1. $x_0=-1,\ x_1=0,\ x_2=1 \quad f(x_0)=5 \quad f(x_1)=1 \quad f(x_2)=3 \quad f'(x_2)=4.$

(1) 差商表

x_i	$f(x_i)$	$f[x_{i-1},x_i]$	$f[x_{i-2},x_{i-1},x_i]$	$f[x_{i-3},x_{i-2},x_{i-1},x_i]$
-1	5	-4	3	$-\frac{1}{2}$
0	1	2	2	
1	3	4		
1	3			

(2) $f(x)=5-4(x+1)+3(x+1)x-\frac{1}{2}(x-1)x(x+1)=-\frac{1}{2}x^3+3x^2-\frac{1}{2}x+1$

2. 函数 f 在 $x=-2,-1,0,1,2$ 处值 $P_4(x)=1+2x+3x^2+4x^3+5x^4$.

$x=0$ 处值应为 $y=3$，而不是之前的 $y=1$. 求纠正后的插值多项式 $P_4^C(x)$

(法一): 记在 $x_i=-2,-1,0,1,2 \quad i=1\sim5$ 处的Lagrange基函数为 $l_i(x) \quad i=1\sim5$

$\because P_4(x)=\sum_{i=1}^{5}f(x_i)l_i(x)=f(x_1)l_1(x)+f(x_2)l_2(x)+1\cdot l_3(x)+f(x_4)l_4(x)+f(x_5)l_5(x)$

$P_4^C(x)=f(x_1)l_1(x)+f(x_2)l_2(x)+3\cdot l_3(x)+f(x_4)l_4(x)+f(x_5)l_5(x)$

$=P_4(x)+2\cdot l_3(x)$

$l_3(x)=\dfrac{(x+2)(x+1)(x-1)(x-2)}{2\cdot1\cdot(-1)\cdot(-2)}=\frac{1}{4}(x^2-4)(x^2-1)$

$\therefore P_4^C(x)=1+2x+3x^2+4x^3+5x^4+2\cdot\frac{1}{4}(x^2-4)(x^2-1)$

$=\frac{11}{2}x^4+4x^3+\frac{1}{2}x^2+2x+3$

(法二) 求 $x=-2 \quad f(-2)=57$

$x=-1 \quad f(-1)=3$

$x=1 \quad f(1)=15$

$x=2 \quad f(2)=129$

$P_4^C(x)=\sum l_i(x)f(x_i)+3l_3(x)=\frac{11}{2}x^4+4x^3+\frac{1}{2}x^2+2x+3$

(方法不唯一)

3.

x_i	-2	0	1	2
$f(x_i)$	4	8	12	16

(1) $l_0(x)=\dfrac{x(x-1)(x-2)}{(-2-0)(-2-1)(-2-2)}=\frac{1}{-24}x(x-1)(x-2)=\frac{1}{-24}(x^3-3x^2+2x)$

$l_1(x)=\dfrac{(x+2)(x-1)(x-2)}{(0+2)(0-1)(0-2)}=\frac{1}{4}(x+2)(x-1)(x-2)=\frac{1}{4}(x^3-x^2-4x+4)$

$$l_2(x) = \frac{(x+2)x(x-2)}{3\cdot 1\cdot -1} = -\frac{1}{3}(x-2)x(x+2) = -\frac{1}{3}(x^3-4x)$$

$$l_3(x) = \frac{(x+2)x(x-1)}{4\cdot 2\cdot 1} = \frac{1}{8}(x-1)x(x+2) = \frac{1}{8}(x^3+x^2-2x)$$

$$\therefore f(x) = f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x) + f(x_3)l_3(x)$$

$$= 4\cdot\frac{1}{-24}x(x-1)(x-2) + 8\cdot\frac{1}{4}(x+2)(x-1)(x-2) + 12\cdot -\frac{1}{3}(x-2)x(x+2) + 16\cdot\frac{1}{8}(x-1)x(x+2)$$

$$= -\frac{1}{6}x(x-1)(x-2) + 2(x+2)(x-1)(x-2) - 4(x-2)x(x+2) + 2(x-1)x(x+2)$$

$$= -\frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{11}{3}x + 8$$

(2) $f(\frac{1}{2}) = -\frac{1}{6}\times\frac{1}{8} + \frac{1}{2}\times\frac{1}{4} + \frac{11}{3}\times\frac{1}{2} + 8 = \frac{159}{16}$

4. (1). $P_3(0) = f(0) = 0$

$P_3'(0) = f'(0) = 0$

$P_3(a) = f(a) = a^5$

$P_3'(a) = f'(a) = 5a^4$

差商表:

x_i	$f(x_i)$			
0	0	0	a^3	$3a^2$
0	0	a^4	$4a^3$	
a	a^5	$5a^4$		
a	a^5			

$$\therefore P_3(x) = a^3x^2 + 3a^2(x-a)x^2 = 3a^2x^3 - 2a^3x^2$$

(2). 由多项式插值误差定理知

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}x^2(x-a)^2 \quad 这里 n=3$$

$$\Rightarrow f(x) - p(x) = x^5 - 3a^2x^3 + 2a^3x^2 = x^2(x+2a)(x-a)^2$$

$$\frac{f^{(n+1)}(\xi_x)}{(n+1)!}x^2(x-a)^2 = \frac{f^{(4)}(\xi_x)}{4!}x^2(x-a)^2$$

$$\Rightarrow x^2(x+2a)(x-a)^2 = \frac{f^{(4)}(\xi_x)}{4!}x^2(x-a)^2$$

由 $f(x) = x^5 \Rightarrow f^{(4)}(\xi_x) = 5!\,\xi_x$ 代入 $\Rightarrow \xi_x = \frac{x+2a}{5}$