HW7.

1. (PPT-1116). 讨论 $u_t + a(x,t)\,u_x = 0$，其中 $a(x,t) > 0$ 且 $a_x(x,t)$ 有界。

讨论 Lax-Friedrichs 格式的 L^2 模稳定性：

$$v_j^{n+1} = \frac{1}{2}\left(v_{j+1}^n + v_{j-1}^n\right) - \frac{\Delta t}{2\Delta x} a_j^n \left(v_{j+1}^n - v_{j-1}^n\right)$$

解：改写格式为 $v_j^{n+1} = \left(\frac{1}{2} - \frac{\Delta t}{2\Delta x} a_j^n\right) v_{j+1}^n + \left(\frac{1}{2} + \frac{\Delta t}{2\Delta x} a_j^n\right) v_{j-1}^n$

记 $A = \max a(x,t)$ $\quad B = \max |a_x(x,t)|$. 由CFL条件 $\frac{\Delta t}{\Delta x}\cdot A \le 1$

故 $\frac{1}{2} \pm \frac{\Delta t}{2\Delta x} a_j^n \ge 0$ 且 $\left(\frac{1}{2} - \frac{\Delta t}{2\Delta x} a_j^n\right) + \left(\frac{1}{2} - \frac{\Delta t}{2\Delta x} a_j^n\right) = 1$.

对凸函数"x^2"用 Jensen 不等式得

$$\left(v_j^{n+1}\right)^2 \le \left(\frac{1}{2} - \frac{\Delta t}{2\Delta x} a_j^n\right)\left(v_{j+1}^n\right)^2 + \left(\frac{1}{2} + \frac{\Delta t}{2\Delta x} a_j^n\right)\left(v_{j-1}^n\right)^2$$

$$= \left(\frac{1}{2} - \frac{\Delta t}{2\Delta x} a_{j+1}^n\right)\left(v_{j+1}^n\right)^2 + \left(\frac{1}{2} + \frac{\Delta t}{2\Delta x} a_{j-1}^n\right)\left(v_{j-1}^n\right)^2$$

$$+ \frac{\Delta t}{2\Delta x}\left(a_{j+1}^n - a_j^n\right)\left(v_{j+1}^n\right)^2 + \frac{\Delta t}{2\Delta x}\left(a_j^n - a_{j-1}^n\right)\left(v_{j-1}^n\right)^2$$

中值公式 $\left|a_{j\pm1}^n - a_j^n\right| \le B\Delta x$

$$\le \left(\frac{1}{2} - \frac{\Delta t}{2\Delta x} a_{j+1}^n\right)\left(v_{j+1}^n\right)^2 + \left(\frac{1}{2} + \frac{\Delta t}{2\Delta x} a_{j-1}^n\right)\left(v_{j-1}^n\right)^2$$

$$+ \frac{\Delta t}{2} B\left(v_{j+1}^n\right)^2 + \frac{\Delta t}{2} B\left(v_{j-1}^n\right)^2$$

对j求和 $\sum_j$ 并适当平移坐标合并可得

$$\Rightarrow \|V^{n+1}\|_2^2 \le (1+B\Delta t)\|V^n\|_2^2$$

$$\cdots\cdots$$

$$\le (1+B\Delta t)^n \|V^0\|_2^2$$

$$\le e^{BT}\|V^0\|_2^2$$

$\Delta t = \frac{T}{n}$

$$\Rightarrow \|V^{n+1}\|_2 \le e^{\frac{BT}{2}}\|V^0\|_2 .$$

2. (PPT-1125). 考虑 Burgers 方程，假定给定光滑初值 $u_0(x)$，其在某些点的导数 $u_0'(x)<0$，试证明：在 T_b 时刻特征线首次产生相交

$$T_b = \frac{-1}{\min u_0'(x)}$$

此时，方程的解产生无穷斜率，波产生间断 (wave "breaks").

证明：
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u(x,0) = u_0(x). \end{cases}$$

$u_t + uu_x = 0$ 特征线方程满足 $\frac{dx}{dt} = u$

于是沿着特征线有 $\frac{du(x(t),t)}{dt} = u_x\frac{dx}{dt} + u_t = u_t + uu_x = 0.$

即沿着特征线 u 为常数，即 $u(x(t),t) = C$

而此时特征线为直线 $x(t) = x(0) + Ct$

其中 $C = u(x(0),0) = u_0(x(0))$，记 $\xi = x(0)$，则有

$$\begin{cases} x = \xi + u_0(\xi)\cdot t & ① \\ u(x,t) = u_0(\xi) & ② \end{cases}$$

当时间 t 还比较小的时候，即特征线还未相交时，对任点 (x,t)，我们可以先从①式解出 $\boxed{\xi = \xi(x,t)}$，代入②式便得 $u(x,t) = u_0(\xi(x,t))$.

于是由②式得

$$u_t = u_0'(\xi)\cdot \xi_t$$

$$u_x = u_0'(\xi)\cdot \xi_x$$

再对①式两边分别对 t，x 求导得

$$0 = \xi_t + u_0'(\xi)\xi_t\cdot t + u_0(\xi) \Rightarrow \xi_t = -\frac{u_0(\xi)}{1+u_0'(\xi)\cdot t}$$

$$1 = \xi_x + u_0'(\xi)\xi_x\cdot t \Rightarrow \xi_x = \frac{1}{1+u_0'(\xi)\cdot t}$$

$$\Rightarrow u_t = -\frac{u_0(\xi)\,u_0'(\xi)}{1+u_0'(\xi)\cdot t}$$

$$u_x = \frac{u_0'(\xi)}{1+u_0'(\xi)\cdot t} \qquad (u_t + uu_x = 0).$$

于是当分母为 0 时即 $t = \frac{-1}{u_0'(\xi)}$ 时方程的解产生无穷斜率，波产生间断，由于存在某些点使得 $u_0'(x) < 0$ 知这种时间 t 存在，且最小为 $T_b = \frac{-1}{\min u_0'(x)}$.

我们已知特征线方程为直线 $x=\xi+u_0(\xi)\cdot t$　　$\xi=x(0)$.

特征线相交即 $\begin{cases} x=\xi_1+u_0(\xi_1)\cdot t \\ x=\xi_2+u_0(\xi_2)\cdot t \end{cases}$

$$\Rightarrow \quad t=\frac{-1}{\frac{u_0(\xi_2)-u_0(\xi_1)}{\xi_2-\xi_1}} \overset{\text{中值定理}}{=} \frac{-1}{u_0'(\eta)}$$　η 介于 ξ_1,ξ_2 之间

同样由于存在 $u_0'(x)<0$ 的点，故特征线必会相交，且首次相交的时刻为 $T_b=\frac{-1}{\min u_0'(x)}$.

综上，在 $T_b=\frac{-1}{\min u_0'(x)}$ 特征线首次相交　此时方程的解会产生无穷斜率，波发生间断!

HW8.

1.(PPT-1203). 针对非线性方程 $u_t+f(u)_x=0$，利用"流通分裂技术"构造数值算法：

$$u_t+f^+(u)_x+f^-(u)_x=0 \quad , \quad f^\pm(u)=\frac{1}{2}(f(u)\pm\alpha u).$$

其中，$\alpha=\max_u|f'(u)|$，对 $f^\pm(u)$ 分别使用迎风格式：

$$V_j^{n+1}=V_j^n-\frac{\Delta t}{\Delta x}\left(f^+(V_j^n)-f^+(V_{j-1}^n)\right)-\frac{\Delta t}{\Delta x}\left(f^-(V_{j+1}^n)-f^-(V_j^n)\right).$$

试判断格式是否为守恒型格式，并分析其单调性和TVD性质。

解：改写原格式为

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta x}\left[\left(f^-(v_{j+1}^n) + f^+(v_j^n)\right) - \left(f^-(v_j^n) + f^+(v_{j-1}^n)\right)\right]$$

记 $F(v_{j+1}^n, v_j^n) = f^-(v_{j+1}^n) + f^+(v_j^n)$，则

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta x}\left[F(v_{j+1}^n, v_j^n) - F(v_j^n, v_{j-1}^n)\right]$$

且 $F(u,u) = f^-(u) + f^+(u) = f(u)$，局部 Lipschitz 成立！

故该格式是相容的守恒型格式.

记 $v_j^{n+1} = H(v_{j-1}^n, v_j^n, v_{j+1}^n) = \left(1 - \alpha\frac{\Delta t}{\Delta x}\right)v_j^n + \frac{\Delta t}{2\Delta x}\left(f(v_{j-1}^n) - f(v_{j+1}^n) + \alpha(v_{j-1}^n + v_{j+1}^n)\right)$

则 $\dfrac{\partial H}{\partial v_{j-1}^n} = \dfrac{\Delta t}{2\Delta x}\left(\alpha + f'(v_{j-1}^n)\right) \geq 0$

$$\frac{\partial H}{\partial v_j^n} = 1 - \alpha\frac{\Delta t}{\Delta x} \geq 0 \quad \left(\text{CFL条件 } \alpha\frac{\Delta t}{\Delta x} \leq 1\right)$$

$$\frac{\partial H}{\partial v_{j+1}^n} = \frac{\Delta t}{2\Delta x}\left(\alpha - f'(v_{j+1}^n)\right) \geq 0$$

因此该格式是单调格式，自然也是TVD的（定理）.

亦或直接用 Harten 引理来证明TVD性质，具体的，改写原格式为

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{2\Delta x}\left(\alpha + \frac{f(v_j^n) - f(v_{j-1}^n)}{v_j^n - v_{j-1}^n}\right)(v_j^n - v_{j-1}^n)$$

$$+ \frac{\Delta t}{2\Delta x}\left(\alpha - \frac{f(v_{j+1}^n) - f(v_j^n)}{v_{j+1}^n - v_j^n}\right)(v_{j+1}^n - v_j^n)$$

即 $C_{j-\frac{1}{2}} = \frac{\Delta t}{2\Delta x}\left(a + \frac{f(v_j^n) - f(v_{j-1}^n)}{v_j^n - v_{j-1}^n}\right) \overset{\text{中值公式}}{=} \frac{\Delta t}{2\Delta x}\left(a + f'(\xi)\right)$ ξ 介 v_{j-1}^n, v_j^n

≥ 0

$D_{j+\frac{1}{2}} = \frac{\Delta t}{2\Delta x}\left(a - \frac{f(v_{j+1}^n) - f(v_j^n)}{v_{j+1}^n - v_j^n}\right) = \frac{\Delta t}{2\Delta x}\left(a - f'(\eta)\right)$ η 介 v_j^n, v_{j+1}^n.

≥ 0

又 $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} = a\frac{\Delta t}{\Delta x} \leq 1$ (CFL条件)

因此由 Harten 引理知这个格式是 TVD 的.

(以下作业位于第二本参考书).

2. HW 2.3.5 (C)

初值和右端点边值是精确的, ($k=M$).

$$\frac{u_k^{n+1} - u_k^n}{\Delta t} + a\,\frac{u_{k+1}^{n+1} - u_{k-1}^{n+1}}{2\Delta x} = \nu\,\frac{u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}}{\Delta x^2} \qquad ①$$

对 $k=1,\dots,M-1$, $\tau_k^{n+1} = \frac{v_k^{n+1} - v_k^n}{\Delta t} + a\,\frac{v_{k+1}^{n+1} - v_{k-1}^{n+1}}{2\Delta x} - \nu\,\frac{v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}}{\Delta x^2}$

$$= \left((v_t)_k^{n+1} + O(\Delta t)\right) + a\left((v_x)_k^{n+1} + O(\Delta x^2)\right) - \nu\left((v_{xx})_k^{n+1} + O(\Delta x^2)\right)$$

$$= \cancel{\left(v_t + a v_x - \nu v_{xx}\right)_k^{n+1}} + O(\Delta t + \Delta x^2)$$

$$= O(\Delta t + \Delta x^2)$$

左端点边值: $\frac{u_1^n - u_0^{n+1}}{\Delta x} = a((n+1)\Delta t)$. ②

($k=0$).

$$\tau_0^{n+1} = \frac{V_1^n - V_0^{n+1}}{\Delta x} - \alpha((n+1)\Delta t) \left(= (V_x)_0^{n+1}\right).$$

其中 $V_1^n = V_1^{n+1} - \Delta t (V_t)_1^{n+1} + O(\Delta t^2)$

$$= \left(V_0^{n+1} + \Delta x (V_x)_0^{n+1} + \frac{\Delta x^2}{2}(V_{xx})_0^{n+1} + O(\Delta x^3)\right)$$

$$- \Delta t\left((V_t)_0^{n+1} + \Delta x (V_{tx})_0^{n+1} + O(\Delta x^2)\right) + \cancel{O(\Delta t^2)}$$

($V_t = V_{xx}$, $r=\frac{\Delta t}{\Delta x^2}$ 近似为常数)

$$= V_0^{n+1} + \Delta x (V_x)_0^{n+1} + \left(\frac{\Delta x^2}{2} - \Delta t\right)(V_{xx})_0^{n+1} + O(\Delta t\Delta x + \Delta x^3)$$

$$\Rightarrow \tau_0^{n+1} = \left(\frac{\Delta x}{2} - \frac{\Delta t}{\Delta x}\right)(V_{xx})_0^{n+1} + O(\Delta t + \Delta x^2)$$

因此当 $r = \frac{\Delta t}{\Delta x^2} = \frac{1}{2}$ 时 $\tau_0^{n+1} = O(\Delta t + \Delta x^2)$

故当 $r = \frac{\Delta t}{\Delta x^2} = \frac{1}{2}$ 时 该格式是逐点相容的，精度阶为 $(1, 2)$.

否则，该格式不相容，没有精度。

①式 $\Rightarrow$ $-\left(\frac{ar}{2}\Delta x + \nu r\right) u_{k-1}^{n+1} + (1+2\nu r) u_k^{n+1} + \left(\frac{ar}{2}\Delta x - \nu r\right) u_{k+1}^{n+1} = u_k^n$ ③

②式 $\Rightarrow$ $u_0^{n+1} = u_1^n - \Delta x\,\alpha((n+1)\Delta t)$ ④

在③式中令 $k=1$ 并代入④式得

$$(1+2\nu r)u_1^{n+1} + \left(\frac{ar}{2}\Delta x - \nu r\right) u_2^{n+1} = \left[1 + \left(\frac{ar}{2}\Delta x + \nu r\right)\right] u_1^n - \left(\frac{ar}{2}\Delta x + \nu r\right)\Delta x\,\alpha((n+1)\Delta t)$$ ⑤

④ $\rightarrow$ u_0^{n+1}, $u_M^{n+1} = 0$, + ③ $\begin{pmatrix} u_1^{n+1} \\ \vdots \\ u_{M-1}^{n+1} \end{pmatrix}$ / ⑤

实际计算　　矩阵形式　　(见下题).

3. (PPT-1209 补充作业, 例 1.1).

解: 同样初值和右端点边值是精确的; $(x_{\frac{1}{2}}=0,\ x_{N_x}=1)$

逐点相容性: $$\frac{V_j^{n+1}-V_j^n}{\Delta t}=\frac{V_{j+1}^n-2V_j^n+V_{j-1}^n}{\Delta x^2} \quad ①$$

$$T_j^n=\frac{u_j^{n+1}-u_j^n}{\Delta t}-\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}$$

$$=\left((u_t)_j^n+O(\Delta t)\right)-\left((u_{xx})_j^n+O(\Delta x^2)\right)$$

$$=\cancel{(u_t-u_{xx})_j^{n+1}}+O(\Delta t+\Delta x^2)$$

$$=O(\Delta t+\Delta x^2).$$

左端点边值 $$\frac{V_1^{n+1}-V_0^{n+1}}{\Delta x}=0 \quad ②$$

$$T_{\frac{1}{2}}^{n+1}=\frac{u_1^{n+1}-u_0^{n+1}}{\Delta x}=(u_x)_{\frac{1}{2}}^{n+1}+O(\Delta x^2)=O(\Delta x^2).$$

其中 由 $u_x(0,t)=0\ \forall t\geq 0$ 且 $\boxed{x_{\frac{1}{2}}=0}$ 知 $(u_x)_{\frac{1}{2}}^{n+1}=0$

因此该格式是逐点相容的, 精度阶为 (1,2).

模相容性: ①式 $\Leftrightarrow$ $V_j^{n+1}=(1-2r)V_j^n+r(V_{j+1}^n+V_{j-1}^n)$ $\quad r=\frac{\Delta t}{\Delta x^2}$

令 $j=1$ 得 $V_1^{n+1}=(1-2r)V_1^n+r(V_2^n+V_0^n)$

则由②式得 $V_0^{n+1} = V_1^{n+1} = (1-2r)V_1^n + r(V_2^n + V_0^n)$.

$$\begin{pmatrix} V_0 \\ V_1 \\ \vdots \\ V_{M-1} \end{pmatrix}$$

$$\Rightarrow T_0^n = \frac{\overset{\cancel{u_1^{n+1}}}{\boxed{u_0^{n+1}}} - u_1^n}{\Delta t} - \frac{u_2^n - 2u_1^n + u_0^n}{\Delta x^2}$$

$$= \left(\cancel{(u_t)_0^n} + O(\Delta t)\right) - \left(\frac{\Delta x}{\Delta t}\cancel{(u_x)_0^n + \frac{\Delta x^2}{2\Delta t}}(u_{xx})_0^n + \frac{\Delta x^3}{6\Delta t}(u_{xxx})_0^n + O\left(\frac{\Delta x^4}{\Delta t}\right)\right)$$

$$- \left(\cancel{(u_{xx})_0^n} + O(\Delta x^2)\right)$$

由 $0 = (u_x)_{\frac{1}{2}}^n = (u_x)_0^n + \frac{\Delta x}{2}(u_{xx})_0^n + O(\Delta x^2)$ 得

$$(u_x)_0^n = -\frac{\Delta x}{2}(u_{xx})_0^n + O(\Delta x^2)$$

$u_t = u_{xx}$ 代入得 $T_0^n = O(\Delta t + \Delta x)$. ($r = \frac{\Delta t}{\Delta x^2}$ 视为常数)

因此该格式是模相容的、精度阶为 (1,1).

4 HW 2.4.2 (按最大模).

解: $u_k^{n+1} = (1+R)u_k^n - R u_{k+1}^n$ $R = a\frac{\Delta t}{\Delta x}$ $a<0$ 故 $R<0$

$$|R| \le 1 \Rightarrow -1 \le R < 0 \Rightarrow 1+R \ge 0,\ -R > 0$$

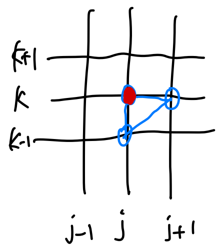
故有 $|u_k^{n+1}| \le (1+R)|u_k^n| - R|u_{k+1}^n| \overset{u_M^{n+1}=0}{\le} (1+R)\|u^n\|_\infty - R\|u^n\|_\infty = \|u^n\|_\infty$

$\Rightarrow \|u^{n+1}\|_\infty \le \|u^n\|_\infty \le \cdots \le \|u^0\|_\infty \le \|f\|_\infty$. 故该格式是稳定的。

HW9.

1. HW 5.8.7 (a), (b).

(a). 1.



$n: [j, j+1] \times [k-1, k]$

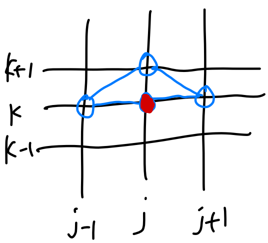
$n-1: [j, j+2] \times [k-2, k]$

$\vdots$

$0: [j, j+n+1] \times [k-n-1, k]$

$\Rightarrow$ 点 $(j\Delta x, k\Delta y, (n+1)\Delta t)$ 的数值依赖区域为 $[j\Delta x, (j+n+1)\Delta x] \times [(k-n-1)\Delta y, k\Delta y]$.

2.



$n: [j-1, j+1] \times [k, k+1]$

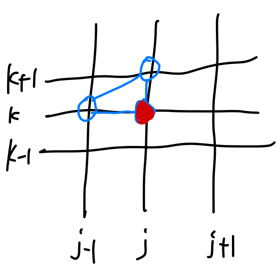
$n-1: [j-2, j+2] \times [k, k+2]$

$\vdots$

$0: [j-n-1, j+n+1] \times [k, k+n+1]$

$\Rightarrow$ 点 $(j\Delta x, k\Delta y, (n+1)\Delta t)$ 的数值依赖区域为 $[(j-n-1)\Delta x, (j+n+1)\Delta x] \times [k\Delta y, (k+n+1)\Delta y]$.

3.



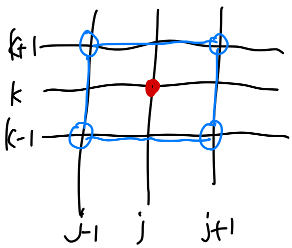
$n: [j-1, j] \times [k, k+1]$

$n-1: [j-2, j] \times [k, k+2]$

$\vdots$

$0: [j-n-1, j] \times [k, k+n+1]$.

$\Rightarrow$ 点 $(j\Delta x, k\Delta y, (n+1)\Delta t)$ 的数值依赖区域为 $[(j-n-1)\Delta x, j\Delta x] \times [k\Delta y, (k+n+1)\Delta y]$.

4. 

$n: [j-1, j+1] \times [k-1, k+1]$

$n-1: [j-2, j+2] \times [k-2, k+2]$

$\vdots$

$0: [j-n-1, j+n+1] \times [k-n-1, k+n+1]$

$\Rightarrow$ 点 $(j\Delta x, k\Delta y, (n+1)\Delta t)$ 的数值依赖区域为 $[(j-n-1)\Delta x, (j+n+1)\Delta x] \times [(k-n-1)\Delta y, (k+n+1)\Delta y]$.

(b) $V_t + aV_x + bV_y = 0$

过任意点 (x, y, t) 的特征线方程为

$$x - at = x_0$$
$$y - bt = y_0$$

于是对于点 $(j\Delta x, k\Delta y, (n+1)\Delta t)$ 有

$$x_0 = j\Delta x - a(n+1)\Delta t = \Delta x[j-(n+1)R_x] \qquad R_x = a\frac{\Delta t}{\Delta x}$$

$$y_0 = k\Delta y - b(n+1)\Delta t = \Delta y[k-(n+1)R_y] \qquad R_y = b\frac{\Delta t}{\Delta y}$$

由CFL条件知数值解的依赖区域包含真解的依赖区域可得

1. $$\begin{cases} j\Delta x \le \Delta x[j-(n+1)R_x] \le (j+n+1)\Delta x & \Rightarrow -1 \le R_x \le 0 \\ (k-n-1)\Delta y \le \Delta y[k-(n+1)R_y] \le k\Delta y & \Rightarrow 0 \le R_y \le 1 \end{cases}$$

其余同理可得，结果罗列如下：

2. $-1 \le R_x \le 1$, $-1 \le R_y \le 0$

3. $0 \le R_x \le 1$, $-1 \le R_y \le 0$

4. $-1 \le R_x \le 1$, $-1 \le R_y \le 1$.

2. (PPT-1214). 构造 $u_t+u_x+u_y=0$ 的ADI格式.

答案不唯一, 例如

$$\begin{cases} \left(1+\frac{R_x}{2}\delta_{x0}\right)u_{jk}^{n+\frac{1}{2}} = u_{jk}^{n} \\ \left(1+\frac{R_y}{2}\delta_{y0}\right)u_{jk}^{n+1} = u_{jk}^{n+\frac{1}{2}} \end{cases} \qquad R_x=\frac{\Delta t}{\Delta x} \quad R_y=\frac{\Delta t}{\Delta y}$$

3. (PPT-1214). 对二维扩散方程的 Douglas-Rachford 格式, 构造不同边界条件的数值方法.

类似对 P-R 格式和 D'Yakonov 格式的处理, 见PPT, 不想誊写了!