QUANTUM GROUP STRUCTURES ASSOCIATED WITH INVARIANCES OF SOME PHYSICAL ALGEBRAS

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ABSTRACT

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In this study, the anticommuting spin algebra is introduced and it is shown to be invariant under the action of the quantum group $SO_{q=-1}(3)$. Furthermore, its representations and Hopf algebra structure are studied and found to be closely resemble the similar results for the angular momentum algebra. The invariance properties of the bosonic and fermionic oscillator algebras under inhomogeneous transformations are also studied. The bosonic inhomogeneous symplectic group, $BISp(2d, \mathbb{R})$, and the fermionic inhomogeneous orthogonal group, $FIO(2d, \mathbb{R})$, are defined as the inhomogeneous invariance quantum groups of these algebras. The sub(quantum)groups and contractions of these quantum groups are studied as a source for new quantum groups. Finally, the fermionic inhomogeneous orthogonal quantum group is defined for odd number of dimensions and its sub(quantum)groups and contractions are studied. inho

ÖZET

BAZI FİZİKSEL CEBİRLERİN DEĞİŞMEZLİĞİ İLE İLGİLİ KUANTUM GRUP YAPILARI

Bu çalışmada, ters-değişmeli spin cebri tanımlanmış ve bu cebrin $SO_{q=-1}(3)$ kuantum grubu altında değişmezliği gösterilmiştir. Bunun ötesinde, bu cebrin temsilleri ve Hopf cebir yapısı incelenmiş ve açısal momentum cebri için bulunmuş olanlara çok benzer yapılara varılmıştır. Bozonik ve fermiyonik osilatör cebirlerinin homojen olmayan değişmezlik özellikleri incelenmiş ve bunların sonucunda bozonik inhomojen simplektik group, $BISp(2d, \mathbb{R})$, ve fermiyonik inhomojen ortogonal group, $FIO(2d, \mathbb{R})$, değişmezlik kuantum grupları olarak tanımlanmıştır. Bu kuantum gruplarının alt(kuantum)grupları ve büzülmeleri yeni kuantum grup kaynakları olarak incelenmiştir. Son olarak, fermiyonik inhomojen ortogonal kuantum grup tek boyutlarda da tanımlanmış ve bu kuantum grubunun da alt(kuantum)grupları ve büzülmeleri incelenmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

a	Annihilation operator
$a^\dagger$	Creation operator
id	Identity map
m	Multiplication map
N	Number operator
S	Coinverse map
Δ	Coproduct map
ϵ	Counit map
η	Unit map
τ	Permutation map
σ_i	Pauli sigma matrices
$\otimes$	Tensor product
$\dot{\otimes}$	Matrix multiplication with tensor product
$\circ$	Function composition
ACSA	The Anticommuting Spin Algebra
BISp	The Bosonic Inhomogeneous Symplectic Quantum Group
FIO	The Fermionic Inhomogeneous Orthogonal Quantum Group

1. INTRODUCTION

1.1. Bosons and Fermions

The concept of bosonic and fermionic particles is one of the most important concepts in modern quantum physics. The behavior of large scale matter, from chemical properties of elements to superconductivity and superfluidity can mostly be understood by referring to the fermionic or bosonic nature of the quantum mechanical particles involved in such phenomena. It is for this reason that understanding the symmetry properties of these phenomena and, motivated by their importance, trying to find other behavior that mimic them is very meaningful.

Furthermore, while bosonic behavior has a classical counterpart, the concept of a fermionic particle is one that can only exist in the quantum domain. This fact makes the study of such behavior even more important. However, what could be more interesting is the study of other such constructs that cannot have a classical counterpart. These constructs would thus belong solely in the quantum domain and could help us understand phenomena that are strictly quantum mechanical in nature.

There is a strong relation between the spin properties of a particle and the particle being a boson or a fermion. In fact, it is a proven fact of quantum physics that integer spin particles are bosons and half-integer spin particles are fermions. This is most often referred to as the *spin-statistics theorem* in quantum mechanics and is a very interesting fact since it implies a relationship between two concepts that seem to be totally unrelated. This strong relation between the bosonic/fermionic nature of a particle and its spin makes the angular momentum algebra also very central in quantum physics.

Before we start investigating such matters, it would be apt to give an overview of the state of bosons and fermions as it has been studied up to now.

When the harmonic oscillator is studied in a quantum mechanical manner [1], one arrives at the relation:

$$aa^\dagger - a^\dagger a = 1 \tag{1.1}$$

to describe the system. The Hamiltonian of this system is given by $\frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a)$. The spectrum of this Hamiltonian, which in turn gives us the allowable energy levels of the quantum harmonic oscillator, can be obtained easily by introducing the hermitian operator $N = a^\dagger a$ which satisfies the following relations with a and $a^\dagger$:

$$[N, a^\dagger] = a^\dagger \tag{1.2}$$

$$[N, a] = a \tag{1.3}$$

where [,] denotes the usual commutator. By observing the fact that the Hamiltonian is nothing but $\hbar\omega(N + \frac{1}{2})$, one can see that one can get the states that correspond to the energy levels as eigenvectors $\mid n \rangle$, of the operator N. The action of $a^\dagger$ and a on such an eigenvector $\mid n \rangle$ is found to be:

$$a^\dagger \mid n \rangle = \sqrt{n+1} \mid n+1 \rangle \tag{1.4}$$

$$a \mid n \rangle = \sqrt{n} \mid n-1 \rangle \tag{1.5}$$

Due to the fact that the operator N is a positive hermitian operator, its eigenvalues, namely n, cannot be negative. For a given positive value of n, however, one can construct states with eigenvalues $n-1$, $n-2$, $n-3$, and so on, by repeatedly applying the operator a on the original state. This sequence of eigenvalues will contain negative values eventually for any given finite n unless it is an integer. In that case, the sequence will end at the eigenvalue 0 since a further application of the operator a on that state will give us the zero vector of the Hilbert space which is not a physically observable state and is thus a state out of our domain.

As a result of this study one finds that the values of n, the eigenvalues of the operator N, begin from 0 and increase by 1 every time $a^\dagger$ is applied on the relevant state and that the energy levels of the quantum harmonic oscillator are given by $\hbar\omega(n+\frac{1}{2})$. The operators $a^\dagger$ and a turn out to be operators that create and destroy, respectively, one quanta of energy and for this reason they are usually called creation and annihilation operators.

Even though this operator algebra seems to only describe the quantum harmonic oscillator, when one studies quantum field theory, this algebra comes up as the algebra of the Fourier coefficients of the field operator describing a bosonic particle. Each normal mode of a quantum field behaves as if it is an independent harmonic oscillator and for that reason we have a separate set of creation and annihilation operators for each of these modes. In that setting, the operators $a_p^\dagger$ and a_p, which now carry a continuous momentum index, are interpreted as the operators that create and destroy, respectively, one bosonic particle of such a field with momentum p.

For fermionic particles the story is a little bit more different. In 1925, Pauli first proposed his *exclusion principle* [2] to explain the behavior of electrons in an atom. According to this principle, no two electrons could exist in the same quantum state and it was for this reason that electrons could not all occupy the lowest energy state in the atomic orbitals but instead had to line up the energy levels in a well ordered manner. The implication of this principle to the electron gas was first considered by Fermi and Dirac and it is for this reason that particles that obey these statistics are called *fermions*. In 1926 Dirac noted [3] that the exclusion principle could also apply to other particles by relating bosons and fermions to the symmetry of the many-particle wavefunction. If the wavefunction changes sign upon exchange of two particles then those particles would be fermions and they would be bosons if the wavefunction did not change sign. This treatment effectively implies the Pauli exclusion principle since if there were to be two fermionic particles occupying the same quantum state, then upon their exchange the wavefunction would change sign; on the other hand, we expect the wavefunction to be identical to the original one before the exchange since nothing must have changed about the quantum state of the system. For this reason the original wavefunction can be nothing but zero if it is to be equal to its negative in this manner. Thus, by contradiction, one can show that no two fermions can exist in the same quantum state. It was only later, in 1928, that Jordan and Wigner proposed [4] that in order to treat fermions in quantum field theory, their field operators had to anticommute so that the wavefunction could be antisymmetric. They showed that a consistent second-quantization of fermions implied anticommutation relations on the field operators. This is turn implies that the Fourier coefficients of the field operators that belong to a normal mode also obey anticommutation relations instead of the commutation relations that the bosonic creation and annihilation operators obey.

In this work, we would like to give an alternative derivation of this algebra by only starting from the Pauli exclusion principle and assuming that fermionic particles also have creation and annihilation operators just like the bosonic particles. If this is the case then Pauli exclusion principle tells us that we cannot create a second fermion in the same quantum state, i.e. that $(a^\dagger)^2$, and in turn a^2, should be 0. This relation, however, is not compatible with the commutation relation (1.1) and thus should be supplemented with another kind of relation. If we define the operator K as the anticommutator of a and $a^\dagger$:

$$K \equiv aa^\dagger + a^\dagger a \tag{1.6}$$

then we find that K is a central element of the algebra, since:

$$\begin{aligned} a^\dagger K &= a^\dagger(aa^\dagger + a^\dagger a) = & a^\dagger a a^\dagger \qquad (1.7) \\ K a^\dagger &= (aa^\dagger + a^\dagger a)a^\dagger = & a^\dagger a a^\dagger \qquad (1.8) \end{aligned}$$

which implies that K commutes with $a^\dagger$ and similarly with a, thus making it a central element of the algebra. The central operator K can be written as a multiple of the identity $k\mathbb{I}$ and if we rescale the operators a and $a^\dagger$ by $1/\sqrt{k}$, we arrive at the fermion anticommutator algebra:

$$
\begin{aligned}
a^2 &= 0 & (1.9)\\
aa^\dagger + a^\dagger a &= 1 & (1.10)
\end{aligned}
$$

This derivation of the fermion algebra also shows clearly that the physically more important relation is the fact that the square of the annihilation operator is zero, since the other relation follows from this fact. In literature, it is often the case that only the anticommutation relation is presented as describing fermionic particles, completely omitting the other, more important, relation. This is usually falsely motivated by the assumption that the anticommutation relation uniquely describes a fermionic system just as the commutation relation alone describes a bosonic system. However, without the first relation, the anticommutation relation alone describes a completely different system which still has two states but is not equivalent to the fermionic system.

A study of this fermion algebra, similar to the boson algebra, shows that, again, a hermitian positive-definite number operator $N = a^\dagger a$ can be defined and has eigenvalues 0 and 1 that correspond to the states $\mid 0 \rangle$ and $\mid 1 \rangle$, respectively. In harmony with our original assumption, the operator $a^\dagger$ takes the state $\mid 0 \rangle$ to the state $\mid 1 \rangle$ thus fulfilling the interpretation of it as a creation operator. Similarly, the operator a acts as an annihilation operator of the algebra.

1.2. Quantum Groups and Hopf Algebras

The discovery of quantum groups has historically been motivated by the study of quantization of non-linear completely integrable systems [5]. The study of such systems has shown that some non-linear completely integrable systems that possess group symmetries, when quantized, acquire a different kind of symmetry; a symmetry under quantum groups. By definition quantum groups are non-commutative and non-cocommutative Hopf algebras and thus the physical importance of quantum groups and Hopf algebras, in general, is very great since the aforementioned discovery.

In order to give an overview of the definition of a Hopf algebra and the motivations behind these definition, we will start from the definition of an associative algebra and starting form that definition give definitions of coalgebra, bialgebra and Hopf algebra.

1.2.1. Associative Algebras

In abstract mathematics, an associative algebra A over a field F is defined to be a vector space over F with an F bilinear multiplication $m : A \otimes A \to A$ (where the image of $(x, y) \in A \otimes A$ which is $m(x, y)$ is usually written as xy) such that the associativity law:

$$(xy)z = x(yz) \quad \text{for all } x, y, z \in A \tag{1.11}$$

is satisfied. This associativity condition can also be written without reference to any of the elements of the algebra A by first considering that the condition is equivalent to:

$$m \circ (m(x, y), z) = m \circ (x, m(y, z)) \quad \text{for all } x, y, z \in A, \tag{1.12}$$

where $\circ$ denotes functional composition, and then realizing that the *for all* condition can be expressed as:

$$m \circ (m(A \otimes A) \otimes A) = m \circ (A \otimes m(A \otimes A)) \quad . \tag{1.13}$$

If we further define the identity operator on A as $id(x) = x$ for all $x \in A$, then we can write the above form as:

$$m \circ (m \otimes id)(A \otimes A \otimes A) = m \circ (id \otimes m)(A \otimes A \otimes A) \quad , \tag{1.14}$$

where it is obvious that we can drop the $A \otimes A \otimes A$ terms from both sides of the equation without losing the expressive power of the relation. Thus we end up with:

$$m \circ (m \otimes id) = m \circ (id \otimes m) \tag{1.15}$$

for the definition of associativity of the product on an algebra A. This form of *element free notation*, where appropriate, will be used in this work from this point on.

An associative algebra is called unital if the algebra A contains an identity element 1 such that $1x = x1 = x$ for all $x \in A$. Such a unital algebra is also a ring and contains all the elements of the field F by identifying an element k of the field with the algebra element $k1$. This identification can be expressed as the existence of a unit map $\eta : F \to A$ which has the property:

$$m \circ (id \otimes \eta) = s = m \circ (\eta \otimes id) \tag{1.16}$$

where s is the scalar multiplication $s : F \otimes A \to A$ such that $s(k, x) = kx$. Since $F \otimes A$ is isomorphic to the original algebra A, the above relation is sometimes written with id in place of s with scalar multiplication being implicitly understood.

As a result, we can see that the definition of a unital associative algebra is a vector space over a field F with two operations, $m : A \otimes A \to A$ and $\eta : F \to A$ defined such that the operations satisfy:

$$m \circ (m \otimes id) = m \circ (id \otimes m) \tag{1.17}$$

$$m \circ (id \otimes \eta) = id = m \circ (\eta \otimes id) \tag{1.18}$$

These relations can also be written as the condition that the following diagrams commute:

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \circ id"] \arrow[d, "id \circ m"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
$$

Figure 1.1. Associativity in an algebra A

$$
\begin{tikzcd}
F \otimes A \cong A \cong A \otimes F \arrow[r, "id \circ \eta"] \arrow[d, "\eta \circ id"] \arrow[dr, "id"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
$$

Figure 1.2. Existence of unit in the algebra A

1.2.2. Coalgebras

The primary motivation for coalgebras stem from the study of the effect of the multiplication and unity operators defined on an algebra on the dual of that algebra. The dual A^* of an algebra A is defined to be the set of all linear maps from A to F. By this definition, the dual of an algebra is a vector space provided that the addition and scalar multiplication is defined as:

$$(\phi + \psi)(x) = \phi(x) + \psi(x) \tag{1.19}$$

$$(k\phi)(x) = k\phi(x) \tag{1.20}$$

for all $\phi, \psi \in A^*$, $x \in A$ and $k \in F$. The dual does not naturally carry any of the algebra structure of the original algebra and, in general, is not itself an algebra. For this reason, it is very natural to inquire about the effect of multiplication in A on the dual A^*. For this we consider:

$$\phi(xy) = \phi(m(x \otimes y)) \tag{1.21}$$

for $\phi \in A^*$ and $x, y \in A$. This form, in general, is not equal to $\phi(x)\phi(y)$ but it should be possible to write it as a tensor product in terms of other elements of A^* valued at $x \otimes y$. The possibility of this can be shown if A is finite-dimensional. In general, the multiplication $m : A \otimes A \to A$ yields a linear map on the dual $\Delta : A^* \to (A \otimes A)^*$. However, if A is finite-dimensional, $(A \otimes A)^*$ is naturally isomorphic to $(A^* \otimes A^*)$ and for that reason the map on the dual can be written as $\Delta : A^* \to A^* \otimes A^*$. This map is called the coproduct. In terms of the coproduct, the above relation becomes:

$$\phi(xy) = \phi(m(x \otimes y)) = \Delta(\phi)(x \otimes y) \tag{1.22}$$

Similarly, the action of the unit map $\eta : F \to A$ yields a linear map on the dual $\epsilon : A^* \to F$, which is called the counit. The action of the counit is as follows:

$$\phi(k1) = \phi(\eta(k)) = \epsilon(\phi)k \tag{1.23}$$

for $\phi \in A^*$ and $k \in F$. Thus, we see that the multiplication and unit maps on A naturally define the coproduct and counit maps on the dual A^*. Furthermore, the associativity and existence of unit conditions on the algebra A implies certain conditions on the maps defined on the dual A^*. The structure we have thus arrived at is called a coalgebra and the dual of an algebra A becomes a coalgebra.

Formally, the definition of a coalgebra C is a vector space over a field F together with two linear maps:

- Coproduct: $\Delta : C \to C \otimes C$

- Counit: $\epsilon : C \to F$

such that the conditions:

$$(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta \tag{1.24}$$

$$(id \otimes \epsilon) \circ \Delta = id = (\epsilon \otimes id) \circ \Delta \tag{1.25}$$

are satisfied. The first of these conditions is called the coassociativity condition and is equivalent to the fact that Figure 1.3 is commutative. Similarly, the second condition

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "id \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes id"] & C \otimes C \otimes C \end{tikzcd}$$

Figure 1.3. Coassociativity in a coalgebra C

is called the existence of counit and is equivalent to the commutativity of Figure 1.4.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "id"] & C \otimes C \arrow[d, "id \otimes \epsilon"] \\ C \otimes C \arrow[r, "\epsilon \otimes id"] & F \otimes C \cong C \cong C \otimes F \end{tikzcd}$$

Figure 1.4. Existence of counit in the coalgebra C

1.2.3. Bialgebras

Formally, a bialgebra B over a field F is both a unital associative algebra and a coalgebra over F such that the coproduct and counit maps are both algebra homomorphisms. In this respect, the coalgebra structure should be compatible with the algebra structure of the bialgebra. We will also show that the same statement can be expressed from the opposite point of view, ie. that the algebra structure of the bialgebra should be compatible with the coalgebra structure. For this reason, the product and the unit maps should, equivalently, be algebra homomorphisms.

Before analyzing the implications of the compatibility condition, we should define $m_{B\otimes B}$ which is the product defined on $B \otimes B$ using the product defined on B. The map $m_{B\otimes B} : (B \otimes B) \otimes (B \otimes B) \to B \otimes B$ is a formalization of the product rule $(a \otimes b)(c \otimes d) = (ac) \otimes (bd)$ and for this reason the action of this map is defined by:

$$m_{B\otimes B}((a \otimes b) \otimes (c \otimes d)) = m(a \otimes c) \otimes m(b \otimes d) \quad . \tag{1.26}$$

One should notice that the definition of this product involves a permutation of the order of the terms b and c. Using this fact and defining the permutation operator $\tau : B \otimes B \to B \otimes B$ by:

$$\tau(a \otimes b) = b \otimes a \quad , \tag{1.27}$$

we can rewrite the action of the product map on $B \otimes B$ as:

$$\begin{aligned} m_{B\otimes B}((a \otimes b) \otimes (c \otimes d)) &= m(a \otimes c) \otimes m(b \otimes d) \\ &= (m \otimes m)(a \otimes c \otimes b \otimes d) \\ &= (m \otimes m) \circ (id \otimes \tau \otimes id)(a \otimes b \otimes c \otimes d) \quad . \end{aligned} \tag{1.28}$$

As a result, we find that $m_{B\otimes B}$ is defined in terms of the product map on B as:

$$m_{B\otimes B} = (m \otimes m) \circ (id \otimes \tau \otimes id) \quad . \tag{1.29}$$

The statement that the coproduct map is a algebra homomorphism implies that:

$$\Delta(ab) = \Delta(m(a \otimes b)) = m_{B\otimes B}(\Delta(a) \otimes \Delta(b)) = \Delta(a)\Delta(b) \tag{1.30}$$

$$\Delta(1) = 1 \otimes 1 \tag{1.31}$$

These two equations say that the action of the coproduct map respects both the product and the unit of the algebra structure in the bialgebra. Similarly, the condition that the counit is an algebra homomorphism implies:

$$\epsilon(ab) = \epsilon(m(a \otimes b)) = m_F(\epsilon(a) \otimes \epsilon(b)) = \epsilon(a)\epsilon(b) \tag{1.32}$$

$$\epsilon(1) = 1 \tag{1.33}$$

where m_F stands for the product on the field F.

The content of these relations which define a bialgebra can also be expressed by the commutative diagrams in Figures 1.5 and 1.6 for the homomorphism conditions on the coproduct and Figures 1.7 and 1.8 for the homomorphism conditions on the counit.

$$\begin{tikzcd}
B \otimes B \arrow[r, "\Delta \otimes \Delta"] \arrow[d, "m"] & (B \otimes B) \otimes (B \otimes B) \arrow[d, "m_{B\otimes B}"] \\
B \arrow[r, "\Delta"] & B \otimes B
\end{tikzcd}$$

Figure 1.5. Compatibility of the coproduct with the product on the bialgebra B

$$\begin{tikzcd}
F \cong F \otimes F \arrow[rr, "\eta \otimes \eta"] \arrow[dr, "\eta"'] & & B \otimes B \\
& B \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

Figure 1.6. Compatibility of the coproduct with the unit on the bialgebra B

$$\begin{tikzcd}
B \otimes B \arrow[rr, "\epsilon \otimes \epsilon"] \arrow[dr, "m"'] & & F \otimes F \cong F \\
& B \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

Figure 1.7. Compatibility of the counit with the product on the bialgebra B

$$\begin{tikzcd}
F \arrow[rr, "id"] \arrow[dr, "\eta"'] & & F \\
& B \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

Figure 1.8. Compatibility of the counit with the unit on the bialgebra B

One can see from these commutative diagrams, that the diagrams are completely symmetric with respect to the coalgebra and algebra maps. In other words, one can see that these diagrams can also be read as the coalgebra homomorphism conditions of the product and the unit maps of the algebra structure of the bialgebra B. The only diagram that does not explicitly exhibit this symmetry is Figure 1.5. This diagram, however, can be written in an explicitly symmetric way by using the definition of $m_{B \otimes B}$ to produce the commutative diagram shown in Figure 1.9. This way the content of all the diagrams can be read both as the compatibility of the coalgebra maps on the algebra structure and the compatibility of the algebra maps on the coalgebra structure of the bialgebra B.

$$
\begin{tikzcd}
 & B \arrow[dr, "\Delta"] & \\
B \otimes B \arrow[ur, "m"] \arrow[d, "\Delta \otimes \Delta"] & & B \otimes B \\
B \otimes B \otimes B \otimes B \arrow[rr, "id \otimes \tau \otimes id"] & & B \otimes B \otimes B \otimes B \arrow[u, "m \otimes m"']
\end{tikzcd}
$$

Figure 1.9. Compatibility of the coproduct with the product on the bialgebra B

1.2.4. Hopf Algebras

A Hopf algebra H is basically a bialgebra, ie. both a unital associative algebra and a coalgebra, with an additional structure called the coinverse (or the antipode) which is a linear map $S : H \to H$ such that the diagram in Figure 1.10 is commutative.

$$
\begin{tikzcd}
H \otimes H \arrow[rr, "S \otimes id"] & & H \otimes H \arrow[d, "m"] \\
H \arrow[u, "\Delta"] \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & F \arrow[r, "\eta"] & H \\
H \otimes H \arrow[rr, "id \otimes S"] & & H \otimes H \arrow[u, "m"']
\end{tikzcd}
$$

Figure 1.10. Definition of coinverse on the Hopf algebra H

In order to write the concept of a coinverse more explicitly, we will introduce Sweedler's [6] notation which can be considered to be the analogue of Einstein summation convention for coproducts. Given an element c of a coalgebra, there exists elements $c^i_{(1)}$ and $c^i_{(2)}$ in the coalgebra such that:

$$\Delta(c) = \sum_i c^i_{(1)} \otimes c^i_{(2)} \quad . \tag{1.34}$$

Using Sweedler's notation, this can be abbreviated to:

$$\Delta(c) = \sum_c c_{(1)} \otimes c_{(2)} \tag{1.35}$$

and in the sumless version of Sweedler's notation, it further becomes:

$$\Delta(c) = c_{(1)} \otimes c_{(2)} \tag{1.36}$$

Thus the coinverse map S can also be expressed as:

$$S(c_{(1)})c_{(2)} = m(S(c_{(1)}) \otimes c_{(2)}) = \epsilon(c)1 = m(c_{(1)} \otimes S(c_{(2)})) = c_{(1)}S(c_{(2)}) \tag{1.37}$$

The notion of commutativity in a Hopf algebra is defined by the commutativity of the product map of the algebra structure. An algebra is commutative if and only if the product map satisfies the relation:

$$m = m \circ \tau \tag{1.38}$$

so that the order of multiplying terms in the product does not matter. In terms of elements of the algebra this relation becomes:

$$m(a \otimes b) = m(b \otimes a) \tag{1.39}$$

for all elements a, b of the algebra. Similarly, the notion of cocommutativity in a Hopf algebra is defined by the cocommutativity of the coproduct map of the coalgebra structure. A coalgebra is cocommutative if and only if the coproduct map satisfies the relation:

$$\Delta = \tau \circ \Delta \tag{1.40}$$

so that the order of terms in the outcome of the coproduct does not matter. In terms of elements of the coalgebra and using sumless Sweedler's notation, this relation implies:

$$\Delta(c) = c_{(1)} \otimes c_{(2)} = c_{(2)} \otimes c_{(1)} \tag{1.41}$$

for all elements c of the coalgebra.

There are various examples of Hopf algebras. Out of these the most important examples are the group algebras and universal enveloping algebras of Lie algebras. Given a group G, the group algebra FG is a unital associative algebra over the field F. It becomes a Hopf algebra, if we define the coproduct, counit and coinverse maps by:

$$\Delta(g) = g \otimes g \tag{1.42}$$

$$\epsilon(g) = 1 \tag{1.43}$$

$$S(g) = g^{-1} \tag{1.44}$$

for all $g \in G$. In this instance the resulting Hopf algebra is always cocommutative (since $g \otimes g = \tau(g \otimes g)$) and is commutative depending on the original group G being abelian or not. If the underlying group G is abelian, the resulting Hopf algebra is both cocommutative and commutative. Otherwise, it is cocommutative but noncommutative.

Similarly, given a Lie algebra g over a field F, its universal enveloping algebra

$U(g)$ is a unital associative algebra. This algebra $U(g)$ becomes a Hopf algebra if we define the coproduct, counit and the coinverse maps as:

$$\Delta(x) = 1 \otimes x + x \otimes 1 \tag{1.45}$$

$$\epsilon(x) = 0 \tag{1.46}$$

$$S(x) = -x \tag{1.47}$$

for all $x \in U(g)$. Notice that the coproduct rule is not only compatible with the product on the universal enveloping algebra but it is also compatible with the antisymmetric product defined on the Lie algebra itself. This Hopf algebra is cocommutative (since $1 \otimes x + x \otimes 1 = x \otimes 1 + 1 \otimes x$) but noncommutative.

Quantum groups are, loosely, defined as Hopf algebras that are neither commutative nor cocommutative. As such, they are important in non-commutative geometry. The reason for this stems from the observation that in order to study geometry on a manifold M, it is possible to work with the algebra of functions $A = C(M)$ on M which is a Hopf algebra. Thus, one can continue studying Hopf algebras, including noncommutative and noncocommutative ones, and do geometry with them even though the underlying manifold does not exist anymore in a conventional sense. They are called quantum groups because of a similar reasoning stating that a standard algebraic group is well described by the Hopf algebra of regular functions on the algebraic group and that a deformed version of the Hopf algebra should, in some sense, describe a deformed, quantized version of the algebraic group. In essence, identifying these quantized algebraic groups with their Hopf algebras one can study them in full generality and make a theory of these quantum groups.

Since it is essentially the noncocommutativity of a Hopf algebra that makes it interesting, it is natural for there to be a mathematical property to quantify the amount of its noncocommutativity. This is analogous to the definition of the commutator to describe the amount of noncommutativity of an associative algebra. Thus, a quasi-triangular Hopf algebra is defined as a Hopf algebra H, where there is an invertible element R in $H \otimes H$, such that it satisfies:

$$\tau \circ \Delta = R \Delta R^{-1} \tag{1.48}$$

$$(\Delta \otimes id)(R) = R^{13} R^{13} \tag{1.49}$$

$$(id \otimes \Delta)(R) = R^{13} R^{12} \tag{1.50}$$

where if $R = a_i \otimes \ b_i$ then R^{12}, R^{13} and R^{23} are defined as:

$$R^{12} = a_i \otimes \ b_i \otimes \ 1 \tag{1.51}$$

$$R^{13} = a_i \otimes \ 1 \otimes \ b_i \tag{1.52}$$

and

$$R^{23} = 1 \otimes \ a_i \otimes \ b_i \quad . \tag{1.53}$$

As can be seen, in a quasitriangular Hopf algebra the coproduct is almost cocommutative up to a conjugation by the invertible element R. Moreover, if one works with the equations given above, one can arrive at a matrix equation for R given by the quantum Yang-Baxter equation:

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12} \tag{1.54}$$

which plays a fundamental role in the theory of completely integrable systems [7]. If one starts from this matrix equation for R, one can start categorizing the solutions to the matrix equation and thus categorize quasitriangular Hopf algebras. Equivalently, every matrix representation of a quasitriangular Hopf algebra, implies a matrix representation of R and as such gives one a solution to the quantum Yang-Baxter equation. Thus, from a single Hopf algebra, it is possible to extract many solutions to this equation by using different matrix representations. This is the reason why the element R of $H \otimes H$ is sometimes called the universal R-matrix. It is mostly for these reasons that commonly studied physical Hopf algebras and quantum groups are generally quasitriangular.

1.3. Quantum Matrix Groups

A quantum matrix group is defined by a set of n x n matrices M:

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix} \tag{1.55}$$

such that every element of the matrix belong to a Hopf algebra H. The matrix group defined in this way naturally becomes a Hopf algebra with the coproduct, counit and coinverse of the matrix algebra being defined as:

$$\triangle(M) = M\dot{\otimes}M \tag{1.56}$$

$$\epsilon(M) = \mathbb{I}_n \tag{1.57}$$

$$S(M) = M^{-1} \tag{1.58}$$

where $\dot{\otimes}$ stands for the operation where when the matrix multiplication is performed the matrix elements are multiplied using the tensor product is instead of the normal product and $\mathbb{I}_n$ stands for the n x n unit matrix. The relations above imply the definitions of the coproduct, counit and coinverse of the matrix elements:

$$\triangle(a_{ij}) = \sum_k a_{ik} \otimes a_{kj} \tag{1.59}$$

$$\epsilon(a_{ij}) = \delta_{ij} \tag{1.60}$$

$$\sum_j S(a_{ij})a_{jk} = \delta_{ij} = \sum_j a_{ij}S(a_{jk}) \tag{1.61}$$

One of the most important examples of quantum matrix groups is the quantum group $GL_q(n)$. This quantum group is a quantum subgroup of the bialgebra $M_q(n)$.

An element T of $M_q(n)$ has matrix entries t_{ij} that satisfy:

$$t_{ik}t_{il} = qt_{il}t_{ik} \qquad \text{for } k < l \tag{1.62}$$

$$t_{ik}t_{jk} = qt_{jk}t_{ik} \qquad \text{for } i < j \tag{1.63}$$

$$t_{il}t_{jk} = t_{jk}t_{il} \qquad \text{for } i < j,\ k < l \tag{1.64}$$

$$t_{ik}t_{jl} - t_{jl}t_{ik} = (q - q^{-1})t_{il}t_{jk} \qquad \text{for } i < j,\ k < l \tag{1.65}$$

for some $q \in \mathbb{C}$. One can immediately see that $M_q(n)$ is a bialgebra with the definitions of coproduct and counit given in equations (1.56) and (1.57). In order to define the coinverse, one needs to define the inverse of such a matrix and for this one should define the quantum analogues of the determinant and the adjoint. For an element T of $M_q(n)$ one defines the *quantum determinant* [8] as:

$$det_q(T) = \sum_{\sigma \in S_n} (-q)^{i(\sigma)} t_{1\sigma(1)} \cdots t_{n\sigma(n)} \tag{1.66}$$

where S_n is the symmetric group on $1, \cdots, n$ and $i(\sigma)$ is the number of adjacent transpositions in the permutation σ. Similar to normal matrices, one can also define the *quantum adjoint* matrix $adj(T) = (a_{ij})$ such that:

$$a_{ij} = (-q)^{j-i} det_q(T^{ij}) \tag{1.67}$$

where T^{ij} stands for the $(n-1)$ x $(n-1)$ matrix obtained from T by deleting the ith row and the jth column. Just as in classical matrices, one has:

$$T \cdot adj(T)^T = adj(T)^T \cdot T = det_q(T)\mathbb{I}_n \tag{1.68}$$

which yields the inverse, and in turn the coinverse, of such a quantum matrix as:

$$S(T) = T^{-1} = det_q^{-1}(T)adj(T)^T \tag{1.69}$$

This coinverse is obviously only defined when $det_q(T)$ is invertible for all T in the quantum matrix group. The quantum subgroup of $M_q(n)$ that satisfies this condition is called $GL_q(n)$, the quantum general linear group of dimension n and this structure is a Hopf algebra since the coinverse is now defined. In analogy with the classical matrix groups, one can further restrict the determinant of such matrices to be equal to 1 and obtain the quantum subgroup $SL_q(n)$, the special linear quantum group of dimension n.

1.4. Quantum Group Invariance of an Algebra

A (left)module over the ring R consists of an abelian group M and the scalar multiplication operation $s : R \otimes M \rightarrow M$, the action of which is usually written as $s(r,x) = rx$ for some r in R and some x in M, and such that:

$$r(x+y) = rx + ry \tag{1.70}$$

$$(r+s)x = rx + sx \tag{1.71}$$

$$(rs)x = r(sx) \tag{1.72}$$

$$1x = x \tag{1.73}$$

for all r, s in R and all x, y in M. Notice that this definition of a module is the same as the definition of a vector space except a module is defined over a ring instead of a field. Thus, every vector space is also a module and a module over K is the same thing as a vector space over K if K is a field.

The reason why the module is defined here is that they gives a representation of a ring or any structure that extends the ring structure. On order to see this, one can consider the scalar multiplication as the action of the ring R on M by sending the element x to rx. This action will be a group endomorphism due to the definition of a module. Thus, if one identifies an element r in R by its action, then one has defined a map from R to $End(M)$ which respects the ring structure. Such a map $R \rightarrow End(M)$ is called a representation of the ring R over the abelian group M. If we consider the representations of a vector space, then these representations also form a vector space such that these representations can be multiplied by the elements of the underlying field and can be added to generate new representations. Since a Hopf algebra is an associative algebra, which itself is a vector space, it also has such representations. However, the interesting fact for Hopf algebras (more specifically bialgebras) is that if U and V are two representations of the Hopf algebra then $U \otimes V$ is also a representation for the Hopf algebra due to the nature of the coproduct. The representation of A in H on $W = U \otimes V$ is given by $\Delta(A) = A_{(1)} \otimes A_{(2)}$ such that $A_{(1)}$ gives the representation on U and $A_{(2)}$ gives the representation on V. Thus, one can form direct products of representations to form new representations for a Hopf algebra.

Starting from this interesting fact for Hopf algebras one can arrive at an even more interesting result. If one were to consider an associative algebra A as a vector representation of an algebra, then the original algebra structure of A would not be preserved since the product of two representations don't even form another representation. However, if we consider A as representation for a Hopf algebra H and if the product m_A on A respects the representation map, then the linear map $\rho : H \otimes A \to A$ is an algebra representation of the Hopf algebra. Thus Hopf algebras accept representations which can also form algebras.

Finally, if A is a representation of a Hopf algebra and X is an element of A such that:

$$c[X] = \epsilon(c)X \tag{1.74}$$

for all c in H, then X is said to invariant under H. The subset of all such invariant elements of a representation A forms a subrepresentation of H. Thus, given an algebra A which is a representation of H, if the invariant elements of A form the whole of the algebra then A is said to be invariant under the action of the Hopf algebra H.

1.5. Summary

This work is divided into four chapters. In the first chapter, an introduction was given to the basic concepts of boson and fermions and basic mathematical structures related to the succeeding chapters were introduced.

In the second chapter, the anticommuting spin algebra (ACSA) will be introduced. In that section it will be shown that the invariance group of ACSA is $SO_{q=-1}(3)$ and that the representations of ACSA show great similarity to the representations of its sister spin algebra. Finally, the exact relationship between ACSA and the spin algebra will be examined and a braided Hopf algebra structure for ACSA will be introduced.

The third chapter deals entirely with the inhomogeneous invariance (quantum) groups of the boson and fermion algebras. The bosonic inhomogeneous symplectic quantum group and the fermionic inhomogeneous orthogonal group will be introduced to describe these invariance conditions. In the subsections of this chapter, the sub(quantum)groups and contractions of these new quantum groups will be studied. Finally, the fermionic inhomogeneous orthogonal group will defined in odd number of dimensions and its sub(quantum)groups will be studied.

The fourth and final chapter is reserved for concluding remarks about the body of work that has been introduced in this study.

2. THE ANTICOMMUTING SPIN ALGEBRA

In recent years quantum groups involving fermions have received widespread attention. These include deformed fermion algebras [9, 10, 11, 12], spin chains [13, 14, 15] and Fermi gases [16]. Motivated by these, we start investigating a fermionic analogue of the angular momentum algebra where the commutator relations are replaced by anticommutator relations.

2.1. Defining Relations

We define the anticommuting spin algebra by the relations:

$$\{J_1, J_2\} = J_3 \tag{2.1}$$

$$\{J_2, J_3\} = J_1 \tag{2.2}$$

$$\{J_3, J_1\} = J_2 \tag{2.3}$$

where J_1, J_2, J_3 are hermitian generators of the algebra. In these expressions the curly bracket denotes the anticommutator

$$\{A, B\} \equiv AB + BA \tag{2.4}$$

so (2.1)-(2.3) should be taken as the definition of an associative algebra. This proposed algebra does not fall into the category of superalgebras in the sense of Berezin-Kac axioms. In particular, the algebra is consistent without grading and there are no (graded) Jacobi relations. As it is defined this algebra falls into the category of a (non-exceptional) Jordan algebra where the Jordan product is defined by:

$$A \circ B \equiv \frac{1}{2}(AB + BA) \quad . \tag{2.5}$$

A formal Jordan algebra, in addition to a commutative Jordan product, also satisfies $A^2 \circ (B \circ A) = (A^2 \circ B) \circ A$. When the Jordan product is given in terms of an anticommutator this relation is automatically satisfied. Just as a Lie algebra where the Lie bracket as defined by the commutator leads to an enveloping associative algebra, a Jordan algebra defined in terms of the above product leads to an enveloping associative algebra which we consider as an algebra of observables.

The physical properties of this system turn out to be similar to those of the angular momentum algebra yet exhibit remarkable differences. Since the angular momentum algebra is alos used to describe various internal symmetries, ACSA could be relevant as well in describing those symmetries.

2.2. The Invariance Quantum Group $SO_{q=-1}(3)$

In order to find the invariance quantum group of this algebra, we transform the generators J_i to J_i' by:

$$J_i' = \sum_j \alpha_{ij} J_j \quad . \tag{2.6}$$

The matrix elements α_{ij} are hermitian since J_i's are hermitian and they commute with J_i's but are not assumed to commute with each other. For the transformed operators to obey the original relations, there should exist some conditions on the α's which define the invariance quantum group of the algebra. It is very convenient at this moment to switch to an index notation that encompasses all three defining relations of the algebra in one index equation. For the angular momentum algebra this is possible by defining the totally anti-symmetric rank 3 pseudo-tensor ϵ_{ijk}. A similar object for ACSA which we will call the fermionic Levi-Civita tensor, u_{ijk}, is defined as:

$$u_{ijk} = \begin{cases} 1, & \text{for } i \neq j \neq k \neq i, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

Thus the defining relations (2.1-2.3) become:

$$\{J_i, J_j\} = \sum_k u_{ijk}\, J_k + 2\delta_{ij}\, J_i^2 \tag{2.8}$$

The second term on the right is needed since when $i = j$ the left-hand side becomes $2J_i^2$. Upon transformation (2.6) we require the algebra relations to remain invariant which means:

$$\{J_i', J_j'\} = J_k' \qquad \text{for } i \neq j \neq k \neq i. \tag{2.9}$$

However, substituting the transformation equations into the left-hand side, we have:

$$\{J_i', J_j'\} = \sum_{k,\, m} (\alpha_{ik}\alpha_{jm} J_k J_m + \alpha_{jm}\alpha_{ik} J_m J_k) \tag{2.10}$$

If one considers the quadratic forms in the universal enveloping algebra of ACSA, then one can see that the symmetric part of these forms resolve to linear forms owing to the defining relations of the algebra. Thus the independent quadratic forms in the algebra are the antisymmetric forms, $[J_m, J_k]$ where $m \neq k$, and the square forms, J_k^2. Using this observation we put the above relation in the form of a linear sum over independent algebra elements:

$$\begin{aligned}
\{J_i', J_j'\} &= \sum_{n,\, m} (\alpha_{in}\alpha_{jm} J_n J_m + \alpha_{jm}\alpha_{in} J_m J_n) \\
&= \frac{1}{2} \sum_{n,\, m} (\alpha_{in}\alpha_{jm}(\{J_n, J_m\} + [J_n, J_m]) + \alpha_{jm}\alpha_{in}(\{J_m, J_n\} + [J_m, J_n])) \\
&= \frac{1}{2} \sum_{\substack{n,\, m \\ n\neq m}} \left(\sum_l (\alpha_{in}\alpha_{jm} + \alpha_{jm}\alpha_{in}) u_{nml} J_l + (\alpha_{in}\alpha_{jm} - \alpha_{jm}\alpha_{in})[J_n, J_m] \right) \\
&\quad + \sum_n (\alpha_{in}\alpha_{jn} + \alpha_{jn}\alpha_{in})\, J_n^2
\end{aligned} \tag{2.11}$$

which should be equal to J'_k for $i \neq j \neq k \neq i$, which in turn gives:

$$\{J'_i, J'_j\} = J'_k \tag{2.12}$$

$$\begin{aligned}
&\sum_n (\alpha_{in}\alpha_{jn} + \alpha_{jn}\alpha_{in})\, J_n^2 \\
&\quad + \frac{1}{2}\sum_{\substack{n,\, m \\ n\neq m}} \left(\sum_l (\alpha_{in}\alpha_{jm} + \alpha_{jm}\alpha_{in}) u_{nml} J_l + (\alpha_{in}\alpha_{jm} - \alpha_{jm}\alpha_{in})[J_n, J_m] \right) \\
&\qquad = \sum_l \alpha_{kl} J_l \qquad \text{for } i \neq j \neq k \neq i.
\end{aligned} \tag{2.13}$$

This final equation yields the following relations among α_{ij} :

$$\alpha_{in}\alpha_{jn} + \alpha_{jn}\alpha_{in} = 0 \qquad \text{for } i \neq j \tag{2.14}$$

$$\alpha_{in}\alpha_{jm} - \alpha_{jm}\alpha_{in} = 0 \qquad \text{for } i \neq j \text{ and } n \neq m \tag{2.15}$$

$$\frac{1}{2}\sum_{\substack{n,\, m \\ n\neq m}} (\alpha_{in}\alpha_{jm} + \alpha_{jm}\alpha_{in}) u_{nml} = \alpha_{kl} \qquad \text{for } i \neq j \neq k \neq i \tag{2.16}$$

However, by virtue of (2.15) and the fact that $u_{ijk} = 0$ if any two indices are the same, the relation (2.16) can be written as:

$$\begin{aligned}
\alpha_{kl} &= \frac{1}{2}\sum_{n,\, m} (\alpha_{in}\alpha_{jm} + \alpha_{jm}\alpha_{in}) u_{nml} \\
&= \frac{1}{2}\sum_{n,\, m} 2\alpha_{in}\alpha_{jm} u_{nml} \\
&= \sum_{n,\, m} \alpha_{in}\alpha_{jm} u_{nml} \quad \text{for } i \neq j \neq k \neq i
\end{aligned} \tag{2.17}$$

Therefore the resulting relations between the α_{ij} that define the invariance group of this algebra becomes:

$$\alpha_{in}\alpha_{jn} + \alpha_{jn}\alpha_{in} = 0 \qquad \text{for } i \neq j \tag{2.18}$$

$$\alpha_{in}\alpha_{jm} - \alpha_{jm}\alpha_{in} = 0 \qquad \text{for } i \neq j \text{ and } n \neq m \tag{2.19}$$

$$\sum_{n,\, m} \alpha_{in}\alpha_{jm} u_{nml} = \alpha_{kl} \qquad \text{for } i \neq j \neq k \neq i \tag{2.20}$$

Before we define the quantum group $SO_q(3)$ and show that the relations above correspond to the case $q = -1$, we first define the quantum general linear group $GL_q(2)$. This group is defined by the elements:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.21}$$

such that the matrix elements satisfy the following relations:

$$\begin{aligned} ab &= qba && (2.22) \\ bd &= qdb && (2.23) \\ ac &= qca && (2.24) \\ cd &= qdc && (2.25) \\ ad - qbc &= da - q^{-1}cb && (2.26) \\ bc &= cb && (2.27) \end{aligned}$$

Using this definition of $GL_q(2)$, we can define $GL_q(3)$ as the set of matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \in GL_q(3) \tag{2.28}$$

where

$$\begin{pmatrix} A_{in} & A_{im} \\ A_{jn} & A_{jm} \end{pmatrix} \in GL_q(2) \quad \text{for } i \neq j \text{ and } m \neq n \quad . \tag{2.29}$$

Note that this definition of $GL_q(3)$ is an alternative but equivalent definition to the one given in the Introduction.

From $GL_q(3)$, one can obtain $SL_q(3)$, the quantum special linear group in 3 dimensions, by imposing the condition:

$$det_q(A) = 1 \tag{2.30}$$

where the quantum determinant is defined as presented in the Introduction. Furthermore, on $SL_q(3)$, one can impose the *reality* condition:

$$A_{ij} = A^*_{ij} \tag{2.31}$$

thus ending up the quantum group $SL_q(3, \mathbb{R})$. On the other hand, one can impose the unitarity condition:

$$A^\dagger = A^{-1} \tag{2.32}$$

on $SL_q(3)$ and obtain the quantum group $SU_q(3)$. The quantum group $SO_q(3)$ is equivalent to the quantum group $SL_q(3, \mathbb{R}) \cap SU_q(3)$. However one can show for $SL_q(3, \mathbb{R})$ that $q = e^{i\beta}$ for some $\beta \in \mathbb{R}$ and similarly for $SU_q(3)$ that $q \in \mathbb{R}$. Thus one finds that $q = \pm 1$ for $SO_q(3)$. When $q = 1$ the quantum group becomes the usual $SO(3)$ group; the interesting case is when $q = -1$ which, as we will show, is the invariance quantum group of ACSA.

By virtue of the relation (2.29) and the relations (2.24) - (2.27) between the matrix elements of a $GL_q(2)$ matrix, we can see that for the case when $q = -1$, we have the following relations between the matrix elements of $SO_q(3)$:

$$\begin{aligned} A_{in}A_{jn} &= -A_{jn}A_{in} & (2.33)\\ A_{in}A_{jm} + A_{im}A_{jn} &= A_{jm}A_{in} + A_{jn}A_{im} & (2.34)\\ A_{im}A_{jn} &= A_{jn}A_{im} & (2.35) \end{aligned}$$

for $i \neq j$ and $n \neq m$. Furthermore, using (2.35) in (2.34) one finds:

$$A_{in}A_{jm} = A_{jm}A_{in} \tag{2.36}$$

again for $i \neq j$ and $n \neq m$.

Thus for a matrix $A \in SO_{q=-1}(3)$, the transformation invariance relation (2.18) is shown to be satisfied by virtue of relation (2.33). Similarly, the elements of such a matrix satisfy the relation (2.19) by virtue of the $GL_q(2)$ relations (2.35) and (2.36).

It is a little harder to show that equation (2.20) is satisfied by elements of $SO_{q=-1}(3)$ matrices. However, if one considers a particular choice of k and l on the right hand side of this equation, one can see that on the left hand side one has the freedom to choose i and j in two different ways. This implies that a particular α_{kl} is equal to two separate forms. Given explicitly, for a given choice of k and l, we get:

$$\alpha_{kl} = \sum_{r,\, q} \alpha_{ir}\alpha_{jq}u_{rql} \tag{2.37}$$

$$\alpha_{kl} = \sum_{r,\, q} \alpha_{jr}\alpha_{iq}u_{rql} \tag{2.38}$$

for a particular choice of i and j such that i, j, k are all different. In each of these sums only two terms survive, one where $r = n$, $q = m$ and the other one where $r = m$, $q = n$, such that n, m, l all different. This is due to the nature of u_{ijk} which is non-zero only if all the indices are different. Finally we arrive at the explicit form of relation (2.20):

$$\alpha_{kl} = \alpha_{in}\alpha_{jm} + \alpha_{im}\alpha_{jn} = \alpha_{jm}\alpha_{in} + \alpha_{jn}\alpha_{im} \tag{2.39}$$

for i, j, k all different and n, m, l all different. Written in this form, it is obvious that due to the $GL_q(2)$ relation (2.34), part of the above equality is satisfied by the matrix elements of a matrix in $SO_{q=-1}(3)$. The fact that both sides of this relation is equal to another matrix element does not rise form the $GL_q(2)$ relations but is due to the fact that the matrix is special and orthogonal, i.e. it is due to the fact that $A^T = A^{-1}$ and that $det_{q=-1}(A) = 1$. In order to show this, we should first note that $\det_q$ where $q = -1$ is the same as the normal determinant except there is no alternation of signs as there is in the normal determinant; this type of determinant with no alternation of signs is also called a permanent. If we refer to the form of the quantum determinant given in equation (1.66), we can see that for $q = -1$, this form turns into a direct sum of all permutations of matrix elements. Given this fact, one can notice that the $GL_{q=-1}(2)$ relation (2.34) is equal to the determinant of the $GL_{q=-1}(2)$ submatrix and is nothing but the statement that this determinant is defined and unique. Using the definition given in equation (1.69), the inverse of a matrix A that is an element of $SO_{q=-1}(3)$ is defined as:

$$A^{-1} = det^{-1}_{q=-1}(A) adj(A)^T \tag{2.40}$$

For a matrix A in $SO_{q=-1}(3)$, however, we also have the fact that $A^{-1} = A^T$ and $det_{q=-1}(A) = 1$, thus for such a matrix, the relation (2.40) becomes:

$$A^T = adj(A)^T \qquad \Rightarrow \qquad A = adj(A) \tag{2.41}$$

where $adj(A)$ stands for the matrix where each matrix element a_{ij} is equal to the determinant of original matrix A without the ith row and jth column, ie. the cofactor of the matrix element. This implies that the matrix elements of A and $adj(A)$ are equal. This further implies that each matrix element of A is equal to the cofactor of that matrix element. For $SO_{q=-1}(3)$ matrices, the cofactor of a matrix element is equal to the $q = -1$ determinant of its $GL_{q=-1}(2)$ minor submatrix. Thus, as a result of this argument, we have:

$$A_{kl} = cof_{kl}(A) = A_{in}A_{jm} + A_{im}A_{jn} = A_{jm}A_{in} + A_{jn}A_{im} \quad , \tag{2.42}$$

thereby, showing that matrices which are elements of $SO_{q=-1}(3)$ fully satisfy the transformation relations that leave ACSA invariant.

Thus, we have found that the invariance quantum group of ACSA is the quantum group $SO_q(3)$ with $q = -1$. Strictly speaking, ACSA is a module of the q-deformed $SO(3)$ quantum algebra with $q = -1$. It is very interesting to note that the invariance group of the angular momentum algebra is also $SO_q(3)$ but with $q = 1$.

2.3. Representations

The Anticommutator Spin Algebra is defined by the relations (2.1-2.3). In order to find the representations of this algebra we define the operators:

$$
\begin{aligned}
J_+ &= J_1 + J_2 && (2.43)\\
J_- &= J_1 - J_2 && (2.44)\\
J^2 &= J_1^2 + J_2^2 + J_3^2 && (2.45)
\end{aligned}
$$

which obey the following relations:

$$
\begin{aligned}
\{J_+, J_3\} &= J_3 && (2.46)\\
\{J_-, J_3\} &= -J_3 && (2.47)\\
J_+^2 &= J^2 - J_3^2 + J_3 && (2.48)\\
J_-^2 &= J^2 - J_3^2 - J_3 && (2.49)
\end{aligned}
$$

Furthermore, it can easily be shown that J^2 is central in the algebra, i.e. that it commutes with all the elements of the algebra, by first observing that:

$$
\begin{aligned}
J_j^2\, J_i &= J_j(J_k - J_i\, J_j)\\
&= J_j J_k - (J_k - J_i\, J_j) J_j\\
&= (J_j\, J_k - J_k\, J_j) + J_i\, J_j^2\\
&= (2J_j\, J_k - J_i) + J_i\, J_j^2 \quad \text{for} \quad i \neq j \neq k \neq i. && (2.50)
\end{aligned}
$$

Using this relation and the fact that $J^2 = \sum_j J_j$, we can see:

$$\begin{aligned} J^2\ J_i &= \sum_j J_j^2\ J_i \\ &= J_i^3 + \sum_{j \neq j} J_j^2\ J_i \\ &= J_i^3 + \sum_{j \neq i} (2J_j\ J_k - J_i + J_i\ J_j^2) \end{aligned} \tag{2.51}$$

However, in the final form of this expression the sum only contains two terms where the two indices j and k are symmetric. Thus the whole expression can be written as:

$$\begin{aligned} J^2\ J_i &= J_i^3 + \sum_{j \neq i} (2J_j\ J_k - J_i + J_i\ J_j^2) \\ &= J_i^3 - 2J_i + 2(J_j\ J_k + J_k\ J_j) + J_i\ J_j^2 + J_i\ J_k^2 \\ &= J_i\ J^2 - 2J_i + 2J_i \\ &= J_i\ J^2 \quad \text{for} \quad i \neq j \neq k \neq i, \end{aligned} \tag{2.52}$$

and therefore showing that J^2 commutes with all the elements of the algebra.

For this reason, we can label the states in our representation with the eigenvalues of J^2 and J_3:

$$J^2 \mid \lambda, \mu \rangle = \lambda \mid \lambda, \mu \rangle \tag{2.53}$$

$$J_3 \mid \lambda, \mu \rangle = \mu \mid \lambda, \mu \rangle \tag{2.54}$$

The action of J_+ and J_- on the states such defined is easily shown to be:

$$J_+ \mid \lambda, \mu \rangle = f(\lambda, \mu) \mid \lambda, -\mu + 1 \rangle \tag{2.55}$$

$$J_- \mid \lambda, \mu \rangle = g(\lambda, \mu) \mid \lambda, -\mu - 1 \rangle \tag{2.56}$$

It is enough to look at the norm of the states $J_+ \mid \lambda, \mu \rangle$ and $J_- \mid \lambda, \mu \rangle$ to find $f(\lambda, \mu)$

and $g(\lambda,\mu)$. Thus:

$$
\begin{aligned}
\langle\, \lambda,\mu \mid J_+^2 \mid \lambda,\mu \,\rangle &= |f(\lambda,\mu)|^2 && (2.57)\\
\langle\, \lambda,\mu \mid J^2 - J_3^2 + J_3 \mid \lambda,\mu \,\rangle &= |f(\lambda,\mu)|^2 && (2.58)\\
\lambda-\mu^2+\mu &= |f(\lambda,\mu)|^2 && (2.59)\\
f(\lambda,\mu) &= \sqrt{\lambda-\mu^2+\mu} && (2.60)
\end{aligned}
$$

and, similarly, $g(\lambda,\mu) = \sqrt{\lambda-\mu^2-\mu}$. These coefficients must be real due to the fact that J_+ and J_- are hermitian operators. This constraint imposes the following conditions on λ and μ:

$$
\begin{aligned}
\lambda-\mu^2+\mu &\geq 0 && (2.61)\\
\lambda-\mu^2-\mu &\geq 0 && (2.62)
\end{aligned}
$$

which can be satisfied by letting $\lambda = j(j+1)$ for some j with:

$$j \geq \mu \geq -j. \tag{2.63}$$

Note that equation (2.55) shows that the action of J_+ is composed of a reflection which changes sign of μ, the eigenvalue of J_3, followed by raising by one unit. Similarly, equation (2.56) shows that J_- reflects and lowers. Thus the highest state $\mu = j$ is annihilated by J_- and *lowered* by J_+. Applying J_+ or J_- twice to any state gives back the same state due to relations (2.48) and (2.49). Thus starting from the highest state we apply J_- and J_+ alternately to get the spectrum:

$$j, -j+1, j-2, -j+3, ... \tag{2.64}$$

This sequence ends so as to satisfy equation (2.63) only for integer or half-integer j. For integer j, it terminates, after an even number of steps, at $-j$ and visits every integer in between only once. For half-integer $j = 2k \pm \frac{1}{2}$ it ends at $j = \pm\frac{1}{2}$ having visited only half the states with μ half-odd integer between j and $-j$. The rest of the states cannot be reached from these but are obtained by starting from the $\mu = -j$ state and applying J_- and J_+ alternately; starting with J_-.

For the first few integer and half-integer values of j, the spectrum and state transitions are depicted in the following figures. In these figures the transitions that take the states to the null state are not shown.

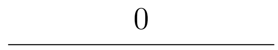

Figure 2.1. State diagram for $j = 0$

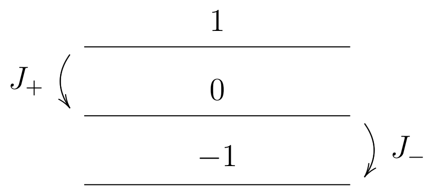

Figure 2.2. State diagram for $j = 1$

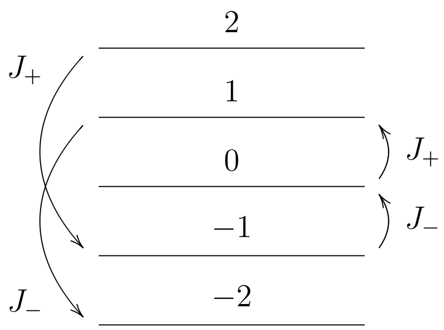

Figure 2.3. State diagram for $j = 2$

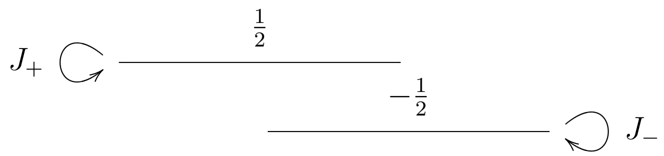

Figure 2.4. State diagram for $j = \frac{1}{2}$

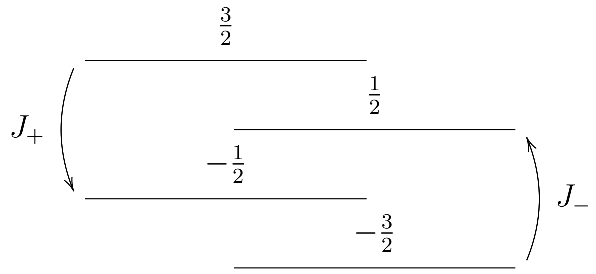

Figure 2.5. State diagram for $j = \frac{3}{2}$

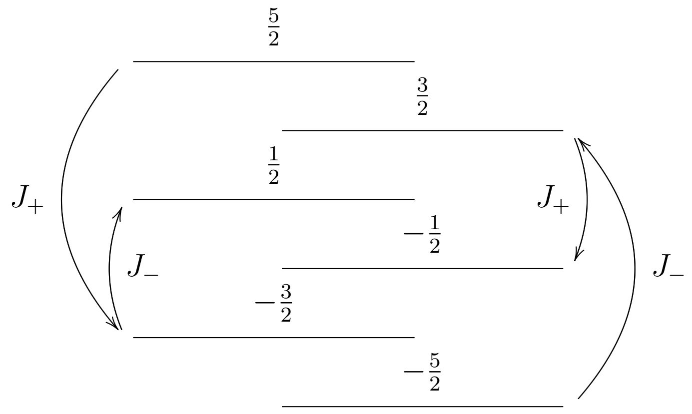

Figure 2.6. State diagram for $j = \frac{5}{2}$

2.4. Hopf Algebra Structure with Braiding

One natural question to ask having considered this associative algebra is whether or not it has a Hopf algebra structure. On the surface, this algebra shares a lot with its sister algebra, the $SU(2)$ Lie algebra, which has a Hopf algebra structure and one would expect ACSA to similarly have one. It turns out, however, that naively trying the same coproduct rule for ACSA does not work due to the symmetric nature of the product defined on ACSA since the product is defined in terms of anticommutators. As was noted in the Introduction of this work, the coproduct of the Lie algebra requires the product on the Lie algebra to be anti-symmetric. For this reason, the coproduct of the $SU(2)$ Lie algebra is not suitable for ACSA.

In our quest for a Hopf algebra structure for ACSA, it would be more fruitful to understand the nature of the relationship of ACSA with the $SU(2)$ Lie algebra. If one names the generators of the $SU(2)$ algebra I_i, then it can easily be shown that $\tilde{J}_i$ defined as

$$\tilde{J}_i = -I_i \otimes \sigma_i \tag{2.65}$$

satisfy the defining relation of ACSA since:

$$\begin{aligned}
\tilde{J}_i\tilde{J}_j + \tilde{J}_j\tilde{J}_i &= I_iI_j \otimes \sigma_i\sigma_j + I_jI_i \otimes \sigma_j\sigma_i \\
&= I_iI_j \otimes i\sigma_k + I_jI_i \otimes -i\sigma_k \\
&= i(I_iI_j - I_jI_i) \otimes \sigma_k \\
&= i(iI_k) \otimes \sigma_k \\
&= -I_k \otimes \sigma_k \\
&= \tilde{J}_k \quad \text{for} \quad i \neq j \neq k \neq i.
\end{aligned}$$

Similarly, the generators satisfying the $SU(2)$ Lie algebra can be written in terms of ACSA generators as:

$$\tilde{I}_i = J_i \otimes \sigma_i \tag{2.66}$$

since:

$$\begin{aligned}
\tilde{I}_i\tilde{I}_j - \tilde{I}_j\tilde{I}_i &= J_iJ_j \otimes \sigma_i\sigma_j - J_jJ_i \otimes \sigma_j\sigma_i \\
&= J_iJ_j \otimes i\sigma_k - J_jJ_i \otimes -i\sigma_k \\
&= i(J_iJ_j + J_jJ_i) \otimes \sigma_k \\
&= i(J_k) \otimes \sigma_k \\
&= \tilde{I}_k \quad \text{for} \quad i \neq j \neq k \neq i.
\end{aligned}$$

These two relations show that the $SU(2)$ algebra and ACSA are so closely related that it is not even possible to identify which one of these algebras is more fundamental. Both of them can be written in terms of the generators of the other and their algebraic structure can be derived from the structure of the other one. However, as mentioned, the Hopf algebra structure of ACSA cannot be derived from the Hopf algebra structure of the $SU(2)$ algebra. Specifically, ACSA does not admit a coproduct defined in a normal way using the usual tensor products. Such a coproduct can be defined if one were to extend the definition of the permutation map τ used in the connecting relation of a bialgebra. Normally the operation of τ is defined as:

$$\tau(A \otimes B) = B \otimes A \quad , \tag{2.67}$$

however, if one considers the algebra to be graded and one were to define a degree operator (deg) which is 0 for bosonic variables and is 1 for fermionic variables, then the natural redefinition of the τ operator is

$$\tau(A \otimes B) = (-1)^{deg\ A\ deg\ B}\, B \otimes A \quad . \tag{2.68}$$

Using this redefined permutation operator, one can still write down the bialgebra and Hopf algebra relations and only the connecting relation will be redefined; thus, one arrives at the definition of a braided Hopf algebra structure.

When the permutation operator is redefined in this way, the product of two tensor product terms is given by $(A \otimes B)(C \otimes D) = (-1)^{deg\ B\ deg\ C}\,(AC \otimes BD)$ where the -1 factor comes in because of the reordering of the B and C terms. Using this rule and defining the degree of 1 as 0 and the degrees of J_1, J_2, J_3 as 1, we can see that the coproduct defined as:

$$\Delta(J_i) = 1 \otimes J_i + J_i \otimes 1 \tag{2.69}$$

$$\Delta(1) = 1 \otimes 1 \tag{2.70}$$

satisfies the algebra structure relations since:

$$\begin{aligned} \Delta(J_i)\Delta(J_j) &= (1 \otimes J_i + J_i \otimes 1)(1 \otimes J_j + J_j \otimes 1) \\ &= 1 \otimes J_iJ_j - 1J_j \otimes J_i1 + J_i1 \otimes 1J_j + J_iJ_j \otimes 1 \\ &= 1 \otimes J_iJ_j - J_j \otimes J_i + J_i \otimes J_j + J_iJ_j \otimes 1 \end{aligned}$$

and

$$\begin{aligned} \Delta(J_i)\Delta(J_j) + \Delta(J_j)\Delta(J_i) &= 1 \otimes J_iJ_j - J_j \otimes J_i + J_i \otimes J_j + J_iJ_j \otimes 1 \\ &\quad +1 \otimes J_jJ_i - J_i \otimes J_j + J_j \otimes J_i + J_jJ_i \otimes 1 \\ &= 1 \otimes J_iJ_j + J_iJ_j \otimes 1 \\ &= 1 \otimes J_k + J_k \otimes 1 \\ &= \Delta(J_k) \quad \text{for} \quad i \neq j \neq k \neq i. \end{aligned}$$

The counit and coinverse are simpler and they match with the definitions for the normal

Lie algebra, i.e.:

$$\epsilon(J_i) = 0 \tag{2.71}$$

$$S(J_i) = -J_i \tag{2.72}$$

These definitions of the coproduct, the counit and the coinverse give us a braided Hopf algebra structure for ACSA.

3. QUANTUM GROUPS ASSOCIATED WITH INVARIANCE OF NON-DEFORMED OSCILLATORS

The concepts of bosons and fermions lie at the heart of microscopic physics. They are described in terms of creation and annihilation operators of the corresponding particle algebra:

$$c_i c_j \mp c_j c_i = 0 \tag{3.1}$$

$$c_i c_j^* \mp c_j^* c_i = \delta_{ij} \tag{3.2}$$

where the upper sign is for the boson algebra $BA(d)$ and the lower sign is for the fermion algebra $FA(d)$.

It has been realized that quantum algebras play an important role in the description of physical phenomena. Some classical physical systems which are invariant under a classical Lie group, when quantized, are invariant under a quantum group [17, 18, 19, 20]. The quantum groups thus considered turn out to be q-deformations of the classical semisimple groups. On the other hand, inhomogeneous quantum groups [21, 22] are perhaps more interesting since classical inhomogeneous groups such as the Poincaré group are more important in physics.

In this paper we will investigate an important class of inhomogeneous quantum groups which are related to the boson algebra $BA(d)$ and the fermion algebra $FA(d)$. Although $BA(d)$ and $FA(d)$ themselves are not quantum groups, by considering quantum group versions of symmetry transformations acting on these algebras, one can arrive at these inhomogeneous quantum groups. Mathematically speaking we are thus interested in constructing left modules of these algebras such that these modules have Hopf algebra structure.

Traditionally the boson algebra has the symmetry group $ISp(2d,\mathbb{R})$, the inhomogeneous symplectic group, which transforms creation and annihilation operators as:

$$c_i \rightarrow \alpha_{ij} c_j + \beta_{ij} c_j^* + \gamma_i \quad . \tag{3.3}$$

In this transformation $\alpha_{ij}, \beta_{ij}, \gamma_i$ are complex numbers satisfying the constraints required by the group $ISp(2d, \mathbb{R})$. One should note that this symmetry group is also the group of linear canonical transformations of a classical dynamical system. An important physical application of this transformation is the Bogoliubov transformation which is crucial in the explanation of many quantum mechanical effects such as the Unruh Effect [23] and Hawking Radiation [24]. In the case of the Hawking Radiation, the physical reinterpretation of the transformed operators imply that the future vacuum state is annihilated by the transformed annihilation operator, which is related to the initial creation and annihilation operators by a Bogoliubov transformation.

Similar to the boson algebra, the fermion algebra has the classical symmetry group $O(2d)$ with the transformation law:

$$c_i \rightarrow \alpha_{ij} c_j + \beta_{ij} c_j^* \quad . \tag{3.4}$$

however, unlike its bosonic counterpart this algebra is not inhomogeneous. This fact is the primary motivation for the generalization that we are going to offer. By relaxing the conditions on the transformation coefficients such as commutativity, one can come up with inhomogeneous invariance (quantum)groups for fermions and for bosons alike. The explicit R-matrices utilizing the quantum group properties of these structures have already been presented [25, 26]. In this paper, after a brief definition of these quantum groups $FIO(2d, \mathbb{R})$, the fermionic inhomogeneous orthogonal quantum group, and $BISp(2d, \mathbb{R})$, the bosonic inhomogeneous symplectic quantum group, in Section 1, we will investigate their sub(quantum)groups and also study the (quantum)groups obtained by their contractions. In the last section, $FIO(2d+1, \mathbb{R})$, the fermionic inhomogeneous quantum orthogonal group in odd number of dimensions, will also be defined and its properties examined.

A general transformation of a particle algebra can be described in the following way:

$$\begin{pmatrix} c' \\ c^{*\prime} \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ 0 & 0 & 1 \end{pmatrix} \dot{\otimes} \begin{pmatrix} c \\ c^* \\ 1 \end{pmatrix} \tag{3.5}$$

where c, c^*, γ, γ^* are column matrices and $\alpha, \beta, \alpha^*, \beta^*$ are $d \times d$ matrices. Thus, in index notation the transformation is given by:

$$c'_i = \alpha_{ij} \otimes c_j + \beta_{ij} \otimes c^*_j + \gamma_i \otimes 1 \quad , \tag{3.6}$$

$$c^{*\prime}_i = \alpha^*_{ij} \otimes c^*_j + \beta^*_{ij} \otimes c_j + \gamma^*_i \otimes 1 \quad . \tag{3.7}$$

Given this transformation, we look for an algebra $\mathcal{A}$ generated by these matrix elements such that the particle algebra remains invariant. Thus, we first write the transformation matrix in the above equation in the following way:

$$M = \left(\begin{array}{cc|c} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & \Gamma \\ \hline 0 & 1 \end{array}\right) \quad . \tag{3.8}$$

We assume that $\alpha_{ij}, \beta_{ij}, \gamma_i$ belong to a possibly noncommutative algebra on which a hermitian conjugation denoted by $*$ is defined.

3.1. The Bosonic Inhomogeneous Symplectic Quantum Group $BISp(2d, \mathbb{R})$

If we consider the transformation matrix (3.8) being applied to the boson algebra given by:

$$\begin{aligned} c_i c_j - c_j c_i &= 0 \qquad (3.9) \\ c_i c_j^* - c_j^* c_i &= \delta_{ij} \qquad (3.10) \end{aligned}$$

then we require that the transformed operators c_i' and ${c^*}_i'$ are required to satisfy the same algebra in order for the transformation to be an algebra invariance. Thus we require that:

$$\begin{aligned} c_i' c_j' - c_j' c_i' &= 0 \qquad (3.11) \\ c_i' {c^*}_j' - {c^*}_j' c_i' &= \delta_{ij} \qquad (3.12) \end{aligned}$$

Explicitly writing out the transformed operators, these relations become:

$$\begin{aligned} &(\alpha_{ik} \otimes c_k + \beta_{ik} \otimes c_k^* + \gamma_i \otimes 1)(\alpha_{jl} \otimes c_l + \beta_{jl} \otimes c_l^* + \gamma_j \otimes 1) \\ &-(\alpha_{jl} \otimes c_l + \beta_{jl} \otimes c_l^* + \gamma_j \otimes 1)(\alpha_{ik} \otimes c_k + \beta_{ik} \otimes c_k^* + \gamma_i \otimes 1) = 0 \qquad (3.13) \\ &(\alpha_{ik} \otimes c_k + \beta_{ik} \otimes c_k^* + \gamma_i \otimes 1)(\alpha_{jl}^* \otimes c_l^* + \beta_{jl}^* \otimes c_l + \gamma_j^* \otimes 1) \\ &-(\alpha_{jl}^* \otimes c_l^* + \beta_{jl}^* \otimes c_l + \gamma_j^* \otimes 1)(\alpha_{ik} \otimes c_k + \beta_{ik} \otimes c_k^* + \gamma_i \otimes 1) = \delta_{ij} \qquad (3.14) \end{aligned}$$

which gives us:

$$\begin{aligned} &[\alpha_{ik}, \alpha_{jl}] c_l c_k + [\beta_{ik}, \beta_{jl}] c_l^* c_k^* \\ &+ [\alpha_{ik}, \gamma_j] c_k + [\beta_{ik}, \gamma_j] c_k^* \\ &+ [\gamma_i, \alpha_{jl}] c_l + [\gamma_i, \beta_{jl}] c_l^* \qquad (3.15) \\ &+ [\alpha_{ik}, \beta_{jl}] c_k c_l^* + [\beta_{ik}, \alpha_{jl}] c_k^* c_l \\ &+ (\alpha_{jk}\beta_{ik} - \beta_{jk}\alpha_{ik} + [\gamma_i, \gamma_j]) = 0 \quad , \end{aligned}$$

and

$$
\begin{aligned}
&[\alpha_{ik}, \beta^*_{jl}]c_l c_k + [\beta_{ik}, \alpha^*_{jl}]c^*_l c^*_k \\
&+ [\alpha_{ik}, \gamma^*_j]c_k + [\beta_{ik}, \gamma^*_j]c^*_k \\
&+ [\gamma_i, \beta^*_{jl}]c_l + [\gamma_i, \alpha_{jl}]c^*_l \\
&+ [\alpha_{ik}, \alpha^*_{jl}]c_k c^*_l + [\beta_{ik}, \beta^*_{jl}]c^*_k c_l \\
&+ (\alpha^*_{jk}\alpha_{ik} - \beta^*_{jk}\beta_{ik} + [\gamma_i, \gamma^*_j]) = \delta_{ij} \quad .
\end{aligned}
\tag{3.16}
$$

In the first of these relations, for the equality to be satisfied, it is sufficient for the coefficients of all the terms on the left hand side to be equal to zero. In the second one, however, we only have a term that is a multiple of the unit element of the boson algebra on the right hand side, thus the coefficient of that term should be equal on both sides and it is sufficient for the coefficients of the other terms on the left hand side to be separately equal to zero.

Thus we have the following relations between the transformation elements:

$$
\begin{aligned}
\gamma_i\gamma^*_j - \gamma^*_j\gamma_i &= \delta_{ij} - \alpha_{ik}\alpha^*_{jk} + \beta_{ik}\beta^*_{jk} && (3.17)\\
\gamma_i\gamma_j - \gamma_j\gamma_i &= \beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} && (3.18)\\
\alpha_{ij}\gamma_k - \gamma_k\alpha_{ij} &= 0 && (3.19)\\
\beta_{ij}\gamma_k - \gamma_k\beta_{ij} &= 0 && (3.20)\\
\alpha_{ij}\gamma^*_k - \gamma^*_k\alpha_{ij} &= 0 && (3.21)\\
\beta_{ij}\gamma^*_k - \gamma^*_k\beta_{ij} &= 0 && (3.22)
\end{aligned}
$$

and any two elements from the set $\alpha_{ij}, \beta_{ij}, \alpha^*_{ij}, \beta^*_{ij}$ commute.

The set of matrices M obeying the above relations form the group of inhomogeneous transformations of bosons. We name this quantum group as the bosonic inhomogeneous symplectic quantum group $BISp(2d,\mathbb{R})$ since it is an inhomogeneous extension of the symplectic group where the inhomogeneous part exhibits bosonic behavior. This symmetry group, however, is not a classical group like the symplectic group but is in fact a quantum group with a Hopf algebra structure. As shown in [26], this Hopf algebra has an explicit R-matrix representation and the coproduct, counit and coinverse are defined as:

$$\Delta(M) = M\dot{\otimes}M \tag{3.23}$$

$$\epsilon(M) = I \tag{3.24}$$

$$S(M) = M^{-1} \quad . \tag{3.25}$$

In Equation (3.23), the symbol $\dot{\otimes}$ denotes the usual matrix multiplication where when elements of the matrices are multiplied, tensor multiplication is used.

The inverse of the matrix M can be defined as:

$$M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\Gamma \\ 0 & 1 \end{pmatrix} \tag{3.26}$$

where A^{-1} is defined in the standard way since matrix elements of A are shown to be commutative.

3.1.1. Subgroups

After having shown that the inhomogeneous transformations of the boson algebra forms the symmetry quantum group $BISp(2d,\mathbb{R})$, one important question is what sub(quantum)groups does this quantum group have. For example, we know that the group $ISp(2d,\mathbb{R})$ is an important special subgroup of $BISp(2d,\mathbb{R})$ and other sub(quantum)groups could turn out to have similarly important physical applications. While searching for sub(quantum)groups, we would also like to find new (quantum)groups allowed by suitable contractions [27] of these quantum groups as well.

The sub(quantum)groups are obtained by imposing additional relations on the matrix elements of M which obey the relations (3.17) - (3.22). The additional relations that we will impose are:

$$\delta_{ij} - \alpha_{ik}\alpha^*_{jk} + \beta_{ik}\beta^*_{jk} = \beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} = 0 \tag{3.27a}$$

$$\gamma_i = 0 \tag{3.27b}$$

$$\beta_{ij} = 0 \tag{3.27c}$$

$$\alpha_{ij} = 0 \tag{3.27d}$$

We would like to study the implication of each relation one by one in the following subsections.

3.1.1.1. Inhomogeneous Subgroup. The relation (3.27a):

$$\delta_{ij} - \alpha_{ik}\alpha^*_{jk} + \beta_{ik}\beta^*_{jk} = \beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} = 0$$

by virtue of (3.17) and (3.18) implies that $\gamma_i\gamma^*_j - \gamma^*_j\gamma_i = 0$ and $\gamma_i\gamma_j - \gamma_j\gamma_i = 0$, i.e. that the inhomogeneous transformation parameters are commutative variables.

For bosonic particles, the fact that the inhomogeneous elements of the quantum group are commutative elements coupled with the fact that the remaining relations between the transformation elements are already commutative gives us a symmetry transformation of the boson algebra where all the elements commute. However, we know that such a transformation is nothing but the classical symmetry group of the bosonic particle algebra $ISp(2d, \mathbb{R})$ in which all the parameters, including the inhomogeneous elements, are commutative.

3.1.1.2. Homogeneous Subgroup. The relation (3.27b):

$$\gamma_i = 0$$

practically gets rid of the inhomogeneous part of the transformation and also implies the relation (3.27a) considered in the previous subsection. Since the previous relation is implied the resulting group will be a subgroup of $ISp(2d,\mathbb{R})$ and since the group is not inhomogeneous anymore the resulting subgroup is the classical symplectic group $Sp(2d,\mathbb{R})$.

3.1.1.3. Bosonic Inhomogeneous Unitary Quantum Group. The relation (3.27c):

$$\beta_{ij} = 0$$

applied to the transformation gets rid of the off-diagonal members of the homogeneous part of it and leaves us with the following relation:

$$\gamma_i\gamma_j^* - \gamma_j^*\gamma_i = \delta_{ij} - \alpha_{ik}\alpha_{jk}^* \tag{3.28}$$

$$\gamma_i\gamma_j - \gamma_j\gamma_i = 0 \tag{3.29}$$

This equation implies for the homogeneous part of the transformation the relation:

$$\delta_{ij} = \alpha_{ik}\alpha_{jk}^* \tag{3.30}$$

which tells us that the submatrices α and α^* in equation (3.8) are members of $U(d)$. The subgroup we have arrived at thus is an inhomogeneous quantum group extension to the classical homogeneous group $U(d)$. Since the inhomogeneous elements of the resulting group obeys the same relations as $BISp(2d,\mathbb{R})$, we will name this quantum group $BIU(d)$, the bosonic inhomogeneous quantum group.

3.1.1.4. Boson Algebra. The relation (3.27d):

$$\alpha_{ij} = 0$$

applied alone onto the transformation gets rid of the diagonal members of the homogeneous part and prevents such transformations from forming a (quantum)group since the homogeneous parts of these set of transformations can never include the identity transformation.

However, if this relation is applied together with the previous one, relation (3.27c), the resulting relation gets rid of the whole homogeneous part of the transformation leaving only the inhomogeneous part and leaves us with two relations:

$$\gamma_i\gamma_j^* - \gamma_j^*\gamma_i = \delta_{ij} \tag{3.31}$$

$$\gamma_i\gamma_j - \gamma_j\gamma_i = 0 \tag{3.32}$$

which gives us back the boson algebra, $BA(d)$.

We should note, however, that after this condition is applied, the resulting set of matrices M, which now form $BA(d)$, is no longer a quantum or classical group since the antipode defined in equation (3.25) no longer exits. For this reason, the boson algebra can be considered to be a boundary for the sub(quantum)groups of $BISp(2d,\mathbb{R})$.

3.1.1.5. Sub(quantum)group Diagram. As a result of the above discussion, we get the sub(quantum)group diagram:

$$\begin{array}{ccccc}
BISp(2d,\mathbb{R}) & \xrightarrow{(3.27a)} & ISp(2d,\mathbb{R}) & \xrightarrow{(3.27b)} & Sp(2d,\mathbb{R}) \\
{\scriptstyle (3.27c)}\downarrow & & {\scriptstyle (3.27c)}\downarrow & & {\scriptstyle (3.27c)}\downarrow \\
BIU(d) & \xrightarrow{(3.27a)} & IU(d) & \xrightarrow{(3.27b)} & U(d) \\
{\scriptstyle (3.27d))}\downarrow & & & & \\
BA(d) & & & &
\end{array}$$

for the sub(quantum)groups of the $BISp(2d,\mathbb{R})$ we have introduced in this section.

3.1.2. Contractions

In order to explore the new (quantum)groups that will come about as the result of a contraction, we replace γ_i by $\gamma_i/\sqrt{\hbar}$ so that we may consider the case $\hbar \to 0$. After this replacement, the equations (3.17) and (3.18) become:

$$\gamma_i\gamma_j^* - \gamma_j^*\gamma_i = \hbar(\delta_{ij} - \alpha_{ik}\alpha_{jk}^* + \beta_{ik}\beta_{jk}^*) \tag{3.33}$$

$$\gamma_i\gamma_j - \gamma_j\gamma_i = \hbar(\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk}) \tag{3.34}$$

When we consider the case $\hbar \to 0$, we get the relations:

$$\gamma_i\gamma_j^* - \gamma_j^*\gamma_i = 0 \tag{3.35}$$

$$\gamma_i\gamma_j - \gamma_j\gamma_i = 0 \tag{3.36}$$

which imply that the inhomogeneous part of the transformation form ordinary complex numbers. What makes this case different from the previous case of subgroups is that the homogeneous part of this transformation forms a matrix A with non-zero determinant. We can transform such a matrix A with a similarity transformation given by the unitary matrix:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{3.37}$$

to put it in a real form. The transformation gives:

$$\begin{aligned} A' &= UAU^\dagger \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\ &= \begin{pmatrix} Re(\alpha) + Re(\beta) & Im(\alpha) - Im(\beta) \\ -Im(\alpha) - Im(\beta) & Re(\alpha) - Re(\beta) \end{pmatrix} \end{aligned} \tag{3.38}$$

which is a real matrix that is a member of the general linear group $GL(2d, \mathbb{R})$. Thus we have the group $IGL(2d, \mathbb{R})$, the inhomogeneous general linear group.

If we consider the contraction of the subgroups as well then we should examine the $\hbar \to 0$ limit after the relations (3.27c) and (3.27d) are applied.

After we apply relation (3.27c), we get the subgroup $BIU(d)$ as discussed previously. After the contraction, again, the inhomogeneous part of this group become complex numbers. However, if we apply the previous similarity transformation on the homogeneous part, we get:

$$\begin{aligned} A' &= UAU^\dagger \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\ &= \begin{pmatrix} Re(\alpha) & Im(\alpha) \\ -Im(\alpha) & Re(\alpha) \end{pmatrix} \\ &= Re(\alpha)\mathbb{I} + Im(\alpha)\mathbb{J} \end{aligned} \tag{3.39}$$

where $\mathbb{I}$ stands for the identity matrix and $\mathbb{J}$ stands for the matrix the square of which is minus the identity matrix. This way we can see that the matrix A' is a actually member of $GL(d, \mathbb{C})$. This gives us $IGL(d, \mathbb{C})$ as the group we arrive at as the contraction of $BIU(d)$.

We have previously shown that we get the boson algebra after applying both of the relations (3.27c) and (3.27d). We have also discussed that in this case only the inhomogeneous part of the transformation survives. After applying the contraction, the surviving inhomogeneous part of the transformation turns into complex variables. Thus in this case, the contraction of $BA(d)$ gives us $\mathbb{C}^d$.

As a summary, for the contraction considered in this section applied onto the subgroups obtained in the previous subsection we get the following group diagram:

$$\begin{array}{ccc}
BISp(2d,\mathbb{R}) & \xrightarrow{\hbar\to 0} & IGL(2d,\mathbb{R}) \\
{\scriptstyle (3.27c)}\Big\downarrow & & {\scriptstyle (3.27c)}\Big\downarrow \\
BIU(d) & \xrightarrow{\hbar\to 0} & IGL(d,\mathbb{C}) \\
{\scriptstyle (3.27d)}\Big\downarrow & & {\scriptstyle (3.27d)}\Big\downarrow \\
BA(d) & \xrightarrow{\hbar\to 0} & \mathbb{C}^d
\end{array}$$

3.2. The Fermionic Inhomogeneous Group $FIO(2d, \mathbb{R})$

Similarly to how it was done in the bosonic case one can also consider the transformation matrix (3.8) being applied to the fermion algebra given by:

$$c_i c_j + c_j c_i = 0 \tag{3.40}$$

$$c_i c_j^* + c_j^* c_i = \delta_{ij} \tag{3.41}$$

and then require that the transformed operators c_i' and ${c^*}'_i$ satisfy the same algebra in order for the transformation to be an algebra invariance. Thus the requirement is that:

$$c_i' c_j' + c_j' c_i' = 0 \tag{3.42}$$

$$c_i' {c^*}'_j + {c^*}'_j c_i' = \delta_{ij} \tag{3.43}$$

Explicitly writing out the transformed operators, these relations become:

$$\begin{gathered}(\alpha_{ik} \otimes c_k + \beta_{ik} \otimes c_k^* + \gamma_i \otimes 1)(\alpha_{jl} \otimes c_l + \beta_{jl} \otimes c_l^* + \gamma_j \otimes 1) \\ +(\alpha_{jl} \otimes c_l + \beta_{jl} \otimes c_l^* + \gamma_j \otimes 1)(\alpha_{ik} \otimes c_k + \beta_{ik} \otimes c_k^* + \gamma_i \otimes 1) = 0\end{gathered} \tag{3.44}$$

$$\begin{gathered}(\alpha_{ik} \otimes c_k + \beta_{ik} \otimes c_k^* + \gamma_i \otimes 1)(\alpha_{jl}^* \otimes c_l^* + \beta_{jl}^* \otimes c_l + \gamma_j^* \otimes 1) \\ +(\alpha_{jl}^* \otimes c_l^* + \beta_{jl}^* \otimes c_l + \gamma_j^* \otimes 1)(\alpha_{ik} \otimes c_k + \beta_{ik} \otimes c_k^* + \gamma_i \otimes 1) = \delta_{ij}\end{gathered} \tag{3.45}$$

which gives us:

$$
\begin{aligned}
&[\alpha_{jl}, \alpha_{ik}]c_l c_k + [\beta_{jl}, \beta_{ik}]c_l^* c_k^* \\
&+ \{\alpha_{ik}, \gamma_j\}c_k + \{\beta_{ik}, \gamma_j\}c_k^* \\
&+ \{\gamma_i, \alpha_{jl}\}c_l + \{\gamma_i, \beta_{jl}\}c_l^* \\
&+ [\alpha_{ik}, \beta_{jl}]c_k c_l^* + [\beta_{ik}, \alpha_{jl}]c_k^* c_l \\
&+ (\alpha_{jk}\beta_{ik} + \beta_{jk}\alpha_{ik} + \{\gamma_i, \gamma_j\}) = 0 \quad ,
\end{aligned}
\tag{3.46}
$$

and

$$
\begin{aligned}
&[\beta_{jl}^*, \alpha_{ik}]c_l c_k + [\alpha_{jl}^*, \beta_{ik}]c_l^* c_k^* \\
&+ \{\alpha_{ik}, \gamma_j^*\}c_k + \{\beta_{ik}, \gamma_j^*\}c_k^* \\
&+ \{\gamma_i, \beta_{jl}^*\}c_l + \{\gamma_i, \alpha_{jl}\}c_l^* \\
&+ [\alpha_{ik}, \alpha_{jl}^*]c_k c_l^* + [\beta_{ik}, \beta_{jl}^*]c_k^* c_l \\
&+ (\alpha_{jk}^*\alpha_{ik} + \beta_{jk}^*\beta_{ik} + \{\gamma_i, \gamma_j^*\}) = \delta_{ij} \quad .
\end{aligned}
\tag{3.47}
$$

In the first of these relations, for the equality to be satisfied, it is sufficient for the coefficients of all the terms on the left hand side to be equal to zero. In the second one, however, we only have a term that is a multiple of the unit element of the boson algebra on the right hand side, thus the coefficient of that term should be equal on both sides and it is sufficient for the coefficients of the other terms on the left hand side to be separately equal to zero.

Thus we have the following relations between the transformation elements:

$$\begin{aligned}
\gamma_i\gamma_j^* + \gamma_j^*\gamma_i &= \delta_{ij} - \alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^* && (3.48)\\
\gamma_i\gamma_j + \gamma_j\gamma_i &= -\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} && (3.49)\\
\alpha_{ij}\gamma_k + \gamma_k\alpha_{ij} &= 0 && (3.50)\\
\beta_{ij}\gamma_k + \gamma_k\beta_{ij} &= 0 && (3.51)\\
\alpha_{ij}\gamma_k^* + \gamma_k^*\alpha_{ij} &= 0 && (3.52)\\
\beta_{ij}\gamma_k^* + \gamma_k^*\beta_{ij} &= 0 && (3.53)
\end{aligned}$$

and any two elements from the set $\alpha_{ij}, \beta_{ij}, \alpha_{ij}^*, \beta_{ij}^*$ commute.

The set of matrices M obeying the above relations form the group of inhomogeneous transformations of fermions. We call this quantum group the fermionic inhomogeneous orthogonal quantum group $FIO(2d, \mathbb{R})$ since it is an inhomogeneous extension of the orthogonal group where the inhomogeneous part exhibits fermionic behavior. This symmetry group, like its sister $BISp(2d, \mathbb{R})$, is not a classical group but is a quantum group with a Hopf algebra structure. Similar to the case with $BISp(2d, \mathbb{R})$, this Hopf algebra has an explicit R-matrix representation and the coproduct, counit and coinverse are defined as:

$$\begin{aligned}
\Delta(M) &= M\dot{\otimes}M && (3.54)\\
\epsilon(M) &= I && (3.55)\\
S(M) &= M^{-1} \quad . && (3.56)
\end{aligned}$$

3.2.1. Subgroups

We have shown that there is a rich sub(quantum)group structure for $BISp(2d, \mathbb{R})$ and it should naturally follow that there should be a similarly rich sub(quantum)group structure for the fermionic counterpart $FIO(2d, \mathbb{R})$.

In this subsection this sub(quantum)group structure will be explored using relations similar to the ones considered for $BISp(2d, \mathbb{R})$:

$$\begin{aligned} \delta_{ij} - \alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk} = -\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} &= 0 \quad &(3.57a)\\ \gamma_i &= 0 &(3.57b)\\ \beta_{ij} &= 0 &(3.57c)\\ \alpha_{ij} &= 0 \quad . &(3.57d) \end{aligned}$$

The implication of each of these relations will be explored in the following subsections.

<u>3.2.1.1. Inhomogeneous Subsupergroup.</u> The relation (3.57a):

$$\delta_{ij} - \alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk} = -\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} = 0$$

by virtue of (3.48) and (3.49) implies that $\gamma_i\gamma^*_j + \gamma^*_j\gamma_i = 0$ and $\gamma_i\gamma_j + \gamma_j\gamma_i = 0$, i.e. that the inhomogeneous transformation parameters are anticommutative variables.

Thus we end up with an inhomogeneous orthogonal algebra where the inhomogeneous parameters are grassmannian variables giving us the Grassmannian inhomogeneous orthogonal group, $GrIO(2d, \mathbb{R})$, as the resulting subgroup of $FIO(2d, \mathbb{R})$. This subgroup of $FIO(2d, \mathbb{R})$ can also be considered as an inhomogeneous supergroup. Actually, more generally, the transformation elements α_{ij}, β_{ij}, α^*_{ij} and β^*_{ij} anticommute with γ_i, γ^*_i and the $FIO(2d, \mathbb{R})$ matrices M are multiplied with each other using the standard tensor product. One can also show that α_{ij}, β_{ij}, α^*_{ij} and β^*_{ij} can be taken to commute with γ_i, γ^*_i provided that the matrices M are multiplied with a braided [28] tensor product, eg.

$$(A \otimes B)(C \otimes D) = -AC \otimes BD \tag{3.58}$$

whenever B and C are both fermionic. This approach is similar to the approach that was taken with ACSA to obtain a Hopf algebra structure. As a result of this redefinition the treatment of the transformation elements corresponds to the usual superalgebra approach, i.e. that the elements α_{ij}, β_{ij}, α^*_{ij} and β^*_{ij} are bosonic and the elements γ_i, γ^*_i are fermionic.

<u>3.2.1.2. Homogeneous Subgroup.</u> The relation (3.57b):

$$\gamma_i = 0$$

practically gets rid of the inhomogeneous part of the transformation and also implies the relation (3.57a) considered in the previous subsection.

This gives us the subgroup in the previous subsection without the inhomogeneous part which is basically the classical orthogonal group $O(2d, \mathbb{R})$.

<u>3.2.1.3. Fermionic Inhomogeneous Unitary Quantum Group.</u> The relation (3.57c):

$$\beta_{ij} = 0$$

applied to the transformation gets rid of the off-diagonal members of the homogeneous part of it and leaves us with the following relation:

$$\gamma_i\gamma^*_j + \gamma^*_j\gamma_i = \delta_{ij} - \alpha_{ik}\alpha^*_{jk} \tag{3.59}$$

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 0 \tag{3.60}$$

For the homogeneous part of the transformation, this equation implies $\delta_{ij} = \alpha_{ik}\alpha^*_{jk}$ which tells us that the submatrices α and α^* in equation (3.8) are both members of $U(d)$. The subgroup we have arrived at thus is an inhomogeneous quantum group whose homogeneous part is $U(d)$. For fermions we will name this group the fermionic inhomogeneous quantum group, $FIU(d)$, since the inhomogeneous part of the transformation exhibits fermionic behavior.

<u>3.2.1.4. Fermion Algebra.</u> The relation (3.57d):

$$\alpha_{ij} = 0$$

applied alone onto the transformation gets rid of the diagonal members of the homogeneous part and prevents such transformations from forming a (quantum)group since the homogeneous parts of these set of transformations can never include the identity transformation.

However, if this relation is applied together with the previous one, relation (3.57c), the resulting relation gets rid of the whole homogeneous part of the transformation leaving only the inhomogeneous part and gives us a single relation:

$$\gamma_i \gamma_j^* + \gamma_j^* \gamma_i = \delta_{ij} \tag{3.61}$$

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 0 \tag{3.62}$$

which gives us back the fermion algebra, $FA(d)$.

It should be noted that the fermion algebra in d dimensions is isomorphic to the Clifford algebra in $2d$ dimensions. If one considers operators ψ_i defined as:

$$\psi_i = \begin{cases} i(c_{i/2} - c_{i/2}^*) & \text{when } i \text{ is even} \\ (c_{(i+1)/2} + c_{(i+1)/2}^*) & \text{when } i \text{ is odd} \end{cases} \tag{3.63}$$

where $i = 1, 2, \ldots, 2d$ and c_j are elements of the fermion algebra, then one can show that ψ_i satisfy the Clifford algebra rule:

$$\{\psi_i, \psi_j\} = 2\delta_{ij} \tag{3.64}$$

In order to show this, one only needs to prove the cases:

$$\{\psi_i, \psi_j\} = \begin{cases} 2\delta_{ij} & i \text{ and } j \text{ odd} \\ 2\delta_{ij} & i \text{ and } j \text{ even} \\ 0 & i \text{ odd, } j \text{ even} \end{cases} \tag{3.65}$$

For i and j both odd, $\{\psi_i, \psi_j\}$ becomes:

$$\begin{aligned} \{\psi_i, \psi_j\} &= (c_{(i+1)/2} + c^*_{(i+1)/2})(c_{(j+1)/2} + c^*_{(j+1)/2}) \\ &\quad + (c_{(j+1)/2} + c^*_{(j+1)/2})(c_{(i+1)/2} + c^*_{(i+1)/2}) \\ &= \{c_{(i+1)/2}, c_{(j+1)/2}\} + \{c^*_{(i+1)/2}, c^*_{(j+1)/2}\} \\ &\quad + \{c_{(i+1)/2}, c^*_{(j+1)/2}\} + \{c^*_{(i+1)/2}, c_{(j+1)/2}\} \\ &= 0 + 0 + \delta_{ij} + \delta_{ij} \\ &= 2\delta_{ij} \end{aligned} \tag{3.66}$$

Similarly, when i and j are both even, one gets:

$$\begin{aligned} \{\psi_i, \psi_j\} &= i(c_{i/2} - c^*_{i/2})i(c_{j/2} - c^*_{j/2}) \\ &\quad + i(c_{j/2} - c^*_{j/2})i(c_{i/2} - c^*_{i/2}) \\ &= -\{c_{i/2}, c_{j/2}\} - \{c^*_{i/2}, c^*_{j/2}\} \\ &\quad + \{c_{i/2}, c^*_{j/2}\} + \{c^*_{i/2}, c_{j/2}\} \\ &= -0 - 0 + \delta_{ij} + \delta_{ij} \\ &= 2\delta_{ij} \end{aligned} \tag{3.67}$$

Finally, when i is odd and j is even, the anticommutator becomes:

$$\begin{aligned}
\{\psi_i, \psi_j\} &= (c_{(i+1)/2} + c^*_{(i+1)/2}) i (c_{j/2} - c^*_{j/2}) \\
&\quad + i(c_{j/2} - c^*_{j/2})(c_{(i+1)/2} + c^*_{(i+1)/2}) \\
&= i\{c_{(i+1)/2}, c_{j/2}\} - i\{c^*_{(i+1)/2}, c^*_{j/2}\} \\
&\quad - i\{c_{(i+1)/2}, c^*_{j/2}\} + i\{c^*_{(i+1)/2}, c_{j/2}\} \\
&= 0 - 0 - \delta_{i+1,j} + \delta_{i+1,j} \\
&= 0
\end{aligned} \tag{3.68}$$

thereby completing the proof that the operators ψ_i form the elements of a Clifford algebra of $2d$ dimensions. If one also considers that fact that the definition of ψ_i is invertible and that c_i can also be defined in terms of ψ_i, one can conclude that the algebras $FA(d)$ and $Cliff(2d)$ are isomorphic.

3.2.1.5. Sub(quantum)group Diagram. As a result of the above discussion, we get the sub(quantum)group diagram:

$$\begin{array}{ccccc}
FIO(2d,\mathbb{R}) & \xrightarrow{(3.57a)} & GrIO(2d,\mathbb{R}) & \xrightarrow{(3.57b)} & O(2d,\mathbb{R}) \\
{\scriptstyle (3.57c)}\big\downarrow & & {\scriptstyle (3.57c)}\big\downarrow & & {\scriptstyle (3.57c)}\big\downarrow \\
FIU(d) & \xrightarrow{(3.57a)} & GrIU(d) & \xrightarrow{(3.57b)} & U(d) \\
{\scriptstyle (3.57d)}\big\downarrow & & & & \\
FA(d) \approx Cliff(2d) & & & &
\end{array}$$

for the for the sub(quantum)groups of $FIO(2d,\mathbb{R})$ that have been introduced in this section.

3.2.2. Contractions

It was observed for $BISp(2d,\mathbb{R})$ that using a suitable contraction one can obtain new sub(quantum)groups. This should also be possible for $FIO(2d,\mathbb{R})$ and the resulting structures will be examined in this section.

Similar to the bosonic treatment, we replace γ_i by $\gamma_i/\sqrt{\hbar}$ so that we may consider the case $\hbar \to 0$. After this replacement, the equations (3.48) and (3.49) become:

$$\gamma_i\gamma_j^* + \gamma_j^*\gamma_i = \hbar(\delta_{ij} - \alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*) \tag{3.69}$$

$$\gamma_i\gamma_j + \gamma_j\gamma_i = \hbar(-\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk}) \tag{3.70}$$

and we consider the case $\hbar \to 0$, we get the relations:

$$\gamma_i\gamma_j^* + \gamma_j^*\gamma_i = 0 \tag{3.71}$$

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 0 \tag{3.72}$$

which imply that the inhomogeneous part of the transformation are Grassmannian elements. What makes this case different from the previous case of subgroups is that the homogeneous part of this transformation forms a matrix A with non-zero determinant. We have previously shown that this matrix can be put in real form that is a member of the general linear group $GL(2d,\mathbb{R})$. Since the homogeneous part of the transformation is the general linear group and the inhomogeneous part is Grassmannian, we have the group $GrIGL(2d,\mathbb{R})$, the Grassmannian inhomogeneous general linear group where the inhomogeneous part of the group are Grassmannian.

If we consider the contraction of the subgroups as well then we should examine the $\hbar \to 0$ limit after the relations (3.57c) and (3.57d) are applied.

After we apply relation (3.57c), we get the subgroup $FIU(d)$ as discussed previously. After the contraction, again, the inhomogeneous part of this group become Grassmannian variables. However, as was shown during the contraction of $BISp(2d,\mathbb{R})$, if we apply the previous similarity transformation on the homogeneous part of the resulting transformation matrix after contraction, one can see that the homogeneous part of the transformation is a member of the general linear group $GL(d,\mathbb{C})$. Similarly this gives us $GrIGL(d,\mathbb{C})$.

We have previously shown that we get the fermion algebra after applying both of the relations (3.57c) and (3.57d). We have also discussed that in this case only the inhomogeneous part of the transformation survives. After applying the contraction, the surviving inhomogeneous part of the transformation turns into Grassmannian variables. Thus in this case, the contraction of $FA(d)$ gives us $Gr(d, \mathbb{C})$, the d dimensional Grassmann algebra.

As a summary, for the contraction considered combined with the remaining subgroup relations we get the table:

$$\begin{array}{ccc} FIO(2d,\mathbb{R}) & \xrightarrow{\hbar\to 0} & GrIGL(2d,\mathbb{R}) \\ {\scriptstyle (3.57c)}\big\downarrow & & {\scriptstyle (3.57c)}\big\downarrow \\ FIU(d) & \xrightarrow{\hbar\to 0} & GrIGL(d,\mathbb{C}) \\ {\scriptstyle (3.57d)}\big\downarrow & & {\scriptstyle (3.57d)}\big\downarrow \\ FA(d) \approx Cliff(2d) & \xrightarrow{\hbar\to 0} & Gr(d,\mathbb{C}) \end{array}$$

3.3. The Fermionic Inhomogeneous Orthogonal Quantum Group of Odd Dimension

The bosonic transformation quantum group $BISp(2d, \mathbb{R})$ can only be defined in even dimensions and it is not possible to extend this definition to odd dimension. However, as will be shown in this section, it is possible to define the fermionic inhomogeneous orthogonal quantum group of odd dimension. In order to show this, one should first consider a unitary transformation of the $FIO(2d, \mathbb{R})$ matrix:

$$M \to UMU^{-1}$$

using the unitary matrix:

$$U = \left(\begin{array}{cc|c} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{3.73}$$

If one applies this unitary transformation, it can be seen that M can be put in the real form:

$$\left(\begin{array}{cc|c} Re(\alpha+\beta) & Im(\alpha-\beta) & \sqrt{2}Re(\gamma) \\ -Im(\alpha+\beta) & Re(\alpha-\beta) & -\sqrt{2}Im(\gamma) \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & \Gamma \\ \hline 0 & 1 \end{array}\right) \tag{3.74}$$

where A and Γ matrices are defined as in (3.8), and Re and Im denote the hermitian and anti-hermitian parts.

Using this form it is not too hard to show that for $FIO(2d, \mathbb{R})$, the transformation relations (3.48) and (3.49) together transform into the single equation:

$$\{\Gamma_i, \Gamma_j\} = \delta_{ij} - A_{ik}A_{jk} \quad , \quad i, j = 1, 2, \ldots, 2d. \tag{3.75}$$

By extending the range of the indices in this relation to odd-dimensions it is possible to define $FIO(2d + 1, \mathbb{R})$, the fermionic inhomogeneous orthogonal algebra of odd dimension.

In order to show the validity of (3.75), one needs to consider the three cases:

$$\{\Gamma_i, \Gamma_j\} = \begin{cases} 2\{Re(\gamma_i), Re(\gamma_j)\} & \text{for } 1 \le i, j \le d \\ 2\{Im(\gamma_i), Im(\gamma_j)\} & \text{for } d+1 \le i, j \le 2d \\ -2\{Re(\gamma_i), Im(\gamma_j)\} & \text{for } 1 \le i \le d \text{ and } d+1 \le j \le 2d \end{cases}$$

For the case when $1 \leq i, j \leq d$, the above form becomes:

$$
\begin{aligned}
\{\Gamma_i, \Gamma_j\} &= 2\{Re(\gamma_i), Re(\gamma_j)\} \\
&= \frac{1}{2}\left[\{\gamma_i, \gamma_j\} + \{\gamma_i, \gamma_j^*\} + \{\gamma_i^*, \gamma_j\} + \{\gamma_i^*, \gamma_j^*\}\right] \\
&= \frac{1}{2}\left[(-\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk}) + (\delta_{ij} - \alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*)\right. \\
&\qquad \left. +(\delta_{ij} - \alpha_{ik}^*\alpha_{jk} - \beta_{ik}^*\beta_{jk}) + (-\beta_{ik}^*\alpha_{jk}^* - \alpha_{ik}^*\beta_{jk}^*)\right] \\
&= \frac{1}{2}\left[2\delta_{ij} - (\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk}^* + \beta_{jk}) - (\beta_{ik} + \alpha_{ik}^*)(\alpha_{jk} + \beta_{jk})\right] \\
&= \delta_{ij} - \frac{1}{2}\left[(\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk}^* + \beta_{jk}) + (\beta_{ik} + \alpha_{ik}^*)(\alpha_{jk} + \beta_{jk})\right]
\end{aligned}
\tag{3.76}
$$

however, for this case, the form of $A_{ik}A_{jk}$ is:

$$
\begin{aligned}
\sum_{k=1}^{2d} A_{ik}A_{jk} &= \sum_{k=1}^{d} A_{ik}A_{jk} + \sum_{k=d+1}^{2d} A_{ik}A_{jk} \\
&= \sum_{k=1}^{d}\left[Re(\alpha_{ik} + \beta_{ik})Re(\alpha_{jk} + \beta_{jk}) + Im(\alpha_{ik} - \beta_{ik})Im(\alpha_{jk} - \beta_{jk})\right] \\
&= \frac{1}{4}\left[(\alpha_{ik} + \beta_{ik} + \alpha_{ik}^* + \beta_{ik}^*)(\alpha_{jk} + \beta_{jk} + \alpha_{jk}^* + \beta_{jk}^*)\right. \\
&\qquad \left. -(\alpha_{ik} - \beta_{ik} - \alpha_{ik}^* + \beta_{ik}^*)(\alpha_{jk} - \beta_{jk} - \alpha_{jk}^* + \beta_{jk}^*)\right] \\
&= \frac{1}{4}\left[(\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk}^* + \beta_{jk}) + (\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk} + \beta_{jk}^*)\right. \\
&\qquad +(\alpha_{ik}^* + \beta_{ik})(\alpha_{jk} + \beta_{jk}) + (\alpha_{ik}^* + \beta_{ik})(\alpha_{jk}^* + \beta_{jk}^*) \\
&\qquad +(\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk}^* + \beta_{jk}) - (\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk} + \beta_{jk}^*) \\
&\qquad \left. +(\alpha_{ik}^* + \beta_{ik})(\alpha_{jk} + \beta_{jk}) - (\alpha_{ik}^* + \beta_{ik})(\alpha_{jk}^* + \beta_{jk}^*)\right] \\
&= \frac{1}{2}\left[(\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk}^* + \beta_{jk}) + (\alpha_{ik}^* + \beta_{ik})(\alpha_{jk} + \beta_{jk})\right]
\end{aligned}
\tag{3.77}
$$

and using this in equation (3.76) gives us:

$$
\begin{aligned}
\{\Gamma_i, \Gamma_j\} &= \delta_{ij} - \frac{1}{2}\left[(\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk}^* + \beta_{jk}) + (\beta_{ik} + \alpha_{ik}^*)(\alpha_{jk} + \beta_{jk})\right] \\
&= \delta_{ij} - A_{ik}A_{jk} \qquad \text{for } 1 \leq i, j \leq d
\end{aligned}
\tag{3.78}
$$

For the case when $d+1 \leq i, j \leq 2d$, the above form becomes:

$$\begin{aligned}
\{\Gamma_i, \Gamma_j\} &= 2\{Im(\gamma_i), Im(\gamma_j)\} \\
&= -\frac{1}{2}\left[\{\gamma_i, \gamma_j\} - \{\gamma_i, \gamma_j^*\} - \{\gamma_i^*, \gamma_j\} + \{\gamma_i^*, \gamma_j^*\}\right] \\
&= -\frac{1}{2}\left[(-\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk}) - (\delta_{ij} - \alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*)\right. \\
&\qquad \left. -(\delta_{ij} - \alpha_{ik}^*\alpha_{jk} - \beta_{ik}^*\beta_{jk}) + (-\beta_{ik}^*\alpha_{jk}^* - \alpha_{ik}^*\beta_{jk}^*)\right] \\
&= -\frac{1}{2}\left[-2\delta_{ij} + (\alpha_{ik}^* - \beta_{ik})(\alpha_{jk} - \beta_{jk}^*) + (\beta_{ik}^* - \alpha_{ik})(\beta_{jk} - \alpha_{jk}^*)\right] \\
&= \delta_{ij} - \frac{1}{2}\left[(\alpha_{ik}^* - \beta_{ik})(\alpha_{jk} - \beta_{jk}^*) + (\alpha_{ik} - \beta_{ik}^*)(\alpha_{jk}^* - \beta_{jk})\right]
\end{aligned} \tag{3.79}$$

however, for this case, the form of $A_{ik}A_{jk}$ is:

$$\begin{aligned}
\sum_{k=1}^{2d} A_{ik}A_{jk} &= \sum_{k=1}^{d} A_{ik}A_{jk} + \sum_{k=d+1}^{2d} A_{ik}A_{jk} \\
&= \sum_{k=1}^{d}\left[Im(\alpha_{ik} + \beta_{ik})Im(\alpha_{jk} + \beta_{jk}) + Re(\alpha_{ik} - \beta_{ik})Re(\alpha_{jk} - \beta_{jk})\right] \\
&= \frac{1}{4}\left[-(\alpha_{ik} + \beta_{ik} - \alpha_{ik}^* - \beta_{ik}^*)(\alpha_{jk} + \beta_{jk} - \alpha_{jk}^* - \beta_{jk}^*)\right. \\
&\qquad \left. +(\alpha_{ik} - \beta_{ik} + \alpha_{ik}^* - \beta_{ik}^*)(\alpha_{jk} - \beta_{jk} + \alpha_{jk}^* - \beta_{jk}^*)\right] \\
&= \frac{1}{4}\left[(\alpha_{ik}^* - \beta_{ik})(\alpha_{jk} - \beta_{jk}^*) - (\alpha_{ik}^* - \beta_{ik})(\alpha_{jk}^* - \beta_{jk})\right. \\
&\qquad +(\alpha_{ik} - \beta_{ik}^*)(\alpha_{jk}^* - \beta_{jk}) - (\alpha_{ik} - \beta_{ik}^*)(\alpha_{jk} - \beta_{jk}^*) \\
&\qquad +(\alpha_{ik}^* - \beta_{ik})(\alpha_{jk} - \beta_{jk}^*) + (\alpha_{ik}^* - \beta_{ik})(\alpha_{jk}^* - \beta_{jk}) \\
&\qquad \left. +(\alpha_{ik} - \beta_{ik}^*)(\alpha_{jk}^* - \beta_{jk}) + (\alpha_{ik} - \beta_{ik}^*)(\alpha_{jk} - \beta_{jk}^*)\right] \\
&= \frac{1}{2}\left[(\alpha_{ik}^* - \beta_{ik})(\alpha_{jk} - \beta_{jk}^*) + (\alpha_{ik} - \beta_{ik}^*)(\alpha_{jk}^* - \beta_{jk})\right]
\end{aligned} \tag{3.80}$$

and using this in equation (3.79) gives us:

$$\begin{aligned}
\{\Gamma_i, \Gamma_j\} &= \delta_{ij} - \frac{1}{2}\left[(\alpha_{ik}^* - \beta_{ik})(\alpha_{jk} - \beta_{jk}^*) + (\alpha_{ik} - \beta_{ik}^*)(\alpha_{jk}^* - \beta_{jk})\right] \\
&= \delta_{ij} - A_{ik}A_{jk} \qquad \text{for } d+1 \leq i, j \leq 2d
\end{aligned} \tag{3.81}$$

Finally, for the case when $1 \le i \le d$ and $d+1 \le j \le 2d$, the above form becomes:

$$\begin{aligned}
\{\Gamma_i, \Gamma_j\} &= -2\{Re(\gamma_i), Im(\gamma_j)\} \\
&= -\frac{1}{2i}\left[\{\gamma_i, \gamma_j\} - \{\gamma_i, \gamma_j^*\} + \{\gamma_i^*, \gamma_j\} - \{\gamma_i^*, \gamma_j^*\}\right] \\
&= -\frac{1}{2i}\left[(-\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk}) - (\delta_{ij} - \alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*)\right. \\
&\qquad \left. +(\delta_{ij} - \alpha_{ik}^*\alpha_{jk} - \beta_{ik}^*\beta_{jk}) - (-\beta_{ik}^*\alpha_{jk}^* - \alpha_{ik}^*\beta_{jk}^*)\right] \\
&= -\frac{1}{2i}\left[(\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk}^* - \beta_{jk}) - (\alpha_{ik}^* + \beta_{ik})(\alpha_{jk} - \beta_{jk}^*)\right]
\end{aligned} \tag{3.82}$$

however, for this case, the form of $A_{ik}A_{jk}$ is:

$$\begin{aligned}
\sum_{k=1}^{2d} A_{ik}A_{jk} &= \sum_{k=1}^{d} A_{ik}A_{jk} + \sum_{k=d+1}^{2d} A_{ik}A_{jk} \\
&= \sum_{k=1}^{d}\left[-Re(\alpha_{ik} + \beta_{ik})Im(\alpha_{jk} + \beta_{jk}) + Im(\alpha_{ik} - \beta_{ik})Re(\alpha_{jk} - \beta_{jk})\right] \\
&= \frac{1}{4i}\left[-(\alpha_{ik} + \beta_{ik} + \alpha_{ik}^* + \beta_{ik}^*)(\alpha_{jk} + \beta_{jk} - \alpha_{jk}^* - \beta_{jk}^*)\right. \\
&\qquad \left. +(\alpha_{ik} - \beta_{ik} - \alpha_{ik}^* + \beta_{ik}^*)(\alpha_{jk} - \beta_{jk} + \alpha_{jk}^* - \beta_{jk}^*)\right] \\
&= \frac{1}{4i}\left[(\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk}^* - \beta_{jk}) - (\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk} - \beta_{jk}^*)\right. \\
&\qquad -(\alpha_{ik}^* + \beta_{ik})(\alpha_{jk} - \beta_{jk}^*) + (\alpha_{ik}^* + \beta_{ik})(\alpha_{jk}^* - \beta_{jk}) \\
&\qquad +(\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk}^* - \beta_{jk}) + (\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk} - \beta_{jk}^*) \\
&\qquad \left. -(\alpha_{ik}^* + \beta_{ik})(\alpha_{jk} - \beta_{jk}^*) - (\alpha_{ik}^* + \beta_{ik})(\alpha_{jk}^* - \beta_{jk})\right] \\
&= \frac{1}{2}\left[(\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk}^* - \beta_{jk}) - (\alpha_{ik}^* + \beta_{ik})(\alpha_{jk} - \beta_{jk}^*)\right]
\end{aligned} \tag{3.83}$$

and using this in equation (3.82) gives us:

$$\begin{aligned}
\{\Gamma_i, \Gamma_j\} &= -\frac{1}{2i}\left[(\alpha_{ik} + \beta_{ik}^*)(\alpha_{jk}^* - \beta_{jk}) - (\alpha_{ik}^* + \beta_{ik})(\alpha_{jk} - \beta_{jk}^*)\right] \\
&= -A_{ik}A_{jk} \qquad \text{for } 1 \le i \le d \text{ and } d+1 \le j \le 2d
\end{aligned} \tag{3.84}$$

which completes the derivation of equation (3.75).

Similar to the analysis that went into finding the sub(quantum)groups of $FIO(2d,\mathbb{R})$, we can also investigate the sub(quantum)groups of $FIO(2d{+}1,\mathbb{R})$. The sub(quantum)group relations in this case, however, are more restricted owing to the fact that the algebra is not described anymore by submatrices of the A matrix but is rather described by the whole matrix itself. Thus, we cannot set α or β to zero on their own, we can only restrict the algebra by setting the whole of A to zero. Thus the resulting sub(quantum)algebra relations are:

$$\delta_{ij} - A_{ik}A_{jk} = 0 \tag{3.85a}$$

$$\Gamma_i = 0 \tag{3.85b}$$

$$A_{ij} = 0 \tag{3.85c}$$

which, through a similar analysis to the even dimensional case, gives us the following sub(quantum)group diagram:

$$\begin{array}{ccccc} FIO(2d+1,\mathbb{R}) & \xrightarrow{(3.85a)} & GrIO(2d+1,\mathbb{R}) & \xrightarrow{(3.85b)} & O(2d+1,\mathbb{R}) \\ {\scriptstyle (3.85c)}\big\downarrow & & & & \\ Cliff(2d+1) & & & & \end{array}$$

4. CONCLUSIONS

The importance of Lie groups in physics arises from the fact that they are invariance groups of classical physical systems. Thus, for example, the 3 dimensional position space, the 3 dimensional momentum space and the 3 dimensional angular momentum space are all transformed under the same Lie group $SO(3)$. When the classical system is quantized one realizes that although the resulting quantum system is invariant under the classical group $SO(3)$, one should also perhaps generalize the definition of a Lie group such that the transformation matrix may have non-commuting entities. This is precisely what has been considered in this work. If one considers the angular momentum algebra and tries to find such a non-commutative quantum group which leaves the commutation relations of the Lie algebra invariant, one finds that the elements of the transformation matrix should be commutative and reobtain the classical group $SO(3)$. On the other hand, as we have shown, when one considers the anticommuting spin algebra, its invariance quantum group becomes $SO_{-1}(3)$. It is also interesting to note that more algebras like the anticommuting spin algebra can be constructed where the original Lie algebra is turned into a similar Jordan algebra. These might also have invariance quantum groups that is the same as the invariance group of the original Lie algebra in the limit $q = 1$. This possibility is open to investigation in a more general framework.

As far as the momentum and position are concerned, one realizes that the Heisenberg algebra inevitably contains the unit operator and therefore the transformations considered on that algebra should be inhomogeneous. It was shown in this work that this approach indeed makes sense by explicitly calculating the invariance quantum groups of the bosonic and fermionic oscillator algebras. In 3 dimensions, the hermitian and antihermitian parts of the annihilation operator can be identified with the position and i times the momentum operator, respectively. For this reason, the invariance quantum group $BISp(2d, \mathbb{R})$ that was introduced becomes the invariance quantum group of the quantum phase space in 3 dimensions. Both the fermionic and the bosonic inhomogeneous quantum groups considered in this work are relevant for field theoretical systems; especially since they can be made infinite dimensional. This is achieved by extending the discrete indices i, j, k in $BISp(2d, \mathbb{R})$ and $FIO(2d, \mathbb{R})$ to continuous variables together with a replacement of the Krönecker deltas to Dirac delta functions.

As was shown, the boson and fermion algebras can be obtained as a limit of the inhomogeneous quantum groups $BISp(2d, \mathbb{R})$ and $FIO(2d, \mathbb{R})$. We can understand why these boson and fermion algebras are not quantum groups from this construction, since in this limit the quantum group becomes singular and the antipode does not exist. Thus we can consider the invariance quantum groups as deformations, with a Hopf algebra structure, of their respective particle algebras. This construction is similar to q-deforming the bosonic oscillator to obtain Pusz-Woronowicz [29] oscillators and then constructing the q-deformed quantum unitary groups as their left modules. Similarly, in that construction, the q-deformed oscillator can be reobtained as a limit of these q-deformed quantum unitary groups. However, in contrast to that construction the quantum groups presented in this paper are inhomogeneous quantum groups.

Lastly, we would like to remark on the definition of a quantum group. Although in most works, quantum groups are defined as noncocommutative and noncommutative Hopf algebras, this definition does not produce any physical insight. It makes more sense to define a quantum group as a Hopf algebra which is a left and/or right module of a physical algebra obtained by quantizing a classical system. This thesis has directly dealt with such quantum groups.

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