QUANTUM GROUP INVARIANCE OF SOME PHYSICAL ALGEBRAS

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For my loving wife Emi for all her support and my wise and understanding mentor Metin Bey for teaching me a lot of what I know.

ABSTRACT

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ÖZET

BAZI FİZİKSEL CEBİRLERIN KUVANTUM GRUP DEĞİŞMEZLİĞİ

Türkçe tez özetini buraya yazınız.

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LIST OF SYMBOLS/ABBREVIATIONS

a_{ij}	Description of a_{ij}
α	Description of α
DA	Description of abbreviation

1. INTRODUCTION

The concept of bosonic and fermionic particles is one of the most important concepts in modern quantum physics. The behavior of large scale matter, from chemical properties of elements to superconductivity and superfluidity can mostly be understood by referring to the fermionic or bosonic nature of the quantum mechanical particles involved in such processes. It is for this reason that understanding the symmetry properties of these phenomena and, motivated by their importance, trying to find other behavior that mimic them is very meaningful.

Furthermore, while bosonic behavior has a classical counterpart, the concept of a fermionic particle is one that can only exist in the quantum domain. This fact makes the study of such behavior even more important. However, what could be more interesting is the study of other such constructs that cannot have a classical counterpart. These constructs would thus belong solely in the quantum domain and could help us understand phenomena that are strictly quantum mechanical in nature.

There is a strong relation between the spin properties of a particle and the particle being a boson or a fermion. In fact, it is a proven fact of quantum physics that integer spin particles are bosons and half-integer spin particles are fermions. This is most often referred to as the "spin-statistics theorem" in quantum mechanics and is a very interesting fact since it implies a relationship between two concepts that seems to be totally unrelated. This strong relation between the bosonic/fermionic nature of a particle and its spin make the angular momentum algebra also very central in quantum physics.

Before we start investigating such matters, it would be apt to give an overview of the state of bosons and fermions and the angular momentum algebra as it has been studied up to now.

When the harmonic oscillator is studied in a quantum mechanical manner, one arrives at the relation:

$$aa^{\dagger} - a^{\dagger}a = 1 \tag{1.1}$$

for the system the Hamiltonian of which is given by $\hbar\omega(aa^{\dagger} + a^{\dagger}a)$. The spectrum of this Hamiltonian, which in turn gives us the allowable energy levels of the quantum harmonic oscillator, can be obtained easily by introducing the hermitian operator $N = a^{\dagger}a$ which has the following relations with a and $a^{\dagger}$:

$$[N, a^{\dagger}] = a^{\dagger} \tag{1.2}$$

$$[N, a] = a \tag{1.3}$$

where [,] denotes the usual commutator. By observing the fact that the Hamiltonian is nothing but $N+\frac{1}{2}$, one can see that one can get the energy levels states as eigenvectors $\mid n \rangle$, of the operator N. The action of $a^{\dagger}$ and a on such an eigenvector $\mid n \rangle$ can be found to be:

$$a^{\dagger} \mid n \rangle = \sqrt{n+1} \mid n+1 \rangle \tag{1.4}$$

$$a \mid n \rangle = \sqrt{n} \mid n-1 \rangle \tag{1.5}$$

which in turn implies that the values of n, the eigenvalues of the operator N, begin from 0 and increase by 1 every time $a^{\dagger}$ is applied on the relevant state. As a result of this study one finds that the energy levels of the quantum harmonic oscillator is given by $\hbar\omega(n + \frac{1}{2})$ and that the operators $a^{\dagger}$ and a are operators that create and destroy, respectively, one quanta of energy. For this reason they are usually called creation and annihilation operators.

Even though this operator algebra seems to only describe the quantum harmonic oscillator, when one studies quantum field theory, this algebra comes up as the algebra of of the Fourier coefficients of the field operator of a bosonic particle. In that setting, the operators $a_p^{\dagger}$ and a_p, which now carry a continuous momentum index, are interpreted as the operators that create and destroy, respectively, one bosonic particle of such a field with momentum p.

For fermionic particle fields as similar treatment again yields creation and annihilation operators as the Fourier coefficients of the field operator; however, the algebra obeyed by these operators is not the same as the algebra of the quantum harmonic oscillator. Instead, the defining relations of this algebra are:

$$aa^\dagger + a^\dagger a = 1 \tag{1.6}$$

$$aa = 0 \tag{1.7}$$

which, through a similar treatment, yields the eigenvalues 0 and 1 for the operator N.

2. The Anticommuting Spin Algebra

2.1. Introduction

The algebra of observables in quantum theory plays a fundamental role. When classical systems are quantized, their classical symmetry algebra acting on a set of physical observables, in simplest examples, remains the same. For some completely integrable non-linear models, consistent quantization requires that the classical symmetry group be replaced by a quantum group [1, 2, 3, 4] via a deformation parameter $q = 1 + O(\hbar)$. In recent years quantum groups involving fermions have received widespread attention. These include deformed fermion algebras [5, 6, 7, 8], spin chains [9, 10, 11] and Fermi gases [12]. At the same time, some quantum systems, most notably fermionic quantum systems do not have any classical analogues. Nevertheless, fermions are perhaps the most important sector of quantum phenomena. Motivated by these considerations, we define a fermionic version of the angular momentum algebra by the relations

$$\{J_1, J_2\} = J_3 \tag{2.1}$$

$$\{J_2, J_3\} = J_1 \tag{2.2}$$

$$\{J_3, J_1\} = J_2 \tag{2.3}$$

where J_1, J_2, J_3 are hermitian generators of the algebra. We will name this algebra ACSA, the anticommutator spin algebra. In these expressions the curly bracket denotes the anticommutator

$$\{A, B\} \equiv AB + BA \tag{2.4}$$

so (2.1-2.3) should be taken as the definition of an associative algebra. This proposed algebra does not fall into the category of superalgebras in the sense of Berezin-Kac axioms. In particular, the algebra is consistent without grading and there are no (graded) Jacobi relations. As it is defined this algebra falls into the category of a (non-exceptional) Jordan algebra where the Jordan product is defined by:

$$A \circ B \equiv \frac{1}{2}(AB + BA) \quad . \tag{2.5}$$

A formal Jordan algebra, in addition to a commutative Jordan product, also satisfies $A^2 \circ (B \circ A) = (A^2 \circ B) \circ A$. When the Jordan product is given in terms of an anticommutator this relation is automatically satisfied. Just as a Lie algebra where the Lie bracket as defined by the commutator leads to an enveloping associative algebra, a Jordan algebra defined in terms of the above product leads to an enveloping associative algebra which we consider as an algebra of observables.

The physical properties of this system turn out to be similar to those of the angular momentum algebra yet exhibit remarkable differences. Since the angular momentum algebra is used to describe various internal symmetries, ACSA could be relevant in describing those symmetries.

In section 2 we will show that ACSA is invariant under the action of the quantum group $SO_q(3)$ with $q = -1$. Here, $SO_q(3)$ is defined as the quantum subgroup of $SU_q(3)$ where each of the (non-commuting) matrix elements of the 3x3 matrix is hermitian. We note that this defines a quantum group only for $q = \pm 1$. For $q = 1$ one has the real orthogonal group $SO(3)$.

In section 3, we will construct all representations of ACSA and show that the representations can be labelled by a quantum number j corresponding to the eigenvalue of J_3 whose absolute value is maximum. For integer j, spectrum of J_3 is given by $j, j-1, \ldots, -j$ whereas for half-integer j there are two representations. These two representations are such that for $j = 2k \pm \frac{1}{2}$ spectrum of J_3 is respectively given by $j, j-2, \ldots, \pm\frac{1}{2}$ and $-j, j+2, \ldots, \mp\frac{1}{2}$. Section 4 is reserved for conclusions and discussion.

2.2. The invariance quantum group $SO_q(3)$, $q = -1$

In order to find the invariance quantum group of this algebra, we transform the generators J_i to J_i' by:

$$J_i' = \sum_j \alpha_{ij} J_j \quad . \tag{2.6}$$

The matrix elements α_{ij} are hermitian since J_i's are hermitian and they commute with J_i's but do not commute with each other. For the transformed operators to obey the original relations, there should exist some conditions on the α's which define the invariance quantum group of the algebra. It is very convenient at this moment to switch to an index notation that encompasses all three defining relations of the algebra in one index equation. For the angular momentum algebra this is possible by defining the totally anti-symmetric rank 3 pseudo-tensor ϵ_{ijk}. A similar object for ACSA which we will call the fermionic Levi-Civita tensor, u_{ijk}, is defined as:

$$u_{ijk} = \begin{cases} 1, & i \neq j \neq k, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

Thus the defining relations (2.1-2.3) become:

$$\{J_i, J_j\} = \sum_k u_{ijk}\, J_k + 2\delta_{ij}\, J_i^2 \tag{2.8}$$

The second term on the right is needed since when $i = j$ the left-hand side becomes $2J_i^2$. When we apply the transformation (2.6) on this relation we get:

$$\{J'_k, J'_m\} = \sum_p u_{kmp} J'_p = \sum_{p,\, j} u_{kmp}\, \alpha_{pj}\, J_j \qquad \text{for} \quad k \neq m. \tag{2.9}$$

However, substituting the transformation equations into the left-hand side, we have:

$$\{J'_k, J'_m\} = \sum_{i,\, j} \left([\alpha_{ki}, \alpha_{mj}] J_i J_j + \alpha_{mj}\, \alpha_{ki}\, u_{ijn}\, J_n + 2\alpha_{mj}\, \alpha_{kj}\, J_j^2 \right) \tag{2.10}$$

These two equations yield the following relations among α_{ij} when $k \neq m$:

$$\{\alpha_{mj}, \alpha_{kj}\} \;=\; 0 \tag{2.11}$$

$$[\alpha_{ki}, \alpha_{mj}] \;=\; 0 \qquad \text{for} \quad i \neq j \tag{2.12}$$

$$\sum_{i\, j} \alpha_{mj}\, \alpha_{ki}\, u_{ijn} \;=\; \sum_p u_{mkp}\, \alpha_{pn} \tag{2.13}$$

Now we will define the quantum group $SO_q(3)$ and show that the relations above correspond to the case $q = -1$. The quantum group $SO_q(3)$ can be defined as the quantum subgroup of $SL_q(3, C)$ where an element is given by:

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \tag{2.14}$$

where

$$\alpha_{ij}^* = \alpha_{ij} \tag{2.15}$$

$$A^T = A^{-1} \tag{2.16}$$

and

$$\begin{pmatrix} \alpha_{mj} & \alpha_{mi} \\ \alpha_{kj} & \alpha_{ki} \end{pmatrix} \in GL_q(2) \quad \text{for} \quad k \neq m, i \neq j. \tag{2.17}$$

The quantum group $SO_q(3)$ is equivalent to the quantum group $SL_q(3,R) \cap SU_q(3)$. However one can show for $SL_q(3)$ that $q = e^{i\beta}$ for some $\beta \in R$ and similarly for $SU_q(3)$ that $q \in R$. Thus one finds that $q = \pm 1$ for $SO_q(3)$. When $q = 1$ the quantum group becomes the usual $SO(3)$ group; the interesting case is when $q = -1$ which, as we will show, is the invariance quantum group of ACSA.

Equations (2.11) and (2.12) are easily shown to be satisfied by the matrix $A \in SO_{q=-1}(3)$ by recognizing that the quantities involved belong to a submatrix that is an element of $GL_{q=-1}(2)$, as in equation (2.17). For a general matrix $M \in GL_q(2)$ where:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have the relations:

$$\begin{aligned} ac &= qca && (2.18) \\ ad - qbc &= da - q^{-1}cb && (2.19) \\ bc &= cb && (2.20) \end{aligned}$$

The relation (2.18) implies that $\alpha_{mj}\,\alpha_{kj} = (-1)\alpha_{kj}\,\alpha_{mj}$, which proves equation (2.11) is satisfied, and the relations (2.19) and (2.20) show $ad = da$ for $q = -1$ which implies $\alpha_{ki}\,\alpha_{mj} = \alpha_{mj}\,\alpha_{ki}$ thus proving that equation (2.12) is satisfied by the elements of an $SO_{q=-1}(3)$ matrix.

It is a little harder to show that equation (2.13) is satisfied by elements of $SO_{q=-1}(3)$ matrices. However, if one writes out the indices of the equation, one finds that equation (2.13) implies that each matrix element is equal to the $GL_{q=-1}(2)$-determinant of its minor. This fact is indeed satisfied by $SO_{q=-1}(3)$ matrices since $\det A = 1$ and ${A^{-1}}^T = A$, one can show that $A = Co(A)$ which itself means that every element is equal to the determinant of its minor. Note that since $q = -1$, the cofactor of an element is always equal to the minor without any alternation of signs. This type of determinant with no alternation of signs is also called a permanent.

Thus, we have found that the invariance quantum group of ACSA is the quantum group $SO_q(3)$ with $q = -1$. Strictly speaking, ACSA is a module of the q-deformed $SO(3)$ quantum algebra with $q = -1$. It is very interesting to note that the invariance group of the angular momentum algebra is also $SO_q(3)$ but with $q = 1$.

2.3. Representations

The Anticommutator Spin Algebra is defined by the relations (2.1-2.3). In order to find the representations of this algebra we define the operators:

$$J_+ = J_1 + J_2 \tag{2.21}$$
$$J_- = J_1 - J_2 \tag{2.22}$$
$$J^2 = J_1^2 + J_2^2 + J_3^2 \tag{2.23}$$

which obey the following relations:

$$\{J_+, J_3\} = J_3 \tag{2.24}$$
$$\{J_-, J_3\} = -J_3 \tag{2.25}$$
$$J_+^2 = J^2 - J_3^2 + J_3 \tag{2.26}$$
$$J_-^2 = J^2 - J_3^2 - J_3 \tag{2.27}$$

Furthermore, it can easily be shown that J^2 is central in the algebra, ie that it commutes with all the elements of the algebra. For this reason, we can label the states in our representation with the eigenvalues of J^2 and J_3:

$$J^2 \mid \lambda, \mu \rangle = \lambda \mid \lambda, \mu \rangle \tag{2.28}$$

$$J_3 \mid \lambda, \mu \rangle = \mu \mid \lambda, \mu \rangle \tag{2.29}$$

The action of J_+ and J_- on the states such defined is easily shown to be:

$$J_+ \mid \lambda, \mu \rangle = f(\lambda, \mu) \mid \lambda, -\mu + 1 \rangle \tag{2.30}$$

$$J_- \mid \lambda, \mu \rangle = g(\lambda, \mu) \mid \lambda, -\mu - 1 \rangle \tag{2.31}$$

It is enough to look at the norm of the states $J_+ \mid \lambda, \mu \rangle$ and $J_- \mid \lambda, \mu \rangle$ to find $f(\lambda, \mu)$ and $g(\lambda, \mu)$. Thus:

$$\langle \lambda, \mu \mid J_+^2 \mid \lambda, \mu \rangle = |f(\lambda, \mu)|^2 \tag{2.32}$$

$$\langle \lambda, \mu \mid J^2 - J_3^2 + J_3 \mid \lambda, \mu \rangle = |f(\lambda, \mu)|^2 \tag{2.33}$$

$$\lambda - \mu^2 + \mu = |f(\lambda, \mu)|^2 \tag{2.34}$$

$$f(\lambda, \mu) = \sqrt{\lambda - \mu^2 + \mu} \tag{2.35}$$

and, similarly, $g(\lambda, \mu) = \sqrt{\lambda - \mu^2 - \mu}$. These coefficients must be real due to the fact that J_+ and J_- are hermitian operators. This constraint imposes the following conditions on λ and μ:

$$\lambda - \mu^2 + \mu \geq 0 \tag{2.36}$$

$$\lambda - \mu^2 - \mu \geq 0 \tag{2.37}$$

which can be satisfied by letting $\lambda = j(j+1)$ for some j with:

$$j \geq \mu \geq -j. \tag{2.38}$$

Note that equation (2.30) shows that the action of J_+ is composed of a reflection which changes sign of μ, the eigenvalue of J_3, followed by raising by one unit. Similarly, equation (2.31) shows that J_- reflects and lowers. Thus the highest state $\mu = j$ is annihilated by J_- and "lowered" by J_+. Applying J_+ or J_- twice to any state gives back the same state due to relations (2.26) and (2.27). Thus starting from the highest state we apply J_- and J_+ alternately to get the spectrum:

$$j, -j+1, j-2, -j+3, ... \tag{2.39}$$

This sequence ends so as to satisfy equation (2.38) only for integer or half-integer j. For integer j, it terminates, after an even number of steps, at $-j$ and visits every integer in between only once. For half-integer $j = 2k \pm \frac{1}{2}$ it ends at $j = \pm\frac{1}{2}$ having visited only half the states with μ half-odd integer between j and $-j$. The rest of the states cannot be reached from these but are obtained by starting from the $\mu = -j$ state and applying J_- and J_+ alternately; starting with J_-.

We now give a few examples:

- For $\mathbf{j} = \mathbf{2}$ the states follow the sequence:

$$\mu = \mathbf{2}, -\mathbf{1}, \mathbf{0}, \mathbf{1}, -\mathbf{2} \quad .$$

- For $\mathbf{j} = \frac{\mathbf{3}}{\mathbf{2}}$ there exist two irreducible representations one with:

$$\mu = \frac{\mathbf{3}}{\mathbf{2}}, -\frac{\mathbf{1}}{\mathbf{2}} \quad ,$$

  and the other with:

$$\mu = -\frac{\mathbf{3}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}} \quad .$$

- For $\mathbf{j} = \frac{\mathbf{5}}{\mathbf{2}}$ the two representations are given by:

$$\mu = \frac{\mathbf{5}}{\mathbf{2}}, -\frac{\mathbf{3}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}} \quad ,$$

and by:

$$\mu = -\frac{\mathbf{5}}{\mathbf{2}}, \frac{\mathbf{3}}{\mathbf{2}}, -\frac{\mathbf{1}}{\mathbf{2}} \quad .$$

2.4. Conclusions

The Anticommutator Spin Algebra, which is a special Jordan algebra, has many implications. The first of these is the fact that this algebra is a consistent fermionic algebra which is not a superalgebra. For possible physical applications the right-hand side of the defining relations (2.1-2.3) must also be supplied with an $\hbar$. In a superalgebra approach where the J_i are regarded as odd operators, the $\hbar$ on the right-hand side should also be regarded as an operator anticommuting with the J_i. These models [13, 14] result from the quantization of the odd Poisson bracket. In our approach however, the concept of grading and therefore an underlying Poisson bracket formalism does not exist. In particular, there is no Jacobi identity. Nevertheless, the associative algebra we consider is consistent with quantum mechanics where physical observables correspond to hermitian operators and their eigenvalues to possible results of physical measurement of these observables. It is for this reason that ACSA suggests a new kind of statistics which, we believe, will be useful in physics.

The second implication is the important role of quantum groups in mathematical physics. As we have shown in this paper, the invariance group of ACSA turns out to be a quantum group. Given the fact that ACSA is very similar to normal spin algebra and that the invariance group of spin algebra plays an important role in physics, the invariance quantum group of ACSA, $SO_{q=-1}(3)$, becomes a prime example of how central quantum groups have become in mathematical physics. It is also interesting to note that more algebras like ACSA can be constructed where the commutators of the original Lie algebra are turned into anticommutators and that such algebras might also have invariance quantum groups that is the same as the invariance group of the original Lie algebra with $q = -1$. This possibility is open to investigation in a more general framework.

3. Hopf Algebras associated with invariance of non-deformed oscillators

The concepts of bosons and fermions lie at the heart of microscopic physics. They are described in terms of creation and annihilation operators of the corresponding particle algebra:

$$c_i c_j \mp c_j c_i = 0 \tag{3.1}$$

$$c_i c_j^* \mp c_j^* c_i = \delta_{ij} \tag{3.2}$$

where the upper sign is for the boson algebra $BA(d)$ and the lower sign is for the fermion algebra $FA(d)$.

It has been realized that quantum algebras play an important role in the description of physical phenomena. Some classical physical systems which are invariant under a classical Lie group, when quantized, are invariant under a quantum group [15]. The quantum groups thus considered turn out to be q-deformations of the classical semisimple groups. On the other hand, inhomogeneous quantum groups [16, 17] are perhaps more interesting since classical inhomogeneous groups such as the Poincaré group are more important in physics.

In this paper we will investigate an important class of inhomogeneous quantum groups which are related to the boson algebra $BA(d)$ and the fermion algebra $FA(d)$. Although $BA(d)$ and $FA(d)$ themselves are not quantum groups, by considering quantum group versions of symmetry transformations acting on these algebras, one can arrive at these inhomogeneous quantum groups. Mathematically speaking we are thus interested in constructing left modules of these algebras such that these modules have Hopf algebra structure.

Traditionally the boson algebra has the symmetry group $ISp(2d,\mathbb{R})$, the inhomogeneous symplectic group, which transforms creation and annihilation operators as:

$$c_i \rightarrow \alpha_{ij} c_j + \beta_{ij} c_j^* + \gamma_i \quad . \tag{3.3}$$

In this transformation $\alpha_{ij}, \beta_{ij}, \gamma_i$ are complex numbers satisfying the constraints required by the group $ISp(2d, \mathbb{R})$. One should note that this symmetry group is also the group of linear canonical transformations of a classical dynamical system. An important physical application of this transformation is the Bogoliubov transformation which is crucial in the explanation of many quantum mechanical effects such as the Unruh Effect [18] and Hawking Radiation [19]. In the case of the Hawking Radiation, the physical reinterpretation of the transformed operators imply that the future vacuum state is annihilated by the transformed annihilation operator, which is related to the initial creation and annihilation operators by a Bogoliubov transformation.

Similar to the boson algebra, the fermion algebra has the classical symmetry group $O(2d)$ with the transformation law:

$$c_i \rightarrow \alpha_{ij} c_j + \beta_{ij} c_j^* \quad . \tag{3.4}$$

however, unlike its bosonic counterpart this algebra is not inhomogeneous. This fact is the primary motivation for the generalization that we are going to offer. By relaxing the conditions on the transformation coefficients such as commutativity, one can come up with inhomogeneous invariance (quantum)groups for fermions and for bosons alike. The explicit R-matrices utilizing the quantum group properties of these structures have already been presented [20, 21]. In this paper, after a brief definition of these quantum groups $FIO(2d, \mathbb{R})$, the fermionic inhomogeneous orthogonal quantum group, and $BISp(2d, \mathbb{R})$, the bosonic inhomogeneous symplectic quantum group, in Section 1, we will investigate their sub(quantum)groups and also study the (quantum)groups obtained by their contractions. In the last section, $FIO(2d + 1, \mathbb{R})$, the fermionic inhomogeneous quantum orthogonal group in odd number of dimensions, will also be defined and its properties examined.

3.1. $FIO(2d,\mathbb{R})$ and $BISp(2d,\mathbb{R})$

A general transformation of a particle algebra can be described in the following way:

$$\begin{pmatrix} c' \\ c^{*\prime} \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ 0 & 0 & 1 \end{pmatrix} \dot{\otimes} \begin{pmatrix} c \\ c^* \\ 1 \end{pmatrix} \tag{3.5}$$

where c, c^*, γ, γ^* are column matrices and $\alpha, \beta, \alpha^*, \beta^*$ are $d \times d$ matrices. Thus, in index notation the transformation is given by:

$$\begin{aligned} c'_i &= \alpha_{ij} \otimes c_j + \beta_{ij} \otimes c^*_j + \gamma_i \otimes 1 \quad , & (3.6) \\ {c^*}'_i &= \alpha^*_{ij} \otimes c^*_j + \beta^*_{ij} \otimes c_j + \gamma^*_i \otimes 1 \quad . & (3.7) \end{aligned}$$

Given this transformation, we look for an algebra $\mathcal{A}$ generated by these matrix elements such that the particle algebra remains invariant. Thus, we first write the transformation matrix in the above equation in the following way:

$$M = \left(\begin{array}{cc|c} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & \Gamma \\ \hline 0 & 1 \end{array}\right) \quad . \tag{3.8}$$

We assume that $\alpha_{ij}, \beta_{ij}, \gamma_i$ belong to a possibly noncommutative algebra on which a hermitian conjugation denoted by $*$ is defined. We also assume that the matrix elements of A form a commutative subalgebra.

Applying this transformation and requiring that the bosonic/fermionic particle algebra remains invariant after the transformation, we arrive at the following relations that the transformation parameters should obey:

$$\gamma_i\gamma_j^* \mp \gamma_j^*\gamma_i = \delta_{ij} - \alpha_{ik}\alpha_{jk}^* \pm \beta_{ik}\beta_{jk}^* \quad (3.9)$$

$$\gamma_i\gamma_j \mp \gamma_j\gamma_i = \pm\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} \quad (3.10)$$

$$\alpha_{ij}\gamma_k \mp \gamma_k\alpha_{ij} = 0 \quad (3.11)$$

$$\beta_{ij}\gamma_k \mp \gamma_k\beta_{ij} = 0 \quad (3.12)$$

$$\alpha_{ij}\gamma_k^* \mp \gamma_k^*\alpha_{ij} = 0 \quad (3.13)$$

$$\beta_{ij}\gamma_k^* \mp \gamma_k^*\beta_{ij} = 0 \quad (3.14)$$

together with the $*$-conjugates of these relations. In these relations the upper and lower signs are for the transformation of bosons and fermions, respectively. Furthermore according to our assumption above, the set $\alpha_{ij}, \beta_{i,j}, \alpha_{ij}^*, \beta_{ij}^*$ forms a commutative algebra.

The set of matrices M obeying the above relations form the group of inhomogeneous transformations of bosons and fermions. For bosons, we name this group $BISp(2d, \mathbb{R})$ and for fermions $FIO(2d, \mathbb{R})$. These symmetry groups, however, are not classical groups but are in fact quantum groups with a Hopf algebra structure. As shown in [21], this Hopf algebra has an explicit R-matrix representation and the coproduct, counit and antipode are defined as:

$$\Delta(M) = M\dot{\otimes}M \quad (3.15)$$

$$\epsilon(M) = I \quad (3.16)$$

$$S(M) = M^{-1} \quad . \quad (3.17)$$

In Equation 3.15, the symbol $\dot{\otimes}$ denotes the usual matrix multiplication where when elements of the matrices are multiplied, tensor multiplication is used.

The inverse of the matrix M can be defined as:

$$M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\Gamma \\ 0 & 1 \end{pmatrix} \tag{3.18}$$

where A^{-1} is defined in the standard way since matrix elements of A were assumed to be commutative.

3.2. Subgroups and Contractions

After having shown that the inhomogeneous transformations form the symmetry groups $FIO(2d,\mathbb{R})$ and $BISp(2d,\mathbb{R})$ and that they are quantum groups, one important question is what sub(quantum)groups do these quantum groups have. For example, we know that the group $ISp(2d,\mathbb{R})$ is an important special subgroup of $BISp(2d,\mathbb{R})$ and other sub(quantum)groups could turn out to have similarly important physical applications. While searching for sub(quantum)groups, we would also like to find new (quantum)groups allowed by suitable contractions [22] of these quantum groups as well.

The sub(quantum)groups of these algebras are obtained by imposing additional relations on the matrix elements of M which obey the relations (3.9) - (3.14). The additional relations that we will impose are:

(a) $\delta_{ij} - \alpha_{ik}\alpha^*_{jk} \pm \beta_{ik}\beta^*_{jk} = \pm\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} = 0$

(b) $\gamma_i = 0$

(c) $\beta_{ij} = 0$

(d) $\alpha_{ij} = 0$

We would like to study the implication of each relation one by one in the bosonic and the fermionic case.

3.2.1. Inhomogeneous Sub(super)groups

The relation **(a)**:

$$\delta_{ij} - \alpha_{ik}\alpha^*_{jk} \pm \beta_{ik}\beta^*_{jk} = \pm\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} = 0$$

by virtue of (3.9) and (3.10) implies that $\gamma_i\gamma^*_j \mp \gamma^*_j\gamma_i = 0$ and $\gamma_i\gamma_j \mp \gamma_j\gamma_i = 0$, i.e. that the inhomogeneous transformation parameters are commutative or anticommutative variables.

In the case of the fermionic particles, we end up with an inhomogeneous orthogonal algebra where the inhomogeneous parameters are grassmanian variables thus giving us the group $GrIO(2d,\mathbb{R})$ as the resulting subgroup of $FIO(2d,\mathbb{R})$. This group can be considered to be an inhomogeneous supergroup.

More generally, in the fermionic case, α_{ij}, β_{ij}, α^*_{ij}, β^*_{ij} anticommute with γ_i, γ^*_i and the $FIO(2d,\mathbb{R})$ matrices M are multiplied with each other using the standard tensor product. It can additionally be shown that α_{ij}, β_{ij}, α^*_{ij}, β^*_{ij} can be taken to commute with γ_i, γ^*_i provided that the matrices M are multiplied with a braided [23] tensor product, eg.

$$(A \otimes B)(C \otimes D) = -AC \otimes BD \tag{3.19}$$

whenever B and C are both fermionic. This also corresponds to the usual superalgebra approach.

For bosonic particles, applying the relation **(a)** gives us the classical symmetry group of the bosonic particle algebra $ISp(2d,\mathbb{R})$ in which all the parameters of the inhomogeneous transformation are commutative.

3.2.2. Homogeneous Subgroups

The relation **(b)**:

$$\gamma_i = 0$$

practically gets rid of the inhomogeneous part of the transformation and also implies the relation **(a)** considered in the previous subsection.

For the fermionic particles, this gives us the subgroup in the previous subsection without the inhomogeneous part which is the classical orthogonal group $O(2d, \mathbb{R})$ and similarly, for bosonic particles, gives us the classical symplectic group $Sp(2d, \mathbb{R})$.

3.2.3. Fermionic and Bosonic Inhomogeneous Unitary Quantum Groups

The relation **(c)**:

$$\beta_{ij} = 0$$

applied to the transformation gets rid of the off-diagonal members of the homogeneous part of it and leaves us with the following relation:

$$\gamma_i \gamma_j^* \mp \gamma_j^* \gamma_i = \delta_{ij} - \alpha_{ik} \alpha_{jk}^* \tag{3.20}$$

For the homogeneous part of the transformation, this equation implies $\delta_{ij} = \alpha_{ik}\alpha_{jk}^*$ which tells us that the submatrices α and α^* in equation (3.8) are both members of $U(d)$. The subgroup we have arrived at thus is an inhomogeneous quantum group whose homogeneous part is $U(d)$. For fermions we will name this group $FIU(d)$, the fermionic inhomogeneous quantum group, and for bosons we will similarly name it $BIU(d)$, the bosonic inhomogeneous quantum group.

3.2.4. Fermion and Boson Algebra

The relation **(d)**:

$$\alpha_{ij} = 0$$

applied alone onto the transformation gets rid of the diagonal members of the homogeneous part and prevents such transformations from forming a (quantum)group since the homogeneous parts of these set of transformations can never include the identity transformation.

However, if this relation is applied together with the previous one, relation **(c)**, the resulting relation gets rid of the whole homogeneous part of the transformation leaving only the inhomogeneous part and gives us a single relation:

$$\gamma_i \gamma_j^* \mp \gamma_j^* \gamma_i = \delta_{ij} \tag{3.21}$$

which gives us back the fermion algebra, $FA(d)$, for the fermionic case and the boson algebra, $BA(d)$, for the bosonic case.

We should note, however, that after this condition is applied, the resulting set of matrices M, which form $FA(d)$ or $BA(d)$, are no longer quantum or classical groups since the antipode defined in equation (3.17) no longer exits. For this reason, these algebras can be considered to be a boundary for the subgroups of their corresponding quantum groups.

3.2.5. Sub(quantum)group Diagram

As a result of the above discussion, we get the sub(quantum)group diagram:

$$\begin{tikzcd} FIO(2d,\mathbb{R}) \arrow[r, "\textbf{(a)}"] \arrow[d, "\textbf{(c)}"'] & GrIO(2d,\mathbb{R}) \arrow[r, "\textbf{(b)}"] \arrow[d, "\textbf{(c)}"'] & O(2d,\mathbb{R}) \arrow[d, "\textbf{(c)}"'] \\ FIU(d) \arrow[r, "\textbf{(a)}"] \arrow[d, "\textbf{(d)}"'] & GrIU(d) \arrow[r, "\textbf{(b)}"] & U(d) \\ FA(d) & & \end{tikzcd}$$

for the fermionic case and the diagram:

$$\begin{tikzcd} BISp(2d,\mathbb{R}) \arrow[r, "\textbf{(a)}"] \arrow[d, "\textbf{(c)}"'] & ISp(2d,\mathbb{R}) \arrow[r, "\textbf{(b)}"] \arrow[d, "\textbf{(c)}"'] & Sp(2d,\mathbb{R}) \arrow[d, "\textbf{(c)}"'] \\ BIU(d) \arrow[r, "\textbf{(a)}"] \arrow[d, "\textbf{(d)}"'] & IU(d) \arrow[r, "\textbf{(b)}"] & U(d) \\ BA(d) & & \end{tikzcd}$$

for the bosonic case.

3.2.6. Contractions of $FIO(2d,\mathbb{R})$ and $BISp(2d,\mathbb{R})$

In order to explore the new (quantum)groups that will come about as the result of a contraction, we replace γ_i by $\gamma_i/\hbar$ so that we may consider the case $\hbar \to 0$. After this replacement, the equations (3.9) and (3.10) become:

$$\gamma_i\gamma_j^* \mp \gamma_j^*\gamma_i = \hbar(\delta_{ij} - \alpha_{ik}\alpha_{jk}^* \pm \beta_{ik}\beta_{jk}^*) \tag{3.22}$$

$$\gamma_i\gamma_j \mp \gamma_j\gamma_i = \hbar(\pm\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk}) \tag{3.23}$$

When we consider the case $\hbar \to 0$, we get the relations:

$$\gamma_i\gamma_j^* \mp \gamma_j^*\gamma_i = 0 \tag{3.24}$$

$$\gamma_i\gamma_j \mp \gamma_j\gamma_i = 0 \tag{3.25}$$

which imply that the inhomogeneous part of the transformation form grassmanian variables for the fermionic case and ordinary complex numbers for the bosonic case. What makes this case different from the previous case of sub(super)groups is that the homogeneous part of this transformation forms a matrix A with non-zero determinant. We can transform such a matrix A with a similarity transformation given by the unitary matrix:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{3.26}$$

to put it in a real form. The transformation gives:

$$\begin{aligned} A' &= UAU^\dagger && (3.27)\\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} && (3.28)\\ &= \begin{pmatrix} Re(\alpha)+Re(\beta) & Im(\alpha)-Im(\beta) \\ -Im(\alpha)-Im(\beta) & Re(\alpha)-Re(\beta) \end{pmatrix} && (3.29) \end{aligned}$$

which is a real matrix that is a member of the general linear group $GL(2d,\mathbb{R})$. Thus for the fermionic case we have the group $GrIGL(2d,\mathbb{R})$, the grassmanian inhomogeneous general linear group, and for the bosonic case we have $IGL(2d,\mathbb{R})$, the inhomogeneous general linear group.

If we consider the contraction of the subgroups as well then we should examine the $\hbar \to 0$ limit after the relations **(c)** and **(d)** are applied.

After we apply relation **(c)**, we get the subgroups, $FIU(d)$ and $BIU(d)$ as discussed previously. After, the contraction, again the inhomogeneous part of these groups become grassmanian variables and complex numbers for the fermionic and bosonic cases respectively. However, if we apply the previous similarity transformation on the homogeneous part, we get:

$$\begin{aligned}
A' &= UAU^{\dagger} && (3.30)\\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} && (3.31)\\
&= \begin{pmatrix} Re(\alpha) & Im(\alpha) \\ -Im(\alpha) & Re(\alpha) \end{pmatrix} && (3.32)\\
&= Re(\alpha)\mathbb{I} + Im(\alpha)\mathbb{J} && (3.33)
\end{aligned}$$

where $\mathbb{I}$ stands for the identity matrix and $\mathbb{J}$ stands for the matrix the square of which is minus the identity matrix. This way we can see that the matrix A' is a actually member of $GL(d,\mathbb{C})$. This gives us $GrIGL(d,\mathbb{C})$ as the group for the contraction in the fermionic case and $IGL(d,\mathbb{C})$ for the bosonic case.

We have previously shown that we get the fermionic and bosonic algebras after applying both of the relations **(c)** and **(d)**. We have also discussed that in this case only the inhomogeneous part of the transformation survives and after applying the contraction the inhomogeneous part of the transformation turns into grassmanian or complex variables. Thus in this case, the contraction of $FA(d)$ gives us $Gr(d,\mathbb{C})$ and the contraction of $BA(d)$ gives us $\mathbb{C}^d$.

As a summary, for the contraction considered combined with the remaining subgroup relations we get the tables:

$$
\begin{array}{ccc}
FIO(2d,\mathbb{R}) & \xrightarrow{\textbf{(e)}} & GrIGL(2d,\mathbb{R}) \\
\textbf{(c)}\Big\downarrow & & \textbf{(c)}\Big\downarrow \\
FIU(d) & \xrightarrow{\textbf{(e)}} & GrIGL(d,\mathbb{C}) \\
\textbf{(d)}\Big\downarrow & & \textbf{(d)}\Big\downarrow \\
FA(d) & \xrightarrow{\textbf{(e)}} & Gr(d,\mathbb{C})
\end{array}
$$

for the fermionic case and:

$$
\begin{array}{ccc}
BISp(2d,\mathbb{R}) & \xrightarrow{\textbf{(e)}} & IGL(2d,\mathbb{R}) \\
\textbf{(c)}\Big\downarrow & & \textbf{(c)}\Big\downarrow \\
BIU(d) & \xrightarrow{\textbf{(e)}} & IGL(d,\mathbb{C}) \\
\textbf{(d)}\Big\downarrow & & \textbf{(d)}\Big\downarrow \\
BA(d) & \xrightarrow{\textbf{(e)}} & \mathbb{C}^d
\end{array}
$$

for the bosonic case.

3.3. Fermionic Inhomogeneous Orthogonal Algebra of Odd Dimension

When we consider a unitary transformation of the $FIO(2d,\mathbb{R})$ matrix as:

$$M \to UMU^{-1}$$

using the unitary matrix:

$$U = \left(\begin{array}{cc|c} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{3.34}$$

we can see that we can put M in the real form:

$$\left(\begin{array}{cc|c} Re(\alpha+\beta) & Im(\alpha-\beta) & \sqrt{2}Re(\gamma) \\ Im(-\alpha-\beta) & Re(\alpha-\beta) & \sqrt{2}Im(\gamma) \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & \Gamma \\ \hline 0 & 1 \end{array}\right) \tag{3.35}$$

where A and Γ matrices are defined as in (3.8), and Re and Im denote the hermitian and anti-hermitian parts.

Using this form it is found that for $FIO(2d, \mathbb{R})$, the transformation relations (3.9) and (3.10) together become:

$$[\Gamma_i, \Gamma_j]_+ = \delta_{ij} - A_{ik}A_{jk} \qquad , i, j = 1, 2, \ldots, 2d. \tag{3.36}$$

By extending the range of the indices in this relation to odd-dimensions it is possible to define $FIO(2d+1, \mathbb{R})$, the fermionic inhomogeneous orthogonal algebra of odd dimension.

Similar to the analysis that went into finding the sub(quantum)groups of $FIO(2d, \mathbb{R})$, we can also investigate the sub(quantum)groups of $FIO(2d+1, \mathbb{R})$. The sub(quantum)group relations in this case, however, are more restricted owing to the fact that the algebra is not described anymore by submatrices of the A matrix but is rather described by the whole matrix itself. Thus, we cannot set α or β to zero on their own, we can only restrict the algebra by setting the whole of A to zero. Thus the resulting sub(quantum)algebra relations are:

(a) $\delta_{ij} - A_{ik}A_{jk} = 0$

(b) $\Gamma_i = 0$

(c) $A_{ij} = 0$

which, through a similar analysis to the even dimensional case, gives us the following sub(quantum)group diagram:

$$
\begin{array}{ccccc}
FIO(2d+1,\mathbb{R}) & \xrightarrow{\textbf{(a)}} & GrIO(2d+1,\mathbb{R}) & \xrightarrow{\textbf{(b)}} & O(2d+1,\mathbb{R}) \\
\Big\downarrow{\scriptstyle \textbf{(c)}} & & & & \\
Clif(2d+1) & & & &
\end{array}
$$

3.4. Discussion

As we have shown, the boson and fermion algebras can be obtained as a limit of the inhomogeneous quantum groups $BISp(2d,\mathbb{R})$ and $FIO(2d,\mathbb{R})$. We can understand why these boson and fermion algebras are not quantum groups from this construction, since in this limit the quantum group becomes singular and the antipode does not exist. Thus we can consider these quantum groups as deformations with a Hopf algebra structure of their respective particle algebras. This construction is similar to q-deforming the bosonic oscillator to obtain Pusz-Woronowicz [24] oscillators and then constructing the q-deformed quantum unitary groups as their left modules. Similarly, in that construction, the q-deformed oscillator can be reobtained as a limit of these q-deformed quantum unitary groups. However, unlike that construction the quantum groups presented in this paper are inhomogeneous quantum groups.

Finally, we would like to remark that the widely used field theoretical generalization achieved by extending the discrete indices i, j, k to continuous variables together with a replacement of the Krönecker deltas to Dirac delta functions is also applicable to the quantum groups we have presented. In this respect, these quantum groups are also different from the Pusz-Woronowicz oscillators which cannot be extended to continuous indices.

We believe that the establishment of these and similar quantum groups in field theory will be helpful in generalizing methods of quantization. These approaches will yield a more consistent approach to interacting field theory and will be the subject of further investigations.

4. Conclusions

APPENDIX A: Explicit calculation of ACSA invariance relations

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