4.1.1. $\tilde{u}$：实际计算得到的解；

u：精确解；

由题意 $\dfrac{\|\tilde{u}-u\|}{\|u\|} \leq 1\%$

$\Rightarrow \|\tilde{u}-u\| \leq 1\% \cdot \|u\|$

P_{112}： $\|u(\cdot,t)\| \leq e^{100t}\|f(\cdot)\|$

$0 \leq t \leq 2$

$\Rightarrow \|\tilde{u}-u\| \leq \dfrac{e^{200}}{100}\|f\|$.

4.2.1. 类似 The 4.2.1. 只是将 k 替换，此时 $k = \sum_{j=0}^{4} a_j (i\omega)^4$

$\Rightarrow \mathrm{Re}\,k = \mathrm{Re}\,a_4\omega^4 + \mathrm{Im}\,a_3\omega^3 - \mathrm{Re}\,a_2\omega^2 - \mathrm{Im}\,a_1\omega + \mathrm{Re}\,a_0$

若 $\mathrm{Re}\,a_4<0$，则当 $\omega \to \pm\infty$ 时，$\mathrm{Re}\,k \to -\infty$.

$\Rightarrow$ 存在实数 α 使得 $\mathrm{Re}\,k \leq \alpha$，$\forall \omega$.

故当 $\mathrm{Re}\,a_4<0$ 时该问题适定。

4.3.1 首先考察谐波解；$u(x,0)=f(x)=\frac{1}{\sqrt{2\pi}}e^{i\omega x}\hat{f}(\omega)$

"范数定义" $u(x,t)=\frac{1}{\sqrt{2\pi}}e^{i\omega x}\hat{u}(\omega,t) \Rightarrow \hat{u}(\omega,0)=\hat{f}(\omega)$.

此时 $\|u(\cdot,t)\|^2=|\hat{u}(\omega,t)|^2$

$\Rightarrow \|u(\cdot,t)\|=\|u(\cdot,0)\| \Leftrightarrow |\hat{u}(\omega,t)|^2=|\hat{u}(\omega,0)|^2$

即要求 $|\hat{u}(\omega,t)|^2=$ 常数，或等价地，

$\frac{\partial}{\partial t}|\hat{u}|^2 = 0.$

而 $\hat{u}_t \overset{\hat{\partial}=0}{=} (i\omega A + B)\hat{u}$

于是 $\frac{\partial}{\partial t}|\hat{u}|^2 = \langle \hat{u}_t, \hat{u} \rangle + \langle \hat{u}, \hat{u}_t \rangle$

$= \langle (i\omega A + B)\hat{u}, \hat{u} \rangle + \langle \hat{u}, (i\omega A + B)\hat{u} \rangle$

向量内积定义

$= \langle \hat{u}, (-i\omega A^* + B^*)\hat{u} \rangle + \langle \hat{u}, (i\omega A + B)\hat{u} \rangle$

$= \langle \hat{u}, (i\omega(A - A^*) + (B + B^*))\hat{u} \rangle$

$= 0$

$\Rightarrow \quad A^* = A, \quad B^* = -B.$

对于一般解, $u(x,0) = f(x) = \frac{1}{\sqrt{2\pi}} \sum_{\omega} e^{i\omega x} \hat{f}(\omega)$

$u(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{\omega} e^{i\omega x} \hat{u}(\omega, t) \quad \Rightarrow \hat{u}(\omega,0) = \hat{f}(\omega)$

由 Parseval 等式 $\|u(\cdot,t)\|^2 = \sum_{\omega} |\hat{u}(\omega,t)|^2$

同样地若要 $\|u(\cdot,t)\| = \|u(\cdot,0)\|$, 只需 $|\hat{u}(\omega,t)|^2 = |\hat{u}(\omega,0)|^2$

亦即要求 $A^* = A, \quad B^* = -B.$

4.4.1. 首先考察谐波解. $u(x,0)=f(x)=\frac{1}{\sqrt{2\pi}}e^{i\omega x}\hat{f}(\omega)$

$$u(x,t)=\frac{1}{\sqrt{2\pi}}e^{i\omega x}\hat{u}(\omega,t) \quad \Rightarrow \hat{u}(\omega,0)=\hat{f}(\omega).$$

$$\Rightarrow u_x(x,t)=i\omega\frac{1}{\sqrt{2\pi}}e^{i\omega x}\hat{u}(\omega,t)=i\omega u(x,t)$$

$$\Rightarrow \|u(\cdot,t)\|^2=|\hat{u}(\omega,t)|^2$$

$$\|u_x(\cdot,t)\|^2=\omega^2|\hat{u}(\omega,t)|^2$$

于是原不等式 $\Leftrightarrow |\hat{u}(\cdot,t)|^2+\delta\omega^2\int_0^t|\hat{u}(\cdot,\xi)|^2d\xi \leq K|\hat{u}(\cdot,0)|^2$

注意到 LHS $t=0$ 时便是 RHS, 取 $K=1$

因此只需证明 LHS 不增, 这等价于对 t 求导

$$\frac{\partial}{\partial t}|\hat{u}|^2+\delta\omega^2|\hat{u}|^2\leq 0 \qquad (*)$$

事实上有 $\hat{u}_t=-\omega^2A\hat{u}$

$$\Rightarrow \frac{\partial}{\partial t}|\hat{u}|^2=\langle\hat{u}_t,\hat{u}\rangle+\langle\hat{u},\hat{u}_t\rangle$$

$$=-\omega^2(\langle A\hat{u},\hat{u}\rangle+\langle\hat{u},A\hat{u}\rangle)$$

$$=-\omega^2(\langle\hat{u},A^*\hat{u}\rangle+\langle\hat{u},A\hat{u}\rangle)$$

parabolic: $A+A^*\geq\delta I,\ \delta>0$ $\quad =-\omega^2\langle\hat{u},(A+A^*)\hat{u}\rangle$

$$\leq-\omega^2\delta|\hat{u}|^2$$,即 $(*)$ 式成立.

因此存在正常数 δ, K 使得不等式成立.

对一般解，$u(x,0)=f(x)=\frac{1}{\sqrt{2\pi}}\sum_{\omega}e^{i\omega x}\hat{f}(\omega)$

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\sum_{\omega}e^{i\omega x}\hat{u}(\omega,t),\quad \hat{u}(\omega,0)=\hat{f}(\omega)$$

$$\Rightarrow u_x(x,t)=\frac{1}{\sqrt{2\pi}}\sum_{\omega}i\omega e^{i\omega x}\hat{u}(\omega,t).$$

由Parseval等式

$$\Rightarrow \|u(\cdot,t)\|^2=\sum_{\omega}|\hat{u}(\omega,t)|^2$$

$$\|u_x(\cdot,t)\|^2=\sum_{\omega}\omega^2|\hat{u}(\omega,t)|^2$$

$\Rightarrow$ 若成立

$$|\hat{u}(\cdot,t)|^2+\delta\omega^2\int_0^t|\hat{u}(\cdot,s)|d\xi\le K\|u(\cdot,t)\|^2\qquad(**)$$

则对一般解，原不等式也成立，

由于此时仍然成立 $\hat{u}_t=-\omega^2A\hat{u}$，故前面的推导对 $(**)$ 式仍然适用，故 $(**)$ 式成立，故对一般解，不等式仍成立。

4.4.2. 只需证明 $(*)$ 式仍成立。

事实上 此时有 $\hat{u}_t=(-\omega^2A+i\omega B+C)\hat{u}$.

$$\Rightarrow \frac{d}{dt}|\hat{u}|^2=\langle\hat{u}_t,\hat{u}\rangle+\langle\hat{u},\hat{u}_t\rangle$$

$$=\langle(-\omega^2A+i\omega B+C)\hat{u},\hat{u}\rangle+\langle\hat{u},(-\omega^2A+i\omega B+C)\hat{u}\rangle$$

$$=\langle\hat{u},(-\omega^2(A+A^*)+i\omega(B-\cancel{B^*})+(\cancel{C+C^*}))\hat{u}\rangle$$

$B^*=B,\ C^*=-C$

$$=-\omega^2\langle\hat{u},(A+A^*)\hat{u}\rangle$$

$A+A^*\ge\delta I,\ \delta>0$

$$\le-\omega^2\delta|\hat{u}|^2.$$

补充题：P_{120} Thm 4.3.2.

$0 \le \theta \le 1$

2.5.2. θ scheme : $\dfrac{v_j^{n+1}-v_j^n}{k} = \theta\dfrac{v_{j+1}^{n+1}-2v_j^{n+1}+v_{j-1}^{n+1}}{h^2} + (1-\theta)\dfrac{v_{j+1}^n-2v_j^n+v_{j-1}^n}{h^2}$

小测1 FTCS scheme ($\theta=0$)

取 $v_j^n = \frac{1}{\sqrt{2\pi}}\hat{v}^n(\omega)e^{i\omega x_j}$

$$\Rightarrow \hat{v}^{n+1}_{(\omega)} - \hat{v}^n_{(\omega)} = \lambda\underbrace{\left(e^{i\omega h}-2+e^{-i\omega h}\right)}_{=2(\cos\xi-1)\ =-4\sin^2\frac{\xi}{2}}\left(\theta\hat{v}^{n+1}_{(\omega)}+(1-\theta)\hat{v}^n_{(\omega)}\right), \quad \lambda=\frac{k}{h^2}$$

$\xi=\omega h$

$$\Rightarrow \hat{v}^{n+1}_{(\omega)} = \frac{1-4(1-\theta)\lambda\sin^2\frac{\xi}{2}}{1+4\theta\lambda\sin^2\frac{\xi}{2}}\hat{v}^n_{(\omega)} = \hat{Q}\,\hat{v}^n_{(\omega)}$$

$\hat{Q} = 1-\dfrac{4\lambda\sin^2\frac{\xi}{2}}{1+\theta\cdot4\lambda\sin^2\frac{\xi}{2}} \le 1$ 故要 $|\hat{Q}|\le1$ 只需 $\dfrac{4\lambda\sin^2\frac{\xi}{2}}{1+\theta\cdot4\lambda\sin^2\frac{\xi}{2}} \le 2$

$\Leftrightarrow (1-2\theta)4\lambda\sin^2\frac{\xi}{2} \le 2$ 显然当 $\theta\ge\frac{1}{2}$ 时 $1-2\theta\le0$ 不等式成立.

因此当 $\theta\ge\frac{1}{2}$ 时 θ scheme 无条件稳定.

2.5.3. $u_t=u_{xx}$.

backward Euler scheme ($\theta=1$):

$$\frac{v_j^{n+1}-v_j^n}{k} = \frac{v_{j+1}^{n+1}-2v_j^{n+1}+v_{j-1}^{n+1}}{h^2}$$

$\Rightarrow$ 局部截断误差 $T_j^{n+1} = \dfrac{u_j^{n+1}-u_j^n}{k} - \dfrac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h^2}$

$$=\left((u_t)_j^{n+1}-\frac{k}{2}(u_{tt})_j^{n+1}+O(k^2)\right)-\left((u_{xx})_j^{n+1}+\frac{h^2}{12}(u^{(4)})_j^{n+1}+O(h^4)\right)$$

$$=\cancel{(u_t-u_{xx})_j^{n+1}}+O(h^2+k) = O(h^2+k)$$

Crank-Nicholso scheme $(\theta=\frac{1}{2})$:

$$\frac{V_j^{n+1}-V_j^n}{k}=\frac{1}{2}\,\frac{V_{j+1}^{n+1}-2V_j^{n+1}+V_{j-1}^{n+1}}{h^2}+\frac{1}{2}\,\frac{V_{j+1}^n-2V_j^n+V_{j-1}^n}{h^2}$$

⇒ 局部截断误差 $$\tau_j^n=\frac{u_j^{n+1}-u_j^n}{k}-\frac{1}{2}\,\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h^2}-\frac{1}{2}\,\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}$$

$$=\left((u_t)_j^n+\frac{k}{2}(u_{tt})_j^n+O(k^2)\right)$$

$$-\frac{1}{2}\left((u_{xx})_j^{n+1}+\frac{h^2}{12}(u^{(4)})_j^{n+1}+O(h^4)\right)$$

$$-\frac{1}{2}\left((u_{xx})_j^n+\frac{h^2}{12}(u^{(4)})_j^n+O(h^4)\right)$$

又 $(u_{xx})_j^{n+1}=(u_{xx})_j^n+k(u_{xxt})_j^n+O(k^2)$

而由 $u_t=u_{xx}$ 得 $u_{tt}=u_{xxt}$

即得 $(u_{xx})_j^{n+1}=(u_{xx})_j^n+k(u_{tt})_j^n+O(k^2)$

$$\Rightarrow \tau_j^n=\cancel{(u_t-u_{xx})_j^n}+O(h^2+k^2)=O(h^2+k^2).$$

$\lambda=\frac{k}{h^2}$ 近似常值，满足 CFL 条件.

补充题：1.（均匀剖分）用 u 在三个点处的函数值的线性组合无法得到 u_{xx} 的三阶或高于三阶的近似。

$$(u_{xx})_j\approx a_{-1}u_{j-1}+a_0u_j+a_1u_{j+1}$$

$$u_{j\pm1} = u_j \pm h(u_x)_j + \frac{h^2}{2}(u_{xx})_j \pm \frac{h^3}{3!}(u_{xxx})_j + \frac{h^4}{4!}(u^{(4)})_j \pm \cdots$$

$$\Rightarrow a_{-1}u_{j-1} + a_0 u_j + a_1 u_{j+1} = (a_{-1}+a_0+a_1)u_j$$
$$+ (-a_{-1}+a_1)h(u_x)_j$$
$$+ (a_{-1}+a_1)\frac{h^2}{2}(u_{xx})_j$$
$$+ (-a_{-1}+a_1)\frac{h^3}{3!}(u_{xxx})_j$$
$$+ (a_{-1}+a_1)\frac{h^4}{4!}(u^{(4)})_j + \cdots$$

$$\begin{cases} a_{-1}+a_0+a_1 = 0 \\ -a_{-1}+a_1 = 0 \\ (a_{-1}+a_1)\cdot\frac{h^2}{2} = 1 \end{cases}$$

唯一解 $a_{-1} = a_1 = \frac{1}{h^2}$, $a_0 = \frac{-2}{h^2}$

于是用 u_{j-1}, u_j, u_{j+1} 这3个点来近似 $(u_{xx})_j$ 只有

$$\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = (u_{xx})_j + \frac{h^2}{12}(u^{(4)})_j + \cdots$$
$$= (u_{xx})_j + O(h^2).$$

只有2阶，不能是3阶或更高.

2. 针对 $u_t = u_{xx}$，基于其在控制体 $[t_{n-1}, t_{n+1}] \times [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ 上的积分形式构造有限体积格式.

参考书 1.6.2节.

3. 针对 $u_t + a u_x = 0$，a 为常数，基于其积分形式构造时间1阶，空间3阶的有限差分格式。

参考书1.6节，答案不唯一。

注意：2、3题的区别。

4. 针对偏微分方程 $u_t = (p(x)u_x)_x$，构造有限差分格式，并且分析其截断误差。

答案不唯一。

注意 $p(x)$ 已知，不作近似处理。

小测2. $u_t = b(x,t)u_{xx}$ FTCS格式，整体误差。

$$\frac{v_j^{n+1}-v_j^n}{k} = b_j^n \cdot \frac{u_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2}$$

相容性：

$$T_j^n = \frac{u_j^{n+1}-u_j^n}{k} - b_j^n \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}$$

$$= \left((u_t)_j^n + \frac{k}{2}(u_{tt})_j^n + O(k^2)\right) - b_j^n\left((u_{xx})_j^n + \frac{h^2}{12}(u^{(4)})_j^n + O(h^4)\right)$$

$$= \cancel{(u_t - bu_{xx})_j^n} + O(h^2+k)$$

$$= O(h^2+k).$$

稳定性.

$$\lambda = \frac{k}{h^2} \quad , \xi = wh$$

$$v_j^n = \frac{1}{\sqrt{2\pi}} \hat{v}^n_{(\omega)} e^{i\omega x_j}$$

$$\Rightarrow \hat{v}^{n+1} = \left(1 - b_j^n \cdot 4\lambda \sin^2\frac{\xi}{2}\right) \hat{v}^n = \hat{Q}\, \hat{v}^n$$

$$|\hat{Q}| \leq 1 \iff b_j^n \geq 0 \text{ 且 } 2\lambda b_j^n \leq 1 \quad (\text{CFL条件}).$$

整体误差:

$$u_j^{n+1} = u_j^n + \lambda b_j^n \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) + k\tau_j^n$$

$$v_j^{n+1} = v_j^n + \lambda b_j^n \left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right)$$

记 $e_j^n = u_j^n - v_j^n$

$$\Rightarrow e_j^{n+1} = e_j^n + \lambda b_j^n \left(e_{j+1}^n - 2e_j^n + e_{j-1}^n\right) + k\tau_j^n$$

$$= \left(1 - 2\lambda b_j^n\right) e_j^n + \lambda b_j^n \left(e_{j+1}^n + e_{j-1}^n\right) + k\tau_j^n$$

$$E^n = \max_j \{|e_j^n|\}$$

$$\Longrightarrow E^{n+1} \leq \left(1 - 2\lambda b_j^n\right) E^n + 2\lambda b_j^n E^n + k\tau^n$$

$$= E^n + k\tau^n$$

$$\leq E^{n-1} + 2k\tau^n$$

$$\cdots$$

$$E^0 = 0 . \quad k = \frac{T}{n+1}$$

$$\leq E^0 + (n+1)k\tau^n = T\tau^n \to 0 , \quad h, k \to 0$$