例 3.30 记$f(t)=\frac{4sint-4t\cos t}{t^3}$，则$f(t)$的傅里叶变换是

$$\widehat{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt$$
$$=\begin{cases}\sqrt{2\pi}(1-\lambda^2), & |\lambda|\le 1;\\ 0, & |\lambda|>1.\end{cases}$$

所以$f(t)$是一个频率带限信号，所以只要我们取$\Omega\ge 1$，就可以对原始信号完全重构。

$$\begin{aligned}\hat{f}(\lambda) &= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{-i\lambda x}dx\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-i\lambda x}}{-i\lambda}(1-x^2)\Big|_{x=-1}^{1}-\frac{2}{i\lambda}\int_{-1}^{1}xe^{-i\lambda x}dx\right)\\
&= -\frac{1}{\sqrt{2\pi}}\frac{2}{i\lambda}\left(\frac{e^{-i\lambda x}}{-i\lambda}x\Big|_{x=-1}^{1}+\frac{1}{i\lambda}\int_{-1}^{1}e^{-i\lambda x}dx\right)\\
&= -\frac{1}{\sqrt{2\pi}}\frac{2}{i\lambda}\left(\frac{e^{-i\lambda x}}{-i\lambda}\left(x+\frac{1}{i\lambda}\right)\Big|_{x=-1}^{1}\right)\\
&= -\frac{1}{\sqrt{2\pi}}\frac{2}{i\lambda}\left(\frac{e^{-i\lambda}(1+\frac{1}{i\lambda})-e^{i\lambda}(-1+\frac{1}{i\lambda})}{-i\lambda}\right)\\
&= -\frac{1}{\sqrt{2\pi}}\left(\frac{4\cos\lambda}{\lambda^2}-\frac{4\sin\lambda}{\lambda^3}\right)\end{aligned}$$

我们可以从另外一个角度来看这个采样定理，如图3.17所示，设$a=\frac{\pi}{\Omega}$，记$\phi_a(t)=\frac{\sin(\pi t/a)}{\pi t/a}$，则上式可以写成

$$f(t)=\sum_{j=-\infty}^{\infty} f(ja)\phi_a(t-ja)=\phi_a * \sum_{j=-\infty}^{\infty} f(ja)\delta(t-ja).$$

注意到$\sum_{j=-\infty}^{\infty} f(ja)\delta(t-ja)$对应于一个信号$f$的任意均匀采样$f_d(t)$，

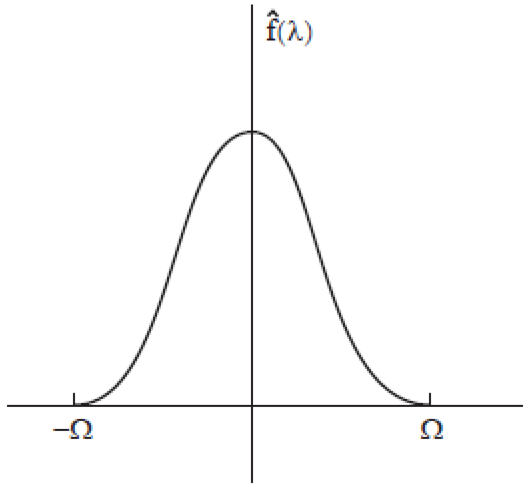
$$f_d(t)=\sum_{j=-\infty}^{\infty} f(ja)\delta(t-ja), \tag{3.48}$$

所以，采样定理说明

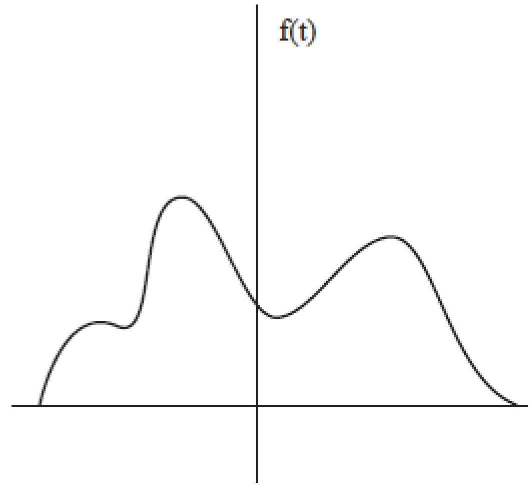
$$\widehat{f}(\lambda)=\sqrt{2\pi}\widehat{\phi}_a(\lambda)\widehat{f}_d(\lambda)=r_{a,\Omega}(\lambda)\widehat{f}_d(\lambda),$$

这里$r_{a,b}(\lambda)=\begin{cases} a, & -b\le\lambda\le b; \\ 0, & \text{其他.}\end{cases}$。事实上可以证明$f_d(t)$的傅里叶变换是

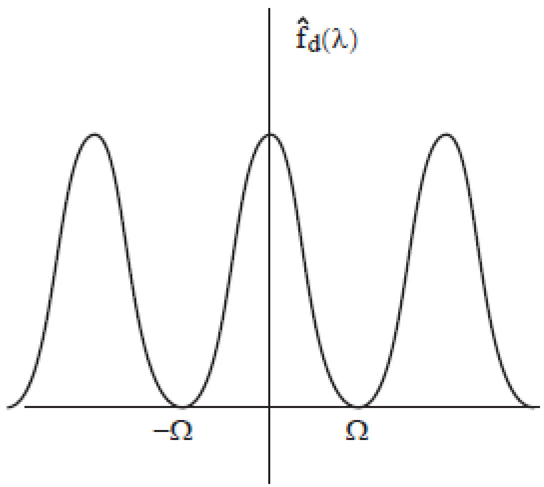
$$\widehat{f}_d(\lambda)=\frac{1}{a}\sum_{j=-\infty}^{\infty}\widehat{f}(\lambda-2k\Omega)$$



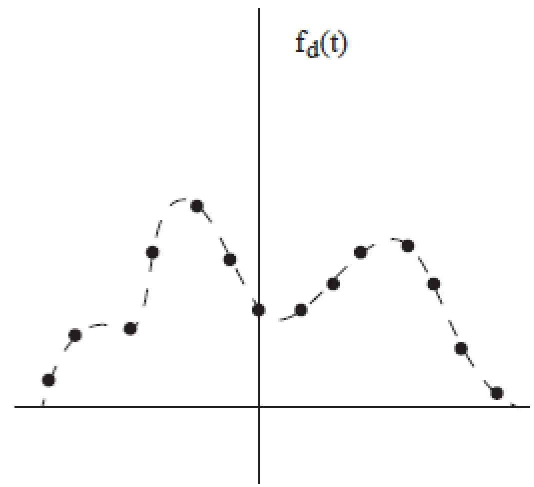
a.f(t)的傅里叶变换



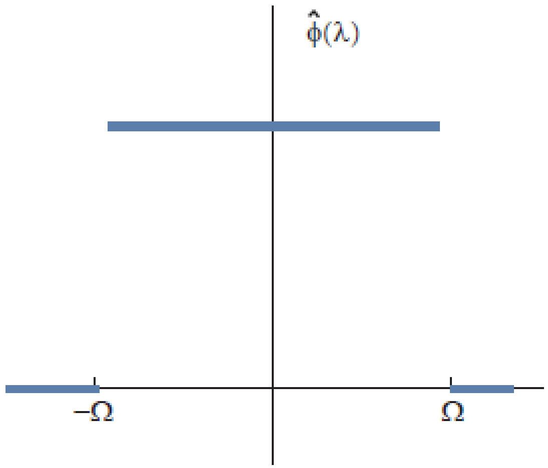
b.信号f(t)的图像



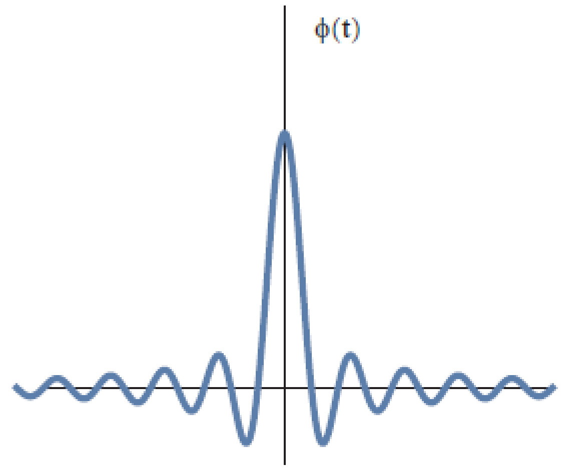
c.f(t)的傅里叶变换周期延拓



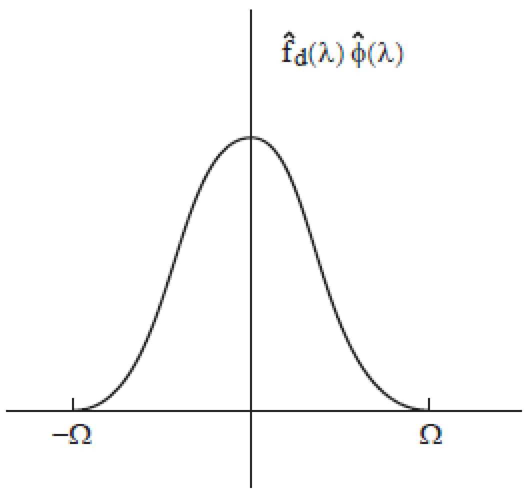
d.信号f(t)的采样



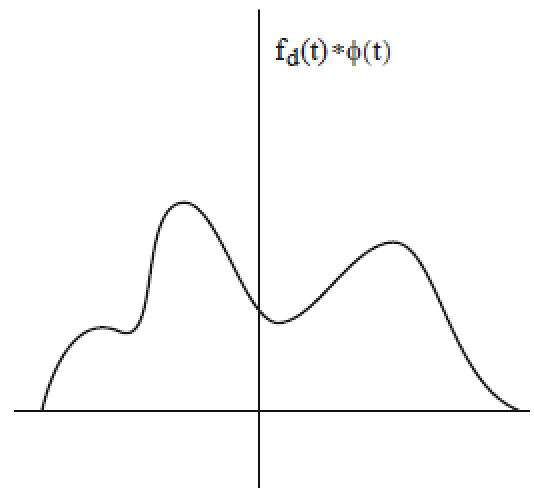
e.频率域中的矩形波函数



f.矩形波的逆傅里叶变换



g.f(t)的傅里叶变换看成两个函数的乘积



h.信号f(t)就是两个函数的卷积

如果$\widehat{f}$的支集超出$[-\Omega,\Omega]$，一般的，对于某些非零的$j\neq 0$，$\widehat{f}(\lambda-2j\Omega)$ 的支集和$[-\Omega,\Omega]$相交，则高频部分折叠到低频区间，这个称之为混叠。由于存在混叠，$\frac{1}{a}\sum_{j=-\infty}^{\infty}\widehat{f}(\lambda-2k\Omega)$完全不同于$\widehat{f}$的支集属于$[-\Omega,\Omega]$的情形，这个过程可以用图3.18来解释。

例 3.31 考虑高频振荡

$$f(t)=\cos(\lambda_0 t)=\frac{e^{i\lambda_0 t}+e^{-i\lambda_0 t}}{2}$$

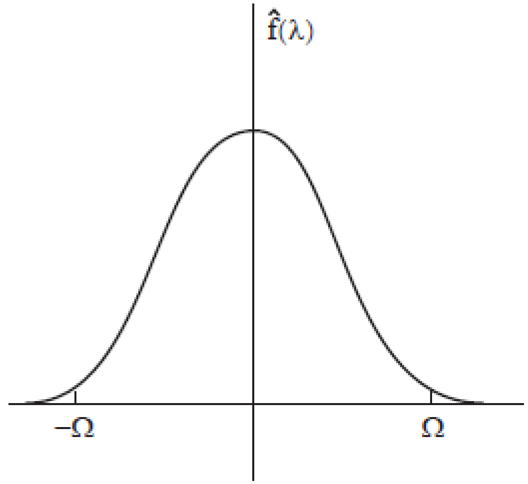
其傅里叶变换是

$$\widehat{f}(\lambda)=\sqrt{\frac{\pi}{2}}(\delta(\lambda-\lambda_0)+\delta(\lambda+\lambda_0))$$

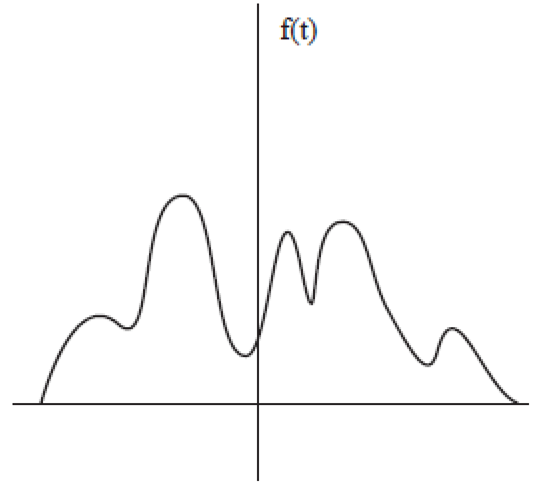
如果$\Omega<\lambda_0<2\Omega$，则

$$\sqrt{2\pi}\widehat{\phi}_a(\lambda)\widehat{f}_d(\lambda)=\sqrt{\frac{\pi}{2}}(\delta(\lambda-2\Omega+\lambda_0)+\delta(\lambda+2\Omega-\lambda_0)) \tag{3.50}$$

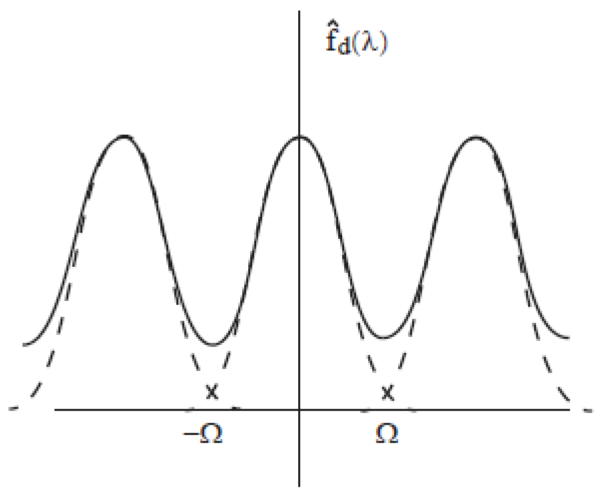
因此，混叠将高频λ_0移至低频$2\Omega-\lambda_0\in[-\Omega,\Omega]$。



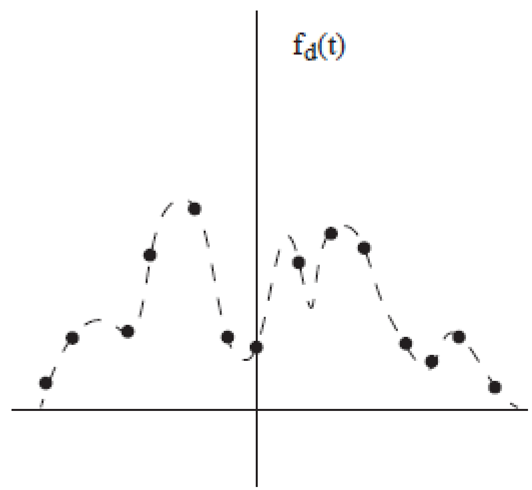
a.f(t)的傅里叶变换



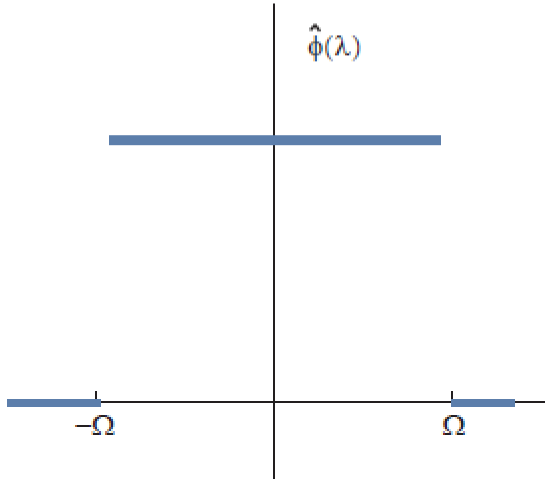
b.信号f(t)的图像



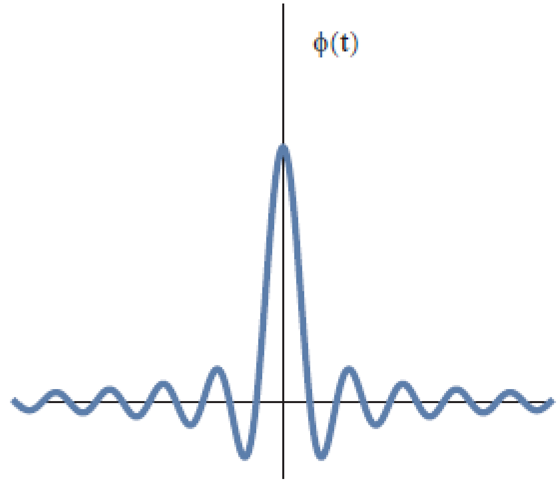
c.f(t)的傅里叶变换周期延拓



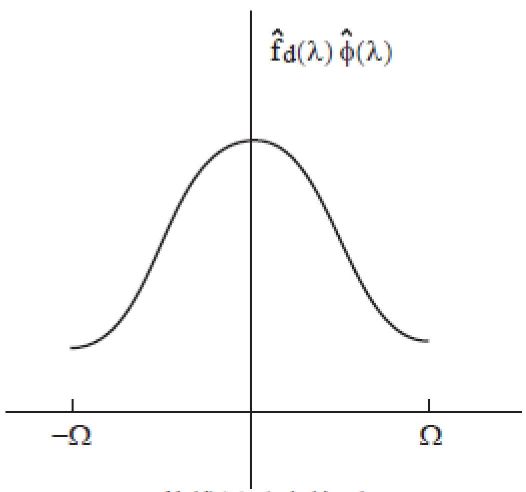
d.信号f(t)的采样



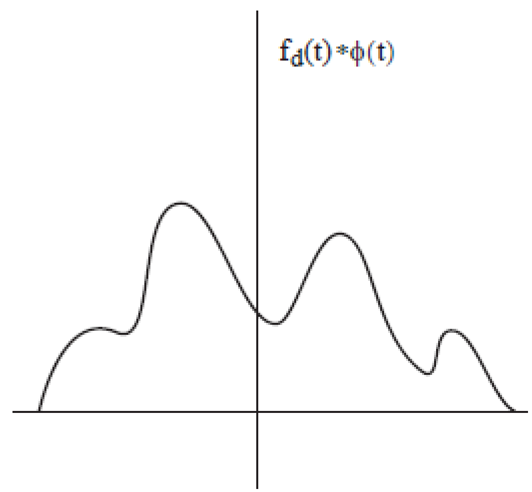
e.频率域中的矩形波函数



f.矩形波的逆傅里叶变换



g.f(t)的傅里叶变换看
成两个函数的乘积



h.信号f(t)就是两个函数的卷积