Twisted soft photon hair implants on Black Holes.

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The Hawking-Perry-Strominger (HPS) work [1] states a new controversial idea about the black hole (BH) information paradox [2–5] where BHs maximally entropize and encode information in their event horizon area [6, 7], with no "hair" were thought to reveal information outside but angular momentum, mass and electric charge only [8, 9] in a unique quantum gravity (QG) vacuum state. This new idea invokes new conservation laws involving gravitation and electromagnetism [10, 11], to generate different QG vacua and preserve more information in hair implants. In the context of black holes and the HPS proposal we find that BH photon hair implants can be spatially shaped ad hoc and encode structured and densely organized information on the event horizon involving novel aspect in the discussion a particular aspect of EM fields, namely the spatial information of the field associated to its orbital angular momentum. BHs can have "curly", twisted, soft–hair implants with vorticity where structured information is holographically encoded in the event horizon in an organized way.

The BH information paradox and information loss together with the so called no–hair theorem were mainly based on the supposed unicity of the quantum gravity vacuum that revealed outside only the macroscopical quantities angular momentum, mass and electric charge. This was changed by HPS invoking new results in GR and QG in terms of new mathematical formulations, with the finding of new field invariants and symmetries such as supertranslations [10, 12] and superrotations [13] with possible observable effects [14–16]. Anyway, at our knowledge, it is still an open question whether the soft hair of HPS can account for the Bekenstein-Hawking entropy and thus capture the information about the black hole microstates.

When one also includes the set of conserved quantities of the electromagnetic (EM) field [11, 17, 18], EM analogs of the supertranslation charges are introduced as a generalization of the electric charge conservation principle, from the set of Lorentz symmetries of the 10-dimensional Poincaré group of the EM field. Also in this case HPS suggest that the unicity of the QG vacuum state is invalidated because of the creation or annihilation of actual physical quantum zero-energy state particles, named "soft" photons. When electromagnetic phenomena act on BHs, BHs are expected to carry a "soft electric hair" implant, one or more soft photons that store the information of the process in the event horizon discriminating one QG vacuum state from another and consequently BHs with the same macroscopical parameters. The complete information is then stored on an holographic plate made of quantum pixels at the future boundary of the horizon, where the excitation of a pixel corresponds to the creation of a spatially localized soft photon on the horizon. One way to excite or modify the quantum state of a pixel is to make one or more charged particles cross the event horizon [1].

To install a soft photon hair implant on the event horizon of a BH, consider the metric obtained from the collapse of neutral matter at advanced time $v = 0$,

$$ds^2 = -\left(1 - \frac{2M\Theta(v)}{r}\right)dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \quad (1)$$

written in terms of the round metric $\gamma_{z\bar{z}}$ on the unit sphere S^2 in complex coordinates $(v, r, z, \bar{z})$, where $v = t + r$ and r the radial coordinate, $z = \tan(\phi/2)\exp(-i\theta)$ and $z = 1/\bar{z}$, M the BH mass and $\Theta = 0$ before the shell at $v = 0$ and $\Theta = 1$ after the shell at $v = 0$. The horizon of the BH, H, has an S^2 boundary in the far future, $\mathscr{H}^+$, working as an holographic plate where soft hair represent quantum pixels.

The simplest example of soft photons excitation is given by a null shock wave with divergence-free charge current carrying an angular momentum eigenstate j falls into the BH at $v = v_0 > 0$ that,

$$j^*_v = \frac{Y_{jm}(z,\bar{z})}{r^2}\delta(v - v_0), \quad (2)$$

where Y_{jm} are spherical harmonics.

This process generates a soft photon on the horizon with polarization vector $\varepsilon_{jm}(\sigma) \propto \partial_z Y_{jm}(z,\bar{z})$ with a multipolar radiation field

$$F_{\mathrm{soft}} = \int_{-\infty}^{+\infty} dv F^{(0)}_{zv} = -\frac{e^2}{j(j+1)}\partial_z Y_{jm} \quad (3)$$

where $F^{(0)}$ indicates the photon term of the field. Superposition of charges and currents can generate fields carrying or-

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bital angular momentum (OAM) of which one easily calculates the orbital angular momentum L with respect to a given point in spacetime [18, 19].

For a generic supertranslation given by an arbitrary combination of higher spherical harmonics, the static metric with a non-trivial asymptotic supertranslation field becomes [15]

$$ds^2 = -\frac{\left(1-\frac{M}{2\rho_s}\right)^2}{\left(1+\frac{M}{2\rho_s}\right)^2}dt^2 + \tag{4}$$
$$\left(1+\frac{M}{2\rho_s}\right)^4 \left(d\rho^2 + \left(((\rho-E)^2+U)\gamma_{z\bar{z}} + \right.\right.$$
$$\left.\left.(\rho-E)C_{z\bar{z}}\right) dz d\bar{z}\right)$$

where $C_{z\bar{z}}(\theta,\phi) \equiv -(2D_zD_{\bar{z}} - \gamma_{z\bar{z}}D^2)C$ are the components of the supertranslation field C, $C_{0,0}$ is the lowest spherical harmonic mode of C and the auxiliary quantities $U(\theta,\phi) \equiv \frac{1}{8}C_{z\bar{z}}C^{z\bar{z}}$, $E(\theta,\phi) \equiv \frac{1}{2}D^2C + C - C_{(0,0)}$, $t_s = t + C_{(0,0)}$ and $\rho_s = \sqrt{(\rho - C + C_{(0,0)})^2 + D_zCD^zC}$, with possible observable effects as to detect the supertranslation field one requires an experiment which takes in account the effects of the entire sphere around the BH. The simple bending of light of a distant star in a finite solid angular range outside of the supertranslation horizon is unaffected by the supertranslation field. To detect these deviations one would need experimental tests such as an array of rulers around the central object to deduce the integrated superrotation charge, e.g. possible deviations from closed null geodesics and strong lensing effects.

ORGANIZED STRUCTURES OF SOFT PHOTONS

In this section we will introduce in the context of black holes and the HPS proposal a possible novel aspect on the properties soft photon fields discussing a particular aspect of real photons, namely the spatial information of light associated to its orbital angular momentum. When currents generating structured EM fields and real photons carrying well-defined values of OAM, fall into a BH and produce supertranslation states and different spacetimes according to the arbitrary combination of higher spherical harmonics so far generated, we expect to find local spatial structures also in the soft photon hair implant.

Electromagnetic fields, but for few exceptions, like a single classical plane wave, transport angular momentum. IN fact, differently from plane waves, which carry linear momentum with no azimuthal component, vortex beams can have nonzero total azimuthal momentum p_ϕ but an identically zero azimuthal component of linear momentum [20].

These fields, generated in a finite (source) volume V' containing an arbitrary distribution of electric (magnetic) charge and current densities $\rho(t',\mathbf{x}')$ and $\mathbf{j}(t',\mathbf{x}')$ are those associated to real photons. Superpositions of multipolar sources, generated by currents involving electric charged particles, generate EM fields that carry precise values of orbital angular momentum [18, 21].

The conserved quantity EM angular momentum $\mathbf{J}$ is known to generate fields with well–defined spatial structures of intensity and phase. The total angular momentum $\mathbf{J}$ of a particle can be decomposed in spin angular momentum (SAM), $\mathbf{S}$, and orbital angular momentum (OAM), $\mathbf{L}$. Whilst the splitting of the total angular momentum in two observables, $J = L + S$, is valid for massive particles, it has not always a precise physical meaning for the photon; in any case SAM and OAM are auxiliary concepts that describe the photon wavefunction properties with respect to rotations and OAM l gives the order of the spherical functions involved in the radiation field together with the parity of the photon state, following precise rules for the composition of L and S from the classical field formulation down to the single photon level, where intensity corresponds to the probability of generating a photon in a specific region of spacetime with precise properties [22–24]. EM–OAM is currently being widely applied in modern physics and technology such as quantum and classical communications [25–27], astrophysics [28], nanotechnology and many other research fields [29].

The most known example of fields carrying precise SAM and OAM states are those described by Laguerre-Gaussian (LG) beams, also known as LG modes. Their importance is that they provide an orthogonal fundamental basis to expand and describe any OAM field. These are cylindrically symmetric structured EM beams that carry $l\hbar$ OAM per photon relative to its symmetry axis with EM field amplitude of a LG beam is in cylindrical coordinates (r,z,φ)

$$u_{lm}^{L-G}(r,\varphi,z) = \sqrt{\frac{2m!}{\pi(m+l)!}}\frac{1}{w(z)}\left[\frac{r\sqrt{2}}{w(z)}\right]^l \tag{5}$$
$$L_m^l\left[\frac{2r^2}{w^2(z)}\right]\exp\left[\frac{-r^2}{w(z)^2}\right]\exp\left[\frac{-\mathrm{i}kr^2}{2R(z)}\right]$$
$$\exp\left[-\mathrm{i}(2m+l+1)\arctan\left(\frac{z}{z_R}\right)\right]e^{-\mathrm{i}l\varphi}$$

where z_R is the Rayleigh range of the beam, $w(z)$ the beam waist, $L_m^l(x)$ an associated Laguerre polynomial and $R(z)$ the radius of curvature. The azimuthal and radial indices l and m describe the OAM of the beam and the number of radial nodes of the associated intensity profile, respectively [19].

Any generic field f can be expressed in terms of multipolar fields expanded in spherical harmonics [30],

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{l,m}Y_{l,m}(\theta,\phi) \tag{6}$$

and the multipolar content can be expressed in terms of paraxial fields, with properties that remain valid down to the single photon level. Thus, also the EM fields carrying OAM and the so-called vortex beams that usually are written as superpositions of LG modes can be expanded in spherical harmonics as superpositions of multipolar fields that depend only on the wavelength. A simple LG beam with $m = 0$ has vector potential A that, in terms of multipolar superpositions of circularly polarized plane waves becomes,

$$\mathbf{A} = 2\pi \sum_{j=\|l+p\|}^{\infty} i^j (2j+1)^{1/2} C_{jlp} \left[\mathbf{A}^{(\mathrm{m})}_{j(l+p)} + i\mathbf{A}^{(\mathrm{e})}_{j(l+p)} \right] \quad (7)$$

where

$$C_{jlp} = \sqrt{\frac{(j+p)!(j-p)!(j+l+p)!(j-l-p)!}{(j-\frac{|l|-|l+2p|}{2})!}} \quad (8)$$
$$k(-)^{j+\frac{l+|l+2p|}{2}} 2^{|l|/2+1} w_0^{-2j-1+|l+2p|} L^{|l+2p|}_{j-\frac{|l|+|l+2p|}{2}} \left((w_0/k)^2\right),$$

where the vector potential $\mathbf{A}$ is decomposed in the magnetic (m) and electric (e) terms, k is the wavenumber, w_0 the waist of the associated Gaussian beam and L^p_l is the associated Laguerre polynomial. We note that the field is composed of multipolar solutions with different total angular momentum, but with the same projection of the angular momentum in the direction z, which is $+1$ or -1 depending on the polarization of the field and $(j, m = l+p)$, p represents the circular polarization operator and k the wavevector. Fields with OAM l and polarization p such as LG fields are decomposed into multipolar modes with a fixed component $m = l+p$ along z of the angular momentum [31].

Suitable superpositions of charged currents j^*_ν can generate multipolar spatially structured EM fields carrying OAM and vorticity that are characterized by precise phase structures in space [18, 32]. Recalling Ref. [1], let us consider such a set of falling currents on a static BH, as in Eq. 1. We find numerically and from simple analytic considerations, that these charged currents induce spatial structures also on the soft photon field in a neighborhood of the BH event horizon.

In the numerical simulations, for the sake of simplicity, we consider a distribution of dipolar currents in a set of azimuthally-dephased radiating dipoles on a circle with radius r_c and center O_c that produce a radiating EM field with a well-defined OAM value l. The circle is orthogonal to the direction $\mathbf{r}_\perp$ connecting 0_c and the center of the BH. We calculate the spatial distribution of the soft photon implant on the event horizon of the BH from Eqs. 2 and 3 in a neighborhood of the point O. We assume that the BH radius, r^*, is $r^* >> r_c$ and approximate the metrics near the horizon $r \sim 2MG$ in a small angular region $\theta = 0$, where a small neighborhood of an observer can be described in first approximation as a Rindler coordinate system and a set of observers on the direction $r_\perp$ identify, at a first approximation, a class of Rindler observers. For these observers and with the approximations here adopted, the radiating field from the currents is not significantly affected by the free-fall radiating process [33, 34] that we neglect in first approximation.

More in detail, in the numerical simulations, we considered the formation of a soft photon hair implant from four falling dipolar currents with linear polarization across the $x^1 = x-$axis of the new accelerated local frame (x^i) that generate an $l = 1$ EM vortex of real photons. The four dipoles, distributed around a circle with radius $r_c = \lambda/10$ have a spatial extension much smaller than the wavelength emitted, $d = \lambda/100$ and are azimuthally dephased of $\delta\varphi = \pi/4$ to generate an EM field with $l = 1$ OAM value calculated with respect to the center of symmetry of the dipole array. The point of observation is located at 300 wavelengths distance in the direction $\mathbf{r}_\perp$. We calculate each photon hair implant and draw the "intensity", which has the meaning of probability of implanting/detecting a soft photon in a specific point of the spacetime and phase profile in the neighborhood of the event horizon where the field has not null intensity yet, neglecting the backreaction of the shell and of the EM field on the geometry. All numerical simulations were performed with Matlab and NEC (Numerical Electromagnetics Code) [35–37] that morphed the standard NEC input and output into the correct shape of the static spacetime geometry here considered through a geometrical optics transformation derived from analog gravity [38, 39].

Figure 1 displays the probability of emission of a photon in a superposition of OAM states and the spatial phase distribution of the EM field generated by the four dipolar currents for an observer located on the radial axis with direction $\mathbf{r}_\perp$ at a distance $D = 300\lambda$ in a window of $25\lambda \times 25\lambda$. The corresponding probability of associating a soft photon state to any point in that window is numerically calculated directly from the distribution of the multipolar radiation fields. On the right side of Fig. 1 are reported the corresponding phase maps of the soft photon hair implant are recovered from the multipolar distribution of the real and of the soft photon field emitted and induced on the BH horizon by the currents, respectively.

Owing to the realistic simulation of the four dipoles, the spiral spectrum of the electric field component along x generated by the dipoles clearly shows a complex structure of OAM components where the dominant term is $l = 1$, as shown in Fig. 2, where are reported the spiral spectra [40] of the EM field emitted by the dipoles and that of the soft photon field induced by them. The spiral spectrum shows the distribution of OAM modes that build up the spatial phase information of the field. Interestingly, the numerical results show that the spatial information of the EM field generated by the four falling dipoles described by the spiral spectrum peaked at the $l = +1$ OAM value, is similar to that of the soft photon hair implant on the BH horizon but for a translation and small modifications, showing a dominant $l = +2$ OAM mode.

Naively, this effect of spiral spectrum shift finds an interesting parallelism with the optical experiments involving OAM beams: the soft photon hair implant F_{soft} behaves as if it were the product of the initial radiation field of the current j^*_ν after crossing a single-bifurcation fork hologram that shifts the spiral spectrum of one OAM unit [19]. Because of this, the "hologram" engraved on the event horizon [1] acquires this spatial structure described by the spiral spectrum of the distribution of the soft photon field implants, involving larger and larger pixel sets on the horizon with the increase of the complexity of the fields as the OAM value increases.

From an analytic point of view, the shift of the spiral spectrum can be explained as follows. Consider a set of charges

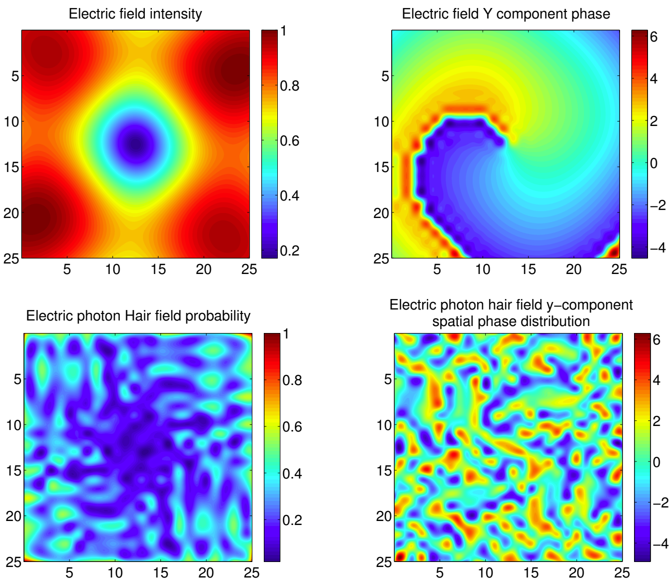

Figure 1. Normalized electric field (E) intensity distribution and phase generated by a distribution of four dipoles oriented across the y-axis, expressed in λ units (upper panel). The four dipoles are much smaller than the wavelength and are distributed on a circle with radius $\lambda/10$. The dipoles are azimuthally dephased to generate a phase pattern of an $l = 1$ vortex. The spiral pattern of the phase indicates that the EM waves, observed at that distance have a dominant spherical wavefront, as expected. In the lower panel are reported the probability of associating a soft photon hair in a region of spacetime and spatial phase information of the soft photon field generated by the currents.

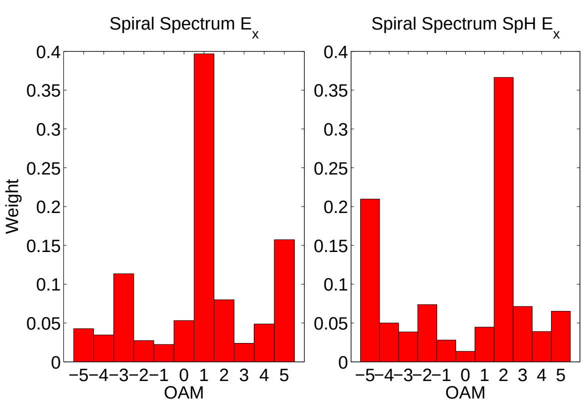

Figure 2. Left: The spiral spectrum of the radiation field emitted by the dipolar currents shows as expected a dominant contribution of the $l = 1$ vorticity of the source, as expected. The additional spectral terms present indicate that the field produced by these four currents is not a pure $l = 1$ Laguerre-Gaussian beam. Right: the soft photon field induced by the currents shows a spiral spectrum peaked on a dominant term that is larger of one unity with respect to the real photon field, namely, $l = 2$.

that radiate an EM field described by a single ideal LG mode with topological charge l as in Eq. 5. This carries a finite value of polarization and OAM, with a precise spatial structure in intensity and phase. The decomposition in spherical modes of this field puts in immediate evidence the angular momentum properties through the eigenfunctions of the angular momentum operators, being Y_{jm} the eigenfunction of the operators J^2 and J_z, where z in this case is $r_\perp$. By applying Eq. 3 and the derivation rules in the Riemann sphere [41, 42], after some algebra, being

$$\frac{d}{dx}L_k^l = -L_{k-1}^{l+1} \tag{9}$$

where l and k are two arbitrary indexes, it is clear that the resulting soft photon field presents terms with increased OAM value of exactly one unit, $l+1$, with respect to the spherical harmonic distribution of the field emitted by the charges of Eqs. 7 and 9, confirming our numerical findings in the spiral spectra.

All information regarding the spatial distribution of currents seems to be encoded on the event horizon and written in the modified metric of Eq. 5. In this way we provide a novel physical interpretation to the *lush head* of “soft hair” of BHs, showing their still hidden potentialities of having not only a modified spacetime but also an organized spatial structure on the event horizon with well–defined OAM states there encoded. BH soft photon hair implants seem to inherit similar local spatial properties of the “regular” photons emitted by the currents.

Conclusions. Following HPS, Black holes encode the spatial information of infalling currents generating OAM beams in twisted hair implants that present a local a spatial structure on the event horizon. All information is encoded in the new spacetime transformed by supertranslation fields. We can also use the symmetries of EM fields to locally spatially structure these implants and encode structured and densely organized information on the event horizon through suitable coherent superpositions of spin and angular momentum states of soft photons. Thus, BHs can have “curly”, twisted, structured soft–hair implants where information can be written onto the horizon in an organized way. This procedure can be extended to more complex field configurations expressed in terms of conserved quantities and symmetries inducing additional supertranslation fields. Interestingly, a coherent and organized superposition of such quantum states might correspond to an organized string of quantum bits of information encoded into the event horizon as in an holographic projection as soft hair has a natural description as quantum pixels in a holographic plate.

Multi-dimensional strings of qbits carrying ordered and structured information can excite pixels in an ordered way, creating structures on the event horizon can be built from the invariants and symmetries of the gravitational and EM fields, in a sort of quantum alphabet. This might represent a novel way to input controlled information on BHs that will evolve according to their computational complexity. Complexity can be encoded through a coherent [43] superposition and combination of states as a string of a quantum computer [44] where, possibly, a non-thermal spectrum characterizes the passage from macroscopic to microscopic information.

This information is thought to be processed by the BH interior as in a quantum computer obeying the rules of quantum computation and complexity; BHs are supposed to be the fastest computers to process complexity in the most optimal way [45–47], encoding information in the structure of spacetime modifying it with the supertranslation fields, with the already discussed observational effects. Any BH would resemble a quantum computer with high coherence and this information encoded so far could be located by the soft–hair implant in a neighborhood of the event horizon and in the modified spacetime. In any case, also by using a quantum field theory approach, because of Shaw theorem that connects information and volume in the phases, any physical system must belong to one of these three classes: systems that preserve information, information dissipating systems and polynomial/exponential information amplifiers [48] and for BHs this seems to be still an open debate.

ACKNOWLEDGMENTS

F.T. gratefully acknowledges the financial support from ZKM and MSC-bw. M.D.L. is supported by ERC Synergy Grant "BlackHoleCam" Imaging the Event Horizon of Black Holes awarded by the ERC in 2013 (Grant No. 610058).

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