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6.x.1 **推广的** Cauchy-Schwarz **不等式：** $|a(v,w)| \le |||v|||_{1+t,k}|||w|||_{1-t,k},\quad \forall v,w \in V_k,\ t \in \mathbb{R}.$

证明. 设 $v = \sum_{i=1}^{n_k} v_i\psi_i,\ w = \sum_{i=1}^{n_k} w_i\psi_i$，其中 ψ_i 为 A_k 的特征向量，那么

$$\begin{aligned}|a(v,w)| &= |(A_k v, w)_k| = |(\sum_{i=1}^{n_k} v_i\lambda_i\psi_i, \sum_{j=1}^{n_k} w_j\psi_j)| \\ &= |\sum_{i=1}^{n_k} v_i w_i \lambda_i| \le \sum_{i=1}^{n_k} |v_i\lambda_i^{\frac{1+t}{2}}|\cdot|w_i\lambda_i^{\frac{1-t}{2}}| \\ &\le (\sum v_i^2\lambda_i^{1+t})^{\frac{1}{2}}(\sum w_i^2\lambda_i^{1-t})^{\frac{1}{2}} = |||v|||_{1+t,k}|||w|||_{1-t,k}.\end{aligned}$$

即证。 □

6.x.3 **证明：** $\forall v \in V_k,\ s \in \mathbb{R}$，**成立** $|||R_k v|||_{s,k} \le |||v|||_{s,k}$，$(R_k v, v)_k \le (v,v)_k$.

证明. 设 λ_i, ϕ_i 为 A_k 的特征值、特征向量，$(\phi_i,\phi_j)_k = \delta_{ij}$，$v = \sum v_i\phi_i$，那么

$$\begin{aligned}R_k v &= \sum v_i(1-\Lambda_k^{-1}A_k)\phi_i = \sum v_i(1-\frac{\lambda_i}{\Lambda_k})\phi_i, \\ |||R_k v|||_{s,k}^2 &= \sum v_i^2(1-\frac{\lambda_i}{\Lambda_k})^2\lambda_i^s < \sum v_i^2\lambda_i^s = |||v|||_{s,k}^2.\end{aligned}$$

另一个，

$$\begin{aligned}(R_k v, v)_k &= (\sum v_i(1-\frac{\lambda_i}{\Lambda_k})\phi_i, \sum v_j\phi_j)_k \\ &= \sum v_i^2(1-\frac{\lambda_i}{\Lambda_k}) < \sum v_i^2 = (v,v)_k.\end{aligned}$$

即证。 □

6.x.6 $\forall v,w \in V_k,\ a(R_k v, w) = a(v, R_k w).$

证明. 设 $v = \sum v_i\phi_i,\ w = \sum w_i\phi_i,\ A_k\phi_i = \lambda_i\phi_i$. 那么

$$\begin{aligned}a(R_k v, w) &= (A_k R_k v, w)_k = (\sum v_i(1-\frac{\lambda_i}{\Lambda_k})\lambda_i\phi_i, \sum w_j\phi_j) = \sum v_i w_i\lambda_i(1-\frac{\lambda_i}{\Lambda_k}), \\ a(v, R_k w) &= \cdots = \sum v_i w_i \lambda_i(1-\frac{\lambda_i}{\Lambda_k}).\end{aligned}$$

结合两式即证。 □

6.x.11 **根据** e_k^I, t_k **的递推关系：**$e_{k+1}^I = 2e_k^I + 3t_k$，$t_{k+1} = 4t_k$ **推出** $n_k \sim \frac{t_1}{8}4^k$.

证明. 因为 $e_{k+1}^I - \frac{3}{2}t_{k+1} = 2e_k^I - 3t_k$，所以 $e_{k+1}^I - \frac{3}{2}t_{k+1} = 2^k(2e_1^I - 3t_1)$，$e_{k+1}^I = 2^k(e_1^I - \frac{3}{2}t_1) + \frac{3}{2}4^{k-1}t_1$. 结合 $n_k = 1 + e_k^I - t_k^I$ 即证。 □