HW 2.2.1 Use the technique used in Example 2.2.1 to prove that the solution to difference equation

$$u_k^{n+1} = \frac{1}{2}(u_{k+1}^n + u_{k-1}^n) - \frac{R}{2}\delta_0 u_k^n$$

(the Lax-Friedrichs scheme) converges in the sup-norm to the solution of the partial differential equation

$$v_t + av_x = 0$$

for $| R |\leq 1$ where $R = a\Delta t/\Delta x$.

HW 2.3.1 Determine the order of accuracy of the following difference equations to the given initial–value problems.
(a) Explicit scheme for heat equation with lower order term (FTCS).

$$u_k^{n+1} = u_k^n - \frac{a\Delta t}{2\Delta x}\delta_0 u_k^n + \frac{\nu\Delta t}{\Delta x^2}\delta^2 u_k^n$$

$$v_t + av_x = \nu v_{xx}$$

(b) Implicit scheme for heat equation with lower order term (BTCS).

$$u_k^{n+1} + \frac{a\Delta t}{2\Delta x}\delta_0 u_k^{n+1} - \frac{\nu\Delta t}{\Delta x^2}\delta^2 u_k^{n+1} = u_k^n$$

$$v_t + av_x = \nu v_{xx}$$

(c) Crank-Nicolson Scheme

$$u_k^{n+1} - \frac{\nu\Delta t}{2\Delta x^2}\delta^2 u_k^{n+1} = u_k^n + \frac{\nu\Delta t}{2\Delta x^2}\delta^2 u_k^n$$

$$v_t = \nu v_{xx}$$

Explain why it is logical to consider the consistency of this scheme at the point $(k\Delta x, (n + 1/2)\Delta t)$ rather than at $(k\Delta x, n\Delta t)$ or $(k\Delta x, (n + 1)\Delta t)$.

HW 2.3.2 (a) Show that the following difference scheme is a $\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^4)$ approximation of $v_t = \nu v_{xx}$ (where $r = \nu\Delta t/\Delta x^2$).

$$u_k^{n+1} = u_k^n + r\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right)$$

Discuss the assumptions that must be made on the derivatives of the solution to the partial differential equation that are necessary to make the above statement true.

HW 2.3.3 Determine the order of accuracy of the following difference equations to the partial differential equation

$$v_t + av_x = 0.$$

(a) Leapfrog scheme $u_k^{n+1} = u_k^{n-1} - R\delta_0 u_k^n$

(b) $u_k^{n+1} = u_k^{n-1} - R\delta_0 u_k^n + \frac{R}{6}\delta^2\delta_0 u_k^n$

(c) $u_k^{n+1} = u_k^{n-1} - R\delta_0 u_k^n + \frac{R}{6}\delta^2\delta_0 u_k^n - \frac{R}{30}\delta^4\delta_0 u_k^n$ where $\delta^4 = \delta^2\delta^2$.

(d) $u_k^{n+2} = u_k^{n-2} - \frac{2R}{3}\left(1 - \frac{1}{6}\delta^2\right)\delta_0\left(2u_k^{n+1} - u_k^n + 2u_k^{n-1}\right)$

HW 2.4.1 Show that for $|R| \le 1$, difference scheme

$$u_k^{n+1} = \frac{1}{2}(u_{k+1}^n + u_{k-1}^n) - \frac{R}{2}\delta_0 u_k^n$$

is stable with respect to the sup-norm.

HW 3.1.2 Show that the following difference schemes for approximating the solution to

$$v_t + av_x = \nu v_{xx}$$

are unconditionally stable.

(a) $u_k^{n+1} + \frac{R}{2}\delta_0 u_k^{n+1} - r\delta^2 u_k^{n+1} = u_k^n$

(b) $u_k^{n+1} + \frac{R}{4}\delta_0 u_k^{n+1} - \frac{r}{2}\delta^2 u_k^{n+1} = u_k^n - \frac{R}{4}\delta_0 u_k^n + \frac{r}{2}\delta^2 u_k^n$

补充作业：

1、试证$|r = \frac{a\Delta t}{\Delta x}| \le 1$时，$u_t + au_x = 0$的$Lax$格式$v_j^{n+1} = \frac{1}{2}(v_{j+1}^n + v_{j-1}^n) - \frac{1}{2}r\delta_0 v_j^n$ 关于L_∞ 是稳定的

2、分析$u_t = u_{xx}$的$FTCS$格式关于$L_{2,\Delta x}$的稳定性

作业：分析偏微分方程$u_t + u_x - \nu_2 u_{xx} + \mu_3 u_{xxx} = 0$的耗散性、色散性，其中$\nu_2$, μ_3分别为常数

作业：

1)分析偏微分方程 $u_t = u_x$ 的FTCS格式耗散性、色散性（用二种方法）

2)利用上述例题的结果，对“大作业4”的结果（对比准确解与数值解的图）进行分析

HW 5.

4.5.1 代入谐波解. 得 $\hat{P}(i\omega) = i\omega A$

故 $\mathrm{Re}\,\lambda = \mathrm{Re}\,\lambda(\hat{P}(i\omega))$

$= \mathrm{Re}\,\lambda(i\omega A)$

$= -\omega\, \mathrm{Im}\,\lambda(A)$

若 $\exists\, \alpha > 0$ s.t

$\mathrm{Re}\,\lambda = -\omega\,\mathrm{Im}\,\lambda(A) \le \alpha \quad, \forall \omega$

则 $\mathrm{Im}\,\lambda(A) = 0$

$\therefore\ \mathrm{Re}\,\lambda = 0 \le \alpha$

对 $\alpha = 0$ 成立.

4.5.2 $u_t = \begin{pmatrix} 1 & 10 \\ 0 & 2 \end{pmatrix} u_x$, 设 $\hat{H}(\omega) = \begin{pmatrix} a(\omega) & c(\omega) + d(\omega)i \\ c(\omega) - d(\omega)i & b(\omega) \end{pmatrix}$, $a(\omega), b(\omega), c(\omega), d(\omega) \in \mathbb{R}$

$\hat{P}(i\omega) = \begin{pmatrix} 1 & 10 \\ 0 & 2 \end{pmatrix} i\omega = \begin{pmatrix} i\omega & 10i\omega \\ 0 & 2i\omega \end{pmatrix}$, $\hat{P}^*(i\omega) = \begin{pmatrix} -i\omega & 0 \\ -10i\omega & -2i\omega \end{pmatrix}$

则 $\hat{H}(\omega)\hat{P}(i\omega) + \hat{P}^*(i\omega)\hat{H}(\omega)$

$= \begin{pmatrix} 0 & i(10a\omega + c\omega) - d\omega \\ -i(10a\omega + c\omega) - d\omega & 20 d\omega \end{pmatrix}$

不妨取 $c = -10a$, $d = 0$

则 $\hat{H}(\omega) = \begin{pmatrix} a(\omega) & -10a(\omega) \\ -10a(\omega) & b(\omega) \end{pmatrix}$

取 $a = 1$, $b = 10000$. $\hat{H} = \begin{pmatrix} 1 & -10 \\ -10 & 10000 \end{pmatrix}$ 正定

$\hat{H}(\omega)\hat{P}(i\omega) + \hat{P}^*(i\omega)\hat{H}(\omega) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \le 2\alpha \hat{H}(\omega)$

取 $\alpha = 1$ 即可.

取 $K=1000$)

则 $KI-\hat{A}(w)=\begin{pmatrix} 10000 & 10 \\ 10 & 1 \end{pmatrix}\geq 0$

$$\hat{A}(w)-K^{-1}I=\begin{pmatrix} \frac{10000}{10001} & -10 \\ -10 & 10000-\frac{1}{10001} \end{pmatrix}\geq 0$$

故 $\hat{A}(w)=\begin{pmatrix} 1 & -10 \\ -10 & 10000 \end{pmatrix}$ 满足要求.

HW 2.2.1 方程 $u_t+a u_x=0$

离散为: $v_k^{n+1}=\frac{1}{2}(v_{k+1}^n+v_{k-1}^n)-\frac{a\Delta t}{2\Delta x}(v_{k+1}^n-v_{k-1}^n)$ (*)

则 $T_j^n=\dfrac{u_k^{n+1}-u_k^n}{\Delta t}-\dfrac{1}{2\Delta t}(u_{k+1}^n+u_{k-1}^n-2u_k^n)+\dfrac{a}{2\Delta x}(u_{k+1}^n-u_{k-1}^n)$

$=u_t|_k^n+\frac{\Delta t}{2}u_{tt}|_k^n+O(\Delta t^2)-\frac{1}{2\Delta t}(\Delta x^2 u_{xx}|_k^n+O(\Delta x^4))$

$+a(u_x|_k^n+\frac{\Delta x^2}{12}u_{xxx}|_k^n+O(\Delta x^4))$

$=O(\Delta t+\Delta x^2+\frac{\Delta x^2}{\Delta t})$ ~~$=O(\Delta x^2+\Delta t)$~~

$u_k^{n+1}=\frac{1}{2}(u_{k+1}^n+u_{k-1}^n)-\frac{a\Delta t}{2\Delta x}(u_{k+1}^n-u_{k-1}^n)+\Delta t\, T_j^n$

$=\frac{1}{2}(u_{k+1}^n+u_{k-1}^n)-\frac{a\Delta t}{2\Delta x}(u_{k+1}^n-u_{k-1}^n)+O(\Delta t^2)+O(\Delta x^2)$ (**)

令 $e_k^n=u_k^n-v_k^n$

由 (*), (**). 有 $e_k^{n+1}=\frac{1}{2}(e_{k+1}^n+e_{k-1}^n)-\frac{a\Delta t}{2\Delta x}(e_{k+1}^n-e_{k-1}^n)+O(\Delta t^2)+O(\Delta x^2)$

$=\frac{1}{2}(1-R)e_{k+1}^n+\frac{1}{2}(1+R)e_{k-1}^n+O(\Delta t^2)+O(\Delta x^2)$

令 $E^n=\sup_j|e_j^n|$. $|e_k^{n+1}|\leq\frac{1}{2}|1-R||e_{k+1}^n|+\frac{1}{2}|1+R||e_{k-1}^n|+O(\Delta t^2+\Delta x^2)$.

$|R|\leq 1 \Rightarrow E^{n+1}\leq\frac{1}{2}(1-R)E^n+\frac{1}{2}(1+R)E^n+O(\Delta t^2+\Delta x^2)$.

$=E^n+O(\Delta t^2+\Delta x^2)=E^n+C(\Delta t^2+\Delta x^2)$.

$\leq\cdots\leq E^0+C(n+1)(\Delta t^2+\Delta x^2)$.

$$E^{n+1} \leq E^0 + C(n+1)\Delta t\left(\Delta t + \frac{\Delta x^2}{\Delta t}\right)$$

$\because E^0 = 0 .\ R = \frac{a\Delta t}{\Delta x}$

$\therefore E^{n+1} \leq C t^{n+1} \cdot \left(\Delta t + \frac{a}{R} \cdot \Delta x\right) \to 0$, as $\Delta t, \Delta x \to 0$

故 $|u_k^{n+1} - v_k^{n+1}| \to 0$. as $\Delta t, \Delta x \to 0$.

HW 2.3.1

(a) $u_t + a u_x = \nu u_{xx}$.

~~$v_k^{n+1} = v_k^n - \frac{a\Delta t}{2\Delta x}(u_{k+1}^n - u_{k-1}^n) + \frac{\nu\Delta t}{\Delta x^2}($~~ .

$$v_k^{n+1} = v_k^n - \frac{a\Delta t}{2\Delta x}(v_{k+1}^n - v_{k-1}^n) + \frac{\nu\Delta t}{\Delta x^2}(v_{k+1}^n - 2v_k^n + v_{k-1}^n).$$

$$T_k^n = \frac{u_k^{n+1} - u_k^n}{\Delta t} + \frac{a}{2\Delta x}(u_{k+1}^n - u_{k-1}^n) - \frac{\nu}{\Delta x^2}(u_{k+1}^n - 2u_k^n + u_{k-1}^n).$$

$$= u_t|_k^n + O(\Delta t) + a(u_x|_k^n + O(\Delta x^2)) - \nu(u_{xx}|_k^n + O(\Delta x^2))$$

$$= O(\Delta t) + O(\Delta x^2)$$

阶为 (1,2).

(c) $u_t = \nu u_{xx}$

$$v_k^{n+1} = v_k^n + \frac{\nu\Delta t}{2\Delta x^2}(v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}) + \frac{\nu\Delta t}{2\Delta x^2}(v_{k+1}^n - 2v_k^n + v_{k-1}^n)$$

$$T_k^n = \frac{u_k^{n+1} - u_k^n}{\Delta t} - \frac{\nu}{2\Delta x^2}(u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}) + \frac{\nu}{2\Delta x^2}(u_{k+1}^n - 2u_k^n + u_{k-1}^n)$$

$$= u_t|_k^n + \frac{\Delta t}{2} u_{tt}|_k^n + O(\Delta t^2) - \frac{\nu}{2}(u_{xx}|_k^{n+1} + u_{xx}|_k^n + O(\Delta x^2))$$

$$= u_t|_k^n + \frac{\Delta t}{2} u_{tt}|_k^n + O(\Delta t^2) - \frac{\nu}{2}(2u_{xx}|_k^n + \Delta t\, u_{xxt}|_k^n + O(\Delta t^2 + \Delta x^2))$$

$$= (u_t - \nu u_{xx})|_k^n + \frac{\Delta t}{2}\underbrace{(u_{tt} - \nu u_{xxt})}_{=0}|_k^n + O(\Delta t^2 + \Delta x^2)$$

$\because u_t = \nu u_{xx}$ $\therefore u_{tt} = \nu u_{xxt}$

故 $T_j^n = O(\Delta t^2 + \Delta x^2)$. 阶为 (2,2).

HW2.3.2(a) 在(x_k, t_n)处展开.

$$T_k^n = \frac{u_k^{n+1}-u_k^n}{\Delta t} - \frac{\nu}{\Delta x^2}\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right)$$

$$= u_t|_k^n + O(\Delta t) - \frac{\nu}{\Delta x^2}\Big(-\frac{1}{12}\big(2u_k^n + 4\Delta x^2 u_{xx}|_k^n + \frac{4}{3}\Delta x^4 u_{xxxx}|_k^n + O(\Delta x^6)\big)$$

$$+\frac{4}{3}\big(2u_k^n + \Delta x^2 u_{xx}|_k^n + \frac{1}{12}\Delta x^4 u_{xxxx}|_k^n + O(\Delta x^6)\big)$$

$$-\frac{5}{2}u_k^n\Big)$$

$$= (u_t - \nu u_{xx})|_k^n + O(\Delta t + \Delta x^4)$$

$$= O(\Delta t) + O(\Delta x^4)$$

对t有2阶导. 对x有6阶导.

HW 2.3.3 (b).

$$T_k^n = \frac{u_k^{n+1}-u_k^{n-1}}{2\Delta t} + \frac{a}{2\Delta x}(u_{k+1}^n - u_{k-1}^n) - \frac{a}{12\Delta x}(u_{k+2}^n - 2u_{k+1}^n + 2u_{k-1}^n - u_{k-2}^n)$$

$$= u_t|_k^n + O(\Delta t^2) + a u_x|_k^n + \frac{\Delta x^2}{6}a u_{xxx}|_k^n + O(\Delta x^4).$$

$$-\frac{a}{12\Delta x}\Big[4\Delta x u_x|_k^n + \frac{8}{3}\Delta x^3 u_{xxx}|_k^n + O(\Delta x^5)$$

$$-4\Delta x u_x|_k^n - \frac{2}{3}\Delta x^3 u_{xxx}|_k^n + O(\Delta x^5)\Big]$$

$$= (u_t + a u_x)|_k^n + \left(\frac{a}{6}\Delta x^2 - \frac{a}{6\Delta x}\cdot\Delta x^3\right)u_{xxx}|_k^n + O(\Delta x^4) + O(\Delta t^2)$$

$$= O(\Delta t^2 + \Delta x^4)$$

故阶为(2,4)

HW 2.4.1 $$u_k^{n+1} = \frac{1}{2}(u_{k+1}^n + u_{k-1}^n) - \frac{R}{2}(u_{k+1}^n - u_{k-1}^n)$$

$$= \frac{1}{2}(1-R)u_{k+1}^n + \frac{1}{2}(1+R)u_{k-1}^n$$

$\because |R|\le 1$ $\therefore$ $|u_k^{n+1}| \le \frac{1-R}{2}|u_{k+1}^n| + \frac{1+R}{2}|u_{k-1}^n|$

故稳定 $\|u^{n+1}\|_\infty \le \frac{1-R}{2}\|u^n\|_\infty + \frac{1+R}{2}\|u^n\|_\infty = \|u^n\|_\infty \le \cdots \le \|u^0\|_\infty$

HW 3.1.2.

(a) 代入 $u_k^n = \frac{1}{\sqrt{2\pi}} \hat{u}^n(w) e^{iwx_k}$

得 $\hat{u}^{n+1}(w) + \frac{R}{2}(e^{iwh} - e^{-iwh})\hat{u}^{n+1}(w) - r(e^{iwh}+e^{-iwh}-2)\hat{u}^{n+1}(w)$
$= \hat{u}^n(w).$

即 $[1+R\sin wh \cdot i - r(2\cos wh - 2)]\hat{u}^{n+1}(w) = \hat{u}^n(w)$

$$\therefore \hat{u}^{n+1}(w) = \frac{1}{1+4r\sin^2\frac{wh}{2} + R\sin wh \cdot i}\hat{u}^n(w)$$

故 $$\hat{Q} = \frac{1}{1+4r\sin^2\frac{wh}{2} + i\cdot R\sin wh}$$

$$|\hat{Q}|^2 = \frac{1}{(1+4r\sin^2\frac{wh}{2})^2 + R^2\sin^2 wh} \le 1$$ 无条件稳定

(b). 代入 $u_k^n = \frac{1}{\sqrt{2\pi}} \hat{u}^n(w) e^{iwx_k}$

得 $[1+\frac{R}{2}\sin wh \cdot i - r(\cos wh - 1)]\hat{u}^{n+1}(w) = [1-\frac{R}{2}\sin wh\, i + r(\cos wh - 1)]\hat{u}^n(w)$

$$\Rightarrow \hat{Q} = \frac{1-2r\sin^2\frac{wh}{2} - i\frac{R}{2}\sin wh}{1+2r\sin^2\frac{wh}{2} + i\frac{R}{2}\sin wh}$$

$$\Rightarrow |\hat{Q}|^2 = \frac{(1-2r\sin^2\frac{wh}{2})^2 + \frac{R^2}{4}\sin^2 wh}{(1+2r\sin^2\frac{wh}{2})^2 + \frac{R^2}{4}\sin^2 wh} \le 1.$$ 无条件稳定

补充 2: 分析 $u_t = u_{xx}$ 的 FTCS 格式关于 $L_{2,\Delta x}$ 的稳定性

$$\frac{u_k^{n+1} - u_k^n}{\Delta t} = \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2}$$

代入 $u_k^n = \frac{1}{\sqrt{2\pi}} \hat{u}^n(w) e^{iwx_k}$ 得 $\hat{u}^{n+1}(w) = (1-4\frac{\Delta t}{\Delta x^2}\sin^2\frac{wh}{2})\hat{u}^n(w)$.

$\Rightarrow \hat{Q} = 1 - \frac{\Delta t}{\Delta x^2} 4\sin^2\frac{wh}{2}$

由命题 2: u^n 在 $L_{2,\Delta x}$ 中稳定 $\Leftrightarrow$ $\hat{u}^n$ 在 $L_2[-\pi,\pi]$ 中稳定

故 $|\hat{Q}| \le 1 \Rightarrow \left|1 - \frac{\Delta t}{\Delta x^2} 4\sin^2\frac{wh}{2}\right| \le 1 \Rightarrow 0 \le \frac{\Delta t}{\Delta x^2} \le \frac{2}{4\sin^2\frac{wh}{2}}$ $\forall w$

故 $r = \frac{\Delta t}{\Delta x^2} \le \frac{1}{2}$ 时，格式稳定

HW 6

1. 分析 $u_t+u_x-\nu_2 u_{xx}+\mu_3 u_{xxx}=0$ 的耗散性、色散性，其中ν_2，μ_3为常数.

代入 $u(x,t)=e^{i(wx+kt)}$，得

$$ik+iw-\nu_2(iw)^2+\mu_3(iw)^3=0$$

$$\Rightarrow\quad k=(\mu_3 w^3-w)+\nu_2 w^2 i$$

$$-\frac{Re(k)}{w}=\frac{w-\mu_3 w^3}{w}=1-\mu_3 w^2$$

故 $\mu_3=0$ 时 $-\frac{Re(k)}{w}\equiv 1$，无色散

$\mu_3\neq 0$ 时 有色散

$$\lambda_e=\frac{u(x,t+\Delta t)}{u(x,t)}=e^{ik\Delta t}=e^{-\nu_2 w^2\Delta t+(\mu_3 w^3-w)\Delta t\, i}$$

$$=|\lambda_e|e^{i\varphi_e}$$

故 $|\lambda_e|=e^{-\nu_2 w^2\Delta t}$

$\nu_2=0$ 时 $|\lambda_e|\equiv 1$，无耗散

$\nu_2>0$ 时 $|\lambda_e|$不随时间增长，且 $\forall w\neq 0$ 振幅衰减，有耗散.

$\nu_2<0$ 时 逆耗散.

2. 分析 $u_t = u_x$ 的 FTCS 格式的耗散性、色散性.(两种方法)

FTCS: $\dfrac{u_j^{n+1}-u_j^n}{\Delta t} = \dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$

对于源方程 $u_t = u_x$. 代入 $u = e^{i(wx+kt)}$ 得

$$ik = iw \Rightarrow k = w$$

$$-\frac{Re(k)}{w} \equiv -1 \quad 无色散.$$

$$\lambda_e = e^{ik\Delta t} = e^{iw\Delta t} = |\lambda_e| e^{i\varphi_e}.$$

$$\Rightarrow |\lambda_e| = 1. \quad \varphi_e = w\Delta t \quad 无耗散.$$

法1:

对于数值格式: $v_j^{n+1} = v_j^n + \dfrac{\Delta t}{2\Delta x}(v_{j+1}^n - v_{j-1}^n)$

代入 $v_j^n = \lambda^n e^{iwx_j}$. 得

$$\lambda = 1 + \frac{\Delta t}{2\Delta x}(e^{iw\Delta x} - e^{-iw\Delta x})$$

$$= 1 + \frac{\Delta t}{\Delta x}\sin w\Delta x\, i$$

$$= 1 + r\sin w\Delta x\, i$$

其中 $r = \dfrac{\Delta t}{\Delta x} > 0$

$$|\lambda|^2 = 1 + r^2\sin^2 w\Delta x > 1 = |\lambda_e|^2$$

故 $\dfrac{|\lambda|}{|\lambda_e|} > 1$, 差分格式为数值逆耗散

$$\lambda = 1 + r\sin w\Delta x\, i = |\lambda| e^{i\varphi}$$

$$|\lambda| = \sqrt{1 + r^2\sin^2 w\Delta x}, \quad \varphi = \operatorname{arctg}\frac{r\sin w\Delta x}{1} = \operatorname{arctg}(r\sin\xi).$$

其中 $\xi = w\Delta x$. $\left(\operatorname{arctg} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots\right)$

$$\varphi = \operatorname{arctg}(r\sin\xi) = r\xi\left[1 - \left(\frac{1}{6} + \frac{1}{3}r^2\right)\xi^2\right] + \cdots$$

故 $\dfrac{\varphi}{\varphi_e} = 1 - \left(\dfrac{1}{6} + \dfrac{1}{3}r^2\right)\xi^2 \le 1$ 差分格式为数值负色散.

法2：MPDE.

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}-\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}=0$$

$$u_t+\frac{\Delta t}{2}u_{tt}+\frac{\Delta t^2}{6}u_{ttt}-u_x-\frac{\Delta x^2}{6}u_{xxx}+\cdots=0$$

	u_t	u_x	u_{tt}	u_{tx}	u_{xx}	u_{ttt}	u_{ttx}	u_{txx}	u_{xxx}
①	1	-1	$\frac{\Delta t}{2}$	0	0	$\frac{\Delta t^2}{6}$	0	0	$-\frac{\Delta x^2}{6}$
$-\frac{\Delta t}{2}\frac{\partial ①}{\partial t}$			$-\frac{\Delta t}{2}$	$+\frac{\Delta t}{2}$		$-\frac{\Delta t^2}{4}$			
②	1	-1	0	$\frac{\Delta t}{2}$	0	$-\frac{\Delta t^2}{12}$	0	0	$-\frac{\Delta x^2}{6}$
$-\frac{\Delta t}{2}\frac{\partial ②}{\partial x}$	$\cancel{\frac{\Delta t}{2}}$	$\cancel{0}$		$-\frac{\Delta t}{2}$	$\frac{\Delta t}{2}$			$-\frac{\Delta t^2}{4}$	
③	1	-1	0	0	$\frac{\Delta t}{2}$	$-\frac{\Delta t^2}{12}$	0	$-\frac{\Delta t^2}{4}$	$-\frac{\Delta x^2}{6}$
$\frac{\Delta t^2}{12}\frac{\partial^2 ③}{\partial t^2}$						$\frac{\Delta t^2}{12}$	$-\frac{\Delta t^2}{12}$		
④	1	-1	0	0	$\frac{\Delta t}{2}$	0	$-\frac{\Delta t^2}{12}$	$-\frac{\Delta t^2}{4}$	$-\frac{\Delta x^2}{6}$
$\frac{\Delta t^2}{12}\frac{\partial^2 ④}{\partial x\partial t}$							$\frac{\Delta t^2}{12}$	$-\frac{\Delta t^2}{12}$	
⑤	1	-1	0	0	$\frac{\Delta t}{2}$	0	0	$-\frac{\Delta t^2}{3}$	$-\frac{\Delta x^2}{6}$
$\frac{\Delta t^2}{3}\frac{\partial^2 ⑤}{\partial x^2}$	$\cancel{0}$	$\cancel{0}$						$\frac{\Delta t^2}{3}$	$-\frac{\Delta t^2}{3}$
⑥	1	-1	0	0	$\frac{\Delta t}{2}$	0	0	0	$-\frac{\Delta x^2}{6}-\frac{\Delta t^2}{3}$

故 $u_t-u_x+\frac{\Delta t}{2}u_{xx}-\left(\frac{\Delta x^2}{6}+\frac{\Delta t^2}{3}\right)u_{xxx}+\cdots=0.$

即 $u_t=u_x+\nu_2 u_{xx}+\mu_3 u_{xxx}+\cdots$

其中 $\nu_2=-\frac{\Delta t}{2}$，$\mu_3=\frac{\Delta x^2}{6}+\frac{\Delta t^2}{3}$

代入 $u(x,t)=e^{i(wx+kt)}$ 得

$$ik=iw-w^2\nu_2-w^3\mu_3 i \cancel{=(w-w^3\mu_3)i-w^2\nu_2}$$

故 $u(x,t)=e^{iwx}\cdot e^{iwt}\, e^{-w^2\nu_2 t}\cdot e^{-iw^3\mu_3 t}\cancel{e^{i(w-w^3\mu_3)t}}=e^{iwx}\cdot e^{iwt}e^{\nu t}\cdot e^{i\mu t}$

其中 $\nu = -w^2 \nu_2 = w^2 \frac{\Delta t}{2} > 0 \Rightarrow$ 数值逆耗散

~~其中 ν_2~~ $\mu = -w^3 \mu_3 = -\left(\frac{\Delta x^2}{6} + \frac{\Delta t^2}{3}\right) w^3 < 0 \Rightarrow$ 数值负色散

小测：求 $u_t = b(x,t) u_{xx}$ FTCS格式的整体误差.

FTCS: $v_j^{n+1} = v_j^n + b_j^n \frac{\Delta t}{\Delta x^2} (v_{j+1}^n - 2v_j^n + v_{j-1}^n)$ $\quad (*)$

令 $r = \frac{\Delta t}{\Delta x^2}$, $B = \max\limits_{j,n} |b_j^n|$, $e_j^n = u_j^n - v_j^n$

$$u_j^{n+1} = u_j^n + b_j^n \frac{\Delta t}{\Delta x^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \Delta t\, T_j^n \quad (**)$$

$(**) - (*)$

$$e_j^{n+1} = e_j^n + b_j^n r (e_{j+1}^n - 2e_j^n + e_{j-1}^n) + \Delta t\, T_j^n$$

$$= (1 - 2b_j^n r) e_j^n + b_j^n r (e_{j+1}^n + e_{j-1}^n) + T_j^n \Delta t$$

$$\Rightarrow \quad E^{n+1} \leq (1 - 2b_j^n r) E^n + b_j^n r (E^n + E^n) + \overset{n}{T} \Delta t$$

$$= E^n + \overset{n}{T} \Delta t$$

CFL: $1 - 2b_j^n \frac{\Delta t}{\Delta x^2} \geq 0$

$\Rightarrow \Delta t \leq \frac{1}{2B} \Delta x^2$

$$\leq \cdots$$

$$\leq E^0 + (n+1)\Delta t\, \overset{n}{T}$$

$$= E^0 + \overset{n}{T} t_{n+1}$$

$\because E^0 = 0$, $\overset{n}{T} = O(\Delta t + \Delta x^2)$

$\therefore \lim\limits_{\Delta t, \Delta x \to 0} E^{n+1} = 0$, $v_j^n \to u_j^n$.