QUANTUM GROUP STRUCTURES ASSOCIATED WITH INVARIANCES OF SOME PHYSICAL ALGEBRAS

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Motivations

- Bosons and fermions are important
- Angular momentum algebra is important
- Quantum groups (Hopf algebras) will be important

Bosons

The boson algebra arises from the quantization of the harmonic oscillator:

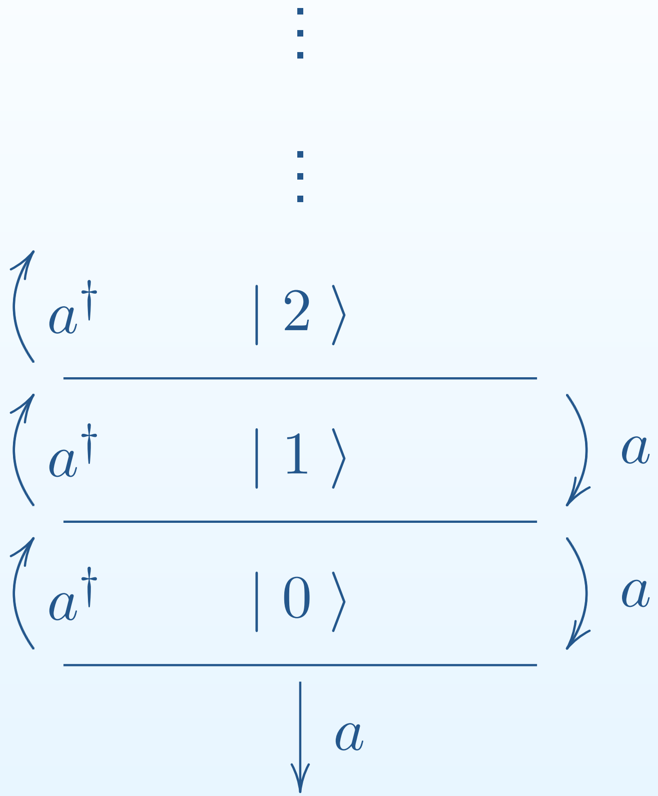
$$aa^\dagger - a^\dagger a = 1$$
$$aa - aa = 0$$

Defining N as $a^\dagger a$ gives:

$$N \mid n \,\rangle = n \mid n \,\rangle$$
$$a^\dagger \mid n \,\rangle = \sqrt{n+1} \mid n+1 \,\rangle$$
$$a \mid n \,\rangle = \sqrt{n} \mid n-1 \,\rangle$$

for $n = 0, 1, 2, \cdots$.

Bosonic Energy Levels



0 - Null State

Fermions

The fermion algebra is invented the other way around.
The algebra is defined by:

$$aa^\dagger + a^\dagger a = 1$$

$$aa + aa = 0$$

Defining N as $a^\dagger a$ gives:

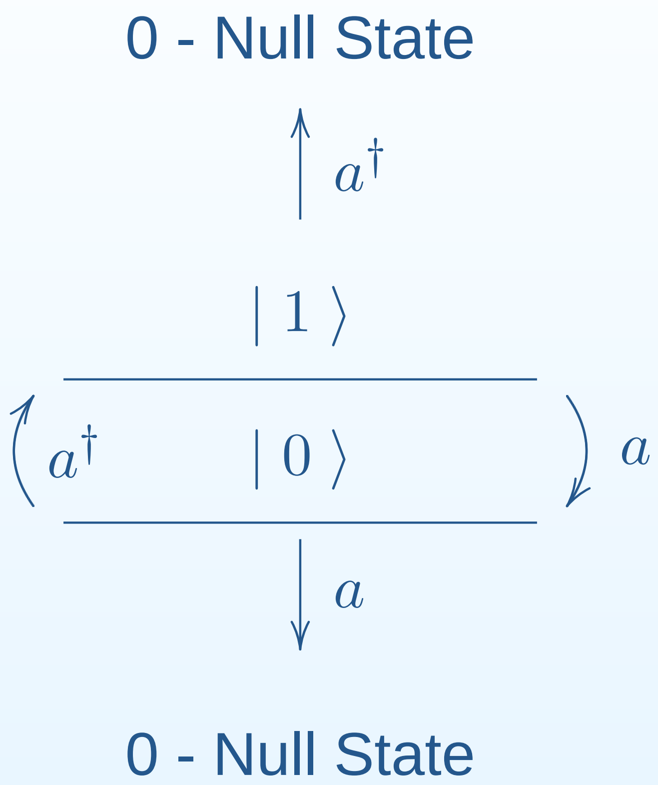
$$N \mid n \rangle = n \mid n \rangle$$

$$a^\dagger \mid n \rangle = \sqrt{n+1} \mid n+1 \rangle$$

$$a \mid n \rangle = \sqrt{n} \mid n-1 \rangle$$

for $n = 0$ and 1.

Fermionic Energy Levels



Associative Algebras

An algebra A is a vector space over F with multiplication m.

$$m(x, y) \equiv m(x \otimes y) \equiv xy$$

We require that, m is associative:

$$(xy)z = x(yz) \quad \text{for all } x, y, z \in A$$

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$$m(m(x, y), z) = m(x, m(y, z)) \quad \text{for all } x, y, z \in A$$

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$$m(m(A \otimes A) \otimes A) = m(A \otimes m(A \otimes A))$$

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$$m \circ (m \otimes id)(A \otimes A \otimes A) = m \circ (id \otimes m)(A \otimes A \otimes A)$$

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$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \circ id"] \arrow[d, "id \circ m"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

Associative Algebras

The algebra A is called unital if there exists an identity

$$x1 = 1x = x \quad \text{for all } x \in A$$

This is the same thing as defining the map $\eta : F \longrightarrow A$
$\eta(k) = k1$ for all $k \in F$ which satisfies:

$$m \circ (id \otimes \eta) = id = m \circ (\eta \otimes id)$$

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$$\begin{tikzcd}
F \otimes A \cong A \cong A \otimes F \arrow[r, "id \circ \eta"] \arrow[d, "\eta \circ id"] \arrow[dr, "id"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

Coalgebras

Motivation: The action of the product on the dual of the algebra

The dual of the algebra A is also a vector space over F. The addition and scalar multiplication are carried over. However, what about the effect of m and η?

$$\phi(xy) = \phi(m(x \otimes y)) = \Delta(\phi)(x \otimes y)$$

$$\phi(k1) = \phi(\eta(k)) = \epsilon(\phi)k$$

Two new maps on A^* are defined:

- The coproduct $\Delta : A^* \longrightarrow A^* \otimes A^*$
- The counit $\epsilon : A^* \longrightarrow F$

The associativity and unity conditions are carried over as coassociativity and counit conditions.

Coalgebras

A coalgebra C is a vector field over F with a coproduct Δ and a counit ϵ such that:

$$(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$$

$$(id \otimes \epsilon) \circ \Delta = id = (\epsilon \otimes id) \circ \Delta$$

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$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "id \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes id"] & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "id"] & C \otimes C \arrow[d, "id \otimes \epsilon"] \\
C \otimes C \arrow[r, "\epsilon \otimes id"] & F \otimes C \cong C \cong C \otimes F
\end{tikzcd}$$

Bialgebras

A bialgebra B is both an algebra and a coalgebra over the field F. The coalgebra and the algebra structure should be compatible.

$$\Delta(ab) = \Delta(m(a \otimes b)) = m_{B\otimes B}(\Delta(a) \otimes \Delta(b))$$
$$\Delta(1) = 1 \otimes 1$$

and

$$\epsilon(ab) = \epsilon(m(a \otimes b)) = m(\epsilon(a) \otimes \epsilon(b)) = \epsilon(a)\epsilon(b)$$
$$\epsilon(1) = 1$$

where $m_{B\otimes B} = (m \otimes m) \circ (id \otimes \tau \otimes id)$ so that:

$$m_{B\otimes B}((a \otimes b) \otimes (c \otimes d)) = m(a \otimes c) \otimes m(b \otimes d) = (ac) \otimes (bd)$$

Bialgebra Diagrams

$$
\begin{tikzcd}
 & B \arrow[dr, "\Delta"] & \\
B \otimes B \arrow[ur, "m"] \arrow[d, "\Delta \otimes \Delta"] & & B \otimes B \\
B \otimes B \otimes B \otimes B \arrow[rr, "id \otimes \tau \otimes id"] & & B \otimes B \otimes B \otimes B \arrow[u, "m \otimes m"']
\end{tikzcd}
$$

Bialgebra Diagrams

$$\begin{tikzcd}
F \cong F \otimes F \arrow[rr, "\eta \otimes \eta"] \arrow[dr, "\eta"'] & & B \otimes B \\
& B \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

Bialgebra Diagrams

$$\begin{tikzcd}[column sep=small]
B \otimes B \arrow[rr, "\epsilon \otimes \epsilon"] \arrow[dr, "m"'] & & F \otimes F \cong F \\
& B \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

Bialgebra Diagrams

$$\begin{tikzcd}
F \arrow[rr, "id"] \arrow[dr, "\eta"'] & & F \\
& B \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

Bialgebra Diagrams

All diagrams are symmetric between m, η and Δ, ϵ.

Replace m and η with Δ and ϵ and reverse the arrows, one gets the same diagrams.

The coalgebra is compatible with the algebra structure $\Leftrightarrow$ The algebra is compatible with the coalgebra structure

Hopf Algebras

A Hopf algebra H is a bialgebra with an additional map $S : H \to H$ such that S satisfies:

$$\begin{tikzcd}
H \otimes H \arrow[rr, "S \otimes id"] & & H \otimes H \arrow[d, "m"] \\
H \arrow[u, "\Delta"] \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & F \arrow[r, "\eta"] & H \\
H \otimes H \arrow[rr, "id \otimes S"] & & H \otimes H \arrow[u, "m"']
\end{tikzcd}$$

In terms of elements, this means that if $\Delta(c) = \sum_c c_{(1)} \otimes c_{(2)}$ then

$$\sum_c S(c_{(1)})c_{(2)} = \epsilon(c)1 = \sum_c c_{(1)}S(c_{(2)})$$

Commutativity

An algebra A is commutative if the input to the product is symmetric.

$$xy = yx$$

which means:

$$m(x \otimes y) = m(y \otimes x)$$

Without reference to elements:

$$m = m \circ \tau$$

Cocommutativity

A coalgebra C is cocommutative if the output of coproduct is symmetric.

If $\Delta(c) = \sum_c c_{(1)} \otimes c_{(2)}$, then for a cocommutative coproduct:

$$\sum_c c_{(1)} \otimes c_{(2)} = \Delta(c) = \sum_c c_{(2)} \otimes c_{(1)}$$

Without reference to elements:

$$\Delta = \tau \circ \Delta$$

Hopf Algebra examples

- **Group Algebra:** For a group G, the group algebra FG over F is a Hopf algebra with:

$$\left.\begin{aligned} \Delta(g) &= g \otimes g \\ \epsilon(g) &= 1 \\ S(g) &= g^{-1} \end{aligned}\right\}$$

Cocommutative
Commutativity depends on G.

- **Lie Algebra:** The universal enveloping algebra $U(g)$ of a Lie algebra g is a Hopf algebra with:

$$\left.\begin{aligned} \Delta(x) &= 1 \otimes x + x \otimes 1 \\ \epsilon(x) &= 0 \\ S(x) &= -x \end{aligned}\right\}$$

Cocommutative
Noncommutative

Quantum Groups

The most important Hopf algebras are noncommutative and noncocommutative ones. These are called quantum groups. Motivations:

- A manifold M can be studied by looking at $C(M)$ which is a cocommutative Hopf algebra. If we study noncocommutative Hopf algebras then we can study noncommutative manifold. Thus, we can study noncommutative geometry by looking at quantum groups.
- A group can be described fully by its Hopf algebra. A "deformed" version of the Hopf algebra will enable one to study "deformed" ("quantized") groups. The study of "quantized" Hopf algebras is a study of "quantum" groups.

Quasitriangular Hopf algebra

A quasitriangular Hopf algebra is almost cocommutative. The coproduct satisfies:

$$\tau \circ \Delta = R\Delta R^{-1}$$

for some element R in $H \otimes H$ such that

$$(\Delta \otimes id)(R) = R^{13}R^{13}$$

$$(id \otimes \Delta)(R) = R^{13}R^{12}$$

where if $R = a_i \otimes \ b_i$

$$R^{12} = a_i \otimes \ b_i \otimes \ 1$$

$$R^{13} = a_i \otimes \ 1 \otimes \ b_i$$

$$R^{23} = 1 \otimes \ a_i \otimes \ b_i$$

R-Matrix

For a quasitriangular Hopf algebra, the condition on R leads to the quantum Yang-Baxter eqaution:

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

Every matrix representation of the Hopf algebra gives a matrix solution to the QYBE and quasitriangular Hopf algebras can be categorized by finding solutions to QYBE.

Most interesting quantum groups are quasitriangular.

Quantum matrix groups

Set of n x n matrices M such that each entry m_{ij} belongs to a Hopf algebra with:

$$\triangle(M) = M\dot{\otimes}M$$
$$\epsilon(M) = \mathbb{I}_n$$
$$S(M) = M^{-1}$$

This is shorthand for:

$$\triangle(m_{ij}) = \sum_k m_{ik} \otimes m_{kj}$$
$$\epsilon(m_{ij}) = \delta_{ij}$$
$$\sum_j S(m_{ij})m_{jk} = \delta_{ij} = \sum_j m_{ij}S(m_{jk})$$

$M_q(n)$

An element T of $M_q(n)$ has matrix entries t_{ij} that satisfy:

$$t_{ik}t_{il} = qt_{il}t_{ik} \qquad \text{for } k < l$$

$$t_{ik}t_{jk} = qt_{jk}t_{ik} \qquad \text{for } i < j$$

$$t_{il}t_{jk} = t_{jk}t_{il} \qquad \text{for } i < j,\ k < l$$

$$t_{ik}t_{jl} - t_{jl}t_{ik} = (q - q^{-1})t_{il}t_{jk} \qquad \text{for } i < j,\ k < l$$

for some $q \in \mathbb{C}$.

The coproduct and counit are defined as in a normal quantum matrix group.

The coinverse might not be defined if T is not invertible.

$GL_q(n)$ and $SL_q(n)$

The q-determinant of T:

$$det_q(T) = \sum_{\sigma \in S_n} (-q)^{i(\sigma)} t_{1\sigma(1)} \cdots t_{n\sigma(n)}$$

and T^{-1} is given by:

$$T^{-1} = det_q^{-1}(T) adj(T)^T$$

$\therefore \quad S(T) = T^{-1}$ is defined $\Leftrightarrow det_q(T)$ is invertible.

This condition gives us $GL_q(n)$.

Further require $det_q(T) = 1$, we get $SL_q(n)$.

Quantum group invariance of an algebra

Interesting facts:

- The tensor product of two vector space representations of a Hopf algebra is also a vector space representation
- The product on the Hopf algebra can be extended to the set of vector space representations of a Hopf algebra. Thus, Hopf algebra have algebra representations

Thus, given an algebra A which is a representation of H, if the invariant elements of A form the whole of the algebra then A is said to be invariant under the action of the Hopf algebra H.

The Anticommuting Spin Algebra

Definition:

$$\{J_1, J_2\} = J_3$$

$$\{J_2, J_3\} = J_1$$

$$\{J_3, J_1\} = J_2$$

A (non-exceptional) Jordan algebra where the Jordan product is defined by:

$$A \circ B \equiv \frac{1}{2}(AB + BA)$$

A formal Jordan algebra, in addition to a commutative Jordan product, also satisfies

$$A^2 \circ (B \circ A) = (A^2 \circ B) \circ A$$

The invariance quantum group of ACSA

Transform the generators J_i to J_i' by:

$$J_i' = \sum_j \alpha_{ij} J_j$$

where α_{ij} do not necessarily commute and define:

$$u_{ijk} = \begin{cases} 1, & \text{for } i \neq j \neq k \neq i, \\ 0, & \text{otherwise.} \end{cases}$$

gives us relations among α_{ij} as:

$$\begin{aligned} \alpha_{in}\alpha_{jn} + \alpha_{jn}\alpha_{in} &= 0 && \text{for } i \neq j \\ \alpha_{in}\alpha_{jm} - \alpha_{jm}\alpha_{in} &= 0 && \text{for } i \neq j \text{ and } n \neq m \\ \sum_{n,\,m} \alpha_{in}\alpha_{jm}u_{nml} &= \alpha_{kl} && \text{for } i \neq j \neq k \neq i \end{aligned}$$

$SO_{q=-1}(3)$

We will show that the previous relations are exactly $SO_{q=-1}(3)$ relations.

One can obtain $SO_q(3)$ by imposing on $SL_q(3)$ two conditions:

- Reality condition: $A_{ij} = A^*_{ij}$
- Unitarity condition: $A^\dagger = A^{-1}$

For $SO_q(3)$ one finds that $q = \pm 1$ only.

The matrix elements of a matrix A in $SO_{q=-1}(3)$ satisfy:

$$\left.\begin{aligned} A_{in}A_{jn} &= -A_{jn}A_{in} \\ A_{in}A_{jm} + A_{im}A_{jn} &= A_{jm}A_{in} + A_{jn}A_{im} \\ A_{im}A_{jn} &= A_{jn}A_{im} \\ A_{in}A_{jm} &= A_{jm}A_{in} \end{aligned}\right\} \text{for } i \neq j \text{ and } n \neq m$$

The invariance group of ACSA = $SO_{q=-1}(3)$

First two relations follow from the first and the last equations from $SO_{q=-1}(3)$.

The last invariance relation gives:

$$\alpha_{kl} = \alpha_{in}\alpha_{jm} + \alpha_{im}\alpha_{jn} = \alpha_{jm}\alpha_{in} + \alpha_{jn}\alpha_{im}$$

for i, j, k all different and n, m, l all different.
The definition of the inverse quantum matrix

$$A^{-1} = det^{-1}_{q=-1}(A)adj(A)^T$$

implies for $A \in SO_{q=-1}(3)$:

$$A^T = adj(A)^T \qquad \Rightarrow \qquad A = adj(A)$$

which results in the relation given above.

Representations of ACSA

Define the operators:

$$J_+ = J_1 + J_2$$
$$J_- = J_1 - J_2$$
$$J^2 = J_1^2 + J_2^2 + J_3^2$$

They obey the following relations:

$$\{J_+, J_3\} = J_3$$
$$\{J_-, J_3\} = -J_3$$
$$J_+^2 = J^2 - J_3^2 + J_3$$
$$J_-^2 = J^2 - J_3^2 - J_3$$

Representations of ACSA

J^2 is central in the algebra and J_+, J_- are hermitian.

Label the states with the eigenvalues of J^2 and J_3:

$$J^2 \mid \lambda, \mu \rangle = \lambda \mid \lambda, \mu \rangle$$
$$J_3 \mid \lambda, \mu \rangle = \mu \mid \lambda, \mu \rangle$$

then we get the action of J_+ and J_- as:

$$J_+ \mid \lambda, \mu \rangle = \sqrt{\lambda - \mu^2 + \mu} \mid \lambda, -\mu + 1 \rangle$$
$$J_- \mid \lambda, \mu \rangle = \sqrt{\lambda - \mu^2 - \mu} \mid \lambda, -\mu - 1 \rangle$$

where $\lambda = j(j+1)$ for some j and $j \geq \mu \geq -j$ in order to have positive square norms.

State diagram of ACSA for $j = 0$

$$0$$

State diagram of ACSA for $j = 0$

State diagram of ACSA for $j = 0$

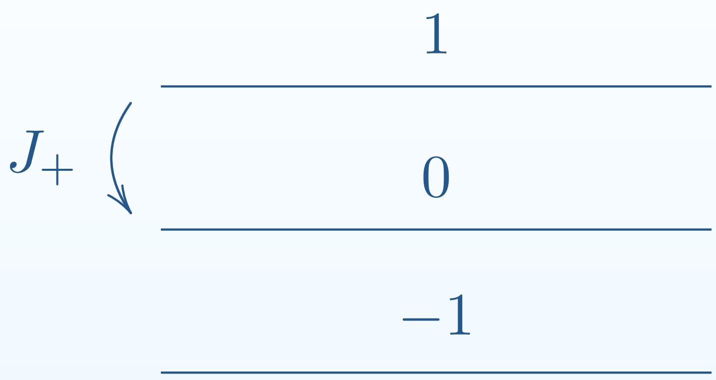
State diagram of ACSA for $j = 1$

1

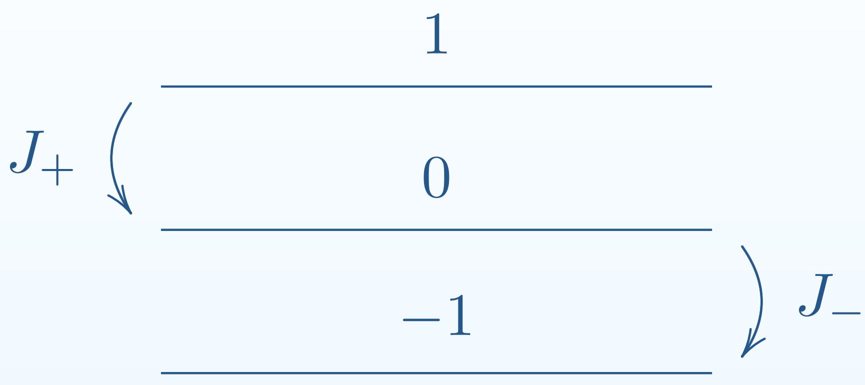
0

-1

State diagram of ACSA for $j = 1$



State diagram of ACSA for $j = 1$



State diagram of ACSA for $j = 2$

2

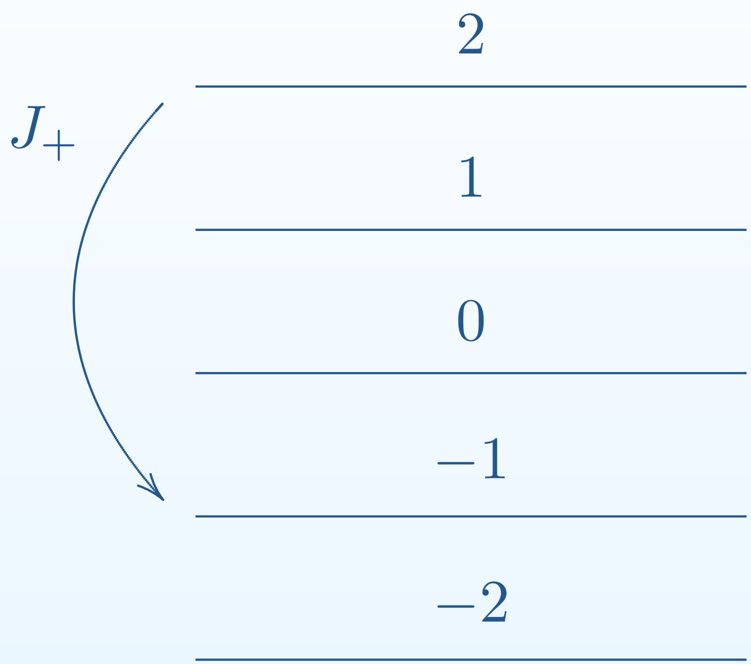
1

0

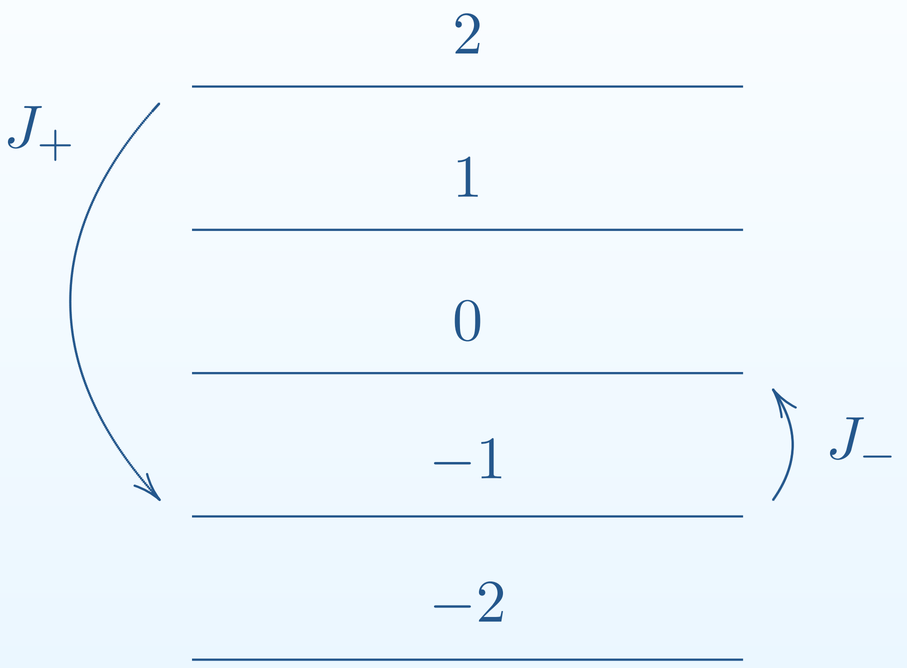
-1

-2

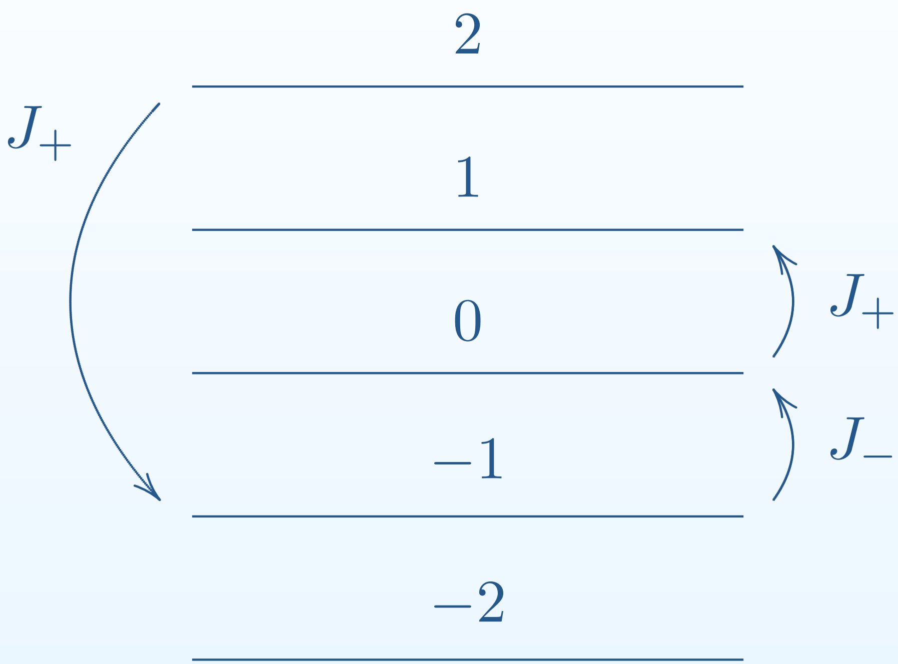
State diagram of ACSA for $j = 2$



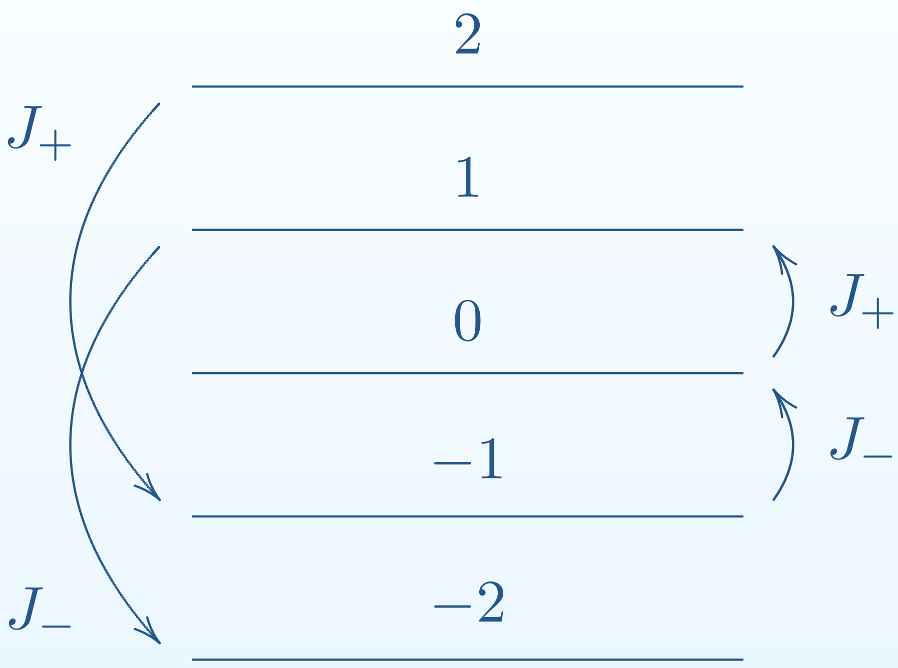
State diagram of ACSA for $j = 2$



State diagram of ACSA for $j = 2$



State diagram of ACSA for $j = 2$

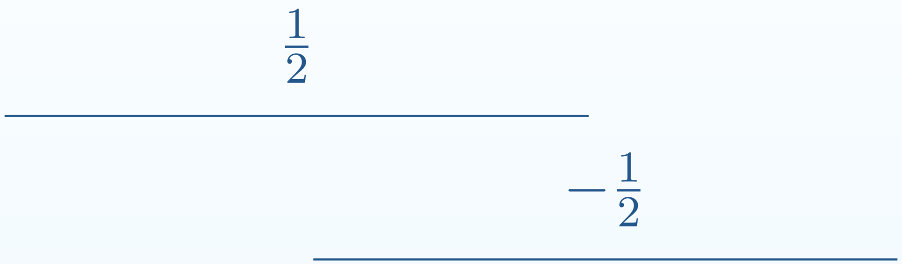


State diagram of ACSA for $j = \frac{1}{2}$

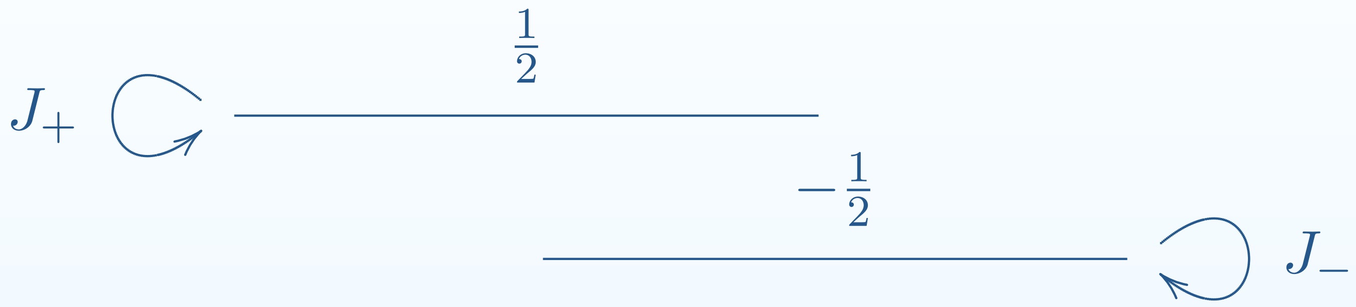
$$\frac{1}{2}$$

$$-\frac{1}{2}$$

State diagram of ACSA for $j = \frac{1}{2}$



State diagram of ACSA for $j = \frac{1}{2}$



State diagram of ACSA for $j = \frac{3}{2}$

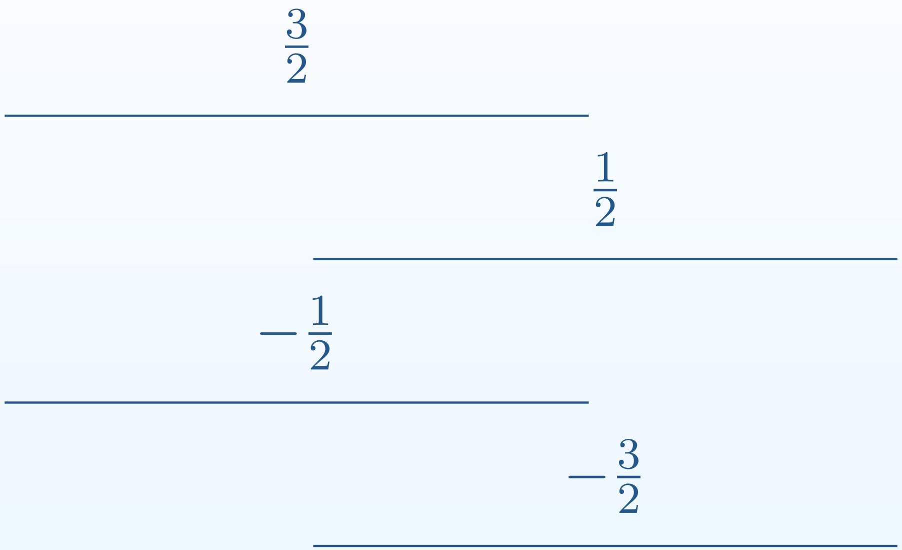
$$\frac{3}{2}$$

$$\frac{1}{2}$$

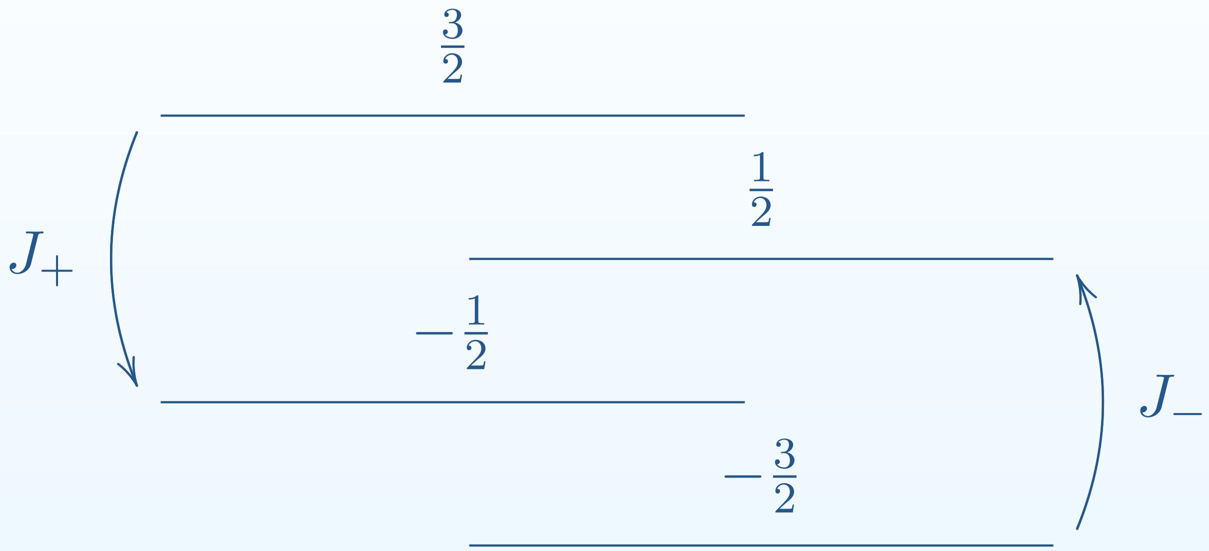
$$-\frac{1}{2}$$

$$-\frac{3}{2}$$

State diagram of ACSA for $j = \frac{3}{2}$



State diagram of ACSA for $j = \frac{3}{2}$



Hopf Algebra Structure with braiding

$SU(2)$ and $ACSA$ are very closely related:

Observe that if I_i is a generator of $SU(2)$ then:

$$\tilde{J}_i = -I_i \otimes \sigma_i$$

satisfies the ACSA relations. Similarly:

$$\tilde{I}_i = J_i \otimes \sigma_i$$

satisfies $SU(2)$ relations.

Try to find a coproduct using the coproduct of $SU(2)$:

$$\Delta(I_i) = 1 \otimes I_i + I_i \otimes 1$$

Hopf Algebra Structure with braiding

Redefine the action of τ:

$$\tau(A \otimes B) = B \otimes A$$

by introducing grading; thus:

$$\tau(A \otimes B) = (-1)^{deg\ A\ deg\ B}\ B \otimes A$$

The Hopf algebra relations remain invariant.

Define the degree of J_1, J_2, J_3 as 1 and degree of 1 as 0. Then, the coproduct:

$$\Delta(J_i) = 1 \otimes J_i + J_i \otimes 1$$

satisfies the Hopf algebra axioms.

The Bosonic Inhom. Symplectic Quantum Group

Consider the multiparticle boson algebra:

$$c_i c_j - c_j c_i = 0$$
$$c_i c_j^* - c_j^* c_i = \delta_{ij}$$

being transformed by:

$$\begin{pmatrix} c' \\ c^{*\prime} \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ 0 & 0 & 1 \end{pmatrix} \dot{\otimes} \begin{pmatrix} c \\ c^* \\ 1 \end{pmatrix}$$

The transformation matrix is inhomogeneous and entries are non-commuting.

The Bosonic Inhom. Symplectic Quantum Group

In order for the boson algebra to be invariant, the matrix elements should satisfy:

$$
\left.
\begin{aligned}
\gamma_i\gamma_j^* - \gamma_j^*\gamma_i &= \delta_{ij} - \alpha_{ik}\alpha_{jk}^* + \beta_{ik}\beta_{jk}^* \\
\gamma_i\gamma_j - \gamma_j\gamma_i &= \beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} \\
\alpha_{ij}\gamma_k - \gamma_k\alpha_{ij} &= 0 \\
\beta_{ij}\gamma_k - \gamma_k\beta_{ij} &= 0 \\
\alpha_{ij}\gamma_k^* - \gamma_k^*\alpha_{ij} &= 0 \\
\beta_{ij}\gamma_k^* - \gamma_k^*\beta_{ij} &= 0
\end{aligned}
\right| \quad \alpha_{ij}, \beta_{ij}, \alpha_{ij}^*, \beta_{ij}^* \text{ commute}
$$

We call this the Bosonic Inhomogeneous Symplectic Quantum Group, $BISp(2d, \mathbb{R})$

$BISp(2d,\mathbb{R})$ - Hopf Algebra structure

In terms of the matrix M:

$$M = \left(\begin{array}{cc|c} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & \Gamma \\ \hline 0 & 1 \end{array}\right)$$

the Hopf algebra structure is given by:

$$\Delta(M) = M\dot{\otimes}M$$
$$\epsilon(M) = I$$
$$S(M) = M^{-1}$$

and as such it is a quantum matrix group. It is also a quasitriangular Hopf algebra with an R-matrix formulation.

$BISp(2d, \mathbb{R})$ - Subgroups

Impose the following conditions to get the subgroups below:

$$\delta_{ij} - \alpha_{ik}\alpha^*_{jk} + \beta_{ik}\beta^*_{jk} = \beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} = 0 \qquad \text{(1a)}$$

$$\gamma_i = 0 \qquad \text{(1b)}$$

$$\beta_{ij} = 0 \qquad \text{(1c)}$$

$$\alpha_{ij} = 0 \qquad \text{(1d)}$$

$$\begin{tikzcd}
BISp(2d,\mathbb{R}) \arrow[r, "\text{(1a)}"] \arrow[d, "\text{(1c)}"'] & ISp(2d,\mathbb{R}) \arrow[r, "\text{(1b)}"] \arrow[d, "\text{(1c)}"'] & Sp(2d,\mathbb{R}) \arrow[d, "\text{(1c)}"'] \\
BIU(d) \arrow[r, "\text{(1a)}"] \arrow[d, "\text{(1d))}"'] & IU(d) \arrow[r, "\text{(1b)}"] & U(d) \\
BA(d) & &
\end{tikzcd}$$

$BISp(2d, \mathbb{R})$ - Contractions

Rescale γ_i to $\gamma_i/\sqrt{\hbar}$ and take the limit $\hbar \to 0$ to study contractions:

$$
\begin{tikzcd}
BISp(2d,\mathbb{R}) \arrow[r, "\hbar\to 0"] \arrow[d, "\text{(1c)}"'] & IGL(2d,\mathbb{R}) \arrow[d, "\text{(1c)}"'] \\
BIU(d) \arrow[r, "\hbar\to 0"] \arrow[d, "\text{(1d)}"'] & IGL(d,\mathbb{C}) \arrow[d, "\text{(1d)}"'] \\
BA(d) \arrow[r, "\hbar\to 0"] & \mathbb{C}^d
\end{tikzcd}
$$

The Fermionic Inhomogeneous Group $FIO(2d, \mathbb{R})$

Study the invariance of the multiparticle fermion algebra:

$$
\begin{aligned}
c_i c_j + c_j c_i &= 0 \\
c_i c_j^* + c_j^* c_i &= \delta_{ij}
\end{aligned}
$$

under the same inhomogeneous transformation to get:

$$
\left.
\begin{aligned}
\gamma_i \gamma_j^* + \gamma_j^* \gamma_i &= \delta_{ij} - \alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^* \\
\gamma_i \gamma_j + \gamma_j \gamma_i &= -\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} \\
\alpha_{ij}\gamma_k + \gamma_k \alpha_{ij} &= 0 \\
\beta_{ij}\gamma_k + \gamma_k \beta_{ij} &= 0 \\
\alpha_{ij}\gamma_k^* + \gamma_k^* \alpha_{ij} &= 0 \\
\beta_{ij}\gamma_k^* + \gamma_k^* \beta_{ij} &= 0
\end{aligned}
\right| \quad \alpha_{ij}, \beta_{ij}, \alpha_{ij}^*, \beta_{ij}^* \text{ commute}
$$

$FIO(2d, \mathbb{R})$ - Subgroups

Impose the following conditions to get the subgroups below:

$$\delta_{ij} - \alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk} = -\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} = 0 \tag{2a}$$

$$\gamma_i = 0 \tag{2b}$$

$$\beta_{ij} = 0 \tag{2c}$$

$$\alpha_{ij} = 0 \quad . \tag{2d}$$

$$\begin{array}{ccccc}
FIO(2d,\mathbb{R}) & \xrightarrow{\text{(2a)}} & GrIO(2d,\mathbb{R}) & \xrightarrow{\text{(2b)}} & O(2d,\mathbb{R}) \\
\downarrow{\scriptstyle (2c)} & & \downarrow{\scriptstyle (2c)} & & \downarrow{\scriptstyle (2c)} \\
FIU(d) & \xrightarrow{\text{(2a)}} & GrIU(d) & \xrightarrow{\text{(2b)}} & U(d) \\
\downarrow{\scriptstyle (2d)} & & & & \\
FA(d) \approx Cliff(2d) & & & &
\end{array}$$

$FIO(2d,\mathbb{R})$ - Contractions

Again rescale γ_i to $\gamma_i/\sqrt{\hbar}$ and take the limit $\hbar \to 0$ to study contractions:

$$\begin{tikzcd}
FIO(2d,\mathbb{R}) \arrow[r, "\hbar\to 0"] \arrow[d, "\text{(2c)}"'] & GrIGL(2d,\mathbb{R}) \arrow[d, "\text{(2c)}"'] \\
FIU(d) \arrow[r, "\hbar\to 0"] \arrow[d, "\text{(2d)}"'] & GrIGL(d,\mathbb{C}) \arrow[d, "\text{(2d)}"'] \\
FA(d) \approx Cliff(2d) \arrow[r, "\hbar\to 0"] & Gr(d,\mathbb{C})
\end{tikzcd}$$

$FIO(2d+1,\mathbb{R})$

Consider the similarity transformation on M as:

$$M \rightarrow UMU^{-1}$$

with

$$U = \left(\begin{array}{cc|c} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

One gets the real form of M after the transformation as:

$$\left(\begin{array}{cc|c} Re(\alpha+\beta) & Im(\alpha-\beta) & \sqrt{2}Re(\gamma) \\ -Im(\alpha+\beta) & Re(\alpha-\beta) & -\sqrt{2}Im(\gamma) \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & \Gamma \\ \hline 0 & 1 \end{array}\right)$$

$FIO(2d+1,\mathbb{R})$

In the real form, for $FIO(2d,\mathbb{R})$, the non-trivial relations:

$$\gamma_i\gamma_j^* + \gamma_j^*\gamma_i = \delta_{ij} - \alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*$$
$$\gamma_i\gamma_j + \gamma_j\gamma_i = -\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk}$$

can be cast into a single equation:

$$\{\Gamma_i, \Gamma_j\} = \delta_{ij} - A_{ik}A_{jk} \quad , \quad i,j = 1,2,\ldots,2d.$$

Using this form as the defining relation, there is no more restrictions on the dimension of the fermionic inhomogeneous algebra and one can extend it to $FIO(2d+1,\mathbb{R})$.

$FIO(2d+1,\mathbb{R})$ - Subgroups

If one considers imposing the relations:

$$\delta_{ij} - A_{ik}A_{jk} = 0 \tag{3a}$$

$$\Gamma_i = 0 \tag{3b}$$

$$A_{ij} = 0 \tag{3c}$$

one gets the subgroup diagram:

$$\begin{tikzcd} FIO(2d+1,\mathbb{R}) \arrow[r, "\text{(3a)}"] \arrow[d, "\text{(3c)}"'] & GrIO(2d+1,\mathbb{R}) \arrow[r, "\text{(3b)}"] & O(2d+1,\mathbb{R}) \\ Cliff(2d+1) & & \end{tikzcd}$$

Conclusions

- Quantized classical systems or non-classical systems have deformed invariance groups.
- Inhomogeneous quantum groups are novel and interesting. In 3 dimensions, the inhomogeneous invariance quantum group of the bosonic oscillator is also a "quantum" symmetry of the phase space.
- Both the fermionic and the bosonic inhomogeneous quantum groups can be extended to infinite dimension by considering their action on fermions-bosons with continuous indices.
- $BISp(2d, \mathbb{R})$ and $FIO(2d, \mathbb{R})$ can be considered to be deformations of bosons and fermions respectively.
- ACSA and its close resemblance to $SU(2)$ is very interesting