Quon 3D language for quantum information

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We present a 3D topological picture-language for quantum information. Our approach combines charged excitations carried by strings, with topological properties that arise from embedding the strings in the interior of a 3D manifold with boundary. A quon is a composite that acts as a particle. Specifically, a quon is a hemisphere containing a neutral pair of open strings with opposite charge. We interpret multiquons and their transformations in a natural way. We obtain a type of relation, a string–genus "joint relation," involving both a string and the 3D manifold. We use the joint relation to obtain a topological interpretation of the C^*-Hopf algebra relations, which are widely used in tensor networks. We obtain a 3D representation of the controlled NOT (CNOT) gate that is considerably simpler than earlier work, and a 3D topological protocol for teleportation.

quon language | picture-language | quantum information | joint relation | topological algebra

Topological quantum information was formulated by Kitaev (1) and Freedman et al. (2). Here, we formulate a 3D topological picture-language that we call the "quon language"—suggesting quantum particles. It leads to strikingly elementary mathematical proofs and insights into quantum information protocols. In our previous work, we represented qudits, the basic unit of quantum information, using charged strings in 2D. This fits naturally into the framework of planar para algebras (3–6). We call this our "two-string model."

We also found a "four-string model" in 2D, in which we represent a 1-qudit vector as a neutral pair of particle–antiparticle charged strings (3, 4). These charged strings have the properties of parafermions. The presence of charges leads to para isotopy relations, which reflect the parafermion multiplication laws. Neutral pairs satisfy isotopy, a very appealing property. However, braiding two strings from different qudits destroys individual qudit neutrality, and this problem seemed unsurmountable for multiqudit states. So can one isolate those transformations that map the neutral pairs into themselves?

Here, we solve this problem by defining "quons." We embed the neutral pairs of charged strings representing qudits into the interior of a 3-manifold. The quon language has the flavor of a topological field theory with strings. The resulting composites of 3-manifolds and strings give us quon states, transformations of quons, and quon measurements. However, the composites contain a further aspect: There are topological relations that involve both the strings and the manifolds. We call them "joint relations." These joint relations provide basic grammatical structure as well as insight into our language.

In *String–Genus Joint Relation*, we see that, if a neutral string surrounds a genus in the manifold, then one can remove them both. In *Topological Relations for* C*-*Hopf Algebras*, we use this joint relation to obtain an elementary understanding of Frobenius and C^*-Hopf algebra relations stated in *Bi-Frobenius Algebras*. These relations are key in tensor network theory. In *Quon Language for a Unitary Modular Tensor Category*, we see that our construction even provides four-string structure for unitary modular tensor categories, with quons a special case.

Basic Grammar

1-Quon Space. We represent a 1-quon by a hemisphere, with no input points and four output points. Transformations of 1-qudits have four input and four output points in a cylinder, so we call this a four-string model. We represent a 1-quon measurement by a hemisphere with four input points and no output point.

In case a quon is a qudit of degree d, one has a simple representation for a 1-quon basis: The interior of a hemisphere contains two charged strings, each linking two of the output points. The value of the charge on one string may equal either $0, 1, \ldots, d-1 \in \mathbb{Z}_d$, while the other string carries the negative of that charge. For $d=2$, the quons reduce to Majorana fermions.

The four-string model for a qudit found in section 5.3 of ref. 3 arose as a natural generalization of Kitaev's picture (7) of a spin as a pair of fermions. The four strings arise as we represent the Pauli matrices X, Y, Z by four parafermions. In our reinterpretation, we replace the two fermions by a pair of parafermion/antiparafermion unitaries with opposite charge. We represent transformations on 1-quons as a box with four input points and four output points, embedded in a 3-manifold. We describe various bases in *1-Quon Bases*.

Multiquon Space. Multiquons have a hemisphere for each 1-quon. A transformation on n-quons has charged strings in a 3-ball with n input handles and n output handles, each containing four strings. This representation leads to a natural multiparticle structure; it allows us to analyze the full Hilbert space for multiquons, with each individual quon remaining a neutral pair.

Quons as Topological Algebra. Picture-language for tensor networks arose in Penrose (8), Deutsch (9), and in Dür et al. (10). The Hopf algebra axioms were studied in tensor networks by Lafont (11). Abramsky, Coecke, and others studied quantum information extensively from a categorical point of view, and found many applications in tensor networks (12–18). Vicary and Reutter applied 2-categories and biunitaries in planar algebras to quantum information (19, 20).

Our quons live in 3D space and thereby capture categorical structures in two directions. We obtain Frobenius algebras in the X direction and the Y direction, corresponding, respectively, to the COPY and SUM maps in tensor networks. We explain these concepts in *Bi-Frobenius Algebras*. They define the underlying Hilbert space as a C^*-Hopf algebra. Moreover, the

Significance

We give a 3D picture-language for quantum information. This language is based on an inherently 3D pictorial representation of particle-like excitations (quons) and of transformations acting on them. Mathematical identities and quantum information protocols are expressed through deformations of these pictures. We explore our language, highlighting conceptual insights, 3D visualizations, and suggestive intuition that it motivates for understanding algebra and quantum information.

string Fourier transform $\mathfrak{F}_S$ is a 90° rotation around the Z axis. Conjugation by $\mathfrak{F}_S$ maps one Frobenius algebra to the other. This gives a topological interpretation of the algebraic axioms of Hopf algebras, as explained in *Topological Relations for C*-Hopf Algebras*.

Algebraic relations arise from the invariance of certain elementary diagrams under topological isotopy. It is significant that different algebraic conditions have the same topological representation. In other words, diagrams that are equivalent up to isotopy can have different algebraic meanings when they are located at different positions. We have already used this philosophy in our two-string model to design protocols in a topological way (5, 6). Although 3D braiding appears in many places (e.g., refs. 1, 2, and 21–25), our present work combines charged strings with 3D manifolds.

Using these diagrams, in *Teleportation* we obtain a 3D representation of the controlled NOT (CNOT) map and the quantum teleportation protocol. The teleportation protocol becomes a topological protocol in the quon language.

Parafermion Algebras

The "parafermion algebra" PF_n of order d is a $*$-algebra with unitary generators c_m, $m = 1, 2, \cdots, n$, which satisfy

$$c_m^d = 1 \text{ and } c_m c_{m'} = q\, c_{m'} c_m \text{ for } 1 \le m < m' \le n. \quad [1]$$

Here, $q \equiv e^{2\pi i/d}$ and $i \equiv \sqrt{-1}$. Consequently, $c_j^* = c_j^{-1} = c_j^{d-1}$, where $*$ denotes the adjoint. Majorana fermions arise from the case $d = 2$. The Jordan–Wigner transformation gives the isomorphism $PF_{2n} \cong M_d(\mathbb{C})^{\otimes n}$ to tensor products of $d \times d$ matrices. The parafermion algebra PF_n has a basis $\{c^\alpha\}$. Here, $c^\alpha = c_1^{\alpha_1} \cdots c_n^{\alpha_n}$, and $\alpha_k \in \mathbb{Z}_d$. The charge of c^α is defined to be $|c^\alpha| = \sum_{k=1}^n \alpha_k$ in Z_d. The zero-charged elements PF_n^0 form a subalgebra of the parafermion algebra PF_n, namely the neutral subalgebra.

Parafermion Planar Para Algebras.
Given $1 \le m \le n$, our diagrammatic representation for $c_m^{\alpha_m}$, with $\alpha_m = k$, as explained in ref. 3, is

$$c_m^k \longleftrightarrow \left| \cdots \left| k \right| \cdots \right|,$$

where the mth string is labeled by k. In our notation, we have the relations

Multiplication:
$$\left| \begin{matrix} \ell \\ k \end{matrix} \right. = k + \ell \Big|, \quad d \Big| = \Big| = 0 \Big|.$$

Para isotopy:
$$\left| \left| \cdots \right| \begin{matrix} \ell \\ \end{matrix} \right| = q^{k\ell} \begin{matrix} k \\ \end{matrix} \left| \cdots \right| \begin{matrix} \\ \ell \end{matrix} \right|. \quad [2]$$

The strings between k-charged and ℓ-charged strings are not charged. Take ζ to be a square root of q, such that

$$\zeta^{d^2} = 1. \quad [3]$$

We can interpolate between the diagrams in Eq. **2** as a

Twisted product:
$$k \left\| \cdots \right\| \ell \Big| := \zeta^{k\ell} \begin{matrix} k \\ \end{matrix} \left\| \cdots \right\| \begin{matrix} \\ \ell \end{matrix} \Big|. \quad [4]$$

In the parafermion planar para algebra (PAPPA) model, the charged strings satisfy the following:

- $\rho_\pi(c^j) = \zeta^{j^2} c^j$, where ρ_π is a rotation by π on the plane. Then $\rho_{2\pi}(c^j) = q^{j^2} c^j$ for the 2π rotation $\rho_{2\pi}$.

- $j \bigcirc = 0$, $1 \le j \le d-1$, and $\bigcirc = \delta = \sqrt{d}$.

- Let $\omega = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \zeta^{j^2}$. This is a phase, as shown in proposition 2.15 of ref. 3. Then

$$\left| \;\; \right| \; = \; \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \begin{matrix} \overset{-j}{\frown} \\ \underset{j}{\frown} \end{matrix}$$

$$\times \; = \; \frac{1}{\sqrt{\omega d}} \sum_{j=0}^{d-1} \zeta^{j^2} \begin{matrix} \overset{-j}{\frown} \\ \underset{j}{\frown} \end{matrix}.$$

Using the above definition of braiding, we can establish the braided relation that any neutral diagram can move above or under the strings (see ref. 5). Therefore, the neutral diagrams can be lifted to the 3D space.

Categorical Approach to the Neutral Part of the Parafermion Algebra.
For readers who are familiar with category theory, one can consider the neutral diagrams as morphisms in a monoidal category. The neutral part of PAPPA is the $\mathbb{Z}_d$ unshaded subfactor planar algebra. It is a Z_2 graded unitary fusion category. Its even part is the monoidal category Vec_{Z_d}, whose simple objects X_g are labeled by group elements g in $\mathbb{Z}_d$, indicating the fusion rule. It has only one odd simple object $\tau = \bar{\tau}$, where $\bar{\tau}$ is the dual of τ, such that $\tau^2 = \gamma = \oplus_{g \in Z_d} X_g$. Thus, γ is a Frobenius algebra. Then the neutral subalgebra of the parafermion algebra is given by $PF_n^0 = \hom(\tau^n, \tau^n)$.

Details of the Quon Model

Quons. An n-quon is represented by n hemispheres. We call the flat disc on the boundary of each hemisphere a boundary disc. Each hemisphere contains a neutral diagram with four boundary points on its boundary disk. The dotted box designates the internal structure that specifies the quon vector. For example, the 3-quon is represented as

$$[5]$$

Here, v_j labels a 1-qudit vector given by neutral diagrams with four boundary points in the hemisphere. We orient the boundary discs of the hemispheres to lie on the $X - Y$ plane in the 3D space.

Transformations. An n-quon transformation is represented by a neutral element T in $PF_{4n} = \hom(\tau^{4n}, \tau^{4n})$ embedded in a 3-manifold, isotopic to a 3D-ball. The 3-manifold has n boundary disks on the top and n at the bottom. Each disk contains four boundary points of T. For example, a 3-quon transformation T will have the representation

$$[6]$$

Isotopy of Neutral Diagrams. In addition to the relations for charged strings, we allow isotopy of strings in 3-manifolds. We define a relation for 3-manifolds: If a 3-ball has no diagram inside, then it can be removed. Moreover, we define a joint relation between diagrams and 3-manifolds.

Suppose $T \in \hom(\tau^m, \tau^n)$. Let B_m be an orthonormal basis (ONB) of $\hom(\tau^m, \tau^m)$, and let B_n be an ONB of $\hom(\tau^n, \tau^n)$. We define the relation

$$ [7] $$

Basic linear algebra shows that the second equality always holds. It says that the relation is well defined up to isotopy of neutral diagrams in 3-manifolds. By this relation, the picture Eq. **6** reduces to a linear sum of pictures of the following form:

$$ [8] $$

If we take v_i, w_i to be elements in an ONB of $\hom(1, \tau^4)$, then these pictures represent matrix units of qudit transformations. Therefore, we obtain a representation of quons and transformations by neutral diagrams in 3-manifolds modulo relations. To simplify the notations, sometimes we ignore the 3-manifold, if there is no confusion.

1-Quon Bases

Qubit Case. The space of 1-qubit states is known as the Bloch sphere. Vector states lie on the surface. The antipodes for a unit 3 vector $\pm \vec{n}$ are assigned the eigenvectors of $n_x X + n_y Y + n_z Z$, where X, Y, Z are the Pauli matrices. The eigenvalues are ± 1. The usual convention is to let the eigenvectors of Z be $|0\rangle = |0_Z\rangle$ at the south pole and $|1\rangle = |1_Z\rangle$ at the north pole. Then there are three fundamental sets of bases of the 1-qubit space for $n_x = 1$, etc. They are $|0_X\rangle = 1/\sqrt{2}\,(|0\rangle + |1\rangle)$, $|1_X\rangle = 1/\sqrt{2}\,(|0\rangle - |1\rangle)$, and likewise $|0_Y\rangle = 1/\sqrt{2}\,(|0\rangle + i\,|1\rangle)$, $|1_Y\rangle = 1/\sqrt{2}\,(|0\rangle - i\,|1\rangle)$. The Bloch sphere can be drawn as
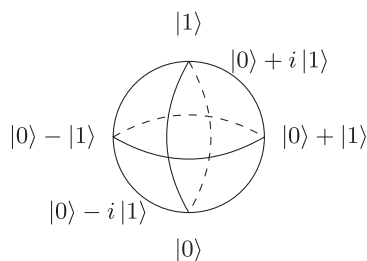

The General 1-Quon Case. In the quon model, the three different ways of connecting the four boundary points give the X, Y, Z basis of the 1-quon space. For $k = 0, 1, \ldots, d-1 \in \mathbb{Z}_d$,

- Z-basis: $|k_Z\rangle = \frac{1}{\sqrt{d}}$
- X-basis: $|k_X\rangle = \frac{1}{\sqrt{d}}$
- Y-basis: $|k_Y\rangle = \frac{1}{\sqrt{d}}$

These are the three eigenbases of the three unitary Pauli matrices. The matrices are given diagrammatically in ref. 3, respectively as

$$ Z = \begin{vmatrix} 1 & -1 & & & \\ & & & & \end{vmatrix}, \quad X = \begin{vmatrix} 1 & & & -1 & \\ & & & & \end{vmatrix}, \quad Y = \begin{vmatrix} -1 & & 1 & \\ & & & \end{vmatrix}. \quad [9] $$

In the quon model, we represent the basis of the 1-quon space by a pair of strings with opposite charges, embedded in a hemisphere, and exiting the bottom. The algebraic adjoint operation is given by a charge-inverting, geometric reflection along the Z direction. Therefore, a measurement is represented by a pair of strings with opposite charges in the reflected hemisphere
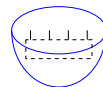
The charge represents the result of the measurement.

1-Quon Clifford Group

The 1-quon transformations $\{X, Y, Z, F, G\}$ are generators of the 1-quon Clifford group. Their algebraic definitions are given by

$$ X|k\rangle = |k+1\rangle, \ Y|k\rangle = \zeta^{1-2k}|k-1\rangle, \ Z|k\rangle = q^k|k\rangle, $$

$$ F|k\rangle = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} q^{kl}|l\rangle, \quad G|k\rangle = \zeta^{k^2}|k\rangle. \quad [10] $$

When we consider a 1-quon as a vector state, the quon transformations are defined up to a phase. These 1-quon transformations form a group $\mathbb{Z}_d^2 \rtimes SL(2, \mathbb{Z}_d)$ as shown in ref. 3. The Pauli matrices are given diagrammatically by Eq. **9**, and

$$ F = \text{}, \quad G = \text{}. \quad [11] $$
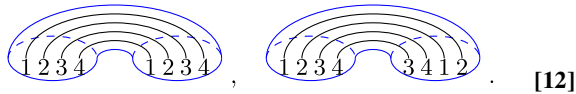
n-Quon Clifford Group

Let C_X be the controlled X transformation. For the qubit case, it becomes CNOT. The n-qudit Clifford group is generated by $\{X, Y, Z, F, G, C_X\}$. We represent the quon transformation C_X by neutral diagrams in 3-manifolds. It is more natural to represent these neutral diagrams in the 3D space. In the 3D space, we label the four boundary points as 1, 2, 3, 4, corresponding to the order of the boundary points of the 2D diagrams. The order indicates the choice of basis in the 3D space. We discuss more about C_X and the 3D representations in *Bi-Frobenius Algebras*, *String–Genus Joint Relation*, and *Topological Relations for* C*-Hopf Algebras.
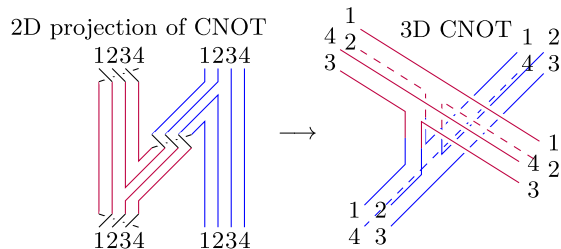
Earlier 2D representations of approximate multiqubit CNOT gates are complicated. Some even resemble a musical score, as in figure 3 of ref. 26; see also ref. 27. Exact 2D qudit CNOT representations appeared in ref. 28 for odd d, where the complexity of the representation depends on d.

Resource States

The generalized Bell states for qudits are given by $B_+ = d^{-1/2} \sum_{k \in \mathbb{Z}_d} |k, k\rangle$ and $B_- = d^{-1/2} \sum_{k \in \mathbb{Z}_d} |k, -k\rangle$. Diagrammatically,

$$\qquad , \qquad . \qquad \textbf{[12]}$$

The order 3, 4, 1, 2 indicates the action of F^2 on the second qudit. One can check the identifications by the joint relation. The corresponding multiple-qudit generalizations of the Bell states are known as the Greenberger–Horne–Zeilinger (GHZ) state (29) and Max. We give their algebraic definitions in *Bi-Frobenius Algebras*, and we give their 3D representations in *Topological Relations for* C*-*Hopf Algebras*.

Teleportation

In the quon language, we represent the teleportation protocol by the following diagrammatic protocol using the X basis:
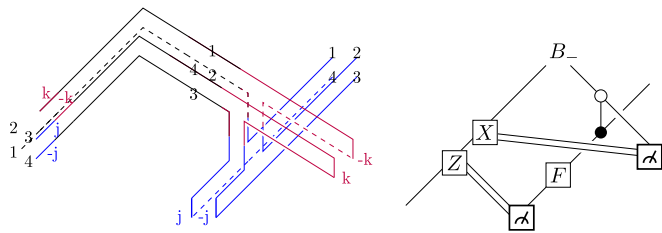
The pair of oppositely-charged strings is neutral; thus it is defined in the 3D space. It represents the teleportation process in a topological way. Moreover, it shows the one-to-one correspondence between the diagrammatic representation of this protocol and the algebraic representation in the teleportation protocol of Bennett et al. (30), illustrated to the right of the quon-language diagram.

Bi-Frobenius Algebras

In tensor networks, one decomposes the qudit CNOT gate into COPY and SUM, defined algebraically by planar diagrams.
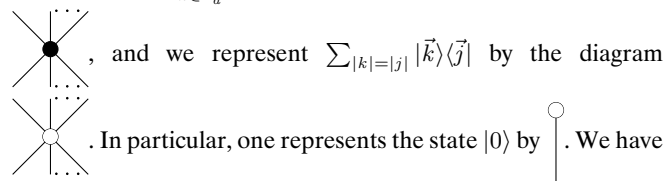
$$CNOT = \sum_{k,j\in\mathbb{Z}_d} |k+j,j\rangle\langle k,j| :$$

$$COPY = \sum_{j\in\mathbb{Z}_d} |j,j\rangle\langle j| :$$
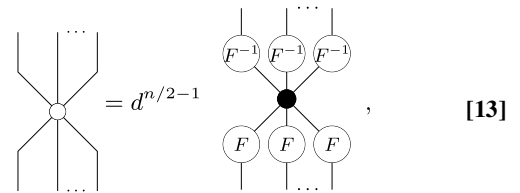
$$SUM = \sum_{j\in\mathbb{Z}_d} |k+j\rangle\langle k,j| :$$

Using the "spider" notation of ref. 15, we represent the qudit transformation $\sum_{k\in\mathbb{Z}_d} \overbrace{|k,k,\ldots,k\rangle}^{n \text{ entries}} \overbrace{\langle k,k,\ldots,k|}^{n' \text{ entries}}$ by the diagram

 , and we represent $\sum_{|k|=|j|} |\vec{k}\rangle\langle\vec{j}|$ by the diagram

 . In particular, one represents the state $|0\rangle$ by . We have

the duality induced by the Fourier transform F between the two spiders.

$$= d^{n/2-1} \qquad , \qquad \textbf{[13]}$$
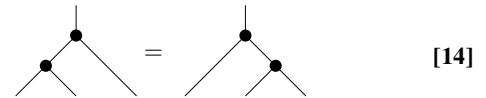
where n is the number of boundary points. This generalizes the duality between the resource states analyzed in ref. 5,

$$|\text{Max}\rangle = d^{\frac{1-n}{2}} \sum_{|\vec{k}|=0} |\vec{k}\rangle, \quad |\text{GHZ}\rangle = \frac{1}{\sqrt{d}} \sum_{k\in\mathbb{Z}_d} |k,k,\cdots,k\rangle.$$

In particular, one represents $\sum_{k=0}^{d-1} |k\rangle = \sqrt{d}$ by .
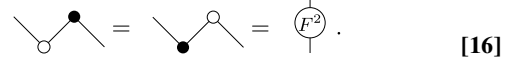
The adjoint transformations are represented by the vertical reflections of these diagrams. It is known that both trivalent vertices are Frobenius algebras. That means the following relations

hold for and , and similarly for and :

$$= \qquad \textbf{[14]}$$


$$= \qquad . \qquad \textbf{[15]}$$

One can flip the boundary points of a black/white spider from top to bottom or the other way using caps or cups labeled by a black/white bullet. Thus, each Frobenius algebra has a compatible pivotal structure. However, the two Frobenius algebras do not share pivotal structures. A composition of the cap and the cup with different colored bullets is not the identity map. Instead, it is the antipode map $\mathfrak{F}_S^2 = F^2$, where $\mathfrak{F}_S$ is the string Fourier transform defined in the introductory section,

$$= \qquad = \quad \boxed{F^2} \quad . \qquad \textbf{[16]}$$
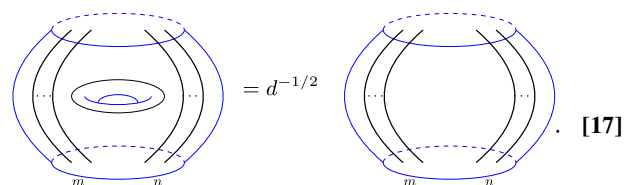
In the quon model, we can represent these maps in a consistent way by strings in 3-manifolds, such that the algebraic relations become topological isotopy. The COPY map is represented by
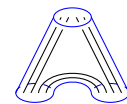
One can check that this diagrammatic definition coincides with the algebraic definition by Eq. **7**. By this representation, the algebraic conditions of the Frobenius algebra become topological isotopy in the quon model.

String–Genus Joint Relation

In case we have a diagram of the form where a closed neutral string surrounds a genus of the manifold, then we can remove both, up to a scalar. When m and n are odd numbers, we have

$$= d^{-1/2} \qquad . \qquad \textbf{[17]}$$

Note that τ^m and τ^n are multiples of τ. It is enough to prove the relation for $m = n = 1$. In this case, it follows from the relation Eq. **7**.

If m or n is even, by relation Eq. **7**, then the diagram is 0. Thus, if the diagram is a part of a nonzero transformation, then the m and n have to be odd numbers. In this case, the relation means that, if there is a circle around a genus of the 3-manifold, then we can remove the circle and the genus by multiplying a scalar $d^{-1/2}$.

## Topological Relations for C^*-Hopf Algebras

A more conceptual way to look at pictures in the quon model is to assign the four boundary points of the quon to the corners of a square in a plane orthogonal to the Z axis. The white and black bullets in the spiders indicate diagrammatic operations in X and Y directions on the 2D plane. For example, if we look at this 3D diagram from the top along the Z direction, then the picture for the Z basis in *1-Quon Bases* is given by

for $k \in \mathbb{Z}_d$. Here we only draw the boundary circle of the 3-manifold to simplify the picture. The pictures for COPY and SUM become, respectively,
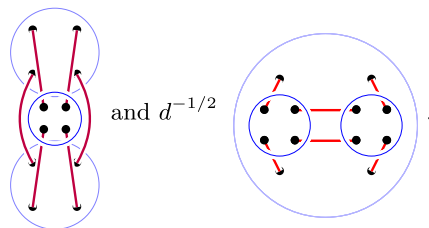
Similarly, we can represent the white and black spiders as strings in 3-manifolds in the quon model. The white spiders are expended in the X direction, and the black spiders are expended in the Y direction. In particular, the resource states $|\mathrm{GHZ}\rangle$ and $|\mathrm{Max}\rangle$ (for 3-quons) are, respectively,
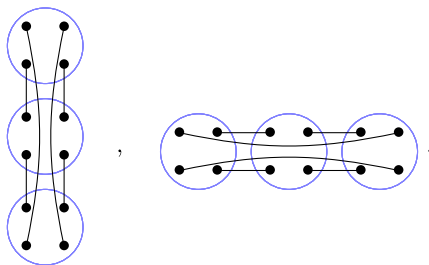
From this point of view, the Fourier transform is a 90° rotation around the Z axis, which explains the duality of the two Frobenius algebras in Eq. **13** in a geometric way. Moreover, one can check that the relation Eq. **16** also becomes an isotopy in the 3D space in the quon language.

Furthermore, this pair of Frobenius algebras satisfies the following additional relations Eqs. **18**–**21**. These relations define $\mathbb{Z}_d$ as a C^*-Hopf algebra, where F^2 is the antipode map of the Hopf algebra, and the involution is an antilinear map that reflects the diagrams vertically. Note that, if a pair of Frobenius algebras satisfies these relations, then the underlying d-dimensional Hilbert space becomes a Hopf algebra. This has been observed in refs. 11 and 15.

$$\text{[18]}$$

Now we give an interpretation of these relations as topological isotopy in 3D. Note that relation Eq. **18** follows from the definition of the COPY map. By isotopy in our quon model, relations Eqs. **19** and **20** are exactly the same diagrams as the relation Eq. **18**. The most interesting relation is Eq. **21**, and we explain that relation in detail.

Relation Eq. **21** becomes topological isotopy, when we use the string–genus joint relation established in *String–Genus Joint Relation*, namely the joint relation Eq. **17**. Then Eq. **21** is given by the isotopy in Eq. **22**. Thus, we have given a topological interpretation for the C^*-Hopf algebra axioms for $\mathbb{Z}_d$.

$$\text{[19]}$$
$$\text{[20]}$$
$$\text{[21]}$$
$$= d^{-1/2} \quad \text{[22]}$$
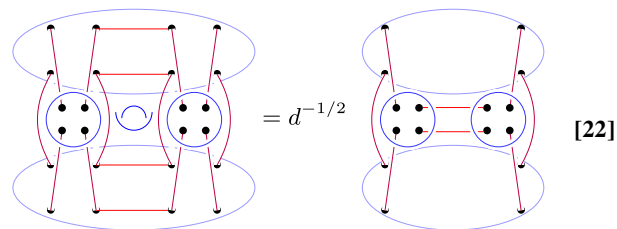
There is a one-to-one correspondence between C^*-Hopf algebras and irreducible, depth-two subfactor planar algebras (31, 32). In this case, d is the global dimension of the C^*-Hopf algebra. Moreover, the even part of the planar algebra is the representation category of the Kac algebra. The odd part has only one simple object, τ, so Eq. **17** also holds. In the above interpretation, we only use (shaded) planar diagrams without braids in 3-manifolds. So this topological interpretation works for any finite dimensional C^*-Hopf algebra. From this point of view, many algebraic properties of C^*-Hopf algebras reduce to topological isotopy.

Quon Language for a Unitary Modular Tensor Category

We can define the quon language for any unitary modular tensor category C, so that the 1-quon basis corresponds to the set of simple objects OB in C. If we take C to be the unitary modular tensor category, such that its fusion ring is Z_d and its modular S matrix is q^{kl}, where $q = e^{2\pi i/d}$, then we get back the quon language for qudits defined by PAPPA.

For each $X \in OB$, we obtain a simple object $\tilde{X} := X \otimes \overline{X}$ in $C \otimes C$, where $\overline{X}$ is the dual of X. Take $\gamma = \oplus_{X \in OB} X \otimes \overline{X}$ in $C \otimes C$. It is known that $\gamma = \oplus_{X \in Ob} X \otimes \overline{X}$ is a Frobenius algebra in $C \otimes C$. Thus $\mathcal{P}_{n,+} = \mathrm{hom}(1, \gamma^n)$ is a subfactor planar algebra generated by τ, such that $\gamma = \tau \otimes \overline{\tau}$. It is proved in ref. 33 that this planar algebra is unshaded. That means $\tau = \overline{\tau}$.

Note that $\mathrm{hom}(1, \tau^4) \cong \mathrm{hom}(1, \gamma^2)$. This space has an ONB given by the canonical inclusion from 1 to $(X \otimes \overline{X}) \otimes (\overline{X} \otimes X)$, for $X \in OB$. We call the generalized single-particle state a 1-quon. Thus, the dimension d of the 1-quon space is the cardinality of OB. Furthermore, the string Fourier transform on the 1-quon space is the S matrix of the unitary modular tensor category C (33).

We generalize the quon language as follows: We label each of the four boundary points by the object τ. The diagrams in the

3-manifolds are given by morphisms in $C \otimes C$. The representation for an n-quon is given by morphisms in $\hom(1, \tau^4)$ in n hemispheres as in Eq. **5**. The n-quon transformations are represented by morphisms in $\hom(\tau^{4n}, \tau^{4n})$ in a 3-manifold as in Eq. **6**. The relations between diagrams and 3-manifolds are also defined by Eq. **7**. Then n-quon transformations also reduce to a linear sum of the form in Eq. **8** which represent matrix units. Therefore, the n-quon transformations are transformations on the d^n-dimensional Hilbert space.

In general, the quon language can be defined for any subfactor planar algebra (21), if we do not require $\tau = \bar{\tau}$. In this case, the diagrams in the 3-manifolds with $4n$ boundary points are given by a shaded planar diagram in the $2n$-box space of planar algebras. We have used this general case to give the topological interpretation of the C^*-Hopf algebra relations.

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