On Dashed Monoids

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ABSTRACT. This paper introduces and formalizes the concept of dashed monoids, which are monoids equipped with a unary operation. We provide a rigorous construction and characterization of the free dashed monoid. Additionally, we define the notion of a dashed monoid basis and demonstrate that the existence of such a basis is equivalent to the dashed monoid being free.

Contents

CHAPTER 1

Introduction

Chapter 2 provides a concise overview of monoid theory, including a formal construction of the free monoid generated by a set S. We use the notion of a *monoid basis*: A subset $X \subseteq M$ such that every element of M can be uniquely expressed as a product of elements from X. We demonstrate that the existence of a monoid basis is equivalent to M being a free monoid.

In Chapter 3, we introduce the algebraic structure of dashed monoids: monoids equipped with a unary operation, *dash*, which need not interact with the monoid multiplication. The motivation for this structure arises from the observation that the objects of a semi-strict categorical group naturally form a dashed monoid. In Section 3.2, we state the universal property of the free dashed monoid and define the concept of a *dashed monoid basis*. Unlike the case for monoids, the structure of a dashed monoid basis is more intricate, and we devote Sections 3.3 through 3.5 to a detailed analysis. In Sections 3.6 and 3.7, we construct the free dashed monoid generated by a set. The technical developments in these sections culminate in a complete characterization of dashed monoid bases, formally established in Section 3.8.

The content of Chapter 3 has been formally encoded using the Lean 4 programming language and theorem prover. Lean 4 enables the formalization and verification of mathematical statements, including theorems and their proofs. The corresponding formalization is available at [**Par25**].

CHAPTER 2

Topics in Monoid Theory

In this chapter, we introduce monoids and provide a construction of the free monoid generated by a set.

2.1. Free Monoids and Monoid Basis

This section defines a monoid and the universal property of a free monoid. We also provide a criterion for determining when a given monoid is free in Theorem 2.1.36.

Definition 2.1.1. A *monoid*

$$(M, I, \bullet)$$

has the following data:

- A set M, called the *underlying set*,
- an element, denoted $I \in M$, called the *unit element*, and
- a binary map

$$- \bullet - : M \times M \longrightarrow M$$

called the *multiplication*.

These satisfy the following conditions:

Unit conditions: For $x \in M$, the equlity

$$x \bullet I = x \qquad \text{and} \qquad I \bullet x = x \tag{2.1.2}$$

holds.

Associativity: For $x, y, z \in M$, the equlity

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z \tag{2.1.3}$$

holds. ◇

Notation 2.1.4. Let $(M, I, \bullet)$ be a monoid. Due to associativity, we omit parentheses when writing products of elements. For $n \in \mathbb{N}$ and $x_i \in M$ for $1 \le i \le n$, we write

$$x_1 \bullet \cdots \bullet x_n$$

to denote the product in that order. By convention, if $n = 0$, the product is defined to be the unit element I:

$$x_1 \bullet \cdots \bullet x_n = I \quad \text{when } n = 0.$$ ◇

Example 2.1.5. The set of natural numbers $\mathbb{N}$ forms a monoid under addition, with 0 as the unit element. When we refer to the natural numbers as a monoid, we mean this structure. We write

$$\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$$

for the set of nonzero natural numbers. ◇

Example 2.1.6. Monoids appear throughout mathematics. Monoids arising from category theory are of particular interest to us. Specifically, if $(\mathscr{M}, I, \bullet)$ is a strict monoidal category, then the set of objects $\mathrm{Obj}(\mathscr{M})$ forms a monoid with unit I and tensor as the multiplication. Moreover, if $\mathscr{M}$ is a small category, the set of morphisms

$$\mathrm{Mor}(\mathscr{M}) := \bigsqcup_{x,y \in \mathrm{Obj}(\mathscr{M})} \mathrm{Hom}_{\mathscr{M}}(x, y)$$

also forms a monoid, where the unit is 1_I and the multiplication is given by the tensor. ◇

Definition 2.1.7. Let $M = (M, \bullet, I)$ and $N = (N, *, J)$ be monoids. A *monoid homomorphism* from M to N is a set map $f : M \longrightarrow N$ that satisfies following conditions:

Unit condition: The equality

$$f(I) = J \tag{2.1.8}$$

holds.

Multiplication condition: For $x, y \in M$, the equality

$$f(x \bullet y) = f(x) * f(y) \tag{2.1.9}$$

holds. ◇

Remark 2.1.10. Monoids and monoid homomorphisms together form a category, denoted by $\mathcal{Mon}$. ◇

Framework 2.1.11. Throughout this section, we fix a monoid $(M, \bullet, I)$ and a subset $X \subseteq M$. ◇

Definition 2.1.12. The *subset generated by X under the multiplication*, denoted by $\langle X;\ \bullet \rangle \subseteq M$, is defined as

$$\langle X;\ \bullet \rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \in \mathbb{N},\ m \geq 1,\ x_i \in X \text{ for } 1 \leq i \leq m\}. \tag{2.1.13}$$

The *subset generated by X under non-trivial multiplication*, denoted by $\langle\langle X;\ \bullet \rangle\rangle \subseteq M$, is defined as

$$\langle\langle X;\ \bullet \rangle\rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \in \mathbb{N},\ m \geq 2,\ x_i \in X \text{ for } 1 \leq i \leq m\}. \tag{2.1.14}$$ ◇

Warning 2.1.15. Observe that the subsets $\langle X;\ \bullet \rangle$ and $\langle\langle X;\ \bullet \rangle\rangle$ may not include the unit I, as the empty product is not considered. As a result, $\langle X;\ \bullet \rangle$ and $\langle\langle X;\ \bullet \rangle\rangle$ are generally not submonoids. ◇

Proposition 2.1.16. *The subset*

$$\langle X;\ \bullet\rangle \cup \{I\}$$

is a submonoid of M. We refer to this as the emphsubmonoid generated by X.

Proof. First, by definition, we have

$$I \in \langle X;\ \bullet\rangle \cup \{I\}.$$

Now, let

$$x, y \in \langle X;\ \bullet\rangle \cup \{I\}.$$

We consider the following cases:

Case I: $x, y \in \langle X;\ \bullet\rangle$. Then there exist $m, n \geq 1$ and $x_i, y_j \in X$ such that

$$x = x_1 \bullet \cdots \bullet x_m, \qquad y = y_1 \bullet \cdots \bullet y_n.$$

Thus,

$$x \bullet y = x_1 \bullet \cdots \bullet x_m \bullet y_1 \bullet \cdots \bullet y_n \in \langle X;\ \bullet\rangle.$$

Case II: ($x = I$, $y \in \langle X;\ \bullet\rangle$): Then

$$x \bullet y = I \bullet y = y \in \langle X;\ \bullet\rangle$$

Case III: ($x \in \langle X;\ \bullet\rangle$, $y = I$): Then

$$x \bullet y = x \bullet I = x \in \langle X;\ \bullet\rangle.$$

Case IV: ($x = I = y$): Then

$$x \bullet y = I \bullet I = I.$$

In all cases, the product $x \bullet y$ belongs to $\langle X;\ \bullet\rangle \cup \{I\}$. Therefore, $\langle X;\ \bullet\rangle \cup \{I\}$ is closed under the multiplication and contains the unit, so it is a submonoid of M. □

Notation 2.1.17. We denote the set of all monoid elements without the unit by M^+. That is,

$$M^+ := M \setminus \{I\}. \tag{2.1.18}$$ ◇

Definition 2.1.19. We say $X \subseteq M$ is *independent with respect to the multiplication* if for every $n, m \in \mathbb{N}^+$, and $x_i, y_j \in X$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, the following implication holds:

$$x_1 \bullet \cdots \bullet x_n = y_1 \bullet \cdots \bullet y_m \qquad \Longrightarrow \qquad n = m \text{ and } x_i = y_j. \tag{2.1.20}$$ ◇

Proposition 2.1.21. *Suppose $X \subseteq M$ is independent with respect to the multiplication, then $I \notin X$.*

Proof. Suppose for contradiction $I \in X$. We have

$$I \bullet I = I.$$

This contradicts multiplicatively independent condition of X. □

Definition 2.1.22. We say $X \subseteq M$ is a *generating set with respect to the multiplication* if

$$M^+ \subseteq \langle X;\ \bullet \rangle. \tag{2.1.23}$$

That is, for every $x \in M^+$ there exits $m \in \mathbb{N}^+$ and $x_i \in X$ for $1 \le i \le m$ such that

◇

$$x_1 \bullet \cdots \bullet x_m = x.$$

Definition 2.1.24. We say $X \subseteq M$ is a *multiplication basis* or a *monoid basis* if X is independent with respect to the multiplication and X is a generating set with respect to the multiplication. ◇

Definition 2.1.25. Let $x \in M^+$. An *X-factorization* of x is a finite ordered collection $(x_1, \ldots, x_m)$ with $m \ge 1$ and $x_i \in X$ for $1 \le i \le m$ such that

$$x_1 \cdots x_n = x. \tag{2.1.26}$$ ◇

Remark 2.1.27. Note that, the subset $X \subseteq M$ is independent with respect to the multiplication if and only if every X-factorization is unique. A subset $X \subseteq M$ is a generating set with respect to multiplication if and only if every $x \in M^+$ has an X-factorization. Thus, a subset $X \subseteq M$ is a multiplicative basis if and only if every $x \in M^+$ has a unique X-factorization. ◇

Proposition 2.1.28. *Let M and N be monoids, and let*

$$f : M \longrightarrow N$$

be a monoid isomorphism. If $X \subseteq M$ is a multiplicative basis of M, then $f(X) \subseteq N$ is a multiplicative basis of N.

Proof. Suppose $X \subseteq M$ is a multiplicative basis of M. We need to show that $f(X)$ is both a multiplicatively independent subset and a generating subset.

First, let $n, m \in \mathbb{N}^+$, and $a_i, b_j \in f(X)$ for $1 \le i \le n$ and $1 \le j \le m$, such that

$$a_1 \bullet \cdots \bullet a_n = b_1 \bullet \cdots \bullet b_m.$$

Since f is a monoid isomorphism, we get $f^{-1}(a_i), f^{-1}(b_j) \in X$, and

$$f^{-1}(a_1) \bullet \cdots \bullet f^{-1}(a_n) = f^{-1}(b_1) \bullet \cdots \bullet f^{-1}(b_m).$$

Because X is a multiplicative basis, it follows that

$$n = m \qquad \text{and} \qquad f^{-1}(a_i) = f^{-1}(b_i).$$

Consequently, we get

$$n = m \qquad \text{and} \qquad a_i = b_i.$$

Next, let $a \in N^+$. Since f is a monoid homomorphism, $f^{-1}(a) \in M^+$. Since X is a multiplicative basis, there exists $n \in \mathbb{N}^+$, and $x_i \in X$ for $1 \le i \le n$ such that

$$f^{-1}(a) = x_1 \bullet \cdots \bullet x_n.$$

Applying f yields

$$a = f(x_1) \bullet \cdots \bullet f(x_n)$$

with each $f(x_i) \in f(X)$. This completes the proof. $\square$

Proposition 2.1.29. *If a monoid M admits a multiplicative basis, then this basis is unique.*

Proof. Let $X, Y \subseteq M$ be multiplicative bases of M. If both are empty, the result is immediate. Suppose X is nonempty and let $x \in X$. By Proposition 2.1.21, $x \in M^+$. Since Y is a multiplicative basis, x has a unique Y-factorization:

$$x = y_1 \bullet \cdots \bullet y_m$$

where $m \geq 1$ and $y_j \in Y$ for $1 \leq j \leq m$. Each $y_j \in M^+$, so each has a unique X-factorization:

$$y_j = x_{j1} \bullet \cdots \bullet x_{ji_j}$$

with $i_j \geq 1$ and $x_{jk} \in X$ for $1 \leq k \leq i_j$. Substituting, we have

$$x = x_{11} \bullet \cdots \bullet x_{1i_1} \bullet \cdots \bullet x_{m1} \bullet \cdots \bullet x_{mi_m}.$$

By uniqueness of X-factorizations, we get

$$m = 1, \qquad i_1 = 1, \qquad \text{and} \qquad x_{11} = x.$$

Thus, $x = y_1$ and $x \in Y$. Therefore, $X \subseteq Y$. By symmetry, $Y \subseteq X$. Hence, $X = Y$. $\square$

Proposition 2.1.30. *Let M be a monoid with a monoid basis X. Then for any $x, y \in M$, if $x \neq I$ and $y \neq I$, it follows that $x \cdot y \neq I$.*

Proof. Suppose $X \subseteq M$ is a monoid basis of M. Let $x, y \in M$ be such that $x \neq I$ and $y \neq I$. Suppose for contradiction that $x \cdot y = I$. Since X is a monoid basis we get $n, m \in \mathbb{N}^+$ and $x_i, y_j \in X$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ such that

$$x = x_1 \cdots x_n \qquad \text{and} \qquad y = y_1 \cdots y_m.$$

Thus, we get

$$I = x \cdot y = x_1 \cdots x_n \cdot y_1 \cdots y_m.$$

Therefore, we have

$$\begin{aligned} x_1 &= x_1 \cdot I \\ &= x_1 \cdot x_1 \cdots x_n \cdot y_1 \cdots y_m. \end{aligned}$$

This contradicts the multiplicative independence of X. Thus, we get $x \cdot y \neq I$. $\square$

Proposition 2.1.31. *Let M be a monoid with a monoid basis X. Then,*

$$M^+ = \langle X;\ \bullet \rangle. \tag{2.1.32}$$

Proof. By Definition 2.1.22, we have $M^+ \subseteq \langle X;\ \bullet \rangle$.

For the reverse direction, let $x \in \langle X;\ \bullet \rangle$. Then there exist $n \in \mathbb{N}^+$ and $x_i \in X$ for $1 \leq i \leq n$ such that

$$x = x_1 \bullet \cdots \bullet x_n.$$

Since X is a monoid basis, Proposition 2.1.21 gives $I \notin X$. Thus, every $x_i \neq I$. By repeated application of Proposition 2.1.30, the product x cannot be I. Thus, $x \in M^+$, as required. $\square$

Definition 2.1.33. Let S be a set and let $\phi : S \longrightarrow M$ be a set map. We say that the pair

$$\big(M, \phi : S \longrightarrow M\big)$$

satisfies the *universal property of the free monoid generated by* S if the following conditions are satisfied:

Existence: Given a monoid N and a set map $u : S \longrightarrow N$ there exists a monoid homomorphism $f_u : M \longrightarrow N$ such that

$$f_u \circ \phi = u. \tag{2.1.34}$$

Uniqueness: Given a monoid N and a pair of monoid homomorphisms $f, g : M \longrightarrow N$, then the following implication holds:

$$f \circ \phi = g \circ \phi \qquad \Longrightarrow \qquad f = g. \tag{2.1.35}$$

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Theorem 2.1.36. *Suppose $X \subseteq M$ is a monoid basis of M. Then, the pair*

$$\big(M, \phi : X \hookrightarrow M\big),$$

where ϕ is the inclusion map, satisfies the universal property of the free monoid generated by X.

Proof. Assume $X \subseteq M$ is a monoid basis of M. Let $(N, \bullet, J)$ be a monoid and $u : X \longrightarrow N$ be a set map.

Define $f_u : M \longrightarrow N$ as follows: Let $x \in M$. If $x = I$ define

$$f_u(I) := J.$$

Now suppose we have $x \in M^+$. Since X is a basis we get a unique $n \in \mathbb{N}^+$ and unique $x_i \in X$ for $1 \le i \le n$ such that $x = x_1 \bullet \cdots \bullet x_n$. Define

$$f_u(x) := u(x_1) \bullet \cdots \bullet u(x_n).$$

Since the factorization $x = x_1 \bullet \cdots \bullet x_n$ is unique, f_u is a well-defined map.

Next, we will show that f_u is a monoid homomorphism. We already have the equality

$$f_u(I) = J.$$

Thus, the unit condition of a monoid homomorphism is satisfied.

Now suppose $x, y \in M$. We will show that

$$f_u(x \cdot y) = f_u(x) \bullet f_u(y).$$

We will consider the following cases:

Case I ($x, y \in M^+$): We get $x = x_1 \bullet \cdots \bullet x_n$ and $y = y_1 \bullet \cdots \bullet y_m$ where $n, m \in \mathbb{N}^+$ and $x_i, y_j \in X$. Then we have

$$x \bullet y = x_1 \bullet \cdots \bullet x_n \bullet y_1 \bullet \cdots \bullet y_m.$$

From Proposition 2.1.30 we get that $x \bullet y \ne I$. Thus, we get

$$f_u(x \bullet y) = u(x_1) \bullet \cdots \bullet u(x_n) \bullet u(y_1) \bullet \cdots \bullet u(y_m) = f_u(x) \bullet f_u(y).$$

Case II ($x = I,\ y \in M$): In this case, we get

$$\begin{aligned} f_u(I \bullet y) &= f_u(y) \\ &= J \bullet f_u(y) \\ &= f_u(I) \bullet f_u(y). \end{aligned}$$

Case III ($x \in M,\ y = I$): In this case, we get

$$\begin{aligned} f_u(x \bullet I) &= f_u(x) \\ &= f_u(x) \bullet J \\ &= f_u(y) \bullet f_u(I). \end{aligned}$$

This shows that the function $f_u : M \longrightarrow N$ satisfies the multiplicative condition of a monoid homomorphism. Thus, $f_u : M \longrightarrow N$ is a monoid homomorphism.

Observe that for $x \in X$, we get $f_u(x) = u(x)$. Thus, the equality

$$f_u \circ \Phi = u$$

holds. This completes the proof of the existence part of the universal property.

Next, we will show the uniqueness part of the universal property. Suppose

$$g, h : M \longrightarrow N$$

are monoid homomorphisms such that

$$f \circ \phi = g \circ \phi.$$

That is, we have $g(z) = h(z)$ for every $z \in X$. We will show that $g = h$.

Let $x \in M$. If $x \in M^+$ then we get unique $n \in \mathbb{N}^+$ and unique $x_i \in X$ such that

$$x = x_1 \bullet \cdots \bullet x_n.$$

We get

$$\begin{aligned} g(x) &= g(x_1 \bullet \cdots \bullet x_n) \\ &= g(x_1) \bullet \cdots \bullet g(x_n) \\ &= h(x_1) \bullet \cdots \bullet h(x_n) \\ &= h(x_1 \bullet \cdots \bullet x_n) \\ &= h(x). \end{aligned}$$

If $x = I$ then we get

$$g(I) = J = h(I).$$

This shows that $g = h$. Thus, the uniqueness part of the universal property is satisfied. This completes the proof. $\square$

2.2. Words as the Free Monoid

In this section, we will describe a standard construction of the free monoid generated by a set S, namely the set of words formed from elements of S as letters. This construction is discussed on page 48 of [**ML78**]. For further information on free monoids and related topics, see [**Lot97**]. Our approach will emphasize a set-theoretic perspective in formalizing the construction of the free monoids.

Framework 2.2.1. Fix a set S. $\diamond$

Notation 2.2.2. Here are some relevant notations that will prove useful throughout our discussion.

- The set of integers are denoted by
$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}.$$
- The set of natural numbers are denoted by
$$\mathbb{N} = \{0, 1, 2, \cdots\}.$$
- The set of positive integers are denoted by
$$\mathbb{N}^+ = \mathbb{Z}^+ = \{1, 2, \cdots\}.$$
- We subscribe to the standard inverval notation for integers. That is, for integers $n, m \in \mathbb{Z}$, we denote
$$(n,m) = \{i \in \mathbb{Z} \mid n < i < m\}, \qquad (n,m] = \{i \in \mathbb{Z} \mid n < i \le m\},$$
$$[n,m) = \{i \in \mathbb{Z} \mid n \le i < m\}, \qquad [n,m] = \{i \in \mathbb{Z} \mid n \le i \le m\}.$$
In addition, we denote
$$[n] = [1, n].$$
In this discussion, an interval will always refer to a finite interval.
- Let $A \subseteq \mathbb{Z}$ be a subset of integers. We use the following notations for the power set of A:
$$\mathcal{P}(A) = \{T \subseteq \mathbb{Z} \mid T \subseteq A\},$$
$$\mathcal{P}^+(A) = \{T \subseteq \mathbb{Z} \mid T \subseteq A,\ T \neq \varnothing\},$$
$$\mathcal{P}_I^+(A) = \{T \subseteq \mathbb{Z} \mid T \subset A,\ T \neq \varnothing,\ T \text{ is a finite interval.}\}$$
We denote an element $T \in \mathcal{P}^+(A)$ by $T \subseteq^+ A$ and an element $T \in \mathcal{P}_I^+(A)$ by $T \subseteq_I^+ A$.
- For $n, m \in \mathbb{Z}$, we denote
$$\mathcal{P}(n,m) = \mathcal{P}([n,m]), \qquad \mathcal{P}(n) = \mathcal{P}([n]),$$
$$\mathcal{P}^+(n,m) = \mathcal{P}^+([n,m]), \qquad \mathcal{P}^+(n) = \mathcal{P}^+([n]),$$
$$\mathcal{P}_I^+(n,m) = \mathcal{P}_I^+([n,m]), \qquad \mathcal{P}_I^+(n) = \mathcal{P}_I^+([n]).$$
$\diamond$

Definition 2.2.3. Let $n \in \mathbb{N}$. An *n-letter word* is a function $u : [n] \longrightarrow S$. We denote the set of all n-letter words by $\mathcal{W}r(n)$. $\diamond$

Definition 2.2.4. Note that the empty function is the only function with a empty domain. Thus, the only 0-letter word is the empty function, $\emptyset : \emptyset \longrightarrow S$. We call this the *unit word*, denoted by $J_{\mathcal{Wr}}$. It follows that

$$\mathcal{Wr}(0) = \{J_{\mathcal{Wr}}\}.$$ ◇

Definition 2.2.5. Let $n, m \in \mathbb{N}$. Let u and v be n and m-letter words respectively. Define the multiplication, $u \cdot v$, to be the following $n+m$-letter word.

$$u \cdot v\ (i) := \begin{cases} u(i) & \text{if } i \leq n \\ v(i-n) & \text{if } i > n. \end{cases} \tag{2.2.6}$$ ◇

Lemma 2.2.7. *Let $n, m, p \in \mathbb{N}$. Let u, v and w be n, m, and p-letter words respectively. Then,*

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w \tag{2.2.8}$$

as $n+m+p$-letter words.

Proof. Observe that for $i \in [n+m+p]$ we get

$$\begin{aligned} u \cdot (v \cdot w)\ (i) &= \begin{cases} u(i) & \text{if } i \leq n \\ v \cdot w\ (i-n) & \text{if } i > n \end{cases} && (2.2.6) \\ &= \begin{cases} u(i) & \text{if } i \leq n \\ v(i-n) & \text{if } 0 < i-n \leq m \\ w(i-n-m) & \text{if } i-n > m \end{cases} && (2.2.6) \\ &= \begin{cases} u(i) & \text{if } i \leq n \\ v(i-n) & \text{if } n < i \leq n+m \\ w(i-n-m) & \text{if } i > n+m \end{cases} \\ &= \begin{cases} u \cdot v\ (i) & \text{if } i \leq n+m \\ w(i-n-m) & \text{if } i > n+m \end{cases} && (2.2.6) \\ &= (u \cdot v) \cdot w\ (i) && (2.2.6). \end{aligned}$$

Since $i \in [n+m+p]$ is arbitrarily chosen, we get

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w$$

as required. □

Lemma 2.2.9. *Let $n \in \mathbb{N}$. Let u be an n-letter word and $J_{\mathcal{Wr}} : [0] \longrightarrow S$ be the empty word. Then we have*

$$u \cdot J_{\mathcal{Wr}} = u \qquad \text{and} \qquad J_{\mathcal{Wr}} \cdot u = u \tag{2.2.10}$$

as n-letter words.

Proof. Observe that for $i \in [n]$ we get

$$\begin{aligned} u \cdot J_{\mathcal{W}r}\ (i) &= \begin{cases} u(i) & \text{if } i \leq n \\ & \text{if } i > n \end{cases} && (2.2.6) \\ &= u(i) && \text{since } i \leq n. \end{aligned}$$

Since $i \in [n]$ is arbitrarily chosen we get

$$u \cdot J_{\mathcal{W}r} = u.$$

Similarly, for $i \in [n]$ we get

$$\begin{aligned} J_{\mathcal{W}r} \cdot u\ (i) &= \begin{cases} & \text{if } i \leq 0 \\ u(i) & \text{if } i > 0 \end{cases} && (2.2.6) \\ &= u(i) && \text{since } i > 0. \end{aligned}$$

Since $i \in [n]$ is arbitrarily chosen we get

$$J_{\mathcal{W}r} \cdot u = u$$

as required. □

Definition 2.2.11. Let $n \in \mathbb{N}$ and u be an n-letter word. Let $(p,q] \subseteq_I [n]$ with $0 \leq p \leq q \leq n$. Define a $q-p$-lenth word, $u_{(p,q]} : [q-p] \longrightarrow S$ as follows:

$$u_{(p,q]}\ (i) := u(p+i). \tag{2.2.12}$$

◇

Lemma 2.2.13. *Let $n \in \mathbb{N}$ and $u : [n] \longrightarrow S$ be an n-letter word. Then, we get*

$$u_{[n]} = u \tag{2.2.14}$$

as n-letter words.

Proof. Let $i \in [n]$. We have

$$u_{[n]}\ (i) = u(i). \qquad (2.2.12)$$

Since $i \in [n]$ is arbitrary, we get

$$u_{[n]} = u$$

as required. □

Lemma 2.2.15. *Let $n, m \in \mathbb{N}$, $u : [n] \longrightarrow S$ be an n-letter word and $v : [m] \longrightarrow S$ be an m-letter word. Let $(p,q] \subseteq_I [n]$ with $0 \leq p \leq q \leq n$. Then, we get*

$$(u \cdot v)_{(p,q]} = u_{(p,q]} \tag{2.2.16}$$

as $q-p$-letter words.

Proof. Let $i \in [q-p]$. Therefore, we get $p < p+i \leq q \leq n$. Now observe

$$\begin{aligned} (u \cdot v)_{(p,q]}\ (i) &= u \cdot v\ (p+i) && (2.2.12) \\ &= u(p+i) && (2.2.6) \\ &= u_{(p,q]}\ (i). && (2.2.12) \end{aligned}$$

Since $i \in [n]$ is arbitrary, we get

$$(u \cdot v)_{(p,q]} = u_{(p,q]}$$

as required. $\square$

Lemma 2.2.17. *Let* $n,m \in \mathbb{N}$, $u : [n] \longrightarrow S$ *be an* n*-letter word and* $v : [m] \longrightarrow S$ *be an* m*-letter word. Let* $(p,q] \subseteq_I (n, n+m]$ *with* $n \le p \le q \le n+m$. *Then, we get*

$$(u \cdot v)_{(p,q]} = v_{(p-n,q-n]} \tag{2.2.18}$$

as $q-p$*-letter words.*

Proof. Let $i \in [q-p]$. Therefore, we get $n \le p < p+i \le q \le n+m$. Now observe

$$\begin{aligned} (u \cdot v)_{(p,q]}\,(i) &= u \cdot v\;(p+i) && (2.2.12) \\ &= v(p+i-n) && (2.2.6) \\ &= u_{(p-n,q-n]}\,(i). && (2.2.12) \end{aligned}$$

Since $i \in [q-p]$ is arbitrary, we get

$$(u \cdot v)_{(p,q]} = u_{(p-n,q-n]}$$

as required. $\square$

Lemma 2.2.19. *Let* $n,m \in \mathbb{N}$ *and* u *and* v *be* n *and* m*-letter words respectively. Then, we get*

$$(u \cdot v)_{[n]} = u \tag{2.2.20}$$

as n*-letter words.*

Proof. Observe that

$$\begin{aligned} (u \cdot v)_{[n]} &= u_{[n]} && (2.2.16) \\ &= u. && (2.2.14) \end{aligned}$$
$\square$

Lemma 2.2.21. *Let* $n,m \in \mathbb{N}$ *and* u *and* v *be* n *and* m*-letter words respectively. Then, we get*

$$(u \cdot v)_{(n,n+m]} = v \tag{2.2.22}$$

as m*-letter words.*

Proof. Observe that

$$\begin{aligned} (u \cdot v)_{(n,n+m]} &= v_{(0,m]} && (2.2.18) \\ &= v. && (2.2.14) \end{aligned}$$
$\square$

Lemma 2.2.23. *Let* $n \in \mathbb{N}$ *be a number and* $u : [n] \longrightarrow S$ *be an* n*-letter word. Let* $0 \le p \le n$. *Then we have*

$$u = u_{[p]} \cdot u_{(p,n]} \tag{2.2.24}$$

as n*-letter words.*

Proof. Let $i \in [n]$. Then we have

$$\begin{aligned} u_{[p]} \cdot u_{(p,n]}\,(i) &= \begin{cases} u_{[p]}(i) & \text{if } i \leq p \\ u_{(p,n]}(i-p) & \text{if } i > p \end{cases} && (2.2.6) \\ &= \begin{cases} u(i) & \text{if } i \leq p \\ u(i-p+p) & \text{if } i > p \end{cases} && (2.2.12) \\ &= u(p). \end{aligned}$$

Since $i \in [n]$ is arbitrary, we get

$$u = u_{[p]} \cdot u_{(p,n]}$$

as required. □

Definition 2.2.25. A *word* with alphabet in S a tuple

$$(n,u)$$

where $n \in \mathbb{N}$ is the length and $u : [n] \longrightarrow S$ is the n-letter word. We will denote the collection of all words with alphabet in S by $\mathcal{M}on\,\langle S\rangle$. ⋄

Remark 2.2.26. Note that, $\mathcal{M}on\,\langle S\rangle$ is the collection of all words,

$$\mathcal{M}on\,\langle S\rangle = \bigsqcup_{n\in\mathbb{N}} \mathcal{W}r(n).$$

⋄

Definition 2.2.27. The monoid of words with alphabet in S, denoted $\mathcal{M}on\,\langle S\rangle$, is defined as follows:

- Let $\mathcal{M}on\,\langle S\rangle$ be the underlying set.
- Let the empty word, $(0, J_{\mathcal{W}r})$, be the unit word.
- For words (n,u) and (m,v), the multiplication is defined as

$$(n,u)\cdot(m,v) := (n+m, u\cdot v) \tag{2.2.28}$$

  where $u \cdot v : [n+m] \longrightarrow S$ is as in Definition 2.2.5.

These data satisfies the monoid conditions as follows:

Associativity: Let $(n,u),(m,v)$ and (p,w) be words. Then we have

$$\begin{aligned} (n,u)\cdot\big((m,v)\cdot(p,w)\big) &= (n,v)\cdot(m+p, v\cdot w) && (2.2.28) \\ &= (n+m+p, u\cdot(v\cdot w)) && (2.2.28) \\ &= (n+m+p, (u\cdot v)\cdot w) && (2.2.8) \\ &= (n+m, u\cdot v)\cdot(p,w) && (2.2.28) \\ &= \big((n,u)\cdot(m,v)\big)\cdot(p,w). && (2.2.28) \end{aligned}$$

Unit condition: Let (n,u) be a word. Then we get

$$\begin{aligned} (n,u)\cdot(0,J_{\mathcal{W}r}) &= (n, u\cdot J_{\mathcal{W}r}) && (2.2.28) \\ &= (n,u). && (2.2.10) \end{aligned}$$

Similarly,

$$\begin{aligned} (0,J_{\mathcal{W}r})\cdot(n,u) &= (n, J_{\mathcal{W}r}\cdot u) && (2.2.28) \\ &= (n,u). && (2.2.10) \end{aligned}$$

⋄

Definition 2.2.29. Define a function

$$\hat{l} : \mathcal{M}on\langle S\rangle \longrightarrow \mathbb{N}$$

as

$$\hat{l}(n,u) = n. \tag{2.2.30}$$

From the monoid structure on $\mathcal{M}on\langle S\rangle$, the function $\hat{l} : \mathcal{M}on\langle S\rangle \longrightarrow \mathbb{N}$ is a monoid homomorphism. ◇

Lemma 2.2.31. *The monoid of words with alphabet in S is a left cancellative monoid. That is, for $x, y, z \in \mathcal{M}on\langle S\rangle$,*

$$x \cdot y = x \cdot z \qquad \textit{implies} \qquad y = z. \tag{2.2.32}$$

Proof. Let $x = (n,u)$, $y = (m,v)$, and $z = (p,w)$ be words. Suppose we have

$$x \cdot y = x \cdot z.$$

From (2.2.28) we get

$$n + m = n + p \qquad \text{and} \qquad u \cdot v = u \cdot w.$$

Thus, we have $m = p$ and

$$\begin{aligned} y = (m,v) &= \big(m, (u \cdot v)_{(n,n+m]}\big) && (2.2.22) \\ &= \big(p, (u \cdot w)_{(n,n+p]}\big) && \text{from above} \\ &= (p,w) = z. && (2.2.22) \end{aligned}$$

□

Lemma 2.2.33. *The monoid of words with alphabet in S is a right cancellative monoid. That is, for $x, y, z \in \mathcal{M}on\langle S\rangle$ we get*

$$y \cdot x = z \cdot x \qquad \textit{implies} \qquad y = z. \tag{2.2.34}$$

Proof. Let $x = (n,u)$, $y = (m,v)$, and $z = (p,w)$ be words. Suppose we have

$$y \cdot x = z \cdot x.$$

From (2.2.28) we get

$$m + n = p + n \qquad \text{and} \qquad v \cdot u = w \cdot u.$$

Thus, we have $m = p$ and

$$\begin{aligned} y = (m,v) &= \big(m, (v \cdot u)_{[m]}\big) && (2.2.20) \\ &= \big(p, (w \cdot u)_{[p]}\big) && \text{from above} \\ &= (p,w) = z. && (2.2.20) \end{aligned}$$

□

Definition 2.2.35. Define a subset, $\mathsf{S} \subseteq \mathcal{M}on\langle S\rangle$ as follows:

$$\mathsf{S} := \{(n,u) \in \mathcal{M}on\langle S\rangle \mid n = 1\}. \tag{2.2.36}$$

◇

Proposition 2.2.37. *The subset S is in bijection with S.*

Proof. Define a function $f : S \longrightarrow \mathtt{S}$ as follows:

$$f(a) := (1, u_a) \in \mathtt{S}$$

where $u_a : [1] \longrightarrow S$ is given by $u(1) = a$. The function $f : S \longrightarrow \mathtt{S}$ is injective since for $a, b \in S$ with $f(a) = f(b)$ we get

$$(1, u_a) = (1, u_b).$$

Therefore, we get $u_a = u_b$. It follows that

$$a = u_a(1) = u_b(1) = b.$$

We will show that the function $f : S \longrightarrow \mathtt{S}$ is surjective. Let $(1, u) \in \mathtt{S}$. Let $a = u[1]$, then we get $u = u_a$ since $u_a[1] = a = u[a]$. Consequently, we get

$$f(a) = (1, u_a) = (1, u). \qquad \square$$

Theorem 2.2.38. *The subset* $\mathtt{S} \subseteq \mathcal{Mon}\langle S\rangle$ *is a monoid basis.*

Proof. First, we will show that $\mathtt{S}$ is a generating set with respect to the multiplication. Let $x = (n, u) \in M^+$ be a non-unit word. We will show, using induction on n, that x has a $\mathtt{S}$-factorization.

Base case ($n = 1$): In this case, we get $x \in \mathtt{S}$. Thus, it has the trivial $\mathtt{S}$-factorization.

Induction case ($n > 1$): Let $x_1 = (1, u_{[1]})$ and $y = (n-1, u_{(1,n]})$. Observe that

$$\begin{aligned} x_1 \cdot y &= (1, u_{[1]}) \cdot (n-1, u_{(1,n]}) \\ &= (n, u_{[1]} \cdot u_{(1,n]}) && (2.2.28) \\ &= (n, u) && (2.2.24) \\ &= x. \end{aligned}$$

From the induction hypothesis, we get an $\mathtt{S}$-factorization of y. That is, we get $x_2, \ldots, x_m \in \mathtt{S}$ such that $y = x_2 \cdots x_m$. Since $x_1 = (1, u_{[1]})$ we have $x_1 \in \mathtt{S}$. Moreover, we get

$$x = x_1 \cdot y = x_1 \cdots x_m.$$

Thus, $(x_1, \ldots, x_m)$ forms an $\mathtt{S}$-factorization of x.

Next, we will show that $\mathtt{S}$ is independent with respect to the multiplication. Let $x = (n, u) \in \langle \mathtt{S}; \bullet \rangle$. We will show, using induction on n, that x has at most one $\mathtt{S}$-factorization. Note that, from the Definition 2.2.35 we conclude that $n \geq 1$.

Base case ($n = 1$): We get $x \in \mathtt{S}$. Thus, x has the trivial $\mathtt{S}$-factorization. Let $x = x_1 \cdots x_m$ be another $\mathtt{S}$-factorization of x. That is, we have $m \geq 1$ and $x_i \in \mathtt{S}$ for $1 \leq i \leq m$. Since $x_i \in X$, we get $\hat{l}(x_i) = 1$. Thus,

$$1 = \hat{l}(x) = \hat{l}(x_1 \cdots x_m) = \hat{l}(x_1) + \cdots + \hat{l}(x_m) = m.$$

Therefore, $x = x_1$ is the only $\mathtt{S}$-factorization of x.

Induction case ($n > 1$): Let

$$x = x_1 \cdots x_m = y_1 \cdots y_p$$

be two S-factorizations of x. That is, we have $x_i, y_j \in \mathtt{S}$ for $1 \le i \le m$ and $1 \le j \le p$. In particular, we have $x_1 = (1, u_1)$ and $y_1 = (1, v_1)$ for some 1-letter words u_1 and v_1. Let $x_R = x_2 \cdots x_m$ and $y_R = y_2 \cdots y_p$. Then, we get

$$x = x_1 \cdot x_R = y_1 \cdot y_R.$$

Observe that

$$\begin{aligned} x_1 &= (x_1 \cdot x_R)_{[1]} && (2.2.16) \\ &= x_{[1]} \\ &= (y_1 \cdot y_R)_{[1]} \\ &= y_1. && (2.2.16) \end{aligned}$$

Since $\mathcal{Mon}\langle S\rangle$ is a left cancellative monoid (Lemma 2.2.31), we get

$$x_R = x_2 \cdots x_m = y_2 \cdots y_p = y_R.$$

We have $\hat{l}(x_R) = n - 1 < n$. Also, we get two $\mathtt{S}$-factorizations of $x_R = y_R$. From the induction hypothesis, we get $m = p$ and $x_i = y_i$ for $2 \le i \le m$. This, along with the fact that $x_1 = y_1$, we conclude that x has at most one $\mathtt{S}$-factorization. □

Theorem 2.2.39. *Let S be a set and let $\mathcal{Mon}\langle S\rangle$ be the monoid defined as in Definition 2.2.25. Let $i : S \longrightarrow \mathcal{Mon}\langle S\rangle$ be the inclusion from Proposition 2.2.37. Then, the pair*

$$(\mathcal{Mon}\langle S\rangle, i : S \longrightarrow \mathcal{Mon}\langle S\rangle)$$

satisfies the universal property of the free monoid.

Proof. By Theorem 2.2.38, $\mathtt{S}$ forms a monoid basis for $\mathcal{Mon}\langle S\rangle$. Therefore, by Theorem 2.1.36, the pair

$$(\mathcal{Mon}\langle S\rangle, i : S \longrightarrow \mathcal{Mon}\langle S\rangle)$$

satisfies the universal property of the free monoid. □

Theorem 2.2.40. *Let M be a monoid, and let $X \subseteq M$ with the inclusion given by*

$$\phi : X \hookrightarrow M.$$

Then, the pair

$$\left(M, \phi : X \hookrightarrow M\right)$$

satisfies the universal property of the free monoid generated by X if and only if X is a multiplicative basis of M.

Proof. We have already shown one direction at Theorem 2.1.36.

For the other direction, suppose the pair

$$\left(M, \phi : X \hookrightarrow M\right)$$

satisfies the universal property of the free monoid generated by X. Consider the construction of the monoid of words $\mathcal{M}on\langle X\rangle$ over the set X as in Definition 2.2.27. From Theorem 2.2.38, we know that $X \subseteq \mathcal{M}on\langle X\rangle$ is a multiplicative basis of $\mathcal{M}on\langle X\rangle$. From Theorem 2.2.39, we know that the pair

$$\left(\mathcal{M}on\langle X\rangle, i : X \longrightarrow \mathcal{M}on\langle X\rangle\right)$$

satisfies the universal property of the free monoid generated by X. It follows that, $\mathcal{M}on\langle X\rangle$ is isomorphic to M via a monoid homomorphism which maps $X \subseteq \mathcal{M}on\langle X\rangle$ to $X \subseteq M$. Since $X \subseteq \mathcal{M}on\langle X\rangle$ is a multiplicative basis of $\mathcal{M}on\langle X\rangle$, from Proposition 2.1.28 we conclude that $X \subseteq M$ is a multiplicative basis of M. $\square$

CHAPTER 3

Topics in Dashed Monoids

We first examine the structure of objects in a semi-strict categorical group. Let $(\mathscr{M}, I, (-)', \otimes)$ be a semi-strict categorical group with the set of objects denoted by M. Then, M forms a monoid where the multiplication is given by the tensor product and I is the unit. Additionally, there is a unary operation $(-)' : M \longrightarrow M$ on objects, which does not interact with the monoid structure except for the condition $I' = I$. This motivates the definition of a new algebraic structure, which we call a *dashed monoid*: a monoid equipped with a unary operation $(-)'$ satisfying $I' = I$.

In Definition 3.2.1, we state the universal property of the free dashed monoid. In Definition 3.2.6 and Theorem 3.2.44, we provide a criterion for when a dashed monoid is free. In Definition 3.6.2, we give an explicit construction of the free dashed monoid generated by a set, and in Theorem 3.7.56, we show that this construction satisfies the universal property.

3.1. Definition of a Dashed Monoid

In this section, we provide the formal definitions of dashed monoids and dashed monoid homomorphisms. We also introduce the notions of multiplicative basis and dash basis for dashed monoids.

Definition 3.1.1. A *dashed monoid*

$$\big(M, I, (-)', \bullet\big)$$

has the following data:

- A monoid,

$$(M, \bullet, I),$$

called the *underlying monoid*.

- A unary set map,

$$(-)' : M \longrightarrow M,$$

called the *dash map*.

These satisfy the following *unit condition*:

$$I' = I. \tag{3.1.2}$$

◇

Definition 3.1.3. A dashed monoid M is called:

- A *monoid with a distributive dash* if it satisfies the distributive property:

$$(x \bullet y)' = y' \bullet x'$$

for all $x, y \in M$.

- A *monoid with convolution* if it satisfies both the distributive property above and the convolution property:

$$x'' = x$$

for all $x \in M$.

- A *group* if it satisfies the full inverse property:

$$x' \bullet x = I = x \bullet x'$$

for all $x \in M$. ◇

Convention 3.1.4. Alternative notations for the dash of $x \in M$ include $x^\dagger$ and $x^{(1)}$. For $n \in \mathbb{N}$, we write $x^{(n)}$ to denote the n-fold application of the dash map to x, with the convention that $x^{(0)} = x$. ◇

Example 3.1.5. The set of natural numbers $\mathbb{N}$ forms a dashed monoid under addition, unit as 0, and the dash operation is the identity map. This will be considered the default dashed monoid structure on $\mathbb{N}$ unless specified otherwise. ◇

Example 3.1.6. Let $(\mathscr{M}, I, (-)', \otimes)$ be a semi-strict categorical group. As discussed earlier, the set of objects of $\mathscr{M}$ naturally forms a dashed monoid. Furthermore, if $\mathscr{M}$ is a small category, then the set of all morphisms

$$\mathrm{Mor}(\mathscr{M}) := \bigsqcup_{x,y \in \mathscr{M}} \mathrm{Hom}_{\mathscr{M}}(x, y)$$

also inherits a dashed monoid structure, where the unit is 1_I, the multiplication is given by the tensor product, and the dash operation is given by the dash functor. ◇

Example 3.1.7. Following are some examples of dashed monoids.

i. We can give dashed monoid structure to a monoid by simply setting the dash map to be the identity map. Let $(M, \cdot, I)$ be a monoid. Then $(M, \cdot, I, \mathrm{Id}_M)$ is a dashed monoid.

ii. A more natural approach is to consider a group as the dashed monoid with the inverse map as the dash map. Let $(G, \cdot, e, (-)^{-1})$ be a group then $(G, \cdot, e, (-)^{-1})$ is a dashed monoid.

iii. The integers form a dashed monoid with identity as a dashed map, $(\mathbb{Z}, +, 0, \mathrm{Id}_{\mathbb{Z}})$ is a dashed monoid. By convention, unless specified otherwise, the above structure will be the dashed monoid structure on the integers.

iv. Let $(M, \cdot, I, D_M)$ and $(N, *, J, D_N)$ be dashed monoids. Then

$$(M \times N,\ \cdot \times *,\ (I, J),\ D_M \times D_N)$$

is also a dashed monoid where the multiplication

$$\cdot \times * : (M \times N) \times (M \times N) \longrightarrow (M \times N)$$

is given by

$$\big((m_1, n_1), (m_2, n_2)\big) \longmapsto (m_1 \cdot m_2,\ n_1 * n_2)$$

and $D_M \times D_N : M \times N \longrightarrow M \times N$ is given by

$$(m,n) \longmapsto (D_M(m), D_N(n)).$$

Since the Cartesian product of monoids is again a monoid,

$$(M \times N, \cdot \times *, (I,J))$$

is a monoid. It remains to check that $D_M \times D_N\ (I,J) = (I,J)$. Indeed, since

$$D_M \times D_N\ (I,J) = (D_M(I), D_N(J)) = (I,J). \qquad \diamond$$

Definition 3.1.8. Let

$$M = (M, \cdot, I, (-)') \qquad \text{and} \qquad N = (N, *, J, (-)^{\dagger})$$

be two dashed monoids. A *morphism of dashed monoids*

$$f : M \longrightarrow N$$

has the following data:

- A monoid homomorphism,

  $$f : M \longrightarrow N,$$

  on the underlying monoidal categories

that satisfies the following dash condition: For every $x \in M$,

$$(f(x))^{\dagger} = f(x'). \tag{3.1.9}$$

$\diamond$

Definition 3.1.10. Let $M = (M, \cdot, I, (-)')$ be a dashed monoid. Define the identity dashed monoid homomorphism, $\mathrm{Id}_M : M \longrightarrow M$ as follows:

- Let $\mathrm{Id}_M : M \longrightarrow M$ be the underlying monoid homomorphism.

This satisfies the dash condition since the equality

$$\mathrm{Id}_M(x)' = x' = \mathrm{Id}_M(x')$$

holds for every $x \in M$. $\diamond$

Definition 3.1.11. Let

$$M = (M, \cdot, I, (-)'), \qquad N = (N, *, J, (-)^{\dagger}), \qquad \text{and} \qquad P = (P, \bullet, K, (-)^{(1)})$$

be dashed monoids, and

$$f : M \longrightarrow N, \qquad \text{and} \qquad g : N \longrightarrow P$$

be dashed monoid homomorphisms. Define the composite dashed monoid homomorphism $g \circ f : M \longrightarrow P$ as follows:

- Let $g \circ f$ be the underlying monoid homomorphism.

The above monoid homomorphism satisfies the dash condition since the chain of equalities

$$\begin{aligned}\left(g\circ f\ (x)\right)^{(1)} &= \left(g(f(x))\right)^{(1)} \\ &= g\left((f(x))^{\dagger}\right) && (3.1.9) \\ &= g\left(f(x')\right) && (3.1.9) \\ &= g\circ f\ (x')\end{aligned}$$

hold for every $x \in M$. ◇

Definition 3.1.12. The *category of dashed monoids*, denoted $\mathcal{DMon}$, is defined as follows:

- The dashed monoids as in Definition 3.1.1 form the class of objects.
- The dashed monoid homomorphisms as in Definition 3.1.8 form the morphisms.
- For a dashed monoid M, the identity homomorphism Id_M, is the identity dashed-monoid homomorphism as in Definition 3.1.10.
- The composition of dashed monoid homomorphism is as in Definition 3.1.11.

Since the composition in dashed monoids is induced from that in monoids, the associativity, the left identity and the right identity conditions are satisfied.◇

Motivation 3.1.13. Recall the notion of a multiplicative basis for a monoid from Definition 2.1.24. Dashed monoids inherit this concept for their multiplication operation. We now introduce analogous definitions for the dash operation. ◇

Framework 3.1.14. For the rest of this section let $(M, I, (-)', \bullet)$ be a dashed monoid, and let $X \subseteq M$ be a subset. ◇

Definition 3.1.15. Define a subset $\langle X;\ (-)'\rangle \subseteq M$ as follows:

$$\langle X;(-)'\rangle := \{x^{(k)} \in M \mid x \in X \ \text{ and } \ k \in \mathbb{N}\}. \tag{3.1.16}$$

We call this the subset generated by X under the dash operation.

Define a subset $\langle\langle X;\ (-)'\rangle\rangle \subseteq M$ as follows:

$$\langle\langle X;(-)'\rangle\rangle := \{x^{(k)} \in M \mid x \in X \ \text{ and } \ k \in \mathbb{N}^{+}\}. \tag{3.1.17}$$ ◇

Definition 3.1.18. We say that the subset $X \subseteq M$ is *independent with respect to the dash operation* if for $r, k \in \mathbb{N}$ and $x, y \in X$ the following implication holds:

$$x^{(r)} = y^{(k)} \qquad \Longrightarrow \qquad r = k \text{ and } x = y. \tag{3.1.19}$$ ◇

Proposition 3.1.20. *Suppose $X \subseteq M$ is independent with respect to the dash operation, then $I \notin X$.*

Proof. Suppose for contradiction $I \in X$. We have

$$I' = I.$$

This contradicts dash independent condition of X. □

Definition 3.1.21. We say that the subset $X \subseteq M$ is a *generating set with respect to the dash operation* if

$$M^+ \subseteq \langle X;\, (-)' \rangle. \tag{3.1.22}$$

In other words, X is a generating set with respect to the dash operation if for every $z \in M^+$ we get $r \in \mathbb{N}$ and $x \in X$ such that

$$x^{(r)} = z.$$
◇

Definition 3.1.23. We say that the subset $X \subseteq M$ is a *dash basis* if X is independent with respect to the dash operation and X is a generating set with respect to the dash operation. ◇

Proposition 3.1.24. *Let M and N be dashed monoids, and let*

$$f : M \longrightarrow N$$

be a dashed monoid isomorphism. If $X \subseteq M$ is a dash basis of M, then $f(X) \subseteq N$ is a dash basis of N.

Proof. Suppose $X \subseteq M$ is a dash basis of M. We need to show that $f(X)$ is both a dash independent subset and a dash generating subset.

First, let $r, k \in \mathbb{N}$, and $a, b \in f(X)$ such that

$$a^{(r)} = b^{(k)}.$$

Since f is a dashed monoid isomorphism, we get $f^{-1}(a), f^{-1}(b) \in X$, and

$$\left(f^{-1}(a)\right)^{(r)} = \left(f^{-1}(b)\right)^{(k)}.$$

Because X is a dash basis, it follows that

$$r = k \qquad \text{and} \qquad f^{-1}(a) = f^{-1}(b).$$

Applying f, we get

$$r = k \qquad \text{and} \qquad a = b.$$

Next, let $a \in N^+$. Since f is a dashed monoid homomorphism, $f^{-1}(a) \in M^+$. Since X is a dash basis, there exists $k \in \mathbb{N}$, and $x \in X$ such that

$$f^{-1}(a) = x^{(k)}.$$

Applying f yields

$$a = (f(x))^{(k)}$$

with $f(x) \in f(X)$. This completes the proof. □

Proposition 3.1.25. *If a dashed monoid M admits a dash basis, then this dash basis is unique.*

Proof. Let $X, Y \subseteq M$ be dash bases of M. If both are empty, the result is immediate. Suppose X is nonempty and let $x \in X$. By Proposition 3.1.20, $x \in M^+$. Since Y is a dash basis, we get unique $r \in \mathbb{N}$ and $y \in Y$ such that

$$x = y^{(r)}.$$

It follows that, $y \in M^+$. Since X is a dash basis, we get unique $k \in \mathbb{N}$ and $x \in X$ such that

$$y = x^{(k)}.$$

Substituting, we have

$$x = x^{(r+k)}.$$

By uniqueness property of dash basis X, we get $r + k = 0$. Thus, $r = k = 0$ and $x = y^{(0)} = y \in Y$. Therefore, we get $X \subseteq Y$. By symmetry, $Y \subseteq X$. Hence, $X = Y$. □

Lemma 3.1.26. *Suppose $X \subseteq M$ is a multiplicatively independent subset. Then, we get*

$$\langle X;\ \bullet\rangle = X \sqcup \langle\langle X;\ \bullet\rangle\rangle. \tag{3.1.27}$$

Proof. Let $x \in M$. Suppose $x \in \langle X;\ \bullet\rangle$. Then, we get $m \in \mathbb{N}$ with $m \geq 1$ and $x_i \in X$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

If $m = 1$ then we get

$$x = x_1 \in X.$$

Otherwise, we have $m \geq 2$ and

$$x = x_1 \bullet x_2 \bullet \cdots \bullet x_m \in \langle\langle X;\ \bullet\rangle\rangle.$$

On the other hand, suppose $x \in X \sqcup \langle\langle X;\ \bullet\rangle\rangle$. If $x \in X$, then we get

$$x \in \langle X;\ \bullet\rangle.$$

Otherwise, we have

$$x \in \langle\langle X;\ \bullet\rangle\rangle \subseteq \langle X;\ \bullet\rangle.$$

Now we will show that

$$X \cap \langle\langle X;\ \bullet\rangle\rangle = \varnothing.$$

Suppose $x \in X \cap \langle\langle X;\ \bullet\rangle\rangle$. Then we have $x \in X$ and $x \in \langle\langle X;\ \bullet\rangle\rangle$. Therefore, we get $m \in \mathbb{N}$ with $m \geq 2$ and $x_i \in X$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet x_2 \bullet \cdots \bullet x_m.$$

This leads to a contradiction since X is independent with respect to the multiplication. Thus, we get

$$X \cap \langle\langle X;\ \bullet\rangle\rangle = \varnothing$$

as required. □

Lemma 3.1.28. *Suppose $X \subseteq M$ is a dash independent subset. Then, we get*

$$\left\langle X;\ (-)'\right\rangle = X \sqcup \left\langle\left\langle X;\ (-)'\right\rangle\right\rangle. \tag{3.1.29}$$

Proof. Let $x \in M$. Suppose $x \in \langle X;\, (-)'\rangle$. Then, we get $k \in \mathbb{N}$ and $y \in X$ such that

$$x = y^{(k)}.$$

If $k = 0$ then we get

$$x = y^{(0)} = y \in X.$$

Otherwise, we get $k \geq 1$ and

$$x = y^{(k)} \in \langle\langle X;\, (-)'\rangle\rangle.$$

On the other hand, suppose $x \in X \sqcup \langle\langle X;\, (-)'\rangle\rangle$. If $x \in X$, then we get

$$x = x^{(0)} \in \langle X;\, (-)'\rangle.$$

Otherwise, we have

$$x \in \langle\langle X;\, (-)'\rangle\rangle \subseteq \langle X;\, (-)'\rangle.$$

Now we will show that

$$X \cap \langle\langle X;\, (-)'\rangle\rangle = \varnothing.$$

Suppose $x \in X \cap \langle\langle X;\, (-)'\rangle\rangle$. Then we have $x \in X$ and $x \in \langle\langle X;\, (-)'\rangle\rangle$. Therefore, we get $y \in X$ and $k \in \mathbb{N}$ with $k \geq 1$ such that $x = y^{(k)}$. Thus, we have

$$x^{(0)} = x = y^{(k)}.$$

This leads to a constradiction since X is independent with respect to the dash operation. Thus, we get

$$X \cap \langle\langle X;\, (-)'\rangle\rangle = \varnothing$$

as required. □

Lemma 3.1.30. *Let $A \subseteq X \subseteq M$ be subsets of M. If X is a multiplicatively independent set then so is A.*

Proof. Suppose X is a multiplicatively independent set. We wish to show that $A \subseteq X$ is a multiplicatively independent set. Let $n, m \geq 1$ and $x_i, y_j \in A$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ such that

$$x_1 \bullet \cdots \bullet x_n = y_1 \bullet \cdots \bullet y_m.$$

Since X is independent with respect to the multiplication and $x_i, y_j \in A \subseteq X$, we get $n = m$ and $x_i = y_j$. This shows that A is independent with respect to the multiplication. □

Definition 3.1.31. Let $A \subseteq X \subseteq M$ be subsets of M. Define a subset

$$e\langle X, A;\, \bullet\rangle \subseteq M$$

as the collection of elements of M that can be written as a product

$$x_1 \bullet \cdots \bullet x_m$$

with $m \geq 1$, each $x_i \in M$, and at least one $x_j \in A$. Specifically,

$$e\langle X, A;\, \bullet\rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \geq 1,\ x_i \in X, \text{ and } \exists\, 1 \leq j \leq m \text{ such that } x_j \in A\}. \tag{3.1.32}$$

Define a subset

$$e\,\langle\langle X, A;\ \bullet\rangle\rangle \subseteq M$$

as as the collection of elements of M that can be written as a product

$$x_1 \bullet \cdots \bullet x_m$$

with $m \geq 2$, each $x_i \in M$, and at least one $x_j \in A$. Specifically

(3.1.33)

$$e\,\langle\langle X, A;\ \bullet\rangle\rangle := \{x_1 \bullet \cdots \bullet x_m \in M \mid m \geq 2,\ x_i \in X,\ \text{and}\ \ \exists\ 1 \leq j \leq m \text{ such that } x_j \in A\}. \quad \diamond$$

Lemma 3.1.34. *Suppose $X \subseteq M$ is a multiplicatively independent subset. Let $A, B \subseteq X$ be disjoint subsets of X. Then, we get*

$$\langle A \sqcup B;\ \bullet\rangle = \langle A;\ \bullet\rangle \bigsqcup e\,\langle A \sqcup B,\ B;\ \bullet\rangle \tag{3.1.35}$$

and

$$\langle\langle A \sqcup B;\ \bullet\rangle\rangle = \langle\langle A;\ \bullet\rangle\rangle \bigsqcup e\,\langle\langle A \sqcup B,\ B;\ \bullet\rangle\rangle \tag{3.1.36}$$

Proof. Let $x \in M$. Suppose $x \in \langle A \sqcup B;\ \bullet\rangle$, then we get $m \geq 1$ and $x_i \in A \sqcup B$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

If for every $1 \leq i \leq m$, we have $x_i \in A$ then we get

$$x = x_1 \bullet \cdots \bullet x_m \in \langle A;\ \bullet\rangle.$$

Otherwise, we get at least one $1 \leq j \leq m$ such that $x_j \in B$. Thus, we have

$$x = x_1 \bullet \cdots \bullet x_m \in e\,\langle A \sqcup B, B;\ \bullet\rangle.$$

On the other hand, suppose $x \in \langle A;\ \bullet\rangle \sqcup e\,\langle A \sqcup B, B;\ \bullet\rangle$. If $x \in \langle A;\ \bullet\rangle$ then we get some $m \geq 1$ and $x_i \in A$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots x_m.$$

Therefore, we have

$$x = x_1 \bullet \cdots \bullet x_m \in \langle A \sqcup B;\ \bullet\rangle.$$

Otherwise, if $x \in e\left\langle A \sqcup B, B;\ (-)'\right\rangle$ then we get some $m \geq 1$ and $x_i \in A \cup B$ with at least one $x_j \in B$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

Therefore, we have

$$x = x_1 \bullet \cdots \bullet x_m \in \langle A \sqcup B;\ \bullet\rangle.$$

Now we will show that $\langle A;\ \bullet\rangle$ and $e\,\langle A \sqcup B, B;\ \bullet\rangle$ are disjoint. Suppose there is $x \in M$ such that

$$x \in \langle A;\ \bullet\rangle \cap e\,\langle A \sqcup B, B;\ \bullet\rangle.$$

We get $n, m \geq 1$, $x_i \in A$ for $1 \leq i \leq n$, and $y_j \in A \cup B$ for $1 \leq j \leq m$ with at least one $y_k \in B$ such that

$$x_1 \bullet \cdots \bullet x_n = x = y_1 \bullet \cdots \bullet x_m.$$

Becasue X is a dash independent set and $x_i, y_j \in A \sqcup B \subseteq X$, we get $n = m$ and $x_i = y_i$. In particular, we get $x_k = y_k \in B$. We already have $x_k \in A$. This is a contradiction since A and B are disjoint. Thus, we get

$$\langle A \sqcup B;\ \bullet\rangle = \langle A;\ \bullet\rangle \bigsqcup e\,\langle A \sqcup B, B;\ \bullet\rangle\,.$$

The proof of equation (3.1.36) is similar to the proof of equation (3.1.35).□

Lemma 3.1.37. *Suppose $A \subseteq X \subseteq M$ are subsets of M. If x is a dash independent subset then so is A.*

Proof. Suppose X is a dash independent set. We wish to show that $A \subseteq X$ is a dash independent set. Let $n, m \in \mathbb{N}$ and $a, b \in A$ such that

$$a^{(n)} = b^{(m)}.$$

Since X is independent with respect to the dash-operation and $a, b \in X$, we get $n = m$ and $a = b$. This shows that A is independent with respect to the dash-operation as well. □

Lemma 3.1.38. *Let $X \subseteq M$ be a dash independent subset. Let $A, B \subseteq X$ be disjoint subsets of X. Then, we get*

$$\big\langle A \sqcup B;\ (-)'\big\rangle = \big\langle A;\ (-)'\big\rangle \bigsqcup \big\langle B;\ (-)'\big\rangle \tag{3.1.39}$$

and

$$\big\langle\big\langle A \sqcup B;\ (-)'\big\rangle\big\rangle = \big\langle\big\langle A;\ (-)'\big\rangle\big\rangle \bigsqcup \big\langle\big\langle B;\ (-)'\big\rangle\big\rangle \tag{3.1.40}$$

Proof. Let $x \in M$. Suppose $x \in \big\langle A \sqcup B;\ (-)'\big\rangle$. Then, we get $r \in \mathbb{N}$ and $y \in A \sqcup B$ such that

$$x = y^{(r)}.$$

If $y \in A$ then we get

$$x = y^{(k)} \in \big\langle\big\langle A;\ (-)'\big\rangle\big\rangle$$

and if $y \in B$ we get

$$x = y^{(k)} \in \big\langle\big\langle B;\ (-)'\big\rangle\big\rangle.$$

On the other hand, suppose $x \in \big\langle A;\ (-)'\big\rangle \sqcup \big\langle B;\ (-)'\big\rangle$. If $x \in \big\langle A;\ (-)'\big\rangle$ then we get some $r \in \mathbb{N}$ and $y \in A$ such that

$$x = y^{(k)}.$$

Therefore, we have

$$x = y^{(k)} \in \big\langle A \sqcup B;\ (-)'\big\rangle.$$

If $x \in \big\langle B;\ (-)'\big\rangle$ then we get some $r \in \mathbb{N}$ and $y \in B$ such that

$$x = y^{(k)}.$$

Therefore, we have

$$x = y^{(k)} \in \big\langle A \sqcup B;\ (-)'\big\rangle.$$

Now we will show that $\big\langle\big\langle A;\ (-)'\big\rangle\big\rangle$ and $\big\langle\big\langle B;\ (-)'\big\rangle\big\rangle$ are disjoint. Suppose there is $x \in M$ such that

$$x \in \big\langle\big\langle A;\ (-)'\big\rangle\big\rangle \cap \big\langle\big\langle B;\ (-)'\big\rangle\big\rangle.$$

We get some $r,k \in \mathbb{N}$, $y \in A$, and $z \in B$ such that

$$y^{(k)} = x = z^{(r)}.$$

Since X is a dash independent set and $y,z \in A \sqcup B \subseteq X$ we get $k = r$ and $y = z$. This is a contradiction since A and B are disjoint. Thus, we get

$$\langle A \sqcup B;\ (-)' \rangle = \langle A;\ (-)' \rangle \bigsqcup \langle B;\ (-)' \rangle.$$

The proof of equation (3.1.40) is similar to the proof of equation (3.1.39). □

3.2. Free Dashed Monoid and Dashed Monoid Basis

In this section, we will define the universal property of a free dashed monoid and provide a criterion for a dashed monoid to be a free dashed monoid.

Definition 3.2.1. Let S be a set, M be a dashed monoid, and $\phi : S \longrightarrow M$ be a function. We say that the pair

$$\big(M, \phi : S \longrightarrow M\big)$$

satisfies the *universal property of the free dashed monoid generated by S* if the following conditions are satisfied:

Existence: For a dashed monoid N and set map $u : S \longrightarrow N$ there exists a dashed monoid homomorphism $f_u : M \longrightarrow N$ such that

$$f_u \circ \phi = u. \tag{3.2.2}$$

Uniqueness: For a dashed monoid N and a pair of dashed monoid homomorphisms $f,g : M \longrightarrow N$ we get

$$f \circ \phi = g \circ \phi \qquad \text{implies} \qquad f = g. \tag{3.2.3}$$ ◇

Construction 3.2.4. Let S be a set. The *free dashed monoid* generated by S, denoted $\mathcal{DMon}\langle S\rangle$, is constructed inductively as follows:

- The unit symbol J and each element of S are elements of $\mathcal{DMon}\langle S\rangle$.
- If $x, y \in \mathcal{DMon}\langle S\rangle$, then $x \bullet y$ is also in $\mathcal{DMon}\langle S\rangle$.
- If $x \in \mathcal{DMon}\langle S\rangle$, then x' is also in $\mathcal{DMon}\langle S\rangle$.

These are subject to the ralations

$$J' = J, \qquad J \bullet x = x = x \bullet J, \qquad \text{and} \qquad x \bullet (y \bullet z) = (x \bullet y) \bullet z.$$ ◇

Remark 3.2.5. We will spend Chapter 3 to understand the algebraic structure behind the above construction. In Section 3.2, we will present a criterion for identifying the free dashed monoid $\mathcal{DMon}\langle S\rangle$. An alternate, formal set-theoretic construction of the free dashed monoid $\mathcal{DMon}\langle S\rangle$ will be given in Section 3.6. Finally, in Section 3.7, we will verify that $\mathcal{DMon}\langle S\rangle$ satisfies the required universal property. ◇

Definition 3.2.6. Let M be a dashed monoid, and let $\mathtt{S} \subseteq M$ be a set. We say $\mathtt{S} \subseteq M$ is a *dashed monoid basis* if M

- admits a multiplicative basis $\mathtt{G}$,
- admits a dash basis $\mathtt{H}$, and

- has a dashed-monoid homomorphism $\hat{l} : M \longrightarrow \mathbb{N}$

such that the following conditions are satisfied:

Length-unit condition: For $x \in M$ we have

$$\hat{l}(x) = 0 \iff x = I \tag{3.2.7}$$

and for $a \in \mathtt{S}$ we have

$$\hat{l}(a) = 1. \tag{3.2.8}$$

Interlocking condition: The equalities

$$\mathtt{G} = \mathtt{S} \sqcup \left\langle\left\langle \mathtt{H};\ (-)' \right\rangle\right\rangle \qquad \text{and} \tag{3.2.9}$$

$$\mathtt{H} = \mathtt{S} \sqcup \langle\langle \mathtt{G};\ \bullet \rangle\rangle \tag{3.2.10}$$

hold. ◇

Proposition 3.2.11. *Let M and N be dashed monoids, and let*

$$f : M \longrightarrow N$$

be a dashed monoid isomorphism. If $\mathtt{S} \subseteq M$ is a dashed monoid basis of M, then $f(\mathtt{S}) \subseteq N$ is a dashed monoid basis of N.

Proof. Suppose $\mathtt{S} \subseteq M$ is a multiplicative basis of M. We need to show that $f(\mathtt{S})$ satisfies the conditions described in Definition 3.2.6.

Since M admits a dashed monoid basis S, it also admits a multiplicative basis $\mathtt{G}$, and a dash basis $\mathtt{H}$. From Propositions 2.1.28 and 3.1.24, $f(\mathtt{G})$ and $f(\mathtt{H})$ are multiplicative and dash basis of N respectively.

Let $\hat{l}_M : M \longrightarrow \mathbb{N}$ be the length function. We define

$$\hat{l}_N : N \longrightarrow \mathbb{N}$$

as

$$\hat{l}_N := \hat{l}_M \circ f^{-1}.$$

Since f is an isomorphism, from the length conditions (3.2.7) and (3.2.8) we get

$$\hat{l}_N(x) = 0 \iff x = I$$

and for $a \in f(\mathtt{S})$

$$\hat{l}_N(a) = 1.$$

Finally, from the interlocking conditions (3.2.9) and (3.2.10), we have

$$\mathtt{G} = \mathtt{S} \sqcup \left\langle\left\langle \mathtt{H};\ (-)' \right\rangle\right\rangle \qquad \text{and}$$

$$\mathtt{H} = \mathtt{S} \sqcup \langle\langle \mathtt{G};\ \bullet \rangle\rangle.$$

Applying f we get

$$f(\mathtt{G}) = f(\mathtt{S}) \sqcup \left\langle\left\langle f(\mathtt{H});\ (-)' \right\rangle\right\rangle \qquad \text{and}$$

$$f(\mathtt{H}) = f(\mathtt{S}) \sqcup \langle\langle f(\mathtt{G});\ \bullet \rangle\rangle.$$

This, completes the proof. □

Proposition 3.2.12. *If a dashed monoid M admits a dashed monoid basis* $\mathtt{S}$, *then this basis is unique and given by*

$$\mathtt{S} = \mathtt{G} \cap \mathtt{H}. \tag{3.2.13}$$

Proof. Suppose $\mathtt{S} \subseteq M$ be dashed monoid bases of M. Then, there exists a multiplicative basis $\mathtt{G}$ and a dash basis $\mathtt{H}$ of M. From Propositions 2.1.29 and 3.1.25, these bases are unique.

From the interlocking conditions (3.2.9) and (3.2.10), we get

$$\mathtt{S} \subseteq \mathtt{G} \cap \mathtt{H}.$$

On the other hand, if $x \in \mathtt{G} \cap \mathtt{H}$ then we get

$$x \in S \qquad \text{or} \qquad x \in \big\langle\big\langle \mathtt{H};\ (-)' \big\rangle\big\rangle.$$

Since $x \in \mathtt{H}$, from Lemma 3.1.28, we conclude that $x \notin \big\langle\big\langle \mathtt{H};\ (-)' \big\rangle\big\rangle$. Thus, $x \in S$. This completes the proof. □

Framework 3.2.14. For the remainder of this section, let $(M, I, (-)', \bullet)$ be a dashed monoid that admits a dashed monoid basis $\mathtt{S}$, with multiplicative basis $\mathtt{G}$, dash basis $\mathtt{H}$, and length function

$$\hat{l} : M \longrightarrow \mathbb{N}.$$ ◇

Definition 3.2.15. Define the *multiplicative composite subset*, denoted

$$\mathtt{R} \subseteq M$$

as follows:

$$\mathtt{R} := \langle\langle \mathtt{G};\ \bullet \rangle\rangle. \tag{3.2.16}$$

Note that, since $\mathtt{G}$ is a multiplicative basis, for every $x \in \mathtt{R}$ we get unique $m \geq 2$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$ ◇

Lemma 3.2.17. *Assume Framework 3.2.14. Then, we get*

$$\mathtt{G} = \big\langle \mathtt{S};\ (-)' \big\rangle \quad \bigsqcup \quad \big\langle\big\langle \mathtt{R};\ (-)' \big\rangle\big\rangle. \tag{3.2.18}$$

Proof. From Definition 3.2.6, we get that $\mathtt{H}$ is a dash basis, and consequently, $\mathtt{H}$ is a dash independent set. We have that $\mathtt{S}, \mathtt{R} \subseteq \mathtt{H}$ are disjoint subsets of $\mathtt{H}$. Thus, applying Lemma 3.1.38 we get

$$\big\langle\big\langle \mathtt{S} \sqcup \mathtt{R};\ (-)' \big\rangle\big\rangle = \big\langle\big\langle \mathtt{S};\ (-)' \big\rangle\big\rangle \sqcup \big\langle\big\langle \mathtt{R};\ (-)' \big\rangle\big\rangle.$$

We note that $\mathtt{S}$ is a dash-independent set because it is a subset of the dash-independent set $\mathtt{H}$, as established by Lemma 3.1.37. Applying Lemma 3.1.28, we get

$$\big\langle \mathtt{S};\ (-)' \big\rangle = \mathtt{S} \sqcup \big\langle\big\langle \mathtt{S};\ (-)' \big\rangle\big\rangle.$$

We will finish the proof with the following direct calculation:

$$\begin{aligned}
\mathtt{G} &= \mathtt{S} \sqcup \left\langle\left\langle \mathtt{H};\ (-)' \right\rangle\right\rangle && (3.2.9)\\
&= \mathtt{S} \sqcup \left\langle\left\langle \mathtt{S} \sqcup \mathtt{R};\ (-)' \right\rangle\right\rangle && (3.2.10) \text{ and } (3.2.16)\\
&= \mathtt{S} \sqcup \left\langle\left\langle \mathtt{S};\ (-)' \right\rangle\right\rangle \sqcup \left\langle\left\langle \mathtt{R};\ (-)' \right\rangle\right\rangle && (3.1.39)\\
&= \left\langle \mathtt{S};\ (-)' \right\rangle \sqcup \left\langle\left\langle \mathtt{R};\ (-)' \right\rangle\right\rangle. && (3.1.29)
\end{aligned}$$

□

Lemma 3.2.19. *Assume Framework 3.2.14. Let $x \in M$. Then, $x \in \mathtt{G}$ if and only if exactly one of the following holds:*

- *We have*
$$x = a^{(k)}$$
for some unique $k \geq 0$ and $a \in \mathtt{S}$. In this case, we also get
$$\hat{l}\left(a^{(k)}\right) = 1. \tag{3.2.20}$$
- *We have*
$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}$$
for some unique $m \geq 2$, $k \geq 1$ and $x_i \in \mathtt{G}$. In this case, we also get
$$\hat{l}(x) \geq 2 \tag{3.2.21}$$
and
$$\hat{l}(x_i) < \hat{l}(x). \tag{3.2.22}$$

Proof. This is a restatement of Lemma 3.2.17:

Let $x \in M$. From Lemma 3.2.17 we get that, $x \in \mathtt{G}$ if and only if either

$$x \in \left\langle \mathtt{S};\ (-)' \right\rangle \qquad \text{or} \qquad x \in \left\langle\left\langle \mathtt{R};\ (-)' \right\rangle\right\rangle.$$

If $x \in \left\langle \mathtt{S};\ (-)' \right\rangle$ then we get $a \in \mathtt{S}$ and $k \in \mathbb{N}$ such that

$$x = a^{(k)}.$$

From Definition 3.2.6 we get $\mathtt{S} \subseteq \mathtt{H}$. Since $\mathtt{H}$ is a dash-basis of M, from Lemma 3.1.37 we get that $\mathtt{S}$ is a dash independent subset of M. Thus, the choice of a and k is unique. Since $a \in \mathtt{S}$, we have $\hat{l}(a) = 1$. Thus, we get

$$\hat{l}(x) = \hat{l}\left(a^{(k)}\right) = (\hat{l}(a))^{(k)} = 1^{(k)} = 1.$$

If $x \in \left\langle\left\langle \mathtt{R};\ (-)' \right\rangle\right\rangle$ we get $k \geq 1$ and $z \in \mathtt{R}$ such that

$$x = z^{(k)}.$$

From the interlocking condition (3.2.10), we get that

$$\mathtt{R} = \langle\langle \mathtt{G};\ \bullet \rangle\rangle \subseteq \mathtt{H}.$$

From Lemma 3.1.37 we get that $\mathtt{R}$ is a dash independent set. Thus, the choice of k and z is unique. Since $\mathtt{G}$ is a multiplicative basis, we get unique $m \geq 2$ and $x_i \in \mathtt{G}$ such that

$$z = x_1 \bullet \cdots \bullet x_m.$$

Thus, we get unique $m \geq 2$, $k \geq 1$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}.$$

Since the length function $\hat{l} : M \longrightarrow \mathbb{N}$ is a dashed-monoid homomorphism, we get

$$\hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i).$$

Since $x_i \in \mathtt{G}$, from Proposition 2.1.21 we get $x_i \neq I$. Therefore, from (3.2.7) we get

$$\hat{l}(x_i) \geq 1.$$

We have

$$\hat{l}(x) = \hat{l}\left((x_1 \bullet x_2 \bullet \cdots \bullet x_m)^{(k)}\right) = \sum_{i=1}^{m} \hat{l}(x_i).$$

Since $m \geq 2$ and $\hat{l}(x_i) \geq 1$ for $1 \leq i \leq m$, we get $\hat{l}(x) \geq 2$ and $\hat{l}(x_i) < \hat{l}(x)$ as required. □

Lemma 3.2.23. *Assume Framework 3.2.14. Then, the multiplicative basis $\mathtt{G} \subseteq M$ is closed under taking dash.*

Proof. Suppose $x \in \mathtt{G}$. We want to show that $x' \in \mathtt{G}$. From Lemma 3.2.17 we get

$$\mathtt{G} = \left\langle \mathtt{S};\ (-)' \right\rangle \sqcup \left\langle\left\langle \mathtt{R};\ (-)' \right\rangle\right\rangle.$$

If $x \in \left\langle \mathtt{S};\ (-)' \right\rangle$ then we get some $k \in \mathbb{N}$ and $y \in \mathtt{S}$ such that

$$x = y^{(k)}.$$

Thus, we get

$$x' = y^{(k+1)} \in \left\langle \mathtt{S};\ (-)' \right\rangle \subseteq \mathtt{G}.$$

Similarly, if $x \in \left\langle\left\langle \mathtt{R};\ (-)' \right\rangle\right\rangle$ then we get some $k \in \mathbb{N}$ with $k \geq 1$ and $y \in \mathtt{R}$ such that

$$x = y^{(k)}.$$

Thus, we get

$$x' = y^{(k+1)} \in \left\langle\left\langle \mathtt{R};\ (-)' \right\rangle\right\rangle \subseteq \mathtt{G}.$$

□

Construction 3.2.24. Assume Framework 3.2.14. Let $u : S \longrightarrow M$ be a function. We will extend the function $u : \mathtt{S} \longrightarrow N$ to a function

$$u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$$

as follows:

We have a length homomorphism $\hat{l} : M \longrightarrow \mathbb{N}$. Suppose $x \in \mathtt{G}$. We will define $u^{\mathtt{G}}(x) \in N$ using induction on $\hat{l}(x)$. Since $x \in \mathtt{G}$, we have $\hat{l}(x) \geq 1$.

Base case $\left(\hat{l}(x) = 1\right)$: From Lemma 3.2.19 we get unique $k \in \mathbb{N}$ and unique $a \in \mathtt{S}$ such that

$$x = a^{(r)}.$$

We define

$$u^{\mathtt{G}}(x) := u(a)^{(k)}. \tag{3.2.25}$$

Induction case $\left(\hat{l}(x) \geq 2\right)$: From Lemma 3.2.19, unique $k \in \mathbb{N}$ with $k \geq 1$, $m \geq 2$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = (x_1 \bullet x_2 \bullet \cdots \bullet x_m)^{(k)}.$$

Furthermore, we have $\hat{l}(x_i) < \hat{l}(x)$. Using the induction hypothesis, we define

$$u^{\mathtt{G}}(x) := \left(u^{\mathtt{G}}(x_1) \bullet u^{\mathtt{G}}(x_2) \bullet \cdots \bullet u^{\mathtt{G}}(x_m)\right)^{(k)}. \tag{3.2.26}$$ ◇

Lemma 3.2.27. *For $a \in \mathtt{S}$, the equality*

$$u^{\mathtt{G}}(a) = u(a). \tag{3.2.28}$$

Proof. Let $a \in \mathtt{S}$. Then, from the length condition (3.2.8), we get $\hat{l}(a) = 1$. It follows that,

$$\begin{aligned} u^{\mathtt{G}}(a) &= (u(a))^{(0)} && (3.2.25) \\ &= u(a). \end{aligned}$$ □

Lemma 3.2.29. *Let $x \in M$. Suppose $x \in \mathtt{G}$. Note that, from Lemma 3.2.23 we have $x' \in \mathtt{G}$. The equality*

$$\left(u^{\mathtt{G}}(x)\right)' = u^{\mathtt{G}}\left(x'\right) \tag{3.2.30}$$

holds.

Proof. Suppose $x \in \mathtt{G}$. We will consider the two cases as in Lemma 3.2.19.

Case I $(\hat{l}(x) = 1)$: We get a unique $k \in \mathbb{N}$ and $a \in \mathtt{S}$ such that

$$x = a^{(k)}.$$

Therefore, we get

$$x' = a^{(k+1)}.$$

Observe that,

$$\begin{aligned} \left(u^{\mathtt{G}}(x)\right)' &= \left(u(a)^{(k)}\right)' && (3.2.25) \\ &= u(a)^{(k+1)} \\ &= u^{\mathtt{G}}\left(x'\right). && (3.2.25) \end{aligned}$$

Case II $(\hat{l}(x) \geq 2)$: We get unique $k \geq 1$, $m \geq 2$, and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = (x_1 \bullet x_2 \bullet \cdots \bullet x_m)^{(k)}.$$

Therefore, we get

$$x' = (x_1 \bullet \cdots \bullet x_m)^{(k+1)}$$

Define $z \in N$ as follows:

$$z := u^{\mathtt{G}}(x_1) \bullet u^{\mathtt{G}}(x_2) \bullet \cdots \bullet u^{\mathtt{G}}(x_m).$$

Observe that

$$\begin{aligned}
\left(u^{\mathtt{G}}(x)\right)' &= \left(z^{(k)}\right)' && (3.2.25)\\
&= z^{(k+1)} \\
&= u^{\mathtt{G}}(x'). && (3.2.25)
\end{aligned}$$

This completes the proof. □

Definition 3.2.31. Let $u : \mathtt{S} \longrightarrow N$ be a function and let $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$ be the function as in Construction 3.2.24. Since $\mathtt{G}$ is a multiplicative basis of M, we can extend the function $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$ to a function

$$\overline{u} : M \longrightarrow N.$$

Explicitly, given $x \in M$, if $x = I$ then we define

$$\overline{u}(I) := J. \tag{3.2.32}$$

Otherwise, we have $x \in M^{+}$. Since $\mathtt{G}$ is a multiplicative basis of M, we get $m \in \mathbb{N}$ with $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

We define

$$\overline{u} := u^{\mathtt{G}}(x_1) \bullet \cdots \bullet u^{\mathtt{G}}(x_m). \tag{3.2.33}$$

◇

Lemma 3.2.34. *The function $\overline{u} : M \longrightarrow N$ as defined in Definition 3.2.31 is a monoid homomorphism.*

Proof. Note that $\mathtt{G}$ is a multiplicative basis of M. Since $\overline{u} : M \longrightarrow N$ defined by multiplicatively extending the function $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$, it follows that $\overline{u}$ is a monoid homomorphism. □

Lemma 3.2.35. *For $x \in \mathtt{G}$ we get*

$$\overline{u}(x) = u^{\mathtt{G}}(x). \tag{3.2.36}$$

Proof. Since $\overline{u} : M \longrightarrow N$ is defined by multiplicatively extending the function $u^{\mathtt{G}} : \mathtt{G} \longrightarrow N$, we get

$$\overline{u}(x) = u^{\mathtt{G}}(x). \qquad (3.2.33)$$

□

Lemma 3.2.37. *For $a \in \mathtt{S}$ we get*

$$\overline{u}(a) = u(a). \tag{3.2.38}$$

Proof. Let $a \in \mathtt{S}$. Then, we get $a \in \mathtt{G}$. Observe that

$$\begin{aligned}
\overline{u}(a) &= u^{\mathtt{G}}(a) && (3.2.36)\\
&= u(a). && (3.2.28)
\end{aligned}$$

□

Lemma 3.2.39. *The function $\overline{u} : M \longrightarrow N$ as defined in Definition 3.2.31 is a dashed monoid homomorphism.*

Proof. Lemma 3.2.34 shows that $\overline{u}: M \longrightarrow N$ is a monoid homomorphism. We will show that $\overline{u}: M \longrightarrow N$ satisfies the dash condition: $\overline{u}(x') = \overline{u}(x)'$ for every $x \in M$. Consider the following calculation:

$$\begin{aligned} M &= \{I\} \sqcup M^+ && (2.1.18) \\ &= \{I\} \sqcup \langle \mathtt{G};\ \bullet \rangle && (2.1.32) \\ &= \{I\} \sqcup \mathtt{G} \sqcup \mathtt{R}. && (3.1.27) \text{ and } (3.2.16) \end{aligned}$$

Thus, we get either $x = I$, $x \in \mathtt{G}$, or $x \in \mathtt{R}$. We consider these three cases.

Case I ($x = I$): We get

$$\begin{aligned} \overline{u}(I)' &= J' && (3.2.32) \\ &= J \\ &= \overline{u}(I) && (3.2.32) \\ &= \overline{u}(I'). \end{aligned}$$

Case II ($x \in \mathtt{G}$): From Lemma 3.2.23 we get $x' \in \mathtt{G}$. We have

$$\begin{aligned} \overline{u}(x)' &= u^{\mathtt{G}}(x)' && (3.2.36) \\ &= u^{\mathtt{G}}(x') && (3.2.30) \\ &= \overline{u}(x'). && (3.2.36) \end{aligned}$$

Case III: ($x \in \mathtt{R}$): We get $m \geq 2$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

Therefore, we have

$$x' = (x_1 \bullet \cdots \bullet x_m)'.$$

It follows that

$$x' \in \left\langle \left\langle \mathtt{R};\ (-)' \right\rangle \right\rangle \subseteq \mathtt{G}.$$

Observe,

$$\begin{aligned} \overline{u}(x') &= u^{\mathtt{G}}(x') && (3.2.36) \\ &= \left(u^{\mathtt{G}}(x_1) \bullet \cdots \bullet u^{\mathtt{G}}(x_m)\right)' && (3.2.26) \\ &= (\overline{u}(x))'. && (3.2.33) \end{aligned}$$

Since we have shown $\overline{u}(x') = \overline{u}(x)'$ in all three cases, we get that $\overline{u}: M \longrightarrow N$ is a dashed monoid homomorphism. □

Lemma 3.2.40. *Let $f: M \longrightarrow N$ be a dashed-monoid homomorphism. Define*

$$u := f\,|_{\mathtt{S}} : \mathtt{S} \longrightarrow N.$$

Then, we have

$$u^{\mathtt{G}} = f\,|_{\mathtt{G}} : \mathtt{G} \longrightarrow N \tag{3.2.41}$$

where $u^{\mathtt{G}}: \mathtt{G} \longrightarrow N$ is as in Construction 3.2.24.

Proof. Let $x \in M$. Suppose $x \in \mathtt{G}$. Using induction on $\hat{l}(x)$, we will show that

$$u^{\mathtt{G}}(x) = f(x).$$

Base case $\left(\hat{l}(x) = 1\right)$: From Lemma 3.2.19 we get unique $k \in \mathbb{N}$ and $a \in \mathtt{S}$ such that

$$x = a^{(k)}.$$

Observe that

$$\begin{aligned} u^{\mathtt{G}}(x) &= u(a)^{(r)} && (3.2.25) \\ &= f(a)^{(r)} && \text{(hypothesis)} \\ &= f\left(a^{(r)}\right) \\ &= f(x). \end{aligned}$$

Here, the third equality follows since f is a dashed monoid homomorphism.

Induction case $\left(\hat{l}(x) \geq 2\right)$: From Lemma 3.2.19 we get unique $k \geq 1$, $m \geq 2$, and $x_i \in \mathtt{G}$ such that

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}.$$

Since $x_i \in \mathtt{G}$, we have $\hat{l}(x_i) \geq 1$. Since $\hat{l} : M \longrightarrow \mathbb{N}$ is a dashed-monoid homomorphism, we get

$$\hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i).$$

Becasue $m \geq 2$, we get $\hat{l}(x_i) < \hat{l}(x)$. Using the induction hypothesis, we get

$$u^{\mathtt{G}}(x_i) = f(x_i).$$

Observe that

$$\begin{aligned} u^{\mathtt{G}}(x) &= \left(u^{\mathtt{G}}(x_1) \bullet \cdots \bullet u^{\mathtt{G}}(x_m)\right)^{(k)} && (3.2.26) \\ &= (f(x_1) \bullet \cdots \bullet f(x_m))^{(k)} && \text{(induction)} \\ &= f\left((x_1 \bullet \cdots \bullet x_m)^{(k)}\right) \\ &= f(x). \end{aligned}$$

Here, the third equality follows because f is a dashed monoid homomorphism.

Thus, by mathematical induction we get

$$u^{\mathtt{G}} = f\,|_{\mathtt{G}} : \mathtt{G} \longrightarrow N$$

as required. $\square$

Lemma 3.2.42. *Suppose $f : M \longrightarrow N$ is a dashed monoid homomorphism. Let*

$$u := f\,|_{\mathtt{S}} : \mathtt{S} \longrightarrow N.$$

Then,

$$\overline{u} = f : M \longrightarrow N, \tag{3.2.43}$$

where $\overline{u} : M \longrightarrow N$ is as in Definition 3.2.31.

Proof. From Lemma 3.2.39, we get that $\overline{u}: M \longrightarrow N$ is a dashed monoid homomorphism. Given that $f: M \longrightarrow N$ is a dashed monoid homomorphism, it follows that $\overline{u}$ and f are both monoid homomorphisms. Since $\mathtt{G}$ is a multiplicative basis of M, it is sufficient to show that

$$\overline{u}\,|_{\mathtt{G}} = f\,|_{\mathtt{G}}\,.$$

Observe that

$$\begin{aligned} \overline{u}\,|_{\mathtt{G}} &= u^{\mathtt{G}} && (3.2.36) \\ &= f\,|_{\mathtt{G}}\,. && (3.2.41) \end{aligned}$$ □

This completes the proof.

Theorem 3.2.44. *Let $M = (M, \bullet, I, (-)')$ be a dashed monoid. Suppose M admits a dashed monoid basis $\mathtt{S}$ as in Definition 3.2.6. Then, the inclusion of sets $\mathtt{S} \subseteq M$ satisfies the universal property of dashed monoids*

Proof. We will show the existence condition (3.2.2) and the uniqueness condition (3.2.3) of the universal property of a free dashed monoid.

Existence: Let N be a dashed monoid and let $u: \mathtt{S} \longrightarrow N$ be a function. Consider the function

$$\overline{u}: M \longrightarrow N$$

as in Definition 3.2.31. From Lemma 3.2.39, we get that $\overline{u}$ is a dashed monoid homomorphism, and Lemma 3.2.37 shows that

$$\overline{u}\,|_{\mathtt{S}} = u.$$

Uniqueness: Let $f, g: M \longrightarrow N$ be dashed monoid homomorphisms such that

$$f\,|_{\mathtt{S}} = g\,|_{\mathtt{S}}\,.$$

Define

$$u := f\,|_{\mathtt{S}} = g\,|_{\mathtt{S}} : \mathtt{S} \longrightarrow N.$$

From Lemma 3.2.42, we have

$$f = \overline{u} = g.$$

This completes the proof. □

3.3. Properties of Integer Sets

Before constructing the free dashed monoid generated by a set, we present some elementary yet nontrivial properties of subsets of integers. The propositions in this section are foundational and will be used repeatedly throughout this article. Given their basic nature, we will refer to them without explicit citation.

Definition 3.3.1. Let $n \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$. Then, $n + A \subseteq \mathbb{Z}$ is defined as follows:

$$n + A := \{n + a \mid a \in A\}. \tag{3.3.2}$$ ◇

Notation 3.3.3. We will, at times, use the notation $A + n$ to refer to $n + A$. ◇

Proposition 3.3.4. *Let $x,n,m,p,q \in \mathbb{Z}$ and $A,B \subseteq \mathbb{Z}$. Then, following are true.*

$$x \in n+A \iff x-n \in A. \tag{3.3.5}$$

$$A \subseteq B \iff n+A \subseteq n+B. \tag{3.3.6}$$

$$A = B \iff n+A = n+B. \tag{3.3.7}$$

$$0+A = A. \tag{3.3.8}$$

$$n+\varnothing = \varnothing. \tag{3.3.9}$$

$$n+(m+A) = (n+m)+A. \tag{3.3.10}$$

$$n+(A\cup B) = (n+A)\cup(n+B). \tag{3.3.11}$$

$$n+(A\cap B) = (n+A)\cap(n+B). \tag{3.3.12}$$

$$n+(p,q] = (n+p,n+q]. \tag{3.3.13}$$

Proof. (3.3.5): Suppose $x \in n+A$. Then, we get $x = n+a$ for some $a \in A$. It follows that $x-n = a \in A$. On the other hand, if $x-n \in A$ then $x = n+(x-n) \in n+A$.

(3.3.6): Suppose $A \subseteq B$. Let $x \in n+A$. From (3.3.5), we get $x-n \in A \subseteq B$. Thus, again from (3.3.5) we get $x \in n+B$. This shows

$$n+A \subseteq n+B.$$

For the reverse direction, assume $n+A \subseteq n+B$. Let $x \in A$. From (3.3.2) we get $n+x \in n+A \subseteq n+B$. Finally, from (3.3.5), we get $x = n+x-n \in B$. Thus, we have

$$A \subseteq B.$$

(3.3.7): We see this from the following chain of double implications.

$$\begin{aligned}
A = B &\iff (A\subseteq B) \text{ and } (B\subseteq A) \\
&\iff (n+A\subseteq n+B) \text{ and } (n+B\subseteq n+A) && (3.3.6) \\
&\iff n+A = n+B.
\end{aligned}$$

(3.3.8): We have

$$\begin{aligned}
0+A &= \{0+a \mid a\in A\} && (3.3.2) \\
&= \{a \mid a\in A\} \\
&= A.
\end{aligned}$$

(3.3.9): For the sake of contradiction assume $n+\varnothing \neq \varnothing$. Let $x \in n+\varnothing$ then, from (3.3.5), we get $x-n \in \varnothing$. This is a contradiction.

(3.3.10): We have

$$\begin{aligned}
n+(m+A) &= n+\{m+a \mid a\in A\} && (3.3.2) \\
&= \{n+m+a \mid a\in A\} && (3.3.2) \\
&= (n+m)+A. && (3.3.2)
\end{aligned}$$

(3.3.11): We get

$$\begin{aligned} n+(A\cup B) &= \{n+a \mid a\in A \ \ \texttt{or} \ \ a\in B\} && (3.3.2)\\ &= \{n+a \mid a\in A\}\cup\{n+a \mid a\in B\} \\ &= (n+A)\cup(n+B). && (3.3.2) \end{aligned}$$

(3.3.12): We have

$$\begin{aligned} n+(A\cap B) &= \{n+a \mid a\in A \ \ \texttt{and} \ \ a\in B\} && (3.3.2)\\ &= \{n+a \mid a\in A\}\cap\{n+a \mid a\in B\} \\ &= (n+A)\cap(n+B). && (3.3.2) \end{aligned}$$

(3.3.13): We see that

$$\begin{aligned} n+[p,q] &= \{n+x \mid p\le x\le q\} && (3.3.2)\\ &= \{y \mid p\le y-n \mid q\} \\ &= \{y \mid n+p\le y\le n+q\} \\ &= [n+p,n+q]. \end{aligned}$$

□

Definition 3.3.14. Let $D\subseteq \mathcal{P}(\mathbb{Z})$. Then, $n+D\subseteq \mathcal{P}(\mathbb{Z})$ is defined as follows:

$$n+D := \{n+A \mid A\in D\}. \tag{3.3.15}$$

◇

Proposition 3.3.16. *Let $n,m\in\mathbb{Z}$, $A,B\subseteq\mathbb{Z}$, and $D,E\subseteq\mathcal{P}(\mathbb{Z})$. Then, following are true.*

$$A\in n+D \iff A-n\in D. \tag{3.3.17}$$

$$D\subseteq E \iff n+D\subseteq n+E. \tag{3.3.18}$$

$$D=E \iff n+D=n+E. \tag{3.3.19}$$

$$0+D=D. \tag{3.3.20}$$

$$n+\varnothing=\varnothing. \tag{3.3.21}$$

$$n+(m+D)=(n+m)+D. \tag{3.3.22}$$

$$n+(D\cup E)=(n+D)\cup(n+E). \tag{3.3.23}$$

$$n+(D\cap E)=(n+D)\cap(n+E). \tag{3.3.24}$$

Proof. (3.3.17): Suppose $X\in n+D$. Then, we get $X=n+A$ for some $A\in D$. It follows that $X-n=A\in D$. On the other hand, if $X-n\in D$ then $X=n+(X-n)\in n+D$.

(3.3.18): Suppose $D\subseteq E$. Let $X\in n+D$. From (3.3.17), we get $X-n\in D\subseteq E$. Thus, again from (3.3.17) we get $X\in n+E$. This shows

$$n+D\subseteq n+E.$$

For the reverse direction, assume $n+D\subseteq n+E$. Let $X\in D$. From (3.3.15) we get $n+X\in n+D\subseteq n+E$. Finally, from (3.3.17), we get $X=n+X-n\in B$. Thus, we have

$$D\subseteq E.$$

(3.3.19): We see this from the following chain of double implications.

$$\begin{aligned} D = E &\iff (D \subseteq E) \text{ and } (E \subseteq D) \\ &\iff (n + D \subseteq n + E) \text{ and } (n + E \subseteq n + D) && (3.3.18) \\ &\iff n + D = n + E. \end{aligned}$$

(3.3.20): We have

$$\begin{aligned} 0 + D &= \{0 + A \mid A \in D\} && (3.3.15) \\ &= \{A \mid A \in D\} \\ &= D. \end{aligned}$$

(3.3.21): For the sake of contradiction assume $n + \varnothing \neq \varnothing$. Let $X \in n + \varnothing$ then, from (3.3.17), we get $X - n \in \varnothing$. This is a contradiction.

(3.3.22): We have

$$\begin{aligned} n + (m + D) &= n + \{m + A \mid A \in D\} && (3.3.15) \\ &= \{n + m + A \mid A \in D\} && (3.3.15) \\ &= (n + m) + D. && (3.3.15) \end{aligned}$$

(3.3.23): We get

$$\begin{aligned} n + (D \cup E) &= \{n + A \mid A \in D \ \text{ or } \ A \in E\} && (3.3.15) \\ &= \{n + A \mid A \in D\} \cup \{n + A \mid A \in E\} \\ &= (n + D) \cup (n + E). && (3.3.15) \end{aligned}$$

(3.3.24): We have

$$\begin{aligned} n + (D \cap E) &= \{n + A \mid A \in D \ \text{ and } \ A \in E\} && (3.3.15) \\ &= \{n + A \mid A \in D\} \cap \{n + A \mid A \in E\} \\ &= (n + D) \cap (n + E). && (3.3.15) \end{aligned}$$

□

Proposition 3.3.25. *Let $n \in \mathbb{Z}$, $B \subseteq \mathbb{Z}$, and $D \subseteq \mathcal{P}(B)$. Then, $n + D \subseteq \mathcal{P}(n + B)$. Moreover, if $D \subseteq \mathcal{P}_I^+(B)$ then $n + D \subseteq \mathcal{P}_I^+(n + B)$.*

Proof. Let $A \in n + D$. We get $-n + A \in D$. Since $D \subseteq \mathcal{P}(B)$ we get $-n + A \subseteq B$. This implies $A \subseteq n + B$. This shows that $n + D \subseteq \mathcal{P}(n + B)$.

Now assume $D \subseteq \mathcal{P}_I^+(B)$. Let $n + A \in n + D$ where $A \in D$. We have that $A \in D$ is a non-empty interval. Therefore, we get that $n + A$ is a non-empty interval. This shows that $n + D \subseteq \mathcal{P}_I^+(n + B)$. □

3.4. Bracketing on a Word

Definition 2.2.25 gives a formal definition of words with letters from a set S. In Theorem 2.2.39, we saw that the set of all such words forms the free monoid generated by S. We now want to extend this construction to include a formal dash operation on words. This dash operation is neither involutive nor does it distribute over multiplication, so we must keep track of which groupings of letters within a word have dashes applied to them.

For example, let $S = \{a, b, c\}$. The word

$$(a''(bc)''')'b$$

should be treated as a distinct element in the free dashed-monoid.

To handle this, we introduce the concept of *bracketings*. A bracketing serves two main purposes: (1) it records the locations of groups of letters that are dashed together, and (2) it keeps track of nested groupings.

In the example above, if we focus only on the locations of grouped letters, we get the following schematic:

$$((*)(**))*$$

Here, each $*$ represents a letter from S, and each bracketed group has at least one dash. To reconstruct the actual word from this schematic, we need to specify which letter each $*$ stands for and how many dashes are on each bracket. The following formal definitions make this precise.

Definition 3.4.1. Let $n \in \mathbb{N}$. A *bracketing* on n letters is a collection D of non-empty sub-intervals of $[n]$, that is,

$$D \subseteq \mathcal{P}_I^+(n),$$

such that for all $A, B \in D$, the following *bracketing condition* holds:

$$A \cap B = \emptyset, \quad \text{or} \quad A \subseteq B, \quad \text{or} \quad B \subseteq A. \tag{3.4.2}$$

The elements of D are called *brackets*. We denote the set of all bracketings on n letters by $\mathcal{B}r(n)$. ◇

Convention 3.4.3. Define

$$\mathcal{B}r := \bigsqcup_{n \in \mathbb{N}} \mathcal{B}r(n)$$

as the set of all bracketings, and

$$\mathcal{B}r^+ := \bigsqcup_{n \in \mathbb{N}^+} \mathcal{B}r(n)$$

as the set of all non-empty bracketings. ◇

Definition 3.4.4. Let $n \in \mathbb{N}$. Then, the empty collection $\varnothing \subseteq \mathcal{P}_I^+(n)$ vacuously satisfies the bracketing condition. We call this the *empty bracketing* on n letters.

We define the *unit bracketing*, denoted $J_{\mathcal{B}r}$, to be the empty bracketing, $\varnothing \subseteq \mathcal{P}_I^+(0)$, on 0 letters. That is,

$$J_{\mathcal{B}r} := \varnothing \in \mathcal{B}r(0). \tag{3.4.5}$$

Note that, the unit bracketing is the only possible bracketing on 0 letters. ◇

Example 3.4.6. Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$, then every $A \in D$ corresponds to a bracket. We require that A is an interval since a bracket can not have gaps within them. Conventionally, we assume that A is non-empty since one can simply omit empty bracket. The bracketing condition ensures that each bracket contains only the closed brackets.

Here are some examples of bracketings on a word with five letters.

Bracketing (D)	Schematic
$\emptyset$	$*\,*\,*\,*\,*$
$\{\{1,2,3,4,5\}\}$	$(*\,*\,*\,*\,*)$
$\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$	$(*)\,(*)\,(*)\,(*)\,(*)$
$\{\{1,2\},\{1,2,3\},\{4,5\}\}$	$((*\,*)\,*)\,(*\,*)$
$\{\{1,2\},\{4,5\},\{1,2,3,4,5\}\}$	$((*\,*)\;*\;(*\,*))$

$\diamond$

Proposition 3.4.7. *Let $n,m \in \mathbb{N}$. Let $D \in \mathcal{Br}(n)$ and $E \in \mathcal{Br}(m)$ be bracketings on n and m letters respectively. Then,*

$$D \cap (n+E) = \emptyset$$

and

$$D \sqcup (n+E) \in \mathcal{Br}(n+m).$$

Proof. First, we will show $D \cap (n+E) = \emptyset$. Let $A \in D \cap (n+E)$, that is, $A \in D$ and $A = n+B$ for some $B \in E$. Then, we get $A \subseteq (0,n]$ and $A \subseteq n+(0,m] = (n,n+m]$. Thus, we get $A = \emptyset$. This is a contradiction since $A \in D$ and D is a bracketing implies $A \neq \emptyset$. Hence, we get $D \cap (n+E) = \emptyset$.

To see that $D \cup (n+E)$ is a bracketing on $n+m$ letters, we need to check following two things:

i. $D \cup (n+E) \subseteq \mathcal{P}_I^+(n+m)$, and
ii. for every $A,B \in D \cup (n+E)$ either $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$.

Since D is a bracketing on n letters, we know that $D \subseteq \mathcal{P}_I^+(n) \subseteq \mathcal{P}_I^+(n+m)$. Since E is a bracketing on m letters, we get that $E \subseteq \mathcal{P}_I^+(m)$. We have

$$n+E \subseteq \mathcal{P}_I^+(n+[m]) = \mathcal{P}_I^+([n+1,n+m]) \subseteq \mathcal{P}_I^+(n+m).$$

Thus, we get

$$D \cup (n+E) \subseteq \mathcal{P}_I^+(n+m).$$

Now for the bracketing condition, let $A,B \in D \cup (n+E)$. We will consider the following cases:

Case 1 ($A,B \in D$): Since D is a bracketing and $A,B \in D$ we get either $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$.

Case 2 ($A,B \in n+E$): That is, we have $X,Y \in E$ such that $A = n+X$ and $B = n+Y$. Since E is a bracketing, from the bracketing condition, we get either $X \cap Y = \emptyset$, $X \subseteq Y$, or $Y \subseteq X$. If $X \cap Y = \emptyset$, then we get

$$\begin{aligned} A \cap B &= (n+X) \cap (n+Y) \\ &= n+(X \cap Y) \\ &= n+\emptyset \\ &= \emptyset. \end{aligned}$$

If $X \subseteq Y$, then we get

$$\begin{aligned} A &= n+X \\ &\subseteq n+Y \\ &= B. \end{aligned}$$

Similarly, we get $B \subseteq A$ if $Y \subseteq X$.

Case 3 ($A \in D, B \in n+E$): That is, $A \in D$ and we have $Y \in E$ such that $B = n+Y$. Since D is a bracketing on n letters, we get $A \subseteq (0,n]$. Similarly, we get $Y \subseteq (0,m]$. This implies $B = n+Y \subseteq (n,m]$. Thus, we get $A \cap B = \varnothing$.

Since in each of the cases we get either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$, we conclude that $D \cup (n+E)$ is a bracketing on $n+m$ letters. $\square$

Definition 3.4.8. Let $n, m \in \mathbb{N}$ and $D \in \mathcal{Br}(n), E \in \mathcal{Br}(m)$ be bracketings. Define the multiplication of the bracketings, $D \bullet E \in \mathcal{Br}(n+m)$, as

$$D \bullet E := D \sqcup (n+E). \tag{3.4.9}$$

From Proposition 3.4.7 we know that $D \bullet E$ is a bracketing on $n+m$ letters. $\diamond$

Example 3.4.10. Let

$$D = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\big\} \in \mathcal{Br}(5)$$

and

$$E = \big\{1\big\} \in \mathcal{Br}(3).$$

Then, we have

$$D \bullet E = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}, \{6\}\big\} \in \mathcal{Br}(8).$$

Using the visual representation, we get

$$D = ((**)\ *)\ (**) \qquad \text{and} \qquad E = (*)\ **.$$

Then, the product is given by the concatenation,

$$D \bullet E = ((**)\ *)\ (**)\ (*)\ **.$$ $\diamond$

Lemma 3.4.11. *Let $n, m, p \in \mathbb{N}$ and $D \in \mathcal{Br}(n), E \in \mathcal{Br}(m), F \in \mathcal{Br}(p)$ be bracketings. Then we have*

$$D \bullet (E \bullet F) = (D \bullet E) \bullet F \tag{3.4.12}$$

as bracketings on $n+m+p$ letters.

Proof. Observe that,

$$\begin{aligned} D \bullet (E \cdot F) &= D \cup (n + (E \bullet F)) && (3.4.9)\\ &= D \cup (n + (E \cup (m+F))) && (3.4.9)\\ &= D \cup (n+E) \cup (n + (m+F)) && (3.3.23)\\ &= D \cup (n+E) \cup ((n+m)+F) && (3.3.22)\\ &= (D \bullet E) \cup ((n+m)+F) && (3.4.9)\\ &= (D \bullet E) \bullet F. && (3.4.9) \end{aligned}$$ $\square$

Remark 3.4.13. Since the bracketing multiplication is associative, we will omit the use of brackets to show the order of multiplication. $\diamond$

Lemma 3.4.14. *Let $n \in \mathbb{N}$ and $D \in \mathcal{Br}(n)$. Recall, $J_{\mathcal{Br}} \in \mathcal{Br}(0)$ is the unit bracketing as in Definition 3.4.4. We have*

$$D \bullet J_{\mathcal{Br}} = D \qquad \text{and} \qquad J_{\mathcal{Br}} \bullet D = D \tag{3.4.15}$$

as bracketings on n letters.

Proof. We get

$$\begin{aligned} D \bullet I &= D \bullet \varnothing && (3.4.5)\\ &= D \cup (n + \varnothing) && (3.4.9)\\ &= D \cup \varnothing && (3.3.21)\\ &= D. \end{aligned}$$

Next, we get

$$\begin{aligned} I \bullet D &= \varnothing \bullet D && (3.4.5)\\ &= \varnothing \cup (0 + D) && (3.4.9)\\ &= \varnothing \cup D && (3.3.20)\\ &= D. \end{aligned}$$

□

Proposition 3.4.16. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then,*

$$E := \{A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ or } \ A = [n]\}$$

is a bracketing on n letters.

Proof. To see that E is a bracketing on n letters, we need to check following two things:

i. $E \subseteq \mathcal{P}_I^+(n)$, and
ii. for every $A, B \in E$ either $A \cap B = \varnothing$, $A \subseteq B$, or $B \subseteq A$.

By the construction, we have $E \subseteq \mathcal{P}_I^+(n)$. Now, let $A, B \in E$. We consider following cases:

Case 1 ($A, B \in D$): In this case, the bracketing condition is satisfied since D itself is a bracketing.

Case 2 ($B = [n]$): In this case, since $E \subseteq \mathcal{P}_I^+(n)$, we have $A \subseteq [n] = B$.

This completes the proof. □

Definition 3.4.17. Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Define the dash of the bracketing, $D' \in \mathcal{B}r(n)$, as follows:

$$D' := \{A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ or } \ A = [n]\}. \tag{3.4.18}$$

From Proposition 3.4.16 we know that D' is a bracketing on n letters. ◇

Proposition 3.4.19. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then, we get*

$$D' = \begin{cases} D & \text{if } \ n = 0 \\ D \cup \{[n]\} & \text{if } \ n \geq 1. \end{cases} \tag{3.4.20}$$

Proof. We will consider following cases:

Case 1($n = 0$): In this case, we have both $D, D' \in \mathcal{B}r(0)$. Since $\mathcal{B}r(0)$ only consists of $J_{\mathcal{B}r}$, we get $D = D'$.

Case 2($n \geq 1$): In this case, we have $[n] \in \mathcal{P}_I^+(n)$. Observe that,

$$\begin{aligned} D' &= \{A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ or } \ A = [n]\} && (3.4.18)\\ &= \{A \in zP_I^+(n) \mid A \in D\} \cup \{A \in \mathcal{P}_I^+(n) \mid A = [n]\}\\ &= D \cup \{[n]\}. \end{aligned}$$

□

Example 3.4.21. Let

$$D = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\big\} \in \mathcal{B}r(5).$$

Then, we have

$$D' = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\},\ \{1,2,3,4,5\}\big\} \in \mathcal{B}r(5).$$

Using the visual representation, we get

$$D = ((**)\ *)\ (**).$$

Then, the dash of D is given by adding the outermost bracket if there is no such bracket,

$$D' = (((**)\ *)\ (**)\ (*)).$$

◇

Lemma 3.4.22. *Consider the unit bracketing, $J_{\mathcal{B}r} \in \mathcal{B}r(0)$. We have*

$$J'_{\mathcal{B}r} = J_{\mathcal{B}r} \tag{3.4.23}$$

as bracketing on 0 *letters.*

Proof. Follows immediately from the case $n = 0$ of Proposition 3.4.19. □

Proposition 3.4.24. *Let $n \geq 1$ and $D \in \mathcal{B}r(n)$ be a bracketing. For $k \geq 1$ let $D^{(k)}$ denote the bracketing on n letters that we get after taking k dashes of D. Then we get*

$$D^{(k)} = D \cup \{[n]\} \tag{3.4.25}$$

as bracketings on n letters.

Proof. We will use induction on k.

Base case ($k = 1$): We get

$$D^{(1)} = D' = D \cup \{[n]\}. \qquad (3.4.20)$$

Induction case ($k \geq 1$): Assume that $D^{(k)} = D \cup \{[n]\}$. We get

$$\begin{aligned} D^{(k+1)} &= \left(D^{(k)}\right)' \\ &= (D \cup \{[n]\})' && \text{Induction} \\ &= D \cup \{[n]\} \cup \{[n]\} && \text{Base case} \\ &= D \cup \{[n]\}. \end{aligned}$$

□

Proposition 3.4.26. *Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing, and $C \subseteq [n]$. Then,*

$$E := \big\{A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ and } \ A \neq C\big\}$$

is a bracketing on n letters.

Proof. To see that E is a bracketing on n letters, we need to check following two things:

i. $E \subseteq \mathcal{P}_I^+(n)$, and
ii. for every $A, B \in E$ either $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$.

By the construction, E is a subset of $\mathcal{P}_I^+(n)$. Now let $A, B \in E$. Then, we get $A, B \in D$. The bracketing condition is satisfied since D itself is a bracketing. This shows that E is a bracketing on n letters. □

Definition 3.4.27. Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Define the *floor* of the bracketing D, denoted $\lfloor D \rfloor \in \mathcal{B}r(n)$, as follows:

$$\lfloor D \rfloor := \left\{ A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ and } \ A \neq [n] \right\}. \tag{3.4.28}$$

From Proposition 3.4.26 we know that $\lfloor D \rfloor$ is a bracketing on n letters. ◇

Proposition 3.4.29. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then, we get*

$$\lfloor D \rfloor = \begin{cases} D & \text{if } \ n = 0 \\ D \setminus \left\{ [n] \right\} & \text{if } \ n \geq 1. \end{cases} \tag{3.4.30}$$

Proof. We will consider following cases:

Case 1($n = 0$): In this case, we have both $D, \lfloor D \rfloor \in \mathcal{B}r(0)$. Since $\mathcal{B}r(0)$ only consists of $J_{\mathcal{B}r}$, we get $D = \lfloor D \rfloor$.

Case 2($n \geq 1$): In this case, we have $[n] \in \mathcal{P}_I^+(n)$. Observe that,

$$\begin{aligned} \lfloor D \rfloor &= \left\{ A \in \mathcal{P}_I^+(n) \mid A \in D \ \text{ and } \ A \neq [n] \right\} && (3.4.28) \\ &= \left\{ A \in z P_I^+(n) \mid A \in D \right\} \cap \left\{ A \in \mathcal{P}_I^+(n) \mid A \neq [n] \right\} \\ &= D \setminus \left\{ [n] \right\}. \end{aligned}$$

□

Example 3.4.31. Let

$$D = \left\{ \{1,2\},\ \{1,2,3\},\ \{4,5\},\ \{1,2,3,4,5\} \right\} \in \mathcal{B}r(5).$$

and

$$E = \left\{ 1 \right\} \in \mathcal{B}r(3).$$

Then, we have

$$\lfloor D \rfloor = \left\{ \{1,2\},\ \{1,2,3\},\ \{4,5\} \right\} \in \mathcal{B}r(5).$$

Using the visual representation, we get

$$D = (((**)\,*)\,(**)).$$

Then, the floor of D is given by removing the outermost bracket if there is one,

$$\lfloor D \rfloor = ((**)\,*)\,(**).$$

◇

Lemma 3.4.32. *Consider the unit bracketing, $J_{\mathcal{B}r} \in \mathcal{B}r(0)$. We have*

$$\lfloor J_{\mathcal{B}r} \rfloor = J_{\mathcal{B}r} \tag{3.4.33}$$

as bracketing on 0 *letters.*

Proof. Follows immediately from the case $n = 0$ of Proposition 3.4.29. □

Proposition 3.4.34. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. We get*

$$\lfloor \lfloor D \rfloor \rfloor := \left\lfloor \left(\lfloor D \rfloor \right) \right\rfloor = \lfloor D \rfloor \tag{3.4.35}$$

as bracketings on n letters.

Proof. If $n = 0$ then, from Proposition 3.4.29, we get $\lfloor D \rfloor = D$. As a result, we get

$$\lfloor \lfloor D \rfloor \rfloor = \lfloor D \rfloor = D.$$

If $n \geq 1$ then, from Proposition 3.4.29, we get $\lfloor D \rfloor = D \setminus \{[n]\}$. Thus, we get
Base case ($k = 1$): We get

$$\begin{aligned} \lfloor \lfloor D \rfloor \rfloor &= \lfloor D \setminus \{[n]\} \rfloor && (3.4.30) \\ &= (D \setminus \{[n]\}) \setminus \{[n]\} && (3.4.30) \\ &= D \setminus \{[n]\} \\ &= \lfloor D \rfloor. && (3.4.30) \end{aligned}$$ □

Proposition 3.4.36. *Let $n \in \mathbb{N}^+$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then, for $k \in \mathbb{N}$ we have*

$$\left\lfloor D^{(k)} \right\rfloor = \lfloor D \rfloor. \tag{3.4.37}$$

Proof. For the case $k = 0$, we get $D^{(0)} = D$. Therefore, the proposition is true for $k = 0$. Now assume $k \geq 1$. We have

$$\begin{aligned} \left\lfloor D^{(k)} \right\rfloor &= \lfloor D \cup \{[n]\} \rfloor && (3.4.25) \\ &= (D \cup \{[n]\}) \setminus \{[n]\} && (3.4.30) \\ &= D \setminus \{[n]\} \\ &= \lfloor D \rfloor. && (3.4.30) \end{aligned}$$ □

Proposition 3.4.38. *Let $n \in \mathbb{N}^+$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then, $\lfloor D \rfloor = D$ if and only if $[n] \notin D$.*

Proof. Suppose $\lfloor D \rfloor = D$. From Proposition 3.4.29, we get $\lfloor D \rfloor = D \setminus \{[n]\}$. Therefore, we have $D = D \setminus \{[n]\}$ implying $[n] \notin D$.

On the other hand, suppose $[n] \notin D$. Then, we get

$$\begin{aligned} \lfloor D \rfloor &= D \setminus \{[n]\} && (3.4.30) \\ &= D. \end{aligned}$$ □

Proposition 3.4.39. *Let $n \in \mathbb{N}^+$ and $D \in \mathcal{B}r(n)$ be a bracketing. Suppose $[n] \in D$. Then, for $k \geq 1$ we have*

$$(\lfloor D \rfloor)^{(k)} = D. \tag{3.4.40}$$

Proof. Observe that

$$\begin{aligned} (\lfloor D \rfloor)^{(k)} &= (D \setminus \{[n]\})^{(k)} && (3.4.30) \\ &= (D \setminus \{[n]\}) \cup \{[n]\} && (3.4.25) \\ &= D. \end{aligned}$$

The last equality follows from the assumption that $[n] \in D$. □

Proposition 3.4.41. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Let $(p,q] \subseteq_I (0,n]$ with $0 \le p \le q \le n$. Then,*

$$E := -p + \{B \in D \mid B \subseteq (p,q]\}$$

is a bracketing on $p-q$ letters.

Proof. To see that E is a bracketing on $p-q$ letters, we need to check following two things:

i. $E \subseteq \mathcal{P}_I^+(p-q)$, and
ii. for every $A,B \in E$ either $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$.

First, we will show that every element of E is a non-empty sub-interval of $(0,q-p]$. Let $A \in E$, then we get $X \in D$ with $X \subseteq (p,q]$ such that $A = -p + X$. Since $X \in D$ we get that X is a non-empty interval and hence $A = -p + X$ is a non-empty interval. Since $X \subseteq (p,q]$ we get

$$A = -p + X \subseteq -p + (p,q] = (0,q-p].$$

This shows that $E \subseteq \mathcal{P}_I^+(q-p)$.

It remains to check that E satisfies the bracketing condition. Let $A,B \in E$. Then we get $X,Y \in D$ with $X,Y \subseteq (p,q]$ such that $A = -p+X$ and $C = -p+Y$. Since D is a bracketing, we get either $X \cap Y = \emptyset$, $X \subseteq Y$, or $Y \subseteq Z$. This implies that either $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$. Thus, E is a bracketing on $q-p$ letters. □

Definition 3.4.42. Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Let $(p,q] \subseteq_I [n]$ with $0 \le p \le q \le n$. Define a bracketing, $D_{(p,q]}$, on $q-p$ letters as follows:

$$D_{(p,q]} := -p + \{B \in D \mid B \subseteq (p,q]\}. \tag{3.4.43}$$

Proposition 3.4.41 shows that $D_{(p,q]}$ is a bracketing on $q-p$ letters. ⋄

Example 3.4.44. Let

$$D = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\big\} \in \mathcal{B}r(5).$$

Then, we get

$$\begin{aligned}
D_{(0,3]} &= \big\{\{1,2\},\ \{1,2,3\}\big\} && \in \mathcal{B}r(3),\\
D_{(0,2]} &= \big\{\{1,2\}\big\} && \in \mathcal{B}r(2),\\
D_{(3,5]} &= \big\{\{1,2\}\big\} && \in \mathcal{B}r(2),\\
D_{(2,5]} &= \big\{\{2,3\}\big\} && \in \mathcal{B}r(3), \text{ and}\\
D_{(1,4]} &= \big\{\big\} && \in \mathcal{B}r(3).
\end{aligned}$$

Using the visual representation, we get

$$D = ((**)\ *)\ (**).$$

Then, the restrictions are given by the,

$$\begin{aligned}
D_{(0,3]} &= ((**)\,*),\\
D_{(0,2]} &= (**),\\
D_{(3,5]} &= (**),\\
D_{(2,5]} &= *\,(**), \quad \text{and}\\
D_{(1,4]} &= ***.
\end{aligned}$$

◇

Proposition 3.4.45. *Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing. Then, we get*

$$D_{[n]} = D. \tag{3.4.46}$$

Proof. We get this from the following:

$$\begin{aligned}
D_{[n]} = D_{(0,n]} &= -0 + \{B \in D \mid B \subseteq (0,n]\} && (3.4.43)\\
&= \{B \in D\}\\
&= D.
\end{aligned}$$

□

Proposition 3.4.47. *Let $n, m \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings. Let $(p,q] \subseteq_I [n]$ with $0 \le p \le q \le n$. Then, we get*

$$\big(D \cdot E\big)_{(p,q]} = D_{(p,q]} \tag{3.4.48}$$

as bracketings on $q - p$ letters.

Proof. We will show $\big(D \cdot E\big)_{(p,q]} = D_{(p,q]}$ by showing that

$$\big(D \cdot E\big)_{(p,q]} \subseteq D_{(p,q]} \qquad \text{and} \qquad D_{(p,q]} \subseteq \big(D \cdot E\big)_{(p,q]}.$$

Let $A \in \mathcal{P}_I^+(q-p)$. Suppose $A \in \big(D \cdot E\big)_{(p,q]}$. We get some $X \in D \cdot E$ with $X \subseteq (p,q]$ such that $-p + X = A$. We have

$$X \in D \cdot E = D \sqcup (n + E).$$

If $X \in n + E$, then we must have $X \subseteq (n,m]$. Since $X \subseteq (p,q] \subseteq [n]$, this case is not possible. Thus, we conclude that $X \in D$. Since $X \in D$ and $X \subseteq (p,q]$, we get that

$$A = -p + X \in D_{(p,q]}.$$

This shows that $\big(D \cdot E\big)_{(p,q]} \subseteq D_{(p,q]}$.

On the other hand, suppose $A \in D_{(p,q]}$. We get some $X \in D$ with $X \subseteq (p,q]$ such that $-p + X = A$. We get $X \in D \sqcup (n+E) = D \cdot E$ and $X \subseteq (p,q]$. Thus,

$$A = p + X \in \big(D \cdot E\big)_{(p,q]}.$$

This shows that $D_{(p,q]} \subseteq \big(D \cdot E\big)_{(p,q]}$.

Therefore, we conclude

$$\big(D \cdot E\big)_{(p,q]} = D_{(p,q]}.$$

Proposition 3.4.49. *Let $n, m \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings. Let $(p,q] \subseteq_I (n, n+m]$ with $n \le p \le q \le n+m$. Then, we get*

$$(D \cdot E)_{(p,q]} = E_{(p-n,q-n]} \tag{3.4.50}$$

as bracketings on $q-p$ letters.

Proof. We will show $(D \cdot E)_{(p,q]} = E_{(p-n,q-n]}$ by showing that

$$(D \cdot E)_{(p,q]} \subseteq E_{(p-n,q-n]} \qquad \text{and} \qquad E_{(p-n,q-n]} \subseteq (D \cdot E)_{(p,q]}.$$

Let $A \in \mathcal{P}_I^+(q-p)$. Suppose $A \in (D \cdot E)_{(p,q]}$. We get some $X \in D \cdot E$ with $X \subseteq (p,q]$ such that $-p + X = A$. We have

$$X \in D \cdot E = D \sqcup (n+E).$$

If $X \in D$, then we must have $X \subseteq [n]$. Since $X \subseteq (p,q] \subseteq (n, n+m]$, this case is not possible. Thus, we conclude that $X \in n+E$. That is, $-n+X \in E$. Moreover, since $X \subseteq (p,q]$ we get $-n + X \subseteq (p-n, q-n]$. This shows that

$$A = -(p-n) + (-n+X) \in E_{(p-n,q-n]}.$$

Thus, we get

$$(D \cdot E)_{(p,q]} \subseteq E_{(p-n,q-n]}.$$

On the other hand, suppose $A \in E_{(p-n,q-n]}$. We get some $X \in E$ with $X \subseteq (p-n, q-n]$ such that $-(p-n) + X = A$. We get

$$n + X \in D \sqcup (n+E) = D \cdot E$$

and $n + X \subseteq (p,q]$. Thus,

$$A = -p + n + X \in (D \cdot E)_{(p,q]}.$$

Consequently, we get

$$E_{(p-n,q-n]} \subseteq (D \cdot E)_{(p,q]}.$$

Therefore, we conclude

$$(D \cdot E)_{(p,q]} = E_{(p-n,q-n]}.$$

Lemma 3.4.51. *Let $n, m \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings. Then, we get*

$$(D \cdot E)_{[n]} = D \tag{3.4.52}$$

as bracketings on n letters.

Proof. We see that

$$\begin{aligned} (D \cdot E)_{[n]} &= D_{[n]} && (3.4.48) \\ &= D. && (3.4.46) \end{aligned}$$

□

Lemma 3.4.53. *Let $n, m \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings. Then, we get*

$$(D \cdot E)_{(n,n+m]} = E \tag{3.4.54}$$

as bracketings on m letters.

Proof. We see that

$$\begin{aligned}(D\cdot E)_{(n,n+m]} &= E_{(0,m]} && (3.4.50)\\ &= E. && (3.4.46)\end{aligned}$$ □

3.5. Dash Assignments on a Bracketing

The bracketing introduced in the previous section gives the structural outline for a dashed word. In this section, we refine that outline by specifying additional information. For instance, let $S = \{a, b, c\}$ and consider the word

$$(a''(bc)''')'b$$

as mentioned at the start of Section 3.4. The corresponding bracketing schematic is

$$((*)(**))*.$$

Here, each bracket represents a group of letters that are dashed together. To indicate how many dashes are applied to each group, we define a function from the set of brackets to the positive integers. This function assigns to each bracket the number of times the dash operation is applied to that group. We call such a function a *dash assignment*. For the example above, the dash assignment schematic is

$$((*)^{(2)}\,(**)^{(3)})^{(1)}\ *.$$

Definition 3.5.1. Let $n \in \mathbb{N}$ and $D \in \mathcal{B}r(n)$ be a bracketing on n letters. A *dash-assignment* on the bracketing D is a function $d : D \longrightarrow \mathbb{N}^+$. We denote the set of all dash-assignments on D by $\mathcal{D}s(D)$. ◇

Definition 3.5.2. Let $n \in \mathbb{N}$ and $\varnothing \in \mathcal{B}r(n)$ be the empty bracketing on n-letters. Then, the empty map $\emptyset : \varnothing \longrightarrow \mathbb{N}^+$ is the only dash-assignment. We call this the *empty dash-assignment* on the empty bracketing on n letters. The *unit dash-assignment*, denoted $J_{\mathcal{D}s}$, is the empty dash-assignment on the unit bracketing. That is,

$$J_{\mathcal{D}s} := \emptyset \in \mathcal{D}s(J_{\mathcal{B}r}). \tag{3.5.3}$$ ◇

Example 3.5.4. Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$, and $d \in \mathcal{D}s(D)$. Then, every $A \in D$ corresponds to a bracket and the positive integer, $d(A)$, corresponds to the number of dashes on that bracket.

For example, let

$$D = \big\{\{1,2\},\ \{1,2,3\},\ \{4,5\}\big\} \in \mathcal{B}r(6)$$

be a bracketing on 6 letters. Then, an example of dash-assignment $d \in \mathcal{D}s(D)$ is given by

$$\begin{aligned}\{1,2\} &\longmapsto 2,\\ \{1,2,3\} &\longmapsto 1,\\ \{4,5\} &\longmapsto 3.\end{aligned}$$

A visual representation of above dash-assignment is given by

$$d = ((**)''\ *)'\ (**)'''\ \ * .$$

Let

$$E = \big\{\{2\},\ \{2,3,4\},\ \{1,2,3,4,5\}\big\} \in \mathcal{B}r(5)$$

be a bracketing on 5 letters. Then, an example of dash-assignment $e \in \mathcal{D}s(D)$ is given by

$$\begin{aligned} \{2\} &\longmapsto 7, \\ \{2,3,4\} &\longmapsto 18, \\ \{1,2,3,4,5\} &\longmapsto 11. \end{aligned}$$

A visual representation of above dash-assignment is given by

$$e = (*\ ((*)^{(7)}\ \ **)^{(18)}\ *)^{(11)}. \qquad \diamond$$

Definition 3.5.5. Let $n, m \in \mathbb{N}$, $D \in \mathcal{B}r(n)$, and $E \in \mathcal{B}r(m)$. Let $d \in \mathcal{D}s(D)$ and $e \in \mathcal{D}s(E)$ be dash-assignments on D and E respectively. Define the multiplication dash-assignments, $d \bullet e \in \mathcal{D}s(D \bullet E)$, as follows:

$$d \bullet e\ (A) = \begin{cases} d(A) & \text{if } A \in D \\ e(-n + A) & \text{if } A \in n + E. \end{cases} \tag{3.5.6}$$

$\diamond$

Example 3.5.7. Let

$$d = ((**)''\ *)'\ (**)'''\ * \in \mathcal{D}s(D) \quad \text{and} \quad e = (*\ ((*)^{(7)}\ \ **)^{(18)}\ *)^{(11)} \in \mathcal{D}s(E)$$

be dash-assignments as in Example 3.5.4. Then, the multiplication of the dash-assignments $d \bullet e \in \mathcal{D}s(D \bullet E)$ is given by

$$\begin{aligned} \{1,2\} &\longmapsto 2, \\ \{1,2,3\} &\longmapsto 1, \\ \{4,5\} &\longmapsto 3, \\ \{8\} &\longmapsto 7, \\ \{8,9,10\} &\longmapsto 18, \\ \{7,8,9,10,11\} &\longmapsto 11. \end{aligned}$$

A visual representation of above dash-assignment is given by the concatenation.

$$d \bullet e = ((**)''\ *)'\ (**)'''\ \ *\ \ (*\ ((*)^{(7)}\ \ **)^{(18)}\ *)^{(11)}. \qquad \diamond$$

Lemma 3.5.8. *Let $n, m, p \in \mathbb{N}$ and $D \in \mathcal{B}r(n), E \in \mathcal{B}r(m), F \in \mathcal{B}r(p)$ be bracketings. Let $d \in \mathcal{D}s(D)$, $e \in \mathcal{D}s(E)$, and $f \in \mathcal{D}s(F)$ be dash-assignments. Then we have*

$$d \bullet (e \bullet f) = (d \bullet e) \bullet f \tag{3.5.9}$$

as dash-assignments on $D \bullet E \bullet F$.

Proof. Let $A \in D \bullet E \bullet F$. Observe that,

$$d \bullet (e \bullet f)\,(A) = \begin{cases} d(A) & \text{if } A \in D \\ e \bullet f\,(-n+A) & \text{if } A \in n + E \cdot F \end{cases} \tag{3.5.6}$$

$$= \begin{cases} d(A) & \text{if } A \in D \\ e(-n+A) & \text{if } A \in n+E \\ f(-n-m+A) & \text{if } A \in n+m+E \end{cases} \tag{3.5.6}$$

$$= \begin{cases} d \bullet e\,(A) & \text{if } A \in D \bullet E \\ f(-(n+m)+A) & \text{if } A \in (n+m)+E \end{cases} \tag{3.5.6}$$

$$= (d \bullet e) \bullet f\,(A). \tag{3.5.6}$$

Since $A \in D \bullet E \bullet F$ is arbitrary, we get

$$d \bullet (e \bullet f) = (d \bullet e) \bullet f$$

as required. □

Lemma 3.5.10. *Let $n \in \mathbb{N}$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Then,*

$$d \bullet J_{\mathcal{Ds}} = d \qquad \text{and} \qquad J_{\mathcal{Ds}} \bullet d = d \tag{3.5.11}$$

as dash-assignments on D.

Proof. Let $A \in D$. We get

$$d \bullet J_{\mathcal{Ds}}\,(A) = \begin{cases} d(A) & \text{if } A \in D \\ & \text{if } A \in J_{\mathcal{Br}} \end{cases} \tag{3.5.6}$$

$$= \begin{cases} d(A) & \text{if } A \in D \\ & \text{if } A \in \varnothing \end{cases} \tag{3.4.5}$$

$$= d.$$

Since $A \in D$ is arbitrary, we get

$$d \bullet J_{\mathcal{Ds}} = d.$$

Next, we get

$$J_{\mathcal{Ds}} \bullet d\,(A) = \begin{cases} & \text{if } A \in J_{\mathcal{Br}} \\ d(A) & \text{if } A \in D \end{cases} \tag{3.5.6}$$

$$= \begin{cases} & \text{if } A \in \varnothing \\ d(A) & \text{if } A \in D \end{cases} \tag{3.4.5}$$

$$= d(A).$$

Since $A \in D$ is arbitrary, we get

$$J_{\mathcal{Ds}} \bullet d = d.$$

□

Definition 3.5.12. Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing. Let $d \in \mathcal{D}s(D)$ be a dash-assignment. Define the dash of the dash-assignment, $d' \in \mathcal{D}s\left(D'\right)$, as follows:

$$d'(A) := \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+1 & \text{if } A = [n] \text{ and } A \in D \\ 1 & \text{if } A = [n] \text{ and } A \notin D. \end{cases} \tag{3.5.13}$$ ◇

Lemma 3.5.14. *Consider the unit dash-assignment, $J_{\mathcal{D}s} \in \mathcal{D}s(J_{\mathcal{B}r})$. We have*

$$J'_{\mathcal{D}s} = J_{\mathcal{D}s} \tag{3.5.15}$$

as dash-assignments on the unit bracketing.

Proof. Since both $J'_{\mathcal{D}s}$ and $J_{\mathcal{D}s}$ are functions with the domain $J_{\mathcal{B}r} = \varnothing$, they are empty functions and thus trivially equal. For the sake of argument, let $A \in J_{\mathcal{B}r}$. Since $J_{\mathcal{B}r}$ is a bracketing, we have $A \neq \varnothing = [0]$. Therefore, from Definition 3.5.12, we get $d'(A) = d(A)$. □

Proposition 3.5.16. *Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing. Let $d \in \mathcal{D}s(D)$ be a dash-assignment. For $k \in \mathbb{N}^+$, let $d^{(k)} \in \mathcal{D}s\left(D^{(k)}\right)$ denote the dash-assignment that we get after taking k dashes. Then, we have*

$$d^{(k)}(A) = \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+k & \text{if } A = [n] \text{ and } A \in D \\ k & \text{if } A = [n] \text{ and } A \notin D. \end{cases} \tag{3.5.17}$$

Proof. We will use induction on k.

Base case ($k = 1$): Observe

$$d^{(1)}(A) = d'(A) = \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+1 & \text{if } A = [n] \text{ and } A \in D \\ 1 & \text{if } A = [n] \text{ and } A \notin D. \end{cases}$$

Induction case ($k \geq 1$): Assume that

$$d^{(k)}(A) = \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+k & \text{if } A = [n] \text{ and } A \in D \\ k & \text{if } A = [n] \text{ and } A \notin D. \end{cases}$$

We get

$$\begin{aligned}
d^{(k+1)}(A) &= \left(d^{(k)}\right)'(A) \\
&= \begin{cases} d^{(k)}(A) & \text{if } A \neq [n] \\ d^{(k)}(A)+1 & \text{if } A = [n] \text{ and } A \in D^{(k)} \\ 1 & \text{if } A = [n] \text{ and } A \notin D^{(k)}. \end{cases} && (3.5.13) \\
&= \begin{cases} d^{(k)}(A) & \text{if } A \neq [n] \\ d^{(k)}(A)+1 & \text{if } A = [n] \text{ and } A \in D \cup \{[n]\} \\ 1 & \text{if } A = [n] \text{ and } A \notin D \cup \{[n]\}. \end{cases} && (3.4.25) \\
&= \begin{cases} d^{(k)}(A) & \text{if } A \neq [n] \\ d^{(k)}(A)+1 & \text{if } A = [n] \end{cases} \\
&= \begin{cases} d(A) & \text{if } A \neq [n] \\ d(A)+k+1 & \text{if } A = [n] \text{ and } A \in D \\ k+1 & \text{if } A = [n] \text{ and } A \notin D. \end{cases} && \text{Induction}
\end{aligned}$$

This completes the proof. □

Example 3.5.18. Consider the dash-assignments

$$d = ((**)''\ *)'\ (**)'''\ * \in \mathcal{D}s(D) \quad \text{and} \quad e = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(11)} \in \mathcal{D}s(E)$$

as described in Example 3.5.4. Then, the dash-assignment $d' \in \mathcal{D}s\left(D'\right)$ is given by:

$$\begin{aligned}
\{1,2\} &\longmapsto 2, \\
\{1,2,3\} &\longmapsto 1, \\
\{4,5\} &\longmapsto 3, \\
\{1,2,3,4,5,6\} &\longmapsto 1.
\end{aligned}$$

A visual representation of this dash-assignment is:

$$d' = (((**)''\ *)'\ (**)'''\ *)'.$$

The dash-assignment $e^{(5)} \in \mathcal{D}s\left(E^{(5)}\right)$ is given by:

$$\begin{aligned}
\{2\} &\longmapsto 7, \\
\{2,3,4\} &\longmapsto 18, \\
\{1,2,3,4,5\} &\longmapsto 16.
\end{aligned}$$

A visual representation of this dash-assignment is:

$$e^{(5)} = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(16)}.$$

In general, applying dashes to a dash-assignment involves adding an outermost bracket if there isn't one already and increasing the number on the outermost bracket by the appropriate dash value. ⋄

Definition 3.5.19. Let $n \in \mathbb{N}$, $D \in \mathcal{Br}(n)$ be a bracketing, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Define the *floor of the dash-assignment*, denoted $\lfloor d \rfloor \in \mathcal{Ds}(\lfloor D \rfloor)$, as follows: Let $A \in \lfloor D \rfloor$ then we have $A \in D$. Define

$$\lfloor d \rfloor (A) := d(A). \tag{3.5.20}$$

◇

Example 3.5.21. Consider the dash-assignments

$$d = ((**)'' \; *)' \; (**)''' \; * \in \mathcal{Ds}(D) \quad \text{and} \quad e = (* \; ((*)^{(7)} \; **)^{(18)} \; *)^{(11)} \in \mathcal{Ds}(E)$$

as described in Example 3.5.4. Then, we have

$$\lfloor d \rfloor = d.$$

The dash-assignment $\lfloor e \rfloor \in \mathcal{Ds}(\lfloor E \rfloor)$ is given by

$$\begin{aligned} \{2\} &\longmapsto 7, \\ \{2,3,4\} &\longmapsto 18. \end{aligned}$$

A visual representation of this dash-assignment is:

$$\lfloor e \rfloor = * \; ((*)^{(7)} \; **)^{(18)} \; *.$$

The floor of a dash-assignment is obtained by removing the outermost bracket and any associated dashes, if present. ◇

Lemma 3.5.22. *Consider the empty dash-assignment, $J_{\mathcal{Ds}} \in \mathcal{Ds}(J_{\mathcal{Br}})$. We have*

$$\lfloor J_{\mathcal{Ds}} \rfloor = J_{\mathcal{Ds}} \tag{3.5.23}$$

as dash-assignments on the unit bracketing.

Proof. From Lemma 3.4.32 we get $\lfloor J_{\mathcal{Br}} \rfloor = J_{\mathcal{Br}}$. Therefore, both $\lfloor J_{\mathcal{Ds}} \rfloor$ and $J_{\mathcal{Ds}}$ are functions with domain $J_{\mathcal{Br}} = \varnothing$. Consequently, both are empty functions and hence vacuously equal. □

Proposition 3.5.24. *Let $n \in \mathbb{N}$, $D \in \mathcal{Br}(n)$ be a bracketing, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Then, we have*

$$\lfloor \lfloor d \rfloor \rfloor = \lfloor d \rfloor \tag{3.5.25}$$

as dash-assignments on $\lfloor D \rfloor$.

Proof. From Proposition 3.4.34 we gave $\lfloor \lfloor D \rfloor \rfloor = \lfloor D \rfloor$. Therefore, it is enough to show that $\lfloor \lfloor d \rfloor \rfloor \; (A) = \lfloor d \rfloor \; (A)$ for $A \in \lfloor D \rfloor$.

Let $A \in \lfloor D \rfloor$. We get

$$\lfloor \lfloor d \rfloor \rfloor \; (A) = \lfloor d \rfloor \; (A) \qquad (3.5.20).$$

Thus, we get

$$\lfloor \lfloor d \rfloor \rfloor = \lfloor d \rfloor .$$

□

Proposition 3.5.26. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{Br}(n)$ be a bracketing, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Then, for $k \in \mathbb{N}$ the equality*

$$\left\lfloor d^{(k)} \right\rfloor = \lfloor d \rfloor \tag{3.5.27}$$

of dash-assignments on $\lfloor D \rfloor$ holds.

Proof. For the case $k = 0$, we get $d^{(0)} = d$. Therefore, the proposition is true for $k = 0$. Now assume $k \geq 1$. From Proposition 3.4.36 we get

$$\left\lfloor D^{(k)} \right\rfloor = \lfloor D \rfloor .$$

Therefore, it is enoguh to show that $\left\lfloor d^{(k)} \right\rfloor (A) = \lfloor d \rfloor (A)$ for $A \in \lfloor D \rfloor$. Let $A \in \lfloor D \rfloor$. Then, we have $A \neq [n]$. Observe that

$$\begin{aligned} \left\lfloor d^{(k)} \right\rfloor (A) &= d^{(k)} (A) && (3.5.20) \\ &= d(A) && (3.5.17) \\ &= \lfloor d \rfloor (A). && (3.5.20) \end{aligned}$$ □

Proposition 3.5.28. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Then, $\lfloor d \rfloor = d$ if and only is $[n] \notin D$.*

Proof. Suppose $\lfloor d \rfloor = d$. Since we have $\lfloor d \rfloor : \lfloor D \rfloor \longrightarrow \mathbb{N}^+$ and $d : D \longrightarrow \mathbb{N}^+$, we get

$$\lfloor D \rfloor = D.$$

From Proposition 3.4.38 we conclude $[n] \notin D$.

On the other hand, suppose $[n] \notin D$. Again from Proposition 3.4.38 we get $\lfloor D \rfloor = D$. Let $A \in \lfloor D \rfloor = D$. From Definition 3.5.19 we get $\lfloor d \rfloor (A) = d(A)$. Since $A \in \lfloor D \rfloor = D$ is arbitrarily chosen, we conclude $\lfloor d \rfloor = d$. □

Proposition 3.5.29. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$ be a dash-assignment. Assign*

$$k = \begin{cases} 0 & \text{if } [n] \notin D \\ d([n]) & \text{if } [n] \in D. \end{cases}$$

Then, the equality

$$(\lfloor d \rfloor)^{(k)} = d \tag{3.5.30}$$

of dash assignments on D holds.

Proof. We will consider the following two cases.

Case 1($[n] \notin D$): In this case we have $k = 0$. Therefore, we get

$$\begin{aligned} (\lfloor d \rfloor)^{(0)} &= \lfloor d \rfloor \\ &= d. && \text{Proposition 3.5.28} \end{aligned}$$

Case 2($[n] \in D$): In this case we have $k = d([n])$. Since $d : D \longrightarrow \mathbb{N}^+$ is a function with codomain $\mathbb{N}^+$ we get $k \geq 1$. From Proposition 3.4.39 we get

$$(\lfloor D \rfloor)^{(k)} = D.$$

Therefore, it is enough to show that $(\lfloor d \rfloor)^{(k)}\ (A) = d(A)$ for $A \in D$. Observe that

$$
\begin{aligned}
(\lfloor d \rfloor)^{(k)}\ (A) &= \begin{cases} \lfloor d \rfloor\ (A) & \text{if } A \neq [n] \\ \lfloor d \rfloor\ (A) + k & \text{if } A = [n] \text{ and } A \in \lfloor D \rfloor \\ k & \text{if } A = [n] \text{ and } A \notin \lfloor D \rfloor \end{cases} && (3.5.17) \\
&= \begin{cases} \lfloor d \rfloor\ (A) & \text{if } A \neq [n] \\ k & \text{if } A = [n] \end{cases} \\
&= \begin{cases} d(A) & \text{if } A \neq [n] \\ k & \text{if } A = [n] \end{cases} && (3.5.20) \\
&= \begin{cases} d(A) & \text{if } A \neq [n] \\ d([n]) & \text{if } A = [n] \end{cases} && (3.5.20) \\
&= d(A).
\end{aligned}
$$

Here, the second equality follows from the fact that $[n] \notin \lfloor D \rfloor$. □

Example 3.5.31. Let

$$d = ((**)''\ *)'\ (**)'''\ * \in \mathcal{D}s(D) \quad \text{and} \quad e = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(11)} \in \mathcal{D}s(E)$$

be dash-assignments as in Example 3.5.4. Since $[6] \notin D$ we get

$$\lfloor d \rfloor = d.$$

Since $[5] \in E$, we get $k = e([6]) = 11$. We have

$$(\lfloor e \rfloor)^{(11)} = e.$$

◇

Definition 3.5.32. Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing, and $d \in \mathcal{D}s(D)$ be a dash-assignment. Let $(p,q] \subseteq_I [n]$ for $0 \le p \le q \le n$. Define a dash-assignment, $d_{(p,q]}$ on $D_{(p,q]}$ as follows:

$$d_{(p,q]}\ (A) := d(p + A) \tag{3.5.33}$$

for $A \in D_{(p,q]}$. This assignment is well-defined since for $A \in D_{(p,q]}$, we get $p + P \in D$. Therefore, $d(p+A) \in \mathbb{N}^+$. ◇

Example 3.5.34. Let

$$d = ((**)''\ *)'\ (**)'''\ * \in \mathcal{D}s(D) \quad \text{and} \quad e = (*\ ((*)^{(7)}\ **)^{(18)}\ *)^{(11)} \in \mathcal{D}s(E)$$

be dash-assignments as described in Example 3.5.4. Then, the dash-assignment $d_{[3]} \in \mathcal{D}s\left(D_{[3]}\right)$ is given by:

$$\{1,2\} \longmapsto 2,$$
$$\{1,2,3\} \longmapsto 1.$$

A visual representation of this dash-assignment is:

$$d_{[3]} = ((**)''\ *)'.$$

The dash-assignment $d_{(3,6]} \in \mathcal{D}s\left(D_{(3,6]}\right)$ is given by:

$$\{1,2\} \longmapsto 3.$$

A visual representation of this dash-assignment is:

$$d_{(3,6]} = (**)''' \;\; * .$$

The dash-assignment $e_{(2,4]} \in \mathcal{D}s\left(D_{(2,4]}\right)$ is given by the empty function since $D_{(2,4]} = \varnothing$. A visual representation of this dash-assignment is:

$$e_{(2,4]} = **. \qquad \diamond$$

Proposition 3.5.35. *Let $n \in \mathbb{N}$, $D \in \mathcal{B}r(n)$ be a bracketing, and $d \in \mathcal{D}s(D)$ be a dash-assignment. Then, we get*

$$d_{[n]} = d \tag{3.5.36}$$

as dash-assignments on D.

Proof. From Proposition 3.4.45 we get $D_{[n]} = D$. Therefore, it is enoguh to show that $d_{[n]}\,(A) = d(A)$ for $A \in D$. Consider the following calculation for $A \in D$:

$$\begin{aligned} d_{[n]}\,(A) = d_{(0,n]}\,(A) &= d(0 + A) && (3.5.33) \\ &= d(A). \end{aligned}$$

Thus, we conclude $d_{[n]} = d$. □

Proposition 3.5.37. *Let $n, m \in \mathbb{N}$, $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings, and $d \in \mathcal{D}s(D)$, $e \in \mathcal{D}s(E)$ be dash assignments. Let $(p,q] \subseteq_I [n]$ with $0 \le p \le q \le n$. Then, we get*

$$\left(d \bullet e\right)_{(p,q]} = d_{(p,q]} \tag{3.5.38}$$

as dash assignments on $D_{(p,q]}$.

Proof. From Proposition 3.4.47 we get $(D \bullet E)_{(p,q]} = D_{(p,q]}$. Therefore, it is enoguh to show that $(d \bullet e)_{(p,q]}\,(A) = d_{(p,q]}(A)$ for $A \in D_{(p,q]}$.

Let $A \in D_{(p,q]}$. We get some $X \in D$ with $X \subseteq (p,q]$ such that $-p + X = A$. It follows that

$$p + A = X \subseteq (p,q] \subseteq [n].$$

Consequently, we get

$$\begin{aligned} \left(d \cdot e\right)_{(p,q]}\,(A) &= d \cdot e\;(p + A) && (3.5.33) \\ &= d(p + A) && (3.5.6) \\ &= d_{(p,q]}\,(A). && (3.5.33) \end{aligned}$$

Since $A \in D_{(p,q]}$ is arbitrary, we get

$$\left(d \cdot e\right)_{(p,q]} = d_{(p,q]}. \qquad \square$$

Proposition 3.5.39. *Let $n, m \in \mathbb{N}$, $D \in \mathcal{B}r(n)$, $E \in \mathcal{B}r(m)$ be bracketings, and $d \in \mathcal{D}s(D)$, $e \in \mathcal{D}s(E)$ be dash assignments. Let $(p,q] \subseteq_I (n, n+m]$ with $n \le p \le q \le n+m$. Then, we get*

$$\left(d \cdot e\right)_{(p,q]} = e_{(p-n,q-n]} \tag{3.5.40}$$

as dash assignments on $E_{(p-n,q-n]}$.

Proof. From Proposition 3.4.49 we get $(D \bullet E)_{(p,q]} = E_{(p-n,q-n]}$. Therefore, it is enoguh to show that $(d \bullet e)_{(p,q]}(A) = e_{(p-n,q-n]}(A)$ for $A \in D_{(p-n,q-n]}$.

Let $A \in E_{(p-n,q-n]}$. We get some $X \in E$ with $X \subseteq (p-n, q-n]$ such that $-(p-n)+X = A$. We have $p+A = n+X \subseteq (p,q] \subseteq (n, n+m]$. Observe that,

$$\begin{aligned}
\left(d \cdot e\right)_{(p,q]}(A) &= d \cdot e\ (p+A) && (3.5.33)\\
&= e(-n+p+A) && (3.5.6)\\
&= e_{(p-n,q-n]}(A). && (3.5.33)
\end{aligned}$$

Since $A \in E_{(p-n,q-n]}$ is arbitrary, we get

$$\left(d \cdot e\right)_{(p,q]} = e_{(p-n,q-n]}. \qquad \square$$

Lemma 3.5.41. *Let $n, m \in \mathbb{N}$, $D \in \mathcal{Br}(n)$, $E \in \mathcal{Br}(m)$ be bracketings. Let $d \in \mathcal{Ds}(D)$ and $e \in \mathcal{Ds}(E)$ be dash-assignments. Then, we get*

$$\left(d \cdot e\right)_{[n]} = d \tag{3.5.42}$$

as dash-assignments on D.

Proof. We see that

$$\begin{aligned}
\left(d \cdot e\right)_{[n]} &= e_{[n]} && (3.5.38)\\
&= d. && (3.5.36)
\end{aligned} \qquad \square$$

Lemma 3.5.43. *Let $n, m \in \mathbb{N}$, $D \in \mathcal{Br}(n)$, $E \in \mathcal{Br}(m)$ be bracketings. Let $d \in \mathcal{Ds}(D)$ and $e \in \mathcal{Ds}(E)$ be dash-assignments. Then, we get*

$$\left(d \cdot e\right)_{(n,n+m]} = e \tag{3.5.44}$$

as dash-assignments on E.

Proof. We see that

$$\begin{aligned}
\left(d \cdot e\right)_{(n,n+m]} &= e_{(0,m]} && (3.5.40)\\
&= e. && (3.5.36)
\end{aligned} \qquad \square$$

3.6. Dashed Words

In Sections 3.4 and 3.5, we introduced the concepts of bracketing and dash assignment for words of length n. These serve as the foundational structures for words in the free dashed monoid. In this section, we complete the construction by explicitly defining the free dashed monoid generated by a set S, denoted $\mathcal{DMon}\langle S \rangle$, utilizing bracketing and dash-assignment.

Building on the groundwork from previous sections, we will demonstrate that this construction indeed yields the free dashed monoid. Specifically, we will show that $\mathcal{DMon}\langle S \rangle$ possesses the structure of a free dashed monoid as described in Definition 3.2.6.

Framework 3.6.1. Throughout this section let S be a set. ⋄

Definition 3.6.2. A *dashed-word* over the set S is a dependent quadruple

$$(n,u,D,d)$$

where:

- n is a natural number representing the length of the dashed-word,
- $u:[n] \longrightarrow S$ (that is, $u \in \mathcal{Wr}(n)$) assigns to each position a letter from S,
- $D \in \mathcal{Br}(n)$ is a bracketing on n letters,
- $d : D \longrightarrow \mathbb{N}^+$ (that is, $d \in \mathcal{Ds}(D)$) is a dash-assignment on the bracketing D.

We denote the set of all dashed words over the set S by $\mathcal{DMon}\langle S\rangle$. ⋄

Remark 3.6.3. We emphasize that the construction in Construction 3.2.4 is also denoted by $\mathcal{DMon}\langle S\rangle$. In Section 3.8, we will demonstrate that both constructions satisfy the universal property of the free dashed monoid generated by S, justifying the use of the same notation. While Construction 3.2.4 offers a more straightforward approach, Definition 3.6.2 provides the detailed structure necessary for the results in Section 3.8. For the remainder of this chapter, $\mathcal{DMon}\langle S\rangle$ will refer to the construction given in Definition 3.6.2. ⋄

Definition 3.6.4. We define the monoid of dashed-words over the set S as follows:

- The underlying set is $\mathcal{DMon}\langle S\rangle$ as in Definition 3.6.2.
- The unit dashed-word, denoted $J \in \mathcal{DMon}\langle S\rangle$, is given by

$$J := (0, J_{\mathcal{Wr}}, J_{\mathcal{Br}}, J_{\mathcal{Ds}}) \tag{3.6.5}$$

  where $J_{\mathcal{Wr}}$ is the unit word as described in Definition 2.2.4, $J_{\mathcal{Br}}$ is the unit bracketing as in Definition 3.4.4, and $J_{\mathcal{Ds}}$ is the unit dash-assignment as in Definition 3.5.2.
- For dashed words (n,u,D,d) and (m,v,E,e), the multiplication is given by

$$(n,u,D,d)\bullet(m,v,E,e) := (n+m, u\bullet v, D\bullet E, d\bullet e) \tag{3.6.6}$$

  where $u\bullet v \in \mathcal{Wr}(n+m)$ is as in Definition 2.2.5, $D\bullet E \in \mathcal{Br}(n+m)$ is as in Definition 3.4.8, and $d\bullet e \in \mathcal{Ds}(D\bullet E)$ is as in Definition 3.5.5.

The associativity for dashed-words follows from equations Lemmas 2.2.7, 3.4.11, and 3.5.8. The unit conditions for dashed-words follow from equations Lemmas 2.2.9, 3.4.14, and 3.5.10. ⋄

Definition 3.6.7. We define the dashed-monoid of dashed-words over the set S as follows:

- The underlying monoid is $\mathcal{DMon}\langle S\rangle$ as in Definition 3.6.4.
- The dash map $(-)' : \mathcal{DMon}\langle S\rangle \longrightarrow \mathcal{DMon}\langle S\rangle$ is given by

$$(n,u,D,d)' := (n,u,D',d') \tag{3.6.8}$$

  where $D' \in \mathcal{Br}(n)$ is as in Definition 3.4.17 and $d' \in \mathcal{Ds}\left(D'\right)$ is as in Definition 3.5.12.

The unit condition for dash follows from Lemmas 3.4.22 and 3.5.14. ◇

Example 3.6.9. Let $S = \{a,b,c\}$. Consider the dashed-word

$$x = (6,u,D,e) \in \mathcal{DMon}\langle S\rangle$$

described as follows:

$$\begin{aligned} u = \{1 &\longmapsto a,\\ 2 &\longmapsto b,\\ 3 &\longmapsto c,\\ 4 &\longmapsto a,\\ 5 &\longmapsto b,\\ 6 &\longmapsto a\}.\end{aligned}$$

The bracketing $D \in \mathcal{Br}(6)$ and the dash-assignment $d \in \mathcal{Ds}(D)$ are same as in Example 3.5.4. A visual representation of the dashed-word is as follows:

$$x = ((ab)''\ c)'\ (ab)''\ a.$$

The dashed-word

$$y = (5,v,E,e) \in \mathcal{DMon}\langle S\rangle$$

is described as follows:

$$\begin{aligned} u = \{1 &\longmapsto b,\\ 2 &\longmapsto b,\\ 3 &\longmapsto c,\\ 4 &\longmapsto a,\\ 5 &\longmapsto a\}.\end{aligned}$$

The bracketing $E \in \mathcal{Br}(5)$ and the dash-assignment $e \in \mathcal{Ds}(E)$ are same as in Example 3.5.4. A visual representation of the dashed-word is as follows:

$$y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

The multiplication of these dashed-words is given by the concatenation:

$$x \bullet y = ((ab)''\ c)'\ (ab)''\ a(b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

The dash of the dashed-words is given by

$$\begin{aligned} x' &= (((ab)''\ c)'\ (ab)''\ a)'\\ y^{(5)} &= (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(16)}.\end{aligned}$$

◇

Definition 3.6.10. Define a *length* function $\hat{l} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N}$ as follows:

$$\hat{l}((n,u,D,d)) = n. \tag{3.6.11}$$

◇

Proposition 3.6.12. *Consider the set of natural numbers* $\mathbb{N}$ *with the dashed-monoid structure as in Example 3.1.5. Then, the length function* $\hat{l} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N}$ *is a dashed-monoid homomorphism. That is, for all* $x, y \in \mathcal{DMon}\langle S\rangle$ *we get*

$$\hat{l}(J) = 0, \tag{3.6.13}$$

$$\hat{l}(x \bullet y) = \hat{l}(x) + \hat{l}(y), \qquad \text{and} \tag{3.6.14}$$

$$\hat{l}\left(x'\right) = \hat{l}(x). \tag{3.6.15}$$

Proof. Let $J = (0, J_{\mathcal{Wr}}, J_{\mathcal{Br}}, J_{\mathcal{Ds}})$ be the unit dashed-word. We get

$$\hat{l}(J) = 0.$$

Now let $x = (n, u, D, d)$ and $y = (m, v, E, e)$ be two dashed-words. From the equation (3.6.6) we get $x \bullet y = (n + m, u \bullet v, D \bullet E, d \bullet e)$. Therefore, we get

$$\hat{l}(x \bullet y) = n + m = \hat{l}(x) + \hat{l}(y).$$

Finally, let $x = (n, u, D, d)$ be a dashed-word. From the equation (3.6.8) we get $x' = (n, u, D', d')$. Therefore, we get

$$\hat{l}\left(x'\right) = n = \hat{l}(x) = (\hat{l}(x))'.$$ □

Proposition 3.6.16. *Let* $x \in \mathcal{DMon}\langle S\rangle$*. Then,* $\hat{l}(x) = 0$ *if and only if* $x = J$*.*

Proof. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle$. Suppose $\hat{l}(x) = 0$, that is, $n = 0$. It follows that, $u : [0] \longrightarrow S$ is the empty function $J_{\mathcal{Wr}}$. Also, we have $D \subseteq \mathcal{P}^{+}_{J}(0) = \varnothing$. Thus, we get $D = J_{\mathcal{Br}}$. It follows that, $d : J_{\mathcal{Br}} \longrightarrow \mathbb{N}$ is the empty function. Therefore, we get $d = J_{\mathcal{Ds}}$. Thus, we have

$$x = (0, J_{\mathcal{Wr}}, J_{\mathcal{Br}}, J_{\mathcal{Ds}}) = J.$$

On the other hand, if $x = J$ then from equation (3.6.13) get $\hat{l}(x) = 0$. □

Notation 3.6.17. Let $\mathcal{DMon}\langle S\rangle^{+}$ denote the collection of all non-zero length dashed-words. That is,

$$\begin{aligned} \mathcal{DMon}\langle S\rangle^{+} &:= \{x \in \mathcal{DMon}\langle S\rangle \mid x \neq J\} \\ &= \{x \in \mathcal{DMon}\langle S\rangle \mid \hat{l}(x) > 0\} \qquad \textit{Proposition } 3.6.16. \end{aligned}$$ ◇

Notation 3.6.18. For $x \in \mathcal{DMon}\langle S\rangle$ and $k \in \mathbb{N}$ let $x^{(k)}$ denote the dashed-word obtained by applying the dash operation k-times. In particular, we have

$$x^{(0)} = x \qquad \text{and} \qquad x^{(1)} = x'.$$ ◇

Proposition 3.6.19. *Let* $k \in \mathbb{N}$ *and* $x \in \mathcal{DMon}\langle S\rangle$*. Then we get*

$$\hat{l}\left(x^{(k)}\right) = \hat{l}(x). \tag{3.6.20}$$

Proof. We will consider the cases $k = 0$ and $k \geq 1$. If $k = 0$, then we have $x^{(0)} = x$. Thus, we get

$$\hat{l}\left(x^{(0)}\right) = \hat{l}(x).$$

Now, let $k \geq 1$. From Proposition 3.6.12, we have $\hat{l}(x') = \hat{l}(x)$. Therefore, applying this fact repeatedly, we get

$$\hat{l}\left(x^{(k)}\right) = \hat{l}(x).$$

□

Definition 3.6.21. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle$. Define *floor* of x, denoted $\lfloor x \rfloor$, as follows:

$$\lfloor x \rfloor := (n, u, \lfloor D \rfloor, \lfloor d \rfloor) \tag{3.6.22}$$

where $\lfloor D \rfloor \in \mathcal{Br}(n)$ is as in Definition 3.4.27 and $\lfloor d \rfloor \in \mathcal{Ds}(\lfloor D \rfloor)$ is as in Definition 3.5.19. ◇

Proposition 3.6.23. *We have*

$$\lfloor J \rfloor = J. \tag{3.6.24}$$

Proof. Observe that

$$\begin{aligned}
\lfloor J \rfloor &= \left\lfloor (0, J_{\mathcal{Wr}}, J_{\mathcal{Br}}, J_{\mathcal{Ds}}) \right\rfloor && (3.6.5)\\
&= (0, J_{\mathcal{Wr}}, \lfloor J_{\mathcal{Br}} \rfloor, \lfloor J_{\mathcal{Ds}} \rfloor) && (3.6.22)\\
&= (0, J_{\mathcal{Wr}}, J_{\mathcal{Br}}, J_{\mathcal{Ds}}) && (3.4.33) \text{ and } (3.5.23)\\
&= J. && (3.6.5)
\end{aligned}$$

□

Example 3.6.25. Let $S = \{a, b, c\}$. Consider the dashed-words

$$xx = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}$$

as in Example 3.6.9. The floor of the dashed-word x is given by

$$\lfloor x \rfloor = x = ((ab)''\ c)'\ (ab)''\ a$$

and the floor of the dashed-word y is given by

$$\lfloor y \rfloor = b\ ((b)^{(7)}\ ca)^{(18)}\ a.$$

◇

Proposition 3.6.26. *Let $x \in \mathcal{DMon}\langle S\rangle$ be a dashed-word. Then, we get*

$$\hat{l}(\lfloor x)\rfloor = \hat{l}(x). \tag{3.6.27}$$

Proof. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle$ where $n \in \mathbb{N}$, $u : [n] \longrightarrow S$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$. Observe that

$$\begin{aligned}
\hat{l}(\lfloor x)\rfloor &= \hat{l}((n, u, \lfloor D), \lfloor d \rfloor)\rfloor && (3.6.22)\\
&= n && (3.6.11)\\
&= \hat{l}(x). && (3.6.11)
\end{aligned}$$

□

Proposition 3.6.28. *Let $x \in \mathcal{DMon}\langle S\rangle$ be a dashed-word. Then, we get*

$$\lfloor \lfloor x \rfloor \rfloor = \lfloor x \rfloor. \tag{3.6.29}$$

Proof. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle$ where $n \in \mathbb{N}$, $u : [n] \longrightarrow S$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$. Observe that

$$\begin{aligned}
\lfloor\lfloor x\rfloor\rfloor &= \lfloor(n, u, \lfloor D\rfloor, \lfloor d\rfloor)\rfloor && (3.6.22)\\
&= (n, u, \lfloor\lfloor D\rfloor\rfloor, \lfloor\lfloor d\rfloor\rfloor) && (3.6.22)\\
&= (n, u, \lfloor D\rfloor, \lfloor d\rfloor) && (3.4.35) \text{ and } (3.5.27)\\
&= \lfloor x\rfloor . && (3.6.22)
\end{aligned}$$ □

Proposition 3.6.30. *Let $x \in \mathcal{DMon}\langle S\rangle$ be a dashed-word and $k \in \mathbb{N}^+$. Then, we get*

$$\left\lfloor x^{(k)}\right\rfloor = \lfloor x\rfloor . \tag{3.6.31}$$

Proof. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle$ where $n \in \mathbb{N}$, $u : [n] \longrightarrow S$, $D \in \mathcal{Br}(n)$, and $d \in \mathcal{Ds}(D)$. Observe that

$$\begin{aligned}
\left\lfloor x^{(k)}\right\rfloor &= \left\lfloor(n, u, D^{(k)}, d^{(k)})\right\rfloor && (3.6.8)\\
&= (n, u, \left\lfloor D^{(k)}\right\rfloor, \left\lfloor d^{(k)})\right\rfloor && (3.6.22)\\
&= (n, u, \lfloor D\rfloor, \lfloor d\rfloor) && (3.4.37) \text{ and } (3.5.27)\\
&= \lfloor x\rfloor . && (3.6.22)
\end{aligned}$$ □

Proposition 3.6.32. *Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle$ be a dashed-word with $n \geq 1$. The, $\lfloor x\rfloor = x$ if and only if $[n] \notin D$.*

Proof. Observe that

$$\lfloor x\rfloor = (n, u, \lfloor D\rfloor, \lfloor d\rfloor).$$

Therefore, from Propositions 3.4.38 and 3.5.28 we get that $\lfloor x\rfloor = x$ if and only if $[n] \notin D$. □

Proposition 3.6.33. *Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle$ be a dashed-word with $n \geq 1$. Assign*

$$k = \begin{cases} 0 & \text{if } [n] \notin D \\ d([n]) & \text{if } [n] \in D. \end{cases}$$

Then, the equality

$$(\lfloor x\rfloor)^{(k)} = x \tag{3.6.34}$$

of dashed-words hold.

Proof. Note that $D \in \mathcal{Br}(n)$ and $d \in \mathcal{Ds}(D)$ satisfy the conditions of Propositions 3.4.39 and 3.5.29 respectively. Therefore, we get

$$\begin{aligned}
(\lfloor x\rfloor)^{(k)} &= \left(n, u, \lfloor D\rfloor, \lfloor d\rfloor\right)^{(k)} && (3.6.22)\\
&= \left(n, u, \lfloor D\rfloor^{(k)}, \lfloor d\rfloor^{(k)}\right) && (3.6.8)\\
&= (n, u, D, d) && (3.4.40) \text{ and } (3.5.30)\\
&= x.
\end{aligned}$$ □

Definition 3.6.35. Let (n,u,D,d) be a dashed word. Let $(p,q] \subseteq_I [n]$ with $0 \le p \le q \le n$. Define a dashed word $(n,u,D,d)_{(p,q]}$ as follows:

$$(n,u,d,D)_{(p,q]} := (q-p, u_{(p,q]}, D_{(p,q]}, d_{(p,q]}) \tag{3.6.36}$$

where $u_{(q,p]} \in \mathcal{Wr}(q-p)$ is as in Definition 2.2.11. The bracketing, $D_{(p,q]} \in \mathcal{Br}(q-p)$, is as in Definition 3.4.42. The dash-assignment, $d_{(p,q]} \in \mathcal{Ds}\left(D_{(p,q]}\right)$ is as in Definition 3.5.32. ◇

Example 3.6.37. Let $S = \{a,b,c\}$. Consider the dashed-words

$$x = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}$$

as in Example 3.6.9. Then, we have

$$\begin{aligned} x_{[3]} &= ((ab)''\ c)' \\ x_{(3,6]} &= (ab)''\ a \\ y_{(2,5]} &= caa. \end{aligned}$$ ◇

Proposition 3.6.38. *Let $x \in \mathcal{DMon}\langle S\rangle$ and $n = \hat{l}(x)$. Let $L \subseteq_I [n]$ be a sub-interval. Then we get*

$$\hat{l}(x_L) = |L|. \tag{3.6.39}$$

Proof. Let $L = (p,q]$ with $0 \le p \le q \le n$. Then from Definition 3.6.35 we get

$$\hat{l}(x_L) = q - p = |L|.$$ □

Proposition 3.6.40. *Let $x = (n,u,D,d)$ be a dashed word. Then, we have*

$$x_{[n]} = x. \tag{3.6.41}$$

Proof. Observe that

$$\begin{aligned} x_{[n]} &= (n, u_{[n]}, D_{[n]}, d_{[n]}) && (3.6.36) \\ &= (n,u,D,d) && (2.2.14), (3.4.46), \text{ and } (3.5.36) \\ &= x. \end{aligned}$$ □

Proposition 3.6.42. *Let $x = (n,u,D,d)$ and $y = (m,v,E,e)$ be dashed-words. Let $(p,q] \subseteq_I [n]$ with $0 \le p \le q \le n$. Then, we get*

$$(x \bullet y)_{(p,q]} = x_{(p,q]}. \tag{3.6.43}$$

Proof. Observe that

$$\begin{aligned} &(x \bullet y)_{(p,q]} \\ &= (n+m, u \bullet v, D \bullet E, d \bullet e)_{(p,q]} && (3.6.6) \\ &= \left(q-p, (u \bullet v)_{(p,q]}, (D \bullet E)_{(p,q]}, (d \bullet e)_{(p,q]}\right) && (3.6.36) \\ &= \left(q-p, u_{(p,q]}, D_{(p,q]}, d_{(p,q]}\right) && (2.2.16), (3.4.48), \text{ and } (3.5.38) \\ &= x_{(p,q]}. && (3.6.36) \end{aligned}$$ □

Proposition 3.6.44. *Let $x = (n,u,D,d)$ and $y = (m,v,E,e)$ be dashed-words. Let $(p,q] \subseteq_I [n]$ with $n \le p \le q \le n+m$. Then, we get*

$$(x \bullet y)_{(p,q]} = y_{(p-n,q-n]}. \tag{3.6.45}$$

Proof. Observe that

$$\begin{aligned}
&(x \bullet y)_{(p,q]} \\
&= (n+m, u \bullet v, D \bullet E, d \bullet e)_{(p,q]} && (3.6.6)\\
&= \big(q-p, (u \bullet v)_{(p,q]}, (D \bullet E)_{(p,q]}, (d \bullet e)_{(p,q]}\big) && (3.6.36)\\
&= \big(q-p, v_{(p-n,q-n]}, E_{(p-n,q-n]}, e_{(p-n,q-n]}\big) && (2.2.18), (3.4.50), \text{and } (3.5.40)\\
&= y_{(p-n,q-n]}. && (3.6.36)
\end{aligned}$$

□

Proposition 3.6.46. *Let $x = (n,u,D,d)$ and $y = (m,v,E,e)$ be two dashed words. Then, we get*

$$(x \bullet y)_{[n]} = x. \tag{3.6.47}$$

Proof. Observe that,

$$\begin{aligned}
(x \bullet y)_{[n]} &= x_{[n]} && (3.6.43)\\
&= x. && (3.6.41)
\end{aligned}$$

□

Proposition 3.6.48. *Let $x = (n,u,D,d)$ and $y = (m,v,E,e)$ be two dashed words. Then, we get*

$$(x \bullet y)_{(n,n+m]} = y. \tag{3.6.49}$$

Proof. Observe that,

$$\begin{aligned}
(x \bullet y)_{(n,n+m]} &= y_{[m]} && (3.6.45)\\
&= y. && (3.6.41)
\end{aligned}$$

□

Lemma 3.6.50. *The dashed monoid of dashed-words over the set S is a left cancellative monoid. That is, for $x,y,z \in \mathcal{DMon}\langle S\rangle$,*

$$x \bullet y = x \bullet z \qquad \textit{implies} \qquad y = z. \tag{3.6.51}$$

Proof. Let $x,y,z \in \mathcal{DMon}\langle S\rangle$ with $\hat{l}(x) = n$, $\hat{l}(y) = m$, and $\hat{l}(z) = p$. Suppose we have

$$x \bullet y = x \bullet z.$$

We get

$$n + m = \hat{l}(x \bullet y) = \hat{l}(x \bullet z) = n + p.$$

Thus, we have $m = p$. Observe that

$$\begin{aligned}
y &= (x \bullet y)_{(n,n+m]} && (3.6.49)\\
&= (x \bullet z)_{(n,n+p]} \\
&= z. && (3.6.49)
\end{aligned}$$

□

Lemma 3.6.52. *The dashed monoid of dashed-words over the set S is a right cancellative monoid. That is, for $x,y,z \in \mathcal{DMon}\langle S\rangle$,*

$$y \bullet x = z \bullet x \qquad \textit{implies} \qquad y = z. \tag{3.6.53}$$

Proof. Let $x, y, z \in \mathcal{DMon}\langle S\rangle$ with $\hat{l}(x) = n$, $\hat{l}(y) = m$, and $\hat{l}(z) = p$. Suppose we have

$$y \bullet x = z \bullet x.$$

We get

$$m + n = \hat{l}(y \bullet x) = \hat{l}(z \bullet x) = p + n.$$

Thus, we have $m = p$. Observe that

$$\begin{aligned} y &= (y \bullet x)_{[m]} && (3.6.47) \\ &= (z \bullet x)_{[p]} \\ &= z. && (3.6.47) \end{aligned}$$

□

3.7. Dashed words as the Free Dashed Monoid

In this section, we will show that the dashed monoid $\mathcal{DMon}\langle S\rangle$ has a free dashed monoid like structure (Definition 3.2.6). To achieve this, we will use $\hat{l} : \mathcal{DMon}\langle S\rangle \longrightarrow \mathbb{N}$ as in Definition 3.6.10 as the length function. We need to provide multiplicative basis $\mathtt{G}$ and dash basis $\mathtt{H}$ of $\mathcal{DMon}\langle S\rangle$. Finally, we will show that these satisfy the interlocking conditions as in (3.2.9) and (3.2.10).

We will start by constructing a multiplicative basis $\mathtt{G}$ for $\mathcal{DMon}\langle S\rangle$.

Definition 3.7.1. Define a subset $\mathtt{G} \subseteq \mathcal{DMon}\langle S\rangle$ as follows:

$$\mathtt{G} := \big\{(n, u, D, d) \mid n = 1 \text{ or } \big(n \geq 2 \text{ and } [n] \in D\big)\big\}. \tag{3.7.2}$$

◇

Remark 3.7.3. Observe that $J \notin \mathtt{G}$ since $\hat{l}(J) = 0$. Therefore, we have

$$\mathtt{G} \subseteq \mathcal{DMon}\langle S\rangle^{+}.$$

◇

Example 3.7.4. Let $S = \{a, b, c\}$. Consider the dashed-words

$$x = (6, u, D, d) \qquad \text{and} \qquad y = (5, u, D, d)$$

as described in Example 3.6.9. These dashed-words are described as follows:

$$x = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

Then, $x \notin \mathtt{G}$ since $6 \geq 2$ and $[6] \notin D$. On the other hand, $y \in \mathtt{G}$ since $5 \geq 2$ and $[5] \in E$. Moreover, $a \in \mathtt{G}$ and $b^{(7)} \in \mathtt{G}$ since we have $\hat{l}(a) = \hat{l}\left(b^{(7)}\right) = 1$. ◇

Definition 3.7.5. Let $n \in \mathbb{N}^{+}$ and $D \in \mathcal{Br}(n)$ be a bracketing. Define the *leading left interval*, $L \subseteq [n]$, as

$$L := \{1\} \cup \left(\bigcup_{A \in D;\ 1 \in A} A\right). \tag{3.7.6}$$

The set L as defined above is an interval since it is a union of intervals containing 1. Suppose $L = [p]$ for some $1 \leq p \leq n$. Define the *trailing right interval*, $R \subseteq [n]$, as the complement of L:

$$R := (p, n]. \tag{3.7.7}$$

◇

Remark 3.7.8. Note that the collection

$$\mathfrak{L} := \{A \in D \mid 1 \in A\}$$

is a finite collection with a linear order with respect to set inclusion. The leading left interval L is the largest interval in the above collection, provided that the collection is non-empty; otherwise, L is equal to $[1]$. ⋄

Proposition 3.7.9. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{B}r(n)$ be a bracketing, and $L = [p]$ for some $1 \leq p \leq n$ be the leading left interval as in Definition 3.7.5. Then, $L \in D$ if and only if there exists $X \in D$ such that $1 \in X$.*

Proof. Suppose $L \in D$. Then, by Definition 3.7.5 we have $1 \in L$. Thus, we can take $X = L$.

On the other hand, suppose there exists $X \in D$ such that $1 \in X$. Since L is the largest such interval in D, we have $L \in D$.

Proposition 3.7.10. *Let $n \in \mathbb{N}^+$, $D \in \mathcal{B}r(n)$ be a bracketing, and L and R be the leading left interval and the trailing right interval as in Definition 3.7.5. Then, for $A \in D$ we get either $A \subseteq L$ or $A \subseteq R$.*

Proof. We will prove this by considering following cases.

Case 1 ($L \in D$): In this case, since D is a bracketing and $A, L \in D$ we get either $A \cap L = \varnothing$, $A \subseteq L$, or $L \subseteq A$. In this first subcase, we get

$$A \subseteq [n] \setminus L = R.$$

In the second subcase, we get $A \subseteq L$ as required. For the third subcase, we get $1 \subseteq A$ since $1 \in L$ and $L \subseteq A$. Since L is the largest interval in D such that $1 \in L$ we get $A \subseteq L$.

Case 2 ($L \notin D$): In this case, from Proposition 3.7.9, we get that for every $X \in D$, it holds that $1 \notin X$. Therefore, we have

$$L = \{1\}.$$ □

Taking $X = A$, we get that $1 \notin A$. Thus, we have

$$A \subseteq [n] \setminus \{1\} = [n] \setminus L = R.$$

Definition 3.7.11. Define functions

$$\texttt{Head} : \mathcal{DMon}\langle S\rangle^+ \longrightarrow \mathcal{DMon}\langle S\rangle^+$$

and

$$\texttt{Tail} : \mathcal{DMon}\langle S\rangle^+ \longrightarrow \mathcal{DMon}\langle S\rangle$$

as follows: Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, we have $n \geq 1$. Let $L, R \subseteq_I [n]$ be the leading left intervals and the trailing right interval as in Definition 3.7.5. Since L is non-empty, we get $x_L \in \mathcal{DMon}\langle S\rangle^+$. Define

$$\texttt{Head}(x) := x_L, \qquad \text{and} \tag{3.7.12}$$

$$\texttt{Tail}(x) := x_R. \tag{3.7.13}$$ ⋄

Example 3.7.14. Let $S = \{a,b,c\}$. Consider the dashed-words

$$x = (6,u,D,d) \qquad \text{and} \qquad y = (5,u,D,d)$$

as described in Example 3.6.9. These dashed-words are described as follows:

$$x = ((ab)''\ c)'\ (ab)''\ a \qquad \text{and} \qquad y = (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)}.$$

Then, we have

$$\begin{aligned} \texttt{Head}(x) &= ((ab)''\ c)' \\ \texttt{Tail}(x) &= (ab)''\ a. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \texttt{Head}(y) &= (b\ ((b)^{(7)}\ ca)^{(18)}\ a)^{(11)} \\ \texttt{Tail}(y) &= J. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \texttt{Head}(abc) &= a && \text{and} && \texttt{Tail}(abc) = bc, \\ \texttt{Head}\left(b^{(7)}\right) &= b^{(7)} && \text{and} && \texttt{Tail}\left(b^{(7)}\right) = J, \\ \texttt{Head}\left(a'a\right) &= a' && \text{and} && \texttt{Tail}\left(a'a\right) = a. \end{aligned} \qquad \diamond$$

Proposition 3.7.15. *Let $x \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, we get*

$$\hat{l}(\texttt{Head}(x)) \le \hat{l}(x) \tag{3.7.16}$$

and

$$\hat{l}(\texttt{Tail}(x)) < \hat{l}(x). \tag{3.7.17}$$

Proof. Let $n = \hat{l}(x)$ and $L = [p], R = (p,n] \subseteq_I [n]$ be the leading left inverval and the trailing right interval where $1 \le p \le n$. We see that

$$\begin{aligned} \hat{l}(\texttt{Head}(x)) &= \hat{l}\left(x_{[p]}\right) && (3.7.12) \\ &= p \\ &\le n \\ &= \hat{l}(x) \end{aligned}$$

and

$$\begin{aligned} \hat{l}(\texttt{Tail}(x)) &= \hat{l}\left(x_{(p,n]}\right) && (3.7.13) \\ &= n - p \\ &< n \\ &= \hat{l}(x). \end{aligned} \qquad \square$$

Proposition 3.7.18. *Let $x \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, we get*

$$\texttt{Head}(\texttt{Head}(x)) = \texttt{Head}(x). \tag{3.7.19}$$

Proof. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle^+$ and $L = [p]$ be the leading left interval of x, where $1 \le p \le n$. Thus, we get

$$y := \mathtt{Head}(x) = x_{[p]} = (p, u_{[p]}, D_{[p]}, d_{[p]}).$$

Let $M = [q]$ be the leading left interval for y, where $1 \le q \le p$. Thus, we get

$$\mathtt{Head}(\mathtt{Head}(x)) = y_{[q]} = \left(x_{[p]}\right)_{[q]}.$$

We claim that $p = q$. If $p = 1$ then we get $q = 1$, and we are done. Assume $2 \le p$. Therefore, we get $L = [p] \ne \{1\}$. Recall that,

$$L = \{1\} \cup \left(\bigcup_{A \in D; 1 \in A} A \right). \qquad (3.7.6)$$

Since $L \ne \{1\}$, there exists $A \in D$ with $1 \in A$. From Proposition 3.7.9 we get $L = [p] \in D$. From Definition 3.4.42, we conclude $[p] \in D_{[p]}$.

We have

$$M = \{1\} \cup \left(\bigcup_{A \in D_{[p]}; 1 \in A} A \right). \qquad (3.7.6)$$

Since $[p] \in D_{[p]}$ and $1 \in [p]$, we get $[p] \subseteq [q] = M$ implying $p \le q$. Therefore, we conclude $p = q$.

Finally, we get

$$\begin{aligned} \mathtt{Head}(\mathtt{Head}(x)) &= \left(x_{[p]}\right)_{[q]} \\ &= \left(x_{[p]}\right)_{[p]} \\ &= x_{[p]} && (3.6.41) \\ &= \mathtt{Head}(x). \end{aligned}$$

□

Proposition 3.7.20. *Let $x \in \mathcal{DMon}\langle S\rangle^+$ be non-unit dashed-word and $y \in \mathcal{DMon}\langle S\rangle$. Then,*

$$\mathtt{Head}(x \bullet y) = \mathtt{Head}(x). \qquad (3.7.21)$$

Proof. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle^+$ and $y = (m, v, E, e) \in \mathcal{DMon}\langle S\rangle$. Let L be the leading left interval of $x \bullet y$. Consider the family, $\mathfrak{L}$, of sub-intervals of $[n+m]$ given by

$$\mathfrak{L} = \{A \in D \bullet E \mid 1 \in A\}$$

Then, we get

$$L = \{1\} \cup \left(\bigcup_{A \in \mathfrak{L}} A \right). \qquad (3.7.6)$$

Let M be the leading left interval of x. Consider the family, $\mathfrak{M}$, of sub-intervals of $[n]$ given by

$$\mathfrak{M} = \{A \in D \mid 1 \in A\}$$

Then, we get

$$M = \{1\} \cup \left(\bigcup_{A \in \mathfrak{M}} A \right). \qquad (3.7.6)$$

From Definition 3.7.11 we get

$$\mathtt{Head}(x \bullet y) = (x \bullet y)_L \qquad \text{and} \qquad \mathtt{Head}(x) = x_M.$$

We claim that, $\mathfrak{L} = \mathfrak{M}$. To see this, suppose $A \in \mathfrak{L}$. That means

$$A \in D \bullet E = D \sqcup (n+E) \qquad \text{and} \qquad 1 \in A.$$

If $A \in n+E$ then we have $A \subseteq (n, n+m]$. Since $x \in \mathcal{DMon}\langle S \rangle^+$, we get $n \geq 1$. This implies, $1 \notin A$ which is a contradiction. Thus, we get $A \in D$. We already have $1 \in A$, thus we get $A \in \mathfrak{M}$. This shows

$$\mathfrak{L} \subseteq \mathfrak{M}.$$

On the other hand, suppose $A \in \mathfrak{M}$. Then we get

$$A \in D \qquad \text{and} \qquad 1 \in A.$$

Therefore,

$$A \in D \subseteq D \sqcup (n+E) = D \bullet E.$$

That is, $A \in \mathfrak{L}$. This shows

$$\mathfrak{L} \supseteq \mathfrak{M}.$$

As a result, we get $\mathfrak{L} = \mathfrak{M}$ implying $L = M$.

Noting $M \subseteq [n]$, we conclude

$$\begin{aligned}
\mathtt{Head}(x \bullet y) &= (x \bullet y)_L \\
&= (x \bullet y)_M \\
&= x_M && (3.6.43) \\
&= \mathtt{Head}(x).
\end{aligned}$$

□

Proposition 3.7.22. *Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S \rangle^+$ be a non-unit dashed-word. Let $L, R \subseteq [n]$ be the left leading interval and the right leading inverval as in Definition 3.7.5. Then, we get*

$$D = D_L \bullet D_R \tag{3.7.23}$$

as bracketings on n letters.

Proof. Let $L = [p]$ and $R = (p, n]$ for where $1 \leq p \leq n$. Thus, we wish to show that

$$D = D_{[p]} \bullet D_{(p,n]}.$$

Suppose $A \in D$. Then, from Proposition 3.7.10, we get $A \subseteq [p]$ or $A \subseteq (p, n]$. We will consider these two cases.

<u>Case 1</u> ($A \subseteq [p]$): We get

$$\begin{aligned}
A &\in D_{[p]} && (3.4.43) \\
&\subseteq D_{[p]} \sqcup p + D_{(p,n]} \\
&= D_{[p]} \bullet D_{(p,n]}. && (3.4.9)
\end{aligned}$$

Case 2 ($A \subseteq (p,n]$): Since $A \in D$ and $A \subseteq (p,n]$, we get $-p+A \in D_{(p,n]}$. Thus, we get $A \in p+D_{(p,n]}$. As a consequence we get

$$A \in D_{[p]} \sqcup (p+D_{(p,n]}) = D_{[p]} \bullet D_{(p,n]}.$$

Since we have shown $A \in D_{[p]} \bullet D_{(p,n]}$ in both the cases, we conclude

$$D \subseteq D_{[p]} \bullet D_{(p,n]}.$$

On the other hand, suppose

$$A \in D_{[p]} \bullet D_{(p,n]} = D_{[p]} \sqcup p + D_{(p,n]}$$

. We will consider these two cases.

Case 1 ($A \in D_{[p]}$): We get that $A \in D$ and $A \subseteq [p]$. In particular, we get $A \in D$.

Case 2 ($A \in p+D_{(p,n]}$): We get $X \in D_{(p,n]}$ such that $A = p+X$. Since $X \in D_{(p,n]}$, we get $Y \in D$ with $Y \subseteq (p,n]$ such that $X = -p+Y$. Observe that

$$A = p+X = p+(-p+Y) = Y.$$

Since $Y \in D$, we conclude $A \in D$.

We have shown $A \in D$ in both the cases, therefore we get

$$D \supseteq D_{[p]} \bullet D_{(p,n]}.$$

Thus, we conclude

$$D = D_{[p]} \bullet D_{(p,n]} = D_L \bullet D_R. \qquad \square$$

Proposition 3.7.24. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Let $L,R \subseteq [n]$ be the leading left interval and the leading right interval as in Definition 3.7.5. Then, we get*

$$d = d_L \bullet d_R \tag{3.7.25}$$

as dash-assignments on D.

Proof. Let $L = [p]$ and $R = (p,n]$ where $1 \le p \le n$. We will show that

$$d = d_{[p]} \bullet d_{(p,n]}.$$

From Proposition 3.7.22, we have

$$D = D_{[p]} \bullet D_{(p,n]} = D_{[p]} \sqcup (p+D_{(p,n]}).$$

Therefore, it is enough to show

$$d(A) = d_{[p]} \bullet d_{(p,n]}\,(A)$$

for $A \in D_{[p]} \sqcup (p+D_{(p,n]})$.

We will consider the following cases.

Case 1 ($A \subseteq D_{[p]}$): Observe that

$$\begin{aligned} d_{[p]} \bullet d_{(p,n]}\,(A) &= d_{(0,p]}\,(A) && (3.5.6)\\ &= d(A). && (3.5.33)\end{aligned}$$

Case 2 ($A \subseteq p + D_{(p,n]}$): Observe that

$$\begin{aligned} d_{(0,p]} \bullet d_{(p,n]}(A) &= d_{(p,n]}(-p+A) && (3.5.6) \\ &= d(p+(-p+A)) && (3.5.33) \\ &= d(A). \end{aligned}$$

We have shown

$$d_{(0,p]} \bullet d_{(p,n]}(A) = d(A)$$

in both the cases, we conclude that

$$d_{(0,p]} \bullet d_{(p,n]} = d. \qquad \square$$

Lemma 3.7.26. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then,*

$$x = \mathtt{Head}(x) \bullet \mathtt{Tail}(x). \tag{3.7.27}$$

Proof. Let $L = [p], R = (p,n] \subseteq_I [n]$ be the leading left interval and the right left interval as in Definition 3.7.5 where $1 \leq p \leq n$. Then, we have

$$\mathtt{Head}(x) = x_L \qquad \text{and} \qquad \mathtt{Tail}(x) = x_R.$$

Observe that

$$\begin{aligned} x &= (n,u,D,d) \\ &= (p+(n-p),\ u_L \bullet u_R,\ D_L \bullet D_R,\ d_L \bullet d_R) && (2.2.24), (3.7.23), \text{ and } (3.7.25) \\ &= (p,u_L,D_L,d_L) \bullet (n-p,u_R,D_R,d_R) && (3.6.6) \\ &= x_L \bullet x_R && (3.6.36) \\ &= \mathtt{Head}(x) \bullet \mathtt{Tail}(x). && (3.7.12) \text{ and } (3.7.13) \square \end{aligned}$$

Proposition 3.7.28. *Let $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then, $x \in \mathtt{G}$ if and only if $\mathtt{Head}(x) = x$.*

Proof. Let $L \subseteq [n]$ be the leading left interval as in Definition 3.7.5, that is

$$L = \{1\} \cup \left(\bigcup_{A \in D; 1 \in A} A \right).$$

We get

$$\mathtt{Head}(x) = x_L. \qquad (3.7.12)$$

Suppose $x \in \mathtt{G}$, then we get $n = 1$ or $[n] \in D$. We will consider these two cases:

Case 1 ($n = 1$): We get $L = \{1\} = [n]$. Thus, we have

$$\begin{aligned} \mathtt{Head}(x) &= x_L \\ &= x_{[n]} \\ &= x. && (3.6.41) \end{aligned}$$

Case 2 ($[n] \in D$): For every $A \in D$ we have $A \subseteq [n]$. Since $[n] \in D$ and $1 \in [n]$, we get $L = [n]$. Observe that

$$\begin{aligned}\mathtt{Head}(x) &= x_L \\ &= x_{[n]} \\ &= x. && (3.6.41)\end{aligned}$$

On the other hand, suppose $\mathtt{Head}(x) = x_L = x$. If $n = 1$ then we get $x \in \mathtt{G}$, and we are done. Assume $n \geq 2$. We get

$$|L| = \hat{l}(x_L) = \hat{l}(x) = n.$$

Thus, L is a length n sub-interval of $[n]$. Therefor, we conclude $L = [n]$. Since $n \geq 1$, we get $[n] = L$ is the largest interval in D such that $1 \in L$. In particular, we get $[n] = L \in D$. This shows that $x \in \mathtt{G}$. □

Lemma 3.7.29. *Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. Then we get $\mathtt{Head}(x) \in \mathtt{G}$.*

Proof. From Proposition 3.7.18 we get

$$\mathtt{Head}(\mathtt{Head}(x)) = \mathtt{Head}(x).$$

From Lemma 3.7.29 we get $\mathtt{Head}(x) \in \mathtt{G}$. □

Theorem 3.7.30. *The subset $\mathtt{G} \subseteq \mathcal{DMon}\langle S\rangle$ is a multiplicative basis of the dashed monoid $\mathcal{DMon}\langle S\rangle$.*

Proof. We will show that $\mathtt{G}$ is a generating set with respect to the multiplication and an independent set with respect to the multiplication.

First, we will show $\mathtt{G}$ is a generating set with respect to the multiplication. Let $x \in \mathcal{DMon}\langle S\rangle^+$ be a non-unit dashed-word. We will show, using induction on $\hat{l}(x)$, that there exists $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

Base case ($\hat{l}(x) = 1$): From Definition 3.7.1 we get $x \in \mathtt{G}$. We take $m = 1$ and $x_1 = x$.

Induction case ($\hat{l}(x) \geq 2$): From Lemma 3.7.26 we get

$$x = \mathtt{Head}(x) \bullet \mathtt{Tail}(x).$$

If $\mathtt{Tail}(x) = J$, then we get $\mathtt{Head}(x) = x$. From Proposition 3.7.28, we get $x \in \mathtt{G}$. Now assume $\mathtt{Tail}(x)$ is a non-unit dashed word. From Proposition 3.7.15 we get

$$\hat{l}(\mathtt{Tail}(x)) < \hat{l}(x).$$

Thus, from the induction hypothesis we get $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$\mathtt{Tail}(x) = x_1 \bullet \cdots \bullet x_m.$$

From Lemma 3.7.29 we have $\mathtt{Head}(x) \in \mathtt{G}$. Therefore, we get

$$x = \mathtt{Head}(x) \bullet \mathtt{Tail}(x) = \mathtt{Head}(x) \bullet x_1 \cdots x_m$$

where $\mathtt{Head}(x), x_i \in \mathtt{G}$ for $1 \leq i \leq m$.

From the mathematical induction, we conclude that $\mathtt{G}$ is a generating set with respect to the multiplication.

Next, we will show that $\mathtt{G}$ is an independent set with respect to the multiplication. Let $x \in \mathcal{DMon}\langle S\rangle^+$. Using induction on $\hat{l}(x)$, we will show that given $m, l \geq 1$ and $x_i, y_j \in \mathtt{G}$ for $1 \leq i \leq m$ and $1 \leq j \leq l$ such that

$$x = x_1 \bullet \cdots \bullet x_m = y_1 \bullet \cdots \bullet y_l,$$

we get $m = l$ and $x_i = y_i$ for $1 \leq i \leq m$.

Base case ($\hat{l}(x) = 1$): Suppose we have $m, l \geq 1$ and $x_i, y_j \in \mathtt{G}$ for $1 \leq i \leq m$ and $1 \leq j \leq l$ such that

$$x = x_1 \bullet \cdots \bullet x_m = y_1 \bullet \cdots \bullet y_l.$$

We get

$$1 = \hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i). \tag{3.6.14}$$

Since $x_i \in \mathtt{G}$, we have $\hat{l}(x_i) \geq 1$. It follows that $m = 1$ and $x_1 = x$. Similarly, we get $l = 1$ and $y_1 = x$. This completes the base case.

Induction case ($\hat{l}(x) \geq 2$): Suppose we have $m, l \geq 1$ and $x_i, y_j \in \mathtt{G}$ for $1 \leq i \leq m$ and $1 \leq j \leq l$ such that

$$x = x_1 \bullet \cdots \bullet x_m = y_1 \bullet \cdots \bullet y_l.$$

In particular, we have $x_1, y_1 \in \mathtt{G}$. Let $z = x_2 \cdots x_m$ and $w = y_2 \cdots y_l$. Then, we get

$$x = x_1 \bullet z = y_1 \bullet w.$$

Observe that

$$\begin{aligned}
x_1 &= \mathtt{Head}(x_1) && \text{Proposition 3.7.28}\\
&= \mathtt{Head}(x_1 \bullet z) && (3.7.21)\\
&= \mathtt{Head}(y_1 \bullet w) && \\
&= \mathtt{Head}(y_1) && (3.7.21)\\
&= y_1. && \text{Proposition 3.7.28}
\end{aligned}$$

Since $\mathcal{DMon}\langle S\rangle$ is a left cancellative monoid (Lemma 3.6.50), we get

$$x_2 \cdots x_m = y = z = y_2 \cdots y_l.$$

If $y = J$ then we get $m = 1$ and $x = x_1$. Since $z = y$, we get $z = J$. Consequently, we have $l = 1$ and $x = y_1$. This gives the required condition in $y = J$ case. Now assume $y \in \mathcal{DMon}\langle S\rangle^+$. Since $x_1 \in \mathtt{G}$ we get $\hat{l}(x_1) \geq 1$. It follows that

$$\hat{l}(y) < \hat{l}(x_1) + \hat{l}(y) = \hat{l}(x).$$

From the induction hypothesis we get $m = l$ and $x_i = y_i$ for $2 \leq i \leq m$. Above, along with the fact that $x_1 = y_1$, gives us $m = l$ and $x_i = y_i$ for $1 \leq i \leq m$.

From the mathematical induction, we conclude that $\mathtt{G}$ is an independent set with respect to the multiplication. Consequently, $\mathtt{G}$ is a multiplicative basis of $\mathcal{DMon}\langle S\rangle$. $\square$

Definition 3.7.31. Define a subset $\mathtt{S} \subseteq \mathcal{DMon}\langle S \rangle$ as follows:

$$\mathtt{S} := \{(n,u,D,d) \in \mathcal{DMon}\langle S \rangle \mid n = 1 \ \text{ and } \ D = \varnothing\}. \tag{3.7.32}$$

Define a function $i_S : S \longrightarrow \mathcal{DMon}\langle S \rangle$ as follows:

$$i_S(a) := (1,\ \overline{a},\ \varnothing,\ \emptyset) \tag{3.7.33}$$

where $\overline{a} : [1] \longrightarrow S$ is given by

$$\overline{a}(1) = a.$$

◇

Proposition 3.7.34. *The set S is in bijection with the subset $\mathtt{S} \subseteq \mathcal{DMon}\langle S \rangle$ via the function i_S.*

Proof. Let $a, b \in S$ and suppose

$$i_S(a) = i_S(b).$$

Then, we get $\overline{a} = \overline{b}$. Thus,

$$a = \overline{a}(1) = \overline{b}(1) = b.$$

This shows that $i_S : S \longrightarrow \mathcal{DMon}\langle S \rangle$ is injective.

Now, let $x = (1,u,\varnothing,\emptyset) \in \mathtt{S}$. Let $a = u(1) \in S$. Observe that

$$i_S(a) = (1,\overline{a},\varnothing,\emptyset)$$

where

$$\overline{a}(1) = a = u(1).$$

Thus, we get $\overline{a} = u$ therefore

$$i_S(a) = x.$$

This shows that $\mathtt{S} \subseteq i_S(S)$.

Finally, let $a \in S$. Then, we have

$$i_S(a) = (1,\overline{a},\varnothing,\emptyset) \in \mathtt{S}.$$

Thus, we have $i_S(S) \subseteq \mathtt{S}$. Therefore, we conclude that i_S is a bijection between S and $\mathtt{S}$. □

Definition 3.7.35. Define a subset $\mathtt{R} \subseteq \mathcal{DMon}\langle S \rangle$ as follows:

$$\begin{aligned} \mathtt{R} &:= \langle\langle \mathtt{G};\ \bullet \rangle\rangle \\ &= \{x_1 \bullet \cdots \bullet x_m \mid m \geq 2,\ x_i \in \mathtt{G} \text{ for } 1 \leq i \leq m\}. \qquad (2.1.14) \end{aligned} \tag{3.7.36}$$

◇

Proposition 3.7.37. *Let $\mathtt{R} \subseteq \mathcal{DMon}\langle S \rangle$ be as in Definition 3.7.35. Then,*

$$\mathtt{R} = \{(n,u,D,d) \in \mathcal{DMon}\langle S \rangle \mid n \geq 2 \ \text{ and } \ [n] \notin D\}. \tag{3.7.38}$$

Proof. Let $x = (n,u,D,d) \in \mathtt{R}$. Then, we get $m \geq 2$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet x_2 \bullet \cdots \bullet x_m.$$

Since $x_i \in \mathtt{G}$, we get $\hat{l}(x_i) \geq 1$. Moreover, since $m \geq 2$ we get

$$n = \hat{l}(x) = \sum_{i=1}^{m} \hat{l}(x_i) \geq 2.$$

For the sake of contradiction, assume $[n] \in D$. Then, from Definition 3.7.1 we get $x \in \mathtt{G}$. Since $\mathtt{G}$ is a multiplicative basis, it is a multiplicatively independent set. Lemma 3.1.26 asserts that

$$\mathtt{G} \cap \mathtt{R} = \varnothing.$$

This leads to a contradiction since we have $x \in \mathtt{R}$ and $x \in \mathtt{G}$.

On the other hand, suppose $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle$ such that $n \geq 2$ and $[n] \notin D$. Since $n \geq 2$ we get $x \neq J$. Since $\mathtt{G}$ is a multiplicative basis, we get $m \geq 1$ and $x_i \in \mathtt{G}$ for $1 \leq i \leq m$ such that

$$x = x_1 \bullet \cdots \bullet x_m.$$

To show that $x \in \mathtt{R}$, we need to show that $m \geq 2$. If $m = 1$ then we have $x = x_1 \in \mathtt{G}$. Since $n \geq 2$ and $[n] \notin D$, we have $x \notin \mathtt{G}$, proving that $m \neq 1$. Therefore, we must have $m \geq 2$. It follows that, $x \in \mathtt{R}$. $\square$

Proposition 3.7.39. *Let* $\mathtt{S} \subseteq \mathcal{DMon}\langle S\rangle$ *be as in Definition 3.7.31 and* $\mathtt{R} \subseteq \mathcal{DMon}\langle S\rangle$ *be as in Definition 3.7.35. Then, we have*

$$\mathtt{S} \cap \mathtt{R} = \varnothing. \tag{3.7.40}$$

Proof. Suppose $x = (n,u,D,d) \in \mathtt{S} \cap \mathtt{R}$. Since $x \in \mathtt{S}$, we get $n = 1$. Since $x \in \mathtt{R}$, we get $n \geq 2$. This is a contradiction. Therefore, we have

$$\mathtt{S} \cap \mathtt{R} = \varnothing.$$ $\square$

Definition 3.7.41. Define a subset $\mathtt{H} \subseteq \mathcal{DMon}\langle S\rangle$ as follows:

$$\mathtt{H} := \mathtt{S} \sqcup \mathtt{R}. \tag{3.7.42}$$ $\diamond$

Proposition 3.7.43. *We have*

$$\mathtt{H} = \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n \geq 1 \ \text{ and } \ [n] \notin D\}. \tag{3.7.44}$$

Proof. Observe that

$$\begin{aligned}
\mathtt{H} &= \mathtt{S} \sqcup \mathtt{R} && (3.7.42)\\
&= \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n = 1 \ \text{ and } \ [n] \notin D\}\\
&\qquad \sqcup \{(n,u,D,d) \in \mathcal{DMon}\langle S\rangle \mid n \geq 2 \ \text{ and } [n] \notin D\} && (3.7.32) \text{ and } (3.7.42)\\
&= \{(n,u,D,d) \mid n \geq 1 \ \text{ and } \ [n] \notin D\}.
\end{aligned}$$ $\square$

Lemma 3.7.45. *Let* $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ *be a non-unit dashed-word. Then,* $x \in \mathtt{H}$ *if and only if* $\lfloor x \rfloor = x$.

Proof. Since x is a non-unit dashed-word, we get $n \geq 1$. From Proposition 3.6.32 we get $\lfloor x \rfloor = x$ if and only if $[n] \notin D$. From Proposition 3.7.43, this is equivalent to $x \in \mathtt{H}$. $\square$

Lemma 3.7.46. *Let* $x = (n,u,D,d) \in \mathcal{DMon}\langle S\rangle^+$ *be a non-unit dashed-word. Then,* $\lfloor x \rfloor \in \mathtt{H}$.

Proof. Since x is a non-unit dashed-word, we get $\hat{l}(x) = n \geq 1$. We have

$$\begin{aligned} \hat{l}(\lfloor x \rfloor) &= \hat{l}(x) && (3.6.27) \\ &= n \geq 1. \end{aligned}$$

From Proposition 3.6.28 we get

$$\lfloor \lfloor x \rfloor \rfloor = \lfloor x \rfloor .$$

Using Lemma 3.7.45 we get $\lfloor x \rfloor \in \mathtt{H}$. □

Lemma 3.7.47. *Let $x = (n, u, D, d) \in \mathtt{H}$. Suppose*

$$x^{(k)} = x^{(l)}$$

for some $k, l \in \mathbb{N}$. Then, we get $k = l$.

Proof. From Proposition 3.7.43 we get $n \geq 1$ and $[n] \notin D$. Without the loss of generality, we will consider the following cases:

Case 1 ($k = 0$): Suppose $l \geq 1$. Then, we get $x = x^{(l)}$. In particular, we get

$$D = D^{(l)}.$$

From Proposition 3.4.19 we get

$$D(l) = D \cup \{[n]\}.$$

Thus, we get $[n] \in D$. This is a contradiction. Therefore, we conclude $l = 0$.

Case 2 ($k, l \geq 1$): Since $[n] \notin D$, from Proposition 3.5.16 we get

$$d^{(k)}([n]) = k \qquad \text{and} \qquad d^{(l)}([n]) = l.$$

From the equality $x^{(k)} = x^{(l)}$ we get

$$d^{(k)} = d^{(l)}.$$

It follows that $k = l$ as required. □

Theorem 3.7.48. *The subset $\mathtt{H} \subseteq \mathcal{DMon}\langle S \rangle$ is a dash basis of $\mathcal{DMon}\langle S \rangle$.*

Proof. We will show that $\mathtt{H}$ is a generating set and an independent set with respect to the dash operation.

First, we will show that $\mathtt{H}$ is a generating set with respect to the dash operation. Let $x = (n, u, D, d) \in \mathcal{DMon}\langle S \rangle^+$ be a non-unit dashed-word. From Lemma 3.7.46, we get that $\lfloor x \rfloor \in \mathtt{H}$. Assign

$$k = \begin{cases} 0 & \text{if } [n] \notin D \\ d([n]) & \text{if } [n] \in D. \end{cases}$$

Then, from Proposition 3.6.33 we get

$$(\lfloor x \rfloor)^{(k)} = x.$$

This shows that $\mathtt{H}$ is a generating set with respect to the dash operation.

Next, we will show that $\mathtt{H}$ is independent with respect to the dash operation. Let $x, y \in \mathtt{H}$. Suppose we have

$$x^{(k)} = y^{(l)}$$

for some $k, l \in \mathbb{N}$. Since $x, y \in \mathtt{H}$, from Lemma 3.7.45 we get

$$x = \lfloor x \rfloor \qquad \text{and} \qquad y = \lfloor y \rfloor .$$

Observe that

$$\begin{aligned} x &= \lfloor x \rfloor \\ &= \left\lfloor x^{(k)} \right\rfloor && (3.6.31) \\ &= \left\lfloor y^{(k)} \right\rfloor && \text{assumption} \\ &= \lfloor y \rfloor && (3.6.31) \\ &= y. \end{aligned}$$

From Lemma 3.7.47 we get $k = l$. This shows that $\mathtt{H}$ is independent with respect to the dash operation. □

Proposition 3.7.49. *The equality*

$$\langle\langle \mathtt{H};\ (-)' \rangle\rangle = \{(n, u, D, d) \in \mathcal{DMon}\langle S \rangle \mid n \geq 1 \ \text{ and } \ [n] \in D\} \tag{3.7.50}$$

holds.

Proof. Let $x = (n, u, D, d) \in \langle\langle \mathtt{H};\ (-)' \rangle\rangle$. Then, we get $y = (m, v, E, e) \in \mathtt{H}$ and $k \geq 1$ such that

$$x = y^{(k)}.$$

That is,

$$\begin{aligned} (n, u, D, d) &= (m, v, E, e)^{(k)} \\ &= (m, v, E^{(k)}, e^{(k)}) && (3.6.8). \end{aligned}$$

Therefore, we have

$$n = m, \qquad u = v, \qquad D = E^{(k)}, \qquad \text{and} \qquad d = e^{(k)}.$$

Since $y \in \mathtt{H}$, we get $m \geq 1$ and $[m] \notin E$. This implies $n \geq 1$. Since $m \geq 1$ and $k \geq 1$, using Proposition 3.4.24, we get

$$\begin{aligned} D &= E^{(k)} \\ &= E \cup \{[m]\} && (3.4.25) \\ &= E \cup \{[n]\}. \end{aligned}$$

Therefore, we get $[n] \in D$.

On the other hand, suppose $x = (n, u, D, d) \in \mathcal{DMon}\langle S \rangle$ such that $n \geq 1$ and $[n] \in D$. Let $k = d([n])$. Note that $k \geq 1$ since $d : D \longrightarrow \mathbb{N}^+$. From Lemma 3.7.46 we get $\lfloor x \rfloor \in \mathtt{H}$. From Proposition 3.6.33 we get

$$(\lfloor x \rfloor)^{(k)} = x.$$

This shows that $x \in \langle\langle \mathtt{H};\ (-)' \rangle\rangle$. □

Proposition 3.7.51. *We have*

$$\mathtt{S} \cap \langle\langle \mathtt{H};\ (-)' \rangle\rangle = \varnothing. \tag{3.7.52}$$

Proof. Suppose $x = (n,u,D,d) \in \mathtt{S} \cap \langle\langle DBasis;\ (-)' \rangle\rangle$. From Definition 3.7.31 we get $n = 1$ and $D = \varnothing$. From Proposition 3.7.49 we get $n \geq 1$ and $[n] \in D$. This leads to a contradiction. Therefore,

$$\mathtt{S} \cap \langle\langle \mathtt{H};\ (-)' \rangle\rangle = \varnothing. \qquad \square$$

Proposition 3.7.53. *The equality*

$$\mathtt{S} \sqcup \langle\langle \mathtt{H};\ (-)' \rangle\rangle = \mathtt{G} \tag{3.7.54}$$

holds.

Proof. We see that

$$\begin{aligned}
&\mathtt{S} \sqcup \langle\langle \mathtt{H};\ (-)' \rangle\rangle \\
&= \{(n,u,D,d) \mid (n = 1 \text{ and } D = \varnothing) \text{ or} \\
&\qquad\qquad (n \geq 1 \text{ and } [n] \in D)\} && (3.7.32) \text{ and } (3.7.50) \\
&= \{(n,u,D,d) \mid n = 1 \text{ or } (n \geq 2 \text{ and } [n] \in D)\} \\
&= \mathtt{G}. && (3.7.2)
\end{aligned}$$

Here, the third equality follows since for $n = 1$ we have either $D = \{\}$ or $D = \{[1]\}$. $\square$

Theorem 3.7.55. *The subset $\mathtt{S} \subseteq \mathcal{DMon}\langle S \rangle$ is a dashed monoid basis.*

Proof. Let $\hat{l} : \mathcal{DMon}\langle S \rangle \longrightarrow \mathbb{N}$ as in Definition 3.6.10 be the length function. Proposition 3.6.16 gives the required property for the length function.

Let $\mathtt{G}$ be the multiplicative basis as shown in Theorem 3.7.30. Let $\mathtt{H}$ be the dash basis as in Definition 3.7.41. Proposition 3.7.53 and Definition 3.7.41 give the required interlocking conditions. $\square$

Theorem 3.7.56. *The inclusion of sets $\mathtt{S} \subseteq \mathcal{DMon}\langle S \rangle$ satisfies the universal property of dashed monoid.*

Proof. From Theorem 3.7.55 we know that $\mathtt{S} \subseteq \mathcal{DMon}\langle S \rangle$ is a dashed monoid basis. The theorem follows immediately from Theorem 3.2.44 . $\square$

3.8. Key Results

We will conclude this chapter with the following key results.

Theorem 3.8.1. *Let M be a dashed monoid, and let $S \subseteq M$ with the inclusion given by*

$$\phi : S \hookrightarrow M.$$

Then, the pair

$$\left(M, \phi : S \hookrightarrow M\right)$$

satisfies the universal property of the free dashed monoid generated by S if and only if S is a dashed monoid basis of M.

Proof. We have already shown one direction at Theorem 3.2.44.

For the other direction, suppose the pair

$$\left(M, \phi : S \hookrightarrow M\right)$$

satisfies the universal property of the free dashed monoid generated by S. Consider the construction of dashed monoid of dashed words $\mathcal{DMon}\langle S\rangle$ over the set S as in Definition 3.6.7. From Theorem 3.7.55, we know that $S \subseteq \mathcal{DMon}\langle S\rangle$ is a dashed monoid basis of $\mathcal{DMon}\langle S\rangle$. From Theorem 3.7.56, we know that the pair

$$\big(\mathcal{Mon}\langle S\rangle, i : S \longrightarrow \mathcal{Mon}\langle S\rangle\big)$$

satisfies the universal property of the free dashed monoid generated by S. It follows that, $\mathcal{DMon}\langle S\rangle$ is isomorphic to M via a dashed monoid homomorphism which maps $\mathtt{S} \subseteq \mathcal{DMon}\langle S\rangle$ to $S \subseteq M$. Since $\mathtt{S} \subseteq \mathcal{DMon}\langle X\rangle$ is a dashed monoid basis of $\mathcal{DMon}\langle S\rangle$, from Proposition 3.2.11 we conclude that $S \subseteq M$ is a dashed monoid basis of M. □

Theorem 3.8.2. *Let S be a set. The dashed monoid $\mathcal{DMon}\langle S\rangle$ as in Construction 3.2.4 satisfies the universal property of the free dashed monoid generated by S.*

Proof. Let $(M, I, (-)', \cdot)$ be a dashed monoid and let $u : S \longrightarrow M$ be a function. We will construct the induced dashed homomorphism as follows:

Definition 3.8.3. Define

$$F : \mathcal{DMon}\langle S\rangle \longrightarrow M$$

as follows:

$$F(J) = I \tag{3.8.4}$$

$$F(a) = u(a) \qquad \text{for } a \in S \tag{3.8.5}$$

$$F(x \bullet y) = F(x) \cdot F(y) \qquad \text{for } x, y \in \mathcal{DMon}\langle S\rangle, \text{ and} \tag{3.8.6}$$

$$F(x') = F(x)' \qquad \text{for } x \in \mathcal{DMon}\langle S\rangle. \tag{3.8.7}$$

This function is well-defined since M is a dashed monoid. From the definition, F is a dashed monoid homomorphism and and satisfies the inclusion condition (3.2.2). ⋄

Now, suppose $(M, I, (-)', \cdot)$ be a dashed monoid and let

$$F, G : \mathcal{DMon}\langle S\rangle \longrightarrow M$$

be dashed monoid homomorphisms such that

$$F(a) = G(a)$$

for $a \in S$. Since F, G are both dashed monoid homomorphisms and agree on $S \subseteq \mathcal{DMon}\langle S\rangle$, from Construction 3.2.4 we conclude that

$$F = G.$$

□

Theorem 3.8.8. *Let M be a dashed monoid, $S \subseteq M$ be a subset and $\phi : S \hookrightarrow M$ be the inclusion function. Suppose the pair*

$$(M, \phi : S \hookrightarrow M)$$

satisfies the universal property of the free monoid generated by S. Then, M is a free monoid with monoid basis $\mathtt{G} \subseteq M$.

Furthermore, the monoid basis $\mathtt{G}$ *is characterized as follows: Let*

$$\hat{l} : M \longrightarrow \mathbb{N}$$

be the induced dashed monoin homomorphism defined by setting

$$\hat{l}(a) = 1 \qquad \textit{for every } a \in S.$$

Then, $x \in \mathtt{G}$ *if and only if exactly one of the following holds:*

- *We have*

$$x = a^{(k)}$$

  for some unique $k \geq 0$ *and* $a \in \mathtt{S}$*. In this case, we also get*

$$\hat{l}\left(a^{(k)}\right) = 1. \tag{3.8.9}$$

- *We have*

$$x = (x_1 \bullet \cdots \bullet x_m)^{(k)}$$

  for some unique $m \geq 2$*,* $k \geq 1$ *and* $x_i \in \mathtt{G}$*. In this case, we also get*

$$\hat{l}(x) \geq 2 \tag{3.8.10}$$

  and

$$\hat{l}(x_i) < \hat{l}(x). \tag{3.8.11}$$

Proof. From Theorem 3.8.1 we get that $S \subseteq M$ is a dashed monoid basis of M. The theorem follows from Lemma 3.2.19. $\square$

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