有限元方法 2021 秋（12 月 6 日、12 月 8 日作业）

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12.x.2 **证明 $H(\mathrm{div})$ 是 Hilbert 空间。**

证明. 平凡验证 $H(\mathrm{div})$ 为内积空间。设 $\{\underset{\sim}{u}_k\}$ 为 $H(\mathrm{div})$ 中的 Cauchy 列，那么 $\|\underset{\sim}{u}_j - \underset{\sim}{u}_k\|_{L^2(\Omega)^n} \le \|\underset{\sim}{u}_j - \underset{\sim}{u}_k\|_{H(\mathrm{div})} \to 0$, $\|\mathrm{div}(\underset{\sim}{u}_j) - \mathrm{div}(\underset{\sim}{u}_k)\|_{L^2(\Omega)^n} \to 0 \Rightarrow$ 由 $L^2(\Omega)^n$ 完备性，存在 $\underset{\sim}{u}, v \in L^2(\Omega)^n$ s.t. $\|\underset{\sim}{u}_k - \underset{\sim}{u}\|_{L^2(\Omega)^n} \to 0$, $\|\mathrm{div}(\underset{\sim}{u}_k) - v\|_{L^2(\Omega)^n} \to 0$. Claim: $v = \mathrm{div}(\underset{\sim}{u})$.

$\forall \phi \in C_0^\infty(\Omega)$，$\int_\Omega v\phi = \lim\limits_{k\to\infty} \int_\Omega \mathrm{div}(\underset{\sim}{u}_k)\phi = -\lim\limits_{k\to\infty} \underset{\sim}{u}_k \cdot D\phi = -\int_\Omega \tilde{u} \cdot D\phi = \int_\Omega \mathrm{div}(\underset{\sim}{u})\phi$.   //Claim.

所以 $\|\underset{\sim}{u}_k - \underset{\sim}{u}\|_{H(\mathrm{div})} \to 0$，即证完备性。 □

12.x.3 **定义 Stokes 变分形式** $a(\underset{\sim}{u}, \underset{\sim}{v}) = 2\int_\Omega \sum\limits_{i,j} e_{ij}(\underset{\sim}{u}) e_{ij}(\underset{\sim}{v})$，**其中** $e_{ij}(\underset{\sim}{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. **证明对于** $\underset{\sim}{u}, \underset{\sim}{v} \in Z = \{\ \underset{\sim}{v} \in H(\mathrm{div}) : \mathrm{div}\ \underset{\sim}{v} = 0\}$，**成立** $a(\underset{\sim}{u}, \underset{\sim}{v}) = \int_\Omega \sum \underset{\sim}{\mathrm{grad}} u_i \cdot \underset{\sim}{\mathrm{grad}} v_i$.

证明. 容易计算 $\mathrm{LHS} - \mathrm{RHS} = \int_\Omega \sum\limits_{i,j} \frac{\partial u_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} = -\int_\Omega \sum\limits_{i,j} \frac{\partial^2 u_i}{\partial x_i \partial x_j} v_j = \int_\Omega \sum\limits_{i,j} \frac{\partial u_i}{\partial x_i}\frac{\partial v_j}{\partial x_j} = \int_\Omega \mathrm{div}(\underset{\sim}{u})\mathrm{div}(\underset{\sim}{v}) = 0.$ □