal{CatGrp}\langle M\rangle \longrightarrow \mathcal{M}$$

be the strict categorical group functor as in Definition 4.5.22. Suppose $D : \mathcal{D} \longrightarrow \mathcal{M}$ is a formal diagram. By definition, there exists a lift

$$\widehat{D} : \mathcal{D} \longrightarrow \mathcal{CatGrp}\langle M\rangle$$

such that $\mathcal{EV} \circ \widehat{D} = D$. From Theorem 9.3.4, we know that $\mathcal{CatGrp}\langle M\rangle$ is a thin category. Since D factors thorough a thin category, the diagram D commutes. □

Examples and Applications

Chapter 10: Every Categorical Group is Equivalent to a Strict One

In this chapter we will show that given a semi-strict categorical group $(\mathscr{M}, -\otimes -, (-)', I)$, we can construct a strict categorical group $(\mathscr{T}, -\cdot-, (-)^\flat, e)$ such that $\mathscr{M}$ is categorically equivalent to $\mathscr{T}$ via categorical group functors.

10.1 The Setup

In this section, we introduce the notations and definitions that will be used throughout the chapter. This section serves to bridge the concepts and constructions introduced in previous chapters with those that will play a central role in the present discussion. Subsequent sections will refer only to the definitions and notations introduced in this section, aside from the basic notions of categorical groups. Therefore, this chapter can be regarded as largely self-contained.

Framework 10.1.1. Throughout this chapter, let $(\mathscr{M}, \otimes, (-)', I)$ be a semi-strict categorical group. We also introduce the following notation:

$$M := \mathrm{Obj}(\mathscr{M}) \qquad \text{the objects of } \mathscr{M} \tag{10.1.2}$$

$$T := \mathcal{G}rp\langle M\rangle \qquad \text{the free group generated by } M \tag{10.1.3}$$

$$N := \mathcal{DM}on\langle M\rangle \qquad \text{the free dashed monoid generated by } M \tag{10.1.4}$$

$$\mathscr{N} := \mathcal{SSC}at\mathcal{G}rp\langle M\rangle \qquad \text{the free semi-strict categorical group generated by } M \tag{10.1.5}$$

We will use the following notation and results:

- Recall that the n-fold negation is denoted by $(-)^{(n)}$. Since $\mathscr{M}$ is a semi-strict categorical group, M forms a dashed monoid with I as the unit element, the tensor as the monoid multiplication, and $(-)'$ as the dash operation.

- See Construction 4.5.4 for the general construction of a free group. In the free group generated by M, the unit element is denoted by e. Group multiplication is denoted by concatenation: for $x, y \in T$, the product is xy. The inverse operation is denoted by $(-)^{-1}$ and also by $(-)^{\flat}$; for $x \in T$, we write $x^{-1} = x^{\flat}$. The notation $x^{\flat}$ is used to distinguish the group inverse from the morphism inverse when T is promoted to a strict categorical group $\mathscr{T}$ later.

  The inclusion of elements of M into the free group is denoted by $[-]_T$. That is, the inclusion of $x \in M$ in T is $[x]_T$. The subscript T distinguishes this from inclusions into the free dashed monoid.

- See Definition 8.1.17 for the general construction of a free semi-strict categorical group. In the free semi-strict categorical group $\mathscr{N}$, the unit object is J. The tensor product is denoted by $-\bullet-$ and the negator by $(-)^\dagger$, with n-fold negation $(-)^{(\dagger n)}$. For example, the tensor product of $x,y \in \mathcal{N}$ is $x \bullet y$ and the negation of $x \in \mathcal{N}$ is $x^\dagger$. Since $\mathcal{N}$ is the free semi-strict categorical group generated by M, the objects of $\mathcal{N}$, denoted $\mathrm{Obj}(\mathcal{N}) = N$, form the free dashed monoid generated by M as in Construction 4.5.2.

The inclusion of M into the objects of the free semi-strict group N is denoted by $[-]_N$. For $x \in M$, the inclusion is $[x]_N$.

- By Theorem 9.3.2, the free semi-strict categorical group $\mathcal{N}$ is a thin category: for any $x,y \in \mathcal{N}$, there is at most one morphism $x \longrightarrow y$.

  To distinguish morphisms in the free semi-strict categorical group $\mathcal{N}$ from those in other categories, we refer to morphisms in $\mathcal{N}$ as arrows. We denote an arrow $f : x \longrightarrow y$ in $\mathcal{N}$ by the special symbol

  $$f : x \multimap\!\!\!\rightarrow y.$$

  Since $\mathcal{N}$ is thin, an arrow $f : x \multimap\!\!\!\rightarrow y$ is completely determined by its domain x and codomain y, so we may omit the label f and simply write $x \multimap\!\!\!\rightarrow y$.

- Let $Q_{\mathcal{SSCatGrp}} : \mathcal{SSCatGrp}\langle S\rangle \longrightarrow \mathcal{Grp}\langle S\rangle$ be the induced strict functor, where $\mathcal{Grp}\langle S\rangle$ is considered as the discrete strict categorical group; we abbreviate this as Q.

- Definition 8.1.2 characterizes the arrows in $\mathcal{N}$ as follows: for any $x,y \in N$, there exists a unique morphism $x \multimap\!\!\!\rightarrow y$ if and only $Q(x) = Q(y)$. Thus, the cancellation isomorphisms are defined as

$$\eta_x = (J \multimap x \bullet x^\dagger)$$
$$\epsilon_x = (x^\dagger \bullet x \multimap J). \qquad \diamond$$

Definition 10.1.6. Define a function $P : T \longrightarrow N$ as follows: Let

$$x = [a_1]_T^{(\flat\ k_1)} \cdots [a_m]_T^{(\flat\ k_m)} \in T$$

be a reduced word in T. That is, $a_i \in M$, $k_i \in \{0,1\}$ for $1 \le i \le m$, and $a_i = a_{i+1}$ implies $k_i = k_{i+1}$ for $1 \le i \le m-1$. The condition '$a_i = a_{i+1}$ implies $k_i = k_{i+1}$' ensures that the cancellation pairs $a \bullet a^\flat$ or $a^\flat \bullet a$ are not present in the reduced word. Since T is the free group generated by M, we know that every element in T has a unique reduced word representation.

Now, define $P(x)$ as

$$P(x) := [a_1]_N^{(\dagger k_1)} \bullet \cdots \bullet [a_m]_N^{(\dagger k_m)}. \tag{10.1.7}$$

$\diamond$

Remark 10.1.8. It should noted that the function $P : T \longrightarrow N$ is not a dashed-monoid homomorphis, it respects neither multiplication nor the negation. For instance, if $M = \{a\}$ then we get

$$\begin{aligned} \phi([a]_T) \bullet \phi([a]_T^{-1}) &= [a]_N \bullet [a]_N^\dagger \\ &\neq J \\ &= \phi(e) \\ &= \phi([a]_T [a]_T^{-1}). \end{aligned}$$

Let $b = [a]_T^{-1}$, then we have

$$\begin{aligned}\phi(b)^\dagger &= [a]_N^{(\dagger 2)}\\ &\neq [a]_N\\ &= \phi([a]_T)\\ &= \phi(b^{-1}).\end{aligned}$$

◇

Definition 10.1.9. Define a induced dashed monoid homomorphism

$$Q : N \longrightarrow T$$

as follows: Let

$$[-]_T : M \longrightarrow T$$

be the inclusion of M into T. Then, we get a unique induced dashed monoid homomorphism $Q : N \longrightarrow T$ such that

$$Q([a]_N) = [a]_T \tag{10.1.10}$$

for an element $a \in M$.

Since $Q : N \longrightarrow T$ is a dashed monoid homomorphism, we get the following equalities:

$$Q(J) = e \tag{10.1.11}$$

$$Q(x \bullet y) = Q(x)Q(y) \qquad \text{for } x, y \in N \tag{10.1.12}$$

$$\Phi(x^\dagger) = (Q(x))^{-1} \qquad \text{for } x \in N. \tag{10.1.13}$$

◇

Remark 10.1.14. Refer Construction 4.5.5 for the general case. ◇

Proposition 10.1.15. *The equality of functions*

$$Q \circ P = \mathrm{Id}_T \tag{10.1.16}$$

hold.

Proof. Let $x \in T$ and let $x = [a_1]_T^{(\flat k_1)} \cdots [a_m]_T^{(\flat k_m)} \in T$ be the reduced word representation.

$$\begin{aligned} Q(P(x)) &= Q([a_1]_N^{(\dagger k_1)} \bullet \cdots \bullet [a_m]_N^{(\dagger k_m)}) && (10.1.7) \\ &= [a_1]_T^{(\flat k_1)} \cdots [a_m]_T^{(\flat k_m)} && (10.1.10), (10.1.12), \text{ and } (10.1.13) \\ &= x. \end{aligned}$$

This proves the proposition. □

Definition 10.1.17. Define a functor $\mathcal{EV} : \mathcal{N} \longrightarrow \mathcal{M}$ as follows:

Consider the function $\mathrm{Id}_M : M \longrightarrow \mathrm{Obj}(\mathcal{M}) = M$. Since $\mathcal{N}$ is a free semi-strict categorical group, we get a unique strict functor between semi-strict categorical group

$$\mathcal{EV} : \mathcal{N} \longrightarrow \mathcal{M}$$

such that

$$\mathcal{EV}([a]_N) = a \tag{10.1.18}$$

for an element $a \in M$.

The functor $\mathcal{EV} : \mathcal{N} \longrightarrow \mathcal{M}$ is completely determined by the following relations: Define a function $\mathrm{ev} : N \longrightarrow M$ to be the restriction of $\mathcal{EV} : \mathcal{N} \longrightarrow \mathcal{M}$ on objects.

Then for $a \in M$ and $x, y \in N$, the equalities

$$\mathrm{ev}(J) = I \tag{10.1.19}$$

$$\mathrm{ev}([a]_N) = a \tag{10.1.20}$$

$$\mathrm{ev}(x \bullet y) = \mathrm{ev}(x) \otimes \mathrm{ev}(y) \tag{10.1.21}$$

$$\mathrm{ev}(x^\dagger) = \mathrm{ev}(x)' \tag{10.1.22}$$

hold. Moreover, for objects $a, b, x, y, z \in \mathcal{N}$ the equalities

$$\mathcal{EV}(x \mathrel{-\!\!\circ\!\!\to} x) = 1_{\mathrm{ev}(x)} \tag{10.1.23}$$

$$\mathcal{EV}(a \mathrel{-\!\!\circ\!\!\to} x) \otimes \mathcal{EV}(b \mathrel{-\!\!\circ\!\!\to} y) = \mathcal{EV}(a \bullet b \mathrel{-\!\!\circ\!\!\to} x \bullet y) \tag{10.1.24}$$

$$\mathcal{EV}(y \mathrel{-\!\!\circ\!\!\to} z) \circ \mathcal{EV}(x \mathrel{-\!\!\circ\!\!\to} y) = \mathcal{EV}(x \mathrel{-\!\!\circ\!\!\to} z) \tag{10.1.25}$$

$$\mathcal{EV}(J \mathrel{-\!\!\circ\!\!\to} x \bullet x^\dagger) = \eta_{\mathrm{ev}(x)} \tag{10.1.26}$$

$$\mathcal{EV}(x^\dagger \bullet x \mathrel{-\!\!\circ\!\!\to} J) = \epsilon_{\mathrm{ev}(x)} \tag{10.1.27}$$

hold. ◇

Definition 10.1.28. Define a function $F : T \longrightarrow M$ by setting

$$F := \mathrm{ev} \circ P. \tag{10.1.29}$$

That is, for a reduced word $x = [a_1]_T^{(\flat k_1)} \cdots [a_m]_T^{(\flat k_m)} \in T$ where $a_i \in M$ and $k_i \in \{0, 1\}$ we get

$$F(x) = a_1^{(k_1)} \otimes \cdots \otimes a_m^{(k_m)}. \tag{10.1.30}$$

◇

10.2 Construction of The Strict Categorical Group

In this section, using the function $F : T \longrightarrow M$ as in Definition 10.1.28, we will give a strict-categorical group structure to T.

Definition 10.2.1. Define a morphism $F_0 : I \longrightarrow F(e)$ in $\mathscr{M}$ as follows: Define

$$F_0 := 1_I : I \longrightarrow F(e). \tag{10.2.2}$$

This is well-defined since we have

$$F(e) = I. \qquad (10.1.30) \qquad \diamond$$

Definition 10.2.3. For $x, y \in T$, we will define a morphism

$$F_2(x, y) : F(x) \otimes F(y) \longrightarrow F(xy)$$

in $\mathscr{M}$ as follows:

We have objects $P(x) \bullet P(y)$ and $P(xy)$ in $\mathscr{N}$. We get

$$\begin{aligned}
Q(P(x) \bullet P(y)) &= Q(P(x))Q(P(y)) && (10.1.12)\\
&= xy && (10.1.16)\\
&= Q(P(xy)). && (10.1.16)
\end{aligned}$$

Therefore, the two objects, $P(x) \bullet P(y), P(xy) \in \mathscr{N}$, map to the same element in the free group T. Thus, from the construction of $\mathscr{N}$, there is a unique arrow

$$P(x) \bullet P(y) \multimap P(xy).$$

We define $F_2(x, y) : F(x) \otimes F(y) \longrightarrow F(xy)$ as

$$F_2(x, y) := \mathcal{EV}(P(x) \bullet P(y) \multimap P(xy)). \tag{10.2.4}$$

This assignment is well-defined, that is, the domain and codomain of $F_2(x, y)$ match with the required domain and codomain since

$$\begin{aligned}
F(x) \otimes F(y) &= \mathrm{ev}(P(x)) \otimes \mathrm{ev}(P(y)) && (10.1.29)\\
&= \mathrm{ev}(P(x) \bullet P(y)). && (10.1.21)
\end{aligned}$$

and

$$F(xy) = \mathrm{ev}(P(xy)). \qquad (10.1.29) \qquad \diamond$$

Proposition 10.2.5. *For $x, y \in M$, the equality*

$$F_2([x]_T, [y]_T) = 1_{x \otimes y} \tag{10.2.6}$$

holds.

Proof. We have

$$\begin{aligned}
F_2([x]_T, [y]_T) &= \mathcal{EV}\big(P([x]_T) \bullet P([y]_T) \multimap P([x]_T[y]_T)\big) && (10.2.4) \\
&= \mathcal{EV}([x]_N \bullet [y]_N \multimap [x]_N \bullet [y]_N) && (10.1.7) \\
&= 1_{\mathrm{ev}([x]_N \bullet [y]_N)} && (10.1.23) \\
&= 1_{x \otimes y}. && (10.1.20) \text{ and } (10.1.21)
\end{aligned}$$

This proves the proposition. $\square$

Proposition 10.2.7. *For $x \in T$, the diagrams*

$$\begin{tikzcd}
Fx \otimes I \arrow[r, "1_{Fx} \otimes F_0"] \arrow[d, equal] & Fx \otimes Fe \arrow[d, "F_2(x,e)"] \\
Fx \arrow[r, equal] & F(xe)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
I \otimes Fx \arrow[r, "F_0 \otimes 1_{Fx}"] \arrow[d, equal] & Fe \otimes Fx \arrow[d, "F_2(e,x)"] \\
Fx \arrow[r, equal] & F(ex)
\end{tikzcd}$$

commute. That is, the equalities

$$F_2(x, e) \circ 1_{Fx} \otimes F_0 = 1_{F(x)} = F_2(e, x) \circ F_0 \otimes 1_{Fx} \tag{10.2.8}$$

hold.

Proof. Observe that

$$\begin{aligned}
F_2(x,e)\circ 1_{Fx}\otimes F_0 &= F_2(x,e) && (10.2.2)\\
&= \mathcal{EV}(Px\bullet Pe \multimap P(xe)) && (10.2.4)\\
&= \mathcal{EV}(Px\bullet J \multimap Px) && (10.1.7)\\
&= \mathcal{EV}(Px \multimap Px) &&\\
&= 1_{ev(Px)} && (10.1.23)\\
&= 1_{F(x)}. && (10.1.29)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
F_2(e,x)\circ F_0\otimes 1_{Fx} &= F_2(e,x) && (10.2.2)\\
&= \mathcal{EV}(Pe\bullet Px \multimap P(ex)) && (10.2.4)\\
&= \mathcal{EV}(J\bullet Px \multimap Px) && (10.1.7)\\
&= \mathcal{EV}(Px \multimap Px) &&\\
&= 1_{ev(Px)} && (10.1.23)\\
&= 1_{F(x)}. && (10.1.29)
\end{aligned}$$

□

Convention 10.2.9. In a monoidal category, the monoidal product operation bounds stronger than the composition operation. For instance, the notation $f\circ g\otimes h$ stands for $f\circ(g\otimes h)$. ◇

Proposition 10.2.10. *For $x, y, z \in T$ the diagram*

$$\begin{tikzcd}
Fx \otimes Fy \otimes Fz \arrow[r, "F_2(x,y) \otimes 1_{Fz}"] \arrow[d, "1_{Fx} \otimes F_2(y,z)"'] & F(xy) \otimes Fz \arrow[d, "F_2(xy,z)"] \\
Fx \otimes F(yz) \arrow[r, "F_2(x,yz)"'] & F(xyz)
\end{tikzcd}$$

commutes. In other words, the equality

$$F_2(xy,z) \circ F_2(x,y) \otimes 1_{Fz} = F_2(x,yz) \circ 1_{Fx} \otimes F_2(y,z) \tag{10.2.11}$$

holds.

Proof. In the above diagram, the top-right composite is given by

$$\begin{aligned}
&F_2(xy,z) \circ F_2(x,y) \otimes 1_{Fz} \\
&= F_2(xy,z) \circ F_2(x,y) \otimes 1_{\mathrm{ev}(Pz)} && (10.1.29) \\
&= F_2(xy,z) \circ F_2(x,y) \otimes \mathcal{EV}(Pz \multimap Pz) && (10.1.23) \\
&= \mathcal{EV}(Pxy \bullet Pz \multimap Pxyz) \circ \mathcal{EV}(Px \bullet Py \multimap Pxy) \otimes \mathcal{EV}(Pz \multimap Pz) && (10.2.4) \\
&= \mathcal{EV}(Pxy \bullet Pz \multimap Pxyz) \circ \mathcal{EV}(Px \bullet Py \bullet Pz \multimap Pxy \bullet Pz) && (10.1.24) \\
&= \mathcal{EV}(Px \bullet Py \bullet Pz \multimap Pxyz). && (10.1.25)
\end{aligned}$$

The left-bottom composite is given by

$$
\begin{aligned}
& F_2(x,yz)\circ 1_{Fx}\otimes F_2(y,z) \\
&= F_2(x,yz)\circ 1_{\mathrm{ev}(Px)}\otimes F_2(y,z) && (10.1.29) \\
&= F_2(x,yz)\circ \mathcal{EV}(Px \multimap Px)\otimes F_2(y,z) && (10.1.23) \\
&= \mathcal{EV}(Px\bullet Pyz \multimap Pxyz)\circ \mathcal{EV}(Px \multimap Px)\otimes \mathcal{EV}(Py\bullet Pz \multimap Pyz) && (10.2.4) \\
&= \mathcal{EV}(Px\bullet Pyz \multimap Pxyz)\circ \mathcal{EV}(Px\bullet Py\bullet Pz \multimap Px\bullet Pyz) && (10.1.24) \\
&= \mathcal{EV}(Px\bullet Py\bullet Pz \multimap Pxyz). && (10.1.25)
\end{aligned}
$$

Since the above equalities match, the diagram commutes. □

Definition 10.2.12. For $x\in T$ define $F_1(x):(Fx)' \longrightarrow F(x^{-1})$ in $\mathcal{M}$ as follows:

We have objects $(Px)^\dagger$ and $P(x^{-1})$ in $\mathcal{N}$. We get

$$
\begin{aligned}
Q((Px)^\dagger) &= (QP(x))^{-1} && (10.1.13) \\
&= x^{-1} && (10.1.16) \\
&= QP(x^{-1}). && (10.1.16)
\end{aligned}
$$

Therefore, the two objects, $(Px)^\dagger$ and $P(x^{-1})$, map to the same element in the free group T. Thus, from the construction of $\mathcal{N}$, there is a unique arrow

$$(Px)^\dagger \longrightarrow P(x^{-1}).$$

We define $F_1(x):(Fx)' \longrightarrow F(x^{-1})$ as

$$F_1(x) := \mathcal{EV}((Px)^\dagger \multimap P(x^{-1})). \tag{10.2.13}$$

Here, the domain and codomain of $F_1(x)$ match with the required domain and codomain since

$$\begin{aligned}(Fx)' &= (\mathrm{ev}(Px))' && (10.1.29)\\ &= \mathrm{ev}((Px)^\dagger) && (10.1.22)\end{aligned}$$

and

$$F(x^{-1}) = \mathrm{ev}(P(x^{-1})). \qquad (10.1.29) \qquad \diamond$$

Proposition 10.2.14. *For $x \in M$, the equality*

$$F_1([x]_T) = 1_{x'} \tag{10.2.15}$$

holds.

Proof. We have

$$\begin{aligned}F_1([x]_T) &= \mathcal{EV}\big((P[x]_T)^\dagger \multimap P([x]_T^{-1})\big) && (10.2.13)\\ &= \mathcal{EV}([x]_N^\dagger \multimap [x]_N^\dagger) && (10.1.7)\\ &= 1_{\mathrm{ev}([x]_N^\dagger)} && (10.1.23)\\ &= 1_{x'}. && (10.1.20) \text{ and } (10.1.22)\end{aligned}$$

This proves the proposition. $\square$

Proposition 10.2.16. *For $x \in T$, the diagram*

$$\begin{tikzcd}(Fx)' \otimes Fx \arrow[r, "F_1(x) \otimes 1_{Fx}"] \arrow[d, "\epsilon_{Fx}"'] & F(x^{-1}) \otimes Fx \arrow[r, "F_2(x^{-1}, x)"] & F(x^{-1}x) \arrow[d, equal] \\ I \arrow[rr, "F_0"] & & F(e)\end{tikzcd}$$

commutes. That is, the equality

$$F_2(x^{-1},x)\circ F_1(x)\otimes 1_{Fx} = F_0 \circ \epsilon_{Fx} \tag{10.2.17}$$

holds.

Proof. Observe that

$$\begin{aligned}
&F_2(x^{-1},x)\circ F_1(x)\otimes 1_{Fx} \\
&= F_2(x^{-1},x)\circ F_1(x)\otimes 1_{\mathrm{ev}(Px)} && (10.1.29)\\
&= F_2(x^{-1},x)\circ F_1(x)\otimes \mathcal{E}\mathcal{V}(Px \mathrel{-\!\!\circ\!\!\to} Px) && (10.1.23)\\
&= \mathcal{E}\mathcal{V}(Px^{-1}\bullet Px \mathrel{-\!\!\circ\!\!\to} Px^{-1}x)\circ F_1(x)\otimes \mathcal{E}\mathcal{V}(Px \mathrel{-\!\!\circ\!\!\to} Px) && (10.2.4)\\
&= \mathcal{E}\mathcal{V}(Px^{-1}\bullet Px \mathrel{-\!\!\circ\!\!\to} Px^{-1}x)\\
&\qquad\qquad \circ\, \mathcal{E}\mathcal{V}((Px)^{\dagger} \mathrel{-\!\!\circ\!\!\to} Px^{-1})\otimes \mathcal{E}\mathcal{V}(Px \mathrel{-\!\!\circ\!\!\to} Px) && (10.2.13)\\
&= \mathcal{E}\mathcal{V}(Px^{-1}\bullet Px \mathrel{-\!\!\circ\!\!\to} Pe)\circ \mathcal{E}\mathcal{V}((Px)^{\dagger}\bullet Px \mathrel{-\!\!\circ\!\!\to} Px^{-1}\bullet Px) && (10.1.24)\\
&= \mathcal{E}\mathcal{V}((Px)^{\dagger}\bullet Px \mathrel{-\!\!\circ\!\!\to} Pe) && (10.1.25)\\
&= \mathcal{E}\mathcal{V}((Px)^{\dagger}\bullet Px \mathrel{-\!\!\circ\!\!\to} J) && (10.1.7)\\
&= \epsilon_{\mathrm{ev}(Px)} && (10.1.27)\\
&= \epsilon_{Fx} && (10.1.29)\\
&= F_0\circ \epsilon_{Fx}. && (10.2.2)
\end{aligned}$$

Thus, the diagram commutes. □

Proposition 10.2.18. *For $x \in T$, the diagram*

$$\begin{tikzcd}
Fx\otimes (Fx)' \arrow[r, "1_{Fx}\otimes F_1(x)"] \arrow[d, "\eta_{Fx}^{-1}"'] & Fx\otimes F(x^{-1}) \arrow[r, "{F_2(x,x^{-1})}"] & F(xx^{-1}) \arrow[d, equal] \\
I \arrow[rr, "F_0"'] & & F(e)
\end{tikzcd}$$

commutes. That is, the equality

$$F_2(x, x^{-1}) \circ 1_{Fx} \otimes F_1(x) = F_0 \circ \eta_{Fx}^{-1} \tag{10.2.19}$$

holds.

Proof. We have

$$
\begin{aligned}
& F_2(x, x^{-1}) \circ 1_{Fx} \otimes F_1(x) \\
&= F_2(x, x^{-1}) \circ 1_{\mathrm{ev}(Px)} \otimes F_1(x) && (10.1.29) \\
&= F_2(x, x^{-1}) \circ \mathcal{EV}(Px \multimapdotbothvert Px) \otimes F_1(x) && (10.1.23) \\
&= \mathcal{EV}(Px \bullet Px^{-1} \longrightarrow Pxx^{-1}) \circ \mathcal{EV}(Px \longrightarrow Px) \otimes F_1(x) && (10.2.4) \\
&= \mathcal{EV}(Px \bullet Px^{-1} \longrightarrow Pxx^{-1}) \\
&\qquad \circ \mathcal{EV}(Px \longrightarrow Px) \otimes \mathcal{EV}((Px)^{\dagger} \longrightarrow Px^{-1}) && (10.2.13) \\
&= \mathcal{EV}(Px \bullet Px^{-1} \longrightarrow Pe) \circ \mathcal{EV}(Px \bullet (Px)^{\dagger} \longrightarrow Px \bullet Px^{-1}) && (10.1.24) \\
&= \mathcal{EV}(Px \bullet (Px)^{\dagger} \longrightarrow Pe) && (10.1.25) \\
&= \mathcal{EV}(Px \bullet (Px)^{\dagger} \longrightarrow I) && (10.1.7) \\
&= \eta_{\mathrm{ev}(Px)}^{-1} && (10.1.26) \\
&= \eta_{Fx}^{-1} && (10.1.29) \\
&= F_0 \circ \eta_{Fx}^{-1}. && (10.2.2)
\end{aligned}
$$

Thus, the diagram commutes. □

Definition 10.2.20. Define a category $\mathscr{T}$ as follows:

- The objects are given by the free group generated by M. That is,

$$\mathrm{Obj}(\mathscr{T}) := T. \tag{10.2.21}$$

- For $x, y \in T$, the set of morphisms $\mathrm{Hom}_{\mathscr{T}}(x,y)$ is given by

$$\mathrm{Hom}_{\mathscr{T}}(x,y) := \mathrm{Hom}_{\mathscr{M}}(Fx, Fy).$$

  We will distinguish a morphism in $\mathscr{T}$ from the corresponding morphisms in $\mathscr{M}$ by a subscript. That is, a morphism $f_{\mathscr{T}} : x \longrightarrow y$ corresponds to $f : Fx \longrightarrow Fy$ in $\mathscr{M}$. Therefore, we get

$$\mathrm{Hom}_{\mathscr{T}}(x,y) = \{f_{\mathscr{T}} : x \longrightarrow y \mid f \in \mathrm{Hom}_{\mathscr{M}}(Fx, Fy)\}. \tag{10.2.22}$$

- For $x \in \mathscr{T}$, the identity morphism, $1_x : x \longrightarrow x$, is given by

$$1_x := (1_{Fx})_{\mathscr{T}}. \tag{10.2.23}$$

- Let $f_{\mathscr{T}} : x \longrightarrow y$ and $g_{\mathscr{T}} : y \longrightarrow z$ be morphisms in $\mathscr{T}$. We get morphisms $f : Fx \longrightarrow Fy$ and $g : Fy \longrightarrow Fz$ in $\mathscr{M}$. The composite $g_{\mathscr{T}} \circ f_{\mathscr{T}} : x \longrightarrow z$ is given by

$$g_{\mathscr{T}} \circ f_{\mathscr{T}} := (g \circ f)_{\mathscr{T}}. \tag{10.2.24}$$

Above assignment satisfies the following conditions: For a morphism

$$f_{\mathscr{T}} : x \longrightarrow y$$

in $\mathscr{T}$ we get

$$\begin{aligned} f_{\mathscr{T}} \circ 1_x &= f_{\mathscr{T}} \circ (1_{Fx})_{\mathscr{T}} && (10.2.23) \\ &= (f \circ 1_{Fx})_{\mathscr{T}} && (10.2.24) \\ &= f_{\mathscr{T}} \end{aligned}$$

and

$$1_y \circ f_{\mathscr{T}} = (1_{Fy})_{\mathscr{T}} \circ f_{\mathscr{T}} \qquad (10.2.23)$$
$$= \left(1_{Fy} \circ f\right)_{\mathscr{T}} \qquad (10.2.24)$$
$$= f_{\mathscr{T}}.$$

Thus, the identity conditions are satisfied.

For morphisms $f_{\mathscr{T}} : x \longrightarrow y$, $g_{\mathscr{T}} : y \longrightarrow z$, and $h_{\mathscr{T}} : z \longrightarrow w$ be in $\mathscr{T}$ we get

$$h_{\mathscr{T}} \circ (g_{\mathscr{T}} \circ f_{\mathscr{T}}) = h_{\mathscr{T}} \circ (g \circ f)_{\mathscr{T}} \qquad (10.2.24)$$
$$= (h \circ g \circ f)_{\mathscr{T}} \qquad (10.2.24)$$
$$= (h \circ g)_{\mathscr{T}} \circ f_{\mathscr{T}} \qquad (10.2.24)$$
$$= (h_{\mathscr{T}} \circ g_{\mathscr{T}}) \circ f_{\mathscr{T}}. \qquad (10.2.24)$$

Thus, the associativity condition is satisfied. ◇

Definition 10.2.25. Define a functor $F : \mathscr{T} \longrightarrow \mathscr{M}$ as follows:

- The functor F on objects is equal to the function $F : T \longrightarrow M$ as in Definition 10.1.28.
- Let $f_{\mathscr{T}} : x \longrightarrow y$ be a morphism in $\mathscr{T}$ where $f : Fx \longrightarrow Fy$ is a morphism in $\mathscr{M}$. Define a morphism $F(f_{\mathscr{T}}) : Fx \longrightarrow Fy$ as

$$F(f_{\mathscr{T}}) := f. \qquad (10.2.26)$$

Above assignment satisfies the functor conditions as follows: For an object $x \in \mathscr{T}$, we have

$$F(1_x) = F((1_{Fx})_{\mathscr{T}}) \qquad (10.2.23)$$
$$= 1_{Fx}. \qquad (10.2.26)$$

Thus, the identity condition is satisfied. For morphisms $f_{\mathscr{T}} : x \longrightarrow y$ and $g_{\mathscr{T}} : y \longrightarrow z$ in $\mathscr{T}$, we have

$$\begin{aligned} F(g_{\mathscr{T}} \circ f_{\mathscr{T}}) &= F((g \circ f)_{\mathscr{T}}) && (10.2.24) \\ &= g \circ f && (10.2.26) \\ &= F(g_{\mathscr{T}}) \circ F(f_{\mathscr{T}}). && (10.2.26) \end{aligned}$$

Thus, the composition condition is satisfied. ◇

Definition 10.2.27. Define a functor $- \cdot - : \mathscr{T} \times \mathscr{T} \longrightarrow \mathscr{T}$, called the tensor product on $\mathscr{T}$, as follows:

- The tensor product on the objects of $\mathscr{T}$ is same as the group multiplication in T.
- For morphisms $f_{\mathscr{T}} : x \longrightarrow z$ and $g_{\mathscr{T}} : y \longrightarrow w$ the product $fg : xy \longrightarrow zw$ is equal to the composite

$$f_{\mathscr{T}} g_{\mathscr{T}} := \left(F_2(z,w) \circ (f \otimes g) \circ F_2^{-1}(x,y)\right)_{\mathscr{T}}. \tag{10.2.28}$$

  That is, the morphism $f_{\mathscr{T}} g_{\mathscr{T}} : xy \longrightarrow zw$ is equal to the morphism

$$F(xy) \xrightarrow{F_2^{-1}(x,y)} F(x) \otimes F(y) \xrightarrow{f \otimes g} F(z) \otimes F(w) \xrightarrow{F_2(z,w)} F(zw)$$

  in $\mathrm{Hom}_{\mathscr{M}}(F(xy), F(zw))$.

Above assignment satisfies the functor conditions as follows: For $x, y \in \mathscr{T}$ we get

$$
\begin{aligned}
1_x 1_y &= (1_{Fx})_{\mathscr{T}} (1_{Fy})_{\mathscr{T}} && (10.2.23)\\
&= \big(F_2(x,y) \circ (1_{Fx} \otimes 1_{Fy}) \circ F_2^{-1}(x,y)\big)_{\mathscr{T}} && (10.2.28)\\
&= \big(F_2(x,y) \circ 1_{Fx \otimes Fy} \circ F_2^{-1}(x,y)\big)_{\mathscr{T}}\\
&= (1_{F(xy)})_{\mathscr{T}}\\
&= 1_{xy}. && (10.2.23)
\end{aligned}
$$

Thus, the identity condition is satisfied. Now, for morphisms

$$
\begin{aligned}
f_{\mathscr{T}} &: a \longrightarrow b & \qquad g_{\mathscr{T}} &: b \longrightarrow c\\
h_{\mathscr{T}} &: x \longrightarrow y & \qquad k_{\mathscr{T}} &: y \longrightarrow z
\end{aligned}
$$

in $\mathscr{T}$, we get

$$
\begin{aligned}
&(g_{\mathscr{T}} \circ f_{\mathscr{T}})(k_{\mathscr{T}} \circ h_{\mathscr{T}})\\
&\qquad = (g \circ f)_{\mathscr{T}} (k \circ h)_{\mathscr{T}} && (10.2.24)\\
&\qquad = \big(F_2(c,z) \circ \big((g \circ f) \otimes (k \circ h)\big) \circ F_2^{-1}(a,x)\big)_{\mathscr{T}} && (10.2.28)\\
&\qquad = \big(F_2(c,z) \circ g \otimes k \circ f \otimes h \circ F_2^{-1}(a,x)\big)_{\mathscr{T}}\\
&\qquad = \big(F_2(c,z) \circ g \otimes k \circ F_2^{-1}(b,y) \circ F_2(b,y) \circ f \otimes h \circ F_2^{-1}(a,x)\big)_{\mathscr{T}}\\
&\qquad = \big(F_2(c,z) \circ g \otimes k \circ F_2^{-1}(b,y)\big)_{\mathscr{T}}\\
&\qquad\qquad \circ \big(F_2(b,y) \circ f \otimes h \circ F_2^{-1}(a,x)\big)_{\mathscr{T}} && (10.2.24)\\
&\qquad = g_{\mathscr{T}} k_{\mathscr{T}} \circ f_{\mathscr{T}} h_{\mathscr{T}}. && (10.2.28)
\end{aligned}
$$

Thus, the composition condition is satisfied. ◇

Proposition 10.2.29. *For morphisms*

$$
(f)_{\mathscr{T}} : [x]_T \longrightarrow [z]_T \qquad \text{and} \qquad (g)_{\mathscr{T}} : [y]_T \longrightarrow [w]_T
$$

in $\mathscr{T}$ where $x, y, z, w \in M$, the equality

$$(f)_{\mathscr{T}}(g)_{\mathscr{T}} = (f \otimes g)_{\mathscr{T}} \tag{10.2.30}$$

holds.

Proof. We have

$$\begin{aligned}
(f)_{\mathscr{T}}(g)_{\mathscr{T}} &= \big(F_2([z]_T, [w]_T) \circ f \otimes g \circ F_2^{-1}([x]_T, [y]_T)\big)_{\mathscr{T}} && (10.2.28) \\
&= (f \otimes g)_{\mathscr{T}}. && (10.2.6)
\end{aligned}$$

This proves the proposition. □

Proposition 10.2.31. *Consider the functors $L, R : \mathscr{T} \longrightarrow \mathscr{T}$ as follows:*

$$\begin{aligned}
L(x) &= ex && \textit{on objects} \\
L(f) &= 1_e f && \textit{on morphisms,}
\end{aligned}$$

and

$$\begin{aligned}
R(x) &= xe && \textit{on objects} \\
R(f) &= f 1_e && \textit{on morphisms.}
\end{aligned}$$

Then, the equalities

$$L = \mathrm{Id}_{\mathscr{T}} = R$$

of functors hold. In particular, for a morphism $f : x \longrightarrow y$ in $\mathscr{T}$, the equalities

$$1_e f = f = f 1_e \tag{10.2.32}$$

of morphisms in G hold.

Proof. The equalities on object follow from the group structure of T. Let $f_{\mathscr{T}} : x \longrightarrow y$ be a morphism in $\mathscr{T}$ where $f : Fx \longrightarrow Fy$ is a morphism in $\mathscr{M}$. We have

$$\begin{aligned}
1_e f_{\mathscr{T}} &= (1_{Fe})_{\mathscr{T}} f_{\mathscr{T}} && (10.2.23)\\
&= (1_I)_{\mathscr{T}} f_{\mathscr{T}} && (10.1.30)\\
&= \big(F_2(e,y) \circ 1_J \otimes f \circ F_2^{-1}(e,x)\big)_{\mathscr{T}} && (10.2.28)\\
&= (1_{Fy} \circ 1_J \otimes f \circ 1_{Fx})_{\mathscr{T}} && (10.2.8)\\
&= (1_{Fy} \circ f \circ 1_{Fx})_{\mathscr{T}} && (3.1.37)\\
&= f_{\mathscr{T}}.
\end{aligned}$$

Also, we get

$$\begin{aligned}
f_{\mathscr{T}} 1_e &= f_{\mathscr{T}} (1_{Fe})_{\mathscr{T}} && (10.2.23)\\
&= f_{\mathscr{T}} (1_J)_{\mathscr{T}} && (10.1.30)\\
&= \big(F_2(y,e) \circ f \otimes 1_J \circ F_2^{-1}(x,e)\big)_{\mathscr{T}} && (10.2.28)\\
&= (1_{Fy} \circ f \otimes 1_J \circ 1_{Fx})_{\mathscr{T}} && (10.2.8)\\
&= (1_{Fy} \circ f \circ 1_{Fx})_{\mathscr{T}} && (3.1.38)\\
&= f_{\mathscr{T}}.
\end{aligned}$$

This proves the proposition. □

Proposition 10.2.33. *Consider the functors $L, R : \mathscr{T} \times \mathscr{T} \times \mathscr{T} \longrightarrow \mathscr{T}$ as follows:*

$$\begin{aligned}
L(x,y,z) &= x(yz) && \textit{on objects}\\
L(f,g,h) &= f(gh) && \textit{on morphisms,}
\end{aligned}$$

and

$$R(x,y,z) = (xy)z \qquad \textit{on objects}$$

$$R(f,g,h) = (fg)h \qquad \textit{on morphisms.}$$

Then, the equality

$$L = R$$

of functors holds. In particular, for morphisms $f : a \longrightarrow x$*,* $g : b \longrightarrow y$*, and* $h : c \longrightarrow z$ *in* $\mathscr{T}$*, the equality*

$$f(gh) = (fg)h \tag{10.2.34}$$

of morphisms holds.

Proof. The equality $L = R$ on objects follows from the group structure of T. Let

$$f_{\mathscr{T}} : a \longrightarrow x, \qquad g_{\mathscr{T}} : b \longrightarrow y, \qquad \text{and} \qquad h_{\mathscr{T}} : c \longrightarrow z$$

be morphisms in $\mathscr{T}$ where

$$f : Fa \longrightarrow Fx, \qquad g : Fb \longrightarrow Fy, \qquad \text{and} \qquad h : Fc \longrightarrow Fz$$

are morphisms in $\mathscr{M}$. We have

$$\begin{aligned}
& f_{\mathscr{T}}(g_{\mathscr{T}} h_{\mathscr{T}}) \\
&= f_{\mathscr{T}} \left(F_2(y,z) \circ g \otimes h \circ F_2^{-1}(b,c)\right)_{\mathscr{T}} && (10.2.28) \\
&= \left(F_2(x,yz) \circ 1_{Fx} \otimes F_2(y,z) \circ f \otimes g \otimes h \circ 1_{Fa} \otimes F_2^{-1}(b,c) \circ F_2^{-1}(a,bc)\right)_{\mathscr{T}} && (10.2.28) \\
&= \left(F_2(xy,z) \circ F_2(x,y) \otimes 1_{Fz} \circ f \otimes g \otimes h \circ F_2^{-1}(a,b) \otimes 1_{Fc} \circ F_2^{-1}(ab,c)\right)_{\mathscr{T}} && (10.2.11) \\
&= \left(F_2(x,y) \circ f \otimes g \circ F_2^{-1}(a,b)\right)_{\mathscr{T}} h_{\mathscr{T}} && (10.2.28) \\
&= (f_{\mathscr{T}} g_{\mathscr{T}}) h_{\mathscr{T}}. && (10.2.28)
\end{aligned}$$

Thus, the tensor product on $\mathscr{T}$ is associativite. $\square$

Definition 10.2.35. Define a strict monoidal category $\mathscr{T}$ as follows:

- The underlying category $\mathscr{T}$ is as in Definition 10.2.20.
- The unit object of $\mathscr{T}$ is equal to the unit element $e \in T$ of the free group.
- The tensor product on $- \cdot - : \mathscr{T} \times \mathscr{T} \longrightarrow \mathscr{T}$ is as in Definition 10.2.27.

From Propositions 10.2.31 and 10.2.33 we know that the tensor product satisfies the unity conditions and the associativity condition. We define the unit natural isomorphisms and associativite natural isomorphism to be the identity. It follows that the above data satisfy the monoidal category axioms. $\diamond$

This makes the assignments F_2 as in Definition 10.2.3 into a natural isomorphism and functor $F : \mathscr{T} \longrightarrow \mathscr{M}$ into a monoidal functor.

Definition 10.2.36. Consider the functor $L, R : \mathscr{T} \times \mathscr{T} \longrightarrow \mathscr{M}$ as follows:

$$L(x,y) = Fx \otimes Fy \qquad \text{on objects}$$

$$L(f,g) = Ff \otimes Fg \qquad \text{on morphisms,}$$

and

$$R(x,y) = F(xy) \qquad \text{on objects}$$

$$R(f,g) = F(fg) \qquad \text{on morphisms.}$$

Define a natural isomorphism $F_2 : L \longrightarrow R$ as follows:

- For objects $x, y \in \mathscr{T}$, morphisms $F_2(x,y) : Fx \otimes Fy \longrightarrow F(xy)$ in $\mathscr{M}$ is as in Definition 10.2.3.

This assignment satisfies the naturality condition as follows: For a morphisms

$$f_{\mathscr{T}} : x \longrightarrow z \qquad \text{and} \qquad g_{\mathscr{T}} : y \longrightarrow w$$

where

$$f : Fx \longrightarrow Fz \qquad \text{and} \qquad g : Fy \longrightarrow Fw$$

are morphisms in $\mathscr{M}$, we wish to show that the diagram

$$\begin{tikzcd}
Fx \otimes Fy \arrow[r, "Ff_{\mathscr{T}} \otimes Fg_{\mathscr{T}}"] \arrow[d, "F_2(x,y)"'] & Fz \otimes Fw \arrow[d, "F_2(z,w)"] \\
F(xy) \arrow[r, "F(f_{\mathscr{T}} g_{\mathscr{T}})"'] & F(zw)
\end{tikzcd}$$

commutes. We get

$$\begin{aligned}
F(f_{\mathscr{T}} g_{\mathscr{T}}) \circ F_2(x,y) &= F\Big(\big(F_2(z,w) \circ f \otimes g \circ F_2^{-1}(x,y)\big)_{\mathscr{T}} \Big) \circ F_2(x,y) && (10.2.28) \\
&= F_2(z,w) \circ f \otimes g \circ F_2^{-1}(x,y) \circ F_2(x,y) && (10.2.26) \\
&= F_2(z,w) \circ Ff_{\mathscr{T}} \otimes Fg_{\mathscr{T}}. && (10.2.26)
\end{aligned}$$

Thus, the naturality condition is satisfied. ◇

Definition 10.2.37. Define a monoidal functor $F : \mathscr{T} \longrightarrow \mathscr{M}$ as follows:

- The underlying functor $F : \mathscr{T} \longrightarrow \mathscr{M}$ is as in Definition 10.2.25.
- The structure morphism $F_0 : I \longrightarrow F(e)$ is as in Definition 10.2.1.
- The structure natural isomorphism F_2 is as in Definition 10.2.36.

From Propositions 10.2.7 and 10.2.10, the unitor conditions and the associator conditions are satisfied. ◇

Now we will give a strict categorical group structure to $\mathscr{T}$.

Definition 10.2.38. Define a negator functor $(-)^{\flat} : \mathscr{T} \longrightarrow \mathscr{T}$ as follows:

- The negator on the objects is same as the group inverse in T. That is, for $x \in T$ we define
$$x^{\flat} := x^{-1}. \tag{10.2.39}$$
- For a morphism $f_{\mathscr{T}} : x \longrightarrow y$, the negator $f^{\flat}_{\mathscr{T}} : x^{-1} \longrightarrow y^{-1}$ is equal to the composite
$$f^{\flat}_{\mathscr{T}} := \big(F_1(y) \circ f' \circ F_1^{-1}(x)\big)_{\mathscr{T}}. \tag{10.2.40}$$
That is, the morphism $f^{\flat}_{\mathscr{T}} : x^{-1} \longrightarrow y^{-1}$ is equal to the morphism
$$F(x^{-1}) \xrightarrow{F_1^{-1}(x)} (Fx)' \xrightarrow{\quad f' \quad} (Fy)' \xrightarrow{F_1(y)} F(y^{-1})$$
in $\mathrm{Hom}_{\mathscr{M}}(F(x^{-1}), F(y^{-1}))$.

This assignment satisfies the functor conditions as follows: For $x \in T$, we have

$$\begin{aligned}
(1_x)^{\flat} &= \big((1_{Fx})_{\mathscr{T}}\big)^{\flat} && (10.2.23)\\
&= \big(F_1(x) \circ 1'_{Fx} \circ F_1^{-1}(x)\big)_{\mathscr{T}} && (10.2.40)\\
&= \big(F_1(x) \circ 1_{(Fx)'} \circ F_1^{-1}(x)\big)_{\mathscr{T}} && (10.2.40)\\
&= \big(1_{F(x^{-1})}\big)_{\mathscr{T}} &&\\
&= 1_{x^{-1}} && (10.2.23)\\
&= 1_{x^{\flat}}. && (10.2.39)
\end{aligned}$$

Thus, the identity condition is satisfied. For morphisms $f_{\mathscr{T}} : x \longrightarrow y$ and $g_{\mathscr{T}} : y \longrightarrow z$ in $\mathscr{T}$, we get

$$
\begin{aligned}
(g_{\mathscr{T}} \circ f_{\mathscr{T}})^{\flat} &= (g \circ f)^{\flat}_{\mathscr{T}} && (10.2.24) \\
&= \big(F_1(z) \circ (g \circ f)' \circ F_1^{-1}(x)\big)_{\mathscr{T}} && (10.2.40) \\
&= \big(F_1(z) \circ g' \circ f' \circ F_1^{-1}(x)\big)_{\mathscr{T}} \\
&= \big(F_1(z) \circ g' \circ F_1^{-1}(y) \circ F_1(y) \circ f' \circ F_1^{-1}(x)\big)_{\mathscr{T}} \\
&= \big(F_1(z) \circ g' \circ F_1^{-1}(y)\big)_{\mathscr{T}} \circ \big(F_1(y) \circ f' \circ F_1^{-1}(x)\big)_{\mathscr{T}} && (10.2.24) \\
&= g^{\flat}_{\mathscr{T}} \circ f^{\flat}_{\mathscr{T}}. && (10.2.40)
\end{aligned}
$$

Thus, the composition condition is satisfied. ◇

Proposition 10.2.41. *For a morphism $f_{\mathscr{T}} : [x]_T \longrightarrow [y]_T$ where $x, y \in M$, the equality*

$$
(f_{\mathscr{T}})^{\flat} = (f')_{\mathscr{T}} \tag{10.2.42}
$$

holds.

Proof. We have

$$
\begin{aligned}
(f_{\mathscr{T}})^{\flat} &= \big(F_1([y]_T) \circ f' \circ F_1^{-1}([x]_T)\big)_{\mathscr{T}} && (10.2.40) \\
&= (f')_{\mathscr{T}}. && (10.2.15)
\end{aligned}
$$

This proves the proposition. □

Proposition 10.2.43. *Consider the functors $L, R : \mathscr{T} \longrightarrow \mathscr{T}$ as follows:*

$$
\begin{aligned}
L(x) &= x^{\flat} x && \textit{on objects} \\
L(f) &= f^{\flat} f && \textit{on morphisms,}
\end{aligned}
$$

and

$$R(x) = e \qquad \textit{on objects}$$

$$= 1_e \qquad \textit{on morphisms.}$$

Then, the equality

$$L = R$$

of functors holds. In particular, for a morphism $f : x \longrightarrow y$, *the equality*

$$f^\flat f = 1_e \tag{10.2.44}$$

holds.

Proof. The equality on objects follows from the group structure of T. For a morphism $f_{\mathscr{T}} : x \longrightarrow y$ in $\mathscr{T}$ where $f : Fx \longrightarrow Fy$ is a morphism in $\mathscr{M}$, we have the following chain of equalities:

$$\begin{aligned}
& f^\flat_{\mathscr{T}} f_{\mathscr{T}} \\
&= \big(F_1(y) \circ f' \circ F_1^{-1}(x)\big)_{\mathscr{T}} f_{\mathscr{T}} && (10.2.40) \\
&= \big(F_2(y^{-1}, y) \circ F_1(y) \otimes 1_{Fy} \circ f' \otimes f \circ F_1(x)^{-1} \otimes 1_{Fx} \circ F_2^{-1}(x^{-1}, x)\big)_{\mathscr{T}} && (10.2.28) \\
&= \big(\epsilon_{Fy} \circ f' f \circ \epsilon_{Fx}^{-1}\big)_{\mathscr{T}} && (10.2.17) \\
&= (1_J)_{\mathscr{T}} && \text{naturality} \\
&= (1_{Fe})_{\mathscr{T}} \\
&= 1_e. && (10.2.23)
\end{aligned}$$

This gives us the required equality. □

Proposition 10.2.45. *Consider the functors $L, R : \mathscr{T} \longrightarrow \mathscr{T}$ as follows:*

$$L(x) = xx^\flat \qquad \textit{on objects}$$

$$L(f) = ff^\flat \qquad \textit{on morphisms,}$$

and

$$R(x) = e \qquad \textit{on objects}$$

$$R(f) = 1_e \qquad \textit{on morphisms.}$$

Then, the equality

$$L = R$$

of functors holds. In particular, for a morphism $f : x \longrightarrow y$, the equality

$$ff^\flat = 1_e \tag{10.2.46}$$

holds.

Proof. The equality on objects follows from the group structure of T. For a morphism $f_{\mathscr{T}} : x \longrightarrow y$, we have the following chain of equalities.

$$\begin{aligned}
& f_{\mathscr{T}} f^\flat_{\mathscr{T}} \\
&= f_{\mathscr{T}} \left(F_1(y) \circ f' \circ F_1^{-1}(x)\right)_{\mathscr{T}} && (10.2.40) \\
&= \left(F_2(y, y^{-1}) \circ 1_{Fy} \otimes F_1(y) \circ f \otimes f' \circ 1_{Fx} \otimes F_1(x)^{-1} \circ F_2^{-1}(x, x^{-1})\right)_{\mathscr{T}} && (10.2.28) \\
&= \left(\eta^{-1}_{Fy} \circ ff' \circ \eta_{Fx}\right)_{\mathscr{T}} && (10.2.19) \\
&= (1_J)_{\mathscr{T}} && \text{naturality} \\
&= (1_{Fe})_{\mathscr{T}} \\
&= 1_e. && (10.2.23)
\end{aligned}$$

This gives us the required equality. □

Now, we are ready to define the strict categorical group.

Definition 10.2.47. Define a strict categorical group $\mathscr{T}$ as follows:

- The underlying strict monoidal category is as in Definition 10.2.35.
- The negator functor $(-)^{(-1)} : \mathscr{T} \longrightarrow \mathscr{T}$ is as in Definition 10.2.38.

From Propositions 10.2.43 and 10.2.45 we know that the negator functor satisfies the cancellative conditions. We set the cancellation isomorphisms equal to identity natural transformations. Moreover, since $\mathscr{M}$ is a semi-strict categorical group, $\mathscr{M}$ is a groupoid. Thus, $\mathscr{T}$ is a groupoid as well. The unit condition and the cancellation triangle conditions are satisfied since all structure isomorphisms are equal to identity. $\diamond$

Definition 10.2.48. Consider the functors $L, R : \mathscr{T} \longrightarrow \mathscr{M}$ as follows:

$$L(x) = (Fx)' \qquad\qquad \text{on objects}$$

$$L(f) = (Ff)' \qquad\qquad \text{on morphisms,}$$

and

$$R(x) = F(x^\flat) \qquad\qquad \text{on objects}$$

$$R(f) = F(f^\flat) \qquad\qquad \text{on morphisms.}$$

Define a natural transformation $F_1 : L \longrightarrow R$ as follows:

- For an object $x \in \mathscr{T}$, the morphisms $F_1(x) : (Fx)' \longrightarrow F(x^{-1})$ is as in (10.2.13).

This assignment satisfies the naturality condition as follows: For a morphism $f_{\mathscr{T}} : x \longrightarrow y$ in $\mathscr{T}$ where $f : Fx \longrightarrow Fy$ is a morphism in $\mathscr{M}$, we wish to show that the diagram

$$\begin{tikzcd}
(Fx)' \arrow[r, "(Ff_{\mathscr{T}})'"] \arrow[d, "F_1(x)"'] & (Fy)' \arrow[d, "F_1(y)"] \\
F(x^{-1}) \arrow[r, "F(f^{\flat}_{\mathscr{T}})"'] & F(y^{-1})
\end{tikzcd}$$

commutes. We have

$$\begin{aligned}
F(f^{\flat}_{\mathscr{T}}) \circ F_1(x) &= F\left(\left(F_1(y) \circ f' \circ F_1^{-1}(x)\right)_{\mathscr{T}}\right) \circ F_1(x) && (10.2.40) \\
&= F_1(y) \circ f' \circ F_1^{-1}(x) \circ F_1(x) && (10.2.26) \\
&= F_1(y) \circ (Ff_{\mathscr{T}})'. && (10.2.26)
\end{aligned}$$

Thus, the naturality condition is satisfied. ⋄

Definition 10.2.49. Define a categorical group functor $F : \mathscr{T} \longrightarrow \mathscr{M}$ as follows:

- The underlying monoidal functor $F : \mathscr{T} \longrightarrow \mathscr{M}$ is as in Definition 10.2.37.
- The structure natural isomorphism F_1 is as in Definition 10.2.48.

From Propositions 10.2.16 and 10.2.18, the cancellation conditions are satisfied.⋄

10.3 Equivalence between a categorical group and strict categorical group

Now, we will define a functor $H : \mathscr{M} \longrightarrow \mathscr{T}$.

Definition 10.3.1. Define a functor $H : \mathscr{M} \longrightarrow \mathscr{T}$ as follows:

- The functor on objects is equal to the inclusion $[-]_T : M \longrightarrow T$. That is, for an element $x \in M$ define

$$H(x) := [x]_T. \tag{10.3.2}$$

- For a morphism $f : x \longrightarrow y$ in $\mathscr{M}$, the morphism $H(f) : [x]_T \longrightarrow [y]_T$ is defined as

$$H(f) := f_{\mathscr{T}}. \tag{10.3.3}$$

This assignment is well-defined since we require

$$H(f) \in \mathrm{Hom}_{\mathscr{T}}([x]_T, [y]_T).$$

We have

$$\begin{aligned} \mathrm{Hom}_{\mathscr{T}}([x]_T, [y]_T) &= \mathrm{Hom}_{\mathscr{M}}(F([x]_T), F([y]_T)) && (10.2.22) \\ &= \mathrm{Hom}_{\mathscr{M}}(x, y). && (10.1.30) \end{aligned}$$

Thus, for $f \in \mathrm{Hom}_{\mathscr{M}}(x,y)$ we get $f_{\mathscr{T}} \in \mathrm{Hom}_{\mathscr{T}}([x]_T, [y]_T)$

The functor conditions are satisfied as follows: For an objects $x \in \mathscr{M}$ we have

$$\begin{aligned} H(1_x) &= (1_x)_{\mathscr{T}} && (10.3.3) \\ &= \left(1_{F[x]_T}\right)_{\mathscr{T}} && (10.1.30) \\ &= 1_{[x]_T} && (10.2.23) \\ &= 1_{Hx}. && (10.3.2) \end{aligned}$$

Thus, the identity condition is satisfied. For morphisms $f : x \longrightarrow y$ and $g : y \longrightarrow z$ we have

$$\begin{aligned} H(g \circ f) &= (g \circ f)_{\mathscr{T}} && (10.3.3) \\ &= g_{\mathscr{T}} \circ f_{\mathscr{T}} && (10.2.24) \\ &= H(g) \circ H(f). && (10.3.3) \end{aligned}$$

Thus, the composition condition is satisfied. $\diamond$

Proposition 10.3.4. *The equality*

$$F \circ H = \mathrm{Id}_{\mathscr{M}} : \mathscr{M} \longrightarrow \mathscr{M} \tag{10.3.5}$$

of functors holds.

Proof. For an object $x \in \mathscr{M}$, we have

$$\begin{aligned} F(Hx) &= F([x]_T) && (10.3.2) \\ &= x. && (10.1.30) \end{aligned}$$

For a morphism $f : x \longrightarrow y$ in $\mathscr{M}$, we have

$$\begin{aligned} F(Hf) &= F(f_{\mathscr{T}}) && (10.3.3) \\ &= f. && (10.2.26) \end{aligned}$$

□

Definition 10.3.6. Define a morphism $H_0 : e \longrightarrow H(I)$ in $\mathscr{T}$ as

$$H_0 := (1_I)_{\mathscr{T}} : e \longrightarrow H(I). \tag{10.3.7}$$

This assignment is well-defined since

$$\begin{aligned} F(e) &= I && (10.1.30) \\ &= F(H(I)). && (10.3.5) \end{aligned}$$

◇

Definition 10.3.8. Consider the functors $L, R : \mathscr{M} \times \mathscr{M} \longrightarrow \mathscr{T}$ as follows:

$$\begin{aligned} L(x,y) &= H(x)H(y) && \text{on objects} \\ L(f,g) &= H(f)H(g) && \text{on morphisms,} \end{aligned}$$

and

$$R(x,y) = H(x \otimes y) \qquad \text{on objects}$$

$$R(x,y) = H(f \otimes y) \qquad \text{on morphisms.}$$

Define a natural transformation $H_2 : L \Rightarrow R$ as follows:

- For objects $x, y \in \mathscr{M}$, define a morphism $H_2(x,y) : H(x)H(y) \longrightarrow H(x \otimes y)$ as

$$H_2(x,y) = \left(1_{x \otimes y}\right)_{\mathscr{T}}. \tag{10.3.9}$$

  This assignment is well defined since we have

$$\begin{aligned} F((Hx)(Hy)) &= F([x]_T[y]_T) && (10.3.2) \\ &= x \otimes y && (10.1.30) \\ &= FH(x \otimes y). && (10.3.5) \end{aligned}$$

This assignment satisfies the naturality condition as follows: For morphisms $f : x \longrightarrow z$ and $g : y \longrightarrow w$ in $\mathscr{M}$, we wish to show that the diagram

$$\begin{tikzcd} H(x)H(y) \arrow[r, "H(f)H(g)"] \arrow[d, "H_2(x,y)"'] & H(z)H(w) \arrow[d, "H_2(z,w)"] \\ H(x \otimes y) \arrow[r, "H(f \otimes g)"'] & H(z \otimes w) \end{tikzcd}$$

commutes. We have

$$
\begin{aligned}
&H_2(z,w)\circ H(f)H(g)\\
&\qquad = (1_{z\otimes w})_{\mathscr{T}} \circ H(f)H(g) && (10.3.9)\\
&\qquad = (1_{z\otimes w})_{\mathscr{T}} \circ f_{\mathscr{T}}\, g_{\mathscr{T}} && (10.3.3)\\
&\qquad = (1_{z\otimes w})_{\mathscr{T}} \circ \big(F_2([z]_T,[w]_T)\circ f\otimes g\circ F_2^{-1}([x]_T,[y]_T)\big) && (10.2.28)\\
&\qquad = (1_{z\otimes w})_{\mathscr{T}} \circ \big(1_{z\otimes w}\circ f\otimes g\circ 1_{x\otimes y}\big)_{\mathscr{T}} && (10.2.6)\\
&\qquad = (f\otimes g)_{\mathscr{T}}\,. && (10.2.24)
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
H(f\otimes g)\circ H_2(x,y) &= (f\otimes g)_{\mathscr{T}}\circ H_2(x,y) && (10.3.3)\\
&= (f\otimes g)_{\mathscr{T}}\circ (1_{x\bullet y})_{\mathscr{T}} && (10.3.9)\\
&= (f\otimes g)_{\mathscr{T}}. && (10.2.24)
\end{aligned}
$$

Thus, the naturality condition is satisfied. ◇

Proposition 10.3.10. *For $x\in M$, the diagrams*

$$
\begin{tikzcd}
(Hx)e \arrow[r, "1_{Hx}H_0"] \arrow[d, equal] & (Hx)(HI) \arrow[d, "H_2(x,I)"] \\
Hx \arrow[r, equal] & H(x\otimes I)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
e(Hx) \arrow[r, "H_0 1_{Hx}"] \arrow[d, equal] & (HI)(Hx) \arrow[d, "H_2(I,x)"] \\
Hx \arrow[r, equal] & H(I\otimes x)
\end{tikzcd}
$$

commute. That is, the equalities

$$H_2(x,I)\circ 1_{Hx}H_0 = 1_{Hx} = H_2(I,x)\circ H_0 1_{Hx} \tag{10.3.11}$$

hold.

Proof. Observe that

$$
\begin{aligned}
H_2(x,I)\circ 1_{Hx}H_0 &= H_2(x,I)\circ(1_x)_{\mathscr{T}}(1_I)_{\mathscr{T}} && (10.3.7)\\
&= H_2(x,I)\circ\left(F_2([x]_T,[I]_T)\circ(1_x\otimes 1_I)\circ F_2^{-1}([x]_T,e)\right)_{\mathscr{T}} && (10.2.28)\\
&= H_2(x,I)\circ\left(F_2^{-1}([x]_T,e)\right)_{\mathscr{T}} && (10.2.6)\\
&= H_2(x,I)\circ\left(1_x\otimes F_0^{-1}\right)_{\mathscr{T}} && (10.2.8)\\
&= H_2(x,I)\circ(1_x\otimes 1_I)_{\mathscr{T}} && (10.2.2)\\
&= (1_{x\otimes I})_{\mathscr{T}} && (10.3.9)\\
&= (1_x)_{\mathscr{T}}
\end{aligned}
$$

Also,

$$
\begin{aligned}
1_{Hx} &= (1_{FHx})_{\mathscr{T}} && (10.2.23)\\
&= (1_x)_{\mathscr{T}}. && (10.3.5)
\end{aligned}
$$

Thus, the first diagram commutes. Similarly, we get

$$
\begin{aligned}
H_2(I,x)\circ H_0 1_{Hx} &= H_2(I,x)\circ(1_I)_{\mathscr{T}}(1_x)_{\mathscr{T}} && (10.3.7)\\
&= H_2(I,x)\circ\left(F_2([I]_T,[x]_T)\circ(1_I\otimes 1_x)\circ F_2^{-1}(e,[x]_T)\right)_{\mathscr{T}} && (10.2.28)\\
&= H_2(I,x)\circ\left(F_2^{-1}(e,[x]_T)\right)_{\mathscr{T}} && (10.2.6)\\
&= H_2(I,x)\circ\left(F_0^{-1}\otimes 1_x\right)_{\mathscr{T}} && (10.2.8)\\
&= H_2(I,x)\circ(1_I\otimes 1_x)_{\mathscr{T}} && (10.2.2)\\
&= (1_{I\otimes x})_{\mathscr{T}} && (10.3.9)\\
&= (1_x)_{\mathscr{T}}
\end{aligned}
$$

Also,

$$\begin{aligned} 1_{Hx} &= (1_{FHx})_{\mathscr{T}} && (10.2.23) \\ &= (1_x)_{\mathscr{T}}. && (10.3.5) \end{aligned}$$

Thus, the second diagram commutes. □

Proposition 10.3.12. *For $x, y, z \in M$ the diagram*

$$\begin{tikzcd} (Hx)(Hy)(Hz) \arrow[r, "H_2(x,y)1_{Hz}"] \arrow[d, "1_{Hx}H_2(y,z)"'] & H(x \otimes y)Hz \arrow[d, "H_2(x \otimes y, z)"] \\ (Hx)H(y \otimes z) \arrow[r, "H_2(x, y \otimes z)"'] & H(x \otimes y \otimes z) \end{tikzcd}$$

commutes. In other words, the equality

$$H_2(x \otimes y, z) \circ H_2(x,y)1_{Hz} = H_2(x, y \otimes z) \circ 1_{Hx}H_2(y,z) \tag{10.3.13}$$

holds.

Proof. In the above diagram, the top-right composite is given by

$$\begin{aligned} &H_2(x \otimes y, z) \circ H_2(x,y)1_{Hz} \\ &= (1_{x \otimes y \otimes z})_{\mathscr{T}} \circ (1_{x \otimes y})_{\mathscr{T}} (1_{FHz})_{\mathscr{T}} && (10.2.23) \text{ and } (10.3.9) \\ &= (1_{x \otimes y \otimes z})_{\mathscr{T}} \circ (1_{x \otimes y})_{\mathscr{T}} (1_z)_{\mathscr{T}} && (10.3.5) \\ &= (1_{x \otimes y \otimes z})_{\mathscr{T}} \circ (1_{x \otimes y} \otimes 1_z)_{\mathscr{T}} && (10.2.30) \\ &= (1_{x \otimes y \otimes z})_{\mathscr{T}}. && (10.2.24) \end{aligned}$$

The left-bottom composite is given by

$$
\begin{aligned}
& H_2(x, y \otimes z) \circ 1_{Hx} H_2(y,z) \\
&= (1_{x \otimes y \otimes z})_{\mathscr{T}} \circ (1_{FHx})_{\mathscr{T}} (1_{y \otimes z})_{\mathscr{T}} && \text{(10.2.23) and (10.3.9)} \\
&= (1_{x \otimes y \otimes z})_{\mathscr{T}} \circ (1_{x})_{\mathscr{T}} (1_{y \otimes z})_{\mathscr{T}} && \text{(10.3.5)} \\
&= (1_{x \otimes y \otimes z})_{\mathscr{T}} \circ (1_{x} \otimes 1_{y \otimes z})_{\mathscr{T}} && \text{(10.2.30)} \\
&= (1_{x \otimes y \otimes z})_{\mathscr{T}}. && \text{(10.2.24)}
\end{aligned}
$$

Since the above equalities match, the diagram commutes. □

Definition 10.3.14. Define a monoidal functor $H : \mathscr{M} \longrightarrow \mathscr{T}$ as follows:

- The underlying functor $H : \mathscr{M} \longrightarrow \mathscr{T}$ is as in Definition 10.3.1.
- The structure morphism $H_0 : e \longrightarrow H(I)$ is as in Definition 10.3.6.
- The structure natural isomorphism H_2 is as in Definition 10.3.8.

From Propositions 10.3.10 and 10.3.12 we know that the unit conditions and the associator condition are satisfied. ◇

Definition 10.3.15. Consider the functors $L, R : \mathscr{M} \longrightarrow \mathscr{T}$ as follows:

$$
\begin{aligned}
L(x) &= (Hx)^{\flat} && \text{on objects} \\
L(f) &= (Hf)^{\flat} && \text{on morphisms,}
\end{aligned}
$$

and

$$
\begin{aligned}
R(x) &= H(x') && \text{on objects} \\
R(f) &= H(f') && \text{on morphisms.}
\end{aligned}
$$

Define a natural isomorphisms $H_1 : L \Rightarrow R$ as follow:

- For an object $x \in \mathscr{T}$ define $H_1(x) : (Hx)^\flat \longrightarrow H(x')$ in $\mathscr{T}$ as

$$H_1(x) := (1_{x'})_{\mathscr{T}}. \tag{10.3.16}$$

  Above assignment is well defined since we have

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{T}}((Hx)^\flat, H(x')) &= \mathrm{Hom}_{\mathscr{M}}(F((Hx)^\flat), FH(x')) && (10.2.22)\\
&= \mathrm{Hom}_{\mathscr{M}}(F([x]_G^\flat), FH(x')) && (10.3.2)\\
&= \mathrm{Hom}_{\mathscr{M}}(x', FH(x')) && (10.1.30)\\
&= \mathrm{Hom}_{\mathscr{M}}(x', x'). && (10.3.5)
\end{aligned}$$

  Thus, $1_{x'} \in \mathrm{Hom}_{\mathscr{M}}(x', x')$ will correspond to a morphism $H_1(x) : (Hx)^\flat \longrightarrow H(x')$ in $\mathscr{T}$.

This assignment satisfies the naturality condition as follows: For a morphism $f : x \longrightarrow y$ in $\mathscr{M}$, we need to show that the diagram

$$\begin{tikzcd}
(Hx)^\flat \arrow[r, "(Hf)^\flat"] \arrow[d, "H_1(x)"'] & (Hy)^\flat \arrow[d, "H_1(y)"] \\
H(x') \arrow[r, "H(f')"'] & H(y')
\end{tikzcd}$$

commutes. The top right composite is given by

$$\begin{aligned}
H_1(y) \circ (Hf)^\flat &= (1_{y'})_{\mathscr{T}} \circ (Hf)^\flat && (10.3.16)\\
&= (1_{y'})_{\mathscr{T}} \circ (f_{\mathscr{T}})^\flat && (10.3.3)\\
&= (1_{y'})_{\mathscr{T}} \circ (f')_{\mathscr{T}} && (10.2.42)\\
&= (f')_{\mathscr{T}}. && (10.2.24)
\end{aligned}$$

The left bottom composite is given by

$$\begin{aligned} H(f')\circ H_1(x) &= (f')_{\mathscr{T}}\circ H_1(x) && (10.3.3)\\ &= (f')_{\mathscr{T}}\circ (1_{x'})_{\mathscr{T}} && (10.3.16)\\ &= (f')_{\mathscr{T}}. && (10.2.24)\end{aligned}$$

This shows the naturality. ◇

Proposition 10.3.17. *For $x \in M$, the diagram*

$$\begin{tikzcd} (Hx)^{\flat}Hx \arrow[r, "H_1(x)1_{Hx}"] \arrow[d, equal] & H(x')Hx \arrow[r, "{H_2(x',x)}"] & H(x'\otimes x) \\ e \arrow[rr, "H_0"'] & & H(I) \arrow[u, "H(\epsilon_x^{-1})"'] \end{tikzcd}$$

commutes. That is, the equality

$$H_2(x',x)\circ H_1(x)1_{Hx} = H(\epsilon_x^{-1})\circ H_0 \tag{10.3.18}$$

holds.

Proof. First observe that

$$\begin{aligned} H_1(x)1_{Hx} &= H_1(x)(1_{FHx})_{\mathscr{T}} && (10.2.23)\\ &= H_1(x)(1_x)_{\mathscr{T}} && (10.3.5)\\ &= (1_{x'})_{\mathscr{T}}(1_x)_{\mathscr{T}} && (10.3.16)\\ &= \Big(F_2([x']_T,[x]_T)\circ 1_{x'}\otimes 1_x\circ F_2^{-1}([x]_T^{\flat},[x]_T)\Big)_{\mathscr{T}} && (10.2.28)\\ &= \Big(1_{x'\otimes x}\circ 1_{x'}\otimes 1_x\circ F_2^{-1}([x]_T^{\flat},[x]_T)\Big)_{\mathscr{T}} && (10.2.6)\\ &= \Big(F_2^{-1}\Big([x]_T^{\flat},[x]_T\Big)\Big)_{\mathscr{T}} \end{aligned}$$

Moreover, we get

$$
\begin{aligned}
F_2\left([x]_T^\flat,[x]_T\right) &= \mathcal{EV}\left(P([x]_T^{-1})\bullet\ P([x]_T) \multimap P([x]_T^{-1}\ [x]_T)\right) && (10.2.4)\\
&= \mathcal{EV}\left([x]_N^\dagger \bullet\ [x]_N \multimap P(e)\right) && (10.1.7)\\
&= \mathcal{EV}\left([x]_N^\dagger \bullet\ [x]_N \multimap J\right) && (10.1.7)\\
&= \epsilon_{\mathrm{ev}([x]_N)} && (10.1.27)\\
&= \epsilon_x. && (10.1.20)
\end{aligned}
$$

Combining above two, we conclude that the top composite of the diagram is given by

$$
\begin{aligned}
H_2(x',x)\circ H_1(x)1_{Hx} &= H_2(x',x)\circ\left(F_2^{-1}\left([x]_T^\flat,[x]_T\right)\right)_{\mathcal{T}} && \text{from above}\\
&= H_2(x',x)\circ(\epsilon_x^{-1})_{\mathcal{T}} && \text{from above}\\
&= (1_{x'\otimes x})\circ(\epsilon_x^{-1})_{\mathcal{T}} && (10.3.9)\\
&= (\epsilon_x^{-1})_{\mathcal{T}}. && (10.2.24)
\end{aligned}
$$

The bottom composite is given by

$$
\begin{aligned}
H(\epsilon_x^{-1})\circ H_0 &= (\epsilon_x^{-1})_{\mathcal{T}}\circ(1_I)_{\mathcal{T}} && (10.3.3)\text{ and }(10.3.7)\\
&= (\epsilon_x^{-1})_{\mathcal{T}}. && (10.2.24)
\end{aligned}
$$

This shows that the diagram commutes. □

Proposition 10.3.19. *For $x\in M$, the diagram*

$$
\begin{tikzcd}
(Hx)(Hx)^\flat \arrow[r, "1_{Hx}H_1(x)"] \arrow[d, equal] & (Hx)H(x') \arrow[r, "H_2(x,x')"] & H(x\otimes x') \\
e \arrow[rr, "H_0"'] & & H(I) \arrow[u, "H(\eta_x)"']
\end{tikzcd}
$$

commutes. That is, the equality

$$H_2(x,x') \circ 1_{Hx} H_1(x) = H(\eta_x) \circ H_0 \tag{10.3.20}$$

holds.

Proof. First observe that

$$\begin{aligned}
1_{Hx} H_1(x) &= (1_{FHx})_{\mathscr{T}} H_1(x) && (10.2.23)\\
&= (1_x)_{\mathscr{T}} H_1(x) && (10.3.5)\\
&= (1_x)_{\mathscr{T}} (1_{x'})_{\mathscr{T}} && (10.3.16)\\
&= \Big(F_2([x]_T,[x']_T) \circ 1_x \otimes 1_{x'} \circ F_2^{-1}([x]_T,[x]_T^\flat)\Big)_{\mathscr{T}} && (10.2.28)\\
&= \Big(1_{x\otimes x'} \circ 1_x \otimes 1_{x'} \circ F_2^{-1}([x]_T,[x]_T^\flat)\Big)_{\mathscr{T}} && (10.2.6)\\
&= \Big(F_2^{-1}\Big([x]_T,[x]_T^\flat\Big)\Big)_{\mathscr{T}}
\end{aligned}$$

Moreover, we get

$$\begin{aligned}
F_2\Big([x]_T,[x]_T^\flat\Big) &= \mathcal{EV}\Big(P([x]_T) \bullet\ P([x]_T^\flat) \longrightarrow\!\!\!\!\!\!\!\circ\ \ P([x]_T\ [x]_T^\flat)\Big) && (10.2.4)\\
&= \mathcal{EV}\Big([x]_N \bullet\ [x]_N^\dagger \longrightarrow\!\!\!\!\!\!\!\circ\ \ P(e)\Big) && (10.1.7)\\
&= \mathcal{EV}\Big([x]_N \bullet\ [x]_N^\dagger \longrightarrow\!\!\!\!\!\!\!\circ\ \ J\Big) && (10.1.7)\\
&= \eta^{-1}_{\mathrm{ev}([x]_N)} && (10.1.26)\\
&= \eta_x^{-1}. && (10.1.20)
\end{aligned}$$

Combining above two, we conclude that the top composite of the diagram is given by

$$\begin{aligned}
H_2(x,x') \circ 1_{Hx} H_1(x) &= H_2(x,x') \circ \Big(F_2^{-1}\Big([x]_T,[x]_T^\flat\Big)\Big)_{\mathscr{T}} && \text{from above}\\
&= H_2(x,x') \circ (\eta_x)_{\mathscr{T}} && \text{from above}\\
&= (1_{x\otimes x'}) \circ (\eta_x)_{\mathscr{T}} && (10.3.9)\\
&= (\eta_x)_{\mathscr{T}}. && (10.2.24)
\end{aligned}$$

The bottom composite is given by

$$
\begin{aligned}
H(\eta_x) \circ H_0 &= (\eta_x)_{\mathscr{T}} \circ (1_I)_{\mathscr{T}} && \text{(10.3.3) and (10.3.7)} \\
&= (\eta_x)_{\mathscr{T}}. && \text{(10.2.24)}
\end{aligned}
$$

This shows that the diagram commutes. □

Definition 10.3.21. Define a categorical group functor $H : \mathscr{M} \longrightarrow \mathscr{T}$ as follows:

- The underlying monoidal functor $H : \mathscr{M} \longrightarrow \mathscr{T}$ is as in Definition 10.3.14.
- The structure natural isomorphism H_1 is as in Definition 10.3.15.

From Propositions 10.3.17 and 10.3.19 we know that the cancellation conditions are satisfied. ◇

Now we will show that the composition $F \circ H : \mathscr{M} \longrightarrow \mathscr{M}$ is equal to the identity functor $\mathrm{Id}_{\mathscr{M}} : \mathscr{M} \longrightarrow \mathscr{M}$ as categorical group functors. That is, the structure morphisms $(F \circ H)_0$, $(F \circ H)_1$, and $(F \circ H)_2$ are all equal to identity.

Proposition 10.3.22. *The structure morphisms of the composite categorical group functor $F \circ H : \mathscr{M} \longrightarrow \mathscr{M}$ are equal to identity morphisms. That is, for $x, y \in \mathrm{Obj}(\mathscr{M})$, the equalities*

$$(F \circ H)_0 = 1_I \ : \ I \longrightarrow I \tag{10.3.23}$$

$$(F \circ H)_1(x) = 1_{x'} \ : \ x' \longrightarrow x' \tag{10.3.24}$$

$$(F \circ H)_2(x, y) = 1_{x \otimes y} \ : \ x \otimes y \longrightarrow x \otimes y \tag{10.3.25}$$

hold.

Proof. Observe that

$$\begin{aligned}
(F \circ H)_0 &= F(H_0) \circ F_0 && (3.1.23)\\
&= F\big((1_I)_{\mathscr{T}}\big) \circ 1_I && (10.2.2) \text{ and } (10.3.7)\\
&= 1_I. && (10.2.26)
\end{aligned}$$

Next, for $x \in \operatorname{Obj}(\mathscr{M})$ we have

$$\begin{aligned}
(F \circ H)_1(x) &= F(H_1(x)) \circ F_1(H(x)) && (4.1.34)\\
&= F((1_{x'})_{\mathscr{T}}) \circ F_1(H(x)) && (10.3.16)\\
&= 1_{x'} \circ F_1(H(x)) && (10.2.26)\\
&= F_1([x]_T) && (10.3.2)\\
&= 1_{x'}. && (10.2.15)
\end{aligned}$$

Finally, for $x, y \in \operatorname{Obj}(\mathscr{M})$ we have

$$\begin{aligned}
(F \circ H)_2(x, y) &= F(H_2(x, y)) \circ F_2(H(x), H(y)) && (3.1.24)\\
&= F((1_{x \otimes y})_{\mathscr{T}}) \circ F_2(H(x), H(y)) && (10.3.9)\\
&= 1_{x \otimes y} \circ F_2(H(x), H(y)) && (10.2.26)\\
&= F_2([x]_T, [y]_T) && (10.3.2)\\
&= 1_{x \otimes y}. && (10.2.6)
\end{aligned}$$

$\square$

Definition 10.3.26. Define a natural transformation $\Phi : H \circ F \Rightarrow \operatorname{Id}_{\mathscr{T}}$ as follows:

- For an element $x \in T$ define a morphism

$$\Phi_x : HF(x) \longrightarrow x$$

as

$$\Phi_x := (1_{Fx})_{\mathscr{T}}. \tag{10.3.27}$$

This is well-defined since we have

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{T}}(HF(x), x) &= \mathrm{Hom}_{\mathscr{M}}(FHF(x), F(x)) && (10.2.22)\\
&= \mathrm{Hom}_{\mathscr{M}}(F(x), F(x)). && (10.3.5)
\end{aligned}$$

Thus, a morphism $1_{Fx} : F(x) \longrightarrow F(x)$ in $\mathscr{M}$ corresponds to a morphism $(1_{Fx})_{\mathscr{T}} : HF(x) \longrightarrow x$ in $\mathscr{T}$.

We will now show the naturality condition. Let $f_{\mathscr{T}} : x \longrightarrow y$ be a morphism in $\mathscr{T}$ where $f : Fx \longrightarrow Fy$ is a morphism in $\mathscr{M}$. We will show that the diagram

$$\begin{tikzcd}
HF(x) \arrow[r, "HF(f_{\mathscr{T}})"] \arrow[d, "\Phi_x"'] & HF(y) \arrow[d, "\Phi_y"] \\
x \arrow[r, "f_{\mathscr{T}}"'] & y
\end{tikzcd}$$

commutes. Observe that

$$\begin{aligned}
\Phi_y \circ HF(f_{\mathscr{T}}) &= (1_{Fy})_{\mathscr{T}} \circ H(f) && (10.2.26) \text{ and } (10.3.27)\\
&= (1_{Fy})_{\mathscr{T}} \circ f_{\mathscr{T}} && (10.3.3)\\
&= (1_{Fy} \circ f)_{\mathscr{T}} && (10.2.24)\\
&= (f \circ 1_{Fx})_{\mathscr{T}} && \\
&= f_{\mathscr{T}} \circ (1_{Fx})_{\mathscr{T}} && (10.2.24)\\
&= f_{\mathscr{T}} \circ \Phi_x. && (10.3.27)
\end{aligned}$$

This shows the naturality condition. $\diamond$

Proposition 10.3.28. *The equality*

$$(HF)_0 = (1_I)_{\mathscr{T}} : e \longrightarrow [I]_T \tag{10.3.29}$$

holds.

Proof. First observe that

$$HF(e) = H(I) = [I]_T.$$

We have

$$\begin{aligned}
(HF)_0 &= H(F_0) \circ H_0 && (3.1.23)\\
&= H(1_I) \circ (1_I)_{\mathscr{T}} && (10.2.2) \text{ and } (10.3.7)\\
&= (1_I)_{\mathscr{T}}. && \square
\end{aligned}$$

Definition 10.3.30. Define a monoidal natural transformation

$$\Phi : H \circ F \Rightarrow \mathrm{Id}_{\mathscr{T}}$$

as follows:

- The underlying natural transformation

  $$\Phi : H \circ F \Rightarrow \mathrm{Id}_{\mathscr{T}}$$

  is same as in Definition 10.3.26.

This assignment satisfies the unit condition (3.1.26). We have

$$\begin{aligned}
\Phi_e \circ (HF)_0 &= (1_{F(e)})_{\mathscr{T}} \circ (HF)_0 && (10.3.27)\\
&= (1_I)_{\mathscr{T}} \circ (1_I)_{\mathscr{T}} && (10.3.29)\\
&= (1_I)_{\mathscr{T}} && (10.2.24)\\
&= 1_e && (10.2.23)\\
&= (\mathrm{Id}_{\mathscr{T}})_0. && (3.1.20)
\end{aligned}$$

Note that, the morphism $(1_I)_{\mathscr{T}}$ in the third equality is

$$(1_I)_{\mathscr{T}} : e \longrightarrow e.$$

Next, we will show that the tensor condition (3.1.27) is satisfied. For $x, y \in \mathrm{Obj}(\mathscr{T})$, we need to show that the diagram

$$\begin{tikzcd}
HF(x)HF(y) \arrow[r, "(HF)_2(x,y)"] \arrow[d, "\Phi_x\Phi_y"'] & HF(xy) \arrow[d, "\Phi_{xy}"] \\
xy \arrow[r, "(\mathrm{Id}_{\mathscr{T}})_2(x,y)"'] & xy
\end{tikzcd}$$

commutes. We will show that the corresponding diagram in $\mathscr{M}$ commutes. The objects of the diagram will become

$$\begin{aligned}
F(HF(x)HF(y)) &= F([Fx]_T[Fy]_T) && (10.3.2)\\
&= F(x) \otimes F(y) && (10.1.30)
\end{aligned}$$

$$F(HF(xy)) = F(xy). \qquad (10.3.5)$$

The morphisms will become

$$
\begin{aligned}
(HF)_2(x,y) &= H(F_2(x,y)) \circ H_2(F(x),F(y)) && (3.1.24)\\
&= (F_2(x,y))_{\mathscr{T}} \circ (1_{Fx\otimes Fy})_{\mathscr{T}} && (10.3.3) \text{ and } (10.3.9)\\
&= (F_2(x,y))_{\mathscr{T}} && (10.2.24)
\end{aligned}
$$

$$
\Phi_{xy} = (1_{F(xy)})_{\mathscr{T}} \qquad (10.3.27)
$$

$$
\begin{aligned}
\Phi_x\Phi_y &= (1_{Fx})_{\mathscr{T}}(1_{Fy})_{\mathscr{T}} && (10.3.27)\\
&= \big(F_2(x,y)\circ 1_{Fx}\otimes 1_{Fy}\circ F_2^{-1}(HF(x),HF(y))\big)_{\mathscr{T}} && (10.2.28)\\
&= \big(F_2(x,y)\circ F_2^{-1}(HF(x),HF(y))\big)_{\mathscr{T}}
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{Id}_{\mathscr{T}})_2(x,y) &= 1_{xy} && (3.1.21)\\
&= (1_{F(xy)})_{\mathscr{T}}. && (10.2.23)
\end{aligned}
$$

We will now show that the diagram

$$
\begin{tikzcd}
F(x)F(y) \arrow[r, "F_2(x,y)"] \arrow[d, "F_2(x,y)\circ F_2^{-1}(HF(x),HF(y))"'] & F(xy) \arrow[d, "1_{F(xy)}"] \\
F(xy) \arrow[r, "1_{F(xy)}"'] & F(xy)
\end{tikzcd}
$$

commutes. Observe that

$$
\begin{aligned}
F_2(HF(x),HF(y)) &= F_2([Fx]_T,[Fy]_T) && (10.3.2)\\
&= 1_{(Fx)\otimes(Fy)}. && (10.2.6)
\end{aligned}
$$

Thus, the left vertical morphism becomes $F_2(x,y)$. Therefore, the above diagram commutes. ◇

Definition 10.3.31. Define a categorical group natural transformation

$$\Phi : H \circ F \Rightarrow \mathrm{Id}_{\mathcal{T}}$$

as follows:

- The underlying monoidal natural transformation

  $$\Phi : H \circ F \Rightarrow \mathrm{Id}_{\mathcal{T}}$$

  is same as in Definition 10.3.30.

We will show that the negator condition (4.1.36) is satisfied. For $x \in \mathrm{Obj}(\mathcal{T})$, we need to show that the diagram

$$\begin{tikzcd}
(HF(x))^\flat \arrow[r, "(HF)_1(x)"] \arrow[d, "(\Phi_x)^\flat"'] & HF(x^\flat) \arrow[d, "\Phi_{x^\flat}"] \\
x^\flat \arrow[r, "(\mathrm{Id}_{\mathcal{T}})_1(x)"'] & x^\flat
\end{tikzcd}$$

commutes. We will show that the corresponding diagram in $\mathcal{M}$ commutes. The objects of the diagram will be

$$\begin{aligned}
F((HF(x))^\flat) &= F([Fx]_T^\flat) && (10.3.2)\\
&= (Fx)' && (10.1.30)
\end{aligned}$$

$$F(HF(x^\flat)) = F(x^\flat). \qquad (10.3.5)$$

The morphisms will be

$$(HF)_1(x) = H(F_1(x)) \circ H_1(F(x)) \quad (4.1.34)$$

$$= (F_1(x))_{\mathscr{T}} \circ (1_{(Fx)'})_{\mathscr{T}} \quad (10.3.16)$$

$$= (F_1(x))_{\mathscr{T}} \quad (10.2.24)$$

$$\Phi_{x^\flat} = (1_{F(x^\flat)})_{\mathscr{T}} \quad (10.3.27)$$

$$(\Phi_x)^\flat = (1_{Fx})^\flat_{\mathscr{T}} \quad (10.3.27)$$

$$= \left(F_1(x) \circ 1'_{Fx} \circ F_1^{-1}(HF(x))\right)_{\mathscr{T}} \quad (10.2.40)$$

$$= \left(F_1(x) \circ F_1^{-1}(HF(x))\right)_{\mathscr{T}}$$

$$(\mathrm{Id}_{\mathscr{T}})_1(x) = 1_{x^\flat} \quad (4.1.32)$$

$$= (1_{F(x^\flat)})_{\mathscr{T}}. \quad (10.2.23)$$

We will now show that the diagram

$$\begin{tikzcd}
(Fx)' \arrow[r, "F_1(x)"] \arrow[d, "F_1(x) \circ F_1^{-1}(HF(x))"'] & F(x^\flat) \arrow[d, "1_{F(x^\flat)}"] \\
F(x^\flat) \arrow[r, "1_{F(x^\flat)}"'] & F(x^\flat)
\end{tikzcd}$$

commutes. Observe that

$$F_1(HF(x)) = F_1([Fx]_T) \quad (10.3.2)$$

$$= 1_{(Fx)'}. \quad (10.2.15)$$

Thus, the left vertical morphism becomes $F_1(x)$. Therefore, the above diagram commutes. ◇

Theorem 10.3.32. *Let $\mathscr{M} = (\mathscr{M}, I, (-)', - \otimes -)$ be a semi-strict categorical group. Let $\mathscr{T} = (\mathscr{T}, e, (-)^\flat, - \cdot -)$ be the strict categorical group as in Definition 10.2.20. Then, $\mathscr{M}$ is categorically equivalent to $\mathscr{T}$ via categorical group functors.*

Proof. Let $F : \mathscr{T} \longrightarrow \mathscr{M}$ be the categorical functor as in Definition 10.2.49 and $H : \mathscr{M} \longrightarrow \mathscr{T}$ be the categorical functor as in Definition 10.3.21. From Propositions 10.3.4 and 10.3.22 we know that the composition $F \circ H$ is equal to the indentity functor $\mathrm{Id}_{\mathscr{M}} : \mathscr{M} \longrightarrow \mathscr{M}$ as categorical group functors. Consider the categorical group natural transformation $\Phi : H \circ F \Rightarrow \mathrm{Id}_{\mathscr{T}}$ as in Definition 10.3.31. This proves that $\mathscr{M}$ is categorically equivalent to $\mathscr{T}$ via categorical group functors $F : \mathscr{T} \longrightarrow \mathscr{M}$ and $H : \mathscr{M} \longrightarrow \mathscr{T}$. □

Theorem 10.3.33. *A categorical group is equivalent to a strict categorical group via categorical group functors.*

Proof. By Theorem 4.3.28, every categorical group is equivalent to a semi-strict categorical group via categorical group functors. Then, by Theorem 10.3.32, every semi-strict categorical group is equivalent to a strict categorical group via categorical group functors. Therefore, every categorical group is equivalent to a strict categorical group via categorical group functors. □

Chapter 11: Iterates and Canonical Natural Isomorphisms

11.1 Pointed Categorical Group of Functor

In this section, we introduce the category of functors from a category $\mathscr{C}$ to a categorical group $\mathscr{M}$, and show that this functor category naturally inherits a categorical group structure from $\mathscr{M}$.

Framework 11.1.1. Throughout this chapter, let $\mathscr{C}$ be a category, and let $(\mathscr{M}, I, (-)', - \otimes -)$ be a categorical group. We adopt the following notation and conventions:

- We use the *left-associative convention* for the tensor product in any categorical group: for objects $x, y, z \in \mathscr{N}$, we write $x \bullet y \bullet z := (x \bullet y) \bullet z$. Similarly, for morphisms $f, g, h \in \mathscr{N}$, we write $f \bullet g \bullet h := (f \bullet g) \bullet h$. The n-fold negation is denoted by $(-)^{(n)}$.

- For a set S, let $\mathcal{C}at\mathcal{G}rp\langle S\rangle$ denote the free categorical group generated by S (see Construction 4.5.19). As in Framework 10.1.1, we refer to morphisms in the free categorical group as *arrows*, and denote an arrow $f : x \longrightarrow y$ in $\mathcal{C}at\mathcal{G}rp\langle S\rangle$ by $f : x \multimap y$. By Theorem 9.3.4, $\mathcal{C}at\mathcal{G}rp\langle S\rangle$ is thin, so an arrow is determined by its source and target. Thus, we may write $x \multimap y$ without specifying f. In particular, for $x, y, z \in \mathcal{C}at\mathcal{G}rp\langle S\rangle$,

$$\alpha_{x,y,z} = (x \bullet (y \bullet z) \multimap\!\!\!\longrightarrow (x \bullet y) \bullet z),$$

$$\lambda_x = (J \bullet x \multimap\!\!\!\longrightarrow x),$$

$$\rho_x = (x \bullet J \multimap\!\!\!\longrightarrow x),$$

$$\eta_x = (J \multimap\!\!\!\longrightarrow x \bullet x^\dagger),$$

$$\epsilon_x = (x^\dagger \bullet x \multimap\!\!\!\longrightarrow J).$$

Let $\mathcal{Q} : \mathcal{C}at\mathcal{G}rp\langle S\rangle \longrightarrow \mathcal{G}rp\langle S\rangle$ be the induced strict functor as in Definition 4.5.23. By Theorem 4.5.27, there is a unique arrow $x \multimap\!\!\!\longrightarrow y$ in $\mathcal{C}at\mathcal{G}rp\langle S\rangle$ if and only if $\mathcal{Q}(x) = \mathcal{Q}(y)$.

- The set of objects of $\mathscr{M}$ is denoted M. Let

$$\mathcal{EV} : \mathcal{C}at\mathcal{G}rp\langle M\rangle \longrightarrow \mathscr{M}$$

be the induced strict categorical group functor as inDefinition 4.5.22. The functor $\mathcal{EV}$ on objects is completely determined by the following relations: For $a \in M$ and $x, y \in \mathcal{C}at\mathcal{G}rp\langle M\rangle$, we have

$$\mathrm{ev}(J) = I, \tag{11.1.2}$$

$$\mathrm{ev}([a]_N) = a, \tag{11.1.3}$$

$$\mathrm{ev}(x \bullet y) = \mathrm{ev}(x) \otimes \mathrm{ev}(y), \tag{11.1.4}$$

$$\mathrm{ev}(x^\dagger) = \mathrm{ev}(x)'. \tag{11.1.5}$$

- For $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$. We focus on the free categorical group generated by $[n]$, denoted

$$\mathcal{C}at\mathcal{G}rp\langle n\rangle := \mathcal{C}at\mathcal{G}rp\langle [n]\rangle\,.$$

For clarity, we may replace the numbers in $[n]$ with letters; for example, for $n = 3$, we write $[3] = \{X, Y, Z\}$, identifying $1 = X$, $2 = Y$, $3 = Z$. $\diamond$

Definition 11.1.6. Define a *functor category* $\mathcal{F}ct(\mathscr{C}, \mathscr{M})$ as follows:

- The objects are functors
$$F : \mathscr{C} \longrightarrow \mathscr{M}.$$
- The morphisms between two functor $F, G : \mathscr{C} \longrightarrow \mathscr{M}$ are given by the natural transformations.
- For a functor $F : \mathscr{C} \longrightarrow \mathscr{M}$, the identity morphism is given by the identity natural tranformation:
$$\mathrm{Id}_F : F \Rightarrow F.$$
- The composition is given by the composition of natural transformations: For $\phi : F \Rightarrow G$, $\psi : G \Rightarrow H$ we have
$$(\psi \circ \phi)(x) = \psi(x) \circ \phi(x) \tag{11.1.7}$$
for $x \in \mathscr{C}$.

The associativity and the identity conditions follow since $\mathscr{M}$ is a category. $\diamond$

Definition 11.1.8. Let $x \in \mathrm{Obj}(\mathscr{C})$. Define the *evaluation functor*
$$\mathcal{EV}_x : \mathcal{F}ct(\mathscr{C}, \mathscr{M}) \longrightarrow \mathscr{M}$$
by setting,
$$\mathcal{EV}_x(F) := F(x), \qquad \text{for functor } F : \mathscr{C} \longrightarrow \mathscr{M} \tag{11.1.9}$$
$$\mathcal{EV}_x(\phi) := \phi(x) \qquad \text{for natural transformation } \phi : F \Rightarrow G. \tag{11.1.10}$$

The functoriality of $\mathcal{EV}_x$ follows directly from the Definition 11.1.6. $\diamond$

Definition 11.1.11. The category $\mathcal{F}ct(\mathscr{C},\mathscr{M})$ is has a monoidal structure inherited from $\mathscr{M}$ as follows:

- The underlying category is as in Definition 11.1.6.
- The unit object is the constant functor at I:

$$\mathrm{Const}_I : \mathscr{C} \longrightarrow \mathscr{M}. \tag{11.1.12}$$

- The monoidal product is defined pointwise from the monoidal product in $\mathscr{M}$. That is, for $F, G : \mathscr{C} \longrightarrow \mathscr{M}$, define $F \otimes G : \mathscr{C} \longrightarrow \mathscr{M}$ by

$$(F \otimes G)(x) := F(x) \otimes G(x), \qquad \text{on objects,} \tag{11.1.13}$$

$$(F \otimes G)(f) := F(f) \otimes G(f), \qquad \text{on morphisms.} \tag{11.1.14}$$

  For natural transformations $\phi : F \Rightarrow G$ and $\psi : H \Rightarrow K$, define $\phi \otimes \psi : F \otimes H \Rightarrow G \otimes K$ by

$$(\phi \otimes \psi)(x) := \phi(x) \otimes \psi(x) : F(x) \otimes H(x) \longrightarrow G(x) \otimes K(x). \tag{11.1.15}$$

- The unit natural isomorphisms λ and ρ are inherited from those of $\mathscr{M}$. For $F : \mathscr{C} \longrightarrow \mathscr{M}$, define natural isomorphisms

$$\lambda_F : \mathrm{Const}_I \otimes F \Rightarrow F, \qquad \rho_F : F \otimes \mathrm{Const}_I \Rightarrow F$$

  by

$$\lambda_F(x) := \lambda(F(x)) : I \otimes F(x) \longrightarrow F(x), \tag{11.1.16}$$

$$\rho_F(x) := \rho(F(x)) : F(x) \otimes I \longrightarrow F(x), \tag{11.1.17}$$

  for all $x \in \mathscr{C}$.

- The associator natural isomorphism α is inherited from $\mathscr{M}$. For $F,G,H : \mathscr{C} \longrightarrow \mathscr{M}$, define

$$\alpha_{F,G,H} : F \otimes (G \otimes H) \Rightarrow F \otimes G \otimes H$$

by

$$\alpha_{F,G,H}(x) := \alpha(F(x),G(x),H(x)) : F(x) \otimes (G(x) \otimes H(x)) \longrightarrow F(x) \otimes G(x) \otimes H(x). \tag{11.1.18}$$

The monoidal axioms are satisfied since the monoidal structure on $\mathcal{Fct}(\mathscr{C},\mathscr{M})$ is inherited from $\mathscr{M}$. $\diamond$

Definition 11.1.19. Let $x \in \mathrm{Obj}(\mathscr{C})$. Then, the evaluation functor

$$\mathcal{EV}_x : \mathcal{Fct}(\mathscr{C},\mathscr{M}) \longrightarrow \mathscr{M}$$

is a strict monoidal functor as follows: The underlying functor $\mathcal{EV}_x$ is as in Definition 11.1.8. For functors $F,G \in \mathcal{Fct}(\mathscr{C},\mathscr{M})$, we have

$$\begin{aligned} \mathcal{EV}_x(F \otimes G) &= (F \otimes G)(x) && (11.1.9) \\ &= F(x) \otimes G(x) && (11.1.13) \\ &= \mathcal{EV}_x(F) \otimes \mathcal{EV}_x(G). && (11.1.9) \end{aligned}$$

and for natural transformations ϕ,ψ,

$$\begin{aligned} \mathcal{EV}_x(\phi \otimes \psi) &= (\phi \otimes \psi)(x) && (11.1.10) \\ &= \phi(x) \otimes \psi(x) && (11.1.14) \\ &= \mathcal{EV}_x(\phi) \otimes \mathcal{EV}_x(\psi). && (11.1.10) \end{aligned}$$

Moreover,

$$\begin{aligned}\mathcal{EV}_x(\mathrm{Const}_I) &= \mathrm{Const}_I(x) && (11.1.9)\\ &= I. && \diamond\end{aligned}$$

Thus, $\mathcal{EV}_x$ is a strict monoidal functor.

Definition 11.1.20. The functor category $\mathcal{F}ct(\mathscr{C},\mathscr{M})$ is a categorical group as follows:

- The underlying monoidal category is as in Definition 11.1.11.
- The negation functor is inherited from $\mathscr{M}$. Specifically, for a functor $F : \mathscr{C} \longrightarrow \mathscr{M}$, define the functor

  $$F' : \mathscr{C} \longrightarrow \mathscr{M}$$

  by

  $$F'(x) := F(x)' \qquad \text{on objects,} \tag{11.1.21}$$

  $$F'(f) := F(f)' \qquad \text{on morphisms.} \tag{11.1.22}$$

  For a natural transformation $\phi : F \Rightarrow G$, define the natural transformation

  $$\phi' : F' \Rightarrow G'$$

  by

  $$\phi'_x := \phi(x)' : F(x)' \longrightarrow G(x)'. \tag{11.1.23}$$

- The cancellation natural isomorphisms η and ϵ are inherited from those in $\mathscr{M}$. For a functor $F : \mathscr{C} \longrightarrow \mathscr{M}$, define natural isomorphisms

  $$\eta_F : \mathrm{Const}_I \Rightarrow F \otimes F' \qquad \text{and} \qquad \epsilon_F : F' \otimes F \Rightarrow \mathrm{Const}_I$$

by

$$\eta_F(x) := \eta(F(x)) : I \longrightarrow F(x) \otimes F(x)' \tag{11.1.24}$$

and

$$\epsilon_F(x) := \epsilon(F(x)) : F(x)' \otimes F(x) \longrightarrow I \tag{11.1.25}$$

for all $x \in \mathscr{C}$.

The categorical group axioms are satisfied since the structure of $\mathcal{Fct}(\mathscr{C}, \mathscr{M})$ is inherited from $\mathscr{M}$. $\diamond$

Definition 11.1.26. Let $x \in \mathrm{Obj}(\mathscr{C})$. Then, the evaluation functor

$$\mathcal{EV}_x : \mathcal{Fct}(\mathscr{C}, \mathscr{M}) \longrightarrow \mathscr{M}$$

is a strict categorical group functor as follows: The underlying monoidal functor $\mathcal{EV}_x$ is as in Definition 11.1.19. For a functor $F : \mathcal{Fct}(\mathscr{C}, \mathscr{M})$, we have

$$\begin{aligned} \mathcal{EV}_x(F') &= F'(x) && (11.1.9) \\ &= F(x)' && (11.1.21) \\ &= \mathcal{EV}_x(F)' && (11.1.9) \end{aligned}$$

and for a natural transformation $\phi \in \mathcal{Fct}(\mathscr{C}, \mathscr{M})$,

$$\begin{aligned} \mathcal{EV}_x(\phi') &= \phi'(x) && (11.1.10) \\ &= \phi(x)' && (11.1.23) \\ &= \mathcal{EV}_x(\phi)'. && (11.1.10) \end{aligned}$$

$\diamond$

Thus, $\mathcal{EV}_x$ is a strict categorical group functor.

Definition 11.1.27. Let $n \in \mathbb{N}$. The *n-fold direct product* of the categorical group $\mathscr{M}$ is denoted by

$$\mathscr{M}^n$$

and the projection functors

$$\Pi_i : \mathscr{M}^n \longrightarrow \mathscr{M}$$

for $1 \leq i \leq n$, defined by

$$\Pi_i(x_1, \ldots, x_n) = x_i \qquad \text{on objects,} \tag{11.1.28}$$

$$\Pi_i(f_1, \ldots, f_n) = f_i \qquad \text{on morphisms.} \tag{11.1.29}$$

◇

Notation 11.1.30. For $n \in \mathbb{N}$, we define the categorical group

$$\mathscr{F}ct_n(\mathscr{M}) := \mathscr{F}ct\left(\mathscr{M}^n, \mathscr{M}\right). \tag{11.1.31}$$

◇

11.2 Iterates and Canonical Natural Isomorphisms

In this section, we formally define *iterates*: functors constructed from repeated application of the tensor product and negation operations. We also introduce *canonical natural isomorphisms*: natural isomorphisms between iterates that are built from compositions of the structural isomorphisms of the categorical group.

Definition 11.2.1. Let $n \in \mathbb{N}$. Define a function

$$\Pi : [n] \longrightarrow \mathrm{Obj}\big(\mathscr{F}ct_n(\mathscr{M})\big)$$

as follows: for each $i \in [n]$, we set

$$\Pi(i) := \Pi_i : \mathscr{M}^n \longrightarrow \mathscr{M}$$

the projection functor as in Definition 11.1.27.

By the universal property of the free categorical group $\mathcal{C}at\mathcal{G}rp\langle n\rangle$ generated by $[n]$, there exists a unique strict categorical group functor

$$\mathtt{Can} : \mathcal{C}at\mathcal{G}rp\langle n\rangle \longrightarrow \mathcal{F}ct_n(\mathcal{M})$$

such that

$$\mathtt{Can}(i) = \Pi_i \tag{11.2.2}$$

for every $i \in [n]$. ◇

Definition 11.2.3. Let $n \in \mathbb{N}$. A functor $F : \mathcal{M}^n \longrightarrow \mathcal{M}$ is called an *iterate* (of $\otimes$ and $(-)'$) in n variables, or simply an *iterate*, if there exists an object $X \in \mathcal{C}at\mathcal{G}rp\langle n\rangle$ such that

$$\mathtt{Can}(X) = F.$$

A natural isomorphism $\phi : F \Rightarrow G$ between functors $F, G : \mathcal{M}^n \longrightarrow \mathcal{M}$ is called a *canonical natural isomorphism* if there exists, objects $X, Y \in \mathcal{C}at\mathcal{G}rp\langle n\rangle$, and an arrow $f : X \xrightarrow{\circ} Y$ in $\mathcal{C}at\mathcal{G}rp\langle n\rangle$ such that

$$\mathtt{Can}(X) = F, \qquad \mathtt{Can}(Y) = G, \qquad \mathtt{Can}(f) = \phi.$$

A pair of natural isomorphisms $\phi, \psi : F \Rightarrow G$ between functors $F, G : \mathcal{M}^n \longrightarrow \mathcal{M}$ is called a *pair of formal canonical natural isomorphisms* if there exist objects $X, Y \in \mathcal{C}at\mathcal{G}rp\langle n\rangle$, and arrows $f, g : X \xrightarrow{\circ} Y$ in $\mathcal{C}at\mathcal{G}rp\langle n\rangle$ such that

$$\mathtt{Can}(X) = F, \qquad \mathtt{Can}(Y) = G, \qquad \mathtt{Can}(f) = \phi, \qquad \mathtt{Can}(g) = \psi.$$ ◇

Theorem 11.2.4. *Let $n \in \mathbb{N}$, and let $F, G : \mathcal{M}^n \longrightarrow \mathcal{M}$ be iterates in n variables. Then any pair of formal canonical natural isomorphisms $\phi, \psi : F \Rightarrow G$ are equal.*

Proof. Let $\phi, \psi : F \Rightarrow G$ be a pair of formal canonical natural isomorphisms. Then, there exist objects X, Y, and arrows $f, g : X \multimap Y$ in $\mathcal{C}at\mathcal{G}rp\langle n\rangle$ such that

$$\mathtt{Can}(f) = \phi \qquad \text{and} \qquad \mathtt{Can}(g) = \psi.$$

Since $\mathcal{C}at\mathcal{G}rp\langle n\rangle$ is a thin category, it follows that $f = g$. Therefore,

$$\phi = \mathtt{Can}(f) = \mathtt{Can}(g) = \psi. \qquad \square$$

Example 11.2.5. Let $\mathscr{M}$ be a categorical group, and consider the functors $F, G : \mathscr{M}^2 \longrightarrow \mathscr{M}$ defined by

$$\begin{aligned} F(x, y) &= (x \otimes y)' && \text{on objects,} \\ F(f, g) &= (f \otimes g)' && \text{on morphisms,} \end{aligned}$$

and

$$\begin{aligned} G(x, y) &= y' \otimes x' && \text{on objects,} \\ G(f, g) &= g' \otimes f' && \text{on morphisms.} \end{aligned}$$

We claim that the functors F and G are iterates in two variables.

Let $[2] = \{X, Y\}$, identifying

$$1 = X \qquad \text{and} \qquad 2 = Y.$$

Define

$$U := (X \bullet Y)^{\dagger} \qquad \text{and} \qquad V := Y^{\dagger} \bullet X^{\dagger}$$

as objects in $\mathcal{CatGrp}\langle 2\rangle$. Observe that, for objects $(x,y)\in\mathcal{M}^2$, we have

$$\begin{aligned}
\texttt{Can}(U)\ (x,y) &= \texttt{Can}\Big((X\bullet Y)^{\dagger}\Big)\ (x,y) \\
&= (\texttt{Can}(X)\otimes\texttt{Can}(Y))'\ (x,y) \\
&= (\texttt{Can}(X)(x,y)\otimes\texttt{Can}(Y)(x,y))' && \text{(11.1.13) and (11.1.21)} \\
&= (x\otimes y)' && \text{(11.2.2)} \\
&= F(x,y).
\end{aligned}$$

Here, the second equality holds since Can is a strict categorical group functor. Similarly, for morphisms $(f,g)\in\mathcal{M}^2$, we have

$$\texttt{Can}(U)\ (f,g) = F(f,g).$$

Thus,

$$F = \texttt{Can}(U).$$

Analogously, we obtain

$$G = \texttt{Can}(V).$$

This demonstrates that both F and G are iterates in two variables. $\diamond$

Example 11.2.6. Let $\mathcal{M}$ be a categorical group, and let $F,G:\mathcal{M}^2\longrightarrow\mathcal{M}$ be the functors defined as in Example 11.2.5. Consider the natural transformation $\phi:F\Rightarrow G$ given by

$$\begin{tikzcd}
(x\otimes y)' \arrow[r, "\mu\circ\langle\eta_x\rangle\circ\mu"] & (x\otimes y)'\otimes x\otimes x' \arrow[r, "\mu\circ\langle\eta_y\rangle\circ\mu"] & (x\otimes y)'\otimes x\otimes y\otimes y'\otimes x' \arrow[d, "\mu\circ\langle\epsilon_{x\otimes y}\rangle\circ\mu"] \\
 & & y'\otimes x'
\end{tikzcd} \tag{11.2.7}$$

where μ denotes the appropriate composition of associators and unitors in $\mathscr{M}$.

Let $[2] = \{X, Y\}$, by identifying

$$1 = X, \qquad 2 = Y,$$

and let $U = (X \bullet Y)^\dagger$ and $V = Y^\dagger \bullet X^\dagger$ be objects in $\mathcal{CatGrp}\langle 2\rangle$ as in Example 11.2.5. Consider the arrow

$$\widehat{\phi} : (X \bullet Y)^\dagger \longrightarrow Y^\dagger \bullet X^\dagger$$

in $\mathcal{CatGrp}\langle S\rangle$ defined by

$$\begin{tikzcd} (X \bullet Y)^\dagger \arrow[r, "\mu \circ \langle \eta_X \rangle \circ \mu"] & (X \bullet Y)^\dagger \bullet X \bullet X^\dagger \arrow[r, "\mu \circ \langle \eta_Y \rangle \circ \mu"] & (X \bullet Y)^\dagger \bullet X \bullet Y \bullet Y^\dagger \bullet X^\dagger \arrow[d, "\mu \circ \langle \epsilon_{X \bullet Y} \rangle \circ \mu"] \\ & & Y^\dagger \bullet X^\dagger \end{tikzcd} \tag{11.2.8}$$

where μ denotes the appropriate composition of associators and unitors in $\mathcal{CatGrp}\langle 2\rangle$. Since $\mathtt{Can}$ is a strict categorical group functor, we have

$$\mathtt{Can}\left(\eta_X\right) = \eta_{\mathtt{Can}(X)}, \qquad \mathtt{Can}\left(\eta_Y\right) = \eta_{\mathtt{Can}(Y)},$$

and

$$\mathtt{Can}(\epsilon_{X \bullet Y}) = \epsilon_{\mathtt{Can}(X) \otimes \mathtt{Can}(Y)}.$$

Moreover, the compositions of associators and unitors in the diagram (11.2.8) are mapped to the corresponding compositions in the diagram (11.2.7) under $\mathtt{Can}$. It follows that the natural transformation ϕ given by the diagram (11.2.7) is the image under $\mathtt{Can}$ of the arrow $\widehat{\phi}$ given by the diagram (11.2.8); that is,

$$\mathtt{Can}\left(\widehat{\phi}\right) = \phi.$$

Thus, $\phi : F \Rightarrow G$ is a canonical natural transformation in two variables. $\diamond$

Example 11.2.9. Let $\mathscr{M}$ be a categorical group, and let $F, G : \mathscr{M}^2 \longrightarrow \mathscr{M}$ be the functors defined as in Example 11.2.5. Consider the natural transformation $\psi : F \Rightarrow G$ given by

$$\begin{tikzcd}
(x\otimes y)' \arrow[d, "\mu\circ\langle\epsilon_y^{-1}\rangle\circ\mu"'] & & \\
y'\otimes y\otimes(x\otimes y)' \arrow[r, "\mu\circ\langle\epsilon_x^{-1}\rangle\circ\mu"'] & y'\otimes x'\otimes x\otimes y\otimes(x\otimes y)' \arrow[r, "\mu\circ\langle\eta_{x\otimes y}^{-1}\rangle\circ\mu"'] & y'\otimes x'
\end{tikzcd} \tag{11.2.10}$$

where μ denotes the appropriate composition of associators and unitors in $\mathscr{M}$.

Let $[2] = \{X, Y\}$ be as in Example 11.2.6. Consider the arrow

$$\widehat{\psi} : (X \bullet Y)^\dagger \longrightarrow\!\!\!\!\!\!\!\circ\;\;\; Y^\dagger \bullet X^\dagger$$

in $\mathit{CatGrp}\langle 2\rangle$ defined by

$$\begin{tikzcd}
(X\bullet Y)^\dagger \arrow[d, "\mu\circ\langle\epsilon_Y^{-1}\rangle\circ\mu"'] & & \\
Y^\dagger\bullet Y\bullet(X\bullet Y)^\dagger \arrow[r, "\mu\circ\langle\epsilon_X^{-1}\rangle\circ\mu"'] & Y^\dagger\bullet X^\dagger\bullet X\bullet Y\bullet(X\bullet Y)^\dagger \arrow[r, "\mu\circ\langle\eta_{X\bullet Y}^{-1}\rangle\circ\mu"'] & Y^\dagger\bullet X^\dagger
\end{tikzcd} \tag{11.2.11}$$

where μ denotes the appropriate composition of associators and unitors in $\mathit{CatGrp}\langle 2\rangle$. By an analysis analogous to that in Example 11.2.6, we conclude that

$$\mathtt{Can}\left(\widehat{\psi}\right) = \psi.$$

It follows that ϕ and ψ form a pair of formal canonical natural isomorphisms. By Theorem 11.2.4, we deduce that $\phi = \psi$. That is, the diagram

$$\begin{tikzcd}
(x\otimes y)' \arrow[r, "\mu\circ\langle\eta_x\rangle\circ\mu"] \arrow[d, "\mu\circ\langle\epsilon_y^{-1}\rangle\circ\mu"'] & (x\otimes y)'\otimes x\otimes x' \arrow[r, "\mu\circ\langle\eta_y\rangle\circ\mu"] & (x\otimes y)'\otimes x\otimes y\otimes y'\otimes x' \arrow[d, "\mu\circ\langle\epsilon_{x\otimes y}\rangle\circ\mu"'] \\
y'\otimes y\otimes(x\otimes y)' \arrow[r, "\mu\circ\langle\epsilon_x^{-1}\rangle\circ\mu"'] & y'\otimes x'\otimes x\otimes y\otimes(x\otimes y)' \arrow[r, "\mu\circ\langle\eta_{x\otimes y}^{-1}\rangle\circ\mu"'] & y'\otimes x'
\end{tikzcd} \tag{11.2.12}$$

commutes for all $x, y \in \mathrm{Obj}(\mathscr{M})$. We denote this canonical natural transformation by

$$\left(\delta_{x,y}\right)_{(x,y)\in\mathscr{M}^2} := \mathtt{Can}\left((X \bullet Y)^\dagger \multimap Y^\dagger \bullet X^\dagger\right).$$

Note that this natural transformation coincides with that defined in Definition 5.2.6.◇

Example 11.2.13. Let $\mathscr{M}$ be a categorical group, and consider the functors $F, G : \mathscr{M}^3 \longrightarrow \mathscr{M}$ defined by

$$F(x,y,a) = x \otimes y \otimes a \otimes (x \otimes y)' \qquad \text{on objects,}$$

$$F(f,g,h) = f \otimes g \otimes h \otimes (f \otimes g)' \qquad \text{on morphisms,}$$

and

$$G(x,y,a) = x \otimes (y \otimes a \otimes y') \otimes x' \qquad \text{on objects,}$$

$$G(f,g,h) = f \otimes (g \otimes h \otimes g') \otimes f' \qquad \text{on morphisms.}$$

Consider the natural transformation $\phi : F \Rightarrow G$ given by

$$\phi_{x,y,a} := \alpha \circ (1_x \otimes 1_y \otimes 1_a \otimes \delta_{x,y}) : x \otimes y \otimes a \otimes (x \otimes y)' \longrightarrow x \otimes (y \otimes a \otimes y') \otimes x',$$

where δ is as in Example 11.2.9, and α denotes the appropriate composition of associators required to rearrange the parentheses.

We will show that, for $x, y, z, a \in \mathrm{Obj}(\mathscr{M})$ the diagram

$$\begin{tikzcd}
(x\otimes y\otimes z)\otimes a\otimes(x\otimes y\otimes z)' \arrow[rr, "\alpha_{x,y,z}\otimes 1_a\otimes\alpha'_{x,y,z}"] \arrow[d, "\phi_{x\otimes y,z,a}"'] & & (x\otimes(y\otimes z))\otimes a\otimes(x\otimes(y\otimes z))' \arrow[d, "\phi_{x,y\otimes z,a}"] \\
(x\otimes y)\otimes\left(z\otimes a\otimes z'\right)\otimes(x\otimes y)' \arrow[dr, "\phi_{x,y,\, z\otimes a\otimes z'}"'] & & x\otimes\left((y\otimes z)\otimes a\otimes(y\otimes z)'\right)\otimes x' \arrow[dl, "1_x\otimes\phi_{y,z,a}\otimes 1_{x'}"] \\
& x\otimes\left(y\otimes\left(z\otimes a\otimes z'\right)\otimes y'\right)\otimes x' &
\end{tikzcd} \tag{11.2.14}$$

commutes.

Let

$$[4] := \{X, Y, Z, A\}$$

with identification

$$1 = X, \qquad 2 = Y, \qquad 3 = Z, \qquad \text{and} \qquad 4 = A.$$

Since Can is a strict categorical group functor, the diagram (11.2.14) is the image of the following diagram in $\mathcal{C}at\mathcal{G}rp\langle 4\rangle$ evaluated on $(x, y, z, a) \in \mathcal{M}^4$:

$$\begin{tikzcd}
(X \bullet Y \bullet Z) \bullet A \bullet (X \bullet Y \bullet Z)^{\dagger} \arrow[rr, "\alpha_{X,Y,Z} \bullet 1_A \bullet \alpha^{\dagger}_{X,Y,Z}"] \arrow[d, "\phi_{X \bullet Y, Z, A}"'] & & (X \bullet (Y \bullet Z)) \bullet A \bullet (X \bullet (Y \bullet Z))^{\dagger} \arrow[d, "\phi_{X, Y \bullet Z, A}"] \\
(X \bullet Y) \bullet \left(Z \bullet A \bullet Z^{\dagger}\right) \bullet (X \bullet Y)^{\dagger} \arrow[dr, "\phi_{X,Y,\, Z \bullet A \bullet Z^{\dagger}}"'] & & X \bullet \left((Y \bullet Z) \bullet A \bullet (Y \bullet Z)^{\dagger}\right) \bullet X^{\dagger} \arrow[dl, "1_X \bullet \phi_{Y,Z,A} \bullet 1_{X^{\dagger}}"] \\
& X \bullet \left(Y \bullet \left(Z \bullet A \bullet Z^{\dagger}\right) \bullet Y^{\dagger}\right) \bullet X^{\dagger} &
\end{tikzcd}$$

Since $\mathcal{C}at\mathcal{G}rp\langle S\rangle$ is a thin category, the above diagram commutes. It follows that, the diagram (11.2.14) commutes. ◇

Definition 11.2.15. A morphism $\gamma : x \longrightarrow y$ in $\mathcal{M}$ is called a *canonical isomorphism* if there exist objects $a, b \in \mathcal{C}at\mathcal{G}rp\langle M\rangle$, and an arrow $\widehat{\gamma} : a \multimap b$ in $\mathcal{C}at\mathcal{G}rp\langle M\rangle$ such that

$$\mathcal{EV}(a) = x, \qquad \mathcal{EV}(b) = y, \qquad \text{and} \qquad \mathcal{EV}(\widehat{\gamma}) = \gamma : x \longrightarrow y.$$

We denote a canonical morphism $\gamma : x \longrightarrow y$ by $\gamma : x \xrightarrow{\text{can}} y$. In this case, we say that x is *canonically isomorphic* to y.

Furthermore, a pair of parallel morphisms $f, g : x \longrightarrow y$ in $\mathscr{M}$ are called a *pair of formal canonical isomorphisms* if there exist objects $a, b \in \mathcal{C}at\mathcal{G}rp\langle M\rangle$ and arrows $\widehat{f}, \widehat{g} : a \longrightarrow\!\!\!\!\circ\;\; b$ in $\mathcal{C}at\mathcal{G}rp\langle M\rangle$ such that

$$\mathcal{EV}(a) = x, \qquad \mathcal{EV}(b) = y, \qquad \mathcal{EV}(\widehat{f}) = f, \qquad \text{and} \qquad \mathcal{EV}(\widehat{g}) = g. \qquad \diamond$$

Remark 11.2.16. Note that the notion of canonical isomorphism in Definition 11.2.15 is, a priori, distinct from that of canonical natural isomorphism in Definition 11.2.3. The essential difference is that a canonical isomorphism refers to when a morphism in $\mathscr{M}$ is canonical, whereas a canonical natural isomorphism refers to when a natural isomorphism between functors is canonical. However, as we will show in Proposition 11.2.17, these two notions are, in fact, equivalent. $\diamond$

Proposition 11.2.17. *Let $n \in \mathbb{N}$, let $u : [n] \longrightarrow M$ be a function. Set $a_i := u(i)$ for $1 \le i \le n$ and denote*

$$u = (a_1, \ldots, a_n)$$

Then, by the universal property of $\mathcal{C}at\mathcal{G}rp\langle n\rangle$, there exists a unique strict categorical group functor

$$\Phi : \mathcal{C}at\mathcal{G}rp\langle n\rangle \longrightarrow \mathcal{C}at\mathcal{G}rp\langle M\rangle$$

such that

$$\Phi(i) = a_i \tag{11.2.18}$$

for each $1 \le i \le n$. Let

$$\mathcal{EV}_u : \mathcal{F}ct_n(\mathscr{M}) \longrightarrow \mathscr{M}$$

be the strict categorical group functor as defined in Definition 11.1.8. Then, the following diagram of strict categorical group functors commutes:

$$
\begin{tikzcd}
\mathcal{C}at\mathcal{G}rp\langle n\rangle \arrow[r, "\Phi"] \arrow[d, "\mathcal{C}an"'] & \mathcal{C}at\mathcal{G}rp\langle M\rangle \arrow[d, "\mathcal{EV}"] \\
\mathcal{F}ct_n(\mathcal{M}) \arrow[r, "\mathcal{EV}_u"'] & \mathcal{M}.
\end{tikzcd}
\tag{11.2.19}
$$

In other words, every canonical morphism comes from a canonical natural isomorphism.

Proof. Since $\mathcal{C}at\mathcal{G}rp\langle n\rangle$ is the free categorical group generated by $[n]$, it suffices to verify that the diagram commutes for the generators $i \in [n]$ and for the structure morphisms λ_X, ρ_X, $\alpha_{X,Y,Z}$, η_X, and ϵ_Y, where $X, Y, Z \in \mathcal{C}at\mathcal{G}rp\langle n\rangle$.

For each $1 \le i \le n$, we have

$$
\begin{aligned}
\mathcal{EV} \circ \Phi(i) &= \mathcal{EV}(a_i) && (11.2.18)\\
&= a_i && (11.1.3)\\
&= \Pi_i(a_1, \ldots, a_n) && (11.1.28)\\
&= \mathcal{EV}_u(\Pi_i) && (11.1.9)\\
&= \mathcal{EV}_u \circ \mathrm{Can}(i). && (11.2.2)
\end{aligned}
$$

For a structure morphism, for example λ_X with $X \in \mathit{CatGrp}\langle n\rangle$, we have

$$\begin{aligned}
\mathcal{EV} \circ \Phi(\lambda_X) &= \mathcal{EV}(\lambda_{\Phi(X)}) \\
&= \lambda_{\mathcal{EV}\circ\Phi(X)} \\
&= \lambda_{\mathcal{EV}_u \circ \mathtt{Can}(X)} && \text{by the above computation} \\
&= \mathcal{EV}_u(\lambda_{\mathtt{Can}(X)}) \\
&= \mathcal{EV}_u \circ \mathtt{Can}(\lambda_X).
\end{aligned}$$

Here, the first, second, fourth, and fifth equalities follow from the fact that $\mathcal{EV}$, Φ, $\mathcal{EV}_u$, and Can are all strict categorical group functors. The same reasoning applies to the other structure morphisms. Therefore, the diagram (11.2.19) commutes. □

Definition 11.2.20. Let $x \in \mathit{CatGrp}\langle M\rangle$ be an object. Then x induces an iterate $F : \mathcal{M}^n \longrightarrow \mathcal{M}$ for some n as follows: Since x is an object in $\mathit{CatGrp}\langle M\rangle$, it can be expressed using a finite collection of elements $a_1, \ldots, a_n \in M$ via a unique sequence of multiplication and dash operations. Define $u : [n] \longrightarrow M$ by $u(i) = a_i$. Let $X \in \mathit{CatGrp}\langle n\rangle$ be the formal word obtained by applying the same operations to the indices corresponding to a_i. Then,

$$\Phi(X) = x$$

where $\Phi : \mathit{CatGrp}\langle n\rangle \longrightarrow \mathit{CatGrp}\langle M\rangle$ is as in Proposition 11.2.17.

Define the iterate $F : \mathcal{M}^n \longrightarrow \mathcal{M}$ by

$$F := \mathtt{Can}(X). \tag{11.2.21}$$

By Proposition 11.2.17, this construction ensures that

$$F(a_1, \ldots, a_n) = \mathcal{EV}(x). \tag{11.2.22}$$

◇

Remark 11.2.23. Note that in the above definition, not every element from the finite collection $a_i \in M$ must be used in the construction of $x \in \mathcal{C}at\mathcal{G}rp\langle M\rangle$. This flexibility allows for a more general notion of induced iterates, which will also be useful in the subsequent definition of induced canonical natural isomorphisms. ◇

Definition 11.2.24. Let $f : x \longrightarrow y$ be a morphism in $\mathcal{C}at\mathcal{G}rp\langle M\rangle$. Then f induces a canonical natural isomorphism $\phi : F \Rightarrow G$ between iterates $F, G : \mathcal{M}^n \longrightarrow \mathcal{M}$ for some n, constructed as follows. Since x and y are objects in $\mathcal{C}at\mathcal{G}rp\langle M\rangle$, there exist finitely many elements $a_1, \ldots, a_n \in M$ such that x and y can be expressed as formal words in these elements using multiplication and dash operations (possibly using different subsets for x and y).

Define $u : [n] \longrightarrow M$ by $u(i) = a_i$. Let $X, Y \in \mathcal{C}at\mathcal{G}rp\langle n\rangle$ be the formal words corresponding to x and y, respectively, constructed using the same operations but with indices in $[n]$. Thus,

$$\Phi(X) = x \qquad \text{and} \qquad \Phi(Y) = y,$$

where $\Phi : \mathcal{C}at\mathcal{G}rp\langle n\rangle \longrightarrow \mathcal{C}at\mathcal{G}rp\langle M\rangle$ is as in Proposition 11.2.17. There is a unique arrow

$$\widehat{f} : X \longrightarrow Y \quad \text{in } \mathcal{C}at\mathcal{G}rp\langle n\rangle$$

such that

$$\Phi(\widehat{f}) = f.$$

Define $F, G : \mathcal{M}^n \longrightarrow \mathcal{M}$ by

$$F := \mathtt{Can}(X), \qquad G := \mathtt{Can}(Y),$$

as in Definition 11.2.20. Let $\phi : F \Rightarrow G$ be the natural transformation

$$\phi := \mathtt{Can}\left(\widehat{f}\right). \tag{11.2.25}$$

By Proposition 11.2.17, we have

$$\phi(a_1, \dots, a_n) = \mathcal{EV}(f). \tag{11.2.26}$$

◇

Example 11.2.27. Let $a, b \in M$ and consider the arrow

$$a^\dagger \bullet a \bullet b \multimap J \bullet b$$

in $\mathcal{CatGrp}\langle M \rangle$. To express this as a canonical natural transformation, take $[2] = \{A, B\}$ with the identification

$$1 = A \qquad \text{and} \qquad 2 = B.$$

Let $u : [2] \longrightarrow M$ be given by $A \longmapsto a$ and $B \longmapsto b$. Consider the arrow

$$A^\dagger \bullet A \bullet B \multimap J \bullet B$$

in $\mathcal{CatGrp}\langle 2 \rangle$. Then,

$$\mathtt{Can}\left(A^\dagger \bullet A \bullet B \multimap J \bullet B\right) = \left(\epsilon_x \otimes 1_y : x' \otimes x \otimes y \longrightarrow I \otimes y\right)_{(x,y) \in \mathcal{M}^2}.$$

In particular,

$$\mathtt{Can}\left(A^\dagger \bullet A \bullet B \multimap J \bullet B\right)(a, b) = \epsilon_a \otimes 1_b = \mathcal{EV}(a^\dagger \bullet a \bullet b \multimap J \otimes b).$$

◇

Chapter 12: Symmetrization of Categorical Group

In this chapter, we present a construction that produces a symmetric categorical group from a given categorical group. We state the universal property characterizing this construction and provide several examples.

12.1 Symmetric Categorical Groups

In this section, we introduce symmetric categorical groups and state their coherence theorem. We present only the essential definitions and a concise overview, as the full technical details are not needed for our construction. For further details, see [Lap83].

Definition 12.1.1. A braided monoidal category

$$\big(\mathcal{M}, I, \otimes, \lambda, \rho, \alpha, \beta\big)$$

consitst of the following data:

- A monoidal category

  $$(\mathcal{M}, I, \otimes, \lambda, \rho, \alpha)$$

  called the *underlying monoidal category*. We adopt the left associative convention for the tensor product. That is, for $x, y, z \in \mathcal{M}$, we denote

  $$x \otimes y \otimes z := (x \otimes y) \otimes z.$$

- A natural isomorphism,

$$\beta := \left(\beta_{x,y} : x \otimes y \longrightarrow y \otimes x\right)_{(x,y) \in \mathscr{M}^2}$$

called the *braiding*.

These satisfy the following coherence axioms:

Hexgon identities: For objects $x, y, z \in \mathscr{M}$, the diagrams

$$\begin{tikzcd}
x \otimes y \otimes z \arrow[r, "\alpha^{-1}_{x,y,z}"] \arrow[d, "\beta_{x,y} \otimes 1_z"'] & x \otimes (y \otimes z) \arrow[r, "\beta_{x,y\otimes z}"] & y \otimes z \otimes x \arrow[d, "\alpha^{-1}_{y,z,x}"] \\
y \otimes x \otimes z \arrow[r, "\alpha^{-1}_{y,z,x}"'] & y \otimes (x \otimes z) \arrow[r, "1_y \otimes \beta_{x,z}"'] & y \otimes (z \otimes x)
\end{tikzcd} \tag{12.1.2}$$

and

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "\alpha_{x,y,z}"] \arrow[d, "1_x \otimes \beta_{y,z}"'] & x \otimes y \otimes z \arrow[r, "\beta_{x\otimes y,z}"] & z \otimes (x \otimes y) \arrow[d, "\alpha z, x, y"] \\
x \otimes (z \otimes y) \arrow[r, "\alpha_{x,z,y}"'] & x \otimes z \otimes y \arrow[r, "\beta_{x,z} \otimes 1_y"'] & z \otimes x \otimes y
\end{tikzcd} \tag{12.1.3}$$

commute. ◇

Definition 12.1.4. A braided monoidal category $\mathscr{M}$ is called a *symmetric monoidal category* if the following condition is satisfied: For objects $x, y \in \mathscr{M}$ the equality

$$\beta_{y,x} \circ \beta_{x,y} = 1_{x \otimes y} \tag{12.1.5}$$

holds. ◇

Definition 12.1.6. A *symmetric categorical group*

$$\left(\mathscr{M}, I, (-)', \otimes, \lambda, \rho, \eta, \epsilon, \alpha, \beta\right)$$

consists of the following data:

- A categorical group

$$(\mathcal{M}, I, \otimes, \lambda, \rho, \alpha)$$

called the *underlying categorical group*.

- A symmetric monoidal category

$$(\mathcal{M}, I, \otimes, \lambda, \rho, \alpha, \beta)$$

called the *underlying symmetric monoidal category*.

These satisfy the following coherence axioms:

Braiding-cancellation compatibility: For every object $x \in \mathcal{M}$, the diagram

$$\begin{tikzcd} x \otimes x' \arrow[d, "\beta_{x,x'}"'] \arrow[dr, "\eta_x^{-1}"] & \\ x' \otimes x \arrow[r, "\epsilon_x"'] & I \end{tikzcd} \tag{12.1.7}$$

◇

Definition 12.1.8. Let $\mathcal{M} = (\mathcal{M}, I, (-)', \otimes)$ and $\mathcal{N} = (\mathcal{N}, J, (-)^\dagger, \bullet)$ be two categorical groups. A categorical group functor

$$(F, F_0, F_1, F_2) : \mathcal{M} \longrightarrow \mathcal{N}$$

is called a *symmetric categorical group functor* if the following condition is satisfied: For objects $x, y \in \mathcal{M}$ the diagram

$$\begin{tikzcd} F(x) \bullet F(y) \arrow[r, "\beta^{\mathcal{N}}_{F(x),F(y)}"] \arrow[d, "F_2(x,y)"'] & F(y) \bullet F(x) \arrow[d, "F_2(y,x)"] \\ F(x \otimes y) \arrow[r, "F(\beta^{\mathcal{M}}_{x,y})"'] & F(y \otimes x) \end{tikzcd} \tag{12.1.9}$$

commutes.

Furthermore, a symmetric categorical group functor is called a *strict symmetric categorical group functor* if the underlying categorical group functor is a strict categorical group functor. $\diamond$

Definition 12.1.10. Let $\mathscr{M} = (\mathscr{M}, I, (-)', \otimes)$ and $\mathscr{N} = (\mathscr{N}, J, (-)^{\dagger}, \bullet)$ be two categorical groups. Let

$$F, G : \mathscr{M} \longrightarrow \mathscr{N}$$

be symmetric categorical group functors. A *symmetric categorical group natural transformation*

$$\phi : F \Rightarrow G$$

is a categorical group natural transformation on the underlying categorical group. $\diamond$

Construction 12.1.11. Let S be a set. The *free symmetric categorical group* generated by S, denoted $\mathcal{SymCatGrp}\langle S\rangle$, is constructed as follows:

- The free dashed multiplicative set with unit $\mathcal{DMul}\langle S\rangle$ as in Construction 4.5.1 serves as the set of objects in $\mathcal{SymCatGrp}\langle S\rangle$.
- The morphisms in $\mathcal{SymCatGrp}\langle S\rangle$ are formally generated from the structure arrows

$$\begin{aligned}
&1_x : x \longrightarrow x, && \tilde{\alpha}_{x,y,z} : x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z,\\
&\tilde{\lambda}_x : J \otimes x \longrightarrow x, && \tilde{\rho}_x : x \otimes J \longrightarrow x,\\
&\tilde{\eta}_x : J \longrightarrow x \otimes x', && \tilde{\epsilon}_x : x' \otimes x \longrightarrow J,\\
&\tilde{\beta}_{x,y} : x \otimes y \longrightarrow y \otimes x.
\end{aligned}$$

for $x, y, z \in \mathcal{DMul}\langle S\rangle$ by tensoring, inverting, taking dash, and composing, subject to the equivalence relations generated by the axioms of a categorical group.

See [Lap83] for the more details on the construction. This construction satisfies the following universal property. ◇

Definition 12.1.12. Let S be a set, $\mathscr{M}$ be a symmetric categorical group, and

$$i : S \longrightarrow \mathrm{Obj}(\mathscr{M})$$

be a function. We say that the pair

$$\big(\mathscr{M}, i : S \longrightarrow \mathrm{Obj}(\mathscr{M})\big)$$

satisfies the *universal property of a free symmetric categorical group* if the following conditions are satisfied:

Existence (strict functor): For a symmetric categorical group $\mathscr{N}$, and a set map $f : S \longrightarrow \mathrm{Obj}(\mathscr{N})$ there exists a strict symmetric categorical group functor

$$F : \mathscr{M} \longrightarrow \mathscr{N}$$

such that the equality

$$\mathrm{Obj}(F) \circ i = f \tag{12.1.13}$$

holds.

Existence (natural transformation): For a symmetric categorical group $\mathscr{N}$, a pair of symmetric categorical group functors

$$F, G : \mathscr{M} \longrightarrow \mathscr{N},$$

and a collection of isomorphisms

$$\widehat{\phi} = \big(\widehat{\phi}_a : F(i(a)) \longrightarrow G(i(a))\big)_{i \in S}$$

there exists a symmetric categorical group natural transformation

$$\phi : F \Rightarrow G$$

such that for every element $a \in S$, the equality

$$\phi_{i(a)} = \widehat{\phi}_a : F(i(a)) \longrightarrow G(i(a)) \tag{12.1.14}$$

holds.

Uniqueness: For a symmetric categorical group $\mathscr{N}$, a pair of symmetric categorical group functors

$$F, G : \mathscr{M} \longrightarrow \mathscr{N},$$

and a pair of symmetric categorical group natural isomorphisms

$$\phi, \psi : F \Rightarrow G$$

the following implication holds: If for every element $a \in S$ the equality

$$\phi_{i(a)} = \psi_{i(a)} : F(i(a)) \longrightarrow G(i(a))$$

of morphism is satisfied then the equality of natural transformations

$$\phi = \psi : F \Rightarrow G \tag{12.1.15}$$

holds. ◇

Theorem 12.1.16. *Let S be a set. Then, the free symmetric categorical group generated by S as in Construction 12.1.11 is a thin category.*

Proof. This result is showed as Theorem 3.2 on page 318 of [Lap83]. □

Corollary 12.1.17. *Let S be a set, $\mathcal{AbGrp}\langle S\rangle$ be the free Abelian group generated by S, and $\mathcal{SymCatGrp}\langle S\rangle$ be the free symmetric categorical group generated by S. We may view $\mathcal{AbGrp}\langle S\rangle$ as the discrete strict symmetric categorical group. Then, the induced strict symmetric categorical group functor*

$$R : \mathcal{SymCatGrp}\langle S\rangle \longrightarrow \mathcal{AbGrp}\langle S\rangle .$$

is an equivalence.

Thus, there is a morphism $f : x \longrightarrow y$ in $\mathcal{SymCatGrp}\langle S\rangle$ if and only if $R(x) = R(y)$.

12.2 Commutator Sub-categorical Group

Definition 12.2.1. Let M be a group. For elements $x, y \in M$, the *commutator* $[x, y]$ is defined as

$$[x, y] := xyx^{-1}y^{-1}. \tag{12.2.2}$$

The *commutator subgroup* $[M, M]$ is the subgroup of M generated by all commutators. Every element of $[M, M]$ can be written as a finite product of commutators:

$$x = [a_1, b_1] \cdots [a_n, b_n]$$

for some $a_i, b_i \in M$. The *Abelianization* of M is the quotient group $M/[M, M]$.

This construction has the following universal property: For any Abelian group N and any group homomorphism $f : M \longrightarrow N$, there exists a unique group homomorphism $\overline{f} : M/[M, M] \longrightarrow N$ such that

$$\overline{f}([x]) = f(x)$$

for all $x \in M$, where $[x]$ denotes the image of x in $M/[M, M]$. $\diamond$

Proposition 12.2.3. *Let S be a set, $\mathcal{G}rp\langle S\rangle$ denote the free group generated by S, and $\mathcal{A}b\mathcal{G}rp\langle S\rangle$ denoted the free Abelian group generated by S. Then, $\mathcal{A}b\mathcal{G}rp\langle S\rangle$ is isomorphic to the Abelianization of the free group $\mathcal{G}rp\langle S\rangle$.*

Proof. Let M be the Abelianization of the free group $\mathcal{G}rp\langle S\rangle$, so M is an Abelian group. By the universal property of $\mathcal{A}b\mathcal{G}rp\langle S\rangle$, there exists a group homomorphism

$$f : \mathcal{A}b\mathcal{G}rp\langle S\rangle \longrightarrow M.$$

Similarly, by the universal property of Abelianization, there is a group homomorphism

$$g : M \longrightarrow \mathcal{A}b\mathcal{G}rp\langle S\rangle.$$

By the uniqueness in these universal properties, we get the required isomorphism.□

Definition 12.2.4. Let $\mathscr{M}$ be a categorical group. For any objects $x, y \in \mathrm{Obj}(\mathscr{M})$, define their commutator as

$$[x, y] := x \otimes y \otimes x' \otimes y'.$$

For morphisms $f : x \longrightarrow z$ and $g : y \longrightarrow w$ in $\mathscr{M}$, define their commutator as

$$[f, g] := f \otimes g \otimes f' \otimes g' : [x, y] \longrightarrow [z, w]. \qquad \diamond$$

Definition 12.2.5. Let S be a set, and let $\mathcal{G}rp\langle S\rangle$ and $\mathcal{A}b\mathcal{G}rp\langle S\rangle$ denote the free group and free Abelian group generated by S, respectively. Consider the functors

$$\mathcal{C}at\mathcal{G}rp\langle S\rangle \xrightarrow{Q} \mathcal{G}rp\langle S\rangle \xrightarrow{R} \mathcal{A}b\mathcal{G}rp\langle S\rangle$$

where $\mathcal{G}rp\langle S\rangle$ and $\mathcal{A}b\mathcal{G}rp\langle S\rangle$ are regarded as discrete categories. Define the *commutator sub-categorical group*

$$[\mathcal{C}at\mathcal{G}rp\langle S\rangle, \mathcal{C}at\mathcal{G}rp\langle S\rangle] \subseteq \mathcal{C}at\mathcal{G}rp\langle S\rangle$$

to be the full subcategory of $\mathcal{CatGrp}\langle S\rangle$ whose objects are those mapped to the unit $0 \in \mathcal{AbGrp}\langle S\rangle$ under the composition $R \circ Q$. That is,

$$\operatorname{Obj}\left([\mathcal{CatGrp}\langle S\rangle, \mathcal{CatGrp}\langle S\rangle]\right) := \left\{x \in \mathcal{CatGrp}\langle S\rangle \mid R(Q(x)) = 0\right\}. \tag{12.2.6}$$

It follows immediately that the commutator sub-categorical group

$$[\mathcal{CatGrp}\langle S\rangle, \mathcal{CatGrp}\langle S\rangle] \subseteq \mathcal{CatGrp}\langle S\rangle$$

is closed under multiplication and taking dash. Thus, it is a sub categorical group of $\mathcal{CatGrp}\langle S\rangle$.

From Proposition 12.2.3, we have that $\mathcal{AbGrp}\langle S\rangle$ is the Abelianization of $\mathcal{Grp}\langle S\rangle$, it follows that the objects of

$$[\mathcal{CatGrp}\langle S\rangle, \mathcal{CatGrp}\langle S\rangle]$$

are precisely the objects that land inside the commutator subgroup of $\mathcal{Grp}\langle S\rangle$. Therefore, for every object

$$x \in [\mathcal{CatGrp}\langle S\rangle, \mathcal{CatGrp}\langle S\rangle]$$

we get an arrow

$$x \multimap [a_1, b_1] \bullet \cdots \bullet [a_n, b_n] \in [\mathcal{CatGrp}\langle S\rangle, \mathcal{CatGrp}\langle S\rangle]$$

for some $a_i, b_i \in \mathcal{CatGrp}\langle S\rangle$. ◇

Proposition 12.2.7. *Let S be a set, $X \in \mathcal{CatGrp}\langle S\rangle$ an object, and*

$$A \in [\mathcal{CatGrp}\langle S\rangle, \mathcal{CatGrp}\langle S\rangle]$$

an object. Then $X \bullet A \bullet X^\dagger$ is also an object of $[\mathcal{CatGrp}\langle S\rangle, \mathcal{CatGrp}\langle S\rangle]$.

Proof. Since $A \in [\mathcal{CatGrp}\langle S\rangle, \mathcal{CatGrp}\langle S\rangle]$, it follows that

$$R(Q(A)) = 0.$$

Because $\mathcal{AbGrp}\langle S\rangle$ is Abelian, we have

$$R(Q\left(X \bullet A \bullet X^{\dagger}\right)) = R(Q(X)) + R(Q(A)) + (-R(Q(X))) = 0,$$

where addition and negation denote the group operation and inverse in $\mathcal{AbGrp}\langle S\rangle$. Therefore,

$$X \bullet A \bullet X^{\dagger} \in [\mathcal{CatGrp}\langle S\rangle, \mathcal{CatGrp}\langle S\rangle].$$

□

Definition 12.2.8. Let $\mathscr{C}$ be a category. A category $\mathscr{D}$ is called a *subcategory* of $\mathscr{C}$ if the following conditions hold:

- The objects of $\mathscr{D}$ form a subset of the objects of $\mathscr{C}$, that is, $\mathrm{Obj}(\mathscr{D}) \subseteq \mathrm{Obj}(\mathscr{C})$.
- For any $x, y \in \mathrm{Obj}(\mathscr{D})$, the set of morphisms $\mathrm{Hom}_{\mathscr{D}}(x, y)$ is a subset of $\mathrm{Hom}_{\mathscr{C}}(x, y)$.

A *monoidal subcategory* of a monoidal category $\mathscr{M}$ is a subcategory of $\mathscr{M}$ that is itself a monoidal category, with the monoidal structure inherited from $\mathscr{M}$. Similarly, a *sub-categorical group* of a categorical group $\mathscr{M}$ is a subcategory of $\mathscr{M}$ that is a categorical group, with the categorical group structure inherited from $\mathscr{M}$. ⋄

Definition 12.2.9. Let $\mathscr{M}$ be a categorical group, and let (C,D) be a *generator pair*, where $C \subseteq \mathrm{Obj}(\mathscr{M})$ is a subset of objects and, for each $x, y \in C$, $D(x, y) \subseteq \mathrm{Hom}_{\mathscr{M}}(x, y)$ is a subset of morphisms. The *sub-categorical group generated by* (C,D), denoted $\mathscr{M}\langle C,D\rangle$, is defined as the smallest sub-categorical group of $\mathscr{M}$ containing all objects in C and all morphisms in D. More precisely, for any sub-categorical group $\mathscr{N} \subseteq \mathscr{M}$ containing C and D, there is a unique strict categorical group functor

$$F : \mathscr{M}\langle C, D\rangle \longrightarrow \mathscr{N}$$

such that $F(x) = x$ for all $x \in C$ and $F(f) = f$ for all $f \in D$. ◇

Construction 12.2.10. Let $\mathscr{M}$ be a categorical group with $M = \mathrm{Obj}(\mathscr{M})$. Define a generator pair (C,D) as follows:

The generator pair (C,D) consists of all objects and morphisms that appear in the naturality squares of every induced canonical natural isomorphism $\mathtt{Can}(X \multimap\!\!\!\rightarrow Y)$, for every arrow $X \multimap\!\!\!\rightarrow Y$ in $[\mathcal{CatGrp}\langle M\rangle, \mathcal{CatGrp}\langle M\rangle]$. The *commutator sub-categorical group*, denoted by $[\mathscr{M},\mathscr{M}]$ is the sub-categorical group generated by (C,D).

More precisely, for each arrow $X \multimap\!\!\!\rightarrow Y$ in $[\mathcal{CatGrp}\langle M\rangle, \mathcal{CatGrp}\langle M\rangle]$, let $\mathtt{Can}(X), \mathtt{Can}(Y) : \mathscr{M}^n \longrightarrow \mathscr{M}$ be the iterates associated to X and Y as in Definition 11.2.20, and let $\phi := \mathtt{Can}(X \multimap\!\!\!\rightarrow Y) : \mathtt{Can}(X) \Rightarrow \mathtt{Can}(Y)$ be the corresponding canonical natural transformation. Then, for any collection of morphisms $f_i : a_i \longrightarrow b_i$ in $\mathscr{M}$, the objects and morphisms in the following commutative diagram are included in (C,D):

$$\begin{tikzcd}
\mathtt{Can}(X)\,(a_1,\dots,a_n) \arrow[r, "\mathtt{Can}(X)\,(f_1,\dots,f_n)"] \arrow[d, "\phi\,(a_1,\dots,a_n)"'] & \mathtt{Can}(X)\,(b_1,\dots,b_n) \arrow[d, "\phi\,(b_1,\dots,b_n)"] \\
\mathtt{Can}(Y)\,(a_1,\dots,a_n) \arrow[r, "\mathtt{Can}(Y)\,(f_1,\dots,f_n)"'] & \mathtt{Can}(Y)\,(b_1,\dots,b_n).
\end{tikzcd} \tag{12.2.11}$$

In summary, every object in C is an image of an iterate $\mathtt{Can}(X)$ for

$$X \in [\mathcal{CatGrp}\langle M\rangle, \mathcal{CatGrp}\langle M\rangle],$$

every arrow in D is either the image of an iterate $\mathtt{Can}(X)$ for

$$X \in [\mathcal{CatGrp}\langle M\rangle, \mathcal{CatGrp}\langle M\rangle],$$

or the image of a canonical natural isomorphism $\mathtt{Can}(X \multimap Y)$ for $X \multimap Y$ in $[\mathcal{C}at\mathcal{G}rp\langle M\rangle, \mathcal{C}at\mathcal{G}rp\langle M\rangle]$. $\diamond$

Remark 12.2.12. The following two lemmas characterize the objects and morphisms of $[\mathcal{M}, \mathcal{M}]$. $\diamond$

Lemma 12.2.13. *Let $\mathcal{M}$ be a categorical group with $M = \mathrm{Obj}(\mathcal{M})$. Let $x \in [\mathcal{M}, \mathcal{M}]$ be an object. Then, there exists $X \in [\mathcal{C}at\mathcal{G}rp\langle M\rangle, \mathcal{C}at\mathcal{G}rp\langle M\rangle]$ such that x is an image of an interate $\mathtt{Can}(X)$ induced by X.*

Furthermore, there exists $a_i, b_i \in M$ for $1 \leq i \leq n$ such that x is canonically isomorphic to $[a_1, b_1] \otimes \cdots \otimes [a_n, b_n]$.

Proof. Let (C, D) be the generator pair for $[\mathcal{M}, \mathcal{M}]$ as described in Construction 12.2.10. For any $x, y \in C$, there exist objects $X, Y \in [\mathcal{C}at\mathcal{G}rp\langle M\rangle, \mathcal{C}at\mathcal{G}rp\langle M\rangle]$ such that x is an image of an iterate $\mathtt{Can}(X)$ and y is an image of an iterate $\mathtt{Can}(Y)$. Since $[\mathcal{C}at\mathcal{G}rp\langle M\rangle, \mathcal{C}at\mathcal{G}rp\langle M\rangle]$ is a sub-categorical group, it follows that $x \otimes y \in C$ and $x' \in C$. Therefore, the objects of $[\mathcal{M}, \mathcal{M}]$ are exactly the elements of C. This proves the first claim.

Furthermore, by Construction 12.2.10, there exists an arrow

$$X \multimap [A_1, B_1] \bullet \cdots \bullet [A_n, B_n] \qquad \text{in } [\mathcal{C}at\mathcal{G}rp\langle M\rangle, \mathcal{C}at\mathcal{G}rp\langle M\rangle]$$

for some $A_i, B_i \in \mathcal{C}at\mathcal{G}rp\langle M\rangle$. Taking image of the induced canonical natural transformation yields the second part of the lemma. $\square$

Lemma 12.2.14. *Let $\mathcal{M}$ be a categorical group with $M = \mathrm{Obj}(\mathcal{M})$. Let (C, D) be the generator pair for $[\mathcal{M}, \mathcal{M}]$ as in Construction 12.2.10. Let $f : x \longrightarrow y$ be a morphism* *in D. Then, there exists*

$$g_i : a_i \longrightarrow c_i \qquad \text{and} \qquad h_i : b_i \longrightarrow d_i \qquad \textit{in } M$$

such that the diagram

$$\begin{tikzcd} x \arrow[r, "\mathrm{can}"] \arrow[d, "f"'] & {[a_1,b_1]\otimes\cdots\otimes[a_n,b_n]} \arrow[d, "{[g_1,f_1]\otimes\cdots\otimes[g_n,h_n]}"] \\ y \arrow[r, "\mathrm{can}"'] & {[c_1,d_1]\otimes\cdots\otimes[c_n,d_n]} \end{tikzcd} \tag{12.2.15}$$

commutes.

Moreover, any morphism $f : x \longrightarrow y$ in $[\mathcal{M},\mathcal{M}]$ can be expressed as a finite composition of morphisms from D.

Proof. Let $f : x \longrightarrow y$ be a morphism in D. If f arises as the image of an iterate $\mathtt{Can}(X)$ for some $X \in [\mathcal{C}at\mathcal{G}rp\langle M\rangle, \mathcal{C}at\mathcal{G}rp\langle M\rangle]$, then by Construction 12.2.10, there exist $A_i, B_i \in \mathcal{C}at\mathcal{G}rp\langle M\rangle$ for $1 \le i \le n$ such that there is an arrow

$$X \multimap [A_1,B_1] \bullet \cdots \bullet [A_n,B_n]$$

in $[\mathcal{C}at\mathcal{G}rp\langle M\rangle, \mathcal{C}at\mathcal{G}rp\langle M\rangle]$. The naturality square associated to this canonical natural transformation yields the required commutative diagram (12.2.15).

If instead $f : x \longrightarrow y$ is the image of a canonical natural isomorphism $\mathtt{Can}(X \multimap Y)$ for some $X \multimap Y \in [\mathcal{C}at\mathcal{G}rp\langle M\rangle, \mathcal{C}at\mathcal{G}rp\langle M\rangle]$, then by a similar argument, there exists a canonical isomorphism

$$y \xrightarrow{\mathrm{can}} [a_1,b_1]\bullet \ \cdots \bullet \ [a_n,b_n]$$

in D. This gives rise to the following commutative diagram:

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "f"'] & y \arrow[r, "can"] & {[a_1,b_1]\otimes\cdots\otimes[a_n,b_n]} \arrow[d, "{[1_{a_1},1_{b_1}]\otimes\cdots\otimes[1_{a_n},1_{b_n}]}"] \\
y \arrow[rr, "\text{can}"'] & & {[a_1,b_1]\otimes\cdots\otimes[a_n,b_n]}
\end{tikzcd}$$

Since f is a canonical isomorphism, the top composite is also a canonical isomorphism. This establishes the first claim.

Since $[\mathscr{M},\mathscr{M}]$ is generated by (C,D), every morphism in $[\mathscr{M},\mathscr{M}]$ is a combination of composition, multiplication, and dash of arrows from D. Since $[\mathcal{CatGrp}\langle S\rangle, \mathcal{CatGrp}\langle S\rangle]$ is a sub-categorical group, the collection of morphisms in D is closed under multiplication and dash. Thus, every morphism in $[\mathscr{M},\mathscr{M}]$ is a composition of morphisms from D. $\square$

Proposition 12.2.16. *Let $\mathscr{M}$ be a categorical group with $M := \mathrm{Obj}(\mathscr{M})$, $x \in \mathscr{M}$ an object, and $a \in [\mathscr{M},\mathscr{M}]$ an object. Then, the object $x\otimes a\otimes x'$ also belongs to $[\mathscr{M},\mathscr{M}]$.*

Moreover, if $f : x \longrightarrow y$ is a morphism in $\mathscr{M}$ and $h : a \longrightarrow b$ is a morphism in $[\mathscr{M},\mathscr{M}]$, then the morphism $f\otimes h\otimes f'$ is a morphism in $[\mathscr{M},\mathscr{M}]$.

Proof. Let x be the image of an iterate $\mathtt{Can}(X)$ for some $X \in \mathcal{CatGrp}\langle M\rangle$. By Lemma 12.2.13, a is the image of an iterate $\mathtt{Can}(A)$ for some

$$A \in [\mathcal{CatGrp}\langle M\rangle, \mathcal{CatGrp}\langle M\rangle].$$

Then, by Proposition 12.2.7, $X\bullet A\bullet X^{\dagger}$ is an object of $[\mathcal{CatGrp}\langle M\rangle, \mathcal{CatGrp}\langle M\rangle]$. Since $x\otimes a\otimes x'$ is the image of the iterate $\mathtt{Can}\left(X\bullet A\bullet X^{\dagger}\right)$, it follows that $x\otimes a\otimes x'$ is an object of $[\mathscr{M},\mathscr{M}]$.

Now, consider $h \in D$. By Construction 12.2.10, h is either the image of an iterate $\mathtt{Can}(A)$ for some $A \in [\mathcal{CatGrp}\langle M\rangle, \mathcal{CatGrp}\langle M\rangle]$, or the image of a canonical natural isomorphism $\mathtt{Can}(A \multimap B)$ for some $A \multimap B$ in $[\mathcal{CatGrp}\langle M\rangle, \mathcal{CatGrp}\langle M\rangle]$. In the first case, $f \otimes h \otimes f'$ is the image of $\mathtt{Can}\left(X \otimes A \otimes X^{\dagger}\right)$ and thus belongs to D.

In the second case, $f \otimes h \otimes f'$ can be factored as

$$x \otimes a \otimes x' \xrightarrow{f \otimes 1_a \otimes f'} y \otimes a \otimes y' \xrightarrow{1_y \otimes h \otimes 1_{y'}} y \otimes b \otimes y'.$$

Both morphisms are in D, so $f \otimes h \otimes f'$ is in $[\mathcal{M}, \mathcal{M}]$.

Finally, if $h \in [\mathcal{M}, \mathcal{M}]$, then by Lemma 12.2.14, h can be written as a composition of morphisms from D. By the above arguments, $f \otimes h \otimes f'$ is also in $[\mathcal{M}, \mathcal{M}]$. $\square$

Proposition 12.2.17. *Let $\mathcal{M}$ be a categorical group with $M := \mathrm{Obj}(\mathcal{M})$, and let $x, y \in M$. Then, each of the following objects:*

$$x \otimes y \otimes x' \otimes y', \qquad x \otimes y' \otimes x' \otimes y, \qquad x' \otimes y \otimes x \otimes y', \qquad \textit{and} \qquad x' \otimes y' \otimes x \otimes y$$

are members of the commutator sub-categorical group $[\mathcal{M}, \mathcal{M}]$.

Proof. Let $x, y \in M$. Choose $X, Y \in \mathcal{CatGrp}\langle M\rangle$ such that $\mathcal{EV}(X) = x$ and $\mathcal{EV}(Y) = y$. Then, the objects

$$X \bullet Y \bullet X^{\dagger} \bullet Y^{\dagger}, \quad X \bullet Y^{\dagger} \bullet X^{\dagger} \bullet Y, \quad X^{\dagger} \bullet Y \bullet X \bullet Y^{\dagger}, \quad \text{and} \quad X^{\dagger} \bullet Y^{\dagger} \bullet X \bullet Y \qquad \square$$

all belong to $[\mathcal{CatGrp}\langle M\rangle, \mathcal{CatGrp}\langle M\rangle]$. By applying the functor $\mathcal{EV}$, their images are precisely the objects listed in the statement. This completes the proof.

12.3 Categorical Crossed Modules

In this section, we will provide crossed module construction for a categorical group morphism $T : \mathscr{H} \longrightarrow \mathscr{G}$. The definitions in this section are from [CGV06, p. 588-592].

Definition 12.3.1. Let $\mathscr{M}$ be a fixed categorical group. A *$\mathscr{M}$-categorical group* is a quadruple $(\mathscr{H}, ac, \psi, \phi)$ consisting of:

- a categorical group $\mathscr{H}$,
- a functor

$$ac : \mathscr{M} \times \mathscr{H} \longrightarrow \mathscr{H}, \qquad (x,a) \longmapsto ac(x,a) = {}^{x}a,$$

- families of natural isomorphisms

$$\psi = \big(\psi_{x,a,b} : {}^{x}(a \otimes b) \longrightarrow {}^{x}a \otimes {}^{x}b\big)_{(x,a,b) \in \mathscr{M} \times \mathscr{H} \times \mathscr{H}},$$
$$\phi = \big(\phi_{x,y,a} : {}^{x \otimes y}a \longrightarrow {}^{x}({}^{y}a)\big)_{(x,y,a) \in \mathscr{M} \times \mathscr{M} \times \mathscr{H}},$$
$$\phi_0 = \big(\phi_{0,a} : {}^{I}a \longrightarrow a\big)_{a \in \mathscr{H}}.$$

These data are required to satisfy the following coherence conditions. For all $x, y, z \in \mathrm{Obj}(\mathscr{M})$ and $a, b, c \in \mathrm{Obj}(\mathscr{H})$, the following diagrams commute:

$$\begin{tikzcd}
{}^{x\otimes y\otimes z}a \arrow[rr, "\alpha_{x,y,z}a"] \arrow[d, "\phi_{x\otimes y,z,a}"'] & & {}^{x\otimes(y\otimes z)}a \arrow[d, "\phi_{x,y\otimes z,a}"] \\
{}^{x\otimes y}({}^{z}a) \arrow[dr, "\phi_{x,y,{}^{z}a}"'] & & {}^{x}({}^{y\otimes z}a) \arrow[dl, "{}^{x}\phi_{y,z,a}"] \\
& {}^{x}({}^{y}({}^{z}a)) &
\end{tikzcd} \tag{12.3.2}$$

$$
\begin{tikzcd}
{}^{x\otimes I}a \arrow[rr, "\phi_{x,I,a}"] \arrow[dr, "{}^{\rho_x}a"'] & & {}^{x}({}^{I}a) \arrow[dl, "{}^{x}\phi_{0,a}"] \\
& {}^{x}a &
\end{tikzcd}
\tag{12.3.3}
$$

$$
\begin{tikzcd}
& {}^{x}(a\otimes(b\otimes c)) \arrow[r, "\psi_{x,a,b\otimes c}"] \arrow[dl, "{}^{x}\alpha_{a,b,c}"'] & {}^{x}a\otimes{}^{x}(b\otimes c) \arrow[dr, "1\otimes\psi_{x,b,c}"] & \\
{}^{x}(a\otimes b\otimes c) \arrow[dr, "\psi_{x,a\otimes b,c}"'] & & & {}^{x}a\otimes({}^{x}b\otimes{}^{x}c) \arrow[dl, "\alpha_{{}^{x}a,{}^{x}b,{}^{x}c}"] \\
& {}^{x}(a\otimes b)\otimes{}^{x}c \arrow[r, "\psi_{x,a,b}\otimes 1"'] & {}^{x}a\otimes{}^{x}b\otimes{}^{x}c &
\end{tikzcd}
\tag{12.3.4}
$$

$$
\begin{tikzcd}
{}^{x\otimes y}(a\otimes b) \arrow[rr, "\phi_{x,y,a\otimes b}"] \arrow[d, "\psi_{x\otimes y,a,b}"'] & & {}^{x}({}^{y}(a\otimes b)) \arrow[d, "{}^{x}\psi_{y,a,b}"] \\
{}^{x\otimes y}a\otimes{}^{x\otimes y}b \arrow[dr, "\phi_{x,y,a}\otimes\phi_{x,y,b}"'] & & {}^{x}({}^{y}a\otimes{}^{y}b) \arrow[dl, "\psi_{x,{}^{y}a,{}^{y}b}"] \\
& {}^{x}({}^{y}a)\otimes{}^{x}({}^{y}b) &
\end{tikzcd}
\tag{12.3.5}
$$

$\diamond$

Definition 12.3.6. Let $\mathscr{M}$ be a categorical group, and let $\mathscr{G}$ and $\mathscr{H}$ be $\mathscr{M}$-categorical groups. Suppose

$$ac_{\mathscr{G}} : \mathscr{M}\times\mathscr{G} \longrightarrow \mathscr{G} \qquad \text{and} \qquad ac_{\mathscr{H}} : \mathscr{M}\times\mathscr{H} \longrightarrow \mathscr{H}$$

are the respective $\mathscr{M}$-actions. We denote

$$ac_{\mathscr{G}}(x,a) = {}^{x}a \qquad \text{and} \qquad ac_{\mathscr{H}}(x,b) = b^{x}.$$

A morphism $T : \mathscr{G} \longrightarrow \mathscr{H}$ of $\mathscr{M}$-categorical groups consists of:

- a categorical group functor $T : \mathscr{G} \longrightarrow \mathscr{H}$,

- a categorical group natural transformation

$$\begin{tikzcd}
\mathscr{M} \times \mathscr{G} \arrow[r, "ac_{\mathscr{G}}"] \arrow[d, "\mathrm{Id}_{\mathscr{M}} \times T"'] \arrow[dr, phantom, "\Downarrow\mu"] & \mathscr{G} \arrow[d, "T"] \\
\mathscr{M} \times \mathscr{H} \arrow[r, "ac_{\mathscr{H}}"'] & \mathscr{H}
\end{tikzcd}$$

  with components

$$\mu = \left(\mu_{x,a} : T({}^{x}a) \longrightarrow T(a)^{x}\right)_{(x,a) \in \mathscr{M} \times \mathscr{H}}.$$

These data are required to satisfy the following coherence condition. For all $x, y \in \mathrm{Obj}(\mathscr{M})$ and $a, b \in \mathrm{Obj}(\mathscr{G})$, the diagrams

$$\begin{tikzcd}
T\left({}^{x\otimes y}a\right) \arrow[rr, "T(\phi_{x,y,a})"] \arrow[d, "\mu_{x\otimes y,a}"'] & & T\left({}^{x}\left({}^{y}a\right)\right) \arrow[d, "\mu_{x,\, {}^{y}a}"] \\
T(a)^{x\otimes y} \arrow[dr, "\phi_{x,y,a}"'] & & T\left({}^{y}a\right)^{x} \arrow[dl, "\mu^{x}_{y,a}"] \\
& \left(T(a)^{y}\right)^{x} &
\end{tikzcd} \tag{12.3.7}$$

and

$$\begin{tikzcd}
T\left({}^{x}(a \otimes b)\right) \arrow[r, "T(\psi_{x,a,b})"] \arrow[d, "\mu_{x,a\otimes b}"'] & T\left({}^{x}a \otimes {}^{x}b\right) \arrow[d, "(\mu_{x,a} \otimes \mu_{x,b}) \circ T_2^{-1}({}^{x}a,\, {}^{x}b)"] \\
T(a \otimes b)^{x} \arrow[r, "\psi_{x,T(a),T(b)} \circ T_2^{-1}(a,b)^{x}"'] & T(a)^{x} \otimes T(b)^{x}
\end{tikzcd} \tag{12.3.8}$$

commute. ◇

Definition 12.3.9. Let $\mathscr{M}$ be a categorical group. Define a conjugate action

$$\mathrm{conj} : \mathscr{M} \times \mathscr{M} \longrightarrow \mathscr{M}$$

as follows: Let $[2] = \{X, A\}$ with identification

$$1 = X \qquad \text{and} \qquad 2 = Y.$$

Define

$$\mathrm{conj} := \mathtt{Can}\left(X \bullet A \bullet X^{\dagger}\right). \tag{12.3.10}$$

Specifically,

$$\mathrm{conj}(x,a) = x \otimes a \otimes x' \qquad \text{on objects} \tag{12.3.11}$$

$$\mathrm{conj}(f,h) = f \otimes h \otimes f' \qquad \text{on morphisms.} \tag{12.3.12}$$

$\diamond$

Definition 12.3.13. Let $\mathscr{M}$ be a categorical group with conjugate action as in Definition 12.3.9. Define a canonical natural isomorphism

$$\psi = \left(\psi_{x,a,b} : {}^{x}(a \otimes b) \longrightarrow {}^{x}a \otimes {}^{x}b\right)_{(x,a,b) \in \mathscr{M}^3}$$

as follows: Consider the set $[3] = \{X, A, B\}$ with identification

$$1 = X, \qquad 2 = A, \qquad 3 = B.$$

Let

$$\psi := \mathtt{Can}\left(X \bullet (A \bullet B) \bullet X^{\dagger} \multimap X \bullet A \bullet X^{\dagger} \bullet (X \bullet B \bullet X^{\dagger})\right). \tag{12.3.14}$$

$\diamond$

Definition 12.3.15. Let $\mathscr{M}$ be a categorical group with conjugate action as in Definition 12.3.9. Define a canonical natural isomorphism

$$\phi = \left(\phi_{x,y,a} : {}^{x \otimes y}a \longrightarrow {}^{x}({}^{y}a)\right)_{(x,y,a) \in \mathscr{M}^3}$$

as follows: Consider the set $[3] = \{X, Y, A\}$ with identification

$$1 = X, \qquad 2 = Y, \qquad 3 = A.$$

Let

$$\phi := \mathtt{Can}\Big(X \otimes Y \otimes A \otimes (X \bullet Y)^{\dagger} \longrightarrow X \bullet (Y \bullet A \bullet Y^{\dagger}) \bullet X^{\dagger}\Big). \tag{12.3.16}$$

$\diamond$

Proposition 12.3.17. *Let $\mathcal{M}$ be a categorical group with conjugate action as in Definition 12.3.9 and the natural isomorphisms ψ and ϕ be as in Definitions 12.3.13 and 12.3.15 respectively. Then, the diagrams (12.3.2) through (12.3.5) commute.*

Consequently, $(\mathcal{M}, \mathrm{Id}_{\mathcal{M}}, \psi, \phi)$ is a $\mathcal{M}$-categorical group.

Proof. Let $[4] = \{X, Y, Z, A\}$ with identification

$$1 = X, \quad 2 = Y, \quad 3 = Z, \quad 4 = A.$$

Observe that the diagram in (12.3.2) is the image under the functor $\mathtt{Can} : \mathcal{CatGrp}\langle S\rangle \longrightarrow \mathcal{Fct}_4(\mathcal{M})$ of the following diagram in $\mathcal{CatGrp}\langle S\rangle$:

$$\begin{tikzcd}
X \bullet Y \bullet Z \bullet A \bullet (X \bullet Y \bullet Z)^{\dagger} \arrow[rr] \arrow[d] & & X \bullet (Y \bullet Z) \bullet A \bullet (X \bullet (Y \bullet Z))^{\dagger} \arrow[d] \\
X \bullet Y \bullet (Z \bullet A \bullet Z^{\dagger}) \bullet (X \bullet Y)^{\dagger} \arrow[dr] & & X \bullet (Y \bullet Z \bullet A \bullet (Y \bullet Z)^{\dagger}) \bullet X^{\dagger} \arrow[dl] \\
& X \bullet (Y \bullet (Z \bullet A \bullet Z^{\dagger}) \bullet Y^{\dagger}) \bullet X^{\dagger} &
\end{tikzcd}$$

$\square$

Since $\mathcal{CatGrp}\langle S\rangle$ is a thin category, the above diagram commutes. Consequently, the digram (12.3.2) commutes.

Following the similar analysis, the diagrams (12.3.3) through (12.3.5) commute.

Definition 12.3.18. Let $\mathscr{M}$ be a categorical group. A categorical *$\mathscr{M}$-pre-crossed module* $(\mathscr{H}, T, \nu)$ consists of:

- a $\mathscr{M}$-categorical group $(\mathscr{H}, ac, \psi, \phi)$,
- a categorical group functor $T : \mathscr{H} \longrightarrow \mathscr{M}$, and
- a family of natural isomorphisms in $\mathscr{M}$

$$\nu = \left(\nu_{x,a} : T({}^{x}a) \otimes x \longrightarrow x \otimes T(a)\right)_{(x,a) \in \mathscr{M} \times \mathscr{H}}$$

These data are required to satisfy the following precrossed module conditions:

pcr1 For all $x, y \in \mathrm{Obj}(\mathscr{M})$ and $a \in \mathrm{Obj}(\mathscr{H})$, the diagram

$$\begin{tikzcd}
T\left({}^{x}\left({}^{y}a\right)\right) \otimes x \otimes y \arrow[r, "T\left(\phi^{-1}_{x,y,a}\right) \otimes 1 \otimes 1"] \arrow[d, "\nu_{x,{}^{y}a} \otimes 1"'] & T\left({}^{x \otimes y}a\right) \otimes x \otimes y \arrow[d, "\nu_{x \otimes y, a} \circ \alpha^{-1}"] \\
x \otimes T\left({}^{y}a\right) \otimes y \arrow[r, "\alpha \circ (1 \otimes \nu_{y,a}) \circ \alpha^{-1}"'] & x \otimes y \otimes T(a)
\end{tikzcd} \tag{12.3.19}$$

commutes. Here, α denotes the appropriate associator and 1 denotes the identity morphism.

pcr2 For all $x \in \mathrm{Obj}(\mathscr{M})$ and $a, b \in \mathrm{Obj}(\mathscr{H})$, the diagram

$$\begin{tikzcd}
T\left({}^{x}a\right) \otimes T\left({}^{x}b\right) \otimes x \arrow[r, "\alpha \circ (1 \otimes \nu_{x,b}) \circ \alpha^{-1}"] \arrow[d, "T_2({}^{x}a, {}^{x}b) \otimes 1"'] & T\left({}^{x}a\right) \otimes x \otimes T(b) \arrow[d, "\nu_{x,a} \otimes 1"] \\
T\left({}^{x}a \otimes {}^{x}b\right) \otimes x \arrow[d, "T\left(\psi^{-1}_{x,a,b}\right) \otimes 1"'] & x \otimes T(a) \otimes T(b) \arrow[d, "1 \otimes T_2(a,b) \circ \alpha^{-1}"] \\
T\left({}^{x}(a \otimes b)\right) \otimes x \arrow[r, "\nu_{x, a \otimes b}"'] & x \otimes T(a \otimes b)
\end{tikzcd} \tag{12.3.20}$$

commutes. Here, α denotes the appropriate associator and 1 denotes the identity morphism. ◇

Remark 12.3.21. Observe that, the data of a $\mathscr{M}$-precrossed module $(\mathscr{H}, T, \nu)$ is equivalent to specifying a morphism of $\mathscr{M}$-categorical groups $(T, \mu) : \mathscr{H} \longrightarrow \mathscr{M}$

$$\begin{tikzcd}
\mathscr{M} \times \mathscr{H} \arrow[r, "ac"] \arrow[d, "\mathrm{Id}_{\mathscr{M}} \times T"'] \arrow[dr, phantom, "\Downarrow\mu"] & \mathscr{H} \arrow[d, "T"] \\
\mathscr{M} \times \mathscr{M} \arrow[r, "\mathrm{conj}"'] & \mathscr{M}
\end{tikzcd}$$

where conj denotes the conjugation action on $\mathscr{M}$ as in Definition 12.3.9. ◇

Definition 12.3.22. Let $\mathscr{M}$ be a categorical group with conjugate action as in Definition 12.3.9. Define a canonical natural isomorphism

$$\nu = \left(\nu_{x,a} : {}^{x}a \otimes x \longrightarrow x \otimes a\right)_{(x,a) \in \mathscr{M}^2}$$

as follows: Consider the set $[2] = \{X, A\}$ with identification

$$1 = X \qquad \text{and} \qquad 2 = A.$$

Let

$$\nu := \mathtt{Can}\Big((X \bullet A \bullet X^{\dagger}) \bullet X \multimap\!\!\rightarrow X \bullet A\Big). \tag{12.3.23}$$

◇

Proposition 12.3.24. *Let $\mathscr{M}$ be a categorical group with conjugate action as in Definition 12.3.9 and the natural isomorphisms ψ, ϕ, and ν be as in Definitions 12.3.13, 12.3.15, and 12.3.22 respectively. Then, the diagrams (12.3.19) and (12.3.20) commute.*

Consequently, $(\mathscr{M}, \mathrm{Id}_{\mathscr{M}}, \psi, \phi, \nu)$ is a $\mathscr{M}$-precrossed module.

Proof. Let $[3] = \{X, Y, A\}$ with identification

$$1 = X, \quad 2 = Y, \quad 3 = A.$$

Observe that the diagram in (12.3.19) is the image under the functor $\mathtt{Can} : \mathcal{CatGrp}\langle S\rangle \longrightarrow \mathcal{Fct}_3(\mathscr{M})$ of the following diagram in $\mathcal{CatGrp}\langle S\rangle$:

$$\begin{tikzcd}
X \bullet (Y \bullet A \bullet Y^{\dagger}) \bullet X^{\dagger} \bullet X \bullet Y \arrow[r, "\circ"] \arrow[d, "\circ"] & X \bullet Y \bullet A \bullet (X \bullet Y)^{\dagger} \bullet X \bullet Y \arrow[d, "\circ"] \\
X \bullet (Y \bullet A \bullet Y^{\dagger}) \bullet Y \arrow[r, "\circ"] & X \bullet Y \bullet A
\end{tikzcd}$$

Since $\mathcal{CatGrp}\langle S\rangle$ is a thin category, the above diagram commutes. Consequently, the digram (12.3.19) commutes.

Following the similar analysis, the diagram (12.3.20) commutes. $\square$

Definition 12.3.25. [CGV06] Let $\mathscr{M}$ be a categorical group. A categorical $\mathscr{M}$-*crossed module* $(\mathscr{H}, T, \nu, \chi)$ consists of:

- a categorical $\mathscr{M}$-precrossed module $(\mathscr{H}, T, \nu)$, and
- a family of natural isomorphisms in $\mathscr{H}$

$$\chi = \left(\chi_{a,b} : {}^{T(a)}b \otimes a \longrightarrow a \otimes b\right)_{(a,b)\in\mathscr{H}\times\mathscr{H}}$$

The above data must satisfy the following $\mathscr{M}$-crossed module axioms:

cr1 For all $a, b, c \in \mathrm{Obj}(\mathscr{H})$, the diagram

$$\begin{tikzcd}
{}^{T(a\otimes b)}c\otimes a\otimes b \arrow[rr, "\chi_{a\otimes b,c}\circ\alpha^{-1}"] \arrow[d, "{}^{T_2^{-1}(a,b)}c\otimes 1\otimes 1"'] & & a\otimes b\otimes c \\
{}^{T(a)\otimes T(b)}c\otimes a\otimes b \arrow[dr, "\phi_{T(a),T(b),c}\otimes 1\otimes 1"'] & & a\otimes {}^{T(b)}c\otimes b \arrow[u, "(1\otimes\chi_{b,c})\circ\alpha^{-1}"'] \\
& {}^{T(a)}\left({}^{T(b)}c\right)\otimes a\otimes b \arrow[ur, "\chi_{a,{}^{T(b)}c}\otimes 1"'] &
\end{tikzcd} \tag{12.3.26}$$

commutes. Here, α denotes the appropriate associator and 1 denotes the appropriate identity morphism.

cr2 For all $a, b, c \in \mathrm{Obj}(\mathscr{H})$, the diagram

$$\begin{tikzcd}
{}^{T(a)}(b\otimes c)\otimes a \arrow[r, "\chi_{a,b\otimes c}"] \arrow[d, "\psi_{T(a),b,c}\otimes 1"'] & a\otimes b\otimes c \\
{}^{T(a)}b\otimes {}^{T(a)}c\otimes a \arrow[r, "\alpha\circ(1\otimes\chi_{a,c})\circ\alpha^{-1}"'] & {}^{T(a)}b\otimes a\otimes c \arrow[u, "\chi_{a,b}\otimes 1"']
\end{tikzcd} \tag{12.3.27}$$

commutes. Here, α denotes the appropriate associator and 1 denotes the appropriate identity morphism.

cr3 For all $x \in \mathrm{Obj}(\mathscr{M})$ and $a, b \in \mathrm{Obj}(\mathscr{H})$, the diagram

$$
\begin{tikzcd}
{}^{x}\left({}^{T(a)}b \otimes a\right) \arrow[rr, "{}^{x}\chi_{a,b}"] \arrow[d, "\psi_{x,\_,a}"'] & & {}^{x}(a \otimes b) \arrow[d, "\psi_{x,a,b}"] \\
{}^{x}\left({}^{T(a)}b\right) \otimes {}^{x}a \arrow[d, "\phi^{-1}_{x,T(a),b} \otimes 1"'] & & {}^{x}a \otimes {}^{x}b \\
{}^{x \otimes T(a)}b \otimes {}^{x}a \arrow[dr, "\nu^{-1}_{x,a}b \otimes 1"'] & & {}^{T({}^{x}a)}({}^{x}b) \otimes {}^{x}a \arrow[u, "\chi_{\_,\_}"'] \\
& {}^{T({}^{x}a) \otimes x}b \otimes {}^{x}a \arrow[ur, "\phi_{\_,x,b} \otimes 1"'] &
\end{tikzcd}
\tag{12.3.28}
$$

commutes. Here, α denotes the appropriate associator, 1 denotes the appropriate identity morphism, and the symbol _ is a placeholder for the appropriate object in context.

cr4 For all $a, b \in \mathrm{Obj}(\mathscr{H})$, the diagram

$$
\begin{tikzcd}
T\left({}^{T(a)}b \otimes a\right) \arrow[r, "T(\chi_{a,b})"] \arrow[d, "T_2^{-1}({}^{T(a)}b,\, a)"'] & T(a \otimes b) \arrow[d, "T_2^{-1}(a,b)"] \\
T\left({}^{T(a)}b\right) \otimes T(a) \arrow[r, "\nu_{T(a),b}"'] & T(a) \otimes T(b)
\end{tikzcd}
\tag{12.3.29}
$$

commutes. ◇

Remark 12.3.30. Recall from Remark 12.3.30 that a pre-crossed module $(\mathscr{H}, T, \nu)$ is equivalent to a morphism of $\mathscr{M}$-categorical groups $(T, \mu) : \mathscr{H} \longrightarrow \mathscr{M}$. It follows that, a crossed module $(\mathscr{H}, T, \nu, \chi)$ is equivalent to the following compatibility condition between arrows of $\mathscr{M}$-categorical groups.

$$
\begin{tikzcd}
\mathscr{H} \times \mathscr{H} \arrow[r, "\mathrm{conj}"] \arrow[d, "T \times \mathrm{Id}_{\mathscr{H}}"'] \arrow[dr, phantom, "\Downarrow\mu_2"] & \mathscr{H} \arrow[d, "\mathrm{Id}_{\mathscr{H}}"] \\
\mathscr{M} \times \mathscr{H} \arrow[r, "ac"] \arrow[d, "\mathrm{Id}_{\mathscr{M}} \times T"'] \arrow[dr, phantom, "\Downarrow\mu_1"] & \mathscr{H} \arrow[d, "T"] \\
\mathscr{M} \times \mathscr{M} \arrow[r, "\mathrm{conj}"'] & \mathscr{M}
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{H} \times \mathscr{H} \arrow[r, "\mathrm{conj}"] \arrow[d, "T \times T"'] \arrow[dr, phantom, "\Downarrow\mathrm{can}"] & \mathscr{H} \arrow[d, "T"] \\
\mathscr{M} \times \mathscr{M} \arrow[r, "\mathrm{conj}"'] & \mathscr{M}
\end{tikzcd}
$$

◇

Definition 12.3.31. Let $\mathscr{M}$ be a categorical group with conjugate action as in Definition 12.3.9. Define a canonical natural isomorphism

$$\chi = \left(\chi_{a,b} : {}^{a}b \otimes a \longrightarrow a \otimes b\right)_{(a,b)\in\mathscr{M}^2}$$

as follows: Consider the set $[2] = \{A, B\}$ with identification

$$1 = A \qquad \text{and} \qquad 2 = B.$$

Let

$$\chi := \mathtt{Can}\left(A \bullet B \bullet A^{\dagger} \bullet A \longrightarrow A \bullet B\right). \tag{12.3.32}$$

◇

Proposition 12.3.33. *Let $\mathscr{M}$ be a categorical group with conjugate action as in Definition 12.3.9 and the natural isomorphisms ψ, ϕ, ν, and χ be as in Definitions 12.3.13, 12.3.15, 12.3.22, and 12.3.31 respectively. Then, the diagrams* (12.3.26) *through* (12.3.29) *commute.*

Consequently, $(\mathscr{M}, \mathrm{Id}_{\mathscr{M}}, \psi, \phi, \nu, \chi)$ is a $\mathscr{M}$-crossed module.

Proof. Let $[3] = \{A, B, C\}$ with identification

$$1 = A, \quad 2 = B, \quad 3 = C.$$

Observe that the diagram in (12.3.26) is the image under the functor $\mathtt{Can} : \mathcal{CatGrp}\langle S\rangle \longrightarrow \mathcal{Fct}_3(\mathscr{M})$ of the following diagram in $\mathcal{CatGrp}\langle S\rangle$:

$$\begin{tikzcd}
A \bullet B \bullet C \bullet (A \bullet B)^\dagger \bullet A \bullet B \arrow[rr] \arrow[d] & & A \bullet B \bullet C \\
A \bullet B \bullet C \bullet (A \bullet B)^\dagger \bullet A \bullet B \arrow[dr] & & A \bullet (B \bullet C \bullet B^\dagger) \bullet B \arrow[u] \\
& A \bullet (B \bullet C \bullet B^\dagger) \bullet A^\dagger \bullet A \bullet B \arrow[ur] &
\end{tikzcd}$$

Since $\mathcal{CatGrp}\langle S\rangle$ is a thin category, the above diagram commutes. Consequently, the digram (12.3.26) commutes.

Following the similar analysis, the diagrams (12.3.27) through (12.3.29) commutes. □

Construction 12.3.34 ([CGV06]). Let $\mathscr{M}$ and $\mathscr{H}$ be categorical groups, and let $T : \mathscr{H} \longrightarrow \mathscr{M}$ be a categorical group functor. The *quotient groupoid* $\mathscr{M}/(\mathscr{H}, T)$ is defined as follows:

- The objects are those of $\mathscr{M}$.
- A *pre-morphism* from x to y is a pair

$$(a, f) : x \longrightarrow y,$$

  where $a \in \mathrm{Obj}(\mathscr{H})$ and $f : x \longrightarrow T(a) \otimes y$ is a morphism in $\mathscr{M}$.

- Morphisms in $\mathscr{M}/(\mathscr{H},T)$ are equivalence classes of pre-morphisms. Specifically, two pre-morphisms (a,f) and (b,g) from x to y are equivalent if there exists a morphism $h : a \longrightarrow b$ in $\mathscr{H}$ such that

$$g = (T(h) \otimes 1_y) \circ f.$$

We denote the equivalence class of (a,f) by $[a,f] : x \longrightarrow y$.

- For each object x, the identity morphism $[I,f] : x \longrightarrow x$ is represented by the pre-morphism

$$f : x \xrightarrow{\lambda} I \otimes x \xrightarrow{T_0 \otimes 1_x} T(I) \otimes x,$$

where λ is the left unitor in $\mathscr{M}$ and T_0 is the unit isomorphism associated to T.

- Given morphisms $[a,f] : x \longrightarrow y$ and $[b,g] : y \longrightarrow z$, their composition is defined by

$$[a \otimes b, h] : x \longrightarrow z,$$

where $h : x \longrightarrow T(a \otimes b) \otimes z$ is given by

$$h : x \xrightarrow{f} T(a) \otimes y \xrightarrow{1_{T(a)} \otimes g} T(a) \otimes T(b) \otimes z \xrightarrow{T_2(a,b) \circ \alpha^{-1}} T(a \otimes b) \otimes z,$$

with $T_2(a,b)$ the monoidal structure isomorphism for T and α the associator in $\mathscr{M}$.

The monoidal constraints of the functor T ensure that the identity and associativity axioms are satisfied. Moreover, $\mathscr{M}/(\mathscr{H},T)$ is a groupoid: the inverse of a morphism $[a,f] : x \longrightarrow y$ is given by $[a',g] : y \longrightarrow x$, where a' is the negator of a in $\mathscr{H}$, and

$g : y \longrightarrow T(a') \otimes x$ is defined by

$$\begin{tikzcd} g : y \arrow[r, "T_0 \circ \lambda_y^{-1}"] & T(I) \otimes y \arrow[r, "T_2(a',a)^{-1} \circ T(\epsilon_a^{-1}) \otimes 1_y"] & T(a') \otimes T(a) \otimes y \arrow[d, "(1 \otimes f^{-1}) \circ \alpha^{-1}"] \\ & & T(a') \otimes x \end{tikzcd}$$

where ϵ_a is the counit isomorphism in $\mathscr{H}$. ◇

Construction 12.3.35 ([CGV06]). Now suppose, in the above construction, we are given a categorical $\mathscr{M}$-crossed module $(\mathscr{H}, T : \mathscr{H} \longrightarrow \mathscr{M}, \nu, \chi)$. Then, the quotient groupoid $\mathscr{M}/(\mathscr{H}, T)$ can be endowed with the structure of a categorical group as follows:

- The unit object is the unit object of $\mathscr{M}$. The monoidal product on objects is inherited from the monoidal product in $\mathscr{M}$.

- Given morphisms

  $$[a, f] : x \longrightarrow y \qquad \text{and} \qquad [b, g] : z \longrightarrow w,$$

  their monoidal product is defined by

  $$[a \otimes {}^y b, h] : x \otimes z \longrightarrow y \otimes w,$$

  where

  $$h : x \otimes z \longrightarrow T(a \otimes {}^y b) \otimes (y \otimes w)$$

  is given by the composition

  $$\begin{tikzcd} x \otimes z \arrow[r, "\alpha \circ (f \otimes g)"] & T(a) \otimes (y \otimes T(b)) \otimes w \arrow[r, "1 \otimes \nu_{y,b}^{-1} \otimes 1"] & T(a) \otimes T({}^y b) \otimes y \otimes w \arrow[d, "\alpha \circ T_2(a, {}^y b)"] \\ & & T(a \otimes {}^y b) \otimes y \otimes w \end{tikzcd}$$

The natural isomorphism χ and its compatibility with ν as in Definition 12.3.25 are required to prove the functoriality of the above tensor product. Consider the functor

$$P_T : \mathscr{M} \longrightarrow \mathscr{M}/(\mathscr{H}, T)$$

given by

$$P_T(x) := x \qquad \text{on objects} \tag{12.3.36}$$

$$P_T(f : x \longrightarrow y) := [I, f_T] \qquad \text{on morphisms} \tag{12.3.37}$$

where the pre-morphism $f_T : x \longrightarrow T(I) \otimes y$ is given by

$$f_T : x \xrightarrow{f} y \xrightarrow{\lambda_y} I \otimes y \xrightarrow{T_0 \otimes 1_y} T(I) \otimes y.$$

The functor P_T is essentially surjective since every object $x \in \mathscr{M}/(\mathscr{H}, T)$ is an object $x \in \mathscr{M}$ and the morphism

$$P_T(1_x : x \longrightarrow x) : P_T(x) \longrightarrow P_T(x) = x$$

gives the required isomorphism. The unit, associativity, and the cancellation constraints in $\mathscr{M}/(\mathscr{H}, T)$ are borrowed from $\mathscr{M}$ via the functor P_T.

Moreover, there is a categorical group natural transformation

$$\begin{tikzcd}
 & \mathscr{G} \arrow[dr, "P_T"] \arrow[d, phantom, "\Downarrow\pi_T"] & \\
\mathscr{H} \arrow[ur, "T"] \arrow[rr, "\mathrm{Const}_I"'] & \, & \mathscr{M}/(\mathscr{H}, T)
\end{tikzcd}$$

is given by

$$\pi_T(a) := [a, \rho_{T(a)}] : T(a) \longrightarrow I \qquad \text{where} \tag{12.3.38}$$

$$\rho_{T(a)} : T(a) \longrightarrow T(a) \otimes I. \qquad \diamond$$

Theorem 12.3.39. *The previous construction*

$$\big(\mathcal{M}/(\mathcal{H},T),\ \ P_T:\mathcal{M}\longrightarrow \mathcal{M}/(\mathcal{H},T),\ \ \pi_T:P_T\circ T\Rightarrow \mathrm{Const}_I\big)$$

is univeral with respect to triples in $\mathcal{M}$-categorical groups,

$$\big(\mathcal{G},\ \ G:\mathcal{M}\longrightarrow \mathcal{G},\ \ \delta:G\circ T\Rightarrow \mathrm{Const}_I\big)$$

satisfying the following condition: For $x\in\mathcal{M}$ and $a\in\mathcal{H}$ the diagram

$$\begin{tikzcd}
G(x)\otimes I \arrow[d, "\lambda^{-1}_{G(x)}\circ\rho_{G(x)}"'] & G(x)\otimes GT(a) \arrow[l, "1\otimes\delta_a"'] \\
I\otimes G(x) & G(x\otimes T(a)) \arrow[u, "G_2^{-1}(X,T(A))"'] \\
GT({}^x a)\otimes G(x) \arrow[u, "\delta_{{}^x a}\otimes 1"] & G(T({}^x a)\otimes x) \arrow[u, "G(\nu_{x,a})"'] \arrow[l, "G_2^{-1}({}^x a,x)"]
\end{tikzcd} \tag{12.3.40}$$

commutes. More specifically, for each triple

$$\big(\mathcal{G},\ \ G:\mathcal{M}\longrightarrow \mathcal{G},\ \ \delta:G\circ T\Rightarrow \mathrm{Const}_I\big),$$

satisfying the condition (12.3.40) there exists a unique categorical group functor

$$\overline{G}:\mathcal{M}/(\mathcal{H},T)\longrightarrow \mathcal{G}$$

such that

$$\overline{G}\circ P_T = G \qquad\qquad \text{and} \tag{12.3.41}$$

$$\overline{G}\circ \pi_T = \delta. \tag{12.3.42}$$

Proof. This is proved on page 596 of [CGV06]. □

12.4 Symmetrization Construction

Definition 12.4.1. Let $\mathscr{M}$ be a categorical group, and let $[\mathscr{M},\mathscr{M}]$ denote the commutator sub-categorical group. Define a functor

$$ac : \mathscr{M} \times [\mathscr{M},\mathscr{M}] \longrightarrow [\mathscr{M},\mathscr{M}]$$

to be the functor conj as in Definition 12.3.9 precomposed with the inclusion

$$[\mathscr{M},\mathscr{M}] \hookrightarrow \mathscr{M}$$

on the second component. More precisely,

$$ac(x,a) = {}^{x}a := x \otimes a \otimes x' \qquad \text{on objects,} \tag{12.4.2}$$

$$ac(f,h) = {}^{f}h := f \otimes h \otimes f' \qquad \text{on morphisms.} \tag{12.4.3}$$

◇

The above functor is well-defined since from Proposition 12.2.16, both $x \otimes a \otimes x'$ and $f \otimes h \otimes f'$ are in $[\mathscr{M},\mathscr{M}]$.

Definition 12.4.4. Let $\mathscr{M}$ be a categorical group and $[\mathscr{M},\mathscr{M}]$ be the commutator sub-categorical group. Define a $\mathscr{M}$-categorical group

$$([\mathscr{M},\mathscr{M}], ac, \psi, \phi)$$

as follows:

- Let $[\mathscr{M},\mathscr{M}]$ be the underlying categorical group.
- Define a functor

  $$ac : \mathscr{M} \times [\mathscr{M},\mathscr{M}] \longrightarrow [\mathscr{M},\mathscr{M}]$$

  to be the same as in Definition 12.4.1.

- The natural transformation

$$\psi = \big(\psi_{x,a,b} : x \otimes (a \otimes b) \otimes x' \longrightarrow x \otimes a \otimes x' \otimes (x \otimes b \otimes x')\big)_{(x,a,b) \in \mathcal{M} \times [\mathcal{M},\mathcal{M}] \times [\mathcal{M},\mathcal{M}]}$$

  is equal to the natural transformation ψ as in Definition 12.3.13 precomposed with the inclusion

$$[\mathcal{M},\mathcal{M}] \hookrightarrow \mathcal{M}$$

  in the a and b components. From the canonical inclusion condition of $[\mathcal{M},\mathcal{M}]$, we know that $\psi_{x,a,b}$ is a morphism in $[\mathcal{M},\mathcal{M}]$.

- The natural isomorphism

$$\phi = \big(\phi_{x,y,a} : x \otimes y \otimes a \otimes (x \otimes y)' \longrightarrow x \otimes (y \otimes a \otimes y') \otimes x'\big)_{(x,y,a) \in \mathcal{M} \times \mathcal{M} \times [\mathcal{M},\mathcal{M}]}$$

  is equal to the natural tranformation ϕ as in (12.3.16) precomposed with the inclusion

$$[\mathcal{M},\mathcal{M}] \hookrightarrow \mathcal{M}$$

  in the a component. From the canonical inclusion condition of $[\mathcal{M},\mathcal{M}]$, we know that $\phi_{x,y,a}$ is a morphism in $[\mathcal{M},\mathcal{M}]$.

From Proposition 12.3.17, we conclude that the diagrams (12.3.2) through (12.3.5) commute. Consequently,

$$\big([\mathcal{M},\mathcal{M}], ac, \psi, \phi\big)$$

is a $\mathcal{M}$-categorical group. ◇

Definition 12.4.5. Let $\mathcal{M}$ be a categorical group and $[\mathcal{M},\mathcal{M}]$ be the commutator sub-categorical group. Define a $\mathcal{M}$-precrossed module

$$([\mathcal{M},\mathcal{M}], ac, \psi, \phi, T, \nu)$$

as follows:

- The $\mathcal{M}$-categorical group

$$\big([\mathcal{M},\mathcal{M}], ac, \psi, \phi\big)$$

  be as in Definition 12.4.4.

- The functor $T : [\mathcal{M},\mathcal{M}] \longrightarrow \mathcal{M}$ be the inclusion

$$[\mathcal{M},\mathcal{M}] \hookrightarrow \mathcal{M}.$$

- The natural transformation

$$\nu = \big(\nu_{x,a} : {}^{x}a \otimes x \longrightarrow x \otimes a\big)_{(x,a)\in\mathcal{M}\times[\mathcal{M},\mathcal{M}]}$$

  is equal to the natural transformation ν as in Definition 12.3.22 precomposed with the inclusion

$$[\mathcal{M},\mathcal{M}] \hookrightarrow \mathcal{M}$$

  in the a component. From the canonical inclusion condition of $[\mathcal{M},\mathcal{M}]$, we know that $\nu_{x,a}$ is a morphism in $[\mathcal{M},\mathcal{M}]$.

From Proposition 12.3.24, we conclude that the diagrams (12.3.19) and (12.3.20) commute. Consequently,

$$\big([\mathcal{M},\mathcal{M}], ac, \psi, \phi, T, \nu\big)$$

is a $\mathcal{M}$-precrossed module. $\diamond$

Definition 12.4.6. Let $\mathcal{M}$ be a categorical group and $[\mathcal{M},\mathcal{M}]$ be the commutator sub-categorical group. Define a $\mathcal{M}$-crossed module

$$([\mathcal{M},\mathcal{M}], ac, \psi, \phi, T, \nu, \chi)$$

as follows:

- The $\mathscr{M}$-precrossed module

$$\big([\mathscr{M},\mathscr{M}], ac, \psi, \phi, T, \nu\big)$$

be as in Definition 12.4.5.

- The natural transformation

$$\chi = \big(\chi_{a,b} : {}^{a}b \otimes a \longrightarrow a \otimes b\big)_{(a,b)\in[\mathscr{M},\mathscr{M}]\times[\mathscr{M},\mathscr{M}]}$$

is equal to the natural transformation χ as in Definition 12.3.31 precomposed with the inclusion

$$[\mathscr{M},\mathscr{M}] \hookrightarrow \mathscr{M}$$

in both a and b components. From the canonical inclusion condition of $[\mathscr{M},\mathscr{M}]$, we know that $\chi_{x,a}$ is a morphism in $[\mathscr{M},\mathscr{M}]$.

From Proposition 12.3.33, we conclude that the diagrams (12.3.26) through (12.3.29) commute. Consequently,

$$\big([\mathscr{M},\mathscr{M}], ac, \psi, \phi, T, \nu, \chi\big)$$

is a $\mathscr{M}$-crossed module. ◇

Definition 12.4.7. Let $\mathscr{M}$ be a categorical group. Define a *symmetrization categorical group*, denoted $\mathtt{Sym}(\mathscr{M})$, as follows: Let

$$\big([\mathscr{M},\mathscr{M}], ac, \psi, \phi, T, \nu, \chi\big)$$

be a $\mathscr{M}$-crossed module as in Definition 12.4.6. Define

$$\mathtt{Sym}(\mathscr{M}) := \mathscr{M}/([\mathscr{M},\mathscr{M}], T) \tag{12.4.8}$$

as in Construction 12.3.35. This construction satisfies the universal property stated at Theorem 12.3.39. ◇

Theorem 12.4.9. *Let $\mathscr{M}$ be a categorical group with $M := \mathrm{Obj}(\mathscr{M})$. The symmetrization categorical group $\mathit{Sym}(\mathscr{M})$ admits the structure of a symmetric categorical group, with braiding*

$$\beta_{x,y} : x \otimes y \longrightarrow y \otimes x$$

in $\mathit{Sym}(\mathscr{M})$ defined by

$$\beta_{x,y} := [x \otimes y \otimes x' \otimes y',\ h] \tag{12.4.10}$$

where

$$h : x \otimes y \longrightarrow x \otimes y \otimes x' \otimes y' \otimes (y \otimes x)$$

is the canonical morphism given by the composition

$$x \otimes y \xrightarrow{\epsilon_x^{-1}} x \otimes y \otimes x' \otimes x \xrightarrow{\epsilon_y^{-1}} x \otimes y \otimes x' \otimes y' \otimes (y \otimes x).$$

Here, the monoidal structure isomorphisms are omitted for brevity.

Proof. We will show that $\beta_{y,x} \circ \beta_{x,y} = 1_{x \otimes y}$ in $\mathtt{Sym}(\mathscr{M})$. By Construction 12.3.34, the composite

$$\beta_{y,x} \circ \beta_{x,y} : x \otimes y \longrightarrow x \otimes y$$

in $\mathtt{Sym}(\mathscr{M})$ is represented by

$$[a \otimes b, h] : x \otimes y \longrightarrow x \otimes y$$

where $a = x \otimes y \otimes x' \otimes y'$, $b = y \otimes x \otimes y' \otimes x'$, and h is the composition

$$\begin{tikzcd}
x \otimes y \arrow[r, "\epsilon_x^{-1}"] & x \otimes y \otimes x' \otimes x \arrow[r, "\epsilon_y^{-1}"] & x \otimes y \otimes x' \otimes y' \otimes (y \otimes x) \arrow[d, "\epsilon_y^{-1}"] \\
& & x \otimes y \otimes x' \otimes y' \otimes (y \otimes x \otimes y' \otimes y) \arrow[d, "\epsilon_x^{-1}"] \\
& & x \otimes y \otimes x' \otimes y' \otimes y \otimes x \otimes y' \otimes x' \otimes (x \otimes y).
\end{tikzcd}$$

Let $X, Y \in \mathcal{C}at\mathcal{G}rp\langle M\rangle$ with $\mathcal{EV}(X) = x$ and $\mathcal{EV}(Y) = y$. Then h can be written as

$$h = \mathcal{EV}(X \bullet Y \multimap A \bullet B \bullet (X \bullet Y)),$$

where $A = X \bullet Y \bullet X^\dagger \bullet Y^\dagger$ and $B = Y \bullet X \bullet Y^\dagger \bullet X^\dagger$.

Note that

$$\mathcal{Q}(A \bullet B) = e = \mathcal{Q}(J)$$

where $\mathcal{Q} : \mathcal{C}at\mathcal{G}rp\langle M\rangle \longrightarrow \mathcal{G}rp\langle M\rangle$ is the induced functor. By Theorem 4.5.27, there is a unique arrow

$$A \bullet B \multimap J$$

in $\mathcal{C}at\mathcal{G}rp\langle M\rangle$, and in fact $A \bullet B \multimap J$ lies in $[\mathcal{C}at\mathcal{G}rp\langle M\rangle, \mathcal{C}at\mathcal{G}rp\langle M\rangle]$. Let $g : a \otimes b \longrightarrow I$ be

$$g := \mathcal{EV}(A \bullet B \multimap J).$$

Thus, $g \in [\mathscr{M}, \mathscr{M}]$ by Construction 12.2.10.

Now consider the diagram

$$\begin{tikzcd}
x \otimes y \arrow[r, "h"] \arrow[dr, "\lambda_{x\otimes y}"'] & a \otimes b \otimes (x \otimes y) \arrow[d, "g \otimes 1_{x\otimes y}"] \\
& I \otimes (x \otimes y).
\end{tikzcd}$$

This is a formal diagram in $\mathscr{M}$, so it commutes. Therefore,

$$\begin{aligned}
\beta_{y,x} \circ \beta_{x,y} &= [a \otimes b, h] \\
&= [I, \lambda_{x\otimes y}] \\
&= 1_{x\otimes y}.
\end{aligned}$$

The hexagon axioms are verified similarly. □

Bibliography

[BL04] John C. Baez and Aaron D. Lauda. Higher-dimensional algebra V: 2-groups. *Theory Appl. Categ.*, 12:423–491, 2004. (cit. on pp. 53, 56).

[CGV06] P. Carrasco, A. R. Garzón, and E. M. Vitale. On categorical crossed modules. *Theory Appl. Categ.*, 16:No. 22, 585–618, 2006. (cit. on pp. 9, 468, 475, 479, 481, 483).

[JS93] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993. (cit. on pp. 44, 46, 47).

[Kel64] G.M Kelly. On maclane's conditions for coherence of natural associativities, commutativities, etc. *Journal of Algebra*, 1(4):397–402, 1964. (cit. on p. 31).

[Lap83] Miguel L Laplaza. Coherence for categories with group structure: An alternative approach. *Journal of Algebra*, 84(2):305–323, 1983. (cit. on pp. 4, 102, 453, 457, 458).

[Lot97] M. Lothaire. *Combinatorics on Words*. Cambridge Mathematical Library. Cambridge University Press, 2 edition, 1997. (cit. on p. 163).

[ML78] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer, New York, NY, 1978. (cit. on pp. 3, 46, 47, 48, 49, 163, 288).

[Par25] A. V. Parab. Dashed monoids. `https://github.com/parabamoghv/DashedMonoids`, 2025. (cit. on p. 6).