QUANTUM GROUP INVARIANCE OF SOME PHYSICAL ALGEBRAS

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For my loving wife Emi for all her support; my wise and understanding mentor Metin Bey for teaching me a lot of what I know.

ABSTRACT

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Type your abstract here.

ÖZET

BAZI FİZİKSEL CEBİRLERİN KUANTUM GRUP DEĞİŞMEZLİĞİ

Türkçe tez özetini buraya yazınız.

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LIST OF SYMBOLS/ABBREVIATIONS

a_{ij}	Description of a_{ij}
α	Description of α
DA	Description of abbreviation

1. INTRODUCTION

1.1. Bosons and fermions

The concept of bosonic and fermionic particles is one of the most important concepts in modern quantum physics. The behavior of large scale matter, from chemical properties of elements to superconductivity and superfluidity can mostly be understood by referring to the fermionic or bosonic nature of the quantum mechanical particles involved in such processes. It is for this reason that understanding the symmetry properties of these phenomena and, motivated by their importance, trying to find other behavior that mimic them is very meaningful.

Furthermore, while bosonic behavior has a classical counterpart, the concept of a fermionic particle is one that can only exist in the quantum domain. This fact makes the study of such behavior even more important. However, what could be more interesting is the study of other such constructs that cannot have a classical counterpart. These constructs would thus belong solely in the quantum domain and could help us understand phenomena that are strictly quantum mechanical in nature.

There is a strong relation between the spin properties of a particle and the particle being a boson or a fermion. In fact, it is a proven fact of quantum physics that integer spin particles are bosons and half-integer spin particles are fermions. This is most often referred to as the "spin-statistics theorem" in quantum mechanics and is a very interesting fact since it implies a relationship between two concepts that seem to be totally unrelated. This strong relation between the bosonic/fermionic nature of a particle and its spin make the angular momentum algebra also very central in quantum physics.

Before we start investigating such matters, it would be apt to give an overview of the state of bosons and fermions and the angular momentum algebra as it has been studied up to now.

When the harmonic oscillator is studied in a quantum mechanical manner, one arrives at the relation:

$$aa^\dagger - a^\dagger a = 1 \tag{1.1}$$

for the system the Hamiltonian of which is given by $\frac{\hbar\omega}{2}(aa^\dagger + a^\dagger a)$. The spectrum of this Hamiltonian, which in turn gives us the allowable energy levels of the quantum harmonic oscillator, can be obtained easily by introducing the hermitian operator $N = a^\dagger a$ which satisfies the following relations with a and $a^\dagger$:

$$[N, a^\dagger] = a^\dagger \tag{1.2}$$

$$[N, a] = a \tag{1.3}$$

where [,] denotes the usual commutator. By observing the fact that the Hamiltonian is nothing but $\hbar\omega(N + \frac{1}{2})$, one can see that one can get the states that correspond to the energy levels as eigenvectors $\mid n \,\rangle$, of the operator N. The action of $a^\dagger$ and a on such an eigenvector $\mid n \,\rangle$ is found to be:

$$a^\dagger \mid n \,\rangle = \sqrt{n+1} \mid n+1 \,\rangle \tag{1.4}$$

$$a \mid n \,\rangle = \sqrt{n} \mid n-1 \,\rangle \tag{1.5}$$

Due to the fact that the operator N is a positive hermitian operator, its eigenvalues, namely n, cannot be negative. For a given positive value of n, however, one can construct states with eigenvalues $n-1$, $n-2$, $n-3$, and so on, by repeatedly applying the operator a on the original state. This sequence of eigenvalues will contain negative values eventually for any given finite n unless it is an integer. In that case, the sequence will end at the eigenvalue 0 since a further application of the operator a on that state will give us the zero vector of the Hilbert space which is not a physically observable state and is thus a state out of our domain.

As a result of this study one finds that the values of n, the eigenvalues of the operator N, begin from 0 and increase by 1 every time $a^\dagger$ is applied on the relevant state and that the energy levels of the quantum harmonic oscillator are given by $\hbar\omega(n+\frac{1}{2})$. The operators $a^\dagger$ and a turn out to be operators that create and destroy, respectively, one quanta of energy and for this reason they are usually called creation and annihilation operators.

Even though this operator algebra seems to only describe the quantum harmonic oscillator, when one studies quantum field theory, this algebra comes up as the algebra of the Fourier coefficients of the field operator describing a bosonic particle. Each normal mode of a quantum field behaves as if it is an independent harmonic oscillator and for that reason we have a separate set of creation and annihilation operators for each of these modes. In that setting, the operators $a_p^\dagger$ and a_p, which now carry a continuous momentum index, are interpreted as the operators that create and destroy, respectively, one bosonic particle of such a field with momentum p.

For fermionic particles the story is a little bit more different. In 1925, Pauli first proposed his "exclusion principle" to explain the behavior of electrons in an atom. According to this principle, no two electrons could exist in the same quantum state and it was for this reason that electrons could not all occupy the lowest energy state in the atomic orbitals but instead had to line up the energy levels in a well ordered manner. The implication of this principle to the electron gas was first considered by Fermi and Dirac and it is for this reason that particles that obey these statistics are called "fermions". In 1926 Dirac noted that the "exclusion principle" could apply to other particles by relating bosons and fermions to the symmetry of the many-particle wavefunction. If the wavefunction changes sign upon exchange of two particles then those particles would be fermions and they would be bosons if the wavefunction did not change sign. This treatment effectively implies the Pauli exclusion principle since if there were to be two fermionic particles occupying the same quantum state, then upon their exchange the wavefunction would change sign; on the other hand, we expect the wavefunction to be identical to the original one before the exchange since nothing must have changed about the quantum state of the system. For this reason the original wavefunction can be nothing but zero if it is to be equal to its negative in this manner. Thus, by contradiction, one can show that no two fermions can exist in the same quantum state. It was only later, in 1928, that Jordan and Wigner proposed that in order to treat fermions in quantum field theory, their field operators had to anticommute so that the wavefunction could be antisymmetric. They showed that a consistent second-quantization of fermions implied anticommutation relations on the field operators. This is turn implies that the Fourier coefficients of the field operators that belong to a normal mode also obey anticommutation relations instead of the commutation relations that the bosonic creation and annihilation operators obey.

In this work, we would like to give an alternative derivation of this algebra by only starting from the Pauli exclusion principle and assuming that fermionic particles also have creation and annihilation operators just like the bosonic particles. If this is the case then Pauli exclusion principle tells us that we cannot create a second fermion in the same quantum state, i.e. that $(a^\dagger)^2$, and in turn a^2, should be 0. This relation, however, is not compatible with the commutation relation (1.1) and thus should be supplemented with another kind of relation. If we define the operator K as the anticommutator of a and $a^\dagger$:

$$K \equiv aa^\dagger + a^\dagger a \tag{1.6}$$

then we find that K is a central element of the algebra, since:

$$\begin{aligned} a^\dagger K &= a^\dagger(aa^\dagger + a^\dagger a) = & a^\dagger a a^\dagger \qquad (1.7) \\ K a^\dagger &= (aa^\dagger + a^\dagger a)a^\dagger = & a^\dagger a a^\dagger \qquad (1.8) \end{aligned}$$

which implies that K commutes with $a^\dagger$ and similarly with a, thus making it a central element of the algebra. The central operator K can be written as a multiple of the identity $k\mathbb{I}$ and if we rescale the operators a and $a^\dagger$ by $1/\sqrt{k}$, we arrive at the fermion anticommutator algebra:

$$a^2 = 0 \tag{1.9}$$

$$aa^\dagger + a^\dagger a = 1 \tag{1.10}$$

This derivation of the fermion algebra also shows clearly that the physically more important relation is the fact that the square of the annihilation operator is zero, since the other relation follows from this fact. In literature, it is often the case that only the anticommutation relation is presented as describing fermionic particles, completely omitting the other, more important, relation. This is usually falsely motivated by the assumption that the anticommutation relation uniquely describes a fermionic system just as the commutation relation alone describes a bosonic system. As we show in Appendix A of this work, however, without the first relation, the anticommutation relation alone describes a completely different system which still has two states but is not equivalent to the fermionic system.

A study of this fermion algebra, similar to the boson algebra, shows that, again, a hermitian positive-definite number operator $N = a^\dagger a$ can be defined and has eigenvalues 0 and 1 that correspond to the states $\mid 0 \rangle$ and $\mid 1 \rangle$, respectively. In harmony with our original assumption, the operator $a^\dagger$ takes the state $\mid 0 \rangle$ to the state $\mid 1 \rangle$ thus fulfilling the interpretation of it as a creation operator. Similarly, the operator a acts as an annihilation operator of the algebra.

1.2. Quantum groups and Hopf algebras

The discovery of quantum groups has historically been motivated by the study of quantization of non-linear completely integrable systems [1].

1.3. Quantum matrix groups

A quantum matrix group is defined by a set of n x n matrices M:

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix} \tag{1.11}$$

such that every element of the matrix belong to a Hopf Algebra $\mathcal{H}$. The matrix group defined in this way naturally becomes a Hopf algebra with the coproduct, counit and coinverse of the matrix algebra being defined as:

$$\triangle(M) = M\dot{\otimes}M \tag{1.12}$$

$$\epsilon(M) = \mathbb{I} \tag{1.13}$$

$$S(M) = M^{-1} \tag{1.14}$$

where $\dot{\otimes}$ stands for the operation where when the matrix multiplication is performed but the matrix elements are multiplied using the tensor product is instead of the normal product. The relations above imply the definitions of the coproduct, counit and coinverse of the matrix elements:

$$\triangle(\alpha_{ij}) = \sum_k \alpha_{ik} \otimes \alpha_{kj} \tag{1.15}$$

$$\epsilon(\alpha_{ij}) = \delta_{ij} \tag{1.16}$$

$$\sum_j S(\alpha_{ij})\alpha_{jk} = \delta_{ij} = \sum_j \alpha_{ij} S(\alpha_{jk}) \tag{1.17}$$

1.4. Quantum group invariance of an algebra

Given an algebra $\mathcal{A}$ with generators a_1, a_2, …, a_n, one can construct the column matrix:

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{1.18}$$

1.5. Summary

2. THE ANTICOMMUTING SPIN ALGEBRA

2.1. Defining relations

$$\{J_1, J_2\} = J_3 \tag{2.1}$$

$$\{J_2, J_3\} = J_1 \tag{2.2}$$

$$\{J_3, J_1\} = J_2 \tag{2.3}$$

where J_1, J_2, J_3 are hermitian generators of the algebra. We will name this algebra ACSA, the anticommutator spin algebra. In these expressions the curly bracket denotes the anticommutator

$$\{A, B\} \equiv AB + BA \tag{2.4}$$

so (2.1-2.3) should be taken as the definition of an associative algebra. This proposed algebra does not fall into the category of superalgebras in the sense of Berezin-Kac axioms. In particular, the algebra is consistent without grading and there are no (graded) Jacobi relations. As it is defined this algebra falls into the category of a (non-exceptional) Jordan algebra where the Jordan product is defined by:

$$A \circ B \equiv \frac{1}{2}(AB + BA) \quad . \tag{2.5}$$

A formal Jordan algebra, in addition to a commutative Jordan product, also satisfies $A^2 \circ (B \circ A) = (A^2 \circ B) \circ A$. When the Jordan product is given in terms of an anticommutator this relation is automatically satisfied. Just as a Lie algebra where the Lie bracket as defined by the commutator leads to an enveloping associative algebra, a Jordan algebra defined in terms of the above product leads to an enveloping associative algebra which we consider as an algebra of observables.

The physical properties of this system turn out to be similar to those of the angular momentum algebra yet exhibit remarkable differences. Since the angular momentum algebra is used to describe various internal symmetries, ACSA could be relevant in describing those symmetries.

In section 2 we will show that ACSA is invariant under the action of the quantum group $SO_q(3)$ with $q = -1$. Here, $SO_q(3)$ is defined as the quantum subgroup of $SU_q(3)$ where each of the (non-commuting) matrix elements of the 3x3 matrix is hermitian. We note that this defines a quantum group only for $q = \pm 1$. For $q = 1$ one has the real orthogonal group $SO(3)$.

In section 3, we will construct all representations of ACSA and show that the representations can be labelled by a quantum number j corresponding to the eigenvalue of J_3 whose absolute value is maximum. For integer j, spectrum of J_3 is given by $j, j-1, \ldots, -j$ whereas for half-integer j there are two representations. These two representations are such that for $j = 2k \pm \frac{1}{2}$ spectrum of J_3 is respectively given by $j, j-2, \ldots, \pm\frac{1}{2}$ and $-j, j+2, \ldots, \mp\frac{1}{2}$. Section 4 is reserved for conclusions and discussion.

2.2. The invariance quantum group $SO_{q=-1}(3)$

In order to find the invariance quantum group of this algebra, we transform the generators J_i to J_i' by:

$$J_i' = \sum_j \alpha_{ij} J_j \quad . \tag{2.6}$$

The matrix elements α_{ij} are hermitian since J_i's are hermitian and they commute with J_i's but do not commute with each other. For the transformed operators to obey the original relations, there should exist some conditions on the α's which define the invariance quantum group of the algebra. It is very convenient at this moment to switch to an index notation that encompasses all three defining relations of the algebra in one index equation. For the angular momentum algebra this is possible by defining the totally anti-symmetric rank 3 pseudo-tensor ϵ_{ijk}. A similar object for ACSA which we will call the fermionic Levi-Civita tensor, u_{ijk}, is defined as:

$$u_{ijk} = \begin{cases} 1, & i \neq j \neq k, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

Thus the defining relations (2.1-2.3) become:

$$\{J_i, J_j\} = \sum_k u_{ijk}\, J_k + 2\delta_{ij}\, J_i^2 \tag{2.8}$$

The second term on the right is needed since when $i = j$ the left-hand side becomes $2J_i^2$. When we apply the transformation (2.6) on this relation we get:

$$\{J'_k, J'_m\} = \sum_p u_{kmp} J'_p = \sum_{p,\, j} u_{kmp}\, \alpha_{pj}\, J_j \qquad \text{for} \quad k \neq m. \tag{2.9}$$

However, substituting the transformation equations into the left-hand side, we have:

$$\{J'_k, J'_m\} = \sum_{i,\, j} \left([\alpha_{ki}, \alpha_{mj}] J_i J_j + \alpha_{mj}\, \alpha_{ki}\, u_{ijn}\, J_n + 2\alpha_{mj}\, \alpha_{kj}\, J_j^2 \right) \tag{2.10}$$

These two equations yield the following relations among α_{ij} when $k \neq m$:

$$\{\alpha_{mj}, \alpha_{kj}\} \quad = \quad 0 \tag{2.11}$$

$$[\alpha_{ki}, \alpha_{mj}] \quad = \quad 0 \qquad \text{for} \quad i \neq j \tag{2.12}$$

$$\sum_{i\, j} \alpha_{mj}\, \alpha_{ki}\, u_{ijn} \quad = \quad \sum_p u_{mkp}\, \alpha_{pn} \tag{2.13}$$

Now we will define the quantum group $SO_q(3)$ and show that the relations above correspond to the case $q = -1$. The quantum group $SO_q(3)$ can be defined as the quantum subgroup of $SL_q(3,C)$ where an element is given by:

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \tag{2.14}$$

where

$$\alpha^*_{ij} = \alpha_{ij} \tag{2.15}$$

$$A^T = A^{-1} \tag{2.16}$$

and

$$\begin{pmatrix} \alpha_{mj} & \alpha_{mi} \\ \alpha_{kj} & \alpha_{ki} \end{pmatrix} \in GL_q(2) \quad \text{for} \quad k \neq m, i \neq j. \tag{2.17}$$

The quantum group $SO_q(3)$ is equivalent to the quantum group $SL_q(3,R) \cap SU_q(3)$. However one can show for $SL_q(3)$ that $q = e^{i\beta}$ for some $\beta \in R$ and similarly for $SU_q(3)$ that $q \in R$. Thus one finds that $q = \pm 1$ for $SO_q(3)$. When $q = 1$ the quantum group becomes the usual $SO(3)$ group; the interesting case is when $q = -1$ which, as we will show, is the invariance quantum group of ACSA.

Equations (2.11) and (2.12) are easily shown to be satisfied by the matrix $A \in SO_{q=-1}(3)$ by recognizing that the quantities involved belong to a submatrix that is an element of $GL_{q=-1}(2)$, as in equation (2.17). For a general matrix $M \in GL_q(2)$ where:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have the relations:

$$\begin{aligned} ac &= qca \qquad (2.18) \\ ad - qbc &= da - q^{-1}cb \qquad (2.19) \\ bc &= cb \qquad (2.20) \end{aligned}$$

The relation (2.18) implies that $\alpha_{mj}\,\alpha_{kj} = (-1)\alpha_{kj}\,\alpha_{mj}$, which proves equation (2.11) is satisfied, and the relations (2.19) and (2.20) show $ad = da$ for $q = -1$ which implies $\alpha_{ki}\,\alpha_{mj} = \alpha_{mj}\,\alpha_{ki}$ thus proving that equation (2.12) is satisfied by the elements of an $SO_{q=-1}(3)$ matrix.

It is a little harder to show that equation (2.13) is satisfied by elements of $SO_{q=-1}(3)$ matrices. However, if one writes out the indices of the equation, one finds that equation (2.13) implies that each matrix element is equal to the $GL_{q=-1}(2)$-determinant of its minor. This fact is indeed satisfied by $SO_{q=-1}(3)$ matrices since $\det A = 1$ and ${A^{-1}}^T = A$, one can show that $A = Co(A)$ which itself means that every element is equal to the determinant of its minor. Note that since $q = -1$, the cofactor of an element is always equal to the minor without any alternation of signs. This type of determinant with no alternation of signs is also called a permanent.

Thus, we have found that the invariance quantum group of ACSA is the quantum group $SO_q(3)$ with $q = -1$. Strictly speaking, ACSA is a module of the q-deformed $SO(3)$ quantum algebra with $q = -1$. It is very interesting to note that the invariance group of the angular momentum algebra is also $SO_q(3)$ but with $q = 1$.

2.3. Representations

The Anticommutator Spin Algebra is defined by the relations (2.1-2.3). In order to find the representations of this algebra we define the operators:

$$J_+ = J_1 + J_2 \tag{2.21}$$

$$J_- = J_1 - J_2 \tag{2.22}$$

$$J^2 = J_1^2 + J_2^2 + J_3^2 \tag{2.23}$$

which obey the following relations:

$$\{J_+, J_3\} = J_3 \tag{2.24}$$

$$\{J_-, J_3\} = -J_3 \tag{2.25}$$

$$J_+^2 = J^2 - J_3^2 + J_3 \tag{2.26}$$

$$J_-^2 = J^2 - J_3^2 - J_3 \tag{2.27}$$

Furthermore, it can easily be shown that J^2 is central in the algebra, i.e. that it commutes with all the elements of the algebra, by first observing that:

$$\begin{aligned}
J_j^2\ J_i &= J_j(J_k - J_i\ J_j) \\
&= J_j J_k - (J_k - J_i\ J_j)J_j \\
&= (J_j\ J_k - J_k\ J_j) + J_i\ J_j^2 \\
&= (2J_j\ J_k - J_i) + J_i\ J_j^2 \quad \text{for} \quad i \neq j \neq k.
\end{aligned} \tag{2.28}$$

Using this relation and the fact that $J^2 = \sum_j J_j$, we can see:

$$\begin{aligned}
J^2\ J_i &= \sum_j J_j^2\ J_i \\
&= J_i^3 + \sum_{j \neq j} J_j^2\ J_i \\
&= J_i^3 + \sum_{j \neq i} (2J_j\ J_k - J_i + J_i\ J_j^2)
\end{aligned} \tag{2.29}$$

However, in the final form of this expression the sum only contains two terms where the two indices j and k are symmetric. Thus the whole expression can be written as:

$$\begin{aligned} J^2\, J_i &= J_i^3 + \sum_{j\neq i}(2J_j\, J_k - J_i + J_i\, J_j^2) \\ &= J_i^3 - 2J_i + 2(J_j\, J_k + J_k\, J_j) + J_i\, J_j^2 + J_i\, J_k^2 \\ &= J_i\, J^2 - 2J_i + 2J_i \\ &= J_i\, J^2 \quad \text{for} \quad i \neq j \neq k, \end{aligned} \tag{2.30}$$

and therefore showing that J^2 commutes with all the elements of the algebra.

For this reason, we can label the states in our representation with the eigenvalues of J^2 and J_3:

$$J^2 \mid \lambda, \mu \,\rangle = \lambda \mid \lambda, \mu \,\rangle \tag{2.31}$$

$$J_3 \mid \lambda, \mu \,\rangle = \mu \mid \lambda, \mu \,\rangle \tag{2.32}$$

The action of J_+ and J_- on the states such defined is easily shown to be:

$$J_+ \mid \lambda, \mu \,\rangle = f(\lambda, \mu) \mid \lambda, -\mu + 1 \,\rangle \tag{2.33}$$

$$J_- \mid \lambda, \mu \,\rangle = g(\lambda, \mu) \mid \lambda, -\mu - 1 \,\rangle \tag{2.34}$$

It is enough to look at the norm of the states $J_+ \mid \lambda, \mu \,\rangle$ and $J_- \mid \lambda, \mu \,\rangle$ to find $f(\lambda, \mu)$ and $g(\lambda, \mu)$. Thus:

$$\langle\, \lambda, \mu \mid J_+^2 \mid \lambda, \mu \,\rangle = |f(\lambda, \mu)|^2 \tag{2.35}$$

$$\langle\, \lambda, \mu \mid J^2 - J_3^2 + J_3 \mid \lambda, \mu \,\rangle = |f(\lambda, \mu)|^2 \tag{2.36}$$

$$\lambda - \mu^2 + \mu = |f(\lambda, \mu)|^2 \tag{2.37}$$

$$f(\lambda, \mu) = \sqrt{\lambda - \mu^2 + \mu} \tag{2.38}$$

and, similarly, $g(\lambda, \mu) = \sqrt{\lambda - \mu^2 - \mu}$. These coefficients must be real due to the fact that J_+ and J_- are hermitian operators. This constraint imposes the following conditions on λ and μ:

$$\lambda - \mu^2 + \mu \geq 0 \quad (2.39)$$

$$\lambda - \mu^2 - \mu \geq 0 \quad (2.40)$$

which can be satisfied by letting $\lambda = j(j+1)$ for some j with:

$$j \geq \mu \geq -j. \quad (2.41)$$

Note that equation (2.33) shows that the action of J_+ is composed of a reflection which changes sign of μ, the eigenvalue of J_3, followed by raising by one unit. Similarly, equation (2.34) shows that J_- reflects and lowers. Thus the highest state $\mu = j$ is annihilated by J_- and "lowered" by J_+. Applying J_+ or J_- twice to any state gives back the same state due to relations (2.26) and (2.27). Thus starting from the highest state we apply J_- and J_+ alternately to get the spectrum:

$$j, -j+1, j-2, -j+3, ... \quad (2.42)$$

This sequence ends so as to satisfy equation (2.41) only for integer or half-integer j. For integer j, it terminates, after an even number of steps, at $-j$ and visits every integer in between only once. For half-integer $j = 2k \pm \frac{1}{2}$ it ends at $j = \pm\frac{1}{2}$ having visited only half the states with μ half-odd integer between j and $-j$. The rest of the states cannot be reached from these but are obtained by starting from the $\mu = -j$ state and applying J_- and J_+ alternately; starting with J_-.
We now give a few examples:

- For $\mathbf{j} = \mathbf{2}$ the states follow the sequence:

$$\mu = \mathbf{2}, \mathbf{-1}, \mathbf{0}, \mathbf{1}, \mathbf{-2} \quad .$$

- For $\mathbf{j} = \frac{\mathbf{3}}{\mathbf{2}}$ there exist two irreducible representations one with:

$$\mu = \frac{\mathbf{3}}{\mathbf{2}}, -\frac{\mathbf{1}}{\mathbf{2}} \quad ,$$

  and the other with:

$$\mu = -\frac{\mathbf{3}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}} \quad .$$

- For $\mathbf{j} = \frac{\mathbf{5}}{\mathbf{2}}$ the two representations are given by:

$$\mu = \frac{\mathbf{5}}{\mathbf{2}}, -\frac{\mathbf{3}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}} \quad ,$$

  and by:

$$\mu = -\frac{\mathbf{5}}{\mathbf{2}}, \frac{\mathbf{3}}{\mathbf{2}}, -\frac{\mathbf{1}}{\mathbf{2}} \quad .$$

2.4. Hopf Algebra Structure with braiding

One natural question to ask having considered this associative algebra is whether or not it has a Hopf algebra structure. On the surface, this algebra shares a lot with its sister algebra, the $SU(2)$ Lie algebra, which has a Hopf algebra structure and one would expect ACSA to similarly have one. It turns out, however, that naively trying the same coproduct rule for ACSA does not work due to the symmetric nature of the product defined on ACSA since the product is defined in terms of anticommutators. As was noted in the Introduction of this work, the coproduct of the Lie algebra requires the product on the Lie algebra to be anti-symmetric. For this reason, the coproduct of the $SU(2)$ Lie algebra is not suitable for ACSA.

In our quest for a Hopf algebra structure for ACSA, it would be more fruitful to understand the nature of the relationship of ACSA with the $SU(2)$ Lie algebra. If one names the generators of the $SU(2)$ algebra I_i, then it can easily be shown that the generators of ACSA can be written as:

$$J_i = -I_i \otimes \sigma_i \tag{2.43}$$

since:

$$\begin{aligned}
J_iJ_j + J_jJ_i &= I_iI_j \otimes \sigma_i\sigma_j + I_jI_i \otimes \sigma_j\sigma_i \\
&= I_iI_j \otimes i\sigma_k + I_jI_i \otimes -i\sigma_k \\
&= i(I_iI_j - I_jI_i) \otimes \sigma_k \\
&= i(iI_k) \otimes \sigma_k \\
&= -I_k \otimes \sigma_k \\
&= J_k \quad \text{for} \quad i \neq j \neq k.
\end{aligned}$$

Similarly, the $SU(2)$ algebra generators can be written in terms of ACSA generators as:

$$I_i = J_i \otimes \sigma_i \tag{2.44}$$

since:

$$\begin{aligned}
I_iI_j - I_jI_i &= J_iJ_j \otimes \sigma_i\sigma_j - J_jJ_i \otimes \sigma_j\sigma_i \\
&= J_iJ_j \otimes i\sigma_k - J_jJ_i \otimes -i\sigma_k \\
&= i(J_iJ_j + J_jJ_i) \otimes \sigma_k \\
&= i(J_k) \otimes \sigma_k \\
&= I_k \quad \text{for} \quad i \neq j \neq k.
\end{aligned}$$

These two relations show that the $SU(2)$ algebra and ACSA are so closely related that it is not even possible to identify which one of these algebras is more fundamental.

Both of them can be written in terms of the generators of the other and their algebraic structure can be derived from the structure of the other one. However, as mentioned, the Hopf algebra structure of ACSA cannot be derived from the Hopf algebra structure of the $SU(2)$ algebra. Specifically, ACSA does not admit a coproduct defined in a normal way using the usual tensor products. Such a coproduct can be defined if one were to extend the permutation operator τ used in the connecting relation. Normally the operation of τ is defined as:

$$\tau(A \otimes B) = B \otimes A \quad , \tag{2.45}$$

however, if one considers the algebra to be graded and one were to define a degree operator (deg) which is 0 for bosonic variables and is 1 for fermionic variables, then the natural redefinition of the τ operator is

$$\tau(A \otimes B) = (-1)^{deg\ A\ deg\ B}\ B \otimes A \quad . \tag{2.46}$$

Using this redefined permutation operator, one can still write down the Hopf algebra relations and only the connecting relation will be the one that will be redefined. Thus, we arrive at a braided Hopf algebra structure.

When the permutation operator is redefined in this way, the product of two tensor product terms is given by $(A \otimes B)(C \otimes D) = (-1)^{deg\ B\ deg\ C}\ (AB \otimes CD)$ where the -1 factor comes in because of the reordering of the B and C terms. Using this rule and defining the degree of 1 as 0 and the degrees of J_1, J_2, J_3 as 1, we can see that the coproduct defined as:

$$\Delta(J_i) = 1 \otimes J_i + J_i \otimes 1 \tag{2.47}$$

$$\Delta(1) = 1 \otimes 1 \tag{2.48}$$

satisfies the algebra structure relations since:

$$\begin{aligned}
\Delta(J_i)\Delta(J_j) &= (1\otimes J_i + J_i\otimes 1)(1\otimes J_j + J_j\otimes 1) \\
&= 1\otimes J_iJ_j - 1J_j\otimes J_i1 + J_i1\otimes 1J_j + J_iJ_j\otimes 1 \\
&= 1\otimes J_iJ_j - J_j\otimes J_i + J_i\otimes J_j + J_iJ_j\otimes 1
\end{aligned}$$

and

$$\begin{aligned}
\Delta(J_i)\Delta(J_j) + \Delta(J_j)\Delta(J_i) &= 1\otimes J_iJ_j - J_j\otimes J_i + J_i\otimes J_j + J_iJ_j\otimes 1 \\
&\quad +1\otimes J_jJ_i - J_i\otimes J_j + J_j\otimes J_i + J_jJ_i\otimes 1 \\
&= 1\otimes J_iJ_j + J_iJ_j\otimes 1 \\
&= 1\otimes J_k + J_k\otimes 1 \\
&= \Delta(J_k) \quad \text{for} \quad i\neq j\neq k.
\end{aligned}$$

The counit and coinverse are simpler and they match with the definitions for the normal Lie algebra, i.e.:

$$\epsilon(J_i) = 0 \tag{2.49}$$

$$S(J_i) = -J_i \tag{2.50}$$

These definitions of the coproduct, the counit and the coinverse give us a braided Hopf algebra structure for ACSA.

3. QUANTUM GROUPS ASSOCIATED WITH INVARIANCE OF NON-DEFORMED OSCILLATORS

The concepts of bosons and fermions lie at the heart of microscopic physics. They are described in terms of creation and annihilation operators of the corresponding particle algebra:

$$c_i c_j \mp c_j c_i = 0 \tag{3.1}$$

$$c_i c_j^* \mp c_j^* c_i = \delta_{ij} \tag{3.2}$$

where the upper sign is for the boson algebra $BA(d)$ and the lower sign is for the fermion algebra $FA(d)$.

It has been realized that quantum algebras play an important role in the description of physical phenomena. Some classical physical systems which are invariant under a classical Lie group, when quantized, are invariant under a quantum group [16]. The quantum groups thus considered turn out to be q-deformations of the classical semisimple groups. On the other hand, inhomogeneous quantum groups [17, 18] are perhaps more interesting since classical inhomogeneous groups such as the Poincaré group are more important in physics.

In this paper we will investigate an important class of inhomogeneous quantum groups which are related to the boson algebra $BA(d)$ and the fermion algebra $FA(d)$. Although $BA(d)$ and $FA(d)$ themselves are not quantum groups, by considering quantum group versions of symmetry transformations acting on these algebras, one can arrive at these inhomogeneous quantum groups. Mathematically speaking we are thus interested in constructing left modules of these algebras such that these modules have Hopf algebra structure.

Traditionally the boson algebra has the symmetry group $ISp(2d,\mathbb{R})$, the inhomogeneous symplectic group, which transforms creation and annihilation operators as:

$$c_i \to \alpha_{ij} c_j + \beta_{ij} c_j^* + \gamma_i \quad . \tag{3.3}$$

In this transformation $\alpha_{ij}, \beta_{ij}, \gamma_i$ are complex numbers satisfying the constraints required by the group $ISp(2d, \mathbb{R})$. One should note that this symmetry group is also the group of linear canonical transformations of a classical dynamical system. An important physical application of this transformation is the Bogoliubov transformation which is crucial in the explanation of many quantum mechanical effects such as the Unruh Effect [19] and Hawking Radiation [20]. In the case of the Hawking Radiation, the physical reinterpretation of the transformed operators imply that the future vacuum state is annihilated by the transformed annihilation operator, which is related to the initial creation and annihilation operators by a Bogoliubov transformation.

Similar to the boson algebra, the fermion algebra has the classical symmetry group $O(2d)$ with the transformation law:

$$c_i \to \alpha_{ij} c_j + \beta_{ij} c_j^* \quad . \tag{3.4}$$

however, unlike its bosonic counterpart this algebra is not inhomogeneous. This fact is the primary motivation for the generalization that we are going to offer. By relaxing the conditions on the transformation coefficients such as commutativity, one can come up with inhomogeneous invariance (quantum)groups for fermions and for bosons alike. The explicit R-matrices utilizing the quantum group properties of these structures have already been presented [21, 22]. In this paper, after a brief definition of these quantum groups $FIO(2d, \mathbb{R})$, the fermionic inhomogeneous orthogonal quantum group, and $BISp(2d, \mathbb{R})$, the bosonic inhomogeneous symplectic quantum group, in Section 1, we will investigate their sub(quantum)groups and also study the (quantum)groups obtained by their contractions. In the last section, $FIO(2d + 1, \mathbb{R})$, the fermionic inhomogeneous quantum orthogonal group in odd number of dimensions, will also be defined and its properties examined.

A general transformation of a particle algebra can be described in the following way:

$$\begin{pmatrix} c' \\ c^{*\prime} \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ 0 & 0 & 1 \end{pmatrix} \dot{\otimes} \begin{pmatrix} c \\ c^* \\ 1 \end{pmatrix} \tag{3.5}$$

where c, c^*, γ, γ^* are column matrices and $\alpha, \beta, \alpha^*, \beta^*$ are $d \times d$ matrices. Thus, in index notation the transformation is given by:

$$c'_i = \alpha_{ij} \otimes c_j + \beta_{ij} \otimes c^*_j + \gamma_i \otimes 1 \quad , \tag{3.6}$$

$$c^{*\prime}_{\ i} = \alpha^*_{ij} \otimes c^*_j + \beta^*_{ij} \otimes c_j + \gamma^*_i \otimes 1 \quad . \tag{3.7}$$

Given this transformation, we look for an algebra $\mathcal{A}$ generated by these matrix elements such that the particle algebra remains invariant. Thus, we first write the transformation matrix in the above equation in the following way:

$$M = \left(\begin{array}{cc|c} \alpha & \beta & \gamma \\ \beta^* & \alpha^* & \gamma^* \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & \Gamma \\ \hline 0 & 1 \end{array}\right) \quad . \tag{3.8}$$

We assume that $\alpha_{ij}, \beta_{ij}, \gamma_i$ belong to a possibly noncommutative algebra on which a hermitian conjugation denoted by $*$ is defined.

3.1. The Bosonic Inhomogeneous Symplectic Quantum Group $BISp(2d,\mathbb{R})$

If we consider the transformation matrix (3.8) being applied to the boson algebra given by:

$$
\begin{aligned}
c_i c_j - c_j c_i &= 0 && (3.9)\\
c_i c_j^* - c_j^* c_i &= \delta_{ij} && (3.10)
\end{aligned}
$$

then we require that the transformed operators c_i' and ${c^*}_i'$ are required to satisfy the same algebra in order for the transformation to be an algebra invariance. Thus we require that:

$$
\begin{aligned}
c_i' c_j' - c_j' c_i' &= 0 && (3.11)\\
c_i' {c^*}_j' - {c^*}_j' c_i' &= \delta_{ij} && (3.12)
\end{aligned}
$$

Explicitly writing out the transformed operators, these relations become:

$$
\begin{aligned}
&(\alpha_{ik}\otimes c_k + \beta_{ik}\otimes c_k^* + \gamma_i\otimes 1)(\alpha_{jl}\otimes c_l + \beta_{jl}\otimes c_l^* + \gamma_j\otimes 1)\\
&-(\alpha_{jl}\otimes c_l + \beta_{jl}\otimes c_l^* + \gamma_j\otimes 1)(\alpha_{ik}\otimes c_k + \beta_{ik}\otimes c_k^* + \gamma_i\otimes 1) = 0 && (3.13)\\
&(\alpha_{ik}\otimes c_k + \beta_{ik}\otimes c_k^* + \gamma_i\otimes 1)(\alpha_{jl}^*\otimes c_l^* + \beta_{jl}^*\otimes c_l + \gamma_j^*\otimes 1)\\
&-(\alpha_{jl}^*\otimes c_l^* + \beta_{jl}^*\otimes c_l + \gamma_j^*\otimes 1)(\alpha_{ik}\otimes c_k + \beta_{ik}\otimes c_k^* + \gamma_i\otimes 1) = \delta_{ij} && (3.14)
\end{aligned}
$$

which gives us:

$$
\begin{aligned}
&[\alpha_{ik},\alpha_{jl}]c_l c_k + [\beta_{ik},\beta_{jl}]c_l^* c_k^*\\
&+[\alpha_{ik},\gamma_j]c_k + [\beta_{ik},\gamma_j]c_k^*\\
&+[\gamma_i,\alpha_{jl}]c_l + [\gamma_i,\beta_{jl}]c_l^*\\
&+[\alpha_{ik},\beta_{jl}]c_k c_l^* + [\beta_{ik},\alpha_{jl}]c_k^* c_l\\
&+(\alpha_{jk}\beta_{ik} - \beta_{jk}\alpha_{ik} + [\gamma_i,\gamma_j]) = 0 \quad , && (3.15)
\end{aligned}
$$

and

$$\begin{aligned}
[\alpha_{ik},\beta^*_{jl}]c_lc_k+[\beta_{ik},\alpha^*_{jl}]c^*_lc^*_k & \\
+[\alpha_{ik},\gamma^*_j]c_k+[\beta_{ik},\gamma^*_j]c^*_k & \\
+[\gamma_i,\beta^*_{jl}]c_l+[\gamma_i,\alpha_{jl}]c^*_l & \\
+[\beta_{ik},\beta^*_{jl}]c_kc^*_l+[\alpha_{ik},\alpha^*_{jl}]c^*_kc_l & \\
+(\alpha^*_{jk}\alpha_{ik}-\beta^*_{jk}\beta_{ik}+[\gamma_i,\gamma^*_j]) &= \delta_{ij} \quad . \qquad (3.16)
\end{aligned}$$

In the first of these relations, for the equality to be satisfied, it is sufficient for the coefficients of all the terms on the left hand side to be equal to zero. In the second one, however, we only have a term that is a multiple of the unit element of the boson algebra on the right hand side, thus the coefficient of that term should be equal on both sides and it is sufficient for the coefficients of the other terms on the left hand side to be separately equal to zero.

Thus we have the following relations between the transformation elements:

$$\begin{aligned}
\gamma_i\gamma^*_j-\gamma^*_j\gamma_i &= \delta_{ij}-\alpha_{ik}\alpha^*_{jk}+\beta_{ik}\beta^*_{jk} && (3.17)\\
\gamma_i\gamma_j-\gamma_j\gamma_i &= \beta_{ik}\alpha_{jk}-\alpha_{ik}\beta_{jk} && (3.18)\\
\alpha_{ij}\gamma_k-\gamma_k\alpha_{ij} &= 0 && (3.19)\\
\beta_{ij}\gamma_k-\gamma_k\beta_{ij} &= 0 && (3.20)\\
\alpha_{ij}\gamma^*_k-\gamma^*_k\alpha_{ij} &= 0 && (3.21)\\
\beta_{ij}\gamma^*_k-\gamma^*_k\beta_{ij} &= 0 && (3.22)
\end{aligned}$$

and any two elements from the set $\alpha_{ij},\beta_{ij},\alpha^*_{ij},\beta^*_{ij}$ commute.

The set of matrices M obeying the above relations form the group of inhomogeneous transformations of bosons. We name this quantum group as the bosonic inhomogeneous symplectic quantum group $BISp(2d,\mathbb{R})$ since it is an inhomogeneous extension of the symplectic group where the inhomogeneous part exhibits bosonic behavior. This symmetry group, however, is not a classical group like the symplectic group but is in fact a quantum group with a Hopf algebra structure. As shown in [22], this Hopf algebra has an explicit R-matrix representation and the coproduct, counit and coinverse are defined as:

$$\Delta(M) = M\dot{\otimes}M \tag{3.23}$$

$$\epsilon(M) = I \tag{3.24}$$

$$S(M) = M^{-1} \quad . \tag{3.25}$$

In Equation 3.53, the symbol $\dot{\otimes}$ denotes the usual matrix multiplication where when elements of the matrices are multiplied, tensor multiplication is used.

The inverse of the matrix M can be defined as:

$$M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\Gamma \\ 0 & 1 \end{pmatrix} \tag{3.26}$$

where A^{-1} is defined in the standard way since matrix elements of A are shown to be commutative.

3.1.1. Subgroups

After having shown that the inhomogeneous transformations of the boson algebra forms the symmetry quantum group $BISp(2d,\mathbb{R})$, one important question is what sub(quantum)groups does this quantum group have. For example, we know that the group $ISp(2d,\mathbb{R})$ is an important special subgroup of $BISp(2d,\mathbb{R})$ and other sub(quantum)groups could turn out to have similarly important physical applications. While searching for sub(quantum)groups, we would also like to find new (quantum)groups allowed by suitable contractions [23] of these quantum groups as well.

The sub(quantum)groups are obtained by imposing additional relations on the matrix elements of M which obey the relations (3.47) - (3.52). The additional relations that we will impose are:

$$\delta_{ij} - \alpha_{ik}\alpha^*_{jk} + \beta_{ik}\beta^*_{jk} = \beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} = 0 \tag{3.27}$$

$$\gamma_i = 0 \tag{3.28}$$

$$\beta_{ij} = 0 \tag{3.29}$$

$$\alpha_{ij} = 0 \tag{3.30}$$

We would like to study the implication of each relation one by one in the following subsections.

3.1.1.1. Inhomogeneous Subgroup. The relation (3.27):

$$\delta_{ij} - \alpha_{ik}\alpha^*_{jk} + \beta_{ik}\beta^*_{jk} = \beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} = 0$$

by virtue of (3.47) and (3.48) implies that $\gamma_i\gamma^*_j - \gamma^*_j\gamma_i = 0$ and $\gamma_i\gamma_j - \gamma_j\gamma_i = 0$, i.e. that the inhomogeneous transformation parameters are commutative variables.

For bosonic particles, the fact that the inhomogeneous elements of the quantum group are commutative elements coupled with the fact that the remaining relations between the transformation elements are already commutative gives us a symmetry transformation of the boson algebra where all the elements commute. However, we know that such a transformation is nothing but the classical symmetry group of the bosonic particle algebra $ISp(2d, \mathbb{R})$ in which all the parameters, including the inhomogeneous elements, are commutative.

3.1.1.2. Homogeneous Subgroup. The relation (3.28):

$$\gamma_i = 0$$

practically gets rid of the inhomogeneous part of the transformation and also implies the relation (3.27) considered in the previous subsection. Since the previous relation is implied the resulting group will be a subgroup of $ISp(2d,\mathbb{R})$ and since the group is not inhomogeneous anymore the resulting subgroup is the classical symplectic group $Sp(2d,\mathbb{R})$.

3.1.1.3. Bosonic Inhomogeneous Unitary Quantum Group. The relation (3.29):

$$\beta_{ij} = 0$$

applied to the transformation gets rid of the off-diagonal members of the homogeneous part of it and leaves us with the following relation:

$$\gamma_i\gamma_j^* - \gamma_j^*\gamma_i = \delta_{ij} - \alpha_{ik}\alpha_{jk}^* \tag{3.31}$$

This equation implies for the homogeneous part of the transformation the relation:

$$\delta_{ij} = \alpha_{ik}\alpha_{jk}^* \tag{3.32}$$

which tells us that the submatrices α and α^* in equation (3.8) are members of $U(d)$. The subgroup we have arrived at thus is an inhomogeneous quantum group extension to the classical homogeneous group $U(d)$. Since the inhomogeneous elements of the resulting group obeys the same relations as $BISp(2d,\mathbb{R})$, we will name this quantum group $BIU(d)$, the bosonic inhomogeneous quantum group.

3.1.1.4. Boson Algebra. The relation (3.30):

$$\alpha_{ij} = 0$$

applied alone onto the transformation gets rid of the diagonal members of the homogeneous part and prevents such transformations from forming a (quantum)group since the homogeneous parts of these set of transformations can never include the identity transformation.

However, if this relation is applied together with the previous one, relation (3.29), the resulting relation gets rid of the whole homogeneous part of the transformation leaving only the inhomogeneous part and leaves us with two relations:

$$\gamma_i \gamma_j^* - \gamma_j^* \gamma_i = \delta_{ij} \tag{3.33}$$

$$\gamma_i \gamma_j - \gamma_j \gamma_i = 0 \tag{3.34}$$

which gives us back the boson algebra, $BA(d)$.

We should note, however, that after this condition is applied, the resulting set of matrices M, which now form $BA(d)$, is no longer a quantum or classical group since the antipode defined in equation (3.55) no longer exits. For this reason, the boson algebra can be considered to be a boundary for the sub(quantum)groups of $BISp(2d, \mathbb{R})$.

3.1.1.5. Sub(quantum)group Diagram. As a result of the above discussion, we get the sub(quantum)group diagram:

$$\begin{tikzcd}
BISp(2d,\mathbb{R}) \arrow[r, "(3.27)"] \arrow[d, "(3.29)"'] & ISp(2d,\mathbb{R}) \arrow[r, "(3.28)"] \arrow[d, "(3.29)"'] & Sp(2d,\mathbb{R}) \arrow[d, "(3.29)"'] \\
BIU(d) \arrow[r, "(3.27)"] \arrow[d, "(3.30)"'] & IU(d) \arrow[r, "(3.28)"] & U(d) \\
BA(d) & &
\end{tikzcd}$$

for the sub(quantum)groups of the $BISp(2d, \mathbb{R})$ we have introduced in this section.

3.1.2. Contractions

In order to explore the new (quantum)groups that will come about as the result of a contraction, we replace γ_i by $\gamma_i/\sqrt{\hbar}$ so that we may consider the case $\hbar \to 0$. After this replacement, the equations (3.47) and (3.48) become:

$$\gamma_i \gamma_j^* - \gamma_j^* \gamma_i = \hbar(\delta_{ij} - \alpha_{ik}\alpha_{jk}^* + \beta_{ik}\beta_{jk}^*) \tag{3.35}$$

$$\gamma_i \gamma_j - \gamma_j \gamma_i = \hbar(\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk}) \tag{3.36}$$

When we consider the case $\hbar \to 0$, we get the relations:

$$\gamma_i \gamma_j^* - \gamma_j^* \gamma_i = 0 \tag{3.37}$$

$$\gamma_i \gamma_j - \gamma_j \gamma_i = 0 \tag{3.38}$$

which imply that the inhomogeneous part of the transformation form ordinary complex numbers. What makes this case different from the previous case of subgroups is that the homogeneous part of this transformation forms a matrix A with non-zero determinant. We can transform such a matrix A with a similarity transformation given by the unitary matrix:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{3.39}$$

to put it in a real form. The transformation gives:

$$A' = UAU^\dagger \tag{3.40}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \tag{3.41}$$

$$= \begin{pmatrix} Re(\alpha) + Re(\beta) & Im(\alpha) - Im(\beta) \\ -Im(\alpha) - Im(\beta) & Re(\alpha) - Re(\beta) \end{pmatrix} \tag{3.42}$$

which is a real matrix that is a member of the general linear group $GL(2d, \mathbb{R})$. Thus we have the group $IGL(2d, \mathbb{R})$, the inhomogeneous general linear group.

If we consider the contraction of the subgroups as well then we should examine the $\hbar \to 0$ limit after the relations (3.29) and (3.30) are applied.

After we apply relation (3.29), we get the subgroup $BIU(d)$ as discussed previously. After the contraction, again, the inhomogeneous part of this group become complex numbers. However, if we apply the previous similarity transformation on the homogeneous part, we get:

$$\begin{aligned}
A' &= UAU^{\dagger} && (3.43)\\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} && (3.44)\\
&= \begin{pmatrix} Re(\alpha) & Im(\alpha) \\ -Im(\alpha) & Re(\alpha) \end{pmatrix} && (3.45)\\
&= Re(\alpha)\mathbb{I} + Im(\alpha)\mathbb{J} && (3.46)
\end{aligned}$$

where $\mathbb{I}$ stands for the identity matrix and $\mathbb{J}$ stands for the matrix the square of which is minus the identity matrix. This way we can see that the matrix A' is a actually member of $GL(d, \mathbb{C})$. This gives us $IGL(d, \mathbb{C})$ as the group we arrive at as the contraction of $BIU(d)$.

We have previously shown that we get the boson algebra after applying both of the relations (3.29) and (3.30). We have also discussed that in this case only the inhomogeneous part of the transformation survives. After applying the contraction, the surviving inhomogeneous part of the transformation turns into complex variables. Thus in this case, the contraction of $BA(d)$ gives us $\mathbb{C}^d$.

As a summary, for the contraction considered in this section applied onto the subgroups obtained in the previous subsection we get the following group diagram:

$$\begin{array}{ccc}
BISp(2d,\mathbb{R}) & \xrightarrow{\hbar\to 0} & IGL(2d,\mathbb{R}) \\
{\scriptstyle (3.29)}\big\downarrow & & {\scriptstyle (3.29)}\big\downarrow \\
BIU(d) & \xrightarrow{\hbar\to 0} & IGL(d,\mathbb{C}) \\
{\scriptstyle (3.30)}\big\downarrow & & {\scriptstyle (3.30)}\big\downarrow \\
BA(d) & \xrightarrow{\hbar\to 0} & \mathbb{C}^d
\end{array}$$

3.2. The Fermionic Inhomogenous Group $FIO(2d,\mathbb{R})$

Applying this transformation and requiring that the bosonic/fermionic particle algebra remains invariant after the transformation, we arrive at the following relations that the transformation parameters should obey:

$$\gamma_i\gamma_j^* \mp \gamma_j^*\gamma_i = \delta_{ij} - \alpha_{ik}\alpha_{jk}^* \pm \beta_{ik}\beta_{jk}^* \tag{3.47}$$

$$\gamma_i\gamma_j \mp \gamma_j\gamma_i = \pm\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} \tag{3.48}$$

$$\alpha_{ij}\gamma_k \mp \gamma_k\alpha_{ij} = 0 \tag{3.49}$$

$$\beta_{ij}\gamma_k \mp \gamma_k\beta_{ij} = 0 \tag{3.50}$$

$$\alpha_{ij}\gamma_k^* \mp \gamma_k^*\alpha_{ij} = 0 \tag{3.51}$$

$$\beta_{ij}\gamma_k^* \mp \gamma_k^*\beta_{ij} = 0 \tag{3.52}$$

together with the $*$-conjugates of these relations. In these relations the upper and lower signs are for the transformation of bosons and fermions, respectively. Furthermore according to our assumption above, the set $\alpha_{ij}, \beta_{i,j}, \alpha_{ij}^*, \beta_{ij}^*$ forms a commutative algebra.

The set of matrices M obeying the above relations form the group of inhomogeneous transformations of bosons and fermions. For bosons, we name this group $BISp(2d,\mathbb{R})$ and for fermions $FIO(2d,\mathbb{R})$. These symmetry groups, however, are not classical groups but are in fact quantum groups with a Hopf algebra structure.

As shown in [22], this Hopf algebra has an explicit R-matrix representation and the coproduct, counit and antipode are defined as:

$$\Delta(M) = M\dot{\otimes}M \tag{3.53}$$

$$\epsilon(M) = I \tag{3.54}$$

$$S(M) = M^{-1} \quad . \tag{3.55}$$

In Equation 3.53, the symbol $\dot{\otimes}$ denotes the usual matrix multiplication where when elements of the matrices are multiplied, tensor multiplication is used.

The inverse of the matrix M can be defined as:

$$M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\Gamma \\ 0 & 1 \end{pmatrix} \tag{3.56}$$

where A^{-1} is defined in the standard way since matrix elements of A were assumed to be commutative.

3.2.1. Subgroups

After having shown that the inhomogeneous transformations form the symmetry groups $FIO(2d,\mathbb{R})$ and $BISp(2d,\mathbb{R})$ and that they are quantum groups, one important question is what sub(quantum)groups do these quantum groups have. For example, we know that the group $ISp(2d,\mathbb{R})$ is an important special subgroup of $BISp(2d,\mathbb{R})$ and other sub(quantum)groups could turn out to have similarly important physical applications. While searching for sub(quantum)groups, we would also like to find new (quantum)groups allowed by suitable contractions [23] of these quantum groups as well.

The sub(quantum)groups of these algebras are obtained by imposing additional relations on the matrix elements of M which obey the relations (3.47) - (3.52). The additional relations that we will impose are:

(a) $\delta_{ij} - \alpha_{ik}\alpha^*_{jk} \pm \beta_{ik}\beta^*_{jk} = \pm\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} = 0$

(b) $\gamma_i = 0$

(c) $\beta_{ij} = 0$

(d) $\alpha_{ij} = 0$

We would like to study the implication of each relation one by one in the bosonic and the fermionic case.

3.2.1.1. Inhomogeneous Subsupergroup. The relation **(a)**:

$$\delta_{ij} - \alpha_{ik}\alpha^*_{jk} \pm \beta_{ik}\beta^*_{jk} = \pm\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk} = 0$$

by virtue of (3.47) and (3.48) implies that $\gamma_i\gamma^*_j \mp \gamma^*_j\gamma_i = 0$ and $\gamma_i\gamma_j \mp \gamma_j\gamma_i = 0$, i.e. that the inhomogeneous transformation parameters are commutative or anticommutative variables.

In the case of the fermionic particles, we end up with an inhomogeneous orthogonal algebra where the inhomogeneous parameters are grassmanian variables thus giving us the group $GrIO(2d,\mathbb{R})$ as the resulting subgroup of $FIO(2d,\mathbb{R})$. This group can be considered to be an inhomogeneous supergroup.

More generally, in the fermionic case, α_{ij}, β_{ij}, α^*_{ij}, β^*_{ij} anticommute with γ_i, γ^*_i and the $FIO(2d,\mathbb{R})$ matrices M are multiplied with each other using the standard tensor product. It can additionally be shown that α_{ij}, β_{ij}, α^*_{ij}, β^*_{ij} can be taken to commute with γ_i, γ^*_i provided that the matrices M are multiplied with a braided [24] tensor product, eg.

$$(A \otimes B)(C \otimes D) = -AC \otimes BD \tag{3.57}$$

whenever B and C are both fermionic. This also corresponds to the usual superalgebra approach.

For bosonic particles, applying the relation **(a)** gives us the classical symmetry group of the bosonic particle algebra $ISp(2d, \mathbb{R})$ in which all the parameters of the inhomogeneous transformation are commutative.

3.2.1.2. Homogeneous Subgroup. The relation **(b)**:

$$\gamma_i = 0$$

practically gets rid of the inhomogeneous part of the transformation and also implies the relation **(a)** considered in the previous subsection.

For the fermionic particles, this gives us the subgroup in the previous subsection without the inhomogeneous part which is the classical orthogonal group $O(2d, \mathbb{R})$ and similarly, for bosonic particles, gives us the classical symplectic group $Sp(2d, \mathbb{R})$.

3.2.1.3. Fermionic Inhomogeneous Unitary Quantum Group. The relation **(c)**:

$$\beta_{ij} = 0$$

applied to the transformation gets rid of the off-diagonal members of the homogeneous part of it and leaves us with the following relation:

$$\gamma_i \gamma_j^* \mp \gamma_j^* \gamma_i = \delta_{ij} - \alpha_{ik} \alpha_{jk}^* \tag{3.58}$$

For the homogeneous part of the transformation, this equation implies $\delta_{ij} = \alpha_{ik}\alpha_{jk}^*$ which tells us that the submatrices α and α^* in equation (3.8) are both members of $U(d)$. The subgroup we have arrived at thus is an inhomogeneous quantum group whose homogeneous part is $U(d)$. For fermions we will name this group $FIU(d)$, the fermionic inhomogeneous quantum group, and for bosons we will similarly name it $BIU(d)$, the bosonic inhomogeneous quantum group.

3.2.1.4. Fermion Algebra. The relation **(d)**:

$$\alpha_{ij} = 0$$

applied alone onto the transformation gets rid of the diagonal members of the homogeneous part and prevents such transformations from forming a (quantum)group since the homogeneous parts of these set of transformations can never include the identity transformation.

However, if this relation is applied together with the previous one, relation **(c)**, the resulting relation gets rid of the whole homogeneous part of the transformation leaving only the inhomogeneous part and gives us a single relation:

$$\gamma_i \gamma_j^* \mp \gamma_j^* \gamma_i = \delta_{ij} \tag{3.59}$$

which gives us back the fermion algebra, $FA(d)$, for the fermionic case and the boson algebra, $BA(d)$, for the bosonic case.

We should note, however, that after this condition is applied, the resulting set of matrices M, which form $FA(d)$ or $BA(d)$, are no longer quantum or classical groups since the antipode defined in equation (3.55) no longer exits. For this reason, these algebras can be considered to be a boundary for the subgroups of their corresponding quantum groups.

<u>3.2.1.5. Sub(quantum)group Diagram.</u> As a result of the above discussion, we get the sub(quantum)group diagram:

$$\begin{tikzcd} FIO(2d,\mathbb{R}) \arrow[r, "\textbf{(a)}"] \arrow[d, "\textbf{(c)}"'] & GrIO(2d,\mathbb{R}) \arrow[r, "\textbf{(b)}"] \arrow[d, "\textbf{(c)}"'] & O(2d,\mathbb{R}) \arrow[d, "\textbf{(c)}"'] \\ FIU(d) \arrow[r, "\textbf{(a)}"] \arrow[d, "\textbf{(d)}"'] & GrIU(d) \arrow[r, "\textbf{(b)}"] & U(d) \\ FA(d) & & \end{tikzcd}$$

for the fermionic case and the diagram:

$$\begin{tikzcd} BISp(2d,\mathbb{R}) \arrow[r, "\textbf{(a)}"] \arrow[d, "\textbf{(c)}"'] & ISp(2d,\mathbb{R}) \arrow[r, "\textbf{(b)}"] \arrow[d, "\textbf{(c)}"'] & Sp(2d,\mathbb{R}) \arrow[d, "\textbf{(c)}"'] \\ BIU(d) \arrow[r, "\textbf{(a)}"] \arrow[d, "\textbf{(d)}"'] & IU(d) \arrow[r, "\textbf{(b)}"] & U(d) \\ BA(d) & & \end{tikzcd}$$

for the bosonic case.

3.2.2. Contractions

In order to explore the new (quantum)groups that will come about as the result of a contraction, we replace γ_i by $\gamma_i/\hbar$ so that we may consider the case $\hbar \to 0$. After this replacement, the equations (3.47) and (3.48) become:

$$\gamma_i\gamma_j^* \mp \gamma_j^*\gamma_i = \hbar(\delta_{ij} - \alpha_{ik}\alpha_{jk}^* \pm \beta_{ik}\beta_{jk}^*) \quad (3.60)$$

$$\gamma_i\gamma_j \mp \gamma_j\gamma_i = \hbar(\pm\beta_{ik}\alpha_{jk} - \alpha_{ik}\beta_{jk}) \quad (3.61)$$

When we consider the case $\hbar \to 0$, we get the relations:

$$\gamma_i\gamma_j^* \mp \gamma_j^*\gamma_i = 0 \quad (3.62)$$

$$\gamma_i\gamma_j \mp \gamma_j\gamma_i = 0 \quad (3.63)$$

which imply that the inhomogeneous part of the transformation form grassmanian variables for the fermionic case and ordinary complex numbers for the bosonic case. What makes this case different from the previous case of sub(super)groups is that the homogeneous part of this transformation forms a matrix A with non-zero determinant. We can transform such a matrix A with a similarity transformation given by the unitary matrix:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{3.64}$$

to put it in a real form. The transformation gives:

$$\begin{aligned} A' &= UAU^{\dagger} && (3.65) \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} && (3.66) \\ &= \begin{pmatrix} Re(\alpha) + Re(\beta) & Im(\alpha) - Im(\beta) \\ -Im(\alpha) - Im(\beta) & Re(\alpha) - Re(\beta) \end{pmatrix} && (3.67) \end{aligned}$$

which is a real matrix that is a member of the general linear group $GL(2d, \mathbb{R})$. Thus for the fermionic case we have the group $GrIGL(2d, \mathbb{R})$, the grassmanian inhomogeneous general linear group, and for the bosonic case we have $IGL(2d, \mathbb{R})$, the inhomogeneous general linear group.

If we consider the contraction of the subgroups as well then we should examine the $\hbar \to 0$ limit after the relations **(c)** and **(d)** are applied.

After we apply relation **(c)**, we get the subgroups, $FIU(d)$ and $BIU(d)$ as discussed previously. After, the contraction, again the inhomogeneous part of these groups become grassmanian variables and complex numbers for the fermionic and bosonic cases respectively. However, if we apply the previous similarity transformation on the homogeneous part, we get:

$$
\begin{aligned}
A' &= UAU^{\dagger} && (3.68)\\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} && (3.69)\\
&= \begin{pmatrix} Re(\alpha) & Im(\alpha) \\ -Im(\alpha) & Re(\alpha) \end{pmatrix} && (3.70)\\
&= Re(\alpha)\mathbb{I} + Im(\alpha)\mathbb{J} && (3.71)
\end{aligned}
$$

where $\mathbb{I}$ stands for the identity matrix and $\mathbb{J}$ stands for the matrix the square of which is minus the identity matrix. This way we can see that the matrix A' is a actually member of $GL(d,\mathbb{C})$. This gives us $GrIGL(d,\mathbb{C})$ as the group for the contraction in the fermionic case and $IGL(d,\mathbb{C})$ for the bosonic case.

We have previously shown that we get the fermionic and bosonic algebras after applying both of the relations **(c)** and **(d)**. We have also discussed that in this case only the inhomogeneous part of the transformation survives and after applying the contraction the inhomogeneous part of the transformation turns into grassmanian or complex variables. Thus in this case, the contraction of $FA(d)$ gives us $Gr(d,\mathbb{C})$ and the contraction of $BA(d)$ gives us $\mathbb{C}^d$.

As a summary, for the contraction considered combined with the remaining subgroup relations we get the tables:

$$
\begin{array}{ccc}
FIO(2d,\mathbb{R}) & \xrightarrow{\textbf{(e)}} & GrIGL(2d,\mathbb{R}) \\
{\scriptstyle\textbf{(c)}}\big\downarrow & & {\scriptstyle\textbf{(c)}}\big\downarrow \\
FIU(d) & \xrightarrow{\textbf{(e)}} & GrIGL(d,\mathbb{C}) \\
{\scriptstyle\textbf{(d)}}\big\downarrow & & {\scriptstyle\textbf{(d)}}\big\downarrow \\
FA(d) & \xrightarrow{\textbf{(e)}} & Gr(d,\mathbb{C})
\end{array}
$$

for the fermionic case and:

$$\begin{array}{ccc} BISp(2d,\mathbb{R}) & \xrightarrow{\textbf{(e)}} & IGL(2d,\mathbb{R}) \\ {\scriptstyle\textbf{(c)}}\big\downarrow & & {\scriptstyle\textbf{(c)}}\big\downarrow \\ BIU(d) & \xrightarrow{\textbf{(e)}} & IGL(d,\mathbb{C}) \\ {\scriptstyle\textbf{(d)}}\big\downarrow & & {\scriptstyle\textbf{(d)}}\big\downarrow \\ BA(d) & \xrightarrow{\textbf{(e)}} & \mathbb{C}^d \end{array}$$

for the bosonic case.

3.3. The Fermionic Inhomogeneous Orthogonal Quantum Group of Odd Dimension

When we consider a unitary transformation of the $FIO(2d,\mathbb{R})$ matrix as:

$$M \to UMU^{-1}$$

using the unitary matrix:

$$U = \left(\begin{array}{cc|c} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{3.72}$$

we can see that we can put M in the real form:

$$\left(\begin{array}{cc|c} Re(\alpha+\beta) & Im(\alpha-\beta) & \sqrt{2}Re(\gamma) \\ Im(-\alpha-\beta) & Re(\alpha-\beta) & \sqrt{2}Im(\gamma) \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & \Gamma \\ \hline 0 & 1 \end{array}\right) \tag{3.73}$$

where A and Γ matrices are defined as in (3.8), and Re and Im denote the hermitian and anti-hermitian parts.

Using this form it is found that for $FIO(2d,\mathbb{R})$, the transformation relations

(3.47) and (3.48) together become:

$$[\Gamma_i, \Gamma_j]_+ = \delta_{ij} - A_{ik}A_{jk} \qquad , i, j = 1, 2, \ldots, 2d. \tag{3.74}$$

By extending the range of the indices in this relation to odd-dimensions it is possible to define $FIO(2d+1, \mathbb{R})$, the fermionic inhomogeneous orthogonal algebra of odd dimension.

Similar to the analysis that went into finding the sub(quantum)groups of $FIO(2d, \mathbb{R})$, we can also investigate the sub(quantum)groups of $FIO(2d+1, \mathbb{R})$. The sub(quantum)group relations in this case, however, are more restricted owing to the fact that the algebra is not described anymore by submatrices of the A matrix but is rather described by the whole matrix itself. Thus, we cannot set α or β to zero on their own, we can only restrict the algebra by setting the whole of A to zero. Thus the resulting sub(quantum)algebra relations are:

(a) $\delta_{ij} - A_{ik}A_{jk} = 0$
(b) $\Gamma_i = 0$
(c) $A_{ij} = 0$

which, through a similar analysis to the even dimensional case, gives us the following sub(quantum)group diagram:

$$\begin{tikzcd} FIO(2d+1,\mathbb{R}) \arrow[r, "\textbf{(a)}"] \arrow[d, "\textbf{(c)}"'] & GrIO(2d+1,\mathbb{R}) \arrow[r, "\textbf{(b)}"] & O(2d+1,\mathbb{R}) \\ Clif(2d+1) & & \end{tikzcd}$$

4. CONCLUSIONS

APPENDIX A: A FERMION LIKE ALGEBRA

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