Thermalization and Return to Equilibrium on Finite Quantum Lattice Systems

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Thermal states are the bedrock of statistical physics. Nevertheless, when and how they actually arise in closed quantum systems is not fully understood. We consider this question for systems with local Hamiltonians on finite quantum lattices. In a first step, we show that states with exponentially decaying correlations equilibrate after a quantum quench. Then we show that the equilibrium state is locally equivalent to a thermal state, provided that the free energy of the equilibrium state is sufficiently small and the thermal state has exponentially decaying correlations. As an application, we look at a related important question: When are thermal states stable against noise? In other words, if we locally disturb a closed quantum system in a thermal state, will it return to thermal equilibrium? We rigorously show that this occurs when the correlations in the thermal state are exponentially decaying. All our results come with finite-size bounds, which are crucial for the growing field of quantum thermodynamics and other physical applications.

I. INTRODUCTION

To completely understand the strengths and limitations of statistical physics, it makes sense to try to derive it from physical principles, without relying on ad hoc assumptions. Along these lines, over the past twenty years, ideas from quantum information have led to new insights into the foundations of statistical physics [1–3]. In particular, some progress has been made recently towards understanding how and when thermalization occurs [4–6]. In [5] it was shown that a large class of states of systems with weak intensive interactions (e.g. one dimensional systems) will thermalize. In [6] thermalization was shown to occur, also for a large class of states, for systems in the thermodynamic limit. (We will compare the results of [6] to ours in more detail in Section IV.) More recently, the equivalence of the microcanonical and canonical ensemble (i.e. a thermal state) was proved for finite quantum lattice systems, when correlations in the thermal state decay sufficiently quickly [7] (see also [6, 8]).

Here, we prove thermalization results for closed quantum systems in two parts. First, we build on previous equilibration results (such as Ref. [9]). A requirement for equilibration is that the effective dimension, defined in Section III, is large. While there are physical arguments for this to be true [10] and it is also known to be true for most states drawn from the Haar measure on large subspaces [9], there are no known techniques to decide whether a given initial state will equilibrate under a given Hamiltonian. Here, we prove that a large effective dimension is guaranteed for local Hamiltonian systems if the correlations in the initial state decay sufficiently quickly and the energy variance is sufficiently large. The latter is known for thermal states with intensive specific heat capacity and may, for large classes of states, be computed straightforwardly. The second part of thermalization is to show that the equilibrium state is locally indistinguishable from a thermal state. We prove that this is the case if the correlations in the corresponding thermal state decay sufficiently quickly and the relative entropy difference between the equilibrium state and the corresponding thermal state is sufficiently small.

As an application, we answer the following question. Given a closed quantum system that is initially in a thermal state, and suppose we locally quench or disturb it somehow, will it re-equilibrate to a thermal state? Understanding the extent to which thermal states are robust against local external noise is important, e.g., in decohering quantum simulations implemented in optical lattice systems (see, e.g., Ref. [11] and references therein), where such noise can be caused by the absorption and re-emission of a photon. These questions of re-equilibration have a long tradition and return to equilibrium was shown to occur for infinite lattice systems in the seventies [12, 13], by making what were effectively transport assumptions: On an infinite lattice, it is possible for information to leave a region and never return, which is not true for finite systems. This fundamental difference highlights the importance of finite-size considerations. We discuss the connection to the results on infinite lattices further in Appendix D.

Return to equilibrium has also been shown for finite quantum systems coupled to infinite reservoirs after a coupling has been turned on [14, 15], and a rough argument for stability of thermal states was given recently in terms of energy probability distributions in [16]. Here, we prove that a system in a thermal state, after being locally disturbed, re-equilibrates to a thermal state provided correlations decay sufficiently quickly. In contrast to what is known for infinite lattice systems, our results give finite-size estimates and our methods and assumptions are entirely different. It is important to emphasize that finite-size bounds are crucial for physical applications, particularly those in quantum thermodynamics [2], where thermal states are usually considered to be free resources. Addressing to what extent thermalization occurs will also have impact on the nature of protocols for extracting work using small quantum thermal machines.

II. SETTING AND NOTATION

We consider a d-dimensional hypercubic lattice with $N = n^d$ sites. We suppose that each site is associated with a d_{loc}-dimensional quantum system, e.g., a spin. On this finite quantum spin lattice, we let H denote the Hamiltonian governing the dynamics and assume that it is k-local, i.e., that it is of the form $H = \sum_i h_i$, where h_i acts only on lattice sites that are no more than k sites separated from i, i.e., only on sites j with $\text{dist}(i,j) \leq k$. We further assume that the h_i are bounded in operator norm and use units with $\|h_i\| \leq 1$ and Boltzmann's and Planck's constants set to one, so that $\hbar = k_B = 1$. We write $\rho(t) = e^{-iHt} \rho e^{iHt}$ for the state at time t and denote the time-average state by

$$\langle \rho \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{d}t \, \rho(t). \qquad (1)$$

We let σ^2 denote the energy variance of a state ρ with the respect to a Hamiltonian H, $\sigma^2 = \text{tr}[\rho H^2] - \text{tr}^2[\rho H]$. We will be interested in subsystems, S, of the whole lattice and denote the rest of the system by B – the *bath* or *environment*. We denote their Hilbert space dimensions by $d_S = d_{\text{loc}}^{|S|}$ and $d_B = d_{\text{loc}}^{|B|}$. Given a state of the whole system ρ, we write $\rho_S = \text{tr}_B[\rho]$ and $\rho_B = \text{tr}_S[\rho]$ for the reduced states on the subsystem and environment, respectively.

To discuss whether two states are close we must consider what one can measure in practice. Mostly, we will consider the local distinguishability of two states, ρ and τ, given by

$$\|\rho - \tau\|_S := \|\rho_S - \tau_S\|_{\text{tr}}, \qquad (2)$$

where $\| \cdot \|_{\text{tr}}$ is the trace distance. With this, our results extend naturally to coarse-grained observables. An example of such an observable could be the magnetization of spins on a large region or even the whole lattice. We may write a coarse-grained observable as $M = \frac{1}{m} \sum_{i=1}^m M_{S_i}$, where the S_i are non-overlapping subsystems and M_{S_i} acts only on subsystem S_i. For example, one could take the magnetization per spin $M = \frac{1}{N} \sum_i \sigma_z^i$. Then local indistinguishability also implies that expectation values of such observables are close: Assuming $\|M_{S_i}\| \leq C$ we have

$$\begin{aligned} |\text{tr}[\rho M] - \text{tr}[\sigma M]| &\leq \frac{1}{m} \sum_i \|M_{S_i}\| \|\rho - \sigma\|_{S_i} \\ &\leq C \cdot \mathbb{E}_{S_i} \|\rho - \sigma\|_{S_i}, \end{aligned} \qquad (3)$$

where $\mathbb{E}_{S_i}$ denotes the average over the subsystems S_i. Thus, we cover a large variety of physically realistic measurement possibilities for distinguishing between states.

Throughout, we will often consider states with exponentially decaying correlations. This is guaranteed to be the case for, e.g., thermal states above a critical temperature [17] and for ground states of gapped k-local Hamiltonians [18]. We define exponentially decaying correlations in the following way.

Definition 1. *A state ρ has exponentially decaying correlations if there is a correlation length $\xi > 0$ and a $K \geq 0$ (both independent of the system size N), such that, for any two lattice regions X and Y, one has*

$$\max_{\substack{supp[P] \subset X \\ supp[Q] \subset Y \\ \|P\|, \|Q\| = 1}} \left| \frac{\text{tr}[\rho P Q] - \text{tr}[\rho P] \, \text{tr}[\rho Q]}{|X||Y|} \right| \leq K e^{-\text{dist}(X,Y)/\xi}.$$

Here, the distance between the two regions X and Y is given by

$$\text{dist}(X,Y) = \min_{\substack{i \in X \\ j \in Y}} \text{dist}(i,j), \qquad (4)$$

where $\text{dist}(\cdot, \cdot)$ is some metric on the lattice.

III. EQUILIBRATION

Due to recurrences, a closed finite system will never truly equilibrate, not even locally. Hence, for finite systems one asks a slightly different question instead [9, 10]: Does a system spend most of its time close to some fixed state? If we denote the fixed state by τ, this means that

$$D_S(\tau) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{d}t \, \|\rho(t) - \tau\|_S \qquad (5)$$

is small, i.e., for the majority of times, $\rho(t)$ and τ are locally indistinguishable. The most natural example to look at is equilibration to the time-average state. For this case, it was proved in [9, 19] that

$$D_S(\langle \rho \rangle) \leq \frac{1}{2} \sqrt{\frac{D_G d_S^2}{d_{\text{eff}}}}, \qquad (6)$$

where we recall that d_S denotes the Hilbert space dimension of the subsystem S. Here, D_G is the degeneracy of the most degenerate *energy gap* [36], i.e., $D_G = 1$ if the Hamiltonian has no degenerate energy gaps. Typically, one expects D_G to be small. In fact, even the existence of degenerate energy gaps is a measure zero constraint on the Hamiltonian. Also appearing in equation (6) is d_{eff}, known as the effective dimension, which is defined by

$$\frac{1}{d_{\text{eff}}} = \sum_k \text{tr}^2[P_k \rho], \qquad (7)$$

where P_k is the energy projector corresponding to the energy E_k. If the spectrum of H is non-degenerate, then the above is equal to $\text{tr}[\langle \rho \rangle^2]$, i.e., the purity of the equilibrium state. Equation (6) implies that equilibration occurs when d_{eff} is large. The fraction of times during which $\|\rho(t) - \langle \rho \rangle\|_S \geq \delta$ is at most $(d_S \sqrt{D_G})/(2\delta \sqrt{d_{\text{eff}}})$, which follows from Markov's inequality. This result is quite powerful: It holds for any decomposition of the total system into a subsystem S and bath B. This division

need not correspond to a spatially localized subsystem. For example, one could apply the result to multi-point correlation functions over arbitrary distances.

In Ref. [10] it was argued on physical grounds that we should expect d_{eff} to be exponentially large in the system size. The argument relied on the exponentially increasing density of energy levels for generic physical systems. Furthermore, in Ref. [9] it was shown that most states drawn at random from a large subspace via the Haar measure would typically have large effective dimension.

However, it is not clear that one can take these arguments for granted in all situations of physical interest: E.g., in the setting of a local (or global) quench or measurement. And there are plenty of physical models for which it is known that the initial state will not have an effective dimension that is exponentially large in the system size. For example, take any system with a quadratic fermion Hamiltonian (e.g. the XY model). If the number of excitations m is fixed and there are N modes in total, then the relevant Hilbert subspace has dimension $N!/(m!(N-m)!) \sim N^m$. Therefore, the effective dimension is at most polynomial in N. Furthermore, just calculating the effective dimension for a particular system means computing the overlaps of the state with energy eigenvectors, which is as hard as diagonalizing the Hamiltonian in general.

So there is a need for concrete lower bounds on the effective dimension. Here, we prove such a bound, which holds for k-local Hamiltonians and states with exponentially decaying correlations.

Lemma 1. *Lower bound on the effective dimension: Suppose the initial state ρ (or its time average $\langle\rho\rangle$) has exponentially decaying correlations as in Def. 1. Let the whole system evolve according to a bounded k-local Hamiltonian, and let ρ have energy variance σ^2 with respect to this Hamiltonian. Then there is a constant C independent of N such that*

$$\frac{1}{d_{\text{eff}}} \leq C \frac{\ln^{2d}(N)}{s^3\sqrt{N}}, \tag{8}$$

where $s = \sigma/\sqrt{N}$.

This is proved in Appendix A and uses a Berry-Esseen theorem for quantum lattice systems [7, 20]. By the assumption on the decay of two-point correlations, s is upper bounded independent of N. Often, it is also lower bounded, e.g., for thermal states with intensive specific heat capacity $c(\beta)$, which is given at inverse temperature β by $c(\beta) = \beta^2\sigma^2/N = \beta^2 s^2$. Furthermore, for large classes of states (e.g., product states or matrix product states) it is straightforward to compute σ^2 such that the question of whether a system equilibrates may be answered directly, without knowledge of the overlap of the initial state with the energy eigenstates. The requirement that σ needs to be sufficiently large is reasonable: If the initial state is not sufficiently spread over many eigenstates one can not expect equilibration.

An interesting application of Lemma 1 and Eq. (6) together is to a quench scenario. If the initial state is a ground state of some Hamiltonian with exponentially decaying correlations (e.g., the ground state of a gapped k-local Hamiltonian), then after quenching to any other local Hamiltonian, equilibration will occur provided the energy variance σ^2 (with respect to the post-quench Hamiltonian) is sufficiently large.

Finally, it is important to mention that a full understanding of how long it takes for a quantum system to equilibrate is still lacking. Rigorous estimates of the timescale in a general setting lead to extremely large upper bounds for the timescale [19, 21, 22] (often these bounds also involve d_{eff} such that our results are applicable). For some quadratic models, the timescale is known to be much shorter [23, 24]. Interesting results also exist showing fast timescales for equilibration in the setting of random Hamiltonians, states or measurements [22, 25–30]. On the other hand, examples have been constructed of reasonable translationally invariant models with extremely long equilibration timescales [31].

IV. THERMALIZATION

In the previous section, we discussed the fact that equilibration occurs with great generality to the time-average state $\langle\rho\rangle$. But that is only part of thermalization. The second part is to see whether the time-average state is close to a thermal state. We thus need a practical way to decide whether, locally, $\langle\rho\rangle$ is close to a thermal state. The following Lemma (see Appendix B for a more quantitative version), recently obtained in Ref. [7], achieves this by relating the local trace-norm distance of two states to their difference in relative entropy.

Lemma 2. *Let σ be a state with exponentially decaying correlations as in Def. 1. Let $0 < \alpha < \frac{1}{d+2}$ and let $l \in \mathbb{N}$, $l^d \in o(n^{\frac{1-\alpha}{d+1}})$. Let τ be a state. If*

$$S(\tau\|\sigma) \in o\left(N^{\frac{1-(2+d)\alpha}{d+1}}\right), \tag{9}$$

then there is a constant C, which is independent of N, such that the average local trace distance between σ and τ is bounded as

$$\mathbb{E}_{S\in\mathcal{S}_l}\|\sigma - \tau\|_S \leq \frac{C}{N^{\alpha/2}}, \tag{10}$$

where $\mathbb{E}_{S\in\mathcal{S}_l}$ denotes the average over all hypercubes on the lattice with length of side l.

Note that even if the relative entropy difference between the two states increases with system size (as in Eq. (9)), the two states are locally close (on average over all cubic subsystems of size l^d). Also, maybe surprisingly, the size of the subsystem need not be fixed but may also increase as a power law in N. The bound in Eq. (10) tells us that (if N is sufficiently large) for the vast majority of

subsystems $S \in \mathcal{S}_l$ the states σ_S and τ_S are close. If one is interested in course-grained observables as discussed in Section II one finds, e.g., for the magnetization per spin $M = \frac{1}{N}\sum_i \sigma_i^z$ (for which we choose $l = 1$)

$$|\mathrm{tr}[\sigma M] - \mathrm{tr}[\tau M]| \leq \frac{1}{N}\sum_i \|\sigma - \tau\|_i \quad (11)$$
$$= \mathbb{E}_{S \in \mathcal{S}_l}\|\sigma - \tau\|_S$$

such that the bound in Eq. (10) directly gives a bound on the difference of expectation values in σ and τ. If both states are translationally invariant, the average is obsolete and one has $\|\sigma - \tau\|_S \leq C N^{-\alpha/2}$ for *all* cubic S of size l^d.

Let us now move on to thermalization, i.e., we set out to show that $D_S(\rho_\beta)$ is small and hence that for the majority of times $\rho(t)$ is locally close to the thermal state $\rho_\beta = \mathrm{e}^{-\beta H}/Z$, $Z = \mathrm{tr}[\mathrm{e}^{-\beta H}]$. We do so by combining Eq. (6) with Lemma 1 and 2: Let the initial state ρ (or $\langle\rho\rangle$) have exponentially decaying correlations, evolve according to a bounded k-local Hamiltonian H, and have energy variance σ^2. Fix $l \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, $0 < \alpha < \frac{1}{d+2}$. Let the thermal state ρ_β have exponentially decaying correlations and suppose that

$$S(\langle\rho\rangle\|\rho_\beta) \in o\big(N^{\frac{1-(2+d)\alpha}{d+1}}\big). \quad (12)$$

Then there is a constant C independent of N such that (see Appendix B for details)

$$\mathbb{E}_{S \in \mathcal{S}_l} D_S(\rho_\beta) \leq C \left(\sqrt{\frac{D_G}{s^3 N^{\frac{d}{2d+4}}}} + 1\right) \frac{1}{N^{\alpha/2}}. \quad (13)$$

If ρ_β and $\langle\rho\rangle$ are translationally invariant then $D_S(\rho_\beta)$ is upper bounded by the right-hand side for *all* $S \in \mathcal{S}_l$, i.e., we get thermalization on *every* cubic subsystem of size l^d. This is true, e.g., when the Hamiltonian is translationally invariant and has no degenerate energies. Without requiring translational invariance or making some other transport assumption, we cannot guarantee that every subsystem thermalizes. This is reasonable: We could consider models where small subsystems retain some memory of their initial state.

In fact, we could replace the requirement that $\langle\rho\rangle$ be translationally invariant by the requirement that there is transport in the following sense. Suppose that, in terms of the time-average state, one cannot tell where a localized disturbance of the initial state had occurred. In other words, let Φ_i denote a local quantum channel on some region centred on i. Then we demand that $\|\langle\Phi_i(\rho)\rangle - \langle\Phi_j(\rho)\rangle\|_S \leq \epsilon \ll 1$ for any i, j and some small region S. It follows that, for all intents and purposes, locally the equilibrium state is indistinguishable from $\langle\frac{1}{N}\sum_i^N \Phi_i(\rho)\rangle$, which is translationally invariant. Then the thermalization result Eq. (13) holds for the *individual* subsystem S with an extra ϵ on the right hand side. To see this, one needs to use the triangle inequality. We discuss what one can prove by making transport assumptions further in Appendix D.

It is important to compare Eq. (13) to the results of [6], which proved that thermalization occurs in the thermodynamic limit, with a comparable condition on the time-average state. Here, we prove thermalization for the important case of finite systems and are able to give finite-size estimates. Furthermore, in [6] it is assumed that the thermal state corresponds to a unique phase. Instead we assume that the thermal state has exponentially decaying correlations. This is always satisfied for $d = 1$ [32] and, for $d > 1$, if the temperature is above a certain critical temperature [17].

Finally, we note that the free energy of a state ρ at inverse temperature β is given by $F(\rho) = \mathrm{tr}[H\rho] - S(\rho)/\beta$, so that

$$S(\langle\rho\rangle\|\rho_\beta) = \beta\left(F(\langle\rho\rangle) - F(\rho_\beta)\right). \quad (14)$$

Thus, whenever the free energy of $\langle\rho\rangle$ is sufficiently small, then $\langle\rho\rangle$ and ρ_β are locally close.

V. THE STABILITY OF THERMAL STATES

We can apply the results of the last section to some interesting examples. We will focus on the translationally-invariant setting, i.e., we will assume that the time-averaged state $\langle\rho\rangle$ and the thermal state ρ_β are translationally invariant. This is true, e.g., when the Hamiltonian is translationally invariant and has no degenerate energies. As already discussed above, assuming translational invariance guarantees transport, without which we can not expect all subsystems to thermalize.

For the first example, suppose we have a system that was in a thermal state ρ_β, but was affected by a local process or some localized noise. We can model this by the application of a local quantum channel, i.e., we take $\Phi(\rho_\beta)$ as the initial state, where

$$\Phi(\rho_\beta) = \sum_i K_i^\dagger \rho_\beta K_i, \quad \sum_i K_i K_i^\dagger = \mathbb{I}, \quad (15)$$

and the K_i act only locally. We will see that the system locally returns to thermal equilibrium provided ρ_β had exponentially decaying correlations.

Corollary 3. *Stability of thermal states under a local disturbance: Let H be a bounded k-local Hamiltonian. Let ρ_β be a translationally-invariant thermal state with exponentially decaying correlations as in Def. 1 and energy variance σ^2. Suppose Φ is a quantum channel acting non trivially only on a cubic subsystem of fixed size. Fix $l \in \mathbb{N}$. Let $\rho = \Phi(\rho_\beta)$ evolve under H and let $\langle\rho\rangle$ be translationally invariant. Then the system locally re-thermalizes: There is a constant C independent of N such that*

$$D_S(\rho_\beta) \leq C \left(\sqrt{\frac{D_G}{s^3 N^{\frac{d}{2d+4}}}} + 1\right) \frac{1}{N^{\frac{1}{2d+5}}} \quad (16)$$

for all cubic subsystems S of size l^d, i.e., the system re-thermalizes on any cubic subsystem of fixed size. In particular, this is true for the subsystem on which the channel Φ acted.

We prove this in Appendix C.

Another interesting application is the following. Starting with a system in thermal equilibrium, how much may the Hamiltonian change in order for the system to again equilibrate to a thermal state? In other words, what if, instead of applying a local channel, we start with a thermal state corresponding to a different Hamiltonian? Does the system still thermalize? The following corollary gives a rigorous answer.

Corollary 4. *Let H_0 be a Hamiltonian and $\rho = e^{-\beta H_0}/Z_0$ be the system's initial state, which we assume to have exponentially decaying correlations. Suppose that this state evolves under a bounded k-local Hamiltonian H and has energy variance σ^2 with respect to H. Let $\rho_\beta = e^{-\beta H}/Z$, the thermal state corresponding to H, be translationally invariant and have exponentially decaying correlations. Let $\langle\rho\rangle$ be translationally invariant and $0 < \alpha < \frac{1}{d+2}$. If*

$$\|H - H_0\| \in o\big(N^{\frac{1-(2+d)\alpha}{d+1}}\big) \qquad (17)$$

then there is a constant C independent of N such that

$$D_S(\rho_\beta) \le C \left(\sqrt{\frac{D_G}{s^3 N^{\frac{d}{2d+4}}}} + 1\right) \frac{1}{N^{\alpha/2}} \qquad (18)$$

for all cubic subsystems S of size l^d.

This is also proved in Appendix C. This corollary tells us that we can quench from a Hamiltonian H_0 to a Hamiltonian H, and we get local thermalization to ρ_β for any cubic subsystem of fixed size l^d provided the Hamiltonians are not too different. Maybe surprisingly, we are not restricted to local quenches: The difference between the Hamiltonians may grow as a power law in the system size N.

VI. DISCUSSION

We have seen that, after locally perturbing a quantum system in a thermal state (with exponentially decaying correlations), the system cannot equilibrate to a state distinguishable from the initial thermal state on small subsystems of fixed size. This is not necessarily true if there are long-range correlations in the initial state. Also, notice that one can easily construct counterexamples where an individual small subsystem will not return to thermal equilibrium after being perturbed *without* some form of transport assumption.

In [12] there are infinite lattice analogues of our findings for finite systems. Infinite lattices are a very different setting because information can leave a subsystem and never return to it. Despite this, one could try to construct finite-size analogues of such infinite lattice results, which may be useful for generalizing our work: E.g., it may be possible to go beyond thermal states and show return to equilibrium of more general equilibrium states. The key assumption in [12] is asymptotic abelianness, which effectively guarantees transport. So a natural question to ask is whether there is a finite-size analogue of this assumption leading to similar behaviour. We discuss this further in Appendix D. It is worth mentioning, however, that understanding the transport properties of quantum lattice systems is a difficult task in general.

There are other open questions and possible generalizations. It would be interesting to see whether similar indistinguishability results hold for correlation functions over large distances. In fact, at least for translationally invariant lattice systems, it seems likely that an analogous result to Lemma 2 should hold for two-point correlation functions over long distances on the lattice. Also, there is the question of scaling: Here, we need the free energy difference between the time-average and thermal state to grow slower than a power law in N (with a power strictly smaller than one), see Eq. (12). The intuition from non-interacting systems and [6] would suggest that a free energy difference of $o(N)$ may be sufficient.

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Appendix A: A lower bound for the effective dimension

To lower bound the effective dimension for local Hamiltonian models, we will use a theorem from [7, 20] as a stepping stone. This is a quantum version of the Berry-Esseen theorem. The Berry-Esseen theorem is a more powerful statement than the central limit theorem, as it gives the rate of convergence of a distribution to a Gaussian. Let $H = \sum_\nu E_\nu |\nu\rangle\langle\nu|$ be k-local, i.e., let H be of the form $H = \sum_i h_i$ with h_i acting only on sites j with $\text{dist}(i,j) \leq k$. Let ρ have exponentially decaying correlations as in Def. 1. Then, by Lemma 8 of Ref. [7]

$$
\begin{aligned}
\Delta := \sup_x \big| F(x) - G(x) \big| \\
\leq C_d \left((k+\xi) \left(\frac{\ln(K)}{\ln(N)} + 3 \right) \right)^{2d} \\
\times \left(1 + \frac{s^2}{\ln(N)} \right) \frac{\ln^{2d}(N)}{s^3 \sqrt{N}},
\end{aligned}
\tag{A1}
$$

where C_d depends only on the lattice dimension d and we recall that $s = \sigma/\sqrt{N}$. Here, F and G are the cumulative

distribution functions

$$F(x) = \sum_{\nu: E_\nu \leq x} \langle \nu | \rho | \nu \rangle,$$

$$G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \mathrm{d}y \, e^{-\frac{(y-\mathrm{tr}[\rho H])^2}{2\sigma^2}}. \tag{A2}$$

We may simplify the upper bound on Δ by noting that s is upper-bounded independent of N: Write $v_{k,d} = \max_i |\mathrm{supp}[h_i]|$. Then

$$\begin{aligned}
s^2 &= \frac{1}{N} \sum_{i,j} \langle h_i h_j \rangle - \langle h_i \rangle \langle h_j \rangle \\
&\leq K v_{k,d}^2 \frac{1}{N} \sum_{i,j} e^{-\mathrm{dist}(\mathrm{supp}[h_i],\mathrm{supp}[h_j])/\xi} \\
&\leq K v_{k,d}^2 e^{2k/\xi} \sum_{l=0}^{\infty} e^{-l/\xi} b_{d,l},
\end{aligned} \tag{A3}$$

where $b_{d,l} = \max_i |\{j \,|\, \mathrm{dist}(i,j) = l\}|$ is the maximum surface area of a ball of radius l centred at i. Hence, we have that there is a constant C independent of N such that

$$\Delta \leq C \frac{\ln^{2d}(N)}{s^3 \sqrt{N}}. \tag{A4}$$

To apply the above theorem, we note that for any $\epsilon > 0$

$$\begin{aligned}
\frac{1}{d_{\mathrm{eff}}} &= \sum_\nu \mathrm{tr}^2[\rho P_\nu] \\
&\leq \max_\nu \mathrm{tr}[\rho P_\nu] \\
&\leq \max_x [F(x) - F(x-\epsilon)].
\end{aligned} \tag{A5}$$

Applying the bound in Eq. (A1) and the mean value theorem we thus find

$$\begin{aligned}
\frac{1}{d_{\mathrm{eff}}} &\leq 2\Delta + \max_x [G(x) - G(x-\epsilon)] \\
&\leq 2\Delta + \epsilon.
\end{aligned} \tag{A6}$$

As ϵ was arbitrary we thus have that

$$\frac{1}{d_{\mathrm{eff}}} \leq 2\Delta. \tag{A7}$$

If we instead assume that $\langle \rho \rangle$ has exponentially decaying correlations, we use

$$\mathrm{tr}[\rho P_\nu] = \mathrm{tr}[\langle \rho \rangle P_\nu] \tag{A8}$$

to arrive at the same bound on d_{eff}. Note also that $\mathrm{tr}[\rho H] = \mathrm{tr}[\langle \rho \rangle H]$ and $\mathrm{tr}[\rho H^2] = \mathrm{tr}[\langle \rho \rangle H^2]$.

Appendix B: Lemma 2 and proof of Eq. (13)

We rely on Proposition 2 of [7], which we state in a slightly simplified version:

Lemma 5. *Let σ be a state with exponentially decaying correlations as in Def. 1. Let $\alpha > 0$ and let N and $l \in \mathbb{N}$ such that $l \leq \frac{n+1}{2}$ and*

$$3N^\alpha + \frac{2\xi \ln(d_{\mathrm{loc}}) + 3}{\xi \ln(2)} l^d + \log(N^{\frac{\ln(K)}{\ln(N)}+3}) \leq \frac{1}{4\xi^{\frac{d}{d+1}}} N^{\frac{1-\alpha}{d+1}}. \tag{B1}$$

Denote by $\mathcal{C}_l$ the set of all hypercubes on the lattice with length of side l. Let τ be a state. If

$$S(\tau \| \sigma) \leq \frac{1}{4\xi^{\frac{d}{d+1}}} N^{\frac{1-(2+d)\alpha}{d+1}} \tag{B2}$$

then the average local trace distance between σ and τ is bounded as

$$\mathbb{E}_{C \in \mathcal{C}_l} \|\sigma - \tau\|_C \leq \frac{7}{N^{\alpha/2}}, \tag{B3}$$

where $\mathbb{E}_{C \in \mathcal{C}_l}$ denotes the average over all $C \in \mathcal{C}_l$.

We arrive at the Lemma in the main text by noting that Eq. (B1) is fulfilled for sufficiently large N if $\alpha < \frac{1}{d+2}$ and $l^d \in o(N^{\frac{1-\alpha}{d+1}})$. Combining Eq. (6) with Lemma 1 and 2, we find by the triangle inequality (C denotes a constant independent of N, which may change from line to line)

$$\begin{aligned}
\mathbb{E}_{S \in \mathcal{S}_l} D_S(\rho_\beta) &\leq D_S(\langle \rho \rangle) + \mathbb{E}_{S \in \mathcal{S}_l} \|\langle \rho \rangle - \rho_\beta\|_S \\
&\leq \frac{1}{2} \sqrt{\frac{D_G d_S^2}{d_{\mathrm{eff}}}} + \frac{C}{N^{\alpha/2}} \\
&\leq C \left(\sqrt{\frac{D_G \ln^{2d}(N)}{s^3 \sqrt{N}}} + \frac{1}{N^{\alpha/2}} \right),
\end{aligned} \tag{B4}$$

where

$$\sqrt{\frac{\ln^{2d}(N)}{\sqrt{N}}} = \frac{\ln^d(N)}{N^{\frac{1}{2}(\frac{1}{d+2}-\alpha)}} \frac{1}{N^{\frac{d}{4(d+2)}}} \frac{1}{N^{\alpha/2}}, \tag{B5}$$

which implies Eq. (13) as $\frac{\ln^d(N)}{N^{\frac{1}{2}(\frac{1}{d+2}-\alpha)}} \to 0$ because $\alpha < \frac{1}{d+2}$.

Appendix C: Proof of the corollaries

To arrive at Corollary 3 we need to (i) show that ρ has exponentially decaying correlations, (ii) bound the relative entropy, and (iii) relate the energy variance of ρ to that of ρ_β.

We start with (i): If X, Y are not in the subsystem that the channel acts on, then two-point correlations of operators P and Q with supports in X and Y, respectively, decay exponentially as they do for ρ_β as in Def. 1. Now denote by A the subsystem the channel acts on and by B the rest of the system. If $\mathrm{dist}(X, Y) > \mathrm{diam}(A)$ then either $X, Y \in B$ or one of the supports has overlap with A

and the other is contained in B. Let us consider the case $X \cap A \neq \emptyset$ and $Y \subset B$. Then, if $\rho = \Phi(\sigma) = \sum_i K_i^\dagger \sigma K_i$ with K_i acting only on A and $\sum_i K_i K_i^\dagger = \mathbb{I}$ and with adjoint $\Phi^*(\cdot) = \sum_i K_i \cdot K_i^\dagger$,

$$
\begin{aligned}
\mathrm{tr}[\rho P Q] &- \mathrm{tr}[\rho P]\,\mathrm{tr}[\rho Q] = \mathrm{tr}[\rho P Q] - \mathrm{tr}[\rho P]\,\mathrm{tr}[\rho_\beta Q] \\
&= \|\Phi^*(P)\| \frac{\mathrm{tr}[\rho_\beta \Phi^*(P) Q] - \mathrm{tr}[\rho_\beta \Phi^*(P)]\,\mathrm{tr}[\rho_\beta Q]}{\|\Phi^*(P)\|} \\
&\leq K|X \cup A||Y| e^{-\mathrm{dist}(Z,Y)/\xi} \\
&\leq K|A||X||Y| e^{-\mathrm{dist}(Z,Y)/\xi}
\end{aligned}
\tag{C1}
$$

as ρ_β has exponentially decaying correlations and $\|\Phi^*(P)\| \leq 1$ (via corollary 2.9 in [33]). Here, $Z = \mathrm{supp}[\Phi^*(P)] \subset X \cup A$ such that $\mathrm{dist}(X,Y) \leq \mathrm{dist}(Z,Y) + \mathrm{diam}(A)$. Hence, ρ has exponentially decaying correlations with $K' = |A| e^{\mathrm{diam}(A)/\xi} K$.

We now address (ii): Denote the subsystem on which Φ acts by A and the rest of the system by B. Then $\rho = \Phi(\rho_\beta)$ and ρ_β coincide on B. Writing $H = H_A + H_B$ with H_B acting exclusively on B and H_A collecting the remaining terms we have $\mathrm{tr}[H_B \Phi(\rho_\beta)] = \mathrm{tr}[H_B \rho_\beta]$ such that

$$
\begin{aligned}
S(\langle\rho\rangle\|\rho_\beta) &= \beta\mathrm{tr}[H(\langle\rho\rangle - \rho_\beta)] + S(\rho_\beta) - S(\langle\rho\rangle) \\
&= \beta\mathrm{tr}[H(\Phi(\rho_\beta) - \rho_\beta)] + S(\rho_\beta) - S(\langle\rho\rangle) \\
&\leq 2\beta\|H_A\| + S(\rho_\beta) - S(\langle\rho\rangle),
\end{aligned}
\tag{C2}
$$

where, as $|A|$ is independent of N and H is bounded and k-local, $\|H_A\|$ is bounded independent of N. The entropy difference may be bounded by using $S(\langle\rho\rangle) \geq S(\rho)$ and the Araki–Lieb inequality $|S(\sigma_A) - S(\sigma_B)| \leq S(\sigma) \leq S(\sigma_A) + S(\sigma_B)$ [34], which holds for any state σ. We find

$$
\begin{aligned}
S(\rho_\beta) &- S(\langle\rho\rangle) \\
&\leq S([\rho_\beta]_A) + S([\rho_\beta]_B) - |S(\rho_A) - S(\rho_B)| \quad \text{(C3)} \\
&\leq S([\rho_\beta]_A) + S(\rho_A) \leq 2|A|\ln(d_{\mathrm{loc}})
\end{aligned}
$$

as $[\rho_\beta]_B = \rho_B$. Hence, $S(\langle\rho\rangle\|\rho_\beta)$ is bounded from above by a constant independent of N. Thus, we may set $\alpha = \frac{1}{d+2.5}$ to find

$$
\mathbb{E}_{S \in \mathcal{S}_l} D_S(\rho_\beta) \leq C \left(\sqrt{\frac{D_G}{s_\rho^3 N^{\frac{d}{2d+4}}}} + 1 \right) \frac{1}{N^{\frac{1}{2d+5}}}, \tag{C4}
$$

where $s_\rho^2 = \sigma_\rho^2 / N$ and σ_ρ^2 is the energy variance of the initial state $\rho = \Phi(\rho_\beta)$ with respect to H. Now,

$$
\begin{aligned}
\sigma_\rho^2 = &\sum_{i,j \in A} \left(\langle h_i h_j \rangle_\rho - \langle h_i \rangle_\rho \langle h_j \rangle_\rho \right) \\
&+ \sum_{i \in A, j \in B} \left(\langle h_i h_j \rangle_\rho - \langle h_i \rangle_\rho \langle h_j \rangle_\rho \right) \\
&+ \sum_{j \in A, i \in B} \left(\langle h_i h_j \rangle_\rho - \langle h_i \rangle_\rho \langle h_j \rangle_\rho \right) \\
&+ \sum_{i,j \in B} \left(\langle h_i h_j \rangle_\rho - \langle h_i \rangle_\rho \langle h_j \rangle_\rho \right)
\end{aligned}
\tag{C5}
$$

and similarly for ρ_β. As ρ and ρ_β coincide on B and both states have exponentially decaying correlations and $|A|$ is upper-bounded independently of N, there is hence a constant C independent of N such that $|\sigma^2 - \sigma_\rho^2| \leq C$, i.e.,

$$
s_\rho^2 \geq s^2 \left(1 - \frac{C}{N s^2} \right). \tag{C6}
$$

If $s^{3/2} \leq N^{-\frac{d}{4d+8} - \frac{1}{2d+5}}$ then Eq. (16) holds trivially, i.e., we may w.l.o.g. assume that $s^{3/2} \geq N^{-\frac{d}{4d+8} - \frac{1}{2d+5}}$. Then $\frac{1}{s^2} \leq N^{\frac{d}{3d+6} + \frac{4}{6d+15}} \leq N^{\frac{1}{3}}$, i.e.,

$$
s_\rho^2 \geq s^2 \left(1 - \frac{C}{N^{\frac{2}{3}}} \right) \tag{C7}
$$

such that for sufficiently large N we have $s_\rho^2 \geq s^2/2$, which, combined with Eq. (C4), implies Eq. (16).

To prove Corollary 4 we need only bound the relative entropy. As above, we have

$$
\begin{aligned}
S(\langle\rho\rangle\|\rho_\beta) &= \beta\mathrm{tr}[H(\langle\rho\rangle - \rho_\beta)] + S(\rho_\beta) - S(\langle\rho\rangle) \\
&\leq \beta\mathrm{tr}[H(\langle\rho\rangle - \rho_\beta)] + S(\rho_\beta) - S(\rho) \\
&= \beta\mathrm{tr}[(H - H_0)\rho] + \ln(Z) - \ln(Z_0) \\
&\leq \beta\|H - H_0\| + \ln(Z) - \ln(Z_0),
\end{aligned}
\tag{C8}
$$

where

$$
\ln(Z) - \ln(Z_0) = \int_0^1 dr \frac{1}{Z(r)} \frac{\partial}{\partial r} Z(r) \tag{C9}
$$

and $Z(r) = \mathrm{tr}[e^{-\beta H_r}]$, $H_r = H_0 + r(H - H_0)$. Now, we use the formula (see, e.g., section 6.5 of [35])

$$
\frac{\partial}{\partial r} e^{-\beta H_r} = -\beta \int_0^1 ds\, e^{-\beta s H_r} (H - H_0) e^{-\beta(1-s)H_r} \tag{C10}
$$

such that by the cyclic property of the trace

$$
\frac{\partial}{\partial r} Z(r) = \beta\mathrm{tr}[(H_0 - H) e^{-\beta H_r}] \tag{C11}
$$

and hence

$$
S(\langle\rho\rangle\|\rho_\beta) \leq 2\beta\|H - H_0\|. \tag{C12}
$$

Appendix D: Comparison with Robinson's construction

In this section, we discuss the relation to Robinson's construction in Ref. [12]. There, infinite lattice analogues of our results may be found and the key assumption in [12] is asymptotic abelianness, which effectively guarantees transport. So a natural question to ask is whether there is a finite-size analogue of this assumption leading to similar behaviour.

Suppose we have a state ω that commutes with the Hamiltonian H. And let U_S be a unitary localized on a

subsystem S. Then suppose that at $t = 0$ we apply U_S to the state, getting $U_S \omega U_S^\dagger$. This evolves over time as $e^{-iHt}(U_S \omega U_S^\dagger)e^{iHt}$.

We assume that equilibration occurs to the time average state $\langle U_S \omega U_S^\dagger \rangle$. Let A_S be an observable on S, then the difference between the expectation values is

$$
\begin{aligned}
\mathrm{tr}\left[A_S \left(\omega - \langle U_S \omega U_S^\dagger \rangle \right)\right] &= \mathrm{tr}\left[\langle A_S \rangle \left(\omega - U_S \omega U_S^\dagger \right)\right] \\
&= \mathrm{tr}\left[\left(\langle A_S \rangle - U_S^\dagger \langle A_S \rangle U_S \right) \omega \right] \\
&\leq \|\langle A_S \rangle - U_S^\dagger \langle A_S \rangle U_S\| \\
&= \|\,[U_S, \langle A_S \rangle]\,\|,
\end{aligned}
\tag{D1}
$$

where $\langle A_S \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{d}t\, e^{iHt} A_S e^{-iHt}$ is the time-average observable in the Heisenberg picture. If we assume that the dynamics spreads $A_S(t)$ out over time, so that

$$
\|\,[U_S, \langle A_S \rangle]\,\| \to 0 \text{ as } N \to \infty,
\tag{D2}
$$

then the expectation values coincide as $N \to \infty$. The assumption of asymptotic abelianness in [12] is a little different. Because the setting is an infinite lattice, one can take limits of expectation values as time goes to infinity. So the condition in [12] is essentially

$$
\|\,[U_S, A_S(t)]\,\| \to 0 \text{ as } t \to \infty.
\tag{D3}
$$

There are other technical assumptions that need to be made in the infinite lattice setting, but they are not important here. It is not clear when one can verify that the condition in Eq. (D2) holds, except for simple cases. Take the example of a translationally-invariant non-interacting free fermion model with non-degenerate *single-particle* energies. Then taking an observable like $A_S = a_n^\dagger a_n$, which counts the number of particles on site n, $\langle A_S \rangle = \frac{1}{N} \sum_n a_n^\dagger a_n$. Therefore, $\|\,[U_S, \langle A_S \rangle]\,\| = O(1/N)$. This scaling is probably the best case scenario. More generally, one probably gets slower decay with N when the condition holds.