Coherence and Symmetrization of Categorical Groups

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Jun 27, 2025

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Overview

1. We introduce and formalize the concept of dashed monoids, state the universal property of the free dashed monoid, define dashed monoid bases, and formalize free dashed monoids in Lean4.
2. We discuss categorical groups, state their universal property, and prove the coherence theorem for categorical groups using an alternate approach.
3. We explore symmetric categorical groups and describe a construction for the symmetrization of categorical groups.

Formalizing Dashed Monoids

Background: Monoids

Lean4 is a functional programming language and a theorem prover.

Definition. A **monoid** is a set equipped with an associative multiplication and a unit. In Lean4,

$$\texttt{structure}\ \texttt{Monoid}\ M :$$
$$\texttt{mul} : M \longrightarrow M \longrightarrow M$$
$$\texttt{mul\_assoc} : \forall (a, b, c : M),\ a \bullet (b \bullet c) = (a \bullet b) \bullet c$$
$$\texttt{one} : M$$
$$\texttt{one\_mul} : \forall (x : M),\ 1 \bullet x = x$$
$$\texttt{mul\_one} : \forall (x : M),\ x \bullet 1 = x.$$

Here, $a \bullet b = \texttt{mul}\ a\ b$ and $1 = \texttt{one}$.

The **free monoid generated by a set** S, denoted $\mathtt{Mon}\langle S\rangle$, is universal in the following sense: For every function $f : S \longrightarrow M$ where M is a monoid, there exists a unique monoid homomorphism $\overline{f} : \mathtt{Mon}\langle S\rangle \longrightarrow M$ such that

$$\begin{tikzcd}
S \arrow[d, "i"'] \arrow[dr, "f"] & \\
\mathtt{Mon}\langle S\rangle \arrow[r, "\exists!\,\overline{f}"'] & M
\end{tikzcd}$$

Definition. A subset $G \subseteq M$ of a monoid M is called a **monoid basis** or a **multiplicative basis** if every $x \in M$ can be written a unique product

$$x = x_1 \bullet \cdots \bullet x_n$$

such that $x_i \in G$ for $1 \leq i \leq n$.

Theorem. A monoid M is a free monoid if and only if it admites a monoid basis $X \subseteq M$.

Set-theoretic construction: The free monoid generated by S consists of all functions $u : [n] \longrightarrow S$ for $n \geq 0$, where $[n] = \{1, \ldots, n\}$.

Formalization of Lists

For a set S, the collection of all lists with entries from S defines a free monoid generated by S. This collection, denoted `List` S, can be defined inductively as follows:

> The empty list, written as **nil** or $[]$, belongs to `List` S.
> If $a \in S$ and $L \in$ `List` S, then the list $a :: L$ (that is, a prepended to L) is also in `List` S.

In Lean4, this is formalized as:

```
inductive List S
| nil : List S
| cons (a : S) (L : List S) : List S
```

Examples.

$$[a] = a :: [] \qquad [a, b, c] = a :: b :: c :: []$$

Dashed Monoids

Definition. A **dashed monoid** is a set equipped with an associative multiplication, a unit element I, and a unary operation $(-)'$ (called the dash) satisfying $I' = I$.

Motivation. The set of objects in a semi-strict categorical group forms a dashed monoid in a natural way.

Examples.

Any monoid, where the dash operation is defined as the identity map.

Any group, where the dash operation is defined as the inverse map.

In Lean4,

$$\texttt{structure DMonoid } M \texttt{ extends Monoid } M \texttt{ where}$$

$$\texttt{dash} : M \longrightarrow M$$

$$\texttt{dash\_one} : 1' = 1.$$

Here, $1' = \texttt{dash } 1$.

Definition. The **free dashed monoid generated by a set** S, denoted $\mathsf{DMon}\langle S\rangle$, is universal in the following sense: For every function $f : S \longrightarrow M$ where M is a dashed monoid, there exists a unique dashed monoid homomorphism $\overline{f} : \mathsf{DMon}\langle S\rangle \longrightarrow M$ such that

$$\begin{tikzcd} S \arrow[d, "i"'] \arrow[dr, "f"] & \\ \mathsf{DMon}\langle S\rangle \arrow[r, "\exists!\,\overline{f}"'] & M \end{tikzcd}$$

Examples. Let $S = \{a, b\}$, then

$$x = ((a \bullet b)' \bullet a)', \qquad y = a' \bullet a'' \bullet b', \qquad z = b'$$

are all elements of $\mathsf{DMon}\langle S\rangle$.

Definition. A subset $H \subseteq M$ of a dashed monoid M is called a **dash basis** if every non-unit element $x \in M$ can be uniquely written as $x = y^{(k)}$ for some $y \in H$ and $k \in \mathbb{N}$, where $y^{(k)}$ denotes the k-fold application of the dash operation to y.

Notation. Let M be a dashed monoid and $X \subseteq M$, we denote

$$\begin{aligned} \langle\langle X;\, \bullet \rangle\rangle &:= \left\{ x_1 \bullet \cdots \bullet x_m \mid m \geq 2,\ x_i \in X \right\} \\ \langle\langle X;\, (-)' \rangle\rangle &:= \left\{ y^{(k)} \mid k \geq 1,\ y \in X \right\} \end{aligned}$$

Definition. A subset $S \subseteq M$ of a dashed monoid M is called a **dashed monoid basis** if

i. M has a multiplicative basis G,

ii. M has a dash basis H,

iii. There exists a homomorphism, called **length function**, $\hat{l} : M \longrightarrow \mathbb{N}$ such that $\hat{l}(a) = 1$ for $a \in S$.

iv. The interlocking conditions hold:

$$G = S \sqcup \langle\langle H;\ (-)' \rangle\rangle$$
$$H = S \sqcup \langle\langle G;\ \bullet \rangle\rangle\ .$$

Theorem. A dashed monoid M is a free dashed monoid if and only if it admites a dashed monoid basis $S \subseteq M$.

Formalization of Dashed Lists

The interlocking conditions

$$G = S \sqcup \langle\langle H; (-)' \rangle\rangle$$
$$H = S \sqcup \langle\langle G; \bullet \rangle\rangle .$$

provide a way to formalize free dashed monoids:

mutual

inductive $\texttt{MulBasis}\ S$

| $\texttt{inc}\ (a : S) : \texttt{MulBasis}\ S$

| $\texttt{consDash}\ (k \geq 1)\ (y : \texttt{DashBasis}\ S) : \texttt{MulBasis}\ S$

inductive $\texttt{DashBasis}\ S$

| $\texttt{inc}\ (a : S) : \texttt{DashBasis}\ S$

| $\texttt{consMul}\ (m \geq 2)\ (x_1, \ldots, x_m : \texttt{MulBasis}\ S) : \texttt{DashBasis}\ S$

The interlocking conditions

$$G = S \sqcup \langle\langle H;\ (-)' \rangle\rangle$$
$$H = S \sqcup \langle\langle G;\ \bullet \rangle\rangle\,.$$

provides a complete characterization of G: For every $x \in G$, exactly one of the following is true:

1. There exist unique $a \in S$ and $k \in \mathbb{N}$ such that
$$x = a^{(k)}.$$
2. There exist unique integers $m \geq 2$, $k \geq 1$, and elements $x_i \in G$ such that
$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}.$$

Coherence for Categorical Groups

Background: Monoidal Categories

Definition. A **monoidal category** $(\mathscr{M}, \otimes, I)$ is a category equipped with a multiplication $\otimes$ (usually called tensor), unit I, and natural isomorphisms:

$$\alpha_{x,y,z} : x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z$$

$$\lambda_x : I \otimes x \longrightarrow x \qquad \text{and} \qquad \rho_x : x \otimes I \longrightarrow x,$$

that satisfy the coherence axioms:

Coherence Axioms for Monoidal Categories

$$\begin{tikzcd}
x \otimes (y \otimes (z \otimes w)) \arrow[r] \arrow[d] & (x \otimes y) \otimes (z \otimes w) \arrow[r] & ((x \otimes y) \otimes z) \otimes w \\
x \otimes ((y \otimes z) \otimes w) \arrow[rr] & & (x \otimes (y \otimes z)) \otimes w \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes (I \otimes y) \arrow[rr, "\alpha_{x,I,y}"] \arrow[dr, "1_x \otimes \lambda_y"'] & & (x \otimes I) \otimes y \arrow[dl, "\rho_x \otimes 1_y"] \\
& x \otimes y &
\end{tikzcd}$$

Background: Categorical Groups

A **categorical group** $(\mathscr{M}, \otimes, I, (-)')$ is a monoidal category equipped with a dash functor $(-)' : \mathscr{M} \longrightarrow \mathscr{M}$, and natural isomorphisms:

$$\eta_x : I \longrightarrow x \otimes x' \qquad \text{and} \qquad \epsilon_x : x' \otimes x \longrightarrow I$$

that satisfy the coherence axioms:

Coherence Axioms for Categorical Groups

$$\begin{tikzcd}
x \arrow[r, "\lambda_x^{-1}"] \arrow[d, equal] & I \otimes x \arrow[r, "\eta_x \otimes 1_x"] & (x \otimes x') \otimes x \arrow[d, "\alpha^{-1}_{x,x',x}"] \\
x & x \otimes I \arrow[l, "\rho_x"] & x \otimes (x' \otimes x) \arrow[l, "1_x \otimes \epsilon_x"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
x' \arrow[r, "\rho_{x'}^{-1}"] \arrow[d, equal] & x' \otimes I \arrow[r, "1_{x'} \otimes \eta_x"] & x' \otimes (x \otimes x') \arrow[d, "\alpha_{x',x,x'}"] \\
x' & I \otimes x' \arrow[l, "\lambda_{x'}"] & (x' \otimes x) \otimes x'. \arrow[l, "\epsilon_x \otimes 1_{x'}"]
\end{tikzcd}$$

Note that, the axioms make $(x, x', \eta_x, \epsilon_x)$ into an adjunction.

Definition. The **free** T**-category generated by a set** S, denoted $\mathsf{T}\langle S\rangle$, is universal in the following sense: For every function $f : S \longrightarrow \mathrm{Obj}(\mathscr{M})$ where $\mathscr{M}$ is a T-category, there exists a T-functor $\overline{f} : \mathsf{T}\langle S\rangle \longrightarrow \mathscr{M}$ unique up to a unique T-natural transformation such that the diagram

$$\begin{tikzcd} S \arrow[d, "i"'] \arrow[dr, "f"] & \\ \mathsf{T}\langle S\rangle \arrow[r, "\exists\overline{f}"'] & \mathscr{M} \end{tikzcd}$$

commutes.

Formal Diagrams

Slogan: "Any formal diagram commutes."

Definition. A **diagram** in a T-category $\mathscr{M}$ is a functor $D : \mathscr{D} \longrightarrow \mathscr{M}$ where $\mathscr{D}$ is a small category.
A **formal diagram** is a diagram that lifts to the free T-category generated by the set $M := \mathrm{Obj}(\mathscr{M})$.

$$\begin{tikzcd} & \mathsf{T}\langle M \rangle \arrow[d] \\ \mathscr{D} \arrow[ur, "\tilde{D}"] \arrow[r, "D"] & \mathscr{M} \end{tikzcd}$$

A **diagram commutes** if it factors through a *thin* category.
A **thin category** is a category in which there is at most one morphism between any pair of objects.

Theorem (Coherence)

Coherence for $\mathtt{T}$*-categories refers to any one of the following statements:*

The free $\mathtt{T}$*-category, denoted* $\mathtt{T}\langle S\rangle$*, is a thin category.*

Every formal diagram in a $\mathtt{T}$*-category commutes.*

Every $\mathtt{T}$*-category is equivalent to a strict* $\mathtt{T}$*-category via* $\mathtt{T}$*-functors.*

The coherence theorems for monoidal categories are proved in [ML78, Kel64].

The coherence statement $\mathtt{CatGrp}\langle S\rangle$ is a thin category is proved in [Lap83].

We will prove all three using a different approach.

Main Results

We prove the following:

Theorem

Let S be a set. The canonical functor

$$Q : \mathtt{CatGrp}\langle S\rangle \longrightarrow \mathtt{Grp}\langle S\rangle$$

is an equivalence, where $\mathtt{Grp}\langle S\rangle$ denotes the free group generated by S, regarded as a discrete categorical group. Therefore, $\mathtt{CatGrp}\langle S\rangle$ is a thin category.

Theorem

Every categorical group is equivalent to a strict categorical group via categorical group functors.

Intermediate Step

Definition. A categorical group is called a semi-strict categorical group if the underlying monoidal structure is strict.

Theorem

Every categorical group is equivalent to a semi-strict categorical group via categorical group functors.

Theorem

Let S be a set. The free categorical group generated by S is equivalent to the free semi-strict categorical group generated by S.

Let

$$Q : \mathtt{DMon}\langle S\rangle \longrightarrow \mathtt{Grp}\langle S\rangle$$

be the induced homomorphism.

Define a semi-strict categorical group, denoted $\mathtt{SSCatGrp}\langle S\rangle$, as follows:

> The set of objects is given by
>
> $$\mathrm{Obj}(\mathtt{SSCatGrp}\langle S\rangle) := \mathtt{DMon}\langle S\rangle .$$
>
> For $x, y \in \mathtt{DMon}\langle S\rangle$, the set of morphisms is defined as
>
> $$\mathrm{Hom}_{\mathtt{SSCatGrp}\langle S\rangle}(x, y) := \begin{cases} \{ \star_{x,y} \} & \text{if } Q(x) = Q(y) \\ \varnothing & \text{if } Q(x) \neq Q(y). \end{cases}$$

We claim $\mathtt{SSCatGrp}\langle S\rangle$ is the free semi-strict categorical group.

Idea of the proof

Let $S = \{a, b\}$, and consider

$$x = ((a \bullet b)' \bullet a)' \qquad \text{and} \qquad y = b''.$$

Observe, $Q(x) = Q(y)$. Therefore, we have $f : x \longrightarrow y$. Then,

$$\begin{tikzcd}
((a \bullet b)' \bullet a)' \arrow[r, "\text{type-C}"] \arrow[d, "f"'] & a' \bullet a'' \bullet b'' \arrow[r, "\text{type-B}"] & a' \bullet a \bullet b \arrow[r, "\text{type-A}"] & b \arrow[d, equal] \\
b'' \arrow[r, "\text{type-C}"'] & b'' \arrow[r, "\text{type-B}"'] & b \arrow[r, "\text{type-A}"'] & b
\end{tikzcd}$$

Where

$$\text{type-C}: \quad (u \bullet v)' \xrightarrow{\text{Distribution}} v' \bullet u'$$

$$\text{type-B}: \quad u'' \xrightarrow{\text{Reduction}} u$$

$$\text{type-A}: \quad a \bullet a',\ a' \bullet a \xrightarrow{\text{Simple Cancellation}} I, \quad a \in S$$

Three Subsets

Let S be a set. The objects of the free semi-strict categorical group $\texttt{SSCatGrp}\langle S\rangle$ are given by the free dashed monoid $\texttt{DMon}\langle S\rangle$. Define

$$
\begin{aligned}
C &:= \left\{a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \mid m \in \mathbb{N},\ a_i \in S,\ k_i \in \mathbb{N}\right\} \\
B &:= \left\{a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \mid m \in \mathbb{N},\ a_i \in S,\ k_i \in \{0,1\}\right\} \\
A &:= \left\{a_1^{(k_1)} \bullet \cdots \bullet a_m^{(k_m)} \in B \mid \text{no cancellation pair is present}\right\}.
\end{aligned}
$$

Let $x \in \texttt{SSCatGrp}\langle S \rangle$, then

> there exists unique $x_C \in C$ and an arrow $x \xrightarrow{Type-C} x_C$ constructed using the distribution arrows.
>
> there exists unique $x_B \in B$ and an arrow $x_C \xrightarrow{Type-B} x_B$ constructed using the reduction arrows.
>
> there exists unique $x_A \in A$ and an arrow $x_B \xrightarrow{Type-A} x_A$ constructed using the simple cancellation arrows.

Coherence Proof

Lemma. If there exists an arrow $f : x \longrightarrow y$ in $\texttt{SSCatGrp}\langle S \rangle$ then $x_A = y_A$. Moreover, f factors as follows:

$$\begin{tikzcd}
x \arrow[r, "\text{Type-C}"] \arrow[d, "f"'] & x_C \arrow[r, "\text{Type-B}"] & x_B \arrow[r, "\text{Type-A}"] & x_A \arrow[d, equal] \\
y \arrow[r, "\text{Type-C}"'] & y_C \arrow[r, "\text{Type-B}"'] & y_B \arrow[r, "\text{Type-A}"'] & y_A
\end{tikzcd}$$

Symmetrization of Categorical Groups

Symmetric Categorical Groups

A **symmetric categorical group** $(\mathscr{M}, \otimes, I, (-)')$ is a categorical group equipped with a natural isomorphisms:

$$\beta_{x,y} : x \otimes y \longrightarrow y \otimes x$$

that satisfy the coherence axioms:

$$\begin{tikzcd}
x \otimes y \otimes z \arrow[r, "\alpha^{-1}_{x,y,z}"] \arrow[d, "\beta_{x,y} \otimes 1_z"'] & x \otimes (y \otimes z) \arrow[r, "\beta_{x,y\otimes z}"] & y \otimes z \otimes x \arrow[d, "\alpha^{-1}_{y,z,x}"] \\
y \otimes x \otimes z \arrow[r, "\alpha^{-1}_{y,z,x}"'] & y \otimes (x \otimes z) \arrow[r, "1_y \otimes \beta_{x,z}"'] & y \otimes (z \otimes x)
\end{tikzcd}$$

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "\alpha_{x,y,z}"] \arrow[d, "1_x \otimes \beta_{y,z}"'] & x \otimes y \otimes z \arrow[r, "\beta_{x \otimes y, z}"] & z \otimes (x \otimes y) \arrow[d, "\alpha z, x, y"] \\
x \otimes (z \otimes y) \arrow[r, "\alpha_{x,z,y}"'] & x \otimes z \otimes y \arrow[r, "\beta_{x,z} \otimes 1_y"'] & z \otimes x \otimes y
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \otimes y \arrow[r, "\beta_{x,y}"] \arrow[dr, equal] & y \otimes x \arrow[d, "\beta_{y,x}"] \\
& x \otimes y
\end{tikzcd}$$

Recipe

1. Start with a categorical group $\mathscr{M}$.
2. Adopt the notion of quotient categorical group $\mathscr{M}/\mathscr{H}$ via a categorical crossed module $T : \mathscr{H} \longrightarrow \mathscr{M}$ from [CGV06].
3. Define the commutator categorical subgroup $[\mathscr{M}, \mathscr{M}]$.
4. Analyze the quotient $\mathscr{M}/[\mathscr{M}, \mathscr{M}]$.

Background: $\mathscr{M}$-Categorical Group [CGV06]

Definition. Let $\mathscr{M}$ be a categorical group. A *$\mathscr{M}$-categorical group* consists of

a categorical group $\mathscr{H}$

a functor

$$ac : \mathscr{M} \times \mathscr{H} \longrightarrow \mathscr{H}, \qquad (x, a) \longmapsto ac(x, a) = {}^{x}a$$

families of natural isomorphisms

$$\psi = \left(\psi_{x,a,b} : {}^{x}(a \otimes b) \longrightarrow {}^{x}a \otimes {}^{x}b\right)_{(x,a,b) \in \mathscr{M} \times \mathscr{H} \times \mathscr{H}},$$

$$\phi = \left(\phi_{x,y,a} : {}^{(xy)}a \longrightarrow {}^{x}\left({}^{y}a\right)\right)_{(x,y,a) \in \mathscr{M} \times \mathscr{M} \times \mathscr{H}}$$

satisfying the compatibility condition for

$${}^{(x \otimes y)}(a \otimes b) \longrightarrow {}^{x}\left({}^{y}a\right) \otimes {}^{x}\left({}^{y}b\right).$$

Background: Categorical precrossed module [CGV06]

Let $\mathscr{M}$ be a categorical group. A $\mathscr{M}$-precrossed module consists of

- a $\mathscr{M}$-categorical group $\mathscr{H}$,
- a morphism of categorical groups $\mathbf{T} = (T, \mu) : \mathscr{H} \longrightarrow \mathscr{M}$ and
- a family of natural isomorphisms in $\mathscr{M}$

$$\nu = (\nu_{x,a} : T\left({}^{x}a\right) \otimes x \longrightarrow x \otimes T(a))_{(x,a)\in\mathscr{M}\times\mathscr{H}}$$

satisfying the coherent compatibility conditions for[1]

$$T\left({}^{x}\left({}^{y}a\right)\right) \otimes x \otimes y \longrightarrow x \otimes y \otimes T(a)$$
$$T\left({}^{x}(a \otimes b)\right) \otimes x \longrightarrow x \otimes T(a) \otimes T(b).$$

[1] for this section, $x \otimes y \otimes z := (x \otimes y) \otimes z$

Categorical crossed module [CGV06]

Let $\mathscr{M}$ be a categorical group. A categorical *$\mathscr{M}$-crossed module* consists of

- a categorical $\mathscr{M}$-precrossed module $\mathscr{H}$ and
- a family of natural isomorphisms in $\mathscr{H}$

$$\chi = \left(\chi_{a,b} : {}^{T(a)}b \otimes a \longrightarrow a \otimes b\right)_{(a,b)\in\mathscr{H}\times\mathscr{H}}$$

satisfying the coherent compatibility conditions for

$$\begin{aligned}
{}^{T(a\otimes b)}c \otimes a \otimes b &\longrightarrow a \otimes b \otimes c \\
{}^{T(a)}(b \otimes c) \otimes a &\longrightarrow a \otimes b \otimes c \\
{}^{x}\left({}^{T(a)}b \otimes a\right) &\longrightarrow {}^{x}a \otimes {}^{x}b \\
T\left({}^{T(a)}b \otimes a\right) &\longrightarrow T(a) \otimes T(b).
\end{aligned}$$

Background: Quotient Categorical Group [CGV06]

Let $T : \mathscr{H} \longrightarrow \mathscr{M}$ be $\mathscr{M}$-crossed module. The *quotient categorical group* $\mathscr{M}/\mathscr{H}$ is define as:

Objects are those of $\mathscr{M}$,

Premorphisms are pairs $(a, f) : x \longrightarrow y$, with a in $\mathscr{H}$ and $f : x \longrightarrow T(a) \otimes y$,

Morphisms are classes of premorphisms $[a, f] : x \multimap y$, where two pairs (a, f) and (a', f') are equivalent if there is $p : a \longrightarrow a'$ in $\mathscr{H}$ such that $f' = (T(p) \otimes 1_y) \circ f$.

The tensor product on objects is the same as that in $\mathscr{M}$. For morphisms $[a, f] : x \multimap y$ and $[b, g] : z \multimap w$ their tensor product $[a \otimes {}^y b, \underline{f \otimes g}] : x \otimes z \multimap y \otimes w$ is given by the arrow part

$$
\begin{tikzcd}
x \otimes z \arrow[r, "f \otimes g"] & T(a) \otimes y \otimes T(b) \otimes w \arrow[r, "1 \otimes \nu^{-1} \otimes 1"] & T(a) \otimes T({}^y b) \otimes y \otimes w \arrow[d, "\mu"] \\
& & T(a \otimes {}^y b) \otimes y \otimes w
\end{tikzcd}
$$

The cancellation isomorphisms are given by the arrow parts

$$I \xrightarrow{\eta_x} x \otimes x' \xrightarrow{\lambda^{-1}} I \otimes (x \otimes x') \longrightarrow T(I) \otimes (x \otimes x')$$

$$x' \otimes x \xrightarrow{\epsilon_x} I \xrightarrow{\lambda^{-1}} I \otimes I \longrightarrow T(I) \otimes I.$$

This makes $\mathscr{M}/\mathscr{H}$ into a categorical group.

Commutator Categorical Group

Definition. Let $\mathscr{M}$ be a categorical group. The **commutator categorical subgroup**, denoted $[\mathscr{M}, \mathscr{M}]$ is the intersection of all categorical subgroups containing the objects canonically isomorphic to $[x, y] = x \otimes y \otimes x' \otimes y'$, the canonical isomorphisms between them, and the morphisms of the form $[f, g] = f \otimes g \otimes f' \otimes g'$.

Any object x in $[\mathscr{M}, \mathscr{M}]$ is canonically isomorphic to $[x_1, y_1] \otimes \cdots \otimes [x_n, y_n]$, denoted

$$a \xrightarrow{can} [x_1, y_1] \otimes \cdots \otimes [x_n, y_n].$$

Morphisms in $[\mathscr{M}, \mathscr{M}]$ are generated by composing morphism of the form

$$\begin{tikzcd}
a \arrow[r, "can"] \arrow[d, "h"'] & {[x_1, y_1] \otimes \cdots \otimes [x_n, y_n]} \arrow[d, "{[f_1, g_1] \otimes \cdots \otimes [f_n, g_n]}"] \\
b & {[z_1, w_1] \otimes \cdots \otimes [z_n, w_n]} \arrow[l, "can"]
\end{tikzcd}$$

Proposition. Let $\mathscr{M}$ be a categorical group. Then $[\mathscr{M}, \mathscr{M}]$ is a $\mathscr{M}$-crossed module.

Sketch of the proof. Define the action

$$\begin{gathered} \mathrm{ac} : \mathscr{M} \times [\mathscr{M}, \mathscr{M}] \longrightarrow [\mathscr{M}, \mathscr{M}] \\ \mathrm{ac}(x, a) := x \otimes a \otimes x'. \end{gathered}$$

Note that

$$x \otimes a \otimes x' \xrightarrow{\epsilon_a^{-1}} x \otimes a \otimes x' \otimes a' \otimes a = [x, a] \otimes a.$$

The required natural isomorphisms are canonical. Coherence theorem shows the compatibility conditions.

Proposition. Let $\mathscr{M}$ be a categorical group. Then $S(\mathscr{M}) = \mathscr{M}/[\mathscr{M},\mathscr{M}]$ is a symmetric categorical group.

Sketch of the proof. Note that the arrow part for any morphisms $[a, f] : x \longrightarrow\!\!\circ\!\!\longrightarrow y$ in $S(\mathscr{M})$ is given by $f : x \longrightarrow a \otimes y$ where $a \in [\mathscr{M},\mathscr{M}]$. Let x, y be objects in $\mathscr{M}/[\mathscr{M},\mathscr{M}]$ then the braiding

$$\sigma_{x,y} : x \otimes y \longrightarrow\!\!\circ\!\!\longrightarrow y \otimes x$$

is given by the arrow part

$$\underline{\sigma}_{x,y} : xy \xrightarrow{\epsilon_x^{-1}} xyx'x \xrightarrow{\epsilon_y^{-1}} xyx'y'yx = [x, y] \otimes yx.$$

Coherence theorem again shows the symmetric categorical group axioms.

Thank You!

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Universal Property in Lean4 (1/1)

We show `FreeDMon` S satisfies the universal property in Lean4,

```
theorem : DMon_UP (S ⟶ FreeDMon S) := by
case exist ⇒
    intro N (f : S ⟶ N)
    use FreeDMon.induce f

case unique ⇒
    intro N (f g : FreeDMon S ⟶ N) inc_hyp
    apply FreeDMon.unique f g
```

Here, the commands in purple are called tactics in tactic mode.

CatGrp equivalent to StCatGrp (1/2)

Let $\mathscr{M}$ be a categorical group with $M := \mathrm{Obj}(\mathscr{M})$. Define

$$F : \mathtt{Grp}\langle M \rangle \longrightarrow M$$

by considering the reduced representation and sending a^{-1} to a'.

Define

$$\begin{aligned} F_0 : I &\longrightarrow F(e) \\ F_1 : F(x)' &\longrightarrow F(x^{-1}) \\ F_2(x,y) : F(x) \otimes F(y) &\longrightarrow F(xy) \end{aligned}$$

using the categorical group structure of $\mathscr{M}$.

CatGrp equivalent to StCatGrp (2/2)

Define a strict categorical group $\mathscr{T}$ as

$$\mathrm{Obj}\,(\mathscr{T}) = \mathtt{Grp}\,\langle M \rangle$$
$$\mathrm{Hom}_{\mathscr{T}}(x, y) = \mathrm{Hom}_{\mathscr{M}}(F(x), F(y)).$$

Use F_0, F_1, and F_2 to define a strict categorical group structure.

Universal Property (1/2)

We also get $P_T : \mathscr{M} \longrightarrow \mathscr{M}/(\mathscr{H}, T)$ and $\pi_T : P_T \circ T \Rightarrow \mathrm{Const}_I$.

The quotient construction

$$\left(\mathscr{M}/(\mathscr{H}, T),\ \ P_T : \mathscr{M} \longrightarrow \mathscr{M}/(\mathscr{H}, T),\ \ \pi_T : P_T \circ T \Rightarrow \mathrm{Const}_I\right)$$

is universal with respect to triples in $\mathscr{M}$-categorical groups,

$$\left(\mathscr{G},\ \ G : \mathscr{M} \longrightarrow \mathscr{G},\ \ \delta : G \circ T \Rightarrow \mathrm{Const}_I\right)$$

satisfying the compatibility condition:

Universal Property (2/2)

For $x \in \mathscr{M}$ and $a \in \mathscr{H}$ the diagram

$$\begin{tikzcd}
G(x) \otimes I \arrow[d, "\lambda^{-1}_{G(x)} \circ \rho_{G(x)}"'] & & G(x) \otimes GT(a) \arrow[ll, "1 \otimes \delta_a"'] \\
I \otimes G(x) & & G(x \otimes T(a)) \arrow[u, "G_2^{-1}(X, T(A))"'] \\
GT({}^x a) \otimes G(x) \arrow[u, "\delta_{{}^x a} \otimes 1"] & & G(T({}^x a) \otimes x) \arrow[u, "G(\nu_{x,a})"'] \arrow[ll, "G_2^{-1}({}^x a, x)"]
\end{tikzcd}$$

commutes.