HW 1.

1.1.1

证明: ① $(f,g)=\overline{(g,f)}$

$$(f,g)=\int_0^{2\pi}\bar f g\,dx=\overline{\left(\int_0^{2\pi}\overline{\bar f g}\,dx\right)}=\overline{\int_0^{2\pi}f\bar g\,dx}=\overline{(g,f)}$$

② $(f+g,h)=(f,h)+(g,h)$

$$\begin{aligned}(f+g,h)&=\int_0^{2\pi}(\overline{f+g})h\,dx\\&=\int_0^{2\pi}(\bar f h+\bar g h)dx\\&=\int_0^{2\pi}\bar f h\,dx+\int_0^{2\pi}\bar g h\,dx\\&=(f,h)+(g,h)\end{aligned}$$

③ $(\lambda f,g)=\bar\lambda(f,g)$

$$\begin{aligned}(\lambda f,g)&=\int_0^{2\pi}\overline{\lambda f}\cdot g\,dx\\&=\int_0^{2\pi}\bar\lambda\bar f g\,dx\\&=\bar\lambda\int_0^{2\pi}\bar f g\,dx\\&=\bar\lambda(f,g)\end{aligned}$$

④ $(f,\lambda g)=\lambda(f,g)$.

$$(f,\lambda g)=\int_0^{2\pi}\bar f\lambda g\,dx=\lambda\int_0^{2\pi}\bar f g\,dx=\lambda(f,g).$$

⑤ $\|\lambda f\|=|\lambda|\,\|f\|$.

$$\begin{aligned}\|\lambda f\|&=\left(\int_0^{2\pi}|\lambda f|^2dx\right)^{1/2}\\&=\left(\lambda^2\int_0^{2\pi}|f|^2dx\right)^{1/2}\\&=|\lambda|\left(\int_0^{2\pi}|f|^2dx\right)^{1/2}\\&=|\lambda|\,\|f\|.\end{aligned}$$

⑥ $|(f,g)| \le \|f\|\cdot\|g\|$.　　$g=0$时显然成立，下设$g\neq 0$

$$0 \le \|f-\lambda g\|^2 = (f-\lambda g, f-\lambda g)$$

$$= \|f\|^2 - (f,\lambda g) - (\lambda g, f) + |\lambda|^2\|g\|^2$$

$$= \|f\|^2 - \lambda(f,g) - \bar{\lambda}\,\overline{(f,g)} + |\lambda|^2\|g\|^2$$

取 $\lambda = \dfrac{\overline{(f,g)}}{\|g\|^2}$ $= \|f\|^2 - \dfrac{|(f,g)|^2}{\|g\|^2} - \dfrac{|(f,g)|^2}{\|g\|^2} + \dfrac{|(f,g)|^2}{\|g\|^2}$

$$= \|f\|^2 - \frac{|(f,g)|^2}{\|g\|^2}$$

$\Rightarrow \|f\|^2\|g\|^2 \ge |(f,g)|^2$

$\Rightarrow |(f,g)| \le \|f\|\|g\|$.

⑦ $\|f+g\| \le \|f\|+\|g\|$

$$\|f+g\|^2 = (f+g, f+g)$$

$$= \|f\|^2 + \|g\|^2 + (f,g) + (g,f)$$

$$\le \|f\|^2 + \|g\|^2 + 2|(f,g)|$$

$$\overset{⑥}{\le} \|f\|^2 + \|g\|^2 + 2\|f\|\cdot\|g\|$$

$$= (\|f\|+\|g\|)^2$$

⑧ ~~$\|f-g\|^2$~~ $\big|\|f\| - \|g\|\big| \le \|f-g\|$.

$$\|f-g\|^2 = (f-g, f-g)$$

$$= \|f\|^2 + \|g\|^2 - (f,g) - (g,f)$$

$$\ge \|f\|^2 + \|g\|^2 - 2|(f,g)|$$

$$\overset{⑥}{\ge} \|f\|^2 + \|g\|^2 - 2\|f\|\cdot\|g\|$$

$$= \big|\|f\| - \|g\|\big|^2$$

1.1.2.

$f(x)=\frac{1}{\sqrt{2\pi}}\sum_{-\infty}^{+\infty}\hat{f}(w)e^{iwx}$，其中 $\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}f(t)e^{-iwt}dt$

~~$\hat{f}(w)$~~ 若 f 为实值函数，则

$$\hat{f}(w)e^{iwx}=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}f(t)e^{iwx-iwt}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}f(t)\cos(wx-wt)+i\cdot\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}f(t)\sin(wx-wt)dt.$$

$$\hat{f}(-w)e^{-iwx}=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}f(t)e^{-iwx+iwt}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}f(t)\cos(wx-wt)dt+i\cdot\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}f(t)\sin(-wx+wt)dt$$

故 $\hat{f}(w)e^{iwx}+\hat{f}(-w)e^{-iwx}\in\mathbb{R}$.

又 $\hat{f}(0)=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}f(t)dt\in\mathbb{R}$.

故 $S_N=\frac{1}{\sqrt{2\pi}}\sum_{w=-N}^{N}\hat{f}(w)e^{iwx}=\frac{1}{\sqrt{2\pi}}\hat{f}(0)+\frac{1}{\sqrt{2\pi}}\sum_{w=1}^{N}[\hat{f}(w)e^{iwx}+\hat{f}(-w)e^{-iwx}]\in\mathbb{R}$.

1.2.1. 估计 $|(D-\frac{d^3}{dx^3})e^{iwx}|$

$D=D_+^3,\ D_-D_+^2,\ D_-^2D_+,\ D_-^3,\ D_0D_+D_-$.

① $D=D_+^3=(\frac{E^1-E^0}{h})^3=\frac{E^3-3E^2+3E-E^0}{h^3}$

$$|(D-\frac{d^3}{dx^3})e^{iwx}|=\left|\frac{e^{iw3h}-3e^{iw2h}+3e^{iwh}-1}{h^3}-\frac{(iw)^3h^3}{h^3}\right|$$

$$=\left|\frac{\frac{3}{2}w^4h^4+Cw^5h^5+\cdots}{h^3}\right|$$

$$=O(w^4h)$$

1.2.2. 不矛盾，因为 D_+, D_0 近似的程度不一样

② $D = D_- D_+^2 = \dfrac{E^2 - 3E + 3E^0 - E^{-1}}{h^3}$

$$\left|\left(D - \frac{\partial^3}{\partial x^3}\right)e^{i\omega x}\right| = \left|\frac{e^{i\omega 2h} - 3e^{i\omega h} + 3 - e^{-i\omega h}}{h^3} - \frac{(i\omega)^3 h^3}{h^3}\right|$$

$$= \left|\frac{\frac{1}{2}\omega^4 h^4 + C\omega^5 h^5 + \cdots}{h^3}\right|$$

$$= O(\omega^4 h)$$

③ $D = D_-^2 D_+ = \dfrac{E^1 - 3E^0 + 3E^{-1} - E^{-2}}{h^3}$

$$\left|\left(D - \frac{\partial^3}{\partial x^3}\right)e^{i\omega x}\right| = \left|\frac{e^{i\omega h} - 3 + 3e^{-i\omega h} - e^{-i\omega 2h}}{h^3} - \frac{(i\omega)^3 h^3}{h^3}\right|$$

$$= \left|\frac{-\frac{1}{2}\omega^4 h^4 + C\omega^5 h^5 + \cdots}{h^3}\right|$$

$$= O(\omega^4 h)$$

④ $D = D_-^3 = \dfrac{E^0 - 3E^{-1} + 3E^{-2} - E^{-3}}{h^3}$

$$\left|\left(D - \frac{\partial}{\partial x^3}\right)e^{i\omega x}\right| = \left|\frac{1 - 3e^{-i\omega h} + 3e^{-2i\omega h} - e^{-3i\omega h}}{h^3} - \frac{(i\omega)^3 h^3}{h^3}\right|$$

$$= \left|\frac{-\frac{3}{2}\omega^4 h^4 + C\omega^5 h^5 + \cdots}{h^3}\right|$$

$$= O(\omega^4 h)$$

⑤ $D = D_0 D_+ D_{-1} = \dfrac{E^2 - 2E^1 + 2E^{-1} - E^{-2}}{2h^3}$

$$\left|\left(D - \frac{\partial}{\partial x^3}\right)e^{i\omega x}\right| = \left|\frac{e^{i\omega 2h} - 2e^{i\omega h} + 2e^{-i\omega h} - e^{-i\omega 2h}}{2h^3} - \frac{(i\omega)^3 \cdot 2h^3}{2h^3}\right|$$

$$= \left|\frac{\frac{1}{2}\omega^5 h^5 i + C\omega^6 h^6 + \cdots}{2h^3}\right|$$

$$= O(\omega^5 h^2)$$

1.5.2 对于矩形网格，

$$h_j = 2\pi/(N_j+1) \quad . j=1.2.$$

$$\vec{x_j} = (h_1 j_1 , h_2 j_2) , \ v_j = v(\vec{x_j}) = v(h_1 j_1, h_2 j_2)$$

$$(u,v)_h = \sum_{j_1=0}^{N_1} \sum_{j_2=0}^{N_2} \overline{u_j} v_j h_1 h_2 , \ \|u\|_h^2 = (u,u)_h .$$

与一维推导类似，

$$\|D_{+x_j} u\|_h = \frac{1}{h_j} \|(E_{x_j}^1 - E_{x_j}^0) u\|_h \le \frac{2}{h_j} \|u\|_h$$

故 $\|D_{+x_j}\|_h \le \frac{2}{h_j}$

对于 D_{+x_j}，取 $u_j = (-1)^{j_1+j_2}$

$$\|u\|_h^2 = (N_1+1)(N_2+1) h_1 h_2$$

$$\|D_{+x_j} u\|_h^2 = \sum_{j_1=0}^{N_1} \sum_{j_2=0}^{N_2} \left((-1)^{j_1+1+j_2} - (-1)^{j_1+j_2}\right)^2 \frac{h_1 h_2}{h_j^2}$$

$$= 4(N_1+1)(N_2+1) h_1 h_2 / h_j^2$$

$$= \frac{4}{h_j^2} \|u\|_h^2$$

$$\Rightarrow \|D_{+x_j}\|_h = \frac{2}{h_j}$$

类似有 $\|D_{-x_j}\|_h = \frac{2}{h_j}$

对于 D_{0x_j}，有 $\|D_{0x_j}\|_h = \frac{1}{2h_j} \|E_{x_j}^1 - E_{x_j}^{-1}\|_h \le \frac{1}{h_j}$

取 $u_j = i^{j_1+j_2}$ $(i=\sqrt{-1})$

$$\|u\|_h^2 = (N_1+1)(N_2+1) h_1 h_2$$

$$\|D_{0x_j} u\|_h^2 = \sum_{j_1=0}^{N_1} \sum_{j_2=0}^{N_2} \frac{\left((-i)^{j_1+j_2+1} - (-i)^{j_1+j_2-1}\right)}{2h_j} \cdot \frac{\left(i^{j_1+j_2+1} - i^{j_1+j_2-1}\right)}{2h_j} \cdot h_1 h_2$$

$$= \frac{(N_1+1)(N_2+1) h_1 h_2}{h_j^2} = \frac{\|u\|_h^2}{h_j^2}$$

故 $\|D_{0x_j}\|_h = \frac{1}{h_j}$

(课本 P13)

Thm 1.4. 任意分片连续的函数 f 都能展开成在 L_2 模意义下收敛于 f 的 ~~Fou~~ Fourier 级数，且 Parseval 关系成立.

证明：1° 若 $f \in C^1(-\infty,+\infty)$，$2\pi$ 周期. 则 S_N 一致收敛于 f.

$$\lim_{N\to\infty} \max_{x\in[0,2\pi]} |f(x)-S_N(x)|=0 \Rightarrow \lim_{N\to\infty} \|f-S_N\|=0.$$

2° 任意分片连续函数 f 在 L_2 模意义下，可以被 2π 周期 C 函数 $\{f_\mu\}$ 任意好的逼近，即

$$\lim_{\mu\to\infty} \|f-f_\mu\|=0 \qquad (***)$$

则 $\lim\limits_{\mu\to\infty} \big|\|f\|-\|f_\mu\|\big| \le \lim\limits_{\mu\to\infty} \|f-f_\mu\|=0 \Rightarrow \lim\limits_{\mu\to\infty} \|f_\mu\| = \|f\| \quad (*)$

又 $\left|\sum\limits_{n=-\infty}^{+\infty} (|\hat{f}(n)|^2-|\hat{f}_\mu(n)|^2)\right| \le \sum\limits_{n=-\infty}^{+\infty} (|\hat{f}(n)|+|\hat{f}_\mu(n)|)(|\hat{f}(n)-\hat{f}_\mu(n)|)$

$$\overset{\text{Cauchy}}{\le} \left[\sum_{n=-\infty}^{+\infty} (|\hat{f}(n)|+|\hat{f}_\mu(n)|)^2\right]^{\frac12} \left[\sum_{n=-\infty}^{+\infty} |\hat{f}(n)-\hat{f}_\mu(n)|^2\right]^{\frac12}$$

$$\le 2(\|f\|^2+\|f_\mu\|^2)^{\frac12} \|f-f_\mu\| \qquad (**)$$

故 $\left|\|f\|^2-\sum\limits_{n=-\infty}^{+\infty} |\hat{f}(n)|^2\right| = \lim\limits_{\mu\to\infty} \left[\big|\|f\|^2-\|f_\mu\|^2\big| + \left|\sum\limits_{n=-\infty}^{+\infty} |\hat{f}_\mu|^2 - |\hat{f}(n)|^2\right|\right]$

$$\overset{(**)}{\le} \lim_{\mu\to\infty} \big|\|f\|^2-\|f_\mu\|^2\big| + \lim_{\mu\to\infty} 2(\|f\|^2+\|f_\mu\|^2)^{\frac12}\|f-f_\mu\|$$

$$\overset{(*)}{\le} 0 + \lim_{\mu\to\infty} 2(\|f\|^2+\|f_\mu\|^2)^{\frac12}\|f-f_\mu\|$$

$$\overset{(***)}{\le} 0.$$

故 $\|f\|^2 = \sum\limits_{n=-\infty}^{+\infty} |\hat{f}(n)|^2$

证明定理1.7（N为奇数）

Thm 1.7：若 $\varphi(x)$，$\psi(x)$ 分别为 u_j，v_j，$j=0,1,\dots,N$ 的三角插值，即

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}\sum_{w=-\frac{N+1}{2}+1}^{\frac{N+1}{2}}\tilde{u}(w)e^{iwx}.$$

$$\psi(x)=\frac{1}{\sqrt{2\pi}}\sum_{\mu=-\frac{N+1}{2}+1}^{\frac{N+1}{2}}\tilde{v}(\mu)e^{i\mu x}$$，N为奇数．

则 (1) $(u,v)_h=(\varphi,\psi)=\sum_{w=-\frac{N+1}{2}+1}^{\frac{N+1}{2}}\bar{\tilde{u}}\tilde{v}$

$$(\varphi,\psi)=\left(\frac{1}{\sqrt{2\pi}}\sum_{w=-\frac{N+1}{2}+1}^{\frac{N+1}{2}}\tilde{u}(w)e^{iwx},\ \frac{1}{\sqrt{2\pi}}\sum_{\mu=-\frac{N+1}{2}+1}^{\frac{N+1}{2}}\tilde{v}(\mu)e^{i\mu x}\right)$$

$$=\frac{1}{2\pi}\sum_{w=-\frac{N+1}{2}+1}^{\frac{N+1}{2}}\sum_{\mu}\overline{\tilde{u}(w)}\,\tilde{v}(\mu)\,(e^{iwx},e^{i\mu x})$$

$$\left(\left(\frac{1}{\sqrt{2\pi}}e^{imx},\frac{1}{\sqrt{2\pi}}e^{inx}\right)=\delta_{mn}\right)\quad=\sum_{w=-\frac{N+1}{2}+1}^{\frac{N+1}{2}}\overline{\tilde{u}(w)}\,\tilde{v}(w)$$

$$(u,v)_h=\sum_{j=0}^{N}\bar{u}_j v_j h=\sum_{j=0}^{N}\overline{\varphi(x_j)}\,\psi(x_j)\,h$$

$$=\sum_{j=0}^{N}\overline{\left(\frac{1}{\sqrt{2\pi}}\sum_{w}\tilde{u}(w)e^{iwx_j}\right)}\cdot\left(\frac{1}{\sqrt{2\pi}}\sum_{\mu}\tilde{v}(\mu)e^{i\mu x_j}\right)\cdot h$$

$$=\sum_{w}\sum_{\mu}\overline{\tilde{u}(w)}\,\tilde{v}(\mu)\sum_{j=0}^{N}e^{i(\mu-w)x_j}\cdot\frac{h}{2\pi}$$

$$\because\ \sum_{j=0}^{N}e^{i(\mu-w)x_j}=\begin{cases}0 & \mu\neq w\\ N+1 & \mu=w\end{cases}$$

$$\therefore\ (u,v)_h=\sum_{w}\overline{\tilde{u}(w)}\,\tilde{v}(w)\frac{(N+1)h}{2\pi}$$

$$=\sum_{w}\overline{\tilde{u}(w)}\,\tilde{v}(w)\ =(\varphi,\psi)$$

(2) $\|\varphi\|^2=\sum_{w=-\frac{N+1}{2}+1}^{\frac{N+1}{2}}|\tilde{u}(w)|^2=\|u\|^2$

(1)中取 $\psi=\varphi$，则 $\|\varphi\|^2=(\varphi,\varphi)=\sum_{w=-\frac{N+1}{2}+1}^{\frac{N+1}{2}}\bar{\tilde{u}}\tilde{u}=\sum_{w}|\tilde{u}(w)|^2=\|u\|^2$

(3) $\|D_+^l u\|_h^2\le\left\|\frac{d^l}{dx^l}\varphi\right\|^2\le\left(\frac{\pi}{2}\right)^{2l}\|D_+^l u\|_h^2$

与课本27页类似．

推广 Thm 1.9 . 1.10 至二维

Thm 1.9 ϕ 为满足 $\phi(x^1_{j_1}, x^2_{j_2}) = u_j = u(x^1_{j_1}, x^2_{j_2})$, $j_1 = 0, 1, \dots, N_1$, $j_2 = 0, 1, \dots, N_2$

的三角插值:

$$\phi(x^1, x^2) = \frac{1}{2\pi} \sum_{w_1=-\frac{N_1}{2}}^{\frac{N_1}{2}} \sum_{w_2=-\frac{N_2}{2}}^{\frac{N_2}{2}} \hat{u}(w_1, w_2)\, e^{i(w_1 x^1 + w_2 x^2)}$$

或 $\phi(\vec{x}) = \frac{1}{2\pi} \sum_{w_1=-\frac{N_1}{2}}^{\frac{N_1}{2}} \sum_{w_2=-\frac{N_2}{2}}^{\frac{N_2}{2}} \hat{u}(\vec{w})\, e^{i\langle \vec{w}, \vec{x}\rangle}$ 是唯一的.

其中 $\vec{w} = (w_1, w_2)$, $\vec{x} = (x^1, x^2)$ $\langle \vec{w}, \vec{x}\rangle = w_1 x^1 + w_2 x^2$

$$\hat{u}(\vec{w}) = \frac{1}{2\pi} (\phi(\vec{x}), e^{i\langle \vec{w}, \vec{x}\rangle})_h .$$

Thm 1.10. 若 $\phi(x)$, $\psi(x)$ 分别满足: $\phi(x^1_{j_1}, x^2_{j_2}) = u_j = u(x^1_{j_1}, x^2_{j_2})$. $j_1 = 0, 1, \dots, N_1$

$\psi(x^1_{j_1}, x^2_{j_2}) = v_j = v(x^1_{j_1}, x^2_{j_2})$. $j_2 = 0, 1, \dots, N_2$

的三角插值 即.

$$\phi(x^1, x^2) = \frac{1}{2\pi} \sum_{w_1=-\frac{N_1}{2}}^{\frac{N_1}{2}} \sum_{w_2=-\frac{N_2}{2}}^{\frac{N_2}{2}} \hat{u}(w_1, w_2)\, e^{i\langle \vec{w}, \vec{x}\rangle} .$$

$$\psi(x^1, x^2) = \frac{1}{2\pi} \sum_{w_1=-\frac{N_1}{2}}^{\frac{N_1}{2}} \sum_{w_2=-\frac{N_2}{2}}^{\frac{N_2}{2}} \hat{v}(w_1, w_2)\, e^{i\langle \vec{w}, \vec{x}\rangle} .$$

则有 (1). $(u, v)_h = \sum_{w_1} \sum_{w_2} \bar{\hat{u}}(w_1, w_2)\, \hat{v}(w_1, w_2) = (\phi, \psi)$.

(2) $\|\phi\|^2 = \sum_{w_1} \sum_{w_2} |\hat{u}(w_1, w_2)|^2 = \|u\|^2$

(3). $\|D^{l_1}_{+x_1} D^{l_2}_{+x_2} u\|_h^2 \leq \|\frac{d^{l_1+l_2}}{dx_1^{l_1} dx_2^{l_2}} \phi\|^2 \leq (\frac{\pi}{2})^{2(l_1+l_2)} \|D^{l_1}_{+x_1} D^{l_2}_{+x_2} u\|_h^2$

$l_1, l_2 = 0, 1, \dots$

HW 2

2.1.1

因为初值 $f(x)=\begin{cases} x & , 0\le x\le \pi \\ 2\pi - x & , \pi \le x \le 2\pi \end{cases}$

不光滑. 所以收敛较慢. I最大

2.1.2

对 $v_t=-v_x$. 差分格式为 $v_j^{n+1}=(I-kD_0)v_j^n+\sigma k h D_+D_- v_j^n$

$\hat{Q}=1-i\lambda\sin\xi-4\sigma\lambda\sin^2\frac{\xi}{2}$, $\xi=wh$, $\lambda=\frac{k}{h}$

故 $|\hat{Q}|^2=(1-4\sigma\lambda\sin^2\frac{\xi}{2})^2+\lambda^2\sin^2\xi$

$=1-(8\sigma\lambda-4\lambda^2)\sin^2\frac{\xi}{2}+(16\sigma^2-4)\lambda^2\sin^4\frac{\xi}{2}$

与 $v_t=v_x$ 格式的 $|\hat{Q}|^2$ 相同. 有同样结果

1° $0<\lambda\le 2\sigma\le 1$

2° $\begin{cases} 1\le 2\sigma \\ 2\sigma\lambda\le 1 \end{cases}$

2.1.3. $v_j^{n+1}=(I+kD_0)v_j^n+\sigma k h D_+D_- v_j^n$

$\frac{v_j^{n+1}-v_j^n}{k}=\frac{v_{j+1}^n-v_{j-1}^n}{2h}+\sigma\frac{v_{j+1}^n-2v_j^n+v_{j-1}^n}{h}$

$v_j^{n+1}=(1-\frac{2\sigma k}{h})v_j^n+k(\frac{1}{2h}+\frac{\sigma}{h})v_{j+1}^n+k(-\frac{1}{2h}+\frac{\sigma}{h})v_{j-1}^n$, $\sigma>0$

1° $\sigma=\frac{h}{2k}$, $v_j^{n+1}=\frac{k}{2h}(v_{j+1}^n-v_{j-1}^n)+\frac{1}{2}(v_{j+1}^n+v_{j-1}^n)$

代入 $v_j=\frac{1}{\sqrt{2\pi}}e^{iwx_j}\hat{v}^n(w)$. 得

$\hat{Q}=\cos wh+i\lambda\sin wh$

故 $|\hat{Q}|^2=\cos^2wh+\lambda^2\sin^2wh=1-(1-\lambda^2)\sin^2wh\le 1$.

$\lambda^2\le 1 \Rightarrow 0<\lambda\le 1$

2° $\sigma=\frac{1}{2}$. $v_j^{n+1}=v_j^n+\frac{k}{h}(v_{j+1}^n-v_j^n)$. $\hat{Q}=1+\lambda\cos wh+i\lambda\sin wh-\lambda$

$|\hat{Q}|^2=2\lambda(\lambda-1)(1-\cos wh)+1\le 1$

$\Rightarrow \lambda-1<0 \Rightarrow 0<\lambda\le 1$

2.2.1 由课本 P48 Thm 2.1.1，为了说明蛙跳格式解的收敛性，需验证

1、初值 $f(x)$ 满足 $f(x)=\frac{1}{\sqrt{2\pi}}\sum_{w=-\infty}^{+\infty}\hat{f}(w)e^{iwx}$，$\sum_{w}|\hat{f}(w)|^2<\infty$

$$\lim_{N\to\infty}\|J_{n t_n}f-f\|=0$$

这由 f 的光滑性假设得到.

2° 稳定性： CTCS: $v_j^{n+1}=v_j^{n-1}+\cancel{2kD_0}\lambda(v_{j+1}^n-v_{j-1}^n)$. $\lambda=k/h$

代入谐波解 $v_j^n=\frac{1}{\sqrt{2\pi}}e^{iwx_j}\hat{v}^n(w)$，得

$$\hat{v}^{n+1}(w)=\hat{v}^{n-1}(w)+2i\lambda\sin\xi\,\hat{v}^n(w)$$

令 $\hat{v}^n(w)=z^n$. $\Rightarrow z^{n+1}=z^{n-1}+2i\lambda\sin\xi\, z^n$

得特征方程 $z^2=1+2i\lambda z\sin\xi$.

$$z_{1,2}=i\lambda\sin\xi\pm\sqrt{1-\lambda^2\sin^2\xi}.$$

故 $\hat{v}^n=\sigma_1z_1^n+\sigma_2z_2^n$，由 $\hat{v}^0(w)=\hat{f}(w)$，$\hat{v}^1(w)=(1+i\lambda\sin\xi)\hat{f}(w)$.

得 $\begin{cases}\sigma_1+\sigma_2=\hat{f}(w).\\ \sigma_1z_1+\sigma_2z_2=(1+i\lambda\sin\xi)\hat{f}(w).\end{cases}$

$\Rightarrow \sigma_1=\hat{f}(w)(1+O(w^2k^2))$，$\sigma_2=O(w^2k^2)\hat{f}(w)$.

$$\Rightarrow \hat{v}^n(w)=\hat{f}(w)(1+O(w^2k^2))e^{iwt_n(1+O(w^2k^2))}+O(w^2k^2)\hat{f}(w)(-1)^ne^{-iwt_n(1+O(w^2k^2))} \quad (※)$$

事实上 $\sigma_{1,2}=\dfrac{\pm1+\sqrt{1-\lambda^2\sin^2\xi}}{2\sqrt{1-\lambda^2\sin^2\xi}}\hat{f}(w)$.

$$\left|\frac{\pm1+\sqrt{1-\lambda^2\sin^2\xi}}{2\sqrt{1-\lambda^2\sin^2\xi}}\right|\leq\frac{2}{2\sqrt{1-(1-\delta)^2}}=\frac{1}{\sqrt{-\delta^2+2\delta}},\ |\sigma_{1,2}|\leq\frac{1}{\sqrt{-\delta^2+2\delta}}|\hat{f}(w)|$$

故 $|\hat{v}^n(w)|=|\sigma_1z_1^n+\sigma_2z_2^n|\leq\frac{2}{\sqrt{-\delta^2+2\delta}}|\hat{f}(w)|$

$\quad = |\hat{Q}^n(\xi)\hat{v}^0(w)| \qquad (|z_{1,2}|=1)$

$\quad = |\hat{Q}^n(\xi)||\hat{f}(w)|$，即 $\sup|\hat{Q}^n|\leq K_\delta$.

3° 相容性： 由(※) $|\hat{Q}^n(\xi)-e^{iwt_n}|=|(1+O(w^2k^2))e^{iwt_n(1+O(w^2k^2))}+O(w^2k^2)(-1)^ne^{-iwt_n(1+O(w^2k^2))}-e^{iwt_n}|$

$\to 0$, as $k,h\to0$.

2.2.2 $\lambda=1$ 时. $v_j^{n+1}=v_j^{n-1}+v_{j+1}^n-v_{j-1}^n$

与2.2.1分析类似

$$z_{1,2}=i\sin\xi\pm\sqrt{1-\sin^2\xi}=i\sin\xi\pm\cos\xi$$

故 $\sigma_{1,2}=\dfrac{\pm1+\cos\xi}{2\cos\xi}=\dfrac{1}{2}\pm\dfrac{1}{2}\dfrac{1}{\cos\xi}\to\infty$, $\xi\to\dfrac{\pi}{2}$

不稳定. 格式不适合计算

2.3.1 θ格式: $(I-\theta kD_0)v_j^{n+1}=(I+(1-\theta)kD_0)v_j^n$

代入谐波解, 得放大因子

$$\hat{Q}=\frac{1+i(1-\theta)\lambda\sin\xi}{1-i\theta\lambda\sin\xi}$$

$$|\hat{Q}|^2=\frac{[1-(1-\theta)\theta\lambda^2\sin^2\xi]^2+\lambda^2\sin^2\xi}{(1+\theta^2\lambda^2\sin^2\xi)^2}\le1$$

$\Leftarrow$ $(1-2\theta)\lambda^2\sin^2\xi\,[1+\lambda^2\sin^2\xi]\le0$

$\Leftarrow$ $(1-2\theta)\le0$

$\Leftarrow$ $\theta\ge\frac{1}{2}$.

2.4.1 设截断误差在(x_j,t_n)的Taylor展开式可写成如下形式

$$T_j^n=f(x_j,t_n)h^p+g(x_j,t_n)k^q+O(h^{p+1}+k^{q+1})$$

其中 h为空间步长, k为时间步长.

$$f(x_j,t_n)=f(x_*,t^n)+O(h)=f(x_*,t_*)+O(h)+O(k)$$

类似有 $g(x_j,t_n)=g(x_*,t_*)+O(h+k)$

故 (x_*,t_*)处的截断误差

$$T_{j*}^{n*}=(f(x_*,t_*)+O(h)+O(k))h^p+(g(x_*,t_*)+O(h)+O(k))k^q+O(h^{p+1}+k^{q+1})$$

$$=f(x_*,t_*)h^p+g(x_*,t_*)k^q+O(h^pk+h^{p+1}+hk^q+k^{q+1})$$

若k与h为同一量级, 则.

$$T_{j*}^{n*}=f(x_*,t_*)h^p+g(x_*,t_*)k^q+O(h^{p+1}+k^{q+1}).$$

2.4.2. CTCS: $\dfrac{V_j^{n+1}-V_j^{n-1}}{2\Delta t} = \dfrac{V_{j+1}^n - V_{j-1}^n}{2\Delta x}$.

$$T_j^n = \frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t} - \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

将 $u_j^{n\pm1}$. $u_{j\pm1}^n$ 在 (x_j, t_n) 处展开.

$$u_j^{n+1}-u_j^{n-1} = 2\Delta t\, u_t\big|_j^n + O(\Delta t^3)$$

$$u_{j+1}^n - u_{j-1}^n = 2\Delta x\, u_x\big|_j^n + O(\Delta x^3).$$

故 $T_j^n = u_t\big|_j^n + O(\Delta t^2) - u_x\big|_j^n + O(\Delta x^2)$

$= (u_t - u_x)\big|_j^n + O(\Delta t^2 + \Delta x^2)$

$= O(\Delta t^2 + \Delta x^2)$.

∴ CTCS 为 (2,2) 阶.

C-N: $\dfrac{V_j^{n+1}-V_j^n}{\Delta t} = \frac{1}{2} D_0 V_j^{n+1} + \frac{1}{2} D_0 V_j^n$

$$T_j^n = \frac{u_j^{n+1}-u_j^n}{\Delta t} - \frac{1}{2}\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2\Delta x} - \frac{1}{2}\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

在 (x_j, t_n) 处展开. $u_j^{n+1}-u_j^n = \Delta t\, u_t\big|_j^n + \frac{\Delta t^2}{2} u_{tt}\big|_j^n + O(\Delta t^3)$.

$$\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2\Delta x} = u_x\big|_j^{n+1} + O(\Delta x^2)$$

$$\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = u_x\big|_j^n + O(\Delta x^2)$$

$$u_j^{n+1}-u_j^n = \Delta t\, u_t\big|_j^{n+1} - \frac{\Delta t^2}{2} u_{tt}\big|_j^{n+1} + O(\Delta t^3).$$

故 $T_j^n = \dfrac{u_j^{n+1}-u_j^n}{2\Delta t} - \dfrac{1}{2}\dfrac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2\Delta x} + \dfrac{u_j^{n+1}-u_j^n}{2\Delta t} - \dfrac{1}{2}\dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$

$= \frac{1}{2} u_t\big|_j^{n+1} - \frac{1}{2} u_x\big|_j^{n+1} + \frac{1}{2} u_t\big|_j^n - \frac{1}{2} u_x\big|_j^n + O(\Delta t^2 + \Delta x^2)$

$+ \frac{\Delta t}{2} \underbrace{(u_{tt}\big|_j^n - u_{tt}\big|_j^{n+1})}_{O(\Delta t)}$

$= \frac{1}{2}(u_t - u_x)\big|_j^n + \frac{1}{2}(u_t - u_x)\big|_j^{n+1} + O(\Delta t^2 + \Delta x^2)$

$= O(\Delta t^2 + \Delta x^2)$

故 C-N 为 (2,2) 阶

但当 $\Delta t = \Delta x$ 时. CTCS 为 (4,4) 阶, 比 C-N 更准确 (多展开几项, 易验证)

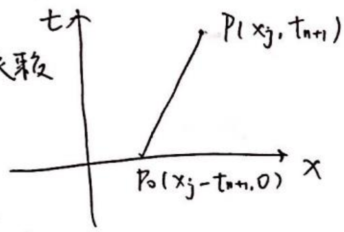
针对方程 $u_t+u_x=0$，导出其解的依赖区；
并求出蛙跳格式的依赖区以及CFL条件.

(1) 对 $u_t+u_x=0$，其特征线为 $\xi = x-t$

沿特征线 $u=u(\xi)$，方程的解不变，即

$$u_t+u_x = u_\xi\cdot\xi_t + u_\xi\cdot\xi_x = u_\xi(-1+1)=0.$$

对于点 $P(x_j,t_{n+1})$，假设其依赖点 $P_0(x_0,0)$ 则



$$\xi = x_j - t_{n+1} = x_0 - 0$$

$$\Rightarrow \quad x_0 = x_j - t_{n+1}$$

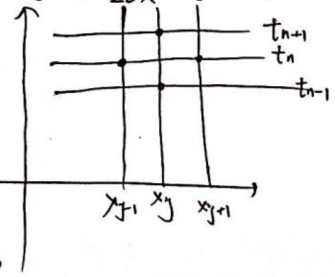
故 PDE的解依赖区为 $D_p=\{P_0\}=\{(x_j-t_{n+1},0)\}$.

(2) CTCS格式：
$$\frac{v_j^{n+1}-v_j^{n-1}}{2\Delta t}+\frac{v_{j+1}^{n}-v_{j-1}^{n}}{2\Delta x}=0$$

$$\Rightarrow \quad v_j^{n+1} = v_j^{n-1} - \frac{\Delta t}{\Delta x}\left(v_{j+1}^{n}-v_{j-1}^{n}\right)$$

第一层用FTCS
$$\frac{v_j^{1}-v_j^{0}}{\Delta t}+\frac{v_{j+1}^{0}-v_{j-1}^{0}}{2\Delta x}=0$$

$$\Rightarrow \quad v_j^{1} = v_j^{0} - \frac{\Delta t}{2\Delta x}\left(v_{j+1}^{0}-v_{j-1}^{0}\right)$$



t_{n+1} : x_j

t_n : x_{j-1}, x_{j+1}

t_{n-1} : x_{j-2}, x_j, x_{j+2}

t_1 : x_{j-n}, x_{j-n+2}, ……, x_{j+n-2}, x_{j+n}.

t_0 : $x_{j-(n+1)}$, x_{j-n}, …, x_{j+n}, $x_{j+(n+1)}$

故 数值解依赖区 $N_p=\{x_{j-(n+1)}, x_{j-n}, \ldots, x_{j+n}, x_{j+(n+1)}\}$.

CFL条件： $D_p \subseteq N_p \Rightarrow x_{j-(n+1)} \leq x_j - t_{n+1} \leq x_{j+(n+1)}$.

$$\Rightarrow -(n+1)\Delta x \leq -(n+1)\Delta t \leq (n+1)\Delta x$$

$$\Rightarrow \quad \lambda = \frac{\Delta t}{\Delta x} \leq 1$$