1. $cond(\lambda)$的定义：对λ的单位特征向量 x，$Ax=\lambda x$，$\|x\|_2=1$

找一个满足 $y^Tx=1$ 的左特征向量 $y^TA=\lambda y^T$, $cond(\lambda)=\|y\|_2$

$\because A=A^T \Rightarrow Ay=\lambda y$，故取 $y=x$ 即可

$\therefore cond(\lambda)=1$

2. pf: A对称 $\Rightarrow \exists$ 正交阵 P s.t. $A=P\Lambda P^T$, $\Lambda=diag\{\lambda_1,\lambda_2,\cdots\lambda_n\}$

对每个 $k\in\{1,2,\cdots n\}$，单独讨论

构造 $B_k=(A-a_{kk}I)v=P(\Lambda-a_{kk}I)P^Tv$，$\|v\|_2=1$ 任取

① 若 $\exists\ \lambda_i=a_{kk}$，则 $\lambda_i\in D_k$，结论成立

② $a_{kk}\notin\lambda(A) \Rightarrow A-a_{kk}I$ 可逆 $\Rightarrow v=(A-a_{kk}I)^{-1}B_k$

$\therefore 1=\|v\|_2=\|(A-a_{kk}I)^{-1}B_k\|_2=\|P(\Lambda-a_{kk}I)^{-1}P^TB_k\|_2\le\|(\Lambda-a_{kk}I)^{-1}\|_2\cdot\|B_k\|_2$

$\therefore \|B_k\|_2\ge 1/\max_i|\lambda_i-a_{kk}|^{-1}=\min|\lambda_i-a_{kk}|$

取 $v=e_k \Rightarrow B_k=(a_{1k},a_{2k},\cdots a_{k-1,k},0,a_{k+1,k},\cdots,a_{nk})$

$\therefore \|B_k\|_2=(\sum_{j\ne k}a_{kj}^2)^{\frac{1}{2}}\ge\min_i|\lambda_i-a_{kk}| \Rightarrow \exists\lambda_i,\ \lambda_i\in D_k$ □

3. pf: A对称 $\Rightarrow\exists$ 正交阵 P，$A=P\Lambda P^T$，$\Lambda=diag\{\lambda_1,\cdots\lambda_n\}$，$\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n>0$

$\therefore \|A^{-1}\|_2=\lambda_n^{-1} \Rightarrow \|E\|_2<\lambda_n$

记 $F\triangleq P^TEP$

故对 $\forall x\in\mathbb{R}^n\setminus\{0\}$，$|x^TFx|\le\|F\|_2x^Tx<\lambda_nx^Tx\le x^TJx$

$\Rightarrow x^T(J+F)x>0 \Rightarrow$ 令 $y=Px$，则 $y^T(A+E)y>0$，$\forall y\in\mathbb{R}^n\setminus\{0\}$ 成立 □

4. $A=P\Sigma Q$ (SVD)，$\therefore AA^T=P(\Sigma\Sigma^T)P^T\sim\Sigma\Sigma^T$

$A^TA=Q^T(\Sigma^T\Sigma)Q\sim\Sigma^T\Sigma$ □

5. $A=P\Lambda P^T$，P正交，$\Lambda=diag(\lambda_1,\lambda_2,\cdots\lambda_n)$

令 $Q=diag\{sgn(\lambda_1),\cdots sgn(\lambda_n)\}$

$\Rightarrow A=P\Lambda(QP^T)$ 为 A的SVD分解

6. 由题4，显然

7. 由$\|\cdot\|_2$的正交不变性，不妨 $A=\Sigma=\begin{pmatrix}\sigma_1 & \cdots & \sigma_n \\ & 0 & \end{pmatrix}_{m\times n}$

令 $U=Span\{e_j\}_{j=i}^n \Rightarrow dim\,U=n-i+1$，$dim\,\mathcal{X}=i \Rightarrow dim\,(\mathcal{X}\cap U)\geq 1$

$\Rightarrow \exists\ u\neq 0$，$u\in\mathcal{X}\cap U \Rightarrow u=\sum_{j=i}^n c_je_j$

$\Rightarrow \frac{\|Au\|_2}{\|u\|_2}=(\sum_{j=i}^n c_j^2\sigma_j^2/\sum_{j=i}^n c_j^2)^{\frac{1}{2}}\leq\sigma_i$

$\therefore \min_{0\neq u\in\mathcal{X}}\frac{\|Au\|_2}{\|u\|_2}\leq i$，$\forall\ \mathcal{X}\in\mathcal{G}_i^n$

又当 $\mathcal{X}=Span\{e_j\}_{j=1}^i$ 时，$\frac{\|Au\|_2}{\|u\|_2}=(\sum_{j=1}^i c_j^2\sigma_j^2/\sum_{j=1}^i c_j^2)^{\frac{1}{2}}\geq\sigma_i$，$\forall u\in\mathcal{X}$

$\Rightarrow \max_{\mathcal{X}\in\mathcal{G}_i^n}\min_{\substack{u\in\mathcal{X}\\u\neq 0}}\frac{\|Au\|_2}{\|u\|_2}=\sigma_i$ 　另一个同理 □

8. 代入即可，具体见学习辅导 P121

9. 设 $Q=[q_1,q_2,\cdots q_n]$，$T=\begin{pmatrix}\alpha_1 & \beta_1 & & \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ & & \beta_{n-1} & \alpha_n\end{pmatrix}$

$\therefore AQ=[Aq_1,\cdots Aq_n]$，$QT=[\alpha_1q_1+\beta_1q_2,\beta_1q_1+\alpha_2q_2+\beta_2q_3,\cdots,\beta_{n-1}q_{n-1}+\alpha_nq_n]$

$AQ=QT \Leftrightarrow (*)\begin{cases}\alpha_1q_1+\beta_1q_2=Aq_1 \\ \beta_iq_i+\alpha_{i+1}q_{i+1}+\beta_{i+1}q_{i+2}=Aq_{i+1} \quad (1\leq i\leq n-2) \\ \beta_{n-1}q_{n-1}+\alpha_nq_n=Aq_n\end{cases}$

任取合适的 q_1，$\|q_1\|_2=1$

由(*)知，q_{k+1} 为 Aq_k 在 $\{q_1,\cdots,q_k\}$ 的垂直空间的投影（Since $q_1,\cdots q_n$正

$\therefore q_{k+1} = Aq_k - \sum_{i=1}^{k} \langle Aq_k, q_i \rangle q_i$ □

另法见 学习辅导 P122

10.

$$A \xrightarrow{U_1} \begin{pmatrix} * & * & \cdots & * \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix} \xrightarrow{V_1} \begin{pmatrix} * & * & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & & * & & \\ 0 & & & & \end{pmatrix} \longrightarrow \cdots$$

即令 $U_k = \begin{pmatrix} I_{k-1} & \\ & U_k' \end{pmatrix}$ 将第k列的 $k+1 \sim m$ 行打至 (k,k) 处

$V_k = \begin{pmatrix} I_k & \\ & V_k' \end{pmatrix}$ 将第k行的 $k+2 \sim n$ 列打至 $(k,k+1)$ 处 □

11.

$$\begin{pmatrix} \alpha_1 & \varepsilon \\ \varepsilon & \alpha_2 \end{pmatrix} - \alpha_2 I = \begin{pmatrix} \alpha_1-\alpha_2 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

~~找~~ 找 $G = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ s.t. $G\begin{pmatrix} \alpha_1-\alpha_2 \\ \varepsilon \end{pmatrix} = \begin{pmatrix} \sqrt{(\alpha_1-\alpha_2)^2+\varepsilon^2} \\ 0 \end{pmatrix}$

$\Rightarrow \sin\theta = \dfrac{-\varepsilon}{\sqrt{(\alpha_1-\alpha_2)^2+\varepsilon^2}}$, $\cos\theta = \dfrac{(\alpha_1-\alpha_2)}{\sqrt{\varepsilon^2+(\alpha_1-\alpha_2)^2}}$

$\therefore GAG^T(2,1) = \dfrac{-\varepsilon^3}{(\alpha_1-\alpha_2)^2+\varepsilon^2} = O(\varepsilon^3)$

改用 Wilkinson 位移时，将 $\mu = a_2 + \delta - \mathrm{sgn}(\delta)\sqrt{\delta^2+\varepsilon^2}$，$\delta = \frac{a_1-a_2}{2}$ 代入

同理可得 $GAG^T(2,1) = O(\varepsilon^3)$ □

过程详见学习辅导 P124

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$$\begin{aligned}
\alpha_{qq}^{(k+1)} &= \alpha_{qq}^{(k)} - c^2\left(-2t\alpha_{pq}^{(k)} + t^2(\alpha_{qq}^{(k)} - \alpha_{pp}^{(k)})\right) \\
&= \alpha_{qq}^{(k)} - c^2\left(-2t\alpha_{pq}^{(k)} + t(1-t^2)\alpha_{pq}^{(k)}\right) \\
&= \alpha_{qq}^{(k)} - \frac{1}{1+t^2}\left(-2t\alpha_{pq}^{(k)} + t(1-t^2)\alpha_{pq}^{(k)}\right) \\
&= \alpha_{qq}^{(k)} + t\,\alpha_{pq}^{(k)}
\end{aligned}$$

13. $\begin{pmatrix} c & s \\ -s & c \end{pmatrix} C = \begin{pmatrix} c\alpha_{11}+s\alpha_{21} & c\alpha_{12}+s\alpha_{22} \\ -s\alpha_{11}+c\alpha_{21} & -s\alpha_{12}+c\alpha_{22} \end{pmatrix} := \tilde{C}$

$\tilde{C}$对称 $\Leftrightarrow c(\alpha_{21}-\alpha_{12}) = s(\alpha_{11}+\alpha_{22})$

$c = \frac{\alpha_{11}+\alpha_{22}}{\sqrt{(\alpha_{11}+\alpha_{22})^2+(\alpha_{21}-\alpha_{12})^2}}$, $s = \frac{\alpha_{21}-\alpha_{12}}{\sqrt{\cdots}}$ 即可

奇异值分解: $\tilde{C} = P\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} P^T$ (不妨 $\lambda_1 \geq \lambda_2$, 若 $\lambda_1 < \lambda_2$, 令 $P = P\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 即可)

$\therefore C = \underbrace{\begin{pmatrix} c & -s \\ s & c \end{pmatrix} P \begin{pmatrix} \text{sgn}\,\lambda_1 & \\ & \text{sgn}\,\lambda_2 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} |\lambda_1| & \\ & |\lambda_2| \end{pmatrix}}_{\Sigma} \underbrace{P^T}_{V^T}$

$\text{sgn}\, x := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ (为了U可逆, sgn x 不能为 0) □

14、利用上题算法找 θ_0, s.t. $J(p,q,\theta_0)A := G$ 满足 $g_{pq} = g_{qp}$

又由Jacobi算法, $\exists \theta_2$, s.t. $J(p,q,\theta_2)^T G J(p,q,\theta_2) = B$ 满足 $B_{pq} = B_{qp}$

令 $\theta_1 = \theta_0 - \theta_2$

$\therefore J(p,q,\theta_1) A J(p,q,\theta_2) = B$ 满足要求

$\because E(G)^2 = E(A)^2 + 2g_{pq}^2 - (\alpha_{pq}^2 + \alpha_{qp}^2)$

$\therefore E(B)^2 = E(G)^2 - 2g_{pq}^2 = E(A)^2 - (\alpha_{pq}^2 + \alpha_{qp}^2)$ □

15. 不妨 $m > n$

① 用 Householder 变换将 A变成 $\begin{pmatrix} \tilde{A} \\ 0 \end{pmatrix}\begin{matrix} n \\ m-n \end{matrix}$ ($\tilde{A}$ 为 $n \times n$)

② 对 $\tilde{A}$ 找最大的 $\alpha_{pq}^2+\alpha_{qp}^2$ 套用上题算法 (直接遍历 p.q 也行) □

16 $[x,y]\begin{pmatrix} c & s \\ -s & c \end{pmatrix} = [cx-sy, sx+cy]$ 正交

$\Leftrightarrow csx^Tx + (c^2-s^2)x^Ty - scy^Ty = 0$ (*)

① $x^Ty=0$, 取 $c=1$, $s=0$ 即可

扫描全能王 创建

② $x^Ty \neq 0$，令 $t=\tan\theta$，则 (*) $\Leftrightarrow (1-t^2)x^Ty + t(x^Tx - y^Ty) = 0$

$\Delta = (x^Tx - y^Ty)^2 + 4(x^Ty)^2 \geq 0$，故取 t 为任一个根即可 □

17. 定义能量项 $E(A) = \sum_{i \neq j} (p_i^H p_j)^2$　　$A = [p_1, p_2, \cdots p_n]$，$E(A) = 0 \Leftrightarrow A$ 正交

下面证，对 p_s、p_t 运用上题算法时，能量下降

令 $[\tilde{p}_s, \tilde{p}_t] = [p_s, p_t]\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$

$\therefore E - \tilde{E} = (p_s^H p_t)^2 + \sum_{i \neq s,t} [(p_i^H p_s)^2 - (p_i^H \tilde{p}_s)^2 + (p_i^H p_t)^2 - (p_i^H \tilde{p}_t)^2]$

$= (p_s^H p_t)^2 > 0$

即每次套用上题算法，均会使 A 变得更"接近"正交阵

故若每次选取 $(s,t) = \underset{\{s,t\}}{\arg\max} (p_i^H p_j)^2$，则

$E^{(k)} = E^{(k-1)} - \max_{i \neq j} (p_i^{(k-1)H} p_j^{(k-1)})^2$

又 $E^{(k-1)} \leq n(n-1) \max_{i \neq j} (p_i^{(k-1)H} p_j^{(k-1)})^2$

$\therefore E^{(k)} \leq (1 - \frac{1}{N}) E^{(k-1)}$　　$N = \frac{1}{2} n(n-1)$

$\Rightarrow \lim_{k \to \infty} E^{(k)} = 0$

$\therefore$ 算法为：每次选取合适的 s、t 套用上题算法（也可直接遍历）直至收敛

18. 设 $D = \mathrm{diag}(d_1, \cdots d_n)$

$\therefore D^{-1}AD$ 的上次对角变为 $\frac{d_2}{d_1}\beta_1, \cdots, \frac{d_n}{d_{n-1}}\beta_{n-1}$

下次对角变为 $\frac{d_1}{d_2}\gamma_1, \cdots \frac{d_{n-1}}{d_n}\gamma_{n-1}$

$\left(\frac{d_{k+1}}{d_k}\right)^2 = \frac{\gamma_k}{\beta_k} > 0$

取 $d_1 = 1$，依次计算 d_k 即可

（详见 学习辅导 P126）

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将 $Tx = \lambda x$ 写为分量形式

$(*)$ $\beta_i \xi_{i-1} + \alpha_i \xi_i + \beta_{i+1}\xi_{i+1} = \lambda \xi_i \quad (i=1,\cdots,n)$, $\beta_1 = \beta_{n+1} = 0$

若 $\xi_1 = 0$ ，则由 $(*)$ 易得 $\xi_i = 0 \quad \forall\, i=1,2,\cdots n$ ， $x=0$, 矛盾

同理可得 $\xi_n \neq 0 \Rightarrow \xi_1 \xi_n \neq 0$

(注：T不可约 $\Rightarrow \beta_i \neq 0$)

(2) 归纳法：

① $i=2$ 时，即 $\beta_2 \xi_2 = -P_1(\lambda) = \lambda - \alpha_1$ ，由 $(*)$ 得证

② 假设 $i<k$ 时成立，$i=k$ 时，$P_{k-1}(\lambda) = (\alpha_{k-1}-\lambda) P_{k-2}(\lambda) - \beta_{k-1}^2 P_{k-3}(\lambda)$

$\therefore P_{k-1}(\lambda) = (\alpha_{k-1}-\lambda)(-1)^{k-2} \prod_{i=2}^{k-1} \beta_i \xi_{k-1} + \beta_{k-1}^2 (-1)^{k-2} \prod_{i=2}^{k-2} \beta_i \xi_{k-2}$

$= (-1)^k \prod_{i=2}^{k-1} \beta_i [(\alpha_{k-1}-\lambda)\xi_{k-1} + \beta_{k-1}\xi_{k-2}]$

$= (-1)^k \prod_{i=2}^{k} \beta_i \xi_k$ □

(详见学习辅导 P127)

~~20 设 $T = diag(T_1, \cdots T_n)$ ， T_i为不可约三对角阵~~

20. ~~反证~~： T对称 $\Rightarrow$ T可对角化 $\Rightarrow \lambda$的几何重数=代数重数

$\therefore rank(T-\lambda I) = n-k$

反证：若 T的次对角元 仅有$k-2$个为0 ，不妨 $\beta_2 = \beta_3 = \cdots \beta_{k-1} = 0$ ， $\beta_k, \beta_{k+1}, \cdots \beta_n \neq 0$

则 $T-\lambda I$中有 $n-k+1$阶子式 $T' = \begin{pmatrix} \beta_k & & * \\ & \ddots & \\ & & \beta_n \end{pmatrix}$ ，$\det(T') \neq 0$

$\therefore rank(T-\lambda I) \geq n-k+1$， 矛盾 □

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(1) $X^T T X = -\sum x_i^2 - (x_1-x_2)^2 - (x_2-x_3)^2 - (x_3-x_4)^2 \leq 0$ 且取等 $\Leftrightarrow x=0$ ，T负定

(2) $\lambda(T) = \{2(\cos\frac{j\pi}{5} - 1) \mid j=1,2,3,4\}$ $\therefore$ 2个

(方法 学习辅导 P127

22 $\begin{cases} (J-\tilde{\lambda} I)\, y_k = z_{k-1} \\ z_k = \cancel{y_k/\|y_k\|_\infty}\ y_k/\mu_k \end{cases}$ ，μ_k为y_k的绝对值最大分量

23 B 二对角 $\Rightarrow A = BB^T$ 三对角对称，对A运用二分法 □

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$C^* = C \Leftrightarrow A^T - iB^T = A + iB \Leftrightarrow A^T = A,\ B^T = -B \Leftrightarrow M^T = M$

设 $\lambda = a+bi$ 对应的特征向量为 $u+vi$

$\therefore (A+Bi)(u+vi) = \lambda(u+vi) \Leftrightarrow \begin{cases} \lambda u = Au - Bv \\ \lambda v = Av + Bu \end{cases}$

$\Leftrightarrow \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$

$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} -v \\ u \end{pmatrix} = \lambda \begin{pmatrix} -v \\ u \end{pmatrix}$

$\therefore C$有特征向量 $u+vi \Leftrightarrow M$有特征向量 $\begin{pmatrix} u \\ v \end{pmatrix}$和$\begin{pmatrix} -v \\ u \end{pmatrix}$，特征值相同

3. 求正交矩阵 R 最小化 $E=\sum_{i=1}^{N}\|q_i'-Rq_i\|^2$，$q_i', q_i \in \mathbb{R}^{3\times 1}$

Lemma：对任意正定矩阵 H 及正交矩阵 B，$Trace(BH) \le Trace(H)$

Lemma pf：对 H 进行 Cholesky 分解 $H=LL^T$，记 l_i 为 L 的第 i 列

$\therefore Trace(BLL^T) = Trace(L^TBL) = \sum_i l_i^T(Bl_i)$

又由 Cachy-Schwarz 不等式 $l_i^T(Bl_i) \le \sqrt{(l_i^Tl_i)(l_i^TB^TBl_i)} = l_i^Tl_i$

$\therefore Trace(BLL^T) \le \sum_i l_i^Tl_i = Trace(LL^T)$

原题 pf：

$\because E = \sum_{i=1}^{N}(q_i'-Rq_i)^T(q_i'-Rq_i)$

$= \sum_{i=1}^{N}(q_i'^Tq_i' + q_i^Tq_i - 2q_i'^TRq_i)$

$\Rightarrow R$ 极小化 $E \Leftrightarrow R$ 极大化 $\sum_{i=1}^{N} q_i'^TRq_i := F$

$F = \sum_{i=1}^{N} q_i'^TRq_i = Trace(\sum_{i=1}^{N} Rq_iq_i'^T)$，记 $H=\sum_{i=1}^{N} q_iq_i'^T$

$\therefore F = Trace(RH)$

对 H 进行 SVD 分解 $H=U\Lambda V^T$，令 $R=VU^T$

$\therefore RH = VU^TU\Lambda V^T = V\Lambda V^T$ 对称正定

故由 Lemma，对 $\forall$ 正交矩阵 B，$F=Trace(RH) \ge Trace(BRH)$

即 R 为最大化 F 的正交矩阵

出处：[Least-Squares Fitting of Two 3D Point Sets, 1987 IEEE]