Quantum Computational Complexity in Curved Spacetime

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***Abstract*—In this paper we examine how computational complexity analysis of quantum algorithms may be compromised when implicit assumptions of flat spacetime are violated. In particular, we show that in curved spacetime, i.e., all practical contexts, the complexity of standard formulations of Grover's algorithm (and other iterative quantum algorithms) may reduce to that of classical alternatives. In addition, we discuss the implications of these results for quantum error correction and fault tolerant quantum computation.**

I. Introduction

The principal goal of quantum information science is to harness quantum phenomena to improve the performance of information processing systems [2], [3]. There has been substantial progress in the development of quantum communication, computation, and sensing devices, and some have even reached a level of technological maturity sufficient to find commercial applications. However, optimism about their potential advantage over classical alternatives is invariably founded on theoretical analysis that implicitly assumes a flat spacetime operating environment, i.e., that their operations will not be influenced by local gravitation.

Gravity affects quantum information [4], especially in the case of spin-based quantum hardware. Indeed, spin is an intrinsically relativistic concept that can only be formally described within the context of relativistic quantum field theory [5]. In this context, spin is simply defined as the degrees of freedom of a quantum state that transform in a non-trivial manner by a Poincare transformation. This group-theoretic definition embodies the invariance of physical laws under special relativity[1]. Because spin emerges from a relativistic concept, any spin-based representation of quantum information is subject to a direct coupling to classical gravitational fields as described by Einstein's general theory of gravity; therefore, any complete analysis of the time evolution of quantum information must consider the effects of gravitation.

The interaction between spin and gravity is expressed through the *Wigner angle* Ω. As a spin-based qubit moves in curved spacetime, its spin will rotate by $\Omega/2$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} \cos\frac{\Omega}{2} & \sin\frac{\Omega}{2} \\ -\sin\frac{\Omega}{2} & \cos\frac{\Omega}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{1}$$

where $|0\rangle$ and $|1\rangle$ are spin eigenstates, and Ω will depend in a non-trivial manner on the gravitational field and the path followed by the qubit[2]. As a consequence, *gravitation rotates spin-based qubits in a non-trivial manner*. This is a critical fact because spin-based qubits in a quantum computer will clearly interact with Earth's gravitational field and their states will therefore change with time even if the computer is in the idle state. In other words, qubit-encoded quantum information will invariably suffer gravitation-induced drift determined by its environment of operation.

It can be shown that *uncorrected* errors can undermine the asymptotic computational complexity advantage of most (if not all) quantum algorithms with respect to classical alternatives [8]. In this paper we examine the perturbative effects of gravitation on the evolution of quantum systems and their implications for quantum computation. Although the effects are small, they cannot generally be mitigated by the use of conventional quantum error correcting codes and can therefore impose limits on the scalability of quantum algorithms.

We begin with an examination of Wigner rotations on the amplitudes of states in the computational basis. We then consider orbiting qubits in Schwarzschild spacetime (i.e. in the static and isotropic curved spacetime produced by a spherically symmetric black hole). From this we examine the effects of gravitational perturbation on Grover's quantum search algorithm and Shor's quantum factoring algorithm.

The principal contribution of this article lies in the last section, which discusses the implications of gravity to quantum error correction and fault tolerant quantum computation. To this end, we show that gravitation induces perturbations that cannot generally be treated using models which assume locally independent errors/noise. Finally, we discuss conclusions that can be drawn and directions for future work.

[1] Note that this definition avoids the problematic conception of spin as an "intrinsic angular momentum". Indeed, for the case of photons moving at the speed of light it is impossible to find an inertial frame where the photon is at rest so the photon's intrinsic angular momentum is undefined.

[2] It is important to remark that this is a general relativistic effect that emerges as a consequence of the qubit moving through a curved manifold. That is, the Wigner angle is strictly zero in Newtonian gravitation.

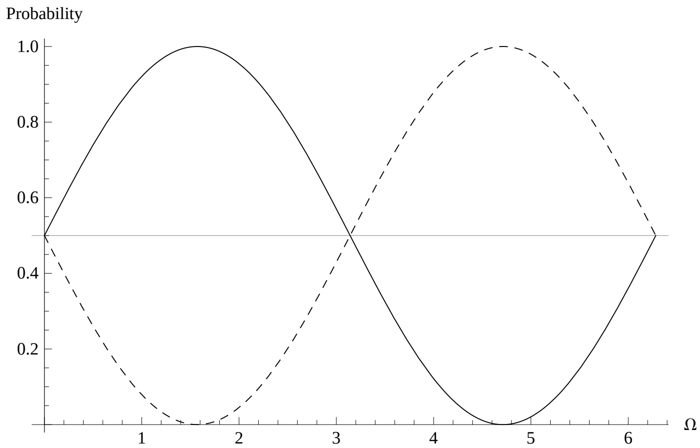

Fig. 1. Drifting of the probability of measuring “1” (solid line) and “0” (dashed line) for a 1 qubit state in the uniform superposition in the presence of a gravitational field described by the Wigner rotation angle $\Omega(t)$.

II. Gravitational Drifting of Qubit States

We begin by considering a single qubit in the uniform superposition, i.e., it will be measured as “0” or “1” with equal probability:

$$|\psi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \tag{2}$$

In the presence of gravity this state will “drift” into:

$$|\psi_1\rangle \quad \longrightarrow \quad \frac{\cos\frac{\Omega}{2} + \sin\frac{\Omega}{2}}{\sqrt{2}}|0\rangle + \frac{\cos\frac{\Omega}{2} - \sin\frac{\Omega}{2}}{\sqrt{2}}|1\rangle \tag{3}$$

The behavior of the probability of measuring “1” and “0” with respect to Ω is shown if Figure 1. Notice that the expressions for the probabilities have a 2π period with the value of Ω determining the relative probability of measuring “1” or “0” (i.e., Ω has the effect of amplifying the probability of measuring one state relative to the other), and of course the sum of the probabilities is unity.

Note also that, in general, the Wigner rotation angle is a function of time, $\Omega = \Omega(t)$, which depends on the existing gravitational fields and the trajectory of the qubit. For example, consider the static and isotropic gravitational field produced by a spherically symmetric body of mass M described by the Schwarzschild metric [4], [6], [7]. If the qubit is following a circular orbit of radius r then the Wigner angle per orbital period is given by:

$$\Omega = 2\pi\sqrt{f}\left(1 - \frac{Kr_s}{2rf}\frac{1}{K+\sqrt{f}}\right) - 2\pi \tag{4}$$

where:

$$K \equiv \frac{1-\frac{r_s}{r}}{\sqrt{1-\frac{3r_s}{2r}}} \qquad f \equiv 1 - \frac{r_s}{r} \qquad r_s \equiv 2M \tag{5}$$

all given in natural units ($G = 1, c = 1$). That is, once a qubit has completed an entire circular orbit in Schwarzschild spacetime, Ω is the total rotation of the spin solely due to the presence of a gravitational field. This is a purely relativistic

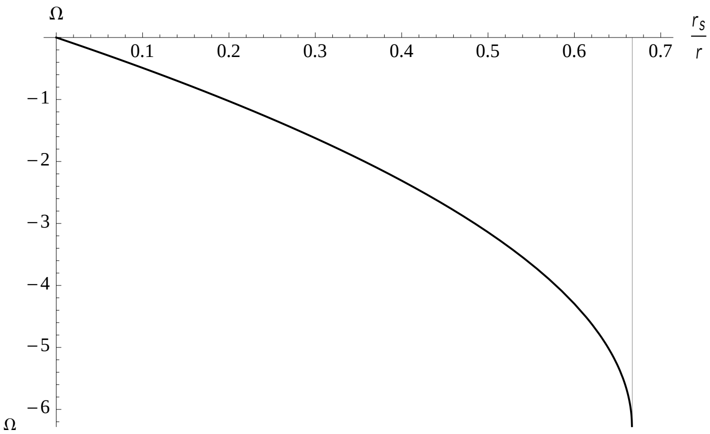

Fig. 2. Wigner angle Ω for a qubit that has completed a circular orbit of radius r in the Schwarzschild spacetime produced by an object of mass $r_s/2$.

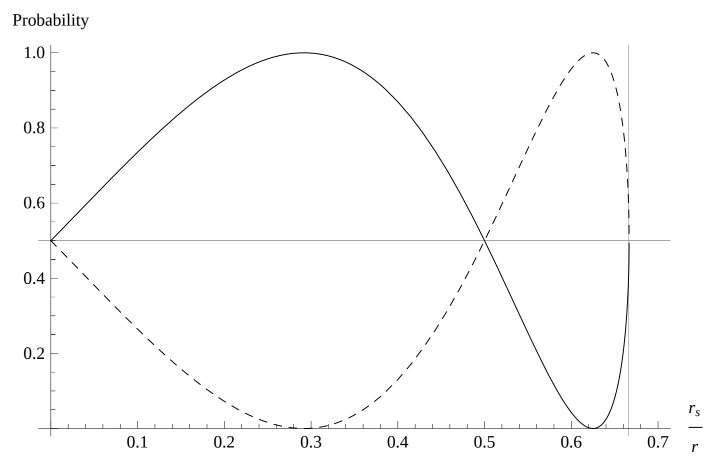

Fig. 3. Drifting of the probability of measuring “1” (solid line) and “0” (dashed line) for a 1 qubit state in the uniform superposition after completion of a circular orbit of radius r in the Schwarzschild spacetime produced by an object of mass $r_s/2$.

effect due to the interaction of a spin-$\frac{1}{2}$ quantum field with a classical gravitational field.

Figure 2 shows the behavior of Ω with respect to r_s/r for a qubit that has completed a circular orbit in Schwarzschild spacetime. On the other hand, figure 3 shows the behavior of the probability of measuring “1” and “0” with respect to r_s/r for a qubit after completion of a single orbit in Schwarzschild spacetime.

In particular, note that if the Wigner rotation angle is very small, $\Omega \ll 1$, then:

$$\begin{aligned} P_0 &= |\langle 0|\psi_1\rangle|^2 \approx \frac{1}{2} + \frac{\Omega}{2} \\ P_1 &= |\langle 1|\psi_1\rangle|^2 \approx \frac{1}{2} - \frac{\Omega}{2} \end{aligned} \tag{6}$$

where P_0 and P_1 are the probabilities of measuring “0” and “1”, respectively. Thus, the measurement error in weak gravitational fields is approximately given by $\Omega/2$.

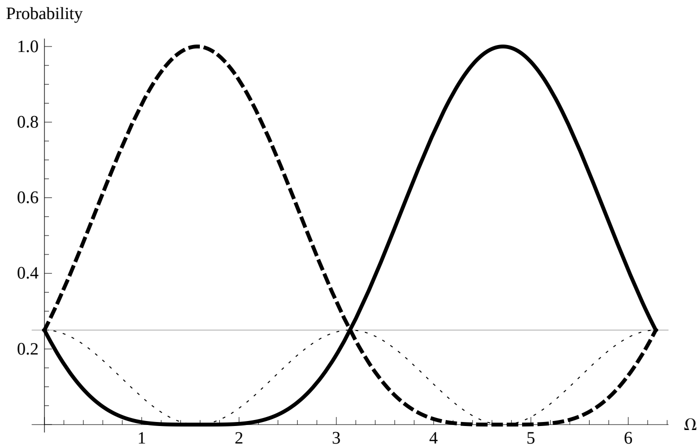

Fig. 4. Drifting of the probability of measuring "00" (solid line), "01" and "10" (dotted line), and "11" (dashed line) for a 2 qubit state in the uniform superposition in the presence of a gravitational field described by the Wigner rotation angle $\Omega(t)$.

Consider now a 2-qubit state uniform superposition:

$$|\psi_2\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \tag{7}$$

which under the presence of gravity is transformed into:

$$\begin{aligned} |\psi_2\rangle \quad \rightarrow \quad & \frac{\left(\cos\frac{\Omega}{2} - \sin\frac{\Omega}{2}\right)^2}{2}|00\rangle + \frac{\cos^2\frac{\Omega}{2} - \sin^2\frac{\Omega}{2}}{2}|01\rangle \\ & + \frac{\cos^2\frac{\Omega}{2} - \sin^2\frac{\Omega}{2}}{2}|10\rangle + \frac{\left(\cos\frac{\Omega}{2} + \sin\frac{\Omega}{2}\right)^2}{2}|11\rangle \end{aligned} \tag{8}$$

where we have assumed that the two qubits have negligible spatial separation so that their Wigner angles are identical.

The drifting of the 2-qubit state in the uniform superposition with respect to the Wigner angle is shown in Figure 4. Note that the probability of measuring "00" and "11" is amplified and/or reduced in a periodic fashion. On the other hand, the probability of measuring "01" or "10" is always smaller than the original $1/4$ probability of the uniform superposition. This suggests the intriguing possibility that drifting qubits could be used to measure gravitational fields. This is certainly the case if the qubits are near a black hole and the Wigner angle is large. Indeed, if $\Omega \approx \pi/2$ then the measurement of the "11" qubit state will be very close to 1.

III. 2-Qubit Grover's Algorithm

Grover's algorithm is the quintessential example of an *amplitude amplification algorithm* [2], [3]. Such algorithms consist of a series of qubit operations that transforms a quantum state in such a way that, after a certain number of iterations, measurements of some states are much more likely than other states. In other words, these quantum algorithms amplify the amplitudes of solution states so that they are more likely to be measured.

Grover's algorithm is designed to identify an item in an unstructured (e.g., unsorted) database. In the case of n qubits spanning a superposition of $N = 2^n$ states, the probability of measuring a specific item is $1/N$. However, after $\sqrt{N}$ sequential applications of the *Grover operator* the item becomes the most likely to be measured (e.g., with probability $\approx 2/3$). Therefore, the complexity of Grover's algorithm is $\mathcal{O}(\sqrt{N})$, as compared to the $\mathcal{O}(N)$ complexity of classical *brute force* exhaustive search [2], [3].

It is instructive to examine the behavior of Grover's algorithm in the presence of gravity for a superposition of 2-qubits because the correct solution is achieved with probability equal to unity after a single iteration of the Grover operator [3].

For the n-qubit case, Grover's algorithm starts in the n-qubit uniform superposition:

$$|\psi_n\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \tag{9}$$

and the n-qubit Grover operator is given by:

$$\hat{G}_n = \hat{M}_n \times \hat{O}_n \tag{10}$$

where $\hat{M}_n$ is the n-qubit *inversion around the mean* and $\hat{O}_n$ is the n-qubit *oracle* [3]. For the 2-qubit case, the inversion operator is given by:

$$\hat{M}_2 = -\frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \tag{11}$$

If the "00" element represents the solution state then the 2-qubit oracle takes the form:

$$\hat{O}_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

so that the oracle "marks" the solution by changing the phase of the corresponding state. In the absence of gravity the state after a single Grover iteration is given by:

$$\hat{G}_2|\psi_2\rangle = |00\rangle \tag{13}$$

and, therefore, the probability of measuring "00" is exactly 1. By contrast, a classical brute force search may require 4 comparison operations, one for each element in the dataset.

In the presence of gravity, however, the operator that must be applied to the uniform state is approximated by:

$$\hat{D}_2 \times \hat{G}_2 \times \hat{D}_2|\psi_2\rangle \tag{14}$$

where $\hat{D}_2$ is the 2-qubit Wigner rotation. That is, the quantum register is initialized to the uniform superposition, then incurs gravitational drift before application of the Grover operator $\hat{G}_2$, and then incurs additional drift prior to the measurement. Therefore, the probability P_{ab} of measuring the state "ab" is given by:

$$P_{00} = \cos^4\left(\frac{\Omega}{2}\right) \qquad P_{11} = \sin^4\left(\frac{\Omega}{2}\right) \qquad P_{01} = P_{10} = \frac{1}{4}\sin^2\Omega$$

where a small nonzero Wigner rotation Ω can be seen to reduce the probability of correctly measuring the desired state "00".

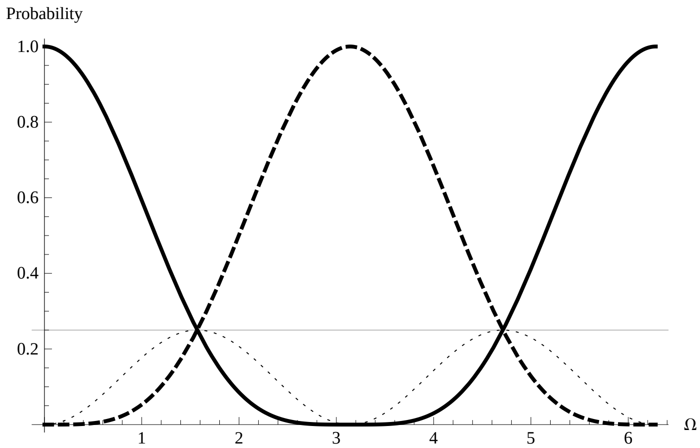

Fig. 5. Probability of measuring "00" (solid line), "01" and "10" (dotted line), and "11" (dashed line) for a 2 qubit state after a single iteration of the Grover operator in the presence of a gravitational field described by the Wigner rotation angle $\Omega(t)$.

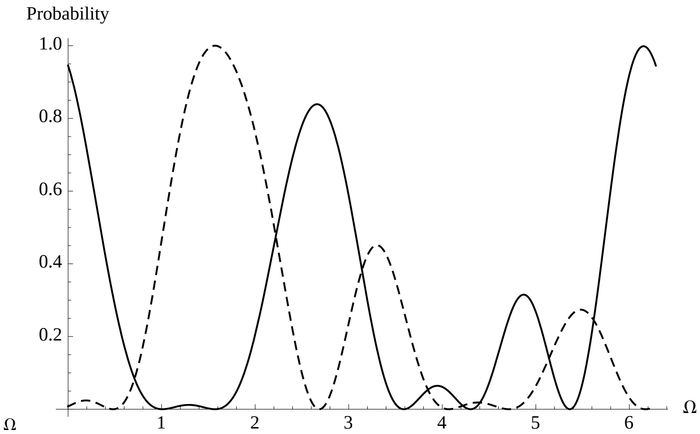

Fig. 6. Probability of measuring "000" (solid line) and "111" (dashed line) for a 3 qubit state after two iterations of the Grover operator in the presence of a gravitational field described by the Wigner rotation angle $\Omega(t)$.

The effect of Ω on the respective probabilities of measuring "00", "01" and "10", and "11" is shown in Figure 5, where Ω represents the Wigner angle accrued during the time between two consecutive operations. That is, if the quantum computer operates at 1 Hz then the Wigner angle is the rotation due to the drifting of the qubit in a gravitational field during the time span of 1 second.

The probability of success of Grover's algorithm diminishes in the presence of a gravitational field. In the case of a quantum computer near a black hole with $\frac{\pi}{2} < \Omega < \frac{3\pi}{2}$ the probability of success is less than $1/n$, i.e., worse than classical, and is near zero for $\Omega \approx \pi$. While Ω may be small near the surface of the earth, its impact on the probability of success for Grover's algorithm can become significant as the number of qubits is increased.

Furthermore, it is worth to remark that the Wigner rotation is an unitary operation, and as a consequence, in principle one could always find the inverse transformation. However, in practice it may be an unfeasible task to find and implement the exact inverse Wigner rotation at each computational step.

IV. 3-Qubit Grover Algorithm

Consider now Grover's algorithm initialized with a 3-qubit state in the uniform superposition. In the absence of gravity the probability of measuring the correct solution (say "000") after two iterations is 0.945313.

In analogy to the 2-qubit case, the Grover operator in the presence of gravity is given by:

$$\hat{D}_3 \times \hat{G}_3 \times \hat{D}_3 \times \hat{G}_3 \times \hat{D}_3|\psi_3\rangle \tag{15}$$

and the probability of measuring the right solution ("000") after two iterations is:

$$P_{000} = \left(\begin{array}{c} (66\sin(x) - 24\sin(3x) - 21\sin(7x) \\ +\sin(9x) + 66\cos(x) + 24\cos(3x) \\ +96\cos(5x) - 3\cos(7x) - 7\cos(9x)) \end{array} \right)^2 \frac{1}{32768} \tag{16}$$

where for notational simplicity we have defined:

$$x \equiv \frac{\Omega}{2} \tag{17}$$

The probability of measuring a non-solution state, e.g., "111", is given by:

$$P_{111} = \left(\begin{array}{c} (-18\sin(x) - 48\sin(3x) + 72\sin(5x) \\ -3\sin(7x) + 7\sin(9x) - 78\cos(x) \\ +48\cos(3x) + 24\cos(5x) + 21\cos(7x) \\ +\cos(9x)) \end{array} \right)^2 \frac{1}{32768} \tag{18}$$

Note that for small Wigner angles $x \ll 1$ we have:

$$\begin{aligned} P_{000} &\approx 0.945313 - 1.54688\ x \\ P_{111} &\approx 0.0078125 + 0.234375\ x \end{aligned} \tag{19}$$

Figure 6 shows the probability of measuring "000" (solid line) and "111" (dashed line) for a 3-qubit state after two iterations of the Grover operator in the presence of a gravitational field described by the Wigner rotation angle Ω. Once again, depending on the value of Ω, Grover's algorithm may not be able to provide the solution to the search problem with high probability.

It is also important to note that, in the case of two qubits, the probabilities P_{00} and P_{11} were the same, but shifted by π in the value of Ω (see Figure 5). On the other hand, for three qubits, the probabilities P_{000} and P_{111} are completely different functions of Ω. This suggests that, in general, we cannot expect that the influence of gravity on the Grover measurement probabilities will posses some easy to identify symmetry.

V. General Iterative Algorithm

Consider now a quantum computer running an algorithm that iteratively applies a quantum operation to an initial n-qubit quantum state represented by the density matrix $\rho^{(0)}$. We can examine the result of this computation in the noiseless (i.e., flat spacetime) and weak-field regimes.

A. Noiseless Operation

Suppose the quantum algorithm performs m sequential applications of a single multi-qubit gate $\hat{U}$. The system has n qubits that represent $N = 2^n$ possible states. In the absence of gravity, and ignoring all possible sources of noise and error, the state's density matrix after the first computational step is:

$$\rho^{(1)} = \hat{U}\rho^{(0)}\hat{U}^{\dagger} \tag{20}$$

and the state after m iterations is given by:

$$\rho^{(m)} = \hat{U}^{m}\rho^{(0)}\hat{U}^{\dagger m} \tag{21}$$

B. Weak Field Approximation

By contrast, the state after one iteration in the presence of a weak gravitational field is:

$$\rho^{(1)} = \hat{D}\hat{U}\rho^{(0)}\hat{U}^{\dagger}\hat{D}^{\dagger} \tag{22}$$

where $\hat{D}$ is the unitary operation that represents the n-qubit Wigner rotation between computational steps. After two iterations the state will be:

$$\rho^{(2)} = \hat{D}\hat{U}\hat{D}\hat{U}\rho^{(0)}\hat{U}^{\dagger}\hat{D}^{\dagger}\hat{U}^{\dagger}\hat{D}^{\dagger} \tag{23}$$

and so on. In this case the Wigner rotation can be written as:

$$\hat{D} = \bigotimes_{j=1}^{n} e^{i\hat{\boldsymbol{\Sigma}}_j\cdot\boldsymbol{\Omega}_j/2} \tag{24}$$

where $\boldsymbol{\Omega}_j$ is the vector of Wigner rotation angles for the j^{th} qubit and $\hat{\boldsymbol{\Sigma}}_j$ is the vector of spin rotation generators that acts on the j^{th} qubit. This expression reflects the fact that, in general, qubits will occupy different positions in spacetime, their spins may be aligned in different directions, and the quantization axes may be different for each of them. To simplify the analysis we assume not only that the gravitational field is weak but also that all qubits undergo the exact same Wigner rotation Ω such that:

$$|\Omega_j - \Omega| \ll 1 \quad \forall j \tag{25}$$

where in the weak field limit $\Omega \ll 1$.

In the weak field limit $\hat{D}$ can be expressed as:

$$\hat{D} \approx \mathbb{I} + \frac{i\Omega}{2}\sum_{j=1}^{n}\hat{\Sigma}_j + \mathcal{O}(\Omega^2) \tag{26}$$

where $\hat{\Sigma}_j$ is a unitary operator that acts in a non-trivial manner only on the j^{th}-qubit:

$$\hat{\Sigma}_j = \underbrace{\mathbb{I}\otimes\mathbb{I}\otimes\ldots\otimes\mathbb{I}}_{j-1}\otimes\sigma\otimes\underbrace{\mathbb{I}\otimes\mathbb{I}\otimes\ldots\otimes\mathbb{I}}_{n-j} \tag{27}$$

where the under-brace denotes the number of single-qubit identity operators in the tensor product and σ is the spin rotation generator.

As a consequence, at order $\mathcal{O}(\Omega)$ the state after one iteration can approximately be written as:

$$\begin{aligned}\rho^{(1)} &\approx \left(1 + \frac{i\Omega}{2}\sum_{j=1}^{n}\hat{\Sigma}_j\right)\hat{U}\rho^{(0)}\hat{U}^{\dagger}\left(1 - \frac{i\Omega}{2}\sum_{k=1}^{n}\hat{\Sigma}_k^{\dagger}\right) + \mathcal{O}(\Omega^2)\\ &\approx \hat{U}\rho^{(0)}\hat{U}^{\dagger} + \frac{i\Omega}{2}\sum_{j=1}^{n}\left(\hat{\Sigma}_j\hat{U}\rho^{(0)}\hat{U}^{\dagger} - \hat{U}\rho^{(0)}\hat{U}^{\dagger}\hat{\Sigma}_j^{\dagger}\right) + \mathcal{O}(\Omega^2)\end{aligned} \tag{28}$$

Defining Ξ as:

$$\begin{aligned}\Xi(\rho) &\equiv \frac{i}{2}\sum_{j=1}^{n}\left(\hat{\Sigma}_j\hat{U}\rho\hat{U}^{\dagger} - \hat{U}\rho\hat{U}^{\dagger}\hat{\Sigma}_j^{\dagger}\right)\\ &= \frac{i}{2}\sum_{j=1}^{n}\left[\hat{\Sigma}_j, U\rho\hat{U}^{\dagger}\right]\end{aligned} \tag{29}$$

allows $\rho^{(1)}$ at first order in Ω to be expressed as:

$$\rho^{(1)} \approx \hat{U}\rho^{(0)}\hat{U}^{\dagger} + \Omega\ \Xi(\rho^{(0)}) \tag{30}$$

Thus, the second iteration is:

$$\begin{aligned}\rho^{(2)} &\approx \hat{U}\rho^{(1)}\hat{U}^{\dagger} + \Omega\ \Xi(\rho^{(1)})\\ &\approx \hat{U}\hat{U}\rho^{(0)}\hat{U}^{\dagger}\hat{U}^{\dagger} + \Omega\hat{U}\ \Xi(\rho^{(0)})\hat{U}^{\dagger} + \Omega\ \Xi(\hat{U}\rho^{(0)}\hat{U}^{\dagger}) + \mathcal{O}\left(\Omega^2\right)\\ &\approx \hat{U}\hat{U}\rho^{(0)}\hat{U}^{\dagger}\hat{U}^{\dagger} + \Omega\left(\hat{U}\ \Xi(\rho^{(0)})\hat{U}^{\dagger} + \Xi(\hat{U}\rho^{(0)}\hat{U}^{\dagger})\right) + \mathcal{O}\left(\Omega^2\right)\end{aligned} \tag{31}$$

and the state after m iterations will be of the form:

$$\rho^{(m)} \approx \left(\hat{U}\right)^{m}\rho^{(0)}\left(\hat{U}^{\dagger}\right)^{m} + \Omega\underbrace{(\ldots)}_{m} + \mathcal{O}(\Omega^2) \tag{32}$$

where the underbrace denotes how many n-sum terms Ξ are contained inside the parentheses.

At this point it is clear that the presence of a gravitational field affects the probability of measuring the correct final state of the system. The exact deviation from the expected performance of the quantum algorithm will depend in a non-trivial manner on the type and number of quantum gates, data qubits, ancillary qubits, and error correction encoding used to implement the computation. While it is notationally straightforward to represent all gravitational effects in the form of a single unitary transformation, the actual identification of that transformation at each computational step may not be feasible.

If it is assumed/conjectured that the gravitational field of the operating environment can only be estimated with bounded accuracy then the probability of measuring the correct result after m iterations can be expressed as:

$$p \gtrsim (1 - n\epsilon)^{m} \approx 1 - nm\epsilon + \mathcal{O}(\epsilon^2) \tag{33}$$

where $\epsilon = \mathcal{O}(\Omega)$ and we have used a small limit approximation $n\epsilon \lesssim 1$. Under these conditions the error probability for the quantum algorithm is:

$$e \lesssim 1 - (1 - n\epsilon)^{m} \tag{34}$$

In principle it would seem that the algorithm could be iterated to increase the probability of success to satisfy any desired threshold. For example, after k runs of the entire algorithm, the bound for the probability of error could be reduced to:

$$e^k \lesssim (1-(1-n\epsilon)^m)^k \tag{35}$$

and k is chosen such that:

$$e^k \approx \delta \tag{36}$$

where δ is the maximum error probability desired for the computational process. That is, k is given by:

$$k \lesssim \frac{\log \delta}{\log\left(1-(1-n\epsilon)^m\right)} \tag{37}$$

In the asymptotic limit for large m we have:

$$k \lesssim -\log \delta \times \left(\frac{1}{1-n\epsilon}\right)^m = \mathcal{O}\left(\left(\frac{1}{1-n\epsilon}\right)^m\right) \tag{38}$$

and the *true complexity* of the algorithm (i.e. the total number of iterations necessary to complete the computational task with error probability δ) is:

$$\mathcal{O}\left(m \times k\right) \approx \mathcal{O}\left(m \times \left(\frac{1}{1-n\epsilon}\right)^m\right). \tag{39}$$

Thus, as the number of qubits n grows - approaching the value $1/\epsilon$ - the algorithmic complexity becomes exponential in the number of iterations m. In other words, the presence of gravity increases the computational dependence on m from linear to exponential. This motivates an examination of the specific consequences of this for practical n-quibit applications of Grover's algorithm and Shor's algorithm.

VI. n-Qubit Grover's Algorithm

In light of the analysis of the previous section, uncorrected gravitation-induced errors increase the complexity of Grover's algorithm to search an N-element database from $\mathcal{O}(\sqrt{N})$ to:

$$\mathcal{O}\left(\sqrt{N} \times \left(\frac{1}{1-n\epsilon}\right)^{\sqrt{N}}\right) \tag{40}$$

which is exponential in N for $n = \log N$ qubits.

There is of course a distinction to be made between behavior in the asymptotic limit of large N and the practically-observed scaling behavior of the algorithm for reasonable values of n and m and gravitation typical at the earth's surface. It is not possible to precisely quantify all possible runtime constants, but we can treat them as *all* being equal to 1 as a rough approximation. If we further assume that $n\epsilon \ll 1$ so that we can use a Taylor series to estimate the value of k, then for finite N we obtain:

$$k \approx \left(\frac{1}{1-n\epsilon}\right)^{\sqrt{N}} \approx 1 + n\epsilon\sqrt{N} \tag{41}$$

Under these assumptions we may estimate that the number of iterations necessary for Grover's algorithm is given by:

$$\mathcal{O}\left(\sqrt{N} + n\epsilon N\right) = \mathcal{O}\left(nN\right) = \mathcal{O}\left(N \log N\right) \tag{42}$$

which is worse than classical. However, $n\epsilon N$ only becomes significant when:

$$n\epsilon N \approx \sqrt{N} \tag{43}$$

which implies:

$$\sqrt{N}\log N \approx \frac{1}{\epsilon} \tag{44}$$

Therefore, if the quantum processor operates at 1 Hz on the surface of Earth, then $\epsilon \approx \Omega \approx 10^{-14}$, which implies that Grover's algorithm could outperform classical brute force for datasets of size up to $N \approx 2^{80}$. This breakeven analysis is of course very crude, but it does suggest the possibility of a practical performance advantage of Grover's algorithm over classical despite what is implied by our asymptotic analysis. On the other hand, a slightly more complete and/or detailed analysis may yield an opposite conclusion.

VII. Shor's Algorithm

For completeness we now consider the other major quantum algorithm, Shor's algorithm, which can factorize an n-bit co-prime number using $\mathcal{O}(\log N)$ quantum computational steps with probability close to one [2]. In the presence of gravity, however, the complexity becomes:

$$\mathcal{O}\left(\log N \times \left(\frac{1}{1-\epsilon\log N}\right)^{\log N}\right) \tag{45}$$

or, equivalently, in terms of number of qubits:

$$\mathcal{O}\left(n \times \left(\frac{1}{1-n\epsilon}\right)^{n}\right) \tag{46}$$

which is exponential in n.

Clearly, Shor's algorithm is more resilient to the effects of gravitation than Grover's algorithm. The reason is that Shor's algorithm has linear dependency on the number of qubits while the dependency of Grover's algorithm is exponential.

VIII. Implications for Quantum Error Correction

The magnitude of gravitation-induced errors clearly can be reduced but will never be zero, so the greater challenge for reliable, scalable quantum computation is to limit their *cumulative* effect.

The cumulative effect of qubit errors induced by gravity is somewhat similar to the case of GPS. Indeed, GPS satellites can maintain bounded-error state estimates in a defined global coordinate frame by measuring and correcting with respect to fixed anchor points which define that coordinate frame. Said another way, the GPS problem admits a Newtonian formulation wherein non-Newtonian sources of error can be corrected to arbitrary precision by a measurement-and-correction process with respect to an absolute coordinate frame. The quantum computation problem, by contrast, can only be formulated to admit an analogous type of "absolute" measurement-and-correction of qubits in the limit of classical information, i.e., bits.

Fundamentally, quantum error correction (QEC) uses redundancy to reduce the magnitude and/or probability of error associated with the result of a computational step [2], [9], [10]. For example, if a computational process is known to produce a correct solution with probability p then the probability of not obtaining a solution from k *independent* executions of the process is reduced to $(1-p)^k$, i.e., the probability of failure diminishes exponentially with the degree of redundancy k. This is why error correction (redundancy) can be applied effectively with minimal effect on the computational complexity of the overall computational process [11], [12], [13], [14], [15], [16], [17].

While some computational processes may produce results that can be unambiguously recognized as either "correct" or "incorrect", the more general problem for error correction requires a determination of what constitutes the "correct" result based on k executions of the process. This determination may be made based on a plurality consensus, weighted average, or any of a multitude of other functions of the k results. Regardless of the specifics, the efficacy of error correction is premised on an assumption that redundant executions provide independent samples from a stationary error process. The errors introduced by gravitation clearly do not satisfy this assumption because each of the k executions will be transformed similarly, i.e., their errors will not be statistically independent, during the time interval of the overall computation.

Gravitational interactions imply non-stationary error processes at all levels of analysis for a given quantum computing system because all components of the system are transformed by the same field, i.e., the net error associated with the system cannot be recursively defined as a linear combination of independent errors from a set of spatially separated subsystems. This would seem to violate an assumption critical for the filtration exploited by all lightweight (polylog overhead) recursive quantum error correction protocols, thus limiting the applicability of QEC. In particular, the celebrated *quantum threshold theorem* for scalable quantum computation relies on assumptions that are violated by gravitation[11], [12], [18]. More specifically, the following is a necessary assumption for fault-tolerance according to the quantum threshold theorem [1]:

> *Faults that act collectively on many qubits are highly suppressed in probability or in amplitude depending on the noise model. Furthermore, the noise strength, ϵ, must be a sufficiently small constant that is independent of the size of the computation.*

The critical element of this assumption is that the noise strength must be sufficiently small that it can be treated analytically as *independent of the size of the computation*. This cannot be assumed in the case of gravitation because it acts coherently on all components of the system throughout the duration of its computation. In other words, the effective noise strength from gravitation increases (integrates) with the size of the computation both in terms of the number of system components involved in the computation and in terms of the temporal length of the computation.

In summary, while the effects of gravitation may be much smaller than those of more mundane sources of error for short-duration computations, errors due to gravitation will asymptotically dominate with time. This implies that many computational complexity results for quantum algorithms technically require a flat-spacetime assumption.

IX. Conclusions

In this paper, we have examined the impact of gravitation on the maintenance and processing of quantum information. The principal conclusion is that uncorrected gravitation-induced errors undermine the theoretical computational complexity advantages of most quantum algorithms, e.g., Grover's and Shor's, with respect to classical alternatives. However, we have also argued that the effective scaling of quantum algorithms in the regime of practical computing problems may be considerably better than what is suggested by asymptotic analysis.

We have discussed the limitations of conventional quantum error correction to mitigate errors introduced when an entire quantum computing system is continuously transformed by a gravitational field. This motivates the need for improved methods to measure this field for purposes of mitigation. To this end we have suggested that instead of viewing the gravitational drift of qubits as an "error" process that it can equally be viewed as a means for measuring the influence of gravitation [4], [19], [20], [21]. Thus, a quantum computer could be augmented with sets of "reference qubits" to determine the inverse transformations necessary to reduce errors during computation. This represents a direction for future research.

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