Linear Optimization: Dantzig-Wolfe Decomposition

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- This technique uses column generation to create "proposed solutions" for the set of easy constraints.
- Dantziq-Wolfe decomposition is generally used to make it possible to solve problems that would otherwise be too large or too difficult to solve.

Example structures

Several structures make good candidates for Dantzig-Wolfe decomposition:

Block diagonal structure

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Such a system may be written as follows:

$$
\begin{array}{lccccccc}
\max & c_1^T x_1 & +c_2^T x_2 & +\cdots & +c_t^T x_t & & \\
s.t. & F_1 x_1 & & & & = & b_1 \\
 & & F_2 x_2 & & & = & b_2 \\
 & & & \ddots & & \vdots & \vdots \\
 & & & & F_t x_t & = & b_t \\
 & D_1 x_1 & +D_2 x_2 & +\cdots & +D_t x_t & = & b_0 \\
 & x_1, & x_2, & \ldots, & x_t & \geq & 0
\end{array}
$$

Note that x_i is a vector for each i. That is not the same as our usual notation.

Economic interpretation

Consider a company with several divisions. Each division has its own planning decisions to make as well as constraints on its decisions. In addition, some constraints depend on all the divisions, such as the overall budget. Based on certain prices attached to the linking constraints, each division is asked to formulate a proposal indicating its optimal plan.

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A central planner takes these proposals (entering columns) and combines them with existing proposals to make an overall plan for the company. This plan gives new prices (dual values) for the linking constraints. The process is repeated with these new prices to see if any division can improve its plan.

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Note that in this interpretation, several proposals from one division can be "mixed" by the central planner to form a hybrid plan for that division.

Solving the subproblem(s)

Consider the subproblem formed by considering one group of variables and the constraints that affect only that group.

$$\begin{array}{rrcl} \max & c_i^T x_i & & \\ s.t. & F_i x_i & = & b_i \\ & x_i & \geq & 0 \end{array}$$

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The result will either be an extreme point x_i^j of the feasible region, or an extreme ray w_i^k of the feasible region that proves the subproblem is unbounded.

What does it mean if the subproblem is infeasible?

Finite basis theorem

Let P_i be the set of feasible solutions to subproblem i. Let the extreme points of the set P_i be $x_i^j, j \in J_i$. Let the set of extreme rays of P_i be $w_i^k, k \in K_i$.
Any feasible solution $x_i \in P_i$ can be written

$$x_i = \sum_{j \in J_i} \lambda_i^j x_i^j + \sum_{k \in K_i} \theta_i^k w_i^k,$$

where $\theta_i^k \geq 0$ and $\lambda_i^j \geq 0, \sum_{j \in J_i} \lambda_i^j = 1$.

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It may be that x_i, the part of the full solution corresponding to subproblem i, is not an extreme point of P_i because of the other "hard" constraints.

The result above indicates that this is OK, since we can represent any feasible solution in terms of extreme points and extreme rays.

Master problem

The master problem (solved by the central planner) is formed by replacing each variable vector x_i in the original problem with the variables λ_i^j and θ_i^k:

$$
\begin{array}{llcll}
\max & \sum_{j \in J_1} c_1^T x_1^j \lambda_1^j + \sum_{k \in K_1} c_1^T w_1^k \theta_1^k + \cdots + \sum_{j \in J_t} c_t^T x_t^j \lambda_t^j + \sum_{k \in K_t} c_t^T w_t^k \theta_t^k & & & \\
s.t. & \sum_{j \in J_i} \lambda_i^j & = & 1 & i = 1, \ldots, t \\
 & \sum_{j \in J_1} D_1 x_1^j \lambda_1^j + \sum_{k \in K_1} D_1 w_1^k \theta_1^k + \cdots + \sum_{j \in J_t} D_t x_t^j \lambda_t^j + \sum_{k \in K_t} D_t w_t^k \theta_t^k & = & b_0 & \\
 & \lambda_i^j & \geq & 0 & \forall i, j \\
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s.t. & \sum_{j \in J_i} \lambda_i^j & = & 1 & i = 1, \ldots, t \\
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 & \lambda_i^j & \geq & 0 & \forall i, j \\
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The "easy" constraints were all removed, but *convexity constraints* were added.

Note that x_i^j and w_i^k are constants—not variables.

Number of constraints and variables

How has the number of constraints changed?

How has the number of variables changed?

Column generation

We will solve the master problem using column generation.
How large is the basis?

How are improving columns generated?

Dantzig-Wolfe decomposition relies on the fact that generating columns can be done much more quickly than solving the original problem.

The solution

1. Add enough extreme points and/or rays to form an initial feasible basis.
2. Solve the master problem.
3. Use the dual prices for the constraints in the master problem to generate one or more favorable extreme points or rays. If none exist, then the current solution is optimal.
4. If one or more favorable extreme points or rays are found, then add them to the original problem and return to step 2.

Dantzig and Wolfe

Unlike many scientific fields, linear optimization and operations research in general is relatively young. Some ideas go back to the nineteenth century (and perhaps earlier), but the field as we know it began during and just after World War II.

George Dantzig plays a prominent role. His primary contribution is that he formalized the problem of linear programming and developed the simplex algorithm for solving LPs. Another contribution (and there were many) was the development, along with Phil Wolfe, of Dantzig-Wolfe decomposition.

Phil Wolfe made other contributions to mathematical programming and game theory, including being one of the founders of the Mathematical Programming Society (`www.mathprog.org`). Both Dantzig and Wolfe have served as chairman of the MPS. George Dantzig is currently a professor at Stanford University. I have not been able to identify Phil Wolfe's current position.