Free Dashed Words

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Introduction

This article explores a novel algebraic structure called *Dashed Monoids* and introduces the associated data structure known as *Free Dashed Monoids*. Definition 2.3.1 outlines the inductive criteria for constructing a free dashed monoid. Theorem 2.3.33 demonstrates that a dashed monoid satisfying these inductive criteria fulfills the universal property of a free dashed monoid. Chapter 3 provides a formal set-theoretic construction of a free dashed monoid.

This article is a work in progress, and I am currently in the process of finalizing it.

CHAPTER 1

Topics in Monoids

1.1. Definition of a Monoids

Definition: (X) factorization, Monoid Basis, Universal property of a free monoid?

Theorems: A monoid satisfies the Universal property of a free monoid if and only if it has a monoid basis, Monoid basis is unique?

Todo: Reorganize and label the equations

Definition 1.1.1. A *monoid* has the following data:

- A set M, called the *underlying set*,
- an element, denoted $I \in M$, called the *unit element*, and
- a binary map
$$- \bullet - : M \times M \longrightarrow M$$
called the *multiplication*.

These satisfy the following conditions:

Unit conditions: For $x \in M$, the equlity

$$x \bullet I = x \qquad \text{and} \qquad I \bullet x = x \tag{1.1.2}$$

holds.

Associativity: For $x, y, z \in M$, the equlity

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z \tag{1.1.3}$$

holds. ◇

Notation 1.1.4. Let $(M, \bullet, I)$ be a monoid. Let $n \in \mathbb{N}$ and $x_i \in M$ for $1 \le i \le n$. We denote the multiplication of the elements as follows:

$$x_1 \bullet \cdots \bullet x_n := (((x_1 \bullet x_2) \bullet x_3) \bullet \cdots \bullet x_n).$$

By convention, we have

$$x_1 \bullet \cdots \bullet x_n = I$$

if $n = 0$. Due to the associativity of multiplication, the order in which we group the terms is irrelevant. ◇

Definition 1.1.5. Let $M = (M, \bullet, I)$ and $N = (N, *, J)$ be monoids. A *monoid homomorphism* from M to N is a set map $f : M \longrightarrow N$ that satisfies following conditions:

Unit condition: The equality

$$f(I) = J \tag{1.1.6}$$

holds.

Multiplication condition: For $x, y \in M$, the equality

{eqn: Mon Hom mul}
$$f(x \bullet y) = f(x) * f(y) \tag{1.1.7}$$

holds. ◇

Let $(M, \bullet, I)$ be a monoid.

defn: Mon mul gen X
Definition 1.1.8. Let $X \subseteq M$ be a subset. The *subset generated by X under the multiplication*, denoted by $\langle X;\ \bullet\rangle \subseteq M$, is defined as

{eqn: Mon mul gen X}
$$\langle X;\ \bullet\rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \in \mathbb{N},\ m \geq 1,\ x_i \in X \text{ for } 1 \leq i \leq m\}. \tag{1.1.9}$$

The *subset generated by X under non-trivial multiplication*, denoted by $\langle\langle X;\ \bullet\rangle\rangle \subseteq M$, is defined as

{eqn: Mon mul genP X}
$$\langle\langle X;\ \bullet\rangle\rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \in \mathbb{N},\ m \geq 2,\ x_i \in X \text{ for } 1 \leq i \leq m\}. \tag{1.1.10}$$ ◇

Remark 1.1.11. It follows from the above definition that

$$\langle\langle X;\ \bullet\rangle\rangle \subseteq \langle X;\ \bullet\rangle.$$ ◇

notation: Mon P
Notation 1.1.12. We denote the set of all monoid elements without the unit by M^+. That is,

{eqn: Mon P}
$$M^+ := M \setminus \{I\}. \tag{1.1.13}$$ ◇

defn: Mon mul independent
Definition 1.1.14. Let $X \subseteq M$ be a subset. We say X is *independent with respect to the multiplication* if for every $z \in \langle X;\ \bullet\rangle$ we get unique $m \in \mathbb{N}$ with $m \geq 1$ and unique $x_i \in X$ for $1 \leq i \leq m$ such that

$$x_1 \bullet \cdots \bullet x_m = z.$$

That is, for $n, m \in \mathbb{N}^+$ and $x_i, y_j \in X$ for $1 \leq i \leq n$ and $1 \leq j \leq m$,

{eqn: Mon mul independent}
$$x_1 \bullet \cdots \bullet x_n = y_1 \bullet \cdots \bullet x_m \qquad \text{implies} \qquad n = m \text{ and } x_i = y_j. \tag{1.1.15}$$ ◇

defn: Mon mul genset
Definition 1.1.16. Let $X \subseteq M$ be a subset. We say X is a *generating set with respect to the multiplication* if

{eqn: Mon mul genset}
$$\langle X;\ \bullet\rangle = M^+. \tag{1.1.17}$$

That is, for every $x \in M^+$ there exits $m \in \mathbb{N}^+$ and $x_i \in X$ for $1 \leq i \leq m$ such that

◇

$$x_1 \bullet \cdots \bullet x_m = x.$$

defn: Mon mul basis
Definition 1.1.18. Let $X \subseteq M$ be a subset. We say X is a *multiplication basis* or a *monoid basis* if X is independent with respect to the multiplication and X is a generating set with respect to the multiplication. ◇

defn: Monoid X factorization
Definition 1.1.19. Let $x \in M^+$ and $X \subseteq M$. An *X-factorization* of x is a finite ordered collection $(x_1, \ldots, x_m)$ with $x_i \in X$ for $1 \leq i \leq m$ such that

{equation: Monoid X factorization}
$$x_1 \cdots x_n = x. \tag{1.1.20}$$ ◇

Remark 1.1.21. Note that, a subset $X \subseteq M$ is independent with respect to the multiplication if and only if every X-factorization is unique. A subset $X \subseteq M$ is a generating set with respect to multiplication if and only if every $x \in M^+$ has an X-factorization. A subset $X \subseteq M$ is a multiplicative basis if and only if every $x \in M^+$ has a unique X-factorization. ◇

prop: Mon x y ne I then xy ne I

Proposition 1.1.22. *Let $(M, \cdot, I)$ be a monoid and let $X \subseteq$ be a subset. Suppose X is a monoid basis then for $x, y \in M$, if $x \neq I$ and $y \neq I$ then $x \cdot y \neq I$.*

Proof. Suppose $X \subseteq M$ is a monoid basis of M. Let $x, y \in M$ be such that $x \neq I$ and $y \neq I$. Suppose for contradiction that $x \cdot y = I$. Since X is a monoid basis we get $n, m \in \mathbb{N}^+$ and $x_i, y_j \in X$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ such that

$$x = x_1 \cdots x_n \qquad \text{and} \qquad y = y_1 \cdots y_m.$$

Thus, we get

$$I = x \cdot y = x_1 \cdots x_n \cdot y_1 \cdots y_m.$$

Therefore, we have

$$\begin{aligned} x_1 &= x_1 \cdot I \\ &= x_1 \cdot x_1 \cdots x_n \cdot y_1 \cdots y_m. \end{aligned}$$

This contradicts the multiplicative independence of X. Thus, we get $x \cdot y \neq I$. □

1.2. Free Monoids

section: Word

Definitions: Word, Word multiplication, length of the word, assoc, mul one, one mul
Theorems: Word multiplication classification
To Do: Can Reorganize

defn: Free monoid generated by S

Definition 1.2.1. Let S be a set. Let M be a monoid and $\phi : S \longrightarrow M$ be a set map. We say that the pair (S, ϕ) satisfies the *universal property of a free monoid* over S if the following conditions are satisfied:

Existence: Given a monoid N and a set map $u : S \longrightarrow N$ there exists a monoid homomorphism $f_u : M \longrightarrow N$ such that

{eqn: Mon freeMon UP exists}
$$f_u \circ \phi = u. \tag{1.2.2}$$

Uniqueness: Given a monoid N and a pair of monoid homomorphisms $f, g : M \longrightarrow N$, then

{eqn: Mon freeMon UP unique}
$$f \circ \phi = g \circ \phi \qquad \text{implies} \qquad f = g. \tag{1.2.3}$$
◇

thm: Monoid has a basis then it is a free monoid

Theorem 1.2.4. *Let $(M, \cdot, I)$ be a monoid and $X \subseteq M$ be a subset of M with the inclusion given by $\phi : X \longrightarrow M$. If X is a monoid basis of M then the pair (X, ϕ) satisfies the universal property of a free monoid.*

Proof. Suppose $X \subseteq M$ is a monoid basis of M. We will show that $\phi : X \longrightarrow M$ satisfies the universal property described in Definition 1.2.1. Let $(N, \bullet, J)$ be a monoid and $u : X \longrightarrow N$ be a set map.

Define $f_u : M \longrightarrow N$ as follows: Let $x \in M$. If $x = I$ define

$$f_u(I) := J.$$

Now suppose we have $x \in M^+$. Since X is a basis we get a unique $n \in \mathbb{N}^+$ and a unique $x_i \in X$ for $1 \le i \le n$ such that $x = x_1 \bullet \cdots \bullet x_n$. Define

$$f_u(x) := u(x_1) \bullet \cdots \bullet u(x_n).$$

Since the factorization $x = x_1 \bullet \cdots \bullet x_n$ is unique, f_u is a well-defined map.

Next, we will show that f_u is a monoid homomorphism. We already have the equality

$$f_u(I) = J.$$

Thus, the unit condition of a monoid homomorphism is satisfied.

Now suppose $x, y \in M$. We will show that

$$f_u(x \cdot y) = f_u(x) \bullet f_u(y).$$

We will consider the following cases:

Case I ($x, y \in M^+$): We get $x = x_1 \bullet \cdots \bullet x_n$ and $y = y_1 \bullet \cdots \bullet y_m$ where $n, m \in \mathbb{N}^+$ and $x_i, y_j \in X$. Then we have

$$x \bullet y = x_1 \bullet \cdots \bullet x_n \bullet y_1 \bullet \cdots \bullet y_m.$$

From Proposition 1.1.22 we get that $x \bullet y \neq I$. Thus, we get

$$f_u(x \bullet y) = u(x_1) \bullet \cdots \bullet u(x_n) \bullet u(y_1) \bullet \cdots \bullet u(y_m) = f_u(x) \bullet f_u(y).$$

Case II ($x = I,\ y \in M$): In this case, we get

$$\begin{aligned} f_u(I \bullet y) &= f_u(y) \\ &= J \bullet f_u(y) \\ &= f_u(I) \bullet f_u(y). \end{aligned}$$

Case III ($x \in M,\ y = I$): In this case, we get

$$\begin{aligned} f_u(x \bullet I) &= f_u(x) \\ &= f_u(x) \bullet J \\ &= f_u(y) \bullet f_u(I). \end{aligned}$$

This shows that the function $f_u : M \longrightarrow N$ satisfies the multiplicative condition of a monoid homomorphism. Thus, $f_u : M \longrightarrow N$ is a monoid homomorphism.

Observe that for $x \in X$, we get $f_u(z) = u(z)$. Thus, the equality

$$f_u \circ \Phi = u$$

holds. This completes the proof of the existence part of the universal property.

Next, we will show the uniqueness part of the universal property. Suppose $g, h : M \longrightarrow N$ are monoid homomorphisms such that $f \circ \phi = g \circ \phi$. That is, we have $g(z) = h(z)$ for every $z \in X$. We will show that $g = h$.

Let $x \in M$. If $x \in M^+$ then we get unique $n \in \mathbb{N}^+$ and unique $x_i \in X$ such that

$$x = x_1 \bullet \cdots \bullet x_n.$$

We get

$$\begin{aligned} g(x) &= g(x_1 \bullet \cdots \bullet x_n) \\ &= g(x_1) \bullet \cdots \bullet g(x_n) \\ &= h(x_1) \bullet \cdots \bullet h(x_n) \\ &= h(x_1 \bullet \cdots \bullet x_n) \\ &= h(x). \end{aligned}$$

If $x = I$ then we get

$$g(I) = J = h(I).$$

This shows that $g = h$. Thus, the uniqueness part of the universal property is satisfied. This completes the proof. □

Now, we will discuss the construction of a free monoid generated by a set. We will first introduce some basic definitions and notations that will be useful in our discussion.

Notation 1.2.5. Here are some relevant notations that will prove useful throughout our discussion.

- The set of integers are denoted by
$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}.$$
- The set of natural numbers are denoted by
$$\mathbb{N} = \{0, 1, 2, \cdots\}.$$
- The set of positive integers are denoted by
$$\mathbb{N}^+ = \mathbb{Z}^+ = \{1, 2, \cdots\}.$$
- We subscribe to the standard inverval notation for integers. That is, for integers $n, m \in \mathbb{Z}$, we denote
$$(n,m) = \{i \in \mathbb{Z} \mid n < i < m\}, \qquad (n,m] = \{i \in \mathbb{Z} \mid n < i \leq m\},$$
$$[n,m) = \{i \in \mathbb{Z} \mid n \leq i < m\}, \qquad [n,m] = \{i \in \mathbb{Z} \mid n \leq i \leq m\}.$$
In addition, we denote
$$[n] = [1, n].$$
In this discussion, an interval will always refer to a finite interval.
- Let $A \subseteq \mathbb{Z}$ be a subset of integers. We use the following notations for the power set of A:
$$\begin{aligned} \mathcal{P}(A) &= \{T \subseteq \mathbb{Z} \mid T \subseteq A\}, \\ \mathcal{P}^+(A) &= \{T \subseteq \mathbb{Z} \mid T \subseteq A,\ T \neq \varnothing\}, \\ \mathcal{P}^+_I(A) &= \{T \subseteq \mathbb{Z} \mid T \subset A,\ T \neq \varnothing,\ T \text{ is a finite interval.}\} \end{aligned}$$
We denote an element $T \in \mathcal{P}^+(A)$ by $T \subseteq^+ A$ and an element $T \in \mathcal{P}^+_I(A)$ by $T \subseteq^+_I A$.

- For $n,m \in \mathbb{Z}$, we denote

$$\mathcal{P}(n,m) = \mathcal{P}([n,m]), \qquad \mathcal{P}(n) = \mathcal{P}([n]),$$
$$\mathcal{P}^+(n,m) = \mathcal{P}^+([n,m]), \qquad \mathcal{P}^+(n) = \mathcal{P}^+([n]),$$
$$\mathcal{P}_I^+(n,m) = \mathcal{P}_I^+([n,m]), \qquad \mathcal{P}_I^+(n) = \mathcal{P}_I^+([n]). \qquad \diamond$$

Let S be a set.

defn: Word letters

Definition 1.2.6. Let $n \in \mathbb{N}$. An n-**letter word** is a function $u : [n] \longrightarrow S$. $\diamond$

convention: Word letters

Convention 1.2.7. For $n \in \mathbb{N}$, let $\mathcal{W}r(n)$ denote the set of all n-letter words. $\diamond$

defn: Word letters unit

Definition 1.2.8. Note that the empty function is the only function with a domain that is an empty set. Thus, the only 0-letter word is the empty function, $\emptyset : \varnothing \longrightarrow S$. We call this the **unit word**, denoted by $I_{\mathcal{W}r}$. It follows that

$$\mathcal{W}r(0) = \{I_{\mathcal{W}r}\}.$$

$\diamond$

defn: Word letters mul

Definition 1.2.9. Let $n,m \in \mathbb{N}$. Let u and v be n and m-letter words respectively. Define the multiplication, $u \cdot v$, to be the following $n+m$-letter word.

{eqn: Word letters mul}

$$u \cdot v\ (i) := \begin{cases} u(i) & \text{if } i \le n \\ v(i-n) & \text{if } i > n. \end{cases} \tag{1.2.10}$$

$\diamond$

lem: Word letters mul assoc

Lemma 1.2.11. *Let $n,m,p \in \mathbb{N}$. Let u, v and w be n, m, and p-letter words respectively. Then,*

{eqn: Word letters mul assoc}

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w \tag{1.2.12}$$

as $n+m+p$-letter words.

Proof. Observe that for $i \in [n+m+p]$ we get

$$\begin{aligned}
u \cdot (v \cdot w)\ (i) &= \begin{cases} u(i) & \text{if } i \le n \\ v \cdot w\ (i-n) & \text{if } i > n \end{cases} && (1.2.10) \\
&= \begin{cases} u(i) & \text{if } i \le n \\ v(i-n) & \text{if } 0 < i-n \le m \\ w(i-n-m) & \text{if } i-n > m \end{cases} && (1.2.10) \\
&= \begin{cases} u(i) & \text{if } i \le n \\ v(i-n) & \text{if } n < i \le n+m \\ w(i-n-m) & \text{if } i > n+m \end{cases} \\
&= \begin{cases} u \cdot v\ (i) & \text{if } i \le n+m \\ w(i-n-m) & \text{if } i > n+m \end{cases} && (1.2.10) \\
&= (u \cdot v) \cdot w\ (i) && (1.2.10).
\end{aligned}$$

Since $i \in [n+m+p]$ is arbitrarily chosen, we get

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w$$

as required. $\square$

lem: Word letters mul unit and **Lemma 1.2.13.** *Let $n \in \mathbb{N}$. Let u be an n-letter word and $I_{\mathcal{W}r} : [0] \longrightarrow S$ be the empty word. Then we have*

{eqn: Word letters mul unit an

$$u \cdot I_{\mathcal{W}r} = u \qquad \text{and} \qquad I_{\mathcal{W}r} \cdot u = u \tag{1.2.14}$$

as n-letter words.

Proof. Observe that for $i \in [n]$ we get

$$\begin{aligned} u \cdot I_{\mathcal{W}r}\ (i) &= \begin{cases} u(i) & \text{if } i \leq n \\ & \text{if } i > n \end{cases} && (1.2.10) \\ &= u(i) && \text{since } i \leq n. \end{aligned}$$

Since $i \in [n]$ is arbitrarily chosen we get

$$u \cdot I_{\mathcal{W}r} = u.$$

Similarly, for $i \in [n]$ we get

$$\begin{aligned} I_{\mathcal{W}r} \cdot u\ (i) &= \begin{cases} & \text{if } i \leq 0 \\ u(i) & \text{if } i > 0 \end{cases} && (1.2.10) \\ &= u(i) && \text{since } i > 0. \end{aligned}$$

Since $i \in [n]$ is arbitrarily chosen we get

$$I_{\mathcal{W}r} \cdot u = u$$

as required. $\square$

defn: Word letters restriction **Definition 1.2.15.** Let $n \in \mathbb{N}$ and u be an n-letter word. Let $(p,q] \subseteq_I [n]$ with $0 \leq p \leq q \leq n$. Define a $q-p$-lenth word, $u_{(p,q]} : [q-p] \longrightarrow S$ as follows:

{eqn: Word letters restriction}

$$u_{(p,q]}\ (i) := u(p+i). \tag{1.2.16}$$

$\diamond$

lem: Word letters restriction **Lemma 1.2.17.** *Let $n \in \mathbb{N}$ and $u : [n] \longrightarrow S$ be an n-letter word. Then, we get*

{eqn: Word letters restriction

$$u_{[n]} = u \tag{1.2.18}$$

as n-letter words.

Proof. Let $i \in [n]$. We have

$$u_{[n]}\ (i) = u(i). \qquad (1.2.16)$$

Since $i \in [n]$ is arbitrary, we get

$$u_{[n]} = u$$

as required. $\square$

lem: Word letters restriction **Lemma 1.2.19.** *Let $n, m \in \mathbb{N}$, $u : [n] \longrightarrow S$ be an n-letter word and $v : [m] \longrightarrow S$ be an m-letter word. Let $(p,q] \subseteq_I [n]$ with $0 \leq p \leq q \leq n$. Then, we get*

{eqn: Word letters restriction

$$(u \cdot v)_{(p,q]} = u_{(p,q]} \tag{1.2.20}$$

as $q-p$-letter words.

Proof. Let $i \in [q-p]$. Therefore, we get $p < p+i \le q \le n$. Now observe

$$\begin{aligned} (u \cdot v)_{(p,q]}\,(i) &= u \cdot v\ (p+i) && (1.2.16) \\ &= u(p+i) && (1.2.10) \\ &= u_{(p,q]}\,(i). && (1.2.16) \end{aligned}$$

Since $i \in [n]$ is arbitrary, we get

$$(u \cdot v)_{(p,q]} = u_{(p,q]}$$

as required. □

lem: Word letters mul restriction right

Lemma 1.2.21. *Let $n, m \in \mathbb{N}$, $u : [n] \longrightarrow S$ be an n-letter word and $v : [m] \longrightarrow S$ be an m-letter word. Let $(p,q] \subseteq_I (n, n+m]$ with $n \le p \le q \le n+m$. Then, we get*

{eqn: Word letters mul restriction}

$$(u \cdot v)_{(p,q]} = v_{(p-n,q-n]} \tag{1.2.22}$$

as $q-p$-letter words.

Proof. Let $i \in [q-p]$. Therefore, we get $n \le p < p+i \le q \le n+m$. Now observe

$$\begin{aligned} (u \cdot v)_{(p,q]}\,(i) &= u \cdot v\ (p+i) && (1.2.16) \\ &= v(p+i-n) && (1.2.10) \\ &= u_{(p-n,q-n]}\,(i). && (1.2.16) \end{aligned}$$

Since $i \in [q-p]$ is arbitrary, we get

$$(u \cdot v)_{(p,q]} = u_{(p-n,q-n]}$$

as required. □

lem: Word letters restriction mul left

Lemma 1.2.23. *Let $n, m \in \mathbb{N}$ and u and v be n and m-letter words respectively. Then, we get*

{eqn: Word letters restriction mul left}

$$(u \cdot v)_{[n]} = u \tag{1.2.24}$$

as n-letter words.

Proof. Observe that

$$\begin{aligned} (u \cdot v)_{[n]} &= u_{[n]} && (1.2.20) \\ &= u. && (1.2.18) \end{aligned}$$

□

lem: Word letters mul restriction right

Lemma 1.2.25. *Let $n, m \in \mathbb{N}$ and u and v be n and m-letter words respectively. Then, we get*

{eqn: Word letters mul restriction right}

$$(u \cdot v)_{(n,n+m]} = v \tag{1.2.26}$$

as m-letter words.

Proof. Observe that

$$\begin{aligned} (u \cdot v)_{(n,n+m]} &= v_{(0,m]} && (1.2.22) \\ &= v. && (1.2.18) \end{aligned}$$

□

lem: Word letters x=LR

Lemma 1.2.27. *Let $n \in \mathbb{N}$ be a number and $u : [n] \longrightarrow S$ be an n-letter word. Let $0 \le p \le n$. Then we have*

{eqn: Word letters x=LR}

$$u = u_{[p]} \cdot u_{(p,n]} \tag{1.2.28}$$

as n-letter words.

Proof. Let $i \in [n]$. Then we have

$$\begin{aligned} u_{[p]} \cdot u_{(p,n]}\,(i) &= \begin{cases} u_{[p]}(i) & \text{if } i \le p \\ u_{(p,n]}(i-p) & \text{if } i > p \end{cases} && (1.2.10) \\ &= \begin{cases} u(i) & \text{if } i \le p \\ u(i-p+p) & \text{if } i > p \end{cases} && (1.2.16) \\ &= u(p). \end{aligned}$$

Since $i \in [n]$ is arbitrary, we get

$$u = u_{[p]} \cdot u_{(p,n]}$$

as required. ☐

Theorems: The monoid of words satisfies the Universal property of a free monoid.

Definition: This construction is the free monoid.

To Do: show this more directly, without using the factorization stuff.

defn: Word

Definition 1.2.29. A **word** with alphabet in S a dependent pair

$$(n, u)$$

where n is the length and $u : [n] \longrightarrow S$ is the n-letter word. We will denote the collection of all words with alphabet in S by $\mathcal{Mon}\langle S\rangle$. ⋄

Remark 1.2.30. Note that, $\mathcal{Mon}\langle S\rangle$ is the collection of all words,

$$\mathcal{Mon}\langle S\rangle = \bigsqcup_{n\in\mathbb{N}} \mathcal{Wr}(n).$$

⋄

defn: Mon word

Definition 1.2.31. The monoid of words with alphabet in S is defined as follows:

- Let $\mathcal{Mon}\langle S\rangle$ be the underlying set.
- Let the empty word, $(0, I_{\mathcal{Wr}})$, be the unit word.
- For words (n,u) and (m,v), the multiplication is defined as

{eqn: Mon word mul}

$$(n,u)\cdot(m,v) := (n+m, u\cdot v) \tag{1.2.32}$$

where $u \cdot v : [n+m] \longrightarrow S$ is as in Definition 1.2.9.

These data satisfies the monoid conditions as follows:

Associativity: Let $(n,u),(m,v)$ and (p,w) be words. Then we have

$$\begin{aligned}
(n,u)\cdot\big((m,v)\cdot(p,w)\big) &= (n,v)\cdot(m+p,v\cdot w) && (1.2.32)\\
&= (n+m+p,u\cdot(v\cdot w)) && (1.2.32)\\
&= (n+m+p,(u\cdot v)\cdot w) && (1.2.12)\\
&= (n+m,u\cdot v)\cdot(p,w) && (1.2.32)\\
&= \big((n,u)\cdot(m,v)\big)\cdot(p,w). && (1.2.32)
\end{aligned}$$

Unit condition: Let (n,u) be a word. Then we get

$$\begin{aligned}
(n,u)\cdot(0,I_{\mathcal{W}r}) &= (n,u\cdot I_{\mathcal{W}r}) && (1.2.32)\\
&= (n,u). && (1.2.14)
\end{aligned}$$

Similarly,

$$\begin{aligned}
(0,I_{\mathcal{W}r})\cdot(n,u) &= (n,I_{\mathcal{W}r}\cdot u) && (1.2.32)\\
&= (n,u). && (1.2.14)
\end{aligned}$$

◇

lem: Mon word left cancellative

Lemma 1.2.33. *The monoid of words with alphabet in S is a left cancellative monoid. That is, for $x,y,z\in \mathcal{M}on\langle S\rangle$,*

{eqn: Mon word left cancellative}

$$x\cdot y = x\cdot z \qquad \textit{implies} \qquad y=z. \tag{1.2.34}$$

Proof. Let $x=(n,u)$, $y=(m,v)$, and $z=(p,w)$ be words. Suppose we have

$$x\cdot y = x\cdot z.$$

From (1.2.32) we get

$$n+m=n+p \qquad \text{and} \qquad u\cdot v = u\cdot w.$$

Thus, we have $m=p$ and

$$\begin{aligned}
y=(m,v) &= \big(m,(u\cdot v)_{(n,n+m]}\big) && (1.2.26)\\
&= \big(p,(u\cdot w)_{(n,n+p]}\big) && \text{from above}\\
&= (p,w)=z. && (1.2.26)
\end{aligned}$$

□

lem: Mon word right cancellative

Lemma 1.2.35. *The monoid of words with alphabet in S is a right cancellative monoid. That is, for $x,y,z\in \mathcal{M}on\langle S\rangle$ we get*

{eqn: Mon word right cancellative}

$$y\cdot x = z\cdot x \qquad \textit{implies} \qquad y=z. \tag{1.2.36}$$

Proof. Let $x=(n,u)$, $y=(m,v)$, and $z=(p,w)$ be words. Suppose we have

$$y\cdot x = z\cdot x.$$

From (1.2.32) we get

$$m+n=p+n \qquad \text{and} \qquad v\cdot u = w\cdot u.$$

Thus, we have $m=p$ and

$$\begin{aligned}
y=(m,v) &= \big(m,(v\cdot u)_{[m]}\big) && (1.2.24)\\
&= \big(p,(w\cdot u)_{[p]}\big) && \text{from above}\\
&= (p,w)=z. && (1.2.24)
\end{aligned}$$

□

defn: Word basis

Definition 1.2.37. Define a subset, $\mathtt{S}$, of $\mathcal{M}on\langle S\rangle$ as follows:

{eqn: Word basis}

$$\mathtt{S} := \{(n,u) \in \mathcal{M}on\langle S\rangle \mid n = 1\}. \tag{1.2.38}$$ ◇

prop: Word basis bijection S

Proposition 1.2.39. *The subset* $\mathtt{S}$ *is in bijection with* S.

Proof. Define a function $f : S \longrightarrow \mathtt{S}$ as follows:

$$f(a) := (1, u_a) \in \mathtt{S}$$

where $u_a : [1] \longrightarrow S$ is given by $u(1) = a$. The function $f : S \longrightarrow \mathtt{S}$ is injective since for $a, b \in S$ with $f(a) = f(b)$ we get

$$(1, u_a) = (1, u_b).$$

Therefore, we get $u_a = u_b$. It follows that

$$a = u_a(1) = u_b(1) = b.$$

We will show that the function $f : S \longrightarrow \mathtt{S}$ is surjective. Let $(1,u) \in \mathtt{S}$. Let $a = u[1]$, then we get $u = u_a$ since $u_a[1] = a = u[a]$. Consequently, we get

$$f(a) = (1, u_a) = (1, u).$$ □

thm: Word basis

Theorem 1.2.40. *The subset* $\mathtt{S} \subseteq \mathcal{M}on\langle S\rangle$ *is a monoid basis.*

Proof. First, we will show that $\mathtt{S}$ is a generating set with respect to the multiplication. Let $x = (n,u) \in M^+$ be a non-unit word. We will show, using induction on n, that x has a $\mathtt{S}$-factorization.

Base case ($n = 1$): In this case, we get $x \in \mathtt{S}$. Thus, it has the trivial $\mathtt{S}$-factorization.

Induction case ($n > 1$): Let $x_1 = (1, u_{[1]})$ and $y = (n-1, u_{(1,n]})$. Observe that

$$\begin{aligned}
x_1 \cdot y &= (1, u_{[1]}) \cdot (n-1, u_{(1,n]}) \\
&= (n, u_{[1]} \cdot u_{(1,n]}) && (1.2.32) \\
&= (n, u) && (1.2.28) \\
&= x.
\end{aligned}$$

From the induction hypothesis, we get an $\mathtt{S}$-factorization of y. That is, we get $x_2, \ldots, x_m \in \mathtt{S}$ such that $y = x_2 \cdots x_m$. Since $x_1 = (1, u_{[1]})$ we have $x_1 \in \mathtt{S}$. Moreover, we get

$$x = x_1 \cdot y = x_1 \cdots x_m.$$

Thus, $(x_1, \ldots, x_m)$ forms an $\mathtt{S}$-factorization of x.

Next, we will show that $\mathtt{S}$ is independent with respect to the multiplication. Let $x = (n,u) \in \langle \mathtt{S}; \bullet\rangle$. We will show, using induction on n, that x has at most one $\mathtt{S}$-factorization. Note that, from the Definition 1.2.37 we conclude that $n \geq 1$.

Base case ($n = 1$): We get $x \in \mathtt{S}$. Thus, x has the trivial $\mathtt{S}$-factorization. Let $x = x_1 \cdots x_m$ be another $\mathtt{S}$-factorization of x. That is, we have $m \geq 1$ and $x_i \in \mathtt{S}$ for $1 \leq i \leq m$. Since $x_i \in X$, we get $\hat{l}(x_i) = 1$. Thus,

$$1 = \hat{l}(x) = \hat{l}(x_1 \cdots x_m) = \hat{l}(x_1) + \cdots + \hat{l}(x_m) = m.$$

Therefore, $x = x_1$ is the only $\mathtt{S}$-factorization of x.

Induction case ($n > 1$): Let

$$x = x_1 \cdots x_m = y_1 \cdots y_p$$

be two S-factorizations of x. That is, we have $x_i, y_j \in \mathtt{S}$ for $1 \le i \le m$ and $1 \le j \le p$. In particular, we have $x_1 = (1, u_1)$ and $y_1 = (1, v_1)$ for some 1-letter words u_1 and v_1. Let $x_R = x_2 \cdots x_m$ and $y_R = y_2 \cdots y_p$. Then, we get

$$x = x_1 \cdot x_R = y_1 \cdot y_R.$$

Observe that

$$\begin{aligned} x_1 &= (x_1 \cdot x_R)_{[1]} && (1.2.20) \\ &= x_{[1]} \\ &= (y_1 \cdot y_R)_{[1]} \\ &= y_1. && (1.2.20) \end{aligned}$$

Since $\mathcal{Mon}\langle S\rangle$ is a left cancellative monoid (Lemma 1.2.33), we get

$$x_R = x_2 \cdots x_m = y_2 \cdots y_p = y_R.$$

We have $\hat{l}(x_R) = n - 1 < n$. Also, we get two $\mathtt{S}$-factorizations of $x_R = y_R$. From the induction hypothesis, we get $m = p$ and $x_i = y_i$ for $2 \le i \le m$. This, along with the fact that $x_1 = y_1$, we conclude that x has at most one $\mathtt{S}$-factorization. $\square$

CHAPTER 2

Introduction to Dashed Monoids

section: DMon

Definitions: Dashed monoids, dashed monoid homomorphism, up of free dashed monoid, map from free dashed monoid to free group?

2.1. Definition of a dashed-monoid

defn: DMon

Definition 2.1.1. A **dashed monoid** has the following data:

- A monoid,
$$(M,\cdot,I),$$
called the *underlying monoid*.
- A unary set map,
$$D : M \longrightarrow M,$$
called the *dash map*.

These satisfy the following *unit condition*:

{eqn: DMon unit}

$$D(I) = I. \tag{2.1.2}$$ ◇

Remark 2.1.3. A dashed monoid M is said to be a *monoid with a distributive dash* if it satisfies the distributive property: $(x \bullet y)' = y' \bullet x'$ for all $x, y \in M$. Furthermore, M is called a *monoid with convolution* if it satisfies both the distributive property and the convolution property: $x'' = x$ for all $x \in M$. Finally, a dashed monoid M is a *group* if it satisfies the full inverse property: $x' \bullet x = I = x \bullet x'$ for all $x \in M$. ◇

convention:Dashed monoid notation

Convention 2.1.4. Some common notations for $D(m)$ are m', $m^\dagger$, or $m^{(1)}$ for an element $m \in M$. Additionally, for a natural number $n \in \mathbb{N}$, we denote $D^n(m)$ as $m^{(n)}$, where $m^{(0)} = m$. ◇

Dashed monoids closely resemble monoids, as the additional dash structure does not interact with the underlying monoid structure, except for the unit-dash condition. This is an intentional design choice since we wish to understand categorical groups using the framework of dashed monoids.

Example 2.1.5. Following are some examples of dashed monoids.

i. We can give dashed monoid structure to a monoid by simply setting the dash map to be the identity map. Let $(M,\cdot,I)$ be a monoid. Then $(M,\cdot,I,\mathrm{Id}_M)$ is a dashed monoid.

ii. A more natural approach to endow a group with a dashed monoid structure is using the inherent inverse map as the dash map. Let $(G,\cdot,e,(-)^{-1})$ be a group then $(G,\cdot,e,(-)^{-1})$ is a dashed monoid.

iii. The natural numbers form a dashed monoid with identity as a dashed map, $(\mathbb{N},+,0,\mathrm{Id}_{\mathbb{N}})$ is a dashed monoid. By convention, unless specified otherwise, the above structure will be the dashed monoid structure on the natural numbers.

iv. The integers form a dashed monoid with identity as a dashed map, $(\mathbb{Z},+,0,\mathrm{Id}_{\mathbb{Z}})$ is a dashed monoid. By convention, unless specified otherwise, the above structure will be the dashed monoid structure on the integers.

v. Let $(M,\cdot,I,D_M)$ and $(N,*,J,D_N)$ be dashed monoids. Then $(M\times N,\cdot\times *,(I,J),D_M\times D_N)$ is also a dashed monoid where $\cdot\times * : (M\times N)\times(M\times N)\longrightarrow (M\times N)$ is given by

$$\big((m_1,n_1),(m_2,n_2)\big)\longmapsto (m_1\cdot m_2,\ n_1 * n_2)$$

and $D_M\times D_N : M\times N \longrightarrow M\times N$ is given by

$$(m,n)\longmapsto (D_M(m),D_N(n)).$$

Since Cartesian product of monoids is again a monoid, $(M\times N,\cdot\times *,(I,J))$ is a monoid. It remains to check that $D_M\times D_N\ (I,J)=(I,J)$. Indeed, since

$$D_M\times D_N\ (I,J)=(D_M(I),D_N(J))=(I,J).$$ ◇

defn: DMon morph

Definition 2.1.6. Let $M=(M,\cdot,I,(-)')$ and $N=(N,*,J,(-)^{\dagger})$ be two dashed monoids. A *morphism of dashed monoids* from M to N has the following data:

- A monoid homomorphism,
$$f : M \longrightarrow N,$$
on the underlying monoidal categories

that satisfies the following dash condition: For every $m\in M$,

{eqn: DMon morph dash}

$$(f(m))^{\dagger}=f(m'). \tag{2.1.7}$$ ◇

defn: DMon morph id

Definition 2.1.8. Let $M=(M,\cdot,I,(-)')$ be a dashed monoid. Define the identity dashed monoid homomorphism, $\mathrm{Id}_M : M\longrightarrow M$ as follows:

- Let $\mathrm{Id}_M : M\longrightarrow M$ be the underlying monoid homomorphism.

The above assignment satisfies the dash condition since the equality

$$\mathrm{Id}_M(m)'=m'=\mathrm{Id}_M(m')$$

holds for every $m\in M$. ◇

defn: DMon morph comp

Definition 2.1.9. Let

$$M=(M,\cdot,I,(-)'),\qquad N=(N,*,J,(-)^{\dagger}),\qquad \text{and}\qquad P=(P,\bullet,K,(-)^{(1)})$$

be dashed monoids, and

$$f : M\longrightarrow N,\qquad \text{and}\qquad g : N\longrightarrow P$$

be dashed monoid homomorphisms. Define the composite dashed monoid homomorphism $g\circ f : M\longrightarrow P$ as follows:

- Let $g\circ f$ be the underlying monoid homomorphism.

The above monoid homomorphism satisfies the dash condition since the chain of equalities

$$\begin{aligned}
\left(g\circ f\ (m)\right)^{(1)} &= \left(g(f(m))\right)^{(1)} \\
&= g\left((f(m))^{\dagger}\right) && (2.1.7) \\
&= g\left(f(m')\right) && (2.1.7) \\
&= g\circ f\ (m')
\end{aligned}$$

hold for every $m \in M$. ⋄

defn: Cat DMon

Definition 2.1.10. The **category of dashed monoids**, denoted $\mathcal{DMon}$, is defined as follows:

- The dashed monoids as in Definition 2.1.1 form the class of objects.
- The dashed monoid homomorphisms as in Definition 2.1.6 form the morphisms.
- For a dashed monoid M, the identity homomorphism Id_M, is the identity dashed-monoid homomorphism as in Definition 2.1.8.
- The composition of dashed monoid homomorphism is as in Definition 2.1.9.

Since the composition in dashed monoids is induced from that in monoids, the associativity, the left identity and the right identity conditions are satisfied.⋄

Now we will state the universal property of a free dashed monoid. It is precisely the universal property of a free object (**??**) in the category of dashed monoids over the forgetful functor $U : \mathcal{DMon} \longrightarrow \mathcal{Set}$.

defn: DMon freeDMon UP

Definition 2.1.11. Let S be a set. We say that a dashed monoid $(M,\cdot,I,(-)')$ with a set map $i : S \longrightarrow M$ satisfies the **universal property** of a free dashed monoid if the following conditions are satisfied:
Existence: For a dashed monoid N and set map $j : S \longrightarrow N$ there exists a dashed monoid homomorphism $f_j : M \longrightarrow N$ such that

{eqn: DMon freeDMon UP exists}
$$f_j \circ i = j. \tag{2.1.12}$$

Uniqueness: For a dashed monoid N and a pair of dashed monoid homomorphisms $f,g : M \longrightarrow N$ we get

{eqn: DMon freeDMon UP unique}
$$f\circ i = g\circ i \qquad \text{implies} \qquad f = g. \tag{2.1.13}$$ ⋄

Rewrite this We wish to have the following universal property for a free dashed monoid $\mathcal{DMon}\langle S\rangle$ over a set S. Given any dashed monoid M and a set map $f : S \longrightarrow M$ there is a unique dashed monoid homomorphism $\tilde{f} : \mathcal{DMon}\langle S\rangle \longrightarrow M$ such that $\tilde{f}\circ \mathrm{Inc}_S^{\mathcal{DMon}} = f$.

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "\mathrm{Inc}_S^{\mathcal{DMon}}"'] & M. \\
\mathcal{DMon}\langle S\rangle \arrow[ur, "\exists!\tilde{f}"'] &
\end{tikzcd}$$

Where $\mathrm{Inc}_S^{\mathcal{DMon}} : S \longrightarrow \mathcal{DMon}\,\langle S\rangle$ is the inclusion map.

2.2. Properties of dashed monoids

In this section we will discuss some properties of dashed monoids. We will define the subset generated by a subset under the multiplication operation and the dash operation. We will also define the notions of independence and generating sets with respect to the multiplication operation and the dash operation. We will also define the notions of multiplication basis and dash basis. We will prove some properties of these notions.

defn: DMon mul gen X

Definition 2.2.1. Let $X \subseteq M$ be a subset. Define a subset $\langle X;\,\bullet\rangle \subseteq M$ as follows:

{eqn: DMon mul gen X}

$$\langle X;\,\bullet\rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \in \mathbb{N},\ m \geq 1, \text{ and } x_i \in X \text{ for } 1 \leq i \leq m\}. \tag{2.2.2}$$

We call this the subset generated by X under the multiplication operation.

Define a subset $\langle\langle X;\,\bullet\rangle\rangle \subseteq M$ as follows:

{eqn: DMon mul genP X}

$$\langle\langle X;\,\bullet\rangle\rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \in \mathbb{N},\ m \geq 2, \text{ and } x_i \in X \text{ for } 1 \leq i \leq m\}. \tag{2.2.3}$$ ⋄

Remark 2.2.4. It is clear from the definition that

$$\langle\langle X;\,\bullet\rangle\rangle \subseteq \langle X;\,\bullet\rangle.$$ ⋄

defn: DMon dash gen X

Definition 2.2.5. Let M be a dashed monoid and $X \subseteq M$ be a subset. Define a subset $\left\langle X;\,(-)'\right\rangle \subseteq M$ as follows:

{eqn: DMon dash gen X}

$$\left\langle X;(-)'\right\rangle := \{x^{(k)} \in M \mid x \in X \ \text{ and } \ k \in \mathbb{N}\}. \tag{2.2.6}$$

We call this the subset generated by X under the dash operation.

Define a subset $\left\langle\left\langle X;\,(-)'\right\rangle\right\rangle \subseteq M$ as follows:

{eqn: DMon dash genP X}

$$\left\langle\left\langle X;(-)'\right\rangle\right\rangle := \{x^{(k)} \in M \mid x \in X \ \text{ and } \ k \in \mathbb{N}^{+}\}. \tag{2.2.7}$$ ⋄

Remark 2.2.8. It is clear from the definition that

$$\left\langle\left\langle X;(-)'\right\rangle\right\rangle \subseteq \left\langle X;(-)'\right\rangle.$$ ⋄

notation: DMon P

Notation 2.2.9. We denote the set of all dashed-monoid elements without the unit by M^{+}. That is,

{eqn: DMon P}

$$M^{+} := M \setminus \{I\}. \tag{2.2.10}$$ ⋄

defn: DMon mul independent

Definition 2.2.11. Let $X \subseteq M$ be a subset. We say X is *independent with respect to the multiplication* if for every $z \in \langle X;\,\bullet\rangle$ we get unique $m \in \mathbb{N}^{+}$ and unique $x_i \in X$ for $1 \leq i \leq m$ such that

$$x_1 \bullet \cdots \bullet x_m = z.$$

In other words, X is independent with respect to the multiplication if for $n, m \in \mathbb{N}^{+}$ and $x_i, y_j \in X$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ we get

{eqn: DMon mul independent}

$$x_1 \bullet \cdots \bullet x_n = y_1 \bullet \cdots \bullet y_m \Rightarrow n = m \text{ and } x_i = y_j. \tag{2.2.12}$$ ⋄

defn: DMon mul genset

Definition 2.2.13. Let $X \subseteq M$ be a subset. We say X is a *generating set with respect to the multiplication* if

{eqn: DMon mul genset}

$$\langle X;\ \bullet \rangle = M^{+}. \tag{2.2.14}$$

In other words, X is a generating set with respect to the multiplication if for every $x \in M^{+}$ we get $m \in \mathbb{N}^{+}$ and $x_i \in X$ for $1 \leq i \leq m$ such that

$$x_1 \bullet \cdots \bullet x_m = x.$$

◇

defn: DMon dash independent

Definition 2.2.15. Let $X \subseteq M$ be a subset. We say X is *independent with respect to the dash operation* if for every $z \in \left\langle X;\ (-)' \right\rangle$ we get unique $r \in \mathbb{N}$ and unique $x \in X$ such that

$$x^{(r)} = z.$$

In other words, X is independent with respect to the dash operation if for $r, k \in \mathbb{N}$ and $x, y \in X$

{eqn: DMon dash independent}

$$x^{(r)} = y^{(k)} \qquad \text{implies} \qquad r = k \text{ and } x = y. \tag{2.2.16}$$

◇

defn: DMon dash genset

Definition 2.2.17. Let $X \subseteq M$ be a subset. We say X is a *generating set with respect to the dash operation* if

{eqn: DMon dash genset}

$$\left\langle X;\ (-)' \right\rangle = M^{+}. \tag{2.2.18}$$

In other words, X is a generating set with respect to the dash operation if for every $z \in M^{+}$ we get $r \in \mathbb{N}$ and $x \in X$ such that

$$x^{(r)} = z.$$

◇

defn: DMon mul basis

Definition 2.2.19. Let $X \subseteq M$ be a subset. We say X is a *multiplication basis* if X is independent with respect to the multiplication and X is a generating set with respect to the multiplication. ◇

defn: DMon dash basis

Definition 2.2.20. Let $X \subseteq M$ be a subset. We say X is a *dash basis* if X is independent with respect to the dash operation and X is a generating set with respect to the dash operation. ◇

lem: DMon mul basis no I

Lemma 2.2.21. *Let $X \subseteq M$ be a subset. Suppose X is a multiplicative basis of M, then $I \notin X$.*

Proof. Suppose for contradiction $I \in X$. We have

$$I \bullet I = I.$$

This gives two different ways to factorize I. This contradicts multiplicatively independent condition of X. □

lem: DMon dash basis no I

Lemma 2.2.22. *Let $X \subseteq M$ be a subset. Suppose X is a dash basis of M, then $I \notin X$.*

Proof. Suppose for contradiction $I \in X$. We have

$$I' = I.$$

This contradicts dash independent condition of X. □

lem: DMon mul independent X cup gen X

Lemma 2.2.23. *Let $X \subseteq M$ be a multiplicatively independent subset. Then, we get*

{eqn: DMon mul independent X cup gen X}

$$\langle X;\ \bullet\rangle = X \sqcup \langle\langle X;\ \bullet\rangle\rangle. \tag{2.2.24}$$

Proof. Let $x \in M$. Suppose $x \in \langle X;\ \bullet\rangle$. Then, we get $m \in \mathbb{N}$ with $m \geq 1$ and $x_i \in X$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

If $m = 1$ then we get

$$x = x_1 \in X.$$

Otherwise, we have $m \geq 2$ and

$$x = x_1 \bullet x_2 \bullet \cdots \bullet x_m \in \langle\langle X;\ \bullet\rangle\rangle.$$

On the other hand, suppose $x \in X \sqcup \langle\langle X;\ \bullet\rangle\rangle$. If $x \in X$, then we get

$$x \in \langle X;\ \bullet\rangle.$$

Otherwise, we have

$$x \in \langle\langle X;\ \bullet\rangle\rangle \subseteq \langle X;\ \bullet\rangle.$$

Now we will show that

$$X \cap \langle\langle X;\ \bullet\rangle\rangle = \varnothing.$$

Suppose $x \in X \cap \langle\langle X;\ \bullet\rangle\rangle$. Then we have $x \in X$ and $x \in \langle\langle X;\ \bullet\rangle\rangle$. Therefore, we get $m \in \mathbb{N}$ with $m \geq 2$ and $x_i \in X$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet x_2 \bullet \cdots \bullet x_m.$$

This leads to a contradiction since X is independent with respect to the multiplication. Thus, we get

$$X \cap \langle\langle X;\ \bullet\rangle\rangle = \varnothing$$

as required. $\square$

lem: DMon dash independent X cup gen X

Lemma 2.2.25. *Let $X \subseteq M$ be a dash independent subset. Then, we get*

{eqn: DMon dash independent X cup gen X}

$$\left\langle X;\ (-)'\right\rangle = X \sqcup \left\langle\left\langle X;\ (-)'\right\rangle\right\rangle. \tag{2.2.26}$$

Proof. Let $x \in M$. Suppose $x \in \left\langle X;\ (-)'\right\rangle$. Then, we get $k \in \mathbb{N}$ and $y \in X$ such that

$$x = y^{(k)}.$$

If $k = 0$ then we get

$$x = y^{(0)} = y \in X.$$

Otherwise, we get $k \geq 1$ and

$$x = y^{(k)} \in \left\langle\left\langle X;\ (-)'\right\rangle\right\rangle.$$

On the other hand, suppose $x \in X \sqcup \left\langle\left\langle X;\ (-)'\right\rangle\right\rangle$. If $x \in X$, then we get

$$x = x^{(0)} \in \left\langle X;\ (-)'\right\rangle.$$

Otherwise, we have

$$x \in \left\langle\left\langle X;\ (-)'\right\rangle\right\rangle \subseteq \left\langle X;\ (-)'\right\rangle.$$

Now we will show that

$$X \cap \langle\langle X;\, (-)' \rangle\rangle = \varnothing.$$

Suppose $x \in X \cap \langle\langle X;\, (-)' \rangle\rangle$. Then we have $x \in X$ and $x \in \langle\langle X;\, (-)' \rangle\rangle$. Therefore, we get $y \in X$ and $k \in \mathbb{N}$ with $k \geq 1$ such that $x = y^{(k)}$. Thus, we have

$$x^{(0)} = x = y^{(k)}.$$

This leads to a constradiction since X is independent with respect to the dash operation. Thus, we get

$$X \cap \langle\langle X;\, (-)' \rangle\rangle = \varnothing$$

as required. □

lem: DMon mul independent sub **Lemma 2.2.27.** *Let $A \subseteq X \subseteq M$ be subsets of M. If X is a multiplicatively independent set then so is A.*

Proof. Suppose X is a multiplicatively independent set. We wish to show that $A \subseteq X$ is a multiplicatively independent set. Let $n, m \geq 1$ and $x_i, y_j \in A$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ such that

$$x_1 \bullet \cdots \bullet x_n = y_1 \bullet \cdots \bullet y_m.$$

Since X is independent with respect to the multiplication and $x_i, y_j \in A \subseteq X$, we get $n = m$ and $x_i = y_j$. This shows that A is independent with respect to the multiplication. □

defn: DMon mul Egen X **Definition 2.2.28.** Let $X, Y \subseteq M$. Define a subset $e\langle X, Y;\, \bullet \rangle \subseteq M$ as follows:

{eqn: DMon mul Egen X} (2.2.29)

$$e\langle X, Y;\, \bullet \rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \geq 1,\ x_i \in X \cup Y, \text{ and } \exists\, 1 \leq j \leq m \text{ such that } x_j \in X\}.$$

We will call this subset the *expanded instance* of X under the multiplication operation.

Define a subset $e\langle\langle X, Y;\, \bullet \rangle\rangle \subseteq M$ as follows:

{eqn: DMon mul EgenP X} (2.2.30)

$$e\langle\langle X, Y;\, \bullet \rangle\rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \geq 2,\ x_i \in X \cup Y, \text{ and } \exists\, 1 \leq j \leq m \text{ such that } x_j \in X\}. \diamond$$

lem: DMon mul independent disjoint **Lemma 2.2.31.** *Let $X \subseteq M$ be a multiplicatively independent subset. Let $A, B \subseteq X$ be disjoint subsets of X. Then, we get*

{eqn: DMon mul independent disjoint} (2.2.32)
$$\langle A \sqcup B;\, \bullet \rangle = \langle A;\, \bullet \rangle \bigsqcup e\langle B, A;\, \bullet \rangle$$

and

{eqn: DMon mul independent disjoint pos} (2.2.33)
$$\langle\langle A \sqcup B;\, \bullet \rangle\rangle = \langle\langle A;\, \bullet \rangle\rangle \bigsqcup e\langle\langle B, A;\, \bullet \rangle\rangle$$

Proof. Let $x \in M$. Suppose $x \in \langle A \sqcup B;\, \bullet \rangle$, then we get $m \geq 1$ and $x_i \in A \sqcup B$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

If for every $1 \leq i \leq m$, we have $x_i \in A$ then we get

$$x = x_1 \bullet \cdots \bullet x_m \in \langle A;\, \bullet \rangle.$$

Otherwise, we get at least one $1 \le j \le m$ such that $x_j \in B$. Thus, we have

$$x = x_1 \bullet \cdots \bullet x_m \in e\langle B, A;\ \bullet\rangle.$$

On the other hand, suppose $x \in \langle A;\ \bullet\rangle \sqcup e\langle B, A;\ \bullet\rangle$. If $x \in \langle A;\ \bullet\rangle$ then we get some $m \ge 1$ and $x_i \in A$ for $1 \le i \le m$ such that

$$x = x_1 \bullet \cdots x_m.$$

Therefore, we have

$$x = x_1 \bullet \cdots \bullet x_m \in \langle A \sqcup B;\ \bullet\rangle.$$

Otherwise, if $x \in e\left\langle B, A;\ (-)'\right\rangle$ then we get some $m \ge 1$ and $x_i \in A \cup B$ with at least one $x_j \in B$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

Therefore, we have

$$x = x_1 \bullet \cdots \bullet x_m \in \langle A \sqcup B;\ \bullet\rangle.$$

Now we will show that $\langle A;\ \bullet\rangle$ and $e\langle B, A;\ \bullet\rangle$ are disjoint. Suppose there is $x \in M$ such that

$$x \in \langle A;\ \bullet\rangle \cap e\langle B, A;\ \bullet\rangle.$$

We get $n, m \ge 1$, $x_i \in A$ for $1 \le i \le n$, and $y_j \in A \cup B$ for $1 \le j \le m$ with at least one $y_k \in B$ such that

$$x_1 \bullet \cdots \bullet x_n = x = y_1 \bullet \cdots \bullet x_m.$$

Becasue X is a dash independent set and $x_i, y_j \in A \sqcup B \subseteq X$, we get $n = m$ and $x_i = y_i$. In particular, we get $x_k = y_k \in B$. We already have $x_k \in A$. This is a contradiction since A and B are disjoint. Thus, we get

$$\langle A \sqcup B;\ \bullet\rangle = \langle A;\ \bullet\rangle \bigsqcup e\langle B, A;\ \bullet\rangle.$$

The proof of equation (2.2.33) is similar to the proof of equation (2.2.32).□

lem: DMon dash independent sub**Lemma 2.2.34.** *Let $A \subseteq X \subseteq M$ be subsets of M. If x is a dash independent subset then so is A.*

Proof. Suppose X is a dash independent set. We wish to show that $A \subseteq X$ is a dash independent set. Let $n, m \in \mathbb{N}$ and $a, b \in A$ such that

$$a^{(n)} = b^{(m)}.$$

Since X is independent with respect to the dash-operation and $a, b \in X$, we get $n = m$ and $a = b$. This shows that A is independent with respect to the dash-operation as well. □

lem: DMon dash independent disj**Lemma 2.2.35.** *Let $X \subseteq M$ be a dash independent subset. Let $A, B \subseteq X$ be disjoint subsets of X. Then, we get*

{eqn: DMon dash independent disjoint}

$$\left\langle A \sqcup B;\ (-)'\right\rangle = \left\langle A;\ (-)'\right\rangle \bigsqcup \left\langle B;\ (-)'\right\rangle \tag{2.2.36}$$

and

{eqn: DMon dash independent disjoint pos}

$$\left\langle\left\langle A \sqcup B;\ (-)'\right\rangle\right\rangle = \left\langle\left\langle A;\ (-)'\right\rangle\right\rangle \bigsqcup \left\langle\left\langle B;\ (-)'\right\rangle\right\rangle \tag{2.2.37}$$

Proof. Let $x \in M$. Suppose $x \in \langle A \sqcup B;\, (-)' \rangle$. Then, we get $r \in \mathbb{N}$ and $y \in A \sqcup B$ such that

$$x = y^{(r)}.$$

If $y \in A$ then we get

$$x = y^{(k)} \in \langle\langle A;\, (-)' \rangle\rangle$$

and if $y \in B$ we get

$$x = y^{(k)} \in \langle\langle B;\, (-)' \rangle\rangle.$$

On the other hand, suppose $x \in \langle A;\, (-)' \rangle \sqcup \langle B;\, (-)' \rangle$. If $x \in \langle A;\, (-)' \rangle$ then we get some $r \in \mathbb{N}$ and $y \in A$ such that

$$x = y^{(k)}.$$

Therefore, we have

$$x = y^{(k)} \in \langle A \sqcup B;\, (-)' \rangle.$$

If $x \in \langle B;\, (-)' \rangle$ then we get some $r \in \mathbb{N}$ and $y \in B$ such that

$$x = y^{(k)}.$$

Therefore, we have

$$x = y^{(k)} \in \langle A \sqcup B;\, (-)' \rangle.$$

Now we will show that $\langle\langle A;\, (-)' \rangle\rangle$ and $\langle\langle B;\, (-)' \rangle\rangle$ are disjoint. Suppose there is $x \in M$ such that

$$x \in \langle\langle A;\, (-)' \rangle\rangle \cap \langle\langle B;\, (-)' \rangle\rangle.$$

We get some $r, k \in \mathbb{N}$, $y \in A$, and $z \in B$ such that

$$y^{(k)} = x = z^{(r)}.$$

Since X is a dash independent set and $y, z \in A \sqcup B \subseteq X$ we get $k = r$ and $y = z$. This is a contradiction since A and B are disjoint. Thus, we get

$$\langle A \sqcup B;\, (-)' \rangle = \langle A;\, (-)' \rangle \bigsqcup \langle B;\, (-)' \rangle.$$

The proof of equation (2.2.37) is similar to the proof of equation (2.2.36).□

2.3. A criterion for a free dashed monoid

In this section we will discuss a criterion for a dashed monoid to be a free dashed monoid.

defn: DMon freeDMon like

Definition 2.3.1. We say that the dashed-monoid M has a *free dashed-monoid like structure* if M has

- a multiplicative basis $\mathtt{G}$,
- a dash basis $\mathtt{H}$,
- a set $\mathtt{S}$, and
- a dashed-monoid homomorphism $\hat{l} : M \longrightarrow \mathbb{N}$

that satisfies the following conditions:

Length function: For $x \in M$ we have

$$\hat{l}(x) = 0 \qquad \Longleftrightarrow \qquad x = I \tag{2.3.2}$$

and for $a \in \mathtt{S}$ we have

$$\hat{l}(a) = 1. \tag{2.3.3}$$

Interlocking condition: The equalities

$$\mathtt{G} = \mathtt{S} \sqcup \big\langle\big\langle \mathtt{H};\, (-)' \big\rangle\big\rangle \qquad \text{and} \tag{2.3.4}$$

$$\mathtt{H} = \mathtt{S} \sqcup \langle\langle \mathtt{G};\, \bullet \rangle\rangle \tag{2.3.5}$$

hold. ◇

Definition 2.3.6. Let M be a dashed-monoid which has a free dashed-monoid like structure as in Definition 2.3.1. We define $\mathtt{R} \subseteq M$ to be the following subset:

$$\mathtt{R} := \langle\langle \mathtt{G};\, \bullet \rangle\rangle\,. \tag{2.3.7}$$

Note that, since $\mathtt{G}$ is a multiplicative basis, for every $x \in \mathtt{R}$ we get unique $m \geq 2$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

◇

Lemma 2.3.8. *Let M be a dashed-monoid, with a free dashed-monoid like structure as in Definition 2.3.1. Then, we get*

$$\mathtt{G} = \big\langle \mathtt{S};\, (-)' \big\rangle \quad \bigsqcup \quad \big\langle\big\langle \mathtt{R};\, (-)' \big\rangle\big\rangle\,. \tag{2.3.9}$$

Proof. From Definition 2.3.1, we get that $\mathtt{H}$ is a dash basis, and consequently, $\mathtt{H}$ is a dash independent set. We have that $\mathtt{S}, \mathtt{R} \subseteq \mathtt{H}$ are disjoint subsets of $\mathtt{H}$. Thus, applying Lemma 2.2.35 we get

$$\big\langle\big\langle \mathtt{S} \sqcup \mathtt{R};\, (-)' \big\rangle\big\rangle = \big\langle\big\langle \mathtt{S};\, (-)' \big\rangle\big\rangle \sqcup \big\langle\big\langle \mathtt{R};\, (-)' \big\rangle\big\rangle\,.$$

We note that $\mathtt{S}$ is a dash-independent set because it is a subset of the dash-independent set $\mathtt{H}$, as established by Lemma 2.2.34. Applying Lemma 2.2.25, we get

$$\big\langle \mathtt{S};\, (-)' \big\rangle = \mathtt{S} \sqcup \big\langle\big\langle \mathtt{S};\, (-)' \big\rangle\big\rangle\,.$$

We will finish the proof with the following direct calculation:

$$\begin{aligned}
\mathtt{G} &= \mathtt{S} \sqcup \big\langle\big\langle \mathtt{H};\, (-)' \big\rangle\big\rangle && (2.3.4)\\
&= \mathtt{S} \sqcup \big\langle\big\langle \mathtt{S} \sqcup \mathtt{R};\, (-)' \big\rangle\big\rangle && (2.3.5) \text{ and } (2.3.7)\\
&= \mathtt{S} \sqcup \big\langle\big\langle \mathtt{S};\, (-)' \big\rangle\big\rangle \sqcup \big\langle\big\langle \mathtt{R};\, (-)' \big\rangle\big\rangle && (2.2.36)\\
&= \big\langle \mathtt{S};\, (-)' \big\rangle \sqcup \big\langle\big\langle \mathtt{R};\, (-)' \big\rangle\big\rangle\,. && (2.2.26)
\end{aligned}$$

□

Lemma 2.3.10. *Let M be a dashed-monoid, with a free dashed-monoid like structure as in Definition 2.3.1. Let $x \in \mathtt{G}$. Then exactly one of the following holds:*

- *We have*
$$x = a^{(k)}$$
for some unique $k \geq 0$ and $a \in \mathtt{S}$.
- *We have*
$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}$$
for some unique $m \geq 2$, $k \geq 1$ and $x_i \in \mathtt{G}$. In this case we also get $\hat{l}(x_i) < \hat{l}(x)$.

Proof. From Lemma 2.3.8 we get either
$$x \in \langle \mathtt{S};\ (-)' \rangle \qquad \text{or} \qquad x \in \langle \langle \mathtt{R};\ (-)' \rangle \rangle .$$
If $x \in \langle \mathtt{S};\ (-)' \rangle$ then we get $a \in \mathtt{S}$ and $k \in \mathbb{N}$ such that
$$x = a^{(k)}.$$
From Definition 2.3.1 we get $\mathtt{S} \subseteq \mathtt{H}$. Since $\mathtt{H}$ is a dash-basis of M, from Lemma 2.2.34 we get that $\mathtt{S}$ is a dash independent subset of M. Thus, the choice of a and k is unique.

On the other hand, if $x \in \langle \langle \mathtt{R};\ (-)' \rangle \rangle$ we get $k \geq 1$ and $z \in \mathtt{R}$ such that
$$x = z^{(k)}.$$
From the interlocking condition (2.3.5), we get that
$$\mathtt{R} = \langle \langle \mathtt{G};\ \bullet \rangle \rangle \subseteq \mathtt{H}.$$
From Lemma 2.2.34 we get that $\mathtt{R}$ is a dash independent set. Thus, the choice of k and z is unique. Since $\mathtt{G}$ is a multiplicative basis, we get unique $m \geq 2$ and $x_i \in \mathtt{G}$ such that
$$z = x_1 \bullet \cdots \bullet x_m.$$
In conclusion, we get unique $m \geq 2$, $k \geq 1$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that
$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}.$$
Since the length function $\hat{l} : M \longrightarrow \mathbb{N}$ is a dashed-monoid homomorphism, we get
$$\hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i).$$
From the length condition (2.3.2), we get $\hat{l}(x_i) \geq 1$. Since $m \geq 2$, we get $\hat{l}(x_i) < \hat{l}(x)$. □

lem: DMon G dash closed

Lemma 2.3.11. *Let M be a dashed-monoid, with a free dashed-monoid like structure as in Definition 2.3.1. Then, the set $\mathtt{G} \subseteq M$ is closed under taking dash.*

Proof. Suppose $x \in \mathtt{G}$. We want to show that $x' \in \mathtt{G}$. From Lemma 2.3.8 we get
$$\mathtt{G} = \langle \mathtt{S};\ (-)' \rangle \sqcup \langle \langle \mathtt{R};\ (-)' \rangle \rangle .$$
If $x \in \langle \mathtt{S};\ (-)' \rangle$ then we get some $k \in \mathbb{N}$ and $y \in \mathtt{S}$ such that
$$x = y^{(k)}.$$

Thus, we get

$$x' = y^{(k+1)} \in \langle \mathtt{S};\, (-)' \rangle \subseteq \mathtt{G}.$$

Similarly, if $x \in \langle\langle \mathtt{R};\, (-)' \rangle\rangle$ then we get some $k \in \mathbb{N}$ with $k \geq 1$ and $y \in \mathtt{R}$ such that

$$x = y^{(k)}.$$

Thus, we get

$$x' = y^{(k+1)} \in \langle\langle \mathtt{R};\, (-)' \rangle\rangle \subseteq \mathtt{G}.$$

□

For the remainder of this section, let M be a dashed-monoid. Assume that M has a free dashed-monoid like structure as in Definition 2.3.1. Therefore, we get a dashed-monoid monoid homomorphism

$$\hat{l} : M \longrightarrow \mathbb{N}$$

that satisfies the length conditions (2.3.2) and (2.3.3). We also get a multiplicative basis $\mathtt{G}$, a dash basis $\mathtt{H}$, and a set $\mathtt{S}$ such that the equations (2.3.4) and (2.3.5) hold. Let N be a dashed-monoid and $u : \mathtt{S} \longrightarrow N$ be a function. Our aim is to extend u to a unique dashed-monoid homomorphism $\overline{u} : M \longrightarrow N$.

lem: DMon freeDMon count 1 and G

Lemma 2.3.12. *Let $x \in M$. Suppose $x \in \mathtt{G}$. Then, $x \in \langle \mathtt{S};\, (-)' \rangle$ if and only if $\hat{l}(x) = 1$.*

Proof. Suppose $x \in \mathtt{G}$. We have

$$\mathtt{G} = \langle \mathtt{S};\, (-)' \rangle \sqcup \langle\langle \mathtt{R};\, (-)' \rangle\rangle. \qquad (2.3.9)$$

Suppose $x \in \langle \mathtt{S};\, (-)' \rangle$. Then, we get $k \in \mathbb{N}$ and $a \in \mathtt{S}$ such that

$$x = a^{(k)}.$$

Since $a \in \mathtt{S}$, we have $\hat{l}(a) = 1$. Thus, we get

$$\hat{l}(x) = \hat{l}\left(a^{(k)}\right) = (\hat{l}(a))^{(k)} = 1^{(k)} = 1.$$

For the other direction, we will show that if $x \notin \langle incS;\, (-)' \rangle$ then $\hat{l}(x) \neq 1$. Suppose $x \notin \langle incS;\, (-)' \rangle$ then from Lemma 2.3.8 we get $x \in \langle\langle \mathtt{R};\, (-)' \rangle\rangle$. Therefore, we get $m \geq 2$, $k \geq 1$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = (x_1 \bullet x_2 \bullet \cdots \bullet x_m)^{(k)}.$$

Since $x_i \in \mathtt{G}$, from Lemma 2.2.21 we get $x_i \neq I$. Therefore, from (2.3.2) we get

$$\hat{l}(x_i) \geq 1.$$

We have

$$\hat{l}(x) = \hat{l}\left((x_1 \bullet x_2 \bullet \cdots \bullet x_m)^{(k)}\right) = \sum_{i=1}^{m} \hat{l}(x_i).$$

Since $m \geq 2$ and $\hat{l}(x_i) \geq 1$ for $1 \leq i \leq m$, we get $\hat{l}(x) \geq 2$. Thus, we have shown that if $x \notin \langle \mathtt{S};\, (-)' \rangle$ then $\hat{l}(x) \neq 1$. □

construction: DMon freeDMon fG

Construction 2.3.13. We will extend the function $u : \mathtt{S} \longrightarrow N$ to a function

$$u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$$

as follows:

Since M has a free dashed-monoid like structure, we get a length homomorphism $\hat{l} : M \longrightarrow \mathbb{N}$. Suppose $x \in \mathtt{G}$. We will define $u^{\mathtt{G}}(x) \in N$ using induction on $\hat{l}(x)$. Since $x \in \mathtt{G}$, we have $\hat{l}(x) \geq 1$.

<u>Base case</u> $\left(\hat{l}(x) = 1\right)$: From Lemma 2.3.12 we get $x \in \langle \mathtt{S};\ (-)' \rangle$. Since $\mathtt{S} \subseteq \mathtt{H}$ is a dash-independent set, we get unique $k \in \mathbb{N}$ and unique $a \in \mathtt{S}$ such that

$$x = a^{(r)}.$$

We define

{eqn: DMon freeDMon fG induct S}

$$u^{\mathtt{G}}(x) := u(a)^{(k)}. \tag{2.3.14}$$

<u>Induction case</u> $\left(\hat{l}(x) \geq 2\right)$: From Lemma 2.3.12, we get $x \notin \langle \mathtt{S};\ (-)' \rangle$. Therefore, from Lemma 2.3.8 we get $x \in \langle \langle \mathtt{R};\ (-)' \rangle \rangle$. As a result we get unique $k \in \mathbb{N}$ with $k \geq 1$, $m \geq 2$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = (x_1 \bullet x_2 \bullet \cdots \bullet x_m)^{(k)}.$$

We have

$$\begin{aligned} \hat{l}(x) &= \hat{l}\left((x_1 \bullet x_2 \bullet \cdots \bullet x_m)^{(k)}\right) \\ &= \sum_{i=1}^{m} \hat{l}(x_i). \end{aligned}$$

Since $x_i \in \mathtt{G}$, from Lemma 2.2.21 we get $x_i \neq I$. Therefore, from (2.3.2) we get $\hat{l}(x_i) \geq 1$. Since $m \geq 2$ we conclude $\hat{l}(x_i) < \hat{l}(x)$. Using the induction hypothesis, we define

{eqn: DMon freeDMon fG induct G}

$$u^{\mathtt{G}}(x) := \left(u^{\mathtt{G}}(x_1) \bullet u^{\mathtt{G}}(x_2) \bullet \cdots \bullet u^{\mathtt{G}}(x_m)\right)^{(k)}. \tag{2.3.15}$$ ◇

lem: DMon freeDMon fG on S

Lemma 2.3.16. *For $a \in \mathtt{S}$, the equality*

{eqn: DMon freeDMon fG on S}

$$u^{\mathtt{G}}(a) = u(a). \tag{2.3.17}$$

Proof. Let $a \in \mathtt{S}$. Then, from the length condition (2.3.3), we get $\hat{l}(a) = 1$. It follows that,

$$\begin{aligned} u^{\mathtt{G}}(a) &= (u(a))^{(0)} && (2.3.14) \\ &= u(a). \end{aligned}$$ □

lem: DMon freeDMon fG dash

Lemma 2.3.18. *Let $x \in M$. Suppose $x \in \mathtt{G}$. Note that, from Lemma 2.3.11 we have $x' \in \mathtt{G}$. The equality*

{eqn: DMon freeDMon fG dash}

$$\left(u^{\mathtt{G}}(x)\right)' = u^{\mathtt{G}}\left(x'\right) \tag{2.3.19}$$

holds.

Proof. Suppose $x \in \mathtt{G}$. From Lemma 2.3.8 we get

$$\mathtt{G} = \langle \mathtt{S};\ (-)' \rangle \sqcup \langle \langle \mathtt{R};\ (-)' \rangle \rangle.$$

Thus, we get that either $x \in \langle \mathtt{S};\ (-)' \rangle$ or $x \in \langle \langle \mathtt{R};\ (-)' \rangle \rangle$. We will do a case analysis.

Case I ($x \in \langle \mathtt{S};\ (-)' \rangle$): We get a unique $k \in \mathbb{N}$ and $a \in \mathtt{S}$ such that

$$x = a^{(k)}.$$

Therefore, we get

$$x' = a^{(k+1)}.$$

Observe that,

$$\begin{aligned}
\left(u^{\mathtt{G}}(x)\right)' &= \left(u(a)^{(k)}\right)' && (2.3.14)\\
&= u(a)^{(k+1)}\\
&= u^{\mathtt{G}}\left(x'\right). && (2.3.14)
\end{aligned}$$

Case II ($x \in \langle \langle \mathtt{R};\ (-)' \rangle \rangle$): We get unique $k \geq 1$, $m \geq 2$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = (x_1 \bullet x_2 \bullet \cdots \bullet x_m)^{(k)}.$$

Therefore, we get

$$x' = (x_1 \bullet \cdots \bullet x_m)^{(k+1)}$$

Define $z \in N$ as follows:

$$z := u^{\mathtt{G}}(x_1) \bullet u^{\mathtt{G}}(x_2) \bullet \cdots \bullet u^{\mathtt{G}}(x_m).$$

Observe that

$$\begin{aligned}
\left(u^{\mathtt{G}}(x)\right)' &= \left(z^{(k)}\right)' && (2.3.14)\\
&= z^{(k+1)}\\
&= u^{\mathtt{G}}(x'). && (2.3.14)
\end{aligned}$$

This completes the proof. □

defn: DMon freeDMon fM

Definition 2.3.20. Let $u : \mathtt{S} \longrightarrow N$ be a function and let $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$ be the function as in Construction 2.3.13. Since $\mathtt{G}$ is a multiplicative basis of M, we can extend the function $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$ to a function

$$\overline{u} : M \longrightarrow N.$$

Explicitly, given $x \in M$, if $x = I$ then we define

{eqn: DMon freeDMon fM base}

$$\overline{u}(I) := J. \tag{2.3.21}$$

Otherwise, we have $x \in M^+$. Since $\mathtt{G}$ is a multiplicative basis of M, we get $m \in \mathbb{N}$ with $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

We define

{eqn: DMon freeDMon fM induct}

$$\overline{u} := u^{\mathtt{G}}(x_1) \bullet \cdots \bullet u^{\mathtt{G}}(x_m). \tag{2.3.22}$$

◇

lem: DMon freeDMon fM Mon

Lemma 2.3.23. *The function $\overline{u} : M \longrightarrow N$ as defined in Definition 2.3.20 is a monoid homomorphism.*

Proof. Note that $\mathtt{G}$ is a multiplicative basis of M. Since $\overline{u} : M \longrightarrow N$ defined by multiplicatively extending the function $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$, it follows that $\overline{u}$ is a monoid homomorphism. □

lem: DMon freeDMon fM on G

Lemma 2.3.24. *For $x \in \mathtt{G}$ we get*

{eqn: DMon freeDMon fM on G}

$$\overline{u}(x) = u^{\mathtt{G}}(x). \tag{2.3.25}$$

Proof. Since $\overline{u} : M \longrightarrow N$ is defined by multiplicatively extending the function $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$, we get

$$\overline{u}(x) = u^{\mathtt{G}}(x). \qquad (2.3.22)$$

□

lem: DMon freeDMon fM on S

Lemma 2.3.26. *For $a \in \mathtt{S}$ we get*

{eqn: DMon freeDMon fM on S}

$$\overline{u}(a) = u(a). \tag{2.3.27}$$

Proof. Let $a \in \mathtt{S}$. Then, we get $a \in \mathtt{G}$. Observe that

$$\begin{aligned} \overline{u}(a) &= u^{\mathtt{G}}(a) && (2.3.25) \\ &= u(a). && (2.3.17) \end{aligned}$$

□

lem: DMon freeDMon fM DMon

Lemma 2.3.28. *The function $\overline{u} : M \longrightarrow N$ as defined in Definition 2.3.20 is a dashed monoid homomorphism.*

Proof. Lemma 2.3.23 shows that $\overline{u} : M \longrightarrow N$ is a monoid homomorphism. We will show that $\overline{u} : M \longrightarrow N$ satisfies the dash condition: $\overline{u}(x') = \overline{u}(x)'$ for every $x \in M$. Consider the following calculation:

$$\begin{aligned} M &= \{I\} \sqcup M^{+} && (2.2.10) \\ &= \{I\} \sqcup \langle \mathtt{G};\ \bullet \rangle && (2.2.14) \\ &= \{I\} \sqcup \mathtt{G} \sqcup \mathtt{R}. && (2.2.24) \text{ and } (2.3.7) \end{aligned}$$

Thus, we get either $x = I$, $x \in \mathtt{G}$, or $x \in \mathtt{R}$. We consider these three cases.

Case I ($x = I$): We get

$$\begin{aligned} \overline{u}(I)' &= J' && (2.3.21) \\ &= J \\ &= \overline{u}(I) && (2.3.21) \\ &= \overline{u}(I'). \end{aligned}$$

Case II ($x \in \mathtt{G}$): From Lemma 2.3.11 we get $x' \in \mathtt{G}$. We have

$$\begin{aligned} \overline{u}(x)' &= u^{\mathtt{G}}(x)' && (2.3.25) \\ &= u^{\mathtt{G}}(x') && (2.3.19) \\ &= \overline{u}(x'). && (2.3.25) \end{aligned}$$

Case III: ($x \in \mathtt{R}$): We get $m \geq 2$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

Therefore, we have

$$x' = (x_1 \bullet \cdots \bullet x_m)'.$$

It follows that

$$x' \in \langle\langle \mathtt{R};\ (-)' \rangle\rangle \subseteq \mathtt{G}.$$

Observe,

$$\begin{aligned} \overline{u}(x') &= u^{\mathtt{G}}(x') && (2.3.25) \\ &= \left(u^{\mathtt{G}}(x_1) \bullet \cdots \bullet u^{\mathtt{G}}(x_m)\right)' && (2.3.15) \\ &= (\overline{u}(x))'. && (2.3.22) \end{aligned}$$

Since we have shown $\overline{u}(x') = \overline{u}(x)'$ in all three cases, we get that $\overline{u} : M \longrightarrow N$ is a dashed monoid homomorphism. □

lem: DMon freeDMon fG unique **Lemma 2.3.29.** *Let $f : M \longrightarrow N$ be a dashed-monoid homomorphism. Define*

$$u := f\,|_{\mathtt{S}} : \mathtt{S} \longrightarrow N.$$

Then, we have

{eqn: DMon freeDMon fG unique} (2.3.30) $$u^{\mathtt{G}} = f\,|_{\mathtt{G}} : \mathtt{G} \longrightarrow N$$

where $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$ is as in Construction 2.3.13.

Proof. Let $x \in M$. Suppose $x \in \mathtt{G}$. Using induction on $\hat{l}(x)$, we will show that

$$u^{\mathtt{G}}(x) = f(x).$$

Base case $\left(\hat{l}(x) = 1\right)$: From Lemma 2.3.12 we get $x \in \langle \mathtt{S};\ (-)' \rangle$. Thus, we get $k \in \mathbb{N}$ and $a \in \mathtt{S}$ such that

$$x = a^{(k)}.$$

Observe that

$$\begin{aligned} u^{\mathtt{G}}(x) &= u(a)^{(r)} && (2.3.14) \\ &= f(a)^{(r)} && \text{(hypothesis)} \\ &= f\left(a^{(r)}\right) \\ &= f(x). \end{aligned}$$

Here, the third equality follows since f is a dashed monoid homomorphism.

Induction case $\left(\hat{l}(x) \geq 2\right)$: From Lemmas 2.3.8 and 2.3.12 we get

$$x \in \langle\langle \mathtt{R}; (-)' \rangle\rangle.$$

Thus, we get $k \geq 1$, $m \geq 2$, and $x_i \in \mathtt{G}$ such that

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}.$$

Since $x_i \in \mathtt{G}$, we have $\hat{l}(x_i) \geq 1$. Since $\hat{l} : M \longrightarrow \mathbb{N}$ is a dashed-monoid homomorphism, we get

$$\hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i).$$

Becasue $m \geq 2$, we get $\hat{l}(x_i) < \hat{l}(x)$. Using the induction hypothesis, we get

$$u^{\mathtt{G}}(x_i) = f(x_i).$$

Observe that

$$\begin{aligned}
u^{\mathtt{G}}(x) &= \left(u^{\mathtt{G}}(x_1) \bullet \cdots \bullet u^{\mathtt{G}}(x_m)\right)^{(k)} && (2.3.15) \\
&= (f(x_1) \bullet \cdots \bullet f(x_m))^{(k)} && \text{(induction)} \\
&= f\left((x_1 \bullet \cdots \bullet x_m)^{(k)}\right) \\
&= f(x).
\end{aligned}$$

Here, the third equality follows because f is a dashed monoid homomorphism.

Thus, by mathematical induction we get

$$u^{\mathtt{G}} = f\,|_{\mathtt{G}} : \mathtt{G} \longrightarrow N$$

as required. □

lem: DMon freeDMon fM unique **Lemma 2.3.31.** *Suppose $f : M \longrightarrow N$ is a dashed monoid homomorphism. Let*

$$u := f\,|_{\mathtt{S}} : \mathtt{S} \longrightarrow N.$$

Then,

{eqn: DMon freeDMon fM unique} (2.3.32)
$$\overline{u} = f : M \longrightarrow N,$$

where $\overline{u} : M \longrightarrow N$ is as in Definition 2.3.20.

Proof. From Lemma 2.3.28, we get that $\overline{u} : M \longrightarrow N$ is a dashed monoid homomorphism. Given that $f : M \longrightarrow N$ is a dashed monoid homomorphism, it follows that $\overline{u}$ and f are both monoid homomorphisms. Since $\mathtt{G}$ is a multiplicative basis of M, it is sufficient to show that

$$\overline{u}\,|_{\mathtt{G}} = f\,|_{\mathtt{G}}\,.$$

Observe that

$$\begin{aligned}
\overline{u}\,|_{\mathtt{G}} &= u^{\mathtt{G}} && (2.3.25) \\
&= f\,|_{\mathtt{G}}\,. && (2.3.30)
\end{aligned}$$ □

This completes the proof.

Recall that the dashed-monoid M has free dashed-monoid like structure. The following theorem will show that M is indeed a free dashed-monoid.

thm: DMon freeDMon like gives **Theorem 2.3.33.** *Let $M = (M, \bullet, I, (-)')$ be a dashed monoid. Suppose M has a free dashed monoid-like structure as in Definition 2.3.1. Then, the inclusion of sets $\mathtt{S} \subseteq M$ satisfies the universal property of dashed monoids*

Proof. We will show the existence condition (2.1.12) and the uniqueness condition (2.1.13) of the universal property of a free dashed monoid.

Existence: Let N be a dashed monoid and let $u : \mathtt{S} \longrightarrow N$ be a function. Consider the function

$$\overline{u} : M \longrightarrow N$$

as in Definition 2.3.20. From Lemma 2.3.28, we get that $\overline{u}$ is a dashed monoid homomorphism, and Lemma 2.3.26 shows that

$$\overline{u}\,|_{\mathtt{S}} = u.$$

Uniqueness: Let $f, g : M \longrightarrow N$ be dashed monoid homomorphisms such that

$$f\mid_{\mathrm{S}} = g\mid_{\mathrm{S}}.$$

Define

$$u := f\mid_{\mathrm{S}} = g\mid_{\mathrm{S}} : \mathrm{S} \longrightarrow N.$$

From Lemma 2.3.31, we have

$$f = \overline{u} = g.$$

This completes the proof. $\square$

CHAPTER 3

Dashed Words

section: DWord

Definitions: Dashed words as (n, u, D, d), multiplication, assoc, mul unit and unit mul, monoid basis, I_1 as the first bracket

Props: $xy = xz$ implies $y = z$, $z = z_I z_{I^C}$, $z = xy$ and $x \neq e$ implies $z_I = x_I$, basis iff $x = x_I$.

Theorems: Dashed words has a monoid basis

To do: Rearrange, needs a lot of work, work around the factorization, look at the notes.

Can break into more sections if needed.

The free monoid generated by a set S consists of finite length words using alphabets from S. The multiplication is given by concatenation and the unit is the empty word. We represented words in the free monoid by finite sequences in S. This gave us a better way to interact with the free monoid. In this section, we want to give a similar treatment to free dashed monoid. Since the dash operation is neither involutive nor it distributes over the multiplication, we have to keep track of the dashes. One can visualize the free dashed monoid as words with alphabet in S along with a clubbing of some letters with a number on top to keep track of dashes. We call this the dashed words. For example, $x = (a^{(2)}(bc)^{(3)})^{(1)}$ is a dashed word in the free dashed monoid where $a, b, c \in S$. The multiplication is again given by concatenation and the unit is the empty word. The dash operation is given by increasing the outermost number by one.

Although, the above way is easy to visualize, it becomes difficult to write precise definitions and propositions. To overcome this, we use finite sequences in S, same as in the free monoid, to keep track of the underlying word. We use a collection of subsets of $\{1, \ldots, n\}$, where n is the length of the word, to keep track of the indices which are clubbed together. We call this the bracketing on the dashed word. Finally, the number of dashed on the clubbed letters is given by a function from the bracketing to the set of positive natural numbers. We call this function the dash assignment.

This chapter is divided into three parts. First, we will define bracketing and derive important properties and definitions. Next, we will do the same for the dash assignment. Finally, we will combine these two to define the set of dashed word and establish some important properties. Mainly, we will show that the set of dashed words is a dashed monoid. In the next chaper we will show that the dashed words form a free dashed monoid.

To begin lets refresh some basic properties of sets.

3.1. Properties of Integer Sets

defn: Int n + A

Definition 3.1.1. Let $n \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$. Then, $n + A \subseteq \mathbb{Z}$ is defined as follows:

{eqn: Int n + A}
$$n + A := \{n + a \mid a \in A\}. \tag{3.1.2}$$
◇

Notation 3.1.3. We will, at times, use the notation $A + n$ to refer to $n + A$. ◇

prop: Int n + A

Proposition 3.1.4. *Let $x, n, m, p, q \in \mathbb{Z}$ and $A, B \subseteq \mathbb{Z}$. Then, following are true.*

{eqn: Int n + A membership}
$$x \in n + A \iff x - n \in A. \tag{3.1.5}$$

{eqn: Int A sub B iff n+A sub n+B}
$$A \subseteq B \iff n + A \subseteq n + B. \tag{3.1.6}$$

{eqn: Int A = B iff n+A = n+B}
$$A = B \iff n + A = n + B. \tag{3.1.7}$$

{eqn: Int 0+A = A}
$$0 + A = A. \tag{3.1.8}$$

{eqn: Int n+empty = empty}
$$n + \varnothing = \varnothing. \tag{3.1.9}$$

{eqn: Int n+(m+A) = (n+m)+A}
$$n + (m + A) = (n + m) + A. \tag{3.1.10}$$

{eqn: Int n + cup}
$$n + (A \cup B) = (n + A) \cup (n + B). \tag{3.1.11}$$

{eqn: Int n + cap}
$$n + (A \cap B) = (n + A) \cap (n + B). \tag{3.1.12}$$

{eqn: Int n + Int}
$$n + (p, q] = (n + p, n + q]. \tag{3.1.13}$$

Proof. (3.1.5): Suppose $x \in n + A$. Then, we get $x = n + a$ for some $a \in A$. It follows that $x - n = a \in A$. On the other hand, if $x - n \in A$ then $x = n + (x - n) \in n + A$.

(3.1.6): Suppose $A \subseteq B$. Let $x \in n + A$. From (3.1.5), we get $x - n \in A \subseteq B$. Thus, again from (3.1.5) we get $x \in n + B$. This shows

$$n + A \subseteq n + B.$$

For the reverse direction, assume $n + A \subseteq n + B$. Let $x \in A$. From (3.1.2) we get $n + x \in n + A \subseteq n + B$. Finally, from (3.1.5), we get $x = n + x - n \in B$. Thus, we have

$$A \subseteq B.$$

(3.1.7): We see this from the following chain of double implications.

$$\begin{aligned}
A = B &\iff (A \subseteq B) \text{ and } (B \subseteq A) \\
&\iff (n + A \subseteq n + B) \text{ and } (n + B \subseteq n + A) && (3.1.6) \\
&\iff n + A = n + B.
\end{aligned}$$

(3.1.8): We have

$$\begin{aligned}
0 + A &= \{0 + a \mid a \in A\} && (3.1.2) \\
&= \{a \mid a \in A\} \\
&= A.
\end{aligned}$$

(3.1.9): For the sake of contradiction assume $n + \varnothing \neq \varnothing$. Let $x \in n + \varnothing$ then, from (3.1.5), we get $x - n \in \varnothing$. This is a contradiction.

(3.1.10): We have

$$
\begin{aligned}
n+(m+A) &= n+\{m+a \mid a\in A\} && (3.1.2)\\
&= \{n+m+a \mid a\in A\} && (3.1.2)\\
&= (n+m)+A. && (3.1.2)
\end{aligned}
$$

(3.1.11): We get

$$
\begin{aligned}
n+(A\cup B) &= \{n+a \mid a\in A \ \text{ or } \ a\in B\} && (3.1.2)\\
&= \{n+a \mid a\in A\}\cup\{n+a \mid a\in B\}\\
&= (n+A)\cup(n+B). && (3.1.2)
\end{aligned}
$$

(3.1.12): We have

$$
\begin{aligned}
n+(A\cap B) &= \{n+a \mid a\in A \ \text{ and } \ a\in B\} && (3.1.2)\\
&= \{n+a \mid a\in A\}\cap\{n+a \mid a\in B\}\\
&= (n+A)\cap(n+B). && (3.1.2)
\end{aligned}
$$

(3.1.13): We see that

$$
\begin{aligned}
n+[p,q] &= \{n+x \mid p\le x\le q\} && (3.1.2)\\
&= \{y \mid p\le y-n \mid q\}\\
&= \{y \mid n+p\le y\le n+q\}\\
&= [n+p,n+q].
\end{aligned}
$$

□

Remark 3.1.14. The above propositions are fundamental and will be used frequently throughout our article. Due to their elementary nature, we will refer to them without providing explicit reference. ⋄

defn: SInt n + D **Definition 3.1.15.** Let $D\subseteq\mathcal{P}(\mathbb{Z})$. Then, $n+D\subseteq\mathcal{P}(\mathbb{Z})$ is defined as follows:

{eqn: SInt n + D} (3.1.16) $$n+D := \{n+A \mid A\in D\}.$$ ⋄

prop: SInt n + D **Proposition 3.1.17.** *Let $n,m\in\mathbb{Z}$, $A,B\subseteq\mathbb{Z}$, and $D,E\subseteq\mathcal{P}(\mathbb{Z})$. Then, following are true.*

{eqn: SInt n + D membership} (3.1.18) $$A\in n+D \iff A-n\in D.$$

{eqn: SInt D sub E iff n+D sub n+E} (3.1.19) $$D\subseteq E \iff n+D\subseteq n+E.$$

{eqn: SInt D = E iff n+D = n+E} (3.1.20) $$D=E \iff n+D=n+E.$$

{eqn: SInt 0+D = D} (3.1.21) $$0+D=D.$$

{eqn: SInt n+empty = empty} (3.1.22) $$n+\varnothing=\varnothing.$$

{eqn: SInt n+(m+D) = (n+m)+D} (3.1.23) $$n+(m+D)=(n+m)+D.$$

{eqn: SInt n + cup} (3.1.24) $$n+(D\cup E)=(n+D)\cup(n+E).$$

{eqn: SInt n + cap} (3.1.25) $$n+(D\cap E)=(n+D)\cap(n+E).$$

Proof. (3.1.18): Suppose $X\in n+D$. Then, we get $X=n+A$ for some $A\in D$. It follows that $X-n=A\in D$. On the other hand, if $X-n\in D$ then $X=n+(X-n)\in n+D$.

(3.1.19): Suppose $D \subseteq E$. Let $X \in n+D$. From (3.1.18), we get $X - n \in D \subseteq E$. Thus, again from (3.1.18) we get $X \in n+E$. This shows

$$n + D \subseteq n + E.$$

For the reverse direction, assume $n+D \subseteq n+E$. Let $X \in D$. From (3.1.16) we get $n+X \in n+D \subseteq n+E$. Finally, from (3.1.18), we get $X = n+X-n \in B$. Thus, we have

$$D \subseteq E.$$

(3.1.20): We see this from the following chain of double implications.

$$\begin{aligned} D = E &\iff (D \subseteq E) \text{ and } (E \subseteq D) \\ &\iff (n+D \subseteq n+E) \text{ and } (n+E \subseteq n+D) && (3.1.19) \\ &\iff n+D = n+E. \end{aligned}$$

(3.1.21): We have

$$\begin{aligned} 0 + D &= \{0 + A \mid A \in D\} && (3.1.16) \\ &= \{A \mid A \in D\} \\ &= D. \end{aligned}$$

(3.1.22): For the sake of contradiction assume $n + \varnothing \neq \varnothing$. Let $X \in n + \varnothing$ then, from (3.1.18), we get $X - n \in \varnothing$. This is a contradiction.

(3.1.23): We have

$$\begin{aligned} n + (m + D) &= n + \{m + A \mid A \in D\} && (3.1.16) \\ &= \{n + m + A \mid A \in D\} && (3.1.16) \\ &= (n+m) + D. && (3.1.16) \end{aligned}$$

(3.1.24): We get

$$\begin{aligned} n + (D \cup E) &= \{n + A \mid A \in D \text{ or } A \in E\} && (3.1.16) \\ &= \{n + A \mid A \in D\} \cup \{n + A \mid A \in E\} \\ &= (n + D) \cup (n + E). && (3.1.16) \end{aligned}$$

(3.1.25): We have

$$\begin{aligned} n + (D \cap E) &= \{n + A \mid A \in D \text{ and } A \in E\} && (3.1.16) \\ &= \{n + A \mid A \in D\} \cap \{n + A \mid A \in E\} \\ &= (n + D) \cap (n + E). && (3.1.16) \end{aligned}$$

□

prop: SInt D is interval iff n+D is interval

Proposition 3.1.26. *Let $n \in \mathbb{Z}$, $B \subseteq \mathbb{Z}$, and $D \subseteq \mathcal{P}(B)$. Then, $n + D \subseteq \mathcal{P}(n+B)$. Moreover, if $D \subseteq \mathcal{P}_I^+(B)$ then $n + D \subseteq \mathcal{P}_I^+(n+B)$.*

Proof. Let $A \in n+D$. We get $-n + A \in D$. Since $D \subseteq \mathcal{P}(B)$ we get $-n + A \subseteq B$. This implies $A \subseteq n + B$. This shows that $n + D \subseteq \mathcal{P}(n+B)$.

Now assume $D \subseteq \mathcal{P}_I^+(B)$. Let $n + A \in n + D$ where $A \in D$. We have that $A \in D$ is a non-empty interval. Therefore, we get that $n + A$ is a non-empty interval. This shows that $n + D \subseteq \mathcal{P}_I^+(n+B)$.

In a monoid, we often omit brackets since the multiplication is associative. However, in mathematical structures with multiple operations, it becomes necessary to use brackets to show the order of operations. For instance in a group $(xy)^{-1}z$ and $xy^{-1}z$ represent different elements. Although groups can be a little forgiving since $(xy)^{-1} = y^{-1}x^{-1}$, one can avoid using brackets. In a dashed monoid, it is essential to show the order of operations since $(xy)'$ and $y'x'$ might represent different elements. Therefore, we need a formal way to represent brackets.

defn: Br

Definition 3.1.27. Let $n \in \mathbb{N}$. A *bracketing* on n letters is a collection of non-empty sub-intervals of $[n]$,

$$D \subseteq \mathcal{P}_I^+(n),$$

called the *brackets*, that satisfy the following bracketing condition: For every $A, B \in D$,

{eqn: Br cond}

$$\text{either } A \cap B = \emptyset,\ A \subseteq B, \text{ or } B \subseteq A. \tag{3.1.28}$$

◇

convention: Br

Convention 3.1.29. For $n \in \mathbb{N}$ let $\mathcal{B}r(n)$ denote the set of all bracketings on n letters. Let

$$\mathcal{B}r := \bigcup_{n \in \mathbb{N}} \mathcal{B}r(n)$$

and

$$\mathcal{B}r^+ := \bigcup_{n \in \mathbb{N}^+} \mathcal{B}r(n).$$

◇

defn: Br unit

Definition 3.1.30. Let $n \in \mathbb{N}$. Then, the empty collection $\emptyset \subseteq \mathcal{P}_I^+(n)$ vacuously satisfies the bracketing condition. We call this the *empty bracketing* on n letters.

We define the *unit bracketing*, denoted $I_{\mathcal{B}r}$, to be the empty bracketing, $\emptyset \subseteq \mathcal{P}_I^+(0)$, on 0 letters. That is,

{eqn: Br unit}

$$I_{\mathcal{B}r} := \emptyset \in \mathcal{B}r(0). \tag{3.1.31}$$

Note that, the unit bracketing is the only bracketing on 0 letters. ◇

Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$, then every $A \in D$ corresponds to a bracket. We require that A is an interval since a bracket can not have gaps within them. Conventionally, we assume that A is non-empty since one can simply omit empty bracket. The bracketing condition ensures that each bracket contains only the closed brackets.

Example 3.1.32. Here are some examples of bracketings on a word with five letters.

Bracketing (D)	Visual
$\emptyset$	$*****$
$\{\{1,2,3,4,5\}\}$	$(*****)$
$\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$	$(*)(*)(*)(*)(*)$
$\{\{1,2\},\{1,2,3\},\{4,5\}\}$	$((**)*)(**)$
$\{\{1,2\},\{4,5\},\{1,2,3,4,5\}\}$	$((**)*(**))$

◇

prop: Br mul

Proposition 3.1.33. *Let $n, m \in \mathbb{N}$. Let $D \in \mathcal{Br}(n)$ and $E \in \mathcal{Br}(m)$ be bracketings on n and m letters respectively. Then,*

$$D \cap (n+E) = \varnothing$$

and

$$D \sqcup (n+E) \in \mathcal{Br}(n+m).$$

Proof. First, we will show $D \cap (n+E) = \varnothing$. Let $A \in D \cap (n+E)$, that is, $A \in D$ and $A = n+B$ for some $B \in E$. Then, we get $A \subseteq (0,n]$ and $A \subseteq n+(0,m] = (n,n+m]$. Thus, we get $A = \varnothing$. This is a contradiction since $A \in D$ and D is a bracketing implies $A \neq \varnothing$. Hence, we get $D \cap (n+E) = \varnothing$.

To see that $D \cup (n+E)$ is a bracketing on $n+m$ letters, we need to check following two things:

i. $D \cup (n+E) \subseteq \mathcal{P}_I^+(n+m)$, and

ii. for every $A, B \in D \cup (n+E)$ either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$.

Since D is a bracketing on n letters, we know that $D \subseteq \mathcal{P}_I^+(n) \subseteq \mathcal{P}_I^+(n+m)$. Since E is a bracketing on m letters, we get that $E \subseteq \mathcal{P}_I^+(m)$. We have

$$n+E \subseteq \mathcal{P}_I^+(n+[m]) = \mathcal{P}_I^+([n+1, n+m]) \subseteq \mathcal{P}_I^+(n+m).$$

Thus, we get

$$D \cup (n+E) \subseteq \mathcal{P}_I^+(n+m).$$

Now for the bracketing condition, let $A, B \in D \cup (n+E)$. We will consider the following cases:

Case 1 ($A, B \in D$): Since D is a bracketing and $A, B \in D$ we get either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$.

Case 2 ($A, B \in n+E$): That is, we have $X, Y \in E$ such that $A = n+X$ and $B = n+Y$. Since E is a bracketing, from the bracketing condition, we get either $X \cap Y = \varnothing$, $X \subseteq Y$, or $Y \subseteq X$. If $X \cap Y = \varnothing$, then we get

$$\begin{aligned} A \cap B &= (n+X) \cap (n+Y) \\ &= n + (X \cap Y) \\ &= n + \varnothing \\ &= \varnothing. \end{aligned}$$

If $X \subseteq Y$, then we get

$$\begin{aligned} A &= n+X \\ &\subseteq n+Y \\ &= B. \end{aligned}$$

Similarly, we get $B \subseteq A$ if $Y \subseteq X$.

Case 3 ($A \in D, B \in n+E$): That is, $A \in D$ and we have $Y \in E$ such that $B = n+Y$. Since D is a bracketing on n letters, we get $A \subseteq (0,n]$. Similarly, we get $Y \subseteq (0,m]$. This implies $B = n+Y \subseteq (n,m]$. Thus, we get $A \cap B = \varnothing$.

Since in each of the cases we get either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$, we conclude that $D \cup (n+E)$ is a bracketing on $n+m$ letters. □

defn: Br mul

Definition 3.1.34. Let $n, m \in \mathbb{N}$ and $D \in \mathcal{B}r(n), E \in \mathcal{B}r(m)$ be bracketings. Define the multiplication of the bracketings, $D \bullet E \in \mathcal{B}r(n+m)$, as

{eqn: Br mul}
$$D \bullet E := D \sqcup (n+E). \tag{3.1.35}$$

From Proposition 3.1.33 we know that $D \bullet E$ is a bracketing on $n+m$ letters.◇

Example 3.1.36. Let

$$D = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\big\} \in \mathcal{B}r(5)$$

and

$$E = \big\{1\big\} \in \mathcal{B}r(3).$$

Then, we have

$$D \bullet E = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}, \{6\}\big\} \in \mathcal{B}r(8).$$

Using the visual representation, we get

$$D = ((**)\ *)\ (**) \qquad \text{and} \qquad E = (*)\ **.$$

Then, the product is given by the concatenation,

$$D \bullet E = ((**)\ *)\ (**)\ (*)\ **.$$ ◇

lem: Br mul assoc

Lemma 3.1.37. *Let $n, m, p \in \mathbb{N}$ and $D \in \mathcal{B}r(n), E \in \mathcal{B}r(m), F \in \mathcal{B}r(p)$ be bracketings. Then we have*

{eqn: Br mul assoc}
$$D \bullet (E \bullet F) = (D \bullet E) \bullet F \tag{3.1.38}$$

as bracketings on $n+m+p$ letters.

Proof. Observe that,

$$\begin{aligned}
D \bullet (E \cdot F) &= D \cup (n + (E \bullet F)) && (3.1.35)\\
&= D \cup (n + (E \cup (m+F))) && (3.1.35)\\
&= D \cup (n+E) \cup (n + (m+F)) && (3.1.24)\\
&= D \cup (n+E) \cup ((n+m)+F) && (3.1.23)\\
&= (D \bullet E) \cup ((n+m)+F) && (3.1.35)\\
&= (D \bullet E) \bullet F. && (3.1.35)
\end{aligned}$$ □

rmk:Bracketing multiplication is associative

Remark 3.1.39. Since the bracketing multiplication is associative, we will omit the use of brackets to show the order of multiplication. ◇

lem: Br mul one and one mul

Lemma 3.1.40. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$. Recall, $I_{\mathcal{B}r} \in \mathcal{B}r(0)$ is the unit bracketing as in Definition 3.1.30. We have*

{eqn: Br mul one and one mul}
$$D \bullet I_{\mathcal{B}r} = D \qquad \text{and} \qquad I_{\mathcal{B}r} \bullet D = D \tag{3.1.41}$$

as bracketings on n letters.

Proof. We get

$$
\begin{aligned}
D \bullet I &= D \bullet \varnothing && (3.1.31)\\
&= D \cup (n + \varnothing) && (3.1.35)\\
&= D \cup \varnothing && (3.1.22)\\
&= D.
\end{aligned}
$$

Next, we get

$$
\begin{aligned}
I \bullet D &= \varnothing \bullet D && (3.1.31)\\
&= \varnothing \cup (0 + D) && (3.1.35)\\
&= \varnothing \cup D && (3.1.21)\\
&= D.
\end{aligned}
$$

□

prop: Br dash

Proposition 3.1.42. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then,*

$$E := \{A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ or } \ A = [n]\}$$

is a bracketing on n letters.

Proof. To see that E is a bracketing on n letters, we need to check following two things:

i. $E \subseteq \mathcal{P}_I^+(n)$, and

ii. for every $A, B \in E$ either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$.

By the construction, we have $E \subseteq \mathcal{P}_I^+(n)$. Now, let $A, B \in E$. We consider following cases:

Case 1 ($A, B \in D$): In this case, the bracketing condition is satisfied since D itself is a bracketing.

Case 2 ($B = [n]$): In this case, since $E \subseteq \mathcal{P}_I^+(n)$, we have $A \subseteq [n] = B$.

This completes the proof. □

defn: Br dash

Definition 3.1.43. Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Define the dash of the bracketing, $D' \in \mathcal{B}r(n)$, as follows:

{eqn: Br dash}

$$D' := \{A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ or } \ A = [n]\}. \tag{3.1.44}$$

From Proposition 3.1.42 we know that D' is a bracketing on n letters. ⋄

prop: Br dash iff

Proposition 3.1.45. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then, we get*

{eqn: Br dash iff}

$$D' = \begin{cases} D & \text{if } \ n = 0 \\ D \cup \{[n]\} & \text{if } \ n \geq 1. \end{cases} \tag{3.1.46}$$

Proof. We will consider following cases:

Case 1($n = 0$): In this case, we have both $D, D' \in \mathcal{B}r(0)$. Since $\mathcal{B}r(0)$ only consists of $I_{\mathcal{B}r}$, we get $D = D'$.

Case 2($n \geq 1$): In this case, we have $[n] \in \mathcal{P}_I^+(n)$. Observe that,

$$
\begin{aligned}
D' &= \{A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ or } \ A = [n]\} && (3.1.44)\\
&= \{A \in zP_I^+(n) \mid A \in D\} \cup \{A \in \mathcal{P}_I^+(n) \mid A = [n]\}\\
&= D \cup \{[n]\}.
\end{aligned}
$$

□

Example 3.1.47. Let

$$D = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\big\} \in \mathcal{B}r(5).$$

Then, we have

$$D' = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\},\ \{1,2,3,4,5\}\big\} \in \mathcal{B}r(5).$$

Using the visual representation, we get

$$D = ((**)\,*)\,(**).$$

Then, the dash of D is given by adding the outermost bracket if there is no such bracket,

$$D' = (((**)\,*)\,(**)\,(*)).$$

◇

lem: Br unit dash

Lemma 3.1.48. *Consider the unit bracketing, $I_{\mathcal{B}r} \in \mathcal{B}r(0)$. We have*

{eqn: Br unit dash}

$$I'_{\mathcal{B}r} = I_{\mathcal{B}r} \tag{3.1.49}$$

as bracketing on 0 letters.

Proof. Follows immediately from the case $n = 0$ of Proposition 3.1.45. □

prop: Br dash k

Proposition 3.1.50. *Let $n \geq 1$ and $D \in \mathcal{B}r(n)$ be a bracketing. For $k \geq 1$ let $D^{(k)}$ denote the bracketing on n letters that we get after taking k dashes of D. Then we get*

{eqn: Br dash k}

$$D^{(k)} = D \cup \{[n]\} \tag{3.1.51}$$

as bracketings on n letters.

Proof. We will use induction on k.

Base case ($k = 1$): We get

$$D^{(1)} = D' = D \cup \{[n]\}. \qquad (3.1.46)$$

Induction case ($k \geq 1$): Assume that $D^{(k)} = D \cup \{[n]\}$. We get

$$\begin{aligned} D^{(k+1)} &= \left(D^{(k)}\right)' \\ &= (D \cup \{[n]\})' && \text{Induction} \\ &= D \cup \{[n]\} \cup \{[n]\} && \text{Base case} \\ &= D \cup \{[n]\}. \end{aligned}$$

□

prop: Br undash

Proposition 3.1.52. *Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing, and $C \subseteq [n]$. Then,*

$$E := \big\{A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ and } \ A \neq C\big\}$$

is a bracketing on n letters.

Proof. To see that E is a bracketing on n letters, we need to check following two things:

i. $E \subseteq \mathcal{P}_I^+(n)$, and
ii. for every $A, B \in E$ either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$.

By the construction, E is a subset of $\mathcal{P}_I^+(n)$. Now let $A, B \in E$. Then, we get $A, B \in D$. The bracketing condition is satisfied since D itself is a bracketing. This shows that E is a bracketing on n letters. □

defn: Br undash

Definition 3.1.53. Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Define the *floor* of the bracketing D, denoted $\lfloor D \rfloor \in \mathcal{B}r(n)$, as follows:

{eqn: Br undash}

$$\lfloor D \rfloor := \left\{ A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ and } \ A \neq [n] \right\}. \tag{3.1.54}$$

From Proposition 3.1.52 we know that $\lfloor D \rfloor$ is a bracketing on n letters. ⋄

prop: Br undash iff

Proposition 3.1.55. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then, we get*

{eqn: Br undash iff}

$$\lfloor D \rfloor = \begin{cases} D & \text{if } \ n = 0 \\ D \setminus \left\{ [n] \right\} & \text{if } \ n \geq 1. \end{cases} \tag{3.1.56}$$

Proof. We will consider following cases:

Case 1($n = 0$): In this case, we have both $D, \lfloor D \rfloor \in \mathcal{B}r(0)$. Since $\mathcal{B}r(0)$ only consists of $I_{\mathcal{B}r}$, we get $D = \lfloor D \rfloor$.

Case 2($n \geq 1$): In this case, we have $[n] \in \mathcal{P}_I^+(n)$. Observe that,

$$\begin{aligned} \lfloor D \rfloor &= \left\{ A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ and } \ A \neq [n] \right\} && (3.1.54) \\ &= \left\{ A \in zP_I^+(n) \mid A \in D \right\} \cap \left\{ A \in \mathcal{P}_I^+(n) \mid A \neq [n] \right\} \\ &= D \setminus \left\{ [n] \right\}. \end{aligned}$$

□

Example 3.1.57. Let

$$D = \left\{ \{1,2\},\ \{1,2,3\},\ \{4,5\},\ \{1,2,3,4,5\} \right\} \in \mathcal{B}r(5).$$

and

$$E = \left\{ 1 \right\} \in \mathcal{B}r(3).$$

Then, we have

$$\lfloor D \rfloor = \left\{ \{1,2\},\ \{1,2,3\},\ \{4,5\} \right\} \in \mathcal{B}r(5).$$

Using the visual representation, we get

$$D = (((**)\,*)\,(**)).$$

Then, the floor of D is given by removing the outermost bracket if there is one,

$$\lfloor D \rfloor = ((**)\,*)\,(**).$$

⋄

lem: Br unit undash

Lemma 3.1.58. *Consider the unit bracketing, $I_{\mathcal{B}r} \in \mathcal{B}r(0)$. We have*

{eqn: Br unit undash}

$$\lfloor I_{\mathcal{B}r} \rfloor = I_{\mathcal{B}r} \tag{3.1.59}$$

as bracketing on 0 letters.

Proof. Follows immediately from the case $n = 0$ of Proposition 3.1.55. □

prop: Br undash undash

Proposition 3.1.60. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. We get*

{eqn: Br undash undash}

$$\lfloor \lfloor D \rfloor \rfloor := \left\lfloor \left(\lfloor D \rfloor \right) \right\rfloor = \lfloor D \rfloor \tag{3.1.61}$$

as bracketings on n letters.

Proof. If $n=0$ then, from Proposition 3.1.55, we get $\lfloor D\rfloor = D$. As a result, we get

$$\lfloor\lfloor D\rfloor\rfloor = \lfloor D\rfloor = D.$$

If $n\geq 1$ then, from Proposition 3.1.55, we get $\lfloor D\rfloor = D\setminus\{[n]\}$. Thus, we get
Base case ($k=1$): We get

$$\begin{aligned}
\lfloor\lfloor D\rfloor\rfloor &= \lfloor D\setminus\{[n]\}\rfloor && (3.1.56)\\
&= (D\setminus\{[n]\})\setminus\{[n]\} && (3.1.56)\\
&= D\setminus\{[n]\}\\
&= \lfloor D\rfloor . && (3.1.56)
\end{aligned}$$

□

prop: Br dash undash

Proposition 3.1.62. *Let $n\in\mathbb{N}^+$ and $D\in\mathcal{Br}(n)$ be a bracketing. Then, for $k\in\mathbb{N}$ we have*

{eqn: Br dash undash}

$$\left\lfloor D^{(k)}\right\rfloor = \lfloor D\rfloor . \tag{3.1.63}$$

Proof. For the case $k=0$, we get $D^{(0)}=D$. Therefore, the proposition is true for $k=0$. Now assume $k\geq 1$. We have

$$\begin{aligned}
\left\lfloor D^{(k)}\right\rfloor &= \lfloor D\cup\{[n]\}\rfloor && (3.1.51)\\
&= (D\cup\{[n]\})\setminus\{[n]\} && (3.1.56)\\
&= D\setminus\{[n]\}\\
&= \lfloor D\rfloor . && (3.1.56)
\end{aligned}$$

□

prop: Br undash eq self iff

Proposition 3.1.64. *Let $n\in\mathbb{N}^+$ and $D\in\mathcal{Br}(n)$ be a bracketing. Then, $\lfloor D\rfloor = D$ if and only if $[n]\notin D$.*

Proof. Suppose $\lfloor D\rfloor = D$. From Proposition 3.1.55, we get $\lfloor D\rfloor = D\setminus\{[n]\}$. Therefore, we have $D=D\setminus\{[n]\}$ implying $[n]\notin D$.

On the other hand, suppose $[n]\notin D$. Then, we get

$$\begin{aligned}
\lfloor D\rfloor &= D\setminus\{[n]\} && (3.1.56)\\
&= D.
\end{aligned}$$

□

prop: Br undash dash

Proposition 3.1.65. *Let $n\in\mathbb{N}^+$ and $D\in\mathcal{Br}(n)$ be a bracketing. Suppose $[n]\in D$. Then, for $k\geq 1$ we have*

{eqn: Br undash dash}

$$(\lfloor D\rfloor)^{(k)} = D. \tag{3.1.66}$$

Proof. Observe that

$$\begin{aligned}
(\lfloor D\rfloor)^{(k)} &= (D\setminus\{[n]\})^{(k)} && (3.1.56)\\
&= (D\setminus\{[n]\})\cup\{[n]\} && (3.1.51)\\
&= D.
\end{aligned}$$

The last equality follows from the assumption that $[n]\in D$. □

New part starts, restriction. label: Br Res prop: Br Res

Proposition 3.1.67. *Let $n \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$ be a bracketing. Let $(p,q] \subseteq_I (0,n]$ with $0 \le p \le q \le n$. Then,*

$$E := -p + \{B \in D \mid B \subseteq (p,q]\}$$

is a bracketing on $p-q$ letters.

Proof. To see that E is a bracketing on $p-q$ letters, we need to check following two things:

i. $E \subseteq \mathcal{P}_I^+(p-q)$, and
ii. for every $A, B \in E$ either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$.

First, we will show that every element of E is a non-empty sub-interval of $(0, q-p]$. Let $A \in E$, then we get $X \in D$ with $X \subseteq (p,q]$ such that $A = -p + X$. Since $X \in D$ we get that X is a non-empty interval and hence $A = -p + X$ is a non-empty interval. Since $X \subseteq (p,q]$ we get

$$A = -p + X \subseteq -p + (p,q] = (0, q-p].$$

This shows that $E \subseteq \mathcal{P}_I^+(q-p)$.

It remains to check that E satisfies the bracketing condition. Let $A, B \in E$. Then we get $X, Y \in D$ with $X, Y \subseteq (p,q]$ such that $A = -p + X$ and $C = -p + Y$. Since D is a bracketing, we get either $X \cap Y = \varnothing$, $X \subseteq Y$, or $Y \subseteq Z$. This implies that either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$. Thus, E is a bracketing on $q-p$ letters. □

defn: Br Res

Definition 3.1.68. Let $n \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$ be a bracketing. Let $(p,q] \subseteq_I [n]$ with $0 \le p \le q \le n$. Define a bracketing, $D_{(p,q]}$, on $q-p$ letters as follows:

{eqn: Br Res}

$$D_{(p,q]} := -p + \{B \in D \mid B \subseteq (p,q]\}. \tag{3.1.69}$$

Proposition 3.1.67 shows that $D_{(p,q]}$ is a bracketing on $q-p$ letters. ⋄

Example 3.1.70. Let

$$D = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\big\} \in \mathcal{Br}(5).$$

Then, we get

$$\begin{aligned}
D_{(0,3]} &= \big\{\{1,2\},\ \{1,2,3\}\big\} && \in \mathcal{Br}(3),\\
D_{(0,2]} &= \big\{\{1,2\}\big\} && \in \mathcal{Br}(2),\\
D_{(3,5]} &= \big\{\{1,2\}\big\} && \in \mathcal{Br}(2),\\
D_{(2,5]} &= \big\{\{2,3\}\big\} && \in \mathcal{Br}(3), \text{ and}\\
D_{(1,4]} &= \big\{\big\} && \in \mathcal{Br}(3).
\end{aligned}$$

Using the visual representation, we get

$$D = ((**)\,*)\,(**).$$

Then, the restrictions are given by the,

$$\begin{aligned}
D_{(0,3]} &= ((**)\ *),\\
D_{(0,2]} &= (**),\\
D_{(3,5]} &= (**),\\
D_{(2,5]} &= *\ (**), \quad \text{and}\\
D_{(1,4]} &= ***.
\end{aligned}$$
◇

prop: Br Res self

Proposition 3.1.71. *Let $n \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$ be a bracketing. Then, we get*

{eqn: Br Res self}

$$D_{[n]} = D. \tag{3.1.72}$$

Proof. We get this from the following:

$$\begin{aligned}
D_{[n]} = D_{(0,n]} &= -0 + \{B \in D \mid B \subseteq (0,n]\} && (3.1.69)\\
&= \{B \in D\}\\
&= D.
\end{aligned}$$
□

prop: Br Res mul

Proposition 3.1.73. *Let $n, m \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$, $E \in \mathcal{Br}(m)$ be bracketings. Let $(p,q] \subseteq_I [n]$ with $0 \le p \le q \le n$. Then, we get*

{eqn: Br Res mul}

$$(D \cdot E)_{(p,q]} = D_{(p,q]} \tag{3.1.74}$$

as bracketings on $q-p$ letters.

Proof. We will show $(D \cdot E)_{(p,q]} = D_{(p,q]}$ by showing that

$$(D \cdot E)_{(p,q]} \subseteq D_{(p,q]} \qquad \text{and} \qquad D_{(p,q]} \subseteq (D \cdot E)_{(p,q]}.$$

Let $A \in \mathcal{P}_I^+(q-p)$. Suppose $A \in (D \cdot E)_{(p,q]}$. We get some $X \in D \cdot E$ with $X \subseteq (p,q]$ such that $-p + X = A$. We have

$$X \in D \cdot E = D \sqcup (n+E).$$

If $X \in n+E$, then we must have $X \subseteq (n,m]$. Since $X \subseteq (p,q] \subseteq [n]$, this case is not possible. Thus, we conclude that $X \in D$. Since $X \in D$ and $X \subseteq (p,q]$, we get that

$$A = -p + X \in D_{(p,q]}.$$

This shows that $(D \cdot E)_{(p,q]} \subseteq D_{(p,q]}$.

On the other hand, suppose $A \in D_{(p,q]}$. We get some $X \in D$ with $X \subseteq (p,q]$ such that $-p + X = A$. We get $X \in D \sqcup (n+E) = D \cdot E$ and $X \subseteq (p,q]$. Thus,

$$A = p + X \in (D \cdot E)_{(p,q]}.$$

This shows that $D_{(p,q]} \subseteq (D \cdot E)_{(p,q]}$.

Therefore, we conclude

$$(D \cdot E)_{(p,q]} = D_{(p,q]}.$$

prop: Br mul Res

Proposition 3.1.75. *Let $n,m \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings. Let $(p,q] \subseteq_I (n,n+m]$ with $n \le p \le q \le n+m$. Then, we get*

{eqn: Br mul Res}

$$(D \cdot E)_{(p,q]} = E_{(p-n,q-n]} \tag{3.1.76}$$

as bracketings on $q-p$ letters.

Proof. We will show $(D \cdot E)_{(p,q]} = E_{(p-n,q-n]}$ by showing that

$$(D \cdot E)_{(p,q]} \subseteq E_{(p-n,q-n]} \qquad \text{and} \qquad E_{(p-n,q-n]} \subseteq (D \cdot E)_{(p,q]}.$$

Let $A \in \mathcal{P}_I^+(q-p)$. Suppose $A \in (D \cdot E)_{(p,q]}$. We get some $X \in D \cdot E$ with $X \subseteq (p,q]$ such that $-p+X = A$. We have

$$X \in D \cdot E = D \sqcup (n+E).$$

If $X \in D$, then we must have $X \subseteq [n]$. Since $X \subseteq (p,q] \subseteq (n,n+m]$, this case is not possible. Thus, we conclude that $X \in n+E$. That is, $-n+X \in E$. Moreover, since $X \subseteq (p,q]$ we get $-n+X \subseteq (p-n,q-n]$. This shows that

$$A = -(p-n) + (-n+X) \in E_{(p-n,q-n]}.$$

Thus, we get

$$(D \cdot E)_{(p,q]} \subseteq E_{(p-n,q-n]}.$$

On the other hand, suppose $A \in E_{(p-n,q-n]}$. We get some $X \in E$ with $X \subseteq (p-n,q-n]$ such that $-(p-n)+X = A$. We get

$$n+X \in D \sqcup (n+E) = D \cdot E$$

and $n+X \subseteq (p,q]$. Thus,

$$A = -p+n+X \in (D \cdot E)_{(p,q]}.$$

Consequently, we get

$$E_{(p-n,q-n]} \subseteq (D \cdot E)_{(p,q]}.$$

Therefore, we conclude

$$(D \cdot E)_{(p,q]} = E_{(p-n,q-n]}.$$

lem: Br Res mul self

Lemma 3.1.77. *Let $n,m \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings. Then, we get*

{eqn: Br Res mul self}

$$(D \cdot E)_{[n]} = D \tag{3.1.78}$$

as bracketings on n letters.

Proof. We see that

$$\begin{aligned} (D \cdot E)_{[n]} &= D_{[n]} && (3.1.74) \\ &= D. && (3.1.72) \end{aligned}$$ □

lem: Br mul Res self

Lemma 3.1.79. *Let $n,m \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings. Then, we get*

{eqn: Br mul Res self}

$$(D \cdot E)_{(n,n+m]} = E \tag{3.1.80}$$

as bracketings on m letters.

Proof. We see that

$$\begin{aligned}\left(D\cdot E\right)_{(n,n+m]} &= E_{(0,m]} && (3.1.76)\\ &= E. && (3.1.72)\end{aligned}$$ □

3.1.1. Dash assignments.

defn: Ds

Definition 3.1.81. Let $n \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$ be a bracketing. A *dash-assignment* on the bracketing D is a function

$$d : D \longrightarrow \mathbb{N}^{+}.$$

convention: Ds

Convention 3.1.82. For $n \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$ let $\mathcal{Ds}(D)$ denote the set of all dash-assignments on the bracketing D. ⋄

defn: Ds unit

Definition 3.1.83. Let $n \in \mathbb{N}$ and $\varnothing \in \mathcal{Br}(n)$ be the empty bracketing on n-letters. Then, the empty map $\emptyset : \varnothing \longrightarrow \mathbb{N}^{+}$ is the only dash-assignment. We call this the *empty dash-assignment* on the empty bracketing on n letters. The *unit dash-assignment*, denoted $I_{\mathcal{Ds}}$, is the empty dash-assignment on the unit bracketing. That is,

{eqn: Ds unit}

$$I_{\mathcal{Ds}} := \emptyset \in \mathcal{Ds}(I_{\mathcal{Br}}). \tag{3.1.84}$$ ⋄

Let $n \in \mathbb{N}$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$. Then, every $A \in D$ corresponds to a bracket and the positive integer, $d(A)$, corresponds to the number of dashes on that bracket.

example: Ds

Example 3.1.85. Let

$$D = \left\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\right\} \in \mathcal{Br}(6)$$

be a bracketing on 6 letters. Then, an example of dash-assignment $d \in \mathcal{Ds}(D)$ is given by

$$\begin{aligned}\{1,2\} &\longmapsto 2,\\ \{1,2,3\} &\longmapsto 1,\\ \{4,5\} &\longmapsto 3.\end{aligned}$$

A visual representation of above dash-assignment is given by

$$d = ((**)''\ *)'\ (**)'''\ \ *\,.$$

Let

$$E = \left\{\{2\},\ \{2,3,4\},\ \{1,2,3,4,5\}\right\} \in \mathcal{Br}(5)$$

be a bracketing on 5 letters. Then, an example of dash-assignment $e \in \mathcal{Ds}(D)$ is given by

$$\begin{aligned}\{2\} &\longmapsto 7,\\ \{2,3,4\} &\longmapsto 18,\\ \{1,2,3,4,5\} &\longmapsto 11.\end{aligned}$$

A visual representation of above dash-assignment is given by

$$e = (*\ ((*)^{(7)}\ \ **)^{(18)}\ \ *)^{(11)}.$$ ⋄

defn: Ds mul

Definition 3.1.86. Let $n, m \in \mathbb{N}$, $D \in \mathcal{B}r(n)$, and $E \in \mathcal{B}r(m)$. Let $d \in \mathcal{D}s(D)$ and $e \in \mathcal{D}s(E)$ be dash-assignments on D and E respectively. Define the multiplication dash-assignments, $d \bullet e \in \mathcal{D}s(D \bullet E)$, as follows:

{eqn: Ds mul}

$$d \bullet e\ (A) = \begin{cases} d(A) & \text{if } A \in D \\ e(-n+A) & \text{if } A \in n+E. \end{cases} \tag{3.1.87}$$ ◇

Example 3.1.88. Let

$$d = ((**)''\ *)'\ (**)'''\ * \in \mathcal{D}s(D) \quad \text{and} \quad e = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(11)} \in \mathcal{D}s(E)$$

be dash-assignments as in Example 3.1.85. Then, the multiplication dash-assignment $d \bullet e \in \mathcal{D}s(D \bullet E)$ is given by

$$\begin{aligned} \{1,2\} &\longmapsto 2, \\ \{1,2,3\} &\longmapsto 1, \\ \{4,5\} &\longmapsto 3, \\ \{8\} &\longmapsto 7, \\ \{8,9,10\} &\longmapsto 18, \\ \{7,8,9,10,11\} &\longmapsto 11. \end{aligned}$$

A visual representation of above dash-assignment is given by the concatenation.

$$d \bullet e = ((**)''\ *)'\ (**)'''\ *\ (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(11)}.$$ ◇

lem: Ds mul assoc

Lemma 3.1.89. *Let $n, m, p \in \mathbb{N}$ and $D \in \mathcal{B}r(n), E \in \mathcal{B}r(m), F \in \mathcal{B}r(p)$ be bracketings. Let $d \in \mathcal{D}s(D)$, $e \in \mathcal{D}s(E)$, and $f \in \mathcal{D}s(F)$ be dash-assignments. Then we have*

{eqn: Ds mul assoc}

$$d \bullet (e \bullet f) = (d \bullet e) \bullet f \tag{3.1.90}$$

as dash-assignments on $D \bullet E \bullet F$.

Proof. Let $A \in D \bullet E \bullet F$. Observe that,

$$\begin{aligned} d \bullet (e \bullet f)\ (A) &= \begin{cases} d(A) & \text{if } A \in D \\ e \bullet f\ (-n+A) & \text{if } A \in n + E \cdot F \end{cases} && (3.1.87) \\ &= \begin{cases} d(A) & \text{if } A \in D \\ e(-n+A) & \text{if } A \in n+E \\ f(-n-m+A) & \text{if } A \in n+m+E \end{cases} && (3.1.87) \\ &= \begin{cases} d \bullet e\ (A) & \text{if } A \in D \bullet E \\ f(-(n+m)+A) & \text{if } A \in (n+m)+E \end{cases} && (3.1.87) \\ &= (d \bullet e) \bullet f\ (A). && (3.1.87) \end{aligned}$$

Since $A \in D \bullet E \bullet F$ is arbitrary, we get

$$d \bullet (e \bullet f) = (d \bullet e) \bullet f$$

as required. □

lem: Ds mul one and one mul

Lemma 3.1.91. *Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$, and $d \in \mathcal{D}s(D)$ be a dash-assignment. Then,*

{eqn: DWord dashes mul one and one mul}

$$d \bullet I_{\mathcal{D}s} = d \qquad \text{and} \qquad I_{\mathcal{D}s} \bullet d = d \tag{3.1.92}$$

as dash-assignments on D.

Proof. Let $A \in D$. We get

$$\begin{aligned} d \bullet I_{\mathcal{D}s}\ (A) &= \begin{cases} d(A) & \text{if } A \in D \\ & \text{if } A \in I_{\mathcal{B}r} \end{cases} && (3.1.87) \\ &= \begin{cases} d(A) & \text{if } A \in D \\ & \text{if } A \in \varnothing \end{cases} && (3.1.31) \\ &= d. \end{aligned}$$

Since $A \in D$ is arbitrary, we get

$$d \bullet I_{\mathcal{D}s} = d.$$

Next, we get

$$\begin{aligned} I_{\mathcal{D}s} \bullet d\ (A) &= \begin{cases} & \text{if } A \in I_{\mathcal{B}r} \\ d(A) & \text{if } A \in D \end{cases} && (3.1.87) \\ &= \begin{cases} & \text{if } A \in \varnothing \\ d(A) & \text{if } A \in D \end{cases} && (3.1.31) \\ &= d(A). \end{aligned}$$

Since $A \in D$ is arbitrary, we get

$$I_{\mathcal{D}s} \bullet d = d.$$

□

defn: Ds dash

Definition 3.1.93. Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing. Let $d \in \mathcal{D}s(D)$ be a dash-assignment. Define the dash of the dash-assignment, $d' \in \mathcal{D}s\left(D'\right)$, as follows:

{eqn: Ds dash}

$$d'\ (A) := \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+1 & \text{if } A = [n] \text{ and } A \in D \\ 1 & \text{if } A = [n] \text{ and } A \notin D. \end{cases} \tag{3.1.94}$$

◇

lem: Ds unit dash

Lemma 3.1.95. *Consider the unit dash-assignment, $I_{\mathcal{D}s} \in \mathcal{D}s(I_{\mathcal{B}r})$. We have*

{eqn: Ds unit dash}

$$I'_{\mathcal{D}s} = I_{\mathcal{D}s} \tag{3.1.96}$$

as dash-assignments on the unit bracketing.

Proof. Since both $I'_{\mathcal{D}s}$ and $I_{\mathcal{D}s}$ are functions with the domain $I_{\mathcal{B}r} = \varnothing$, they are empty functions and thus trivially equal. For the sake of argument, let $A \in I_{\mathcal{B}r}$. Since $I_{\mathcal{B}r}$ is a bracketing, we have $A \neq \varnothing = [0]$. Therefore, from Definition 3.1.93, we get $d'(A) = d(A)$. □

prop: Ds dash k

Proposition 3.1.97. *Let $n \in \mathbb{N}$, $D \in \mathcal{Br}(n)$ be a bracketing. Let $d \in \mathcal{Ds}(D)$ be a dash-assignment. For $k \in \mathbb{N}^+$, let $d^{(k)} \in \mathcal{Ds}\left(D^{(k)}\right)$ denote the dash-assignment that we get after taking k dashes. Then, we have*

{eqn: Ds dash k}

$$d^{(k)}(A) = \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+k & \text{if } A = [n] \text{ and } A \in D \\ k & \text{if } A = [n] \text{ and } A \notin D. \end{cases} \tag{3.1.98}$$

Proof. We will use induction on k.

<u>Base case</u> ($k = 1$): Observe

$$d^{(1)}(A) = d'(A) = \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+1 & \text{if } A = [n] \text{ and } A \in D \\ 1 & \text{if } A = [n] \text{ and } A \notin D. \end{cases}$$

<u>Induction case</u> ($k \geq 1$): Assume that

$$d^{(k)}(A) = \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+k & \text{if } A = [n] \text{ and } A \in D \\ k & \text{if } A = [n] \text{ and } A \notin D. \end{cases}$$

We get

$$\begin{aligned} d^{(k+1)}(A) &= \left(d^{(k)}\right)'(A) \\ &= \begin{cases} d^{(k)}(A) & \text{if } A \neq [n] \\ d^{(k)}(A)+1 & \text{if } A = [n] \text{ and } A \in D^{(k)} \\ 1 & \text{if } A = [n] \text{ and } A \notin D^{(k)}. \end{cases} && (3.1.94) \\ &= \begin{cases} d^{(k)}(A) & \text{if } A \neq [n] \\ d^{(k)}(A)+1 & \text{if } A = [n] \text{ and } A \in D \cup \{[n]\} \\ 1 & \text{if } A = [n] \text{ and } A \notin D \cup \{[n]\}. \end{cases} && (3.1.51) \\ &= \begin{cases} d^{(k)}(A) & \text{if } A \neq [n] \\ d^{(k)}(A)+1 & \text{if } A = [n] \end{cases} \\ &= \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+k+1 & \text{if } A = [n] \text{ and } A \in D \\ k+1 & \text{if } A = [n] \text{ and } A \notin D. \end{cases} && \text{Induction} \end{aligned}$$

This completes the proof. □

Example 3.1.99. Consider the dash-assignments

$$d = ((**)''\ *)'\ (**)'''\ * \in \mathcal{Ds}(D) \quad \text{and} \quad e = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(11)} \in \mathcal{Ds}(E)$$

as described in Example 3.1.85. Then, the dash-assignment $d' \in \mathcal{D}s\big(D'\big)$ is given by:

$$\begin{aligned} \{1,2\} &\longmapsto 2, \\ \{1,2,3\} &\longmapsto 1, \\ \{4,5\} &\longmapsto 3, \\ \{1,2,3,4,5,6\} &\longmapsto 1. \end{aligned}$$

A visual representation of this dash-assignment is:

$$d' = (((**)''\ *)'\ (**)'''\ *)'.$$

The dash-assignment $e^{(5)} \in \mathcal{D}s\big(E^{(5)}\big)$ is given by:

$$\begin{aligned} \{2\} &\longmapsto 7, \\ \{2,3,4\} &\longmapsto 18, \\ \{1,2,3,4,5\} &\longmapsto 16. \end{aligned}$$

A visual representation of this dash-assignment is:

$$e^{(5)} = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(16)}.$$

In general, applying dashes to a dash-assignment involves adding an outermost bracket if there isn't one already and increasing the number on the outermost bracket by the appropriate dash value. ◇

defn: Ds undash

Definition 3.1.100. Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing, and $d \in \mathcal{D}s(D)$ be a dash-assignment. Define the *floor of the dash-assignment*, denoted $\lfloor d \rfloor \in \mathcal{D}s(\lfloor D)\rfloor$, as follows: Let $A \in \lfloor D \rfloor$ then we have $A \in D$. Define

{eqn: Ds undash}

$$\lfloor d \rfloor (A) := d(A). \tag{3.1.101}$$ ◇

Example 3.1.102. Consider the dash-assignments

$$d = ((**)''\ *)'\ (**)'''\ * \in \mathcal{D}s(D) \quad \text{and} \quad e = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(11)} \in \mathcal{D}s(E)$$

as described in Example 3.1.85. Then, we have

$$\lfloor d \rfloor = d.$$

The dash-assignment $\lfloor e \rfloor \in \mathcal{D}s(\lfloor E)\rfloor$ is given by

$$\begin{aligned} \{2\} &\longmapsto 7, \\ \{2,3,4\} &\longmapsto 18. \end{aligned}$$

A visual representation of this dash-assignment is:

$$\lfloor e \rfloor = *\ ((*)^{(7)}\ **)^{(18)}\ *.$$

The floor of a dash-assignment is obtained by removing the outermost bracket and any associated dashes, if present. ◇

lem: Ds unit undash

Lemma 3.1.103. *Consider the empty dash-assignment, $I_{\mathcal{D}s} \in \mathcal{D}s(I_{\mathcal{B}r})$. We have*

{eqn: Ds unit undash}

$$\lfloor I_{\mathcal{D}s} \rfloor = I_{\mathcal{D}s} \tag{3.1.104}$$

as dash-assignments on the unit bracketing.

Proof. From Lemma 3.1.58 we get $\lfloor I_{\mathcal{B}r} \rfloor = I_{\mathcal{B}r}$. Therefore, both $\lfloor I_{\mathcal{D}s} \rfloor$ and $I_{\mathcal{D}s}$ are functions with domain $I_{\mathcal{B}r} = \varnothing$. Consequently, both are empty functions and hence vacuously equal. □

prop: Ds undash undash

Proposition 3.1.105. *Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$, and $d \in \mathcal{D}s(D)$ be a dash-assignment. Then, we have*

{eqn: Ds undash undash}

$$\lfloor \lfloor d \rfloor \rfloor = \lfloor d \rfloor \tag{3.1.106}$$

as dash-assignments on $\lfloor D \rfloor$.

Proof. From Proposition 3.1.60 we gave $\lfloor \lfloor D \rfloor \rfloor = \lfloor D \rfloor$. Therefore, it is enough to show that $\lfloor \lfloor d \rfloor \rfloor (A) = \lfloor d \rfloor (A)$ for $A \in \lfloor D \rfloor$.

Let $A \in \lfloor D \rfloor$. We get

$$\lfloor \lfloor d \rfloor \rfloor (A) = \lfloor d \rfloor (A) \qquad (3.1.101).$$

Thus, we get

$$\lfloor \lfloor d \rfloor \rfloor = \lfloor d \rfloor .$$ □

prop: Ds dash undash

Proposition 3.1.107. *Let $n \in \mathbb{N}^{+}$, $D \in \mathcal{B}r(n)$ be a bracketing, and $d \in \mathcal{D}s(D)$ be a dash-assignment. Then, for $k \in \mathbb{N}$ the equality*

{eqn: Ds dash undash}

$$\left\lfloor d^{(k)} \right\rfloor = \lfloor d \rfloor \tag{3.1.108}$$

of dash-assignments on $\lfloor D \rfloor$ holds.

Proof. For the case $k = 0$, we get $d^{(0)} = d$. Therefore, the proposition is true for $k = 0$. Now assume $k \geq 1$. From Proposition 3.1.62 we get

$$\left\lfloor D^{(k)} \right\rfloor = \lfloor D \rfloor .$$

Therefore, it is enoguh to show that $\left\lfloor d^{(k)} \right\rfloor (A) = \lfloor d \rfloor (A)$ for $A \in \lfloor D \rfloor$. Let $A \in \lfloor D \rfloor$. Then, we have $A \neq [n]$. Observe that

$$\begin{aligned} \left\lfloor d^{(k)} \right\rfloor (A) &= d^{(k)} (A) && (3.1.101) \\ &= d(A) && (3.1.98) \\ &= \lfloor d \rfloor (A). && (3.1.101) \end{aligned}$$ □

prop: Ds undash eq self iff

Proposition 3.1.109. *Let $n \in \mathbb{N}^{+}$, $D \in \mathcal{B}r(n)$, and $d \in \mathcal{D}s(D)$ be a dash-assignment. Then, $\lfloor d \rfloor = d$ if and only is $[n] \notin D$.*

Proof. Suppose $\lfloor d \rfloor = d$. Since we have $\lfloor d \rfloor : \lfloor D \rfloor \longrightarrow \mathbb{N}^{+}$ and $d : D \longrightarrow \mathbb{N}^{+}$, we get

$$\lfloor D \rfloor = D.$$

From Proposition 3.1.64 we conclude $[n] \notin D$.

On the other hand, suppose $[n] \notin D$. Again from Proposition 3.1.64 we get $\lfloor D \rfloor = D$. Let $A \in \lfloor D \rfloor = D$. From Definition 3.1.100 we get $\lfloor d \rfloor\ (A) = d(A)$. Since $A \in \lfloor D \rfloor = D$ is arbitrarily chosen, we conclude $\lfloor d \rfloor = d$. □

prop: Ds undash dash

Proposition 3.1.110. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Assign*

$$k = \begin{cases} 0 & \text{if } [n] \notin D \\ d([n]) & \text{if } [n] \in D. \end{cases}$$

Then, the equality

{eqn: Ds undash dash}

$$(\lfloor d \rfloor)^{(k)} = d \tag{3.1.111}$$

of dash assignments on D holds.

Proof. We will consider the following two cases.

Case 1($[n] \notin D$): In this case we have $k = 0$. Therefore, we get

$$\begin{aligned} (\lfloor d \rfloor)^{(0)} &= \lfloor d \rfloor \\ &= d. && \text{Proposition 3.1.109} \end{aligned}$$

Case 2($[n] \in D$): In this case we have $k = d([n])$. Since $d : D \longrightarrow \mathbb{N}^+$ is a function with codomain $\mathbb{N}^+$ we get $k \geq 1$. From Proposition 3.1.65 we get

$$(\lfloor D \rfloor)^{(k)} = D.$$

Therefore, it is enough to show that $(\lfloor d \rfloor)^{(k)}\ (A) = d(A)$ for $A \in D$. Observe that

$$\begin{aligned} (\lfloor d \rfloor)^{(k)}\ (A) &= \begin{cases} \lfloor d \rfloor\ (A) & \text{if } A \neq [n] \\ \lfloor d \rfloor\ (A) + k & \text{if } A = [n] \text{ and } A \in \lfloor D \rfloor \\ k & \text{if } A = [n] \text{ and } A \notin \lfloor D \rfloor \end{cases} && (3.1.98) \\ &= \begin{cases} \lfloor d \rfloor\ (A) & \text{if } A \neq [n] \\ k & \text{if } A = [n] \end{cases} \\ &= \begin{cases} d(A) & \text{if } A \neq [n] \\ k & \text{if } A = [n] \end{cases} && (3.1.101) \\ &= \begin{cases} d(A) & \text{if } A \neq [n] \\ d([n]) & \text{if } A = [n] \end{cases} && (3.1.101) \\ &= d(A). \end{aligned}$$

Here, the second equality follows from the fact that $[n] \notin \lfloor D \rfloor$. □

Example 3.1.112. Let

$$d = ((**)''\ *)'\ (**)'''\ * \in \mathcal{Ds}(D) \quad \text{and} \quad e = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(11)} \in \mathcal{Ds}(E)$$

be dash-assignments as in Example 3.1.85. Since $[6] \notin D$ we get

$$\lfloor d \rfloor = d.$$

Since $[5] \in E$, we get $k = e([6]) = 11$. We have

$$(\lfloor e \rfloor)^{(11)} = e. \qquad \diamond$$

defn: Ds Res **Definition 3.1.113.** Let $n \in \mathbb{N}$, $D \in \mathcal{Br}(n)$ be a bracketing, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Let $(p,q] \subseteq_I [n]$ for $0 \le p \le q \le n$. Define a dash-assignment, $d_{(p,q]}$ on $D_{(p,q]}$ as follows:

{eqn: Ds Res}

$$d_{(p,q]}(A) := d(p+A) \tag{3.1.114}$$

for $A \in D_{(p,q]}$. This assignment is well-defined since for $A \in D_{(p,q]}$, we get $p+P \in D$. Therefore, $d(p+A) \in \mathbb{N}^+$. ⋄

Example 3.1.115. Let

$$d = ((**)''\ *)'\ (**)'''\ * \in \mathcal{Ds}(D) \quad \text{and} \quad e = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(11)} \in \mathcal{Ds}(E)$$

be dash-assignments as described in Example 3.1.85. Then, the dash-assignment $d_{[3]} \in \mathcal{Ds}\left(D_{[3]}\right)$ is given by:

$$\{1,2\} \longmapsto 2,$$
$$\{1,2,3\} \longmapsto 1.$$

A visual representation of this dash-assignment is:

$$d_{[3]} = ((**)''\ *)'.$$

The dash-assignment $d_{(3,6]} \in \mathcal{Ds}\left(D_{(3,6]}\right)$ is given by:

$$\{1,2\} \longmapsto 3.$$

A visual representation of this dash-assignment is:

$$d_{(3,6]} = (**)'''\ *.$$

The dash-assignment $e_{(2,4]} \in \mathcal{Ds}\left(D_{(2,4]}\right)$ is given by the empty function since $D_{(2,4]} = \varnothing$. A visual representation of this dash-assignment is:

$$e_{(2,4]} = **.$$ ⋄

prop: Ds Res self **Proposition 3.1.116.** *Let $n \in \mathbb{N}$, $D \in \mathcal{Br}(n)$ be a bracketing, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Then, we get*

{eqn: Ds Res self}

$$d_{[n]} = d \tag{3.1.117}$$

as dash-assignments on D.

Proof. From Proposition 3.1.71 we get $D_{[n]} = D$. Therefore, it is enoguh to show that $d_{[n]}(A) = d(A)$ for $A \in D$. Consider the following calculation for $A \in D$:

$$\begin{aligned} d_{[n]}(A) = d_{(0,n]}(A) &= d(0+A) && (3.1.114) \\ &= d(A). \end{aligned}$$

Thus, we conclude $d_{[n]} = d$. □

prop: Ds Res mul **Proposition 3.1.118.** *Let $n,m \in \mathbb{N}$, $D \in \mathcal{Br}(n)$, $E \in \mathcal{Br}(m)$ be bracketings, and $d \in \mathcal{Ds}(D)$, $e \in \mathcal{Ds}(E)$ be dash assignments. Let $(p,q] \subseteq_I [n]$ with $0 \le p \le q \le n$. Then, we get*

{eqn: Ds Res mul}

$$\left(d \bullet e\right)_{(p,q]} = d_{(p,q]} \tag{3.1.119}$$

as dash assignments on $D_{(p,q]}$.

Proof. From Proposition 3.1.73 we get $(D \bullet E)_{(p,q]} = D_{(p,q]}$. Therefore, it is enoguh to show that $(d \bullet e)_{(p,q]}(A) = d_{(p,q]}(A)$ for $A \in D_{(p,q]}$.

Let $A \in D_{(p,q]}$. We get some $X \in D$ with $X \subseteq (p,q]$ such that $-p + X = A$. It follows that

$$p + A = X \subseteq (p,q] \subseteq [n].$$

Consequently, we get

$$\begin{aligned} \left(d \cdot e\right)_{(p,q]}(A) &= d \cdot e\,(p+A) && (3.1.114) \\ &= d(p+A) && (3.1.87) \\ &= d_{(p,q]}(A). && (3.1.114) \end{aligned}$$

Since $A \in D_{(p,q]}$ is arbitrary, we get

$$\left(d \cdot e\right)_{(p,q]} = d_{(p,q]}. \qquad \square$$

prop: Ds mul Res

Proposition 3.1.120. *Let $n, m \in \mathbb{N}$, $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings, and $d \in \mathcal{D}s(D)$, $e \in \mathcal{D}s(E)$ be dash assignments. Let $(p,q] \subseteq_I (n, n+m]$ with $n \le p \le q \le n+m$. Then, we get*

{eqn: Ds mul Res}

$$\left(d \cdot e\right)_{(p,q]} = e_{(p-n,q-n]} \tag{3.1.121}$$

as dash assignments on $E_{(p-n,q-n]}$.

Proof. From Proposition 3.1.75 we get $(D \bullet E)_{(p,q]} = E_{(p-n,q-n]}$. Therefore, it is enoguh to show that $(d \bullet e)_{(p,q]}(A) = e_{(p-n,q-n]}(A)$ for $A \in D_{(p-n,q-n]}$.

Let $A \in E_{(p-n,q-n]}$. We get some $X \in E$ with $X \subseteq (p-n, q-n]$ such that $-(p-n) + X = A$. We have $p + A = n + X \subseteq (p,q] \subseteq (n, n+m]$. Observe that,

$$\begin{aligned} \left(d \cdot e\right)_{(p,q]}(A) &= d \cdot e\,(p+A) && (3.1.114) \\ &= e(-n+p+A) && (3.1.87) \\ &= e_{(p-n,q-n]}(A). && (3.1.114) \end{aligned}$$

Since $A \in E_{(p-n,q-n]}$ is arbitrary, we get

$$\left(d \cdot e\right)_{(p,q]} = e_{(p-n,q-n]}. \qquad \square$$

lem: Ds Res mul self

Lemma 3.1.122. *Let $n, m \in \mathbb{N}$, $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings. Let $d \in \mathcal{D}s(D)$ and $e \in \mathcal{D}s(E)$ be dash-assignments. Then, we get*

{eqn: Ds Res mul self}

$$\left(d \cdot e\right)_{[n]} = d \tag{3.1.123}$$

as dash-assignments on D.

Proof. We see that

$$\begin{aligned} \left(d \cdot e\right)_{[n]} &= e_{[n]} && (3.1.119) \\ &= d. && (3.1.117) \end{aligned} \qquad \square$$

lem: Ds mul Res self

Lemma 3.1.124. *Let $n, m \in \mathbb{N}$, $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings. Let $d \in \mathcal{D}s(D)$ and $e \in \mathcal{D}s(E)$ be dash-assignments. Then, we get*

{eqn: Ds mul Res self}

$$\left(d \cdot e\right)_{(n,n+m]} = e \tag{3.1.125}$$

as dash-assignments on E.

Proof. We see that

$$\begin{aligned}\left(d\cdot e\right)_{(n,n+m]} &= e_{(0,m]} && (3.1.121)\\ &= e. && (3.1.117)\end{aligned}$$ □

CHAPTER 4

Dashed Words as Free Dashed Monoids

section: FreeDMon

Definitions:

Write a summary of the previous section(s). Show that $\mathcal{DMon}\langle S\rangle$ satisfies the UP.

The content is in OldParts Sec5FreeDashedMonoid

4.0.1. The dashed-monoid of dashed words. We have established what is a bracketing of an n letter word. Also, we have dash assignment on the bracketing. In this section, we will give a construction of free dashed monoid generated by a set S, denoted $\mathcal{DMon}\langle S\rangle$, using bracketing and dash-assignment. We will use the foundation laid out in the previous chapters to prove that the given construction is indeed a free construction of a dashed monoid. To achieve this, we will show that $\mathcal{DMon}\langle S\rangle$ has a free dashed monoid like structure as in Definition 2.3.1.

Let S be a set.

defn: DWord

Definition 4.0.1. A *dashed-word* with alphabet in S is a dependent 4-tuple (n,u,D,d) where

- n is a natural number which corresponds to the length of the dashed-word,
- $u:[n]\longrightarrow S$ is a function which corresponds to the letters in the dashed-word,
- $D\in\mathcal{Br}(n)$ is a bracketing on n letters, and
- $d:D\in\mathcal{Ds}(D)$ is a dash-assignment on the bracketing D. ⋄

Notation 4.0.2. We will denote the set of all dashed words with alphabet in S by $\mathcal{DMon}\langle S\rangle$. ⋄

defn: DWord Mon

Definition 4.0.3. We define the monoid of dashed-words with alphabet in S as follows:

- The underlying set is $\mathcal{DMon}\langle S\rangle$ as in Definition 4.0.1.
- The unit dashed-word, denoted $I\in\mathcal{DMon}\langle S\rangle$, is given by

{eqn: DWord unit}

$$I:=(0,I_{\mathcal{Wr}},I_{\mathcal{Br}},I_{\mathcal{Ds}}) \tag{4.0.4}$$

where $I_{\mathcal{Wr}}$ is the unit word as described in Definition 1.2.8, $I_{\mathcal{Br}}$ is the unit bracketing as in Definition 3.1.30, and $I_{\mathcal{Ds}}$ is the unit dash-assignment as in Definition 3.1.83.

- For dashed words (n,u,D,d) and (m,v,E,e), the multiplication is given by

{eqn: DWord mul}

$$(n,u,D,d)\bullet(m,v,E,e) := (n+m, u\bullet v, D\bullet E, d\bullet e) \tag{4.0.5}$$

where $u\bullet v \in \mathcal{Wr}(n+m)$ is as in Definition 1.2.9, $D\bullet E \in \mathcal{Br}(n+m)$ is as in Definition 3.1.34, and $d\bullet e \in \mathcal{Ds}(D\bullet E)$ is as in Definition 3.1.86.

The associativity for dashed-words follows from Lemmas 1.2.11, 3.1.37, and 3.1.89. The unit conditions for dashed-words follow from Lemmas 1.2.13, 3.1.40, and 3.1.91. ⋄

defn: DWord DMon

Definition 4.0.6. We define the dashed-monoid of dashed-words with alphabet in S as follows:

- The underlying monoid is $\mathcal{DMon}\langle S\rangle$ as in Definition 4.0.3.
- The dash map $(-)' : \mathcal{DMon}\langle S\rangle \longrightarrow \mathcal{DMon}\langle S\rangle$ is given by

{eqn: DWord dash}

$$(n,u,D,d)' := (n,u,D',d') \tag{4.0.7}$$

where $D' \in \mathcal{Br}(n)$ is as in Definition 3.1.43 and $d' \in \mathcal{Ds}\left(D'\right)$ is as in Definition 3.1.93.

The unit condition for dash follows from Lemmas 3.1.48 and 3.1.95. ⋄

example: DWord

Example 4.0.8. Let $S = \{a,b,c\}$. Consider the dashed-word

$$x = (6,u,D,e) \in \mathcal{DMon}\langle S\rangle$$

described as follows:

$$\begin{aligned} u = \{1 &\longmapsto a,\\ 2 &\longmapsto b,\\ 3 &\longmapsto c,\\ 4 &\longmapsto a,\\ 5 &\longmapsto b,\\ 6 &\longmapsto a\}. \end{aligned}$$

The bracketing $D \in \mathcal{Br}(6)$ and the dash-assignment $d \in \mathcal{Ds}(D)$ are same as in Example 3.1.85. A visual representation of the dashed-word is as follows:

$$x = ((ab)''\ c)'\ (ab)''\ a.$$

The dashed-word

$$y = (5,v,E,e) \in \mathcal{DMon}\langle S\rangle$$

is described as follows:

$$\begin{aligned} u = \{1 &\longmapsto b,\\ 2 &\longmapsto b,\\ 3 &\longmapsto c,\\ 4 &\longmapsto a,\\ 5 &\longmapsto a\}. \end{aligned}$$

The bracketing $E \in \mathcal{B}r(5)$ and the dash-assignment $e \in \mathcal{D}s(E)$ are same as in Example 3.1.85. A visual representation of the dashed-word is as follows:

$$y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

The multiplication of these dashed-words is given by the concatenation:

$$x \bullet y = ((ab)''\ c)'\ (ab)''\ a(b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

The dash of the dashed-words is given by

$$x' = (((ab)''\ c)'\ (ab)''\ a)'$$
$$y^{(5)} = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(16)}. \qquad \diamond$$

defn: DWord count

Definition 4.0.9. Define a *length* function $\hat{l} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N}$ as follows:

{eqn: DWord count}

$$\hat{l}((n,u,D,d)) = n. \tag{4.0.10}$$ $\diamond$

prop: DWord count DMon

Proposition 4.0.11. *Consider the set of natural numbers, $\mathbb{N}$, to be a dashed-monoid with the addition as the monoid structure and identity as the dash operation. Then, the length function $\hat{l} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N}$ is a dashed-monoid homomorphism. That is, for all $x, y \in \mathcal{DMon}\langle S\rangle$ we get*

{eqn: DWord count unit}

$$\hat{l}(I) = 0, \tag{4.0.12}$$

{eqn: DWord count mul}

$$\hat{l}(x \bullet y) = \hat{l}(x) + \hat{l}(y), \quad \text{and} \tag{4.0.13}$$

{eqn: DWord count dash}

$$\hat{l}(x') = \hat{l}(x). \tag{4.0.14}$$

Proof. Let $I = (0, I_{\mathcal{W}r}, I_{\mathcal{B}r}, I_{\mathcal{D}s})$ be the unit dashed-word. We get

$$\hat{l}(I) = 0.$$

Now let $x = (n,u,D,d)$ and $y = (m,v,E,e)$ be two dashed-words. From the equation (4.0.5) we get $x \bullet y = (n+m, u \bullet v, D \bullet E, d \bullet e)$. Therefore, we get

$$\hat{l}(x \bullet y) = n + m = \hat{l}(x) + \hat{l}(y).$$

Finally, let $x = (n,u,D,d)$ be a dashed-word. From the equation (4.0.7) we get $x' = (n,u,D',d')$. Therefore, we get

$$\hat{l}(x') = n = \hat{l}(x) = (\hat{l}(x))'.$$ $\square$

prop: DWord count I iff

Proposition 4.0.15. *Let $x \in \mathcal{DMon}\langle S\rangle$. Then, $\hat{l}(x) = 0$ if and only if $x = I$.*

Proof. Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$. Suppose $\hat{l}(x) = 0$, that is, $n = 0$. It follows that, $u : [0] \longrightarrow S$ is the empty function $I_{\mathcal{W}r}$. Also, we have $D \subseteq \mathcal{P}_I^+(0) = \varnothing$. Thus, we get $D = I_{\mathcal{B}r}$. It follows that, $d : I_{\mathcal{B}r} \longrightarrow \mathbb{N}$ is the empty function. Therefore, we get $d = I_{\mathcal{D}s}$. Thus, we have

$$x = (0, I_{\mathcal{W}r}, I_{\mathcal{B}r}, I_{\mathcal{D}s}) = I.$$

On the other hand, if $x = I$ then from equation (4.0.12) get $\hat{l}(x) = 0$. $\square$

Notation 4.0.16. Let $\mathcal{DMon}\langle S\rangle^+$ denote the collection of all non-zero length dashed-words. That is,

$$\begin{aligned}\mathcal{DMon}\langle S\rangle^+ &:= \{x \in \mathcal{DMon}\langle S\rangle \mid x \neq I\}\\ &= \{x \in \mathcal{DMon}\langle S\rangle \mid \hat{l}(x) > 0\} && \textit{Proposition } 4.0.15.\end{aligned}$$

$\diamond$

Notation 4.0.17. For $x \in \mathcal{DMon}\langle S\rangle$ and $k \in \mathbb{N}$ let $x^{(k)}$ denote the dashed-word obtained by applying the dash operation k-times. In particular, we have

$$x^{(0)} = x \qquad \text{and} \qquad x^{(1)} = x'.$$

$\diamond$

prop: DWord count dash k

Proposition 4.0.18. *Let $k \in \mathbb{N}$ and $x \in \mathcal{DMon}\langle S\rangle$. Then we get*

{eqn: DWord count dash k}

$$\hat{l}\left(x^{(k)}\right) = \hat{l}(x). \tag{4.0.19}$$

Proof. We will consider the cases $k = 0$ and $k \geq 1$. If $k = 0$, then we have $x^{(0)} = x$. Thus, we get

$$\hat{l}\left(x^{(0)}\right) = \hat{l}(x).$$

Now, let $k \geq 1$. From Proposition 4.0.11, we have $\hat{l}(x') = \hat{l}(x)$. Therefore, applying this fact repeatedly, we get

$$\hat{l}\left(x^{(k)}\right) = \hat{l}(x).$$

$\square$

defn: DWord undash

Definition 4.0.20. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle$. Define *floor* of x, denoted $\lfloor x\rfloor$, as follows:

{eqn: DWord undash}

$$\lfloor x\rfloor := (n, u, \lfloor D\rfloor, \lfloor d\rfloor) \tag{4.0.21}$$

where $\lfloor D\rfloor \in \mathcal{Br}(n)$ is as in Definition 3.1.53 and $\lfloor d\rfloor \in \mathcal{Ds}(\lfloor D)\rfloor$ is as in Definition 3.1.100. $\diamond$

prop: DWord unit undash

Proposition 4.0.22. *We have*

{eqn: DWord unit undash}

$$\lfloor I\rfloor = I. \tag{4.0.23}$$

Proof. Observe that

$$\begin{aligned}\lfloor I\rfloor &= \lfloor(0, I_{\mathcal{Wr}}, I_{\mathcal{Br}}, I_{\mathcal{Ds}})\rfloor && (4.0.4)\\ &= (0, I_{\mathcal{Wr}}, \lfloor I_{\mathcal{Br}}\rfloor, \lfloor I_{\mathcal{Ds}}\rfloor) && (4.0.21)\\ &= (0, I_{\mathcal{Wr}}, I_{\mathcal{Br}}, I_{\mathcal{Ds}}) && (3.1.59) \text{ and } (3.1.104)\\ &= I. && (4.0.4)\end{aligned}$$

$\square$

Example 4.0.24. Let $S = \{a, b, c\}$. Consider the dashed-words

$$xx = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}$$

as in Example 4.0.8. The floor of the dashed-word x is given by

$$\lfloor x\rfloor = x = ((ab)''\ c)'\ (ab)''\ a$$

and the floor of the dashed-word y is given by

$$\lfloor y\rfloor = b\ ((b)^{(7)}\ ca)^{(18)}\ a.$$

$\diamond$

prop: DWord undash count

Proposition 4.0.25. *Let $x \in \mathcal{DMon}\langle S\rangle$ be a dashed-word. Then, we get*

{eqn: DWord undash count}

$$\hat{l}(\lfloor x)\rfloor = \hat{l}(x). \tag{4.0.26}$$

Proof. Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$ where $n \in \mathbb{N}$, $u : [n] \longrightarrow S$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$. Observe that

$$\begin{aligned}
\hat{l}(\lfloor x)\rfloor &= \hat{l}((n,u,\lfloor D),\lfloor d\rfloor)\rfloor && (4.0.21)\\
&= n && (4.0.10)\\
&= \hat{l}(x). && (4.0.10)
\end{aligned}$$
□

prop: DWord undash undash

Proposition 4.0.27. *Let $x \in \mathcal{DMon}\langle S\rangle$ be a dashed-word. Then, we get*

{eqn: DWord undash undash}

$$\lfloor\lfloor x\rfloor\rfloor = \lfloor x\rfloor. \tag{4.0.28}$$

Proof. Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$ where $n \in \mathbb{N}$, $u : [n] \longrightarrow S$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$. Observe that

$$\begin{aligned}
\lfloor\lfloor x\rfloor\rfloor &= \lfloor(n,u,\lfloor D\rfloor,\lfloor d\rfloor)\rfloor && (4.0.21)\\
&= (n,u,\lfloor\lfloor D\rfloor\rfloor,\lfloor\lfloor d\rfloor\rfloor) && (4.0.21)\\
&= (n,u,\lfloor D\rfloor,\lfloor d\rfloor) && (3.1.61) \text{ and } (3.1.108)\\
&= \lfloor x\rfloor. && (4.0.21)
\end{aligned}$$
□

prop: DWord dash undash

Proposition 4.0.29. *Let $x \in \mathcal{DMon}\langle S\rangle$ be a dashed-word and $k \in \mathbb{N}^+$. Then, we get*

{eqn: DWord dash undash}

$$\left\lfloor x^{(k)}\right\rfloor = \lfloor x\rfloor. \tag{4.0.30}$$

Proof. Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$ where $n \in \mathbb{N}$, $u : [n] \longrightarrow S$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$. Observe that

$$\begin{aligned}
\left\lfloor x^{(k)}\right\rfloor &= \left\lfloor(n,u,D^{(k)},d^{(k)})\right\rfloor && (4.0.7)\\
&= (n,u,\left\lfloor D^{(k)}\right\rfloor,\left\lfloor d^{(k)})\right\rfloor && (4.0.21)\\
&= (n,u,\lfloor D\rfloor,\lfloor d\rfloor) && (3.1.63) \text{ and } (3.1.108)\\
&= \lfloor x\rfloor. && (4.0.21)
\end{aligned}$$
□

prop: DWord undash eq self iff

Proposition 4.0.31. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$ be a dashed-word with $n \geq 1$. The, $\lfloor x\rfloor = x$ if and only if $[n] \notin D$.*

Proof. Observe that

$$\lfloor x\rfloor = (n,u,\lfloor D\rfloor,\lfloor d\rfloor).$$

Therefore, from Propositions 3.1.64 and 3.1.109 we get that $\lfloor x\rfloor = x$ if and only if $[n] \notin D$. □

prop: DWord undash dash

Proposition 4.0.32. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$ be a dashed-word with $n \geq 1$. Assign*

$$k = \begin{cases} 0 & \text{if } [n] \notin D\\ d([n]) & \text{if } [n] \in D. \end{cases}$$

Then, the equality

{eqn: DWord = dash undash}

$$(\lfloor x \rfloor)^{(k)} = x \tag{4.0.33}$$

of dashed-words hold.

Proof. Note that $D \in \mathcal{B}r(n)$ and $d \in \mathcal{D}s(D)$ satisfy the conditions of Propositions 3.1.65 and 3.1.110 respectively. Therefore, we get

$$\begin{aligned}
(\lfloor x \rfloor)^{(k)} &= \left(n, u, \lfloor D \rfloor, \lfloor d \rfloor\right)^{(k)} && (4.0.21)\\
&= \left(n, u, \lfloor D \rfloor^{(k)}, \lfloor d \rfloor^{(k)}\right) && (4.0.7)\\
&= (n, u, D, d) && (3.1.66) \text{ and } (3.1.111)\\
&= x.
\end{aligned}$$

□

defn: DWord Res

Definition 4.0.34. Let (n,u,D,d) be a dashed word. Let $(p,q] \subseteq_I [n]$ with $0 \le p \le q \le n$. Define a dashed word $(n,u,D,d)_{(p,q]}$ as follows:

{eqn: DWord Res}

$$(n,u,d,D)_{(p,q]} := (q-p, u_{(p,q]}, D_{(p,q]}, d_{(p,q]}) \tag{4.0.35}$$

where $u_{(q,p]} \in \mathcal{W}r(q-p)$ is as in Definition 1.2.15. The bracketing, $D_{(p,q]} \in \mathcal{B}r(q-p)$, is as in Definition 3.1.68. The dash-assignment, $d_{(p,q]} \in \mathcal{D}s\left(D_{(p,q]}\right)$ is as in Definition 3.1.113. ◇

Example 4.0.36. Let $S = \{a,b,c\}$. Consider the dashed-words

$$x = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}$$

as in Example 4.0.8. Then, we have

$$\begin{aligned}
x_{[3]} &= ((ab)''\ c)'\\
x_{(3,6]} &= (ab)''\ a\\
y_{(2,5]} &= caa.
\end{aligned}$$

◇

prop: DWord restriction count

Proposition 4.0.37. *Let $x \in \mathcal{DM}on\langle S \rangle$ and $n = \hat{l}(x)$. Let $L \subseteq_I [n]$ be a subinterval. Then we get*

{eqn: DWord restriction count}

$$\hat{l}(x_L) = |L|. \tag{4.0.38}$$

Proof. Let $L = (p,q]$ with $0 \le p \le q \le n$. Then from Definition 4.0.34 we get

$$\hat{l}(x_L) = q - p = |L|.$$

□

prop: DWord Res self

Proposition 4.0.39. *Let $x = (n,u,D,d)$ be a dashed word. Then, we have*

{eqn: DWord Res self}

$$x_{[n]} = x. \tag{4.0.40}$$

Proof. Observe that

$$\begin{aligned}
x_{[n]} &= (n, u_{[n]}, D_{[n]}, d_{[n]}) && (4.0.35)\\
&= (n,u,D,d) && (1.2.18), (3.1.72), \text{ and } (3.1.117)\\
&= x.
\end{aligned}$$

□

prop: DWord Res mul

Proposition 4.0.41. *Let $x=(n,u,D,d)$ and $y=(m,v,E,e)$ be dashed-words. Let $(p,q]\subseteq_I [n]$ with $0\le p\le q\le n$. Then, we get*

{eqn: DWord Res mul}

$$(x\bullet y)_{(p,q]}=x_{(p,q]}. \tag{4.0.42}$$

Proof. Observe that

$$\begin{aligned}
&(x\bullet y)_{(p,q]}\\
&=(n+m,u\bullet v,D\bullet E,d\bullet e)_{(p,q]} && (4.0.5)\\
&=\big(q-p,(u\bullet v)_{(p,q]},(D\bullet E)_{(p,q]},(d\bullet e)_{(p,q]}\big) && (4.0.35)\\
&=\big(q-p,u_{(p,q]},D_{(p,q]},d_{(p,q]}\big) && (1.2.20),\ (3.1.74),\ \text{and } (3.1.119)\\
&=x_{(p,q]}. && (4.0.35)
\end{aligned}$$ □

prop: DWord mul Res

Proposition 4.0.43. *Let $x=(n,u,D,d)$ and $y=(m,v,E,e)$ be dashed-words. Let $(p,q]\subseteq_I [n]$ with $n\le p\le q\le n+m$. Then, we get*

{eqn: DWord mul Res}

$$(x\bullet y)_{(p,q]}=y_{(p-n,q-n]}. \tag{4.0.44}$$

Proof. Observe that

$$\begin{aligned}
&(x\bullet y)_{(p,q]}\\
&=(n+m,u\bullet v,D\bullet E,d\bullet e)_{(p,q]} && (4.0.5)\\
&=\big(q-p,(u\bullet v)_{(p,q]},(D\bullet E)_{(p,q]},(d\bullet e)_{(p,q]}\big) && (4.0.35)\\
&=\big(q-p,v_{(p-n,q-n]},E_{(p-n,q-n]},e_{(p-n,q-n]}\big) && (1.2.22),\ (3.1.76),\ \text{and } (3.1.121)\\
&=y_{(p-n,q-n]}. && (4.0.35)
\end{aligned}$$ □

prop: DWord Res mul self

Proposition 4.0.45. *Let $x=(n,u,D,d)$ and $y=(m,v,E,e)$ be two dashed words. Then, we get*

{eqn: DWord Res mul self}

$$(x\bullet y)_{[n]}=x. \tag{4.0.46}$$

Proof. Observe that,

$$\begin{aligned}
(x\bullet y)_{[n]}&=x_{[n]} && (4.0.42)\\
&=x. && (4.0.40)
\end{aligned}$$ □

prop: DWord mul Res self

Proposition 4.0.47. *Let $x=(n,u,D,d)$ and $y=(m,v,E,e)$ be two dashed words. Then, we get*

{eqn: DWord mul Res self}

$$(x\bullet y)_{(n,n+m]}=y. \tag{4.0.48}$$

Proof. Observe that,

$$\begin{aligned}
(x\bullet y)_{(n,n+m]}&=y_{[m]} && (4.0.44)\\
&=y. && (4.0.40)
\end{aligned}$$ □

lem: Dword left cancellative

Lemma 4.0.49. *The dashed monoid of dashed-words with alphabet in S is a left cancellative monoid. That is, for $x,y,z\in \mathcal{DMon}\langle S\rangle$,*

{eqn: Dword left cancellative}

$$x\bullet y=x\bullet z \qquad \textit{implies} \qquad y=z. \tag{4.0.50}$$

Proof. Let $x, y, z \in \mathcal{DMon}\langle S\rangle$ with $\hat{l}(x) = n$, $\hat{l}(y) = m$, and $\hat{l}(z) = p$. Suppose we have

$$x \bullet y = x \bullet z.$$

We get

$$n + m = \hat{l}(x \bullet y) = \hat{l}(x \bullet z) = n + p.$$

Thus, we have $m = p$. Observe that

$$\begin{aligned} y &= (x \bullet y)_{(n,n+m]} && (4.0.48) \\ &= (x \bullet z)_{(n,n+p]} \\ &= z. && (4.0.48) \end{aligned}$$

□

lem: Dword right cancellative **Lemma 4.0.51.** *The dashed monoid of dashed-words with alphabet in S is a right cancellative monoid. That is, for $x, y, z \in \mathcal{DMon}\langle S\rangle$,*

{eqn: Dword right cancellative}

$$y \bullet x = z \bullet x \qquad \textit{implies} \qquad y = z. \tag{4.0.52}$$

Proof. Let $x, y, z \in \mathcal{DMon}\langle S\rangle$ with $\hat{l}(x) = n$, $\hat{l}(y) = m$, and $\hat{l}(z) = p$. Suppose we have

$$y \bullet x = z \bullet x.$$

We get

$$m + n = \hat{l}(y \bullet x) = \hat{l}(z \bullet x) = p + n.$$

Thus, we have $m = p$. Observe that

$$\begin{aligned} y &= (y \bullet x)_{[m]} && (4.0.46) \\ &= (z \bullet x)_{[p]} \\ &= z. && (4.0.46) \end{aligned}$$

□

4.0.2. Multiplicative basis of dashed words. In this section, we will show that the dashed monoid $\mathcal{DMon}\langle S\rangle$ has a free dashed monoid like structure (Definition 2.3.1). To achieve this, we will use $\hat{l} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N}$ as in Definition 4.0.9 as the length function. We need to provide multiplicative basis $\mathtt{G}$ and dash basis $\mathtt{H}$ of $\mathcal{DMon}\langle S\rangle$. Finally, we will show that these satisfy the interlocking conditions as in (2.3.4) and (2.3.5).

We will start by constructing a multiplicative basis $\mathtt{G}$ for $\mathcal{DMon}\langle S\rangle$.

defn: DWord basis **Definition 4.0.53.** Define a subset $\mathtt{G} \subseteq \mathcal{DMon}\langle S\rangle$ as follows:

{eqn: DWord basis}

$$\mathtt{G} := \big\{(n, u, D, d) \mid n = 1 \text{ or } (n \geq 2 \text{ and } [n] \in D)\big\}. \tag{4.0.54}$$

◇

Remark 4.0.55. Observe that $I \notin \mathtt{G}$ since $\hat{l}(I) = 0$. Therefore, we have

$$\mathtt{G} \subseteq \mathcal{DMon}\langle S\rangle^{+}.$$

◇

Example 4.0.56. Let $S = \{a, b, c\}$. Consider the dashed-words

$$x = (6, u, D, d) \qquad \text{and} \qquad y = (5, u, D, d)$$

as described in Example 4.0.8. These dashed-words are described as follows:

$$x = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

Then, $x \notin \mathtt{G}$ since $6 \geq 2$ and $[6] \notin D$. On the other hand, $y \in \mathtt{G}$ since $5 \geq 2$ and $[5] \in E$. Moreover, $a \in \mathtt{G}$ and $b^{(7)} \in \mathtt{G}$ since we have $\hat{l}(a) = \hat{l}\left(b^{(7)}\right) = 1$. ⋄

defn: Br head

Definition 4.0.57. Let $n \in \mathbb{N}^+$ and $D \in \mathcal{B}r(n)$ be a bracketing. Define the *leading left interval*, $L \subseteq [n]$, as

{eqn: Br head}

$$L := \{1\} \cup \left(\bigcup_{A \in D;\ 1 \in A} A \right). \tag{4.0.58}$$

The set L as defined above is an interval since it is a union of intervals containing 1. Suppose $L = [p]$ for some $1 \leq p \leq n$. Define the *trailing right interval*, $R \subseteq [n]$, as the complement of L:

{eqn: Br tail}

$$R := (p, n]. \tag{4.0.59}$$

⋄

Remark 4.0.60. Note that the collection

$$\mathfrak{L} := \{A \in D \mid 1 \in A\}$$

is a finite collection with a linear order with respect to set inclusion. The leading left interval L is the largest interval in the above collection, provided that the collection is non-empty; otherwise, L is equal to $[1]$. ⋄

prop: Br head in D iff

Proposition 4.0.61. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{B}r(n)$ be a bracketing, and $L = [p]$ for some $1 \leq p \leq n$ be the leading left interval as in Definition 4.0.57. Then, $L \in D$ if and only if there exists $X \in D$ such that $1 \in X$.*

Proof. Suppose $L \in D$. Then, by Definition 4.0.57 we have $1 \in L$. Thus, we can take $X = L$.

On the other hand, suppose there exists $X \in D$ such that $1 \in X$. Since L is the largest such interval in D, we have $L \in D$.

prop: Br head union tail = sel

Proposition 4.0.62. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{B}r(n)$ be a bracketing, and L and R be the leading left interval and the trailing right interval as in Definition 4.0.57. Then, for $A \in D$ we get either $A \subseteq L$ or $A \subseteq R$.*

Proof. We will prove this by considering following cases.

Case 1 ($L \in D$): In this case, since D is a bracketing and $A, L \in D$ we get either $A \cap L = \varnothing$, $A \subseteq L$, or $L \subseteq A$. In this first subcase, we get

$$A \subseteq [n] \setminus L = R.$$

In the second subcase, we get $A \subseteq L$ as required. For the third subcase, we get $1 \subseteq A$ since $1 \in L$ and $L \subseteq A$. Since L is the largest interval in D such that $1 \in L$ we get $A \subseteq L$.

Case 2 ($L \notin D$): In this case, from Proposition 4.0.61, we get that for every $X \in D$, it holds that $1 \notin X$. Therefore, we have

$$L = \{1\}.$$

□

Taking $X = A$, we get that $1 \notin A$. Thus, we have

$$A \subseteq [n] \setminus \{1\} = [n] \setminus L = R.$$

defn: DWord head

Definition 4.0.63. Define functions

$$\mathtt{Head} : \mathcal{DMon}\langle S\rangle^{+} \longrightarrow \mathcal{DMon}\langle S\rangle^{+}$$

and

$$\mathtt{Tail} : \mathcal{DMon}\langle S\rangle^{+} \longrightarrow \mathcal{DMon}\langle S\rangle$$

as follows: Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^{+}$ be a non-unit dashed-word. Then, we have $n \geq 1$. Let $L,R \subseteq_I [n]$ be the leading left intervals and the trailing right interval as in Definition 4.0.57. Since L is non-empty, we get $x_L \in \mathcal{DMon}\langle S\rangle^{+}$. Define

{eqn: DWord head}

$$\mathtt{Head}(x) := x_L, \quad \text{and} \tag{4.0.64}$$

{eqn: DWord tail}

$$\mathtt{Tail}(x) := x_R. \tag{4.0.65}$$

◇

Example 4.0.66. Let $S = \{a,b,c\}$. Consider the dashed-words

$$x = (6,u,D,d) \qquad \text{and} \qquad y = (5,u,D,d)$$

as described in Example 4.0.8. These dashed-words are described as follows:

$$x = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

Then, we have

$$\begin{aligned}\mathtt{Head}(x) &= ((ab)''\ c)'\\ \mathtt{Tail}(x) &= (ab)''\ a.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\mathtt{Head}(y) &= (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}\\ \mathtt{Tail}(y) &= I.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\mathtt{Head}(abc) &= a &\quad \text{and} \quad& \mathtt{Tail}(abc) = bc,\\ \mathtt{Head}\left(b^{(7)}\right) &= b^{(7)} &\quad \text{and} \quad& \mathtt{Tail}\left(b^{(7)}\right) = I,\\ \mathtt{Head}\left(a'a\right) &= a' &\quad \text{and} \quad& \mathtt{Tail}\left(a'a\right) = a.\end{aligned}$$

◇

prop: DWord head count

Proposition 4.0.67. *Let $x \in \mathcal{DMon}\langle S\rangle^{+}$ be a non-unit dashed-word. Then, we get*

{eqn: DWord head count}

$$\hat{l}(\mathtt{Head}(x)) \leq \hat{l}(x) \tag{4.0.68}$$

and

{eqn: DWord tail count}

$$\hat{l}(\mathtt{Tail}(x)) < \hat{l}(x). \tag{4.0.69}$$

Proof. Let $n = \hat{l}(x)$ and $L = [p], R = (p,n] \subseteq_I [n]$ be the leading left inverval and the trailing right interval where $1 \le p \le n$. We see that

$$\begin{aligned} \hat{l}(\mathtt{Head}(x)) &= \hat{l}\left(x_{[p]}\right) && (4.0.64) \\ &= p \\ &\le n \\ &= \hat{l}(x) \end{aligned}$$

and

$$\begin{aligned} \hat{l}(\mathtt{Tail}(x)) &= \hat{l}\left(x_{(p,n]}\right) && (4.0.65) \\ &= n - p \\ &< n \\ &= \hat{l}(x). \end{aligned}$$

□

prop: DWord head head **Proposition 4.0.70.** *Let $x \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, we get*

{eqn: DWord bracket head head}

$$\mathtt{Head}(\mathtt{Head}(x)) = \mathtt{Head}(x). \tag{4.0.71}$$

Proof. Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ and $L = [p]$ be the leading left interval of x, where $1 \le p \le n$. Thus, we get

$$y := \mathtt{Head}(x) = x_{[p]} = (p, u_{[p]}, D_{[p]}, d_{[p]}).$$

Let $M = [q]$ be the leading left interval for y, where $1 \le q \le p$. Thus, we get

$$\mathtt{Head}(\mathtt{Head}(x)) = y_{[q]} = \left(x_{[p]}\right)_{[q]}.$$

We claim that $p = q$. If $p = 1$ then we get $q = 1$, and we are done. Assume $2 \le p$. Therefore, we get $L = [p] \ne \{1\}$. Recall that,

$$L = \{1\} \cup \left(\bigcup_{A \in D; 1 \in A} A \right). \qquad (4.0.58)$$

Since $L \ne \{1\}$, there exists $A \in D$ with $1 \in A$. From Proposition 4.0.61 we get $L = [p] \in D$. From Definition 3.1.68, we conclude $[p] \in D_{[p]}$.

We have

$$M = \{1\} \cup \left(\bigcup_{A \in D_{[p]}; 1 \in A} A \right). \qquad (4.0.58)$$

Since $[p] \in D_{[p]}$ and $1 \in [p]$, we get $[p] \subseteq [q] = M$ implying $p \le q$. Therefore, we conclude $p = q$.

Finally, we get

$$\begin{aligned} \mathtt{Head}(\mathtt{Head}(x)) &= \left(x_{[p]}\right)_{[q]} \\ &= \left(x_{[p]}\right)_{[p]} \\ &= x_{[p]} && (4.0.40) \\ &= \mathtt{Head}(x). \end{aligned}$$

□

prop: DWord head mul

Proposition 4.0.72. *Let $x \in \mathcal{DMon}\langle S\rangle^+$ be non-unit dashed-word and $y \in \mathcal{DMon}\langle S\rangle$. Then,*

{eqn: DWord head mul}

$$\mathtt{Head}(x \bullet y) = \mathtt{Head}(x). \tag{4.0.73}$$

Proof. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle^+$ and $y = (m, v, E, e) \in \mathcal{DMon}\langle S\rangle$. Let L be the leading left interval of $x \bullet y$. Consider the family, $\mathfrak{L}$, of sub-intervals of $[n+m]$ given by

$$\mathfrak{L} = \{A \in D \bullet E \mid 1 \in A\}$$

Then, we get

$$L = \{1\} \cup \left(\bigcup_{A \in \mathfrak{L}} A\right). \qquad (4.0.58)$$

Let M be the leading left interval of x. Consider the family, $\mathfrak{M}$, of sub-intervals of $[n]$ given by

$$\mathfrak{M} = \{A \in D \mid 1 \in A\}$$

Then, we get

$$M = \{1\} \cup \left(\bigcup_{A \in \mathfrak{M}} A\right). \qquad (4.0.58)$$

From Definition 4.0.63 we get

$$\mathtt{Head}(x \bullet y) = (x \bullet y)_L \qquad \text{and} \qquad \mathtt{Head}(x) = x_M.$$

We claim that, $\mathfrak{L} = \mathfrak{M}$. To see this, suppose $A \in \mathfrak{L}$. That means

$$A \in D \bullet E = D \sqcup (n+E) \qquad \text{and} \qquad 1 \in A.$$

If $A \in n+E$ then we have $A \subseteq (n, n+m]$. Since $x \in \mathcal{DMon}\langle S\rangle^+$, we get $n \geq 1$. This implies, $1 \notin A$ which is a contradiction. Thus, we get $A \in D$. We already have $1 \in A$, thus we get $A \in \mathfrak{M}$. This shows

$$\mathfrak{L} \subseteq \mathfrak{M}.$$

On the other hand, suppose $A \in \mathfrak{M}$. Then we get

$$A \in D \qquad \text{and} \qquad 1 \in A.$$

Therefore,

$$A \in D \subseteq D \sqcup (n+E) = D \bullet E.$$

That is, $A \in \mathfrak{L}$. This shows

$$\mathfrak{L} \supseteq \mathfrak{M}.$$

As a result, we get $\mathfrak{L} = \mathfrak{M}$ implying $L = M$.

Noting $M \subseteq [n]$, we conclude

$$\begin{aligned}
\mathtt{Head}(x \bullet y) &= (x \bullet y)_L \\
&= (x \bullet y)_M \\
&= x_M && (4.0.42) \\
&= \mathtt{Head}(x).
\end{aligned}$$

□

prop: Br x=LR

Proposition 4.0.74. *Let $x=(n,u,D,d)\in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Let $L,R\subseteq[n]$ be the left leading interval and the right leading inverval as in Definition 4.0.57. Then, we get*

{eqn: Br x=LR}

$$D = D_L \bullet D_R \tag{4.0.75}$$

as bracketings on n letters.

Proof. Let $L=[p]$ and $R=(p,n]$ for where $1\le p\le n$. Thus, we wish to show that

$$D = D_{[p]}\bullet D_{(p,n]}.$$

Suppose $A\in D$. Then, from Proposition 4.0.62, we get $A\subseteq[p]$ or $A\subseteq(p,n]$. We will consider these two cases.

Case 1 ($A\subseteq[p]$): We get

$$\begin{aligned}
A &\in D_{[p]} && (3.1.69)\\
&\subseteq D_{[p]}\sqcup p+D_{(p,n]}\\
&= D_{[p]}\bullet D_{(p,n]}. && (3.1.35)
\end{aligned}$$

Case 2 ($A\subseteq(p,n]$): Since $A\in D$ and $A\subseteq(p,n]$, we get $-p+A\in D_{(p,n]}$. Thus, we get $A\in p+D_{(p,n]}$. As a consequence we get

$$A\in D_{[p]}\sqcup(p+D_{(p,n]}) = D_{[p]}\bullet D_{(p,n]}.$$

Since we have shown $A\in D_{[p]}\bullet D_{(p,n]}$ in both the cases, we conclude

$$D\subseteq D_{[p]}\bullet D_{(p,n]}.$$

On the other hand, suppose

$$A\in D_{[p]}\bullet D_{(p,n]} = D_{[p]}\sqcup p+D_{(p,n]}$$

. We will consider these two cases.

Case 1 ($A\in D_{[p]}$): We get that $A\in D$ and $A\subseteq[p]$. In particular, we get $A\in D$.

Case 2 ($A\in p+D_{(p,n]}$): We get $X\in D_{(p,n]}$ such that $A=p+X$. Since $X\in D_{(p,n]}$, we get $Y\in D$ with $Y\subseteq(p,n]$ such that $X=-p+Y$. Observe that

$$A = p+X = p+(-p+Y) = Y.$$

Since $Y\in D$, we conclude $A\in D$.

We have shown $A\in D$ in both the cases, therefore we get

$$D\supseteq D_{[p]}\bullet D_{(p,n]}.$$

Thus, we conclude

$$D = D_{[p]}\bullet D_{(p,n]} = D_L\bullet D_R.$$

□

prop: Ds x=LR

Proposition 4.0.76. *Let $x=(n,u,D,d)\in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Let $L,R\subseteq[n]$ be the leading left interval and the leading right interval as in Definition 4.0.57. Then, we get*

{eqn: Ds x=LR}

$$d = d_L\bullet d_R \tag{4.0.77}$$

as dash-assignments on D.

Proof. Let $L = [p]$ and $R = (p,n]$ where $1 \le p \le n$. We will show that

$$d = d_{[p]} \bullet d_{(p,n]}.$$

From Proposition 4.0.74, we have

$$D = D_{[p]} \bullet D_{(p,n]} = D_{[p]} \sqcup (p + D_{(p,n]}).$$

Therefore, it is enough to show

$$d(A) = d_{[p]} \bullet d_{(p,n]}\,(A)$$

for $A \in D_{[p]} \sqcup (p + D_{(p,n]})$.

We will consider the following cases.

Case 1 ($A \subseteq D_{[p]}$): Observe that

$$\begin{aligned} d_{[p]} \bullet d_{(p,n]}\,(A) &= d_{(0,p]}\,(A) && (3.1.87) \\ &= d(A). && (3.1.114) \end{aligned}$$

Case 2 ($A \subseteq p + D_{(p,n]}$): Observe that

$$\begin{aligned} d_{(0,p]} \bullet d_{(p,n]}\,(A) &= d_{(p,n]}\,(-p + A) && (3.1.87) \\ &= d(p + (-p + A)) && (3.1.114) \\ &= d(A). \end{aligned}$$

We have shown

$$d_{(0,p]} \bullet d_{(p,n]}\,(A) = d(A)$$

in both the cases, we conclude that

$$d_{(0,p]} \bullet d_{(p,n]} = d. \qquad \square$$

lem: DWord x=LR

Lemma 4.0.78. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then,*

{eqn: DWord x=LR}

$$x = \mathtt{Head}(x) \bullet \mathtt{Tail}(x). \tag{4.0.79}$$

Proof. Let $L = [p], R = (p,n] \subseteq_I [n]$ be the leading left interval and the right left interval as in Definition 4.0.57 where $1 \le p \le n$. Then, we have

$$\mathtt{Head}(x) = x_L \qquad \text{and} \qquad \mathtt{Tail}(x) = x_R.$$

Observe that

$$\begin{aligned} x &= (n,u,D,d) \\ &= (p + (n-p),\ u_L \bullet u_R,\ D_L \bullet D_R,\ d_L \bullet d_R) && (1.2.28), (4.0.75), \text{and } (4.0.77) \\ &= (p, u_L, D_L, d_L) \bullet (n-p, u_R, D_R, d_R) && (4.0.5) \\ &= x_L \bullet x_R && (4.0.35) \\ &= \mathtt{Head}(x) \bullet \mathtt{Tail}(x). && (4.0.64) \text{ and } (4.0.65) \square \end{aligned}$$

prop: DWord basis iff head

Proposition 4.0.80. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, $x \in \mathtt{G}$ if and only if $\mathtt{Head}(x) = x$.*

Proof. Let $L \subseteq [n]$ be the leading left interval as in Definition 4.0.57, that is

$$L = \{1\} \cup \left(\bigcup_{A \in D; 1 \in A} A \right).$$

We get

$$\mathtt{Head}(x) = x_L. \tag{4.0.64}$$

Suppose $x \in \mathtt{G}$, then we get $n = 1$ or $[n] \in D$. We will consider these two cases:

Case 1 ($n = 1$): We get $L = \{1\} = [n]$. Thus, we have

$$\begin{aligned} \mathtt{Head}(x) &= x_L \\ &= x_{[n]} \\ &= x. && (4.0.40) \end{aligned}$$

Case 2 ($[n] \in D$): For every $A \in D$ we have $A \subseteq [n]$. Since $[n] \in D$ and $1 \in [n]$, we get $L = [n]$. Observe that

$$\begin{aligned} \mathtt{Head}(x) &= x_L \\ &= x_{[n]} \\ &= x. && (4.0.40) \end{aligned}$$

On the other hand, suppose $\mathtt{Head}(x) = x_L = x$. If $n = 1$ then we get $x \in \mathtt{G}$, and we are done. Assume $n \geq 2$. We get

$$|L| = \hat{l}(x_L) = \hat{l}(x) = n.$$

Thus, L is a length n sub-interval of $[n]$. Therefor, we conclude $L = [n]$. Since $n \geq 1$, we get $[n] = L$ is the largest interval in D such that $1 \in L$. In particular, we get $[n] = L \in D$. This shows that $x \in \mathtt{G}$. □

lem: DWord head in basis

Lemma 4.0.81. *Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S \rangle^+$ be a non-unit dashed-word. Then we get $\mathtt{Head}(x) \in \mathtt{G}$.*

Proof. From Proposition 4.0.70 we get

$$\mathtt{Head}(\mathtt{Head}(x)) = \mathtt{Head}(x).$$

From Lemma 4.0.81 we get $\mathtt{Head}(x) \in \mathtt{G}$. □

thm: DWord basis

Theorem 4.0.82. *The subset $\mathtt{G} \subseteq \mathcal{DMon}\langle S \rangle$ is a multiplicative basis of $\mathcal{DMon}\langle S \rangle$.*

Proof. We will show that $\mathtt{G}$ is a generating set with respect to the multiplication and an independent set with respect to the multiplication.

First, we will show $\mathtt{G}$ is a generating set with respect to the multiplication. Let $x \in \mathcal{DMon}\langle S \rangle^+$ be a non-unit dashed-word. We will show, using induction on $\hat{l}(x)$, that there exists $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

Base case ($\hat{l}(x) = 1$): From Definition 4.0.53 we get $x \in \mathtt{G}$. We take $m = 1$ and $x_1 = x$.

Induction case ($\hat{l}(x) \geq 2$): From Lemma 4.0.78 we get

$$x = \mathtt{Head}(x) \bullet \mathtt{Tail}(x).$$

If $\mathtt{Tail}(x) = I$, then we get $\mathtt{Head}(x) = x$. From Proposition 4.0.80, we get $x \in \mathtt{G}$. Now assume $\mathtt{Tail}(x)$ is a non-unit dashed word. From Proposition 4.0.67 we get

$$\hat{l}(\mathtt{Tail}(x)) < \hat{l}(x).$$

Thus, from the induction hypothesis we get $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$\mathtt{Tail}(x) = x_1 \bullet \cdots \bullet x_m.$$

From Lemma 4.0.81 we have $\mathtt{Head}(x) \in \mathtt{G}$. Therefore, we get

$$x = \mathtt{Head}(x) \bullet \mathtt{Tail}(x) = \mathtt{Head}(x) \bullet x_1 \cdots x_m$$

where $\mathtt{Head}(x), x_i \in \mathtt{G}$ for $1 \leq i \leq m$.

From the mathematical induction, we conclude that $\mathtt{G}$ is a generating set with respect to the multiplication.

Next, we will show that $\mathtt{G}$ is an independent set with respect to the multiplication. Let $x \in \mathcal{DMon}\langle S \rangle^+$. Using induction on $\hat{l}(x)$, we will show that given $m, l \geq 1$ and $x_i, y_j \in \mathtt{G}$ for $1 \leq i \leq m$ and $1 \leq j \leq l$ such that

$$x = x_1 \bullet \cdots \bullet x_m = y_1 \bullet \cdots \bullet y_l,$$

we get $m = l$ and $x_i = y_i$ for $1 \leq i \leq m$.

Base case ($\hat{l}(x) = 1$): Suppose we have $m, l \geq 1$ and $x_i, y_j \in \mathtt{G}$ for $1 \leq i \leq m$ and $1 \leq j \leq l$ such that

$$x = x_1 \bullet \cdots \bullet x_m = y_1 \bullet \cdots \bullet y_l.$$

We get

$$1 = \hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i). \tag{4.0.13}$$

Since $x_i \in \mathtt{G}$, we have $\hat{l}(x_i) \geq 1$. It follows that $m = 1$ and $x_1 = x$. Similarly, we get $l = 1$ and $y_1 = x$. This completes the base case.

Induction case ($\hat{l}(x) \geq 2$): Suppose we have $m, l \geq 1$ and $x_i, y_j \in \mathtt{G}$ for $1 \leq i \leq m$ and $1 \leq j \leq l$ such that

$$x = x_1 \bullet \cdots \bullet x_m = y_1 \bullet \cdots \bullet y_l.$$

In particular, we have $x_1, y_1 \in \mathtt{G}$. Let $z = x_2 \cdots x_m$ and $w = y_2 \cdots y_l$. Then, we get

$$x = x_1 \bullet z = y_1 \bullet w.$$

Observe that

$$\begin{aligned} x_1 &= \mathtt{Head}(x_1) && \text{Proposition 4.0.80} \\ &= \mathtt{Head}(x_1 \bullet z) && (4.0.73) \\ &= \mathtt{Head}(y_1 \bullet w) && \\ &= \mathtt{Head}(y_1) && (4.0.73) \\ &= y_1. && \text{Proposition 4.0.80} \end{aligned}$$

Since $\mathcal{DMon}\langle S\rangle$ is a left cancellative monoid (Lemma 4.0.49), we get

$$x_2 \cdots x_m = y = z = y_2 \cdots y_l.$$

If $y = I$ then we get $m = 1$ and $x = x_1$. Since $z = y$, we get $z = I$. Consequently, we have $l = 1$ and $x = y_1$. This gives the required condition in $y = I$ case. Now assume $y \in \mathcal{DMon}\langle S\rangle^+$. Since $x_1 \in \mathtt{G}$ we get $\hat{l}(x_1) \geq 1$. It follows that

$$\hat{l}(y) < \hat{l}(x_1) + \hat{l}(y) = \hat{l}(x).$$

From the induction hypothesis we get $m = l$ and $x_i = y_i$ for $2 \leq i \leq m$. Above, along with the fact that $x_1 = y_1$, gives us $m = l$ and $x_i = y_i$ for $1 \leq i \leq m$.

From the mathematical induction, we conclude that $\mathtt{G}$ is an independent set with respect to the multiplication. Consequently, $\mathtt{G}$ is a multiplicative basis of $\mathcal{DMon}\langle S\rangle$. □

In type ABC define type ABC sets first. For x in DMon we want to find xC in C such that there is arrow x to xC which is a composition of simple arrows.

4.0.3. Dash basis of dashed words. Now we will show that $\mathcal{DMon}\langle S\rangle$ has a dash basis.

defn: DWord S

Definition 4.0.83. Define a subset $\mathtt{S} \subseteq \mathcal{DMon}\langle S\rangle$ as follows:

{eqn: DWord S}

$$\mathtt{S} := \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n = 1 \text{ and } D = \varnothing\}. \tag{4.0.84}$$ ◇

prop: DWord S bijection

Proposition 4.0.85. *The subset $\mathtt{S} \subseteq \mathcal{DMon}\langle S\rangle$ is in bijection with the set S.*

Proof. Define a function $\phi : S \longrightarrow \mathtt{S}$ as follows:

$$\phi(a) := (1, \overline{a}, \varnothing, \emptyset)$$

where $\overline{a} : [1] \longrightarrow S$ is given by

$$\overline{a}(1) = a.$$

We claim that the function ϕ is a bijection. Let $a, b \in S$ and suppose

$$\phi(a) = \phi(b).$$

Then, we get $\overline{a} = \overline{b}$. Thus,

$$a = \overline{a}(1) = \overline{b}(1) = b.$$

This shows that $\phi : S \longrightarrow \mathtt{S}$ is injective.

Now, let $x = (1, u, \varnothing, \emptyset) \in \mathtt{S}$. Let $a = u(1) \in S$. Observe that

$$\phi(a) = (1, \overline{a}, \varnothing, \emptyset)$$

where

$$\overline{a}(1) = a = u(1).$$

Thus, we get $\overline{a} = u$ therefore

$$\phi(a) = x.$$

This shows that $\phi : S \longrightarrow \mathtt{S}$ is surjective. $\square$

defn: DWord compos

Definition 4.0.86. Define a subset $\mathtt{R} \subseteq \mathcal{DMon}\langle S \rangle$ as follows:

{eqn: DWord compos}

$$\begin{aligned} \mathtt{R} &:= \langle\langle \mathtt{G};\ \bullet \rangle\rangle \\ &= \{x_1 \bullet \cdots \bullet x_m \mid m \geq 2,\ x_i \in \mathtt{G} \text{ for } 1 \leq i \leq m\}. \quad (2.2.3) \end{aligned} \tag{4.0.87}$$

$\diamond$

prop: DWord compos iff

Proposition 4.0.88. *Let $\mathtt{R} \subseteq \mathcal{DMon}\langle S \rangle$ be as in Definition 4.0.86. Then,*

{eqn: DWord compos iff}

$$\mathtt{R} = \{(n, u, D, d) \in \mathcal{DMon}\langle S \rangle \mid n \geq 2 \ \text{ and } \ [n] \notin D\}. \tag{4.0.89}$$

Proof. Let $x = (n, u, D, d) \in \mathtt{R}$. Then, we get $m \geq 2$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet x_2 \bullet \cdots \bullet x_m.$$

Since $x_i \in \mathtt{G}$, we get $\hat{l}(x_i) \geq 1$. Moreover, since $m \geq 2$ we get

$$n = \hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i) \geq 2.$$

For the sake of contradiction, assume $[n] \in D$. Then, from Definition 4.0.53 we get $x \in \mathtt{G}$. Since $\mathtt{G}$ is a multiplicative basis, it is a multiplicatively independent set. Lemma 2.2.23 asserts that

$$\mathtt{G} \cap \mathtt{R} = \varnothing.$$

This leads to a contradiction since we have $x \in \mathtt{R}$ and $x \in \mathtt{G}$.

On the other hand, suppose $x = (n, u, D, d) \in \mathcal{DMon}\langle S \rangle$ such that $n \geq 2$ and $[n] \notin D$. Since $n \geq 2$ we get $x \neq I$. Since $\mathtt{G}$ is a multiplicative basis, we get $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

To show that $x \in \mathtt{R}$, we need to show that $m \geq 2$. If $m = 1$ then we have $x = x_1 \in \mathtt{G}$. Since $n \geq 2$ and $[n] \notin D$, we have $x \notin \mathtt{G}$, proving that $m \neq 1$. Therefore, we must have $m \geq 2$. It follows that, $x \in \mathtt{R}$. $\square$

prop: DWord S disjoint compos

Proposition 4.0.90. *Let $\mathtt{S} \subseteq \mathcal{DMon}\langle S \rangle$ be as in Definition 4.0.83 and $\mathtt{R} \subseteq \mathcal{DMon}\langle S \rangle$ be as in Definition 4.0.86. Then, we have*

{eqn: DWord S disjoint compos}

$$\mathtt{S} \cap \mathtt{R} = \varnothing. \tag{4.0.91}$$

Proof. Suppose $x = (n, u, D, d) \in \mathtt{S} \cap \mathtt{R}$. Since $x \in \mathtt{S}$, we get $n = 1$. Since $x \in \mathtt{R}$, we get $n \geq 2$. This is a contradiction. Therefore, we have

$$\mathtt{S} \cap \mathtt{R} = \varnothing.$$

$\square$

defn: DWord DBasis

Definition 4.0.92. Define a subset $\mathtt{H} \subseteq \mathcal{DMon}\langle S\rangle$ as follows:

{eqn: DWord DBasis}

$$\mathtt{H} := \mathtt{S} \sqcup \mathtt{R}. \tag{4.0.93}$$ ◇

prop: DWord DBasis iff

Proposition 4.0.94. *We have*

{eqn: DWord DBasis iff}

$$\mathtt{H} = \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n \geq 1 \text{ and } [n] \notin D\}. \tag{4.0.95}$$

Proof. Observe that

$$\begin{aligned}
\mathtt{H} &= \mathtt{S} \sqcup \mathtt{R} && (4.0.93)\\
&= \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n = 1 \text{ and } [n] \notin D\}\\
&\qquad \sqcup \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n \geq 2 \text{ and } [n] \notin D\} && (4.0.84) \text{ and } (4.0.93)\\
&= \{(n,u,D,d) \mid n \geq 1 \text{ and } [n] \notin D\}.
\end{aligned}$$ □

lem: DWord undash eq self iff DBasis

Lemma 4.0.96. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, $x \in \mathtt{H}$ if and only if $\lfloor x \rfloor = x$.*

Proof. Since x is a non-unit dashed-word, we get $n \geq 1$. From Proposition 4.0.31 we get $\lfloor x \rfloor = x$ if and only if $[n] \notin D$. From Proposition 4.0.94, this is equivalent to $x \in \mathtt{H}$. □

lem: DWord undash in dash basis

Lemma 4.0.97. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, $\lfloor x \rfloor \in \mathtt{H}$.*

Proof. Since x is a non-unit dashed-word, we get $\hat{l}(x) = n \geq 1$. We have

$$\begin{aligned}
\hat{l}(\lfloor x)\rfloor &= \hat{l}(x) && (4.0.26)\\
&= n \geq 1.
\end{aligned}$$

From Proposition 4.0.27 we get

$$\lfloor \lfloor x \rfloor \rfloor = \lfloor x \rfloor .$$

Using Lemma 4.0.96 we get $\lfloor x \rfloor \in \mathtt{H}$. □

lem: DWord ceil eq

Lemma 4.0.98. *Let $x = (n,u,D,d) \in \mathtt{H}$. Suppose*

$$x^{(k)} = x^{(l)}$$

for some $k,l \in \mathbb{N}$. Then, we get $k = l$.

Proof. From Proposition 4.0.94 we get $n \geq 1$ and $[n] \notin D$. Without the loss of generality, we will consider the following cases:

Case 1 ($k = 0$): Suppose $l \geq 1$. Then, we get $x = x^{(l)}$. In particular, we get

$$D = D^{(l)}.$$

From Proposition 3.1.45 we get

$$D(l) = D \cup \{[n]\}.$$

Thus, we get $[n] \in D$. This is a contradiction. Therefore, we conclude $l = 0$.

Case 2 ($k,l \geq 1$): Since $[n] \notin D$, from Proposition 3.1.97 we get

$$d^{(k)}([n]) = k \qquad \text{and} \qquad d^{(l)}([n]) = l.$$

From the equality $x^{(k)} = x^{(l)}$ we get

$$d^{(k)} = d^{(l)}.$$

It follows that $k = l$ as required. □

thm: DWord DBasis

Theorem 4.0.99. *The subset* $\mathtt{H} \subseteq \mathcal{DMon}\langle S\rangle$ *is a dash basis of* $\mathcal{DMon}\langle S\rangle$.

Proof. We will show that $\mathtt{H}$ is a generating set and an independent set with respect to the dash operation.

First, we will show that $\mathtt{H}$ is a generating set with respect to the dash operation. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. From Lemma 4.0.97, we get that $\lfloor x \rfloor \in \mathtt{H}$. Assign

$$k = \begin{cases} 0 & \text{if } [n] \notin D \\ d([n]) & \text{if } [n] \in D. \end{cases}$$

Then, from Proposition 4.0.32 we get

$$(\lfloor x \rfloor)^{(k)} = x.$$

This shows that $\mathtt{H}$ is a generating set with respect to the dash operation.

Next, we will show that $\mathtt{H}$ is independent with respect to the dash operation. Let $x, y \in \mathtt{H}$. Suppose we have

$$x^{(k)} = y^{(l)}$$

for some $k, l \in \mathbb{N}$. Since $x, y \in \mathtt{H}$, from Lemma 4.0.96 we get

$$x = \lfloor x \rfloor \qquad \text{and} \qquad y = \lfloor y \rfloor.$$

Observe that

$$\begin{aligned} x &= \lfloor x \rfloor \\ &= \left\lfloor x^{(k)} \right\rfloor && (4.0.30) \\ &= \left\lfloor y^{(k)} \right\rfloor && \text{assumption} \\ &= \lfloor y \rfloor && (4.0.30) \\ &= y. \end{aligned}$$

From Lemma 4.0.98 we get $k = l$. This shows that $\mathtt{H}$ is independent with respect to the dash operation. □

prop: DWord GenP DBasis iff

Proposition 4.0.100. *The equality*

{eqn: DWord GenP DBasis iff}

$$\langle\langle \mathtt{H};\ (-)' \rangle\rangle = \{(n, u, D, d) \in \mathcal{DMon}\langle S\rangle \mid n \geq 1 \ \text{ and } \ [n] \in D\} \tag{4.0.101}$$

holds.

Proof. Let $x = (n, u, D, d) \in \langle\langle \mathtt{H};\ (-)' \rangle\rangle$. Then, we get $y = (m, v, E, e) \in \mathtt{H}$ and $k \geq 1$ such that

$$x = y^{(k)}.$$

That is,

$$\begin{aligned}(n,u,D,d) &= (m,v,E,e)^{(k)} \\ &= (m,v,E^{(k)},e^{(k)}) && (4.0.7).\end{aligned}$$

Therefore, we have

$$n = m, \qquad u = v, \qquad D = E^{(k)}, \qquad \text{and} \qquad d = e^{(k)}.$$

Since $y \in \mathtt{H}$, we get $m \geq 1$ and $[m] \notin E$. This implies $n \geq 1$. Since $m \geq 1$ and $k \geq 1$, using Proposition 3.1.50, we get

$$\begin{aligned}D &= E^{(k)} \\ &= E \cup \{[m]\} && (3.1.51) \\ &= E \cup \{[n]\}.\end{aligned}$$

Therefore, we get $[n] \in D$.

On the other hand, suppose $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$ such that $n \geq 1$ and $[n] \in D$. Let $k = d([n])$. Note that $k \geq 1$ since $d : D \longrightarrow \mathbb{N}^+$. From Lemma 4.0.97 we get $\lfloor x \rfloor \in \mathtt{H}$. From Proposition 4.0.32 we get

$$(\lfloor x \rfloor)^{(k)} = x.$$

This shows that $x \in \langle\langle \mathtt{H};\ (-)' \rangle\rangle$. □

prop: DWord S disjoint GenP DBasis

Proposition 4.0.102. *We have*

{eqn: DWord S disjoint GenP DBasis}

$$\mathtt{S} \cap \langle\langle \mathtt{H};\ (-)' \rangle\rangle = \varnothing. \tag{4.0.103}$$

Proof. Suppose $x = (n,u,D,d) \in \mathtt{S} \cap \langle\langle DBasis;\ (-)' \rangle\rangle$. From Definition 4.0.83 we get $n = 1$ and $D = \varnothing$. From Proposition 4.0.100 we get $n \geq 1$ and $[n] \in D$. This leads to a contradiction. Therefore,

$$\mathtt{S} \cap \langle\langle \mathtt{H};\ (-)' \rangle\rangle = \varnothing.$$ □

prop: DWord S union GenP DBasis = Basis

Proposition 4.0.104. *The equality*

{eqn: DWord S union GenP DBasis = Basis}

$$\mathtt{S} \sqcup \langle\langle \mathtt{H};\ (-)' \rangle\rangle = \mathtt{G} \tag{4.0.105}$$

holds.

Proof. We see that

$$\begin{aligned}&\mathtt{S} \sqcup \langle\langle \mathtt{H};\ (-)' \rangle\rangle \\ &= \{(n,u,D,d) \mid (n = 1 \ \text{ and } D = \varnothing) \ \text{ or} \\ &\qquad\qquad (n \geq 1 \ \text{ and } \ [n] \in D)\} && (4.0.84) \text{ and } (4.0.101) \\ &= \{(n,u,D,d) \mid n = 1 \ \text{ or } \ (n \geq 2 \ \text{ and } \ [n] \in D)\} \\ &= \mathtt{G}. && (4.0.54)\end{aligned}$$

Here, the third equality follows since for $n = 1$ we have either $D = \{\}$ or $D = \{[1]\}$. □

lem: DWord freeDMon like

Lemma 4.0.106. *The dashed monoid of dashed words with alphabet in S, $\mathcal{DMon}\langle S\rangle$, has a free dashed monoid like structure.*

Proof. Let $\hat{l} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N}$ as in Definition 4.0.9 be the length function. Proposition 4.0.15 gives the required property for the length function.

Let $\mathtt{G}$ be the multiplicative basis as shown in Theorem 4.0.82. Let $\mathtt{H}$ be the dash basis as in Definition 4.0.92. Let $\mathtt{S}$ be the set as in Definition 4.0.83. Proposition 4.0.104 and Definition 4.0.92 give the required interlocking conditions. □

thm: DWord freeDMon

Theorem 4.0.107. *The inclusion of sets* $\mathtt{S} \subseteq \mathcal{DMon}\langle S\rangle$ *satisfies the universal property of dashed monoid.*

Proof. From Lemma 4.0.106 we know that $\mathcal{DMon}\langle S\rangle$ has a free dashed monoid like structure with $\mathtt{S}$ serving as the generating set. The theorem follows immediately from Theorem 2.3.33 . □

Bibliography