1. $\because A^TA=\begin{pmatrix}35 & 44\\ 44 & 56\end{pmatrix} \qquad A^Tb=\begin{pmatrix}9\\12\end{pmatrix}$

$\Rightarrow x=\begin{pmatrix}-1\\1\end{pmatrix}$

2. 通解为 $\cancel{\frac{1}{5}\begin{pmatrix}3\\0\\0\\2\end{pmatrix}+a\begin{pmatrix}0\\1\\0\\-3\end{pmatrix}+b\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}} \quad \cancel{a.\ b\in\mathbb{R}}$

$\begin{pmatrix}\frac{3}{5}\\ (\frac{2}{5}-t_1-t_2)/3\\ t_1\\ t_2\end{pmatrix} \qquad t_1, t_2\in\mathbb{R}$

3. H为正交变换 $\Leftrightarrow \|Hx\|_2=\|x\|_2 \Rightarrow \alpha=5$

$w=\frac{x-Hx}{\|x-Hx\|_2}=\frac{1}{5\sqrt{2}}(0,-5,0,0,3,4)^T$

$\Rightarrow H=I-2ww^T=\frac{1}{25}\begin{pmatrix}25 & & & & & \\ & 0 & & & 15 & 20\\ & & 25 & & & \\ & & & 25 & & \\ & 15 & & & 16 & -12\\ & 20 & & & -12 & 9\end{pmatrix}$

4. $\alpha=\pm\frac{\sqrt{5^2+12^2}}{\sqrt{2}}=\pm\frac{13}{\sqrt{2}}$

$\therefore \begin{pmatrix}c\\s\end{pmatrix}=\begin{pmatrix}17\\7\end{pmatrix}\frac{\sqrt{2}}{26}$ or $\begin{pmatrix}-17\\-7\end{pmatrix}\frac{\sqrt{2}}{26}$

5.

$\because \|x\|_2=\sqrt{x^Hx}=\sqrt{|x_1|^2+|x_2|^2}$

由二范数的正交不变性： $\|Qx\|_2=\|x\|_2$

记 $\vec{y}=(y_1,0)=Q\vec{x}$， $\therefore |y_1|=\|x\|_2$

$\vec{x}=Q^H\vec{y}=\begin{pmatrix}c & -\bar{s}\\ s & c\end{pmatrix}\begin{pmatrix}y_1\\0\end{pmatrix}=\begin{pmatrix}cy_1\\ sy_1\end{pmatrix}$

$\therefore |c|=\frac{|x_1|}{|y_1|}=\frac{|x_1|}{\|x\|_2}$， $c\in\mathbb{R}\Rightarrow c=\frac{|x_1|}{\|x\|_2}$

$s=\frac{x_2}{y_1}=\frac{\bar{y}_1x_2}{\bar{y}_1y_1}=\frac{x_2}{\|x\|_2^2}\frac{\bar{x}_1}{c}=\frac{\bar{x}_1x_2}{|x_1|\,\|x\|_2}$

6. 分别寻找 P, Q, $Px = P_n P_{n-1} \cdots P_2 x = e_1$

$$Q'y = Q_n Q_{n-1} \cdots Q_2 y = e_1$$

P_i, Q_i 为 Givens 变换

取 $Q = Q'^T P$ 即可

7.
$$w = \begin{cases} 0 & \vec{x}\text{与}\vec{y}\text{同向} \\ \dfrac{\vec{x} - \alpha\vec{y}}{\|\vec{x} - \alpha\vec{y}\|_2} & \text{else} \end{cases}$$

8.

记 $L = L^{(0)} = \{l_{ij}^{(0)}\}$

第一步：~~找~~ 找 Householder 变换 H_1, s.t.

$$H_1 \begin{pmatrix} 0 \\ \vdots \\ l_{nn}^{(0)} \\ \vdots \\ l_{mn}^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ l_{nn}^{(1)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

，显然 H_1 在消掉第 n 列的后 $m-n$ 行时，不改变 L 的前 $n-1$ 行

第 k 步：找 H_k s.t.

$$H_k \begin{pmatrix} 0 \\ \vdots \\ 0 \\ l_{n-k+1,n-k+1}^{(k-1)} \\ \vdots \\ l_{m,n-k+1}^{(k-1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ l_{n-k+1,n-k+1}^{(k)} \\ l_{n-k+2,n-k+1}^{(k-1)} \\ \vdots \\ l_{n,n-k+1}^{(k-1)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

，H_k 在消掉第 $n-k+1$ 列的后 $m-n$ 行时，不改变 L 的第 $n-k+2 \sim n$ 行

故 $L = H_n H_{n-1} \cdots H_1 L$ 满足条件

9.

由8，∃正交矩阵 Q，s.t. $QL = \begin{array}{c} \begin{pmatrix} L_1 \\ 0 \end{pmatrix} \begin{array}{l} n \\ m-n \end{array} \\ n \end{array}$

记 $QPb = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

故 $\|Ly-Pb\|_2 = \|Q(Ly-Pb)\|_2 = \left\|\begin{pmatrix} L_1y \\ 0 \end{pmatrix} - \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right\|_2$

$= (\|L_1y-c_1\|_2^2 + \|c_2\|_2^2)^{\frac{1}{2}} \geq \|c_2\|_2$

故 $L_1z = c_1$ 为上述LS问题的解

同理：$\|Ax-b\|_2^2 = \min\limits_{y\in\mathbb{R}^n} \|Ay-b\|_2^2$

$= \min\limits_{y\in\mathbb{R}^n} \|LUy - Pb\|_2^2$ （列主元分解）

$= \min\limits_{y\in\mathbb{R}^n} \|L_1Uy - c_1\|_2^2 + \|c_2\|_2^2$

故 $Ux = z$ 为上述LS问题的解

10.

A^TA 不一定可逆，故不能直接 $\tilde{x} = (A^TA)^{-1}A^Tb$

考虑 A 的SVD分解（详见书P204定理7.1.5） $A = P\Sigma Q$，P, Q 正交阵

其中 $\Sigma = \mathrm{diag}(\underbrace{\sigma_1, \sigma_2, \cdots \sigma_k, 0, 0, \cdots}_{m个} 0)$，$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$

$\therefore \|Ax-b\|_2 = \|P\Sigma Qx - b\|_2 = \|\Sigma Qx - P^Tb\|_2$

$AXA = P\Sigma(QXP)\Sigma Q$，$(AX)^T = (P\Sigma QX)^T = X^TQ^T\Sigma P^T$

故设 $Y = QXP$，$c = P^Tb$，$z = Qx$

$\therefore \|Ax-b\|_2 = \|\Sigma z - c\|_2$

$AXA = A \Leftrightarrow \cancel{\Sigma Y \Sigma = \Sigma}$ $\Sigma Y \Sigma = \Sigma$

$(AX)^T = AX \Leftrightarrow (\Sigma Y)^T = \Sigma Y$

$x = Xb \Leftrightarrow z = Yc$

记 $\Sigma = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} \Sigma_1 & \\ & 0 \end{pmatrix} & \begin{array}{l} k \\ m-k \end{array} \\ \begin{array}{cc} k & n-k \end{array} & \end{array}$

由 x 极小化 $\|Ax-b\|_2 \Leftrightarrow z$ 极小化 $\|\Sigma z - c\|_2 \Leftrightarrow \Sigma^T\Sigma Yc = \Sigma^T c$

$\Rightarrow \begin{pmatrix} \sigma_1^2 & & & & & \\ & \ddots & & & & \\ & & \sigma_k^2 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} Yc = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} c$

设 $c = \begin{array}{c} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{array} \begin{array}{l} k \\ n-k \end{array}$ $\quad Y = \begin{array}{cc} \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} & \begin{array}{l} k \\ \cancel{m-k}\ n-k \end{array} \\ \begin{array}{cc} k & m-k \end{array} & \end{array}$，代入上式

$\Sigma_1 (Y_{11} c_1 + Y_{12} c_2) = c_1 \qquad \forall b \in \mathbb{R}^m$ 成立

由b的任意性 $\Rightarrow$ c的任意性 $\Rightarrow \begin{cases} \Sigma_1 Y_{11} = I_k \\ \Sigma_1 Y_{12} = 0 \end{cases}$

$\therefore \Sigma Y \Sigma = \begin{pmatrix} \Sigma_1 Y_{11} \Sigma_1 & \\ & 0 \end{pmatrix} = \begin{pmatrix} \Sigma_1 & \\ & 0 \end{pmatrix} = \Sigma$

$(\Sigma Y)^T = \begin{pmatrix} Y_{11}^T \Sigma_1^T & 0 \\ Y_{12}^T \Sigma_1^T & 0 \end{pmatrix} = \begin{pmatrix} I_k & \\ & 0 \end{pmatrix} = \Sigma Y$

$\Rightarrow AXA = A, \quad (AX)^T = AX$

11、

① 先将A变为如图形状 $\begin{pmatrix} * & & & * \\ * & \ddots & & \\ & \ddots & \ddots & \\ 0 & & * & * \end{pmatrix}$

方法：在第k步时~~找~~找 $\tilde{G}_k \in \mathbb{R}^{2\times 2}$ s.t. $\tilde{G}_k \begin{pmatrix} \beta_{n-k} \\ \sqrt{\beta_n^2 + \cdots + \beta_{n-k+1}^2} \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_n^2 + \cdots + \beta_{n-k}^2} \\ 0 \end{pmatrix}$

$G_k = \begin{pmatrix} I_{n-k-1} & & \\ & \tilde{G}_k & \\ & & I_{k-1} \end{pmatrix}$

则 $A^{(n-2)} = G_{n-2} G_{n-3} \cdots G_1 A = \begin{pmatrix} * & & & * \\ * & \ddots & & \\ & \ddots & \ddots & \\ 0 & & * & * \end{pmatrix} =: B^{(0)} = (b_{ij}^{(0)})_{n\times n}$

再自上而下消去次对角线阵

方法：在第k步时找 $\tilde{G}_k$ s.t. $\tilde{G}_k' \begin{pmatrix} b_{k,k}^{(k-1)} \\ b_{k+1,k}^{(k-1)} \end{pmatrix} = \begin{pmatrix} b_{k,k}^{(k)} \\ 0 \end{pmatrix}$

$$G_k' = \begin{pmatrix} I_{k-1} & & \\ & \tilde{G}_k' & \\ & & I_{n-k-1} \end{pmatrix}$$

n-1步后化为上三角R

故 $Q^T = (G_{n-1}' \cdots G_1')(G_{n-2} \cdots G_1)$

P.S. 学习辅导的方法显然是错的

12. $x \in \mathcal{X}_{LS} \Rightarrow \frac{d}{d\alpha} \|A(x+\alpha w)-b\|_2^2 \Big|_{\alpha=0} = 0$

$\Rightarrow 2w^T A^T (Ax-b) = 0 \quad \forall w \in \mathbb{R}^n$

$\Rightarrow A^T(Ax-b) = 0$