1. $x^TAx - 2b^Tx + x_*^TAx_* \overset{?}{=} (x-x_*)^TA(x-x_*)$

$RHS = x^TAx + x_*^TAx_* - 2x_*^TAx$

$x_*^TA = b^T$ 得证

2. $\varphi(x_k) = \varphi(x_{k-1}) - \frac{(r_{k-1}^Tr_{k-1})^2}{r_{k-1}^TAr_{k-1}} \leq \varphi(x_{k-1}) - \frac{(r_{k-1}^Tr_{k-1})^2}{r_{k-1}^TAr_{k-1}} \cdot \frac{\varphi(x_{k-1})}{r_{k-1}^TA^{-1}r_{k-1}} \leq \varphi(x_{k-1})\left[1 - \frac{1}{\|A\|_2\|A^{-1}\|_2}\right]$

$r_{k-1}^TA^{-1}r_{k-1} = (b-Ax_{k-1})^TA^{-1}(b-Ax_{k-1}) = b^TA^{-1}b + \varphi(x_{k-1}) \geq \varphi(x_{k-1})$

3. $x_* = x_k + \frac{r_k^Tr_k}{r_k^TAr_k} \cdot r_k \Rightarrow b = Ax^* = Ax_k + Ar_k \cdot \frac{r_k^Tr_k}{r_k^TAr_k}$

$\Rightarrow Ar_k = r_k \cdot \frac{r_k^TAr_k}{r_k^Tr_k}$

4. $\underbrace{\begin{pmatrix} r_k^TAr_k & r_k^TAp_{k-1} \\ r_k^TAp_{k-1} & p_{k-1}^TAp_{k-1} \end{pmatrix}}_{=G}$ 可逆 $\Leftrightarrow$ G正定 $\Leftrightarrow (x_1r_k + x_2p_{k-1})^TA(x_1r_k + x_2p_{k-1}) = 0$

当且仅当 $x_1 = x_2 = 0$

$\Leftrightarrow$ r_k与p_{k-1}线性无关

但 $r_k^Tp_{k-1} = 0$ 且 $r_k \neq 0$ $p_{k-1} \neq 0$

故G可逆

5. $\sum_{i=1}^k \lambda_ip_i = 0 \Rightarrow \sum_{i=1}^k\sum_{j=1}^k \lambda_i\lambda_j p_i^TAp_j = (\sum_{i=1}^k\lambda_ip_i)^TA(\sum_{i=1}^k\lambda_ip_i) = 0$

$\Rightarrow \sum_{i=1}^k \lambda_i^2p_i^TAp_i = 0 \Rightarrow \lambda_i = 0 \quad \forall i$

6. $\varphi(x) = x^TAx - 2b^Tx$

$\frac{d\varphi(y_{i-1}+te_i)}{dt} = 2(y_{i-1}+te_i)^TAe_i - 2b^Te_i = 0$

$\Rightarrow t = \frac{1}{a_{ii}}(b - Ay_{i-1})^Te_i$

在GS中第i行的计算为 $\frac{1}{a_{ii}}(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)})$ $(*)$

由于 y_i与y_{i-1}区别仅在第i行，故$\begin{cases} x_j^{(k+1)} = y_i^Te_j & (i \geq j) \\ x_j^{(k)} = y_i^Te_j & (i<j) \end{cases}$ $\Rightarrow (*) = \frac{1}{a_{ii}}(b+(L+U)y_{i-1})^Te_i$

$\Rightarrow$ $(*)$与计算前差别为 $(*) - x_i^{(k)} = (*) - y_{i-1}^Te_i$

$= \frac{1}{a_{ii}}(b - Ay_{i-1})^Te_i$

7. $A = A^T \Rightarrow A$的特征值λ_i (n_i重) $i = 1, \cdots, k$ 实数 $\sum_{i=1}^k n_i = n$

$\Rightarrow d_A(x) = \prod_{i=1}^k (x - \lambda_i)$ 最小多项式

$\Rightarrow d_A(A) = 0 \Rightarrow A^k = \sum_{i<k} c_iA^i \Rightarrow A^kr \in Span\{A^ir\}_{i<k}$

$\Rightarrow A^{k+j}r \in Span\{A^ir\}_{i<k+j} \subset Span\{A^ir\}_{i<k}$ (归纳)

8. 由7.显然 (定理5.2.2)

9. $A = P\Sigma P^T$ P正交 $\Sigma = diag(\lambda_i)$ $\lambda_i \geq \lambda_{i+1} > 0$

$\begin{cases} \lambda_ny^Ty = \lambda_nx^Tx \leq x^T\Sigma x = y^TAy & (y = Px) \\ \lambda_1y^Ty = \lambda_1x^Tx \geq x^T\Sigma x = y^TAy \end{cases}$

$\Rightarrow \sqrt{\lambda_n}\|y\|_2 \leq \|y\|_A \leq \sqrt{\lambda_1}\|y\|_2$ $(*)$

而 $\|A\|_2 = \sqrt{\lambda_1}$, $\|A^{-1}\|_2 = \frac{1}{\sqrt{\lambda_n}}$

$(*)$代入定理5.3.2即可

11. 在 $\langle x, y\rangle = x^TAy$ 下 $\mathbb{R}^n$为内积空间

x_k 最小化 $\|x - A^{-1}b\|_A \Leftrightarrow \forall w \in \mathcal{X}$ $\frac{d}{d\alpha}\|x_k + \alpha w - A^{-1}b\|_A^2\Big|_{\alpha=0} = 0$

因为这样的点存在则唯一

$\Leftrightarrow w^TA(x_k - A^{-1}b) = 0 \quad \forall w \in \mathcal{X}$

$\Leftrightarrow Ax_k - b \perp \mathcal{X}$

12. 将$\begin{cases} p_{k-1}^TA^TAp_{k-1} \\ \alpha_{k-1}A^TAp_{k-1} \end{cases}$ 用先算矩阵乘向量即可