1. $(x-x_*)^TA(x-x_*) = x^TAx + x_*^TAx_* - 2x_*^TAx$

Since $Ax^*=b$ $\Leftarrow$ $= x^TAx + x_*^TAx_* - 2b^Tx = LHS$

2. $\varphi(x_k) = \varphi(x_{k-1}) - \frac{(r_{k-1}^Tr_{k-1})^2}{r_{k-1}^TAr_{k-1}}$

$\because$ $r_{k-1}^TA^{-1}r_{k-1} = (b-Ax_{k-1})^TA^{-1}(b-Ax_{k-1}) = b^TA^{-1}b + \varphi(x_{k-1}) \geq \varphi(x_{k-1})$

$\therefore$ $\varphi(x_k) \leq \varphi(x_{k-1}) - \frac{(r_{k-1}^Tr_{k-1})^2}{r_{k-1}^TAr_{k-1}} \cdot \frac{\varphi(x_{k-1})}{r_{k-1}^TA^{-1}r_{k-1}} \leq \varphi(x_{k-1})\left[1-\frac{1}{\kappa_2(A)}\right]$

3. 即 $x_* = x_k + \underbrace{\frac{r_k^Tr_k}{r_k^TAr_k}}_{\text{记为}C} \cdot r_k$

$\Rightarrow$ $b = Ax_* = Ax_k + Ar_k \cdot C$

$\Rightarrow$ $\cancel{\Leftarrow r_k}$ $r_k = b - Ax_k = \cancel{Ar_k}A\cdot r_k \cdot C$

4. 方程的系数矩阵为 $B = \begin{pmatrix} r_k^TAr_k & r_k^TAP_{k-1} \\ r_k^TAP_{k-1} & P_{k-1}^TAP_{k-1} \end{pmatrix}$

$\because$ B正定 $\Leftrightarrow$ $(x_1r_k+x_2P_{k-1})^TA(x_1r_k+x_2P_{k-1})=0$ ~~相~~ 当且仅当 $x_1=x_2=0$

$\Leftrightarrow$ r_k与P_{k-1}线性无关 (显然)

$\therefore$ B正定 $\Rightarrow$ B可逆 $\Rightarrow$ 解唯一 (亦可用归纳法证B可逆)

5 反证: 若$\exists$ α_i 不全为 0, $\sum_{i=1}^{k}\alpha_i\vec{P_i}=0$

$\Rightarrow$ $0 = \vec{P_i}^TA(\sum_{i=1}^{k}\alpha_i\vec{P_i}) = \alpha_i\vec{P_i}^TA\vec{P_i}$ $\Rightarrow$ $\alpha_i=0$ $\forall i=1,2,\cdots k$

矛盾

6. 令 $\frac{d\,\varphi(y_{i-1}+te_i)}{dt} = 0 \Rightarrow t = \frac{1}{a_{ii}}(b-Ay_{i-1})^Te_i$

$\Rightarrow$ $y_i = y_{i-1} + \frac{1}{a_{ii}}(b-Ay_{i-1})e_i$ 仅第i行发生改变

而对G-S迭代, $x_i^{(k+1)} = \frac{1}{a_{ii}}(b_i - \sum_{j<i}a_{ij}x_j^{(k+1)} - \sum_{j>i}a_{ij}x_j^{(k)})$

$\therefore$ $x_i^{(k+1)} - x_i^{(k)} = \frac{1}{a_{ii}}\cancel{(b_i-}(b-Ay_{i-1})^Te_i$

故经过n次相当于做一次 G-S 迭代

7. A实对称 $\Rightarrow$ A可对角化 $\Rightarrow d_A(\lambda)$ 无重根

$\Rightarrow d_A(\lambda) = \prod_{i=1}^{k}(\lambda-\lambda_k) \Rightarrow \deg d_A(\lambda)=k$

$\Rightarrow \dim \operatorname{span}\{r, \cdots A^m r\} = \dim \operatorname{span}\{r, \cdots, A^{k-1} r\} \le k$

8. 由定理 5.2.2 与上题结论显然

9. $A=A^T \Rightarrow$ A可正交相似到对角阵

设 $A = P\Sigma P^T$，P正交，$\Sigma = \operatorname{diag}(\lambda_1, \cdots \lambda_n)$，$\lambda_i \ge \lambda_{i+1} > 0$

$\because \|x\|_A^2 = x^TAx = x^TP\Sigma P^Tx =: y^T\Sigma y = \sum_{i=1}^{n}\lambda_i y_i^2$ （令 $y = P^Tx$）

$\therefore \lambda_n \|x\|_2^2 = \lambda_n \sum_{i=1}^{n} y_i^2 \le \|x\|_A^2 \le \lambda_1 \sum_{i=1}^{n} y_i^2 = \lambda_1 \|x\|_2^2$，$\forall x \in \mathbb{R}^n$

易得 $\|A\|_2 = \sqrt{\lambda_1}$，$\|A^{-1}\|_2 = \sqrt{\lambda_n}$，代入上式及定理5.3.2 得证

11. x_k 极小化 $\|x - A^{-1}b\|_A \Leftrightarrow \forall w \in \mathcal{X}$，$\frac{d}{d\alpha}\|x_k + \alpha w - A^{-1}b\|_A^2 \big|_{\alpha=0} = 0$

$\Leftrightarrow w^TA(x_k - A^{-1}b) = 0 \quad \forall w \in \mathcal{X}$

$\Leftrightarrow Ax_k - b \perp \mathcal{X}$

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