CS 101 Computer Programming and Utilization

Practice Problems

Param Rathour
https://paramrathour.github.io/CS101

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Disclaimer

These are **optional** problems. As these problems are pretty involving, my advice to you would be to first solve exercises given in slides, lab optional questions and get comfortable with the course content. The taught methods will suffice to solve these problems. (You are free to use 'other' stuff but not recommended)

Good Programming Practices

- Clearly writing documentation explaining what the program does, how to use it, what quantities it takes as input, and what quantities it returns as output.
- Using appropriate variable/function names.
- Extensive internal comments explaining how the program works.
- Complete error handling with informative error messages.
  For example, if $a = b = 0$, then the $\gcd(a, b)$ routine should return the error message "$\gcd(0, 0)$ is undefined" instead of going into an infinite loop or returning a "division by zero" error.

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§1. Practice Problems 1 - Introduction

1.1. Hilbert Curve

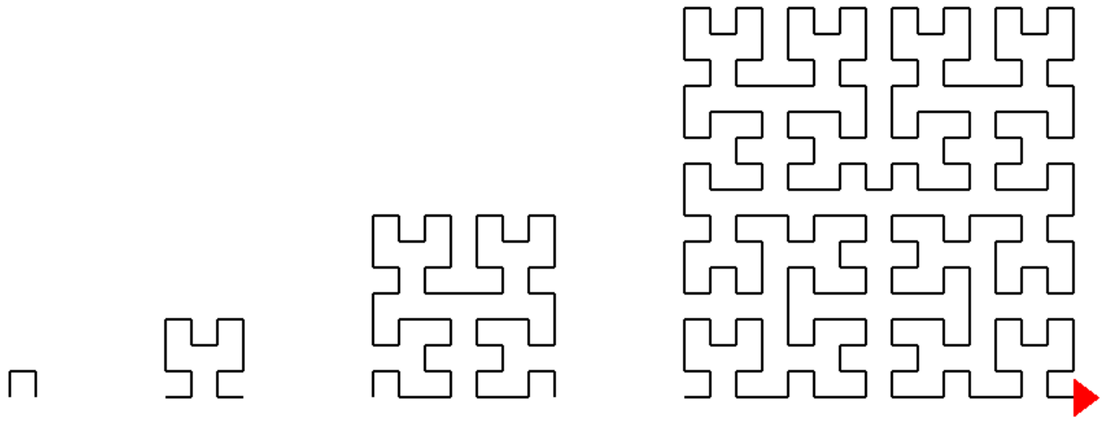


Figure 1: Hilbert Curve

Problem Statement:
Take an integer as input and draw the corresponding iteration of this fractal using turtleSim
You may think along these lines

Step 1 Find a simple pattern in these iterations

Step 2 Think how can you implement this pattern in an efficient way (here think in the number of lines of code you have to write. **Word of caution**: this is just one of the possible definitions of efficient code)

Step 3 Do you think that you need something that will implement/shorten your code?
How will it look like? (it's a feature)

Feel free to discuss your thoughts on this.

Note. *For people comfortable with the basics of C++, this shouldn't be difficult. You may try this*

Fun Videos

Hilbert's Curve: Is infinite math useful?
Recursive PowerPoint Presentations [Gone Fractal!]

Book Chapters for Graphics

Additional chapters of the book on Simplecpp graphics demonstrating its power
(It is just a list, you are not expected to understand/study things, CS101 is for a reason :P)

Chapter 1 Turtle graphics

Chapter 5 Coordinate based graphics, shapes besides turtles

Chapter 15.2.3 Polygons

Chapter 19 Gravitational simulation

Chapter 20 Events, Frames, Snake game

Chapter 24.2 Layout of math formulae

Chapter 26 Composite class

Chapter 28 Airport simulation

§2. Practice Problems 2 - Loops

2.1. Pisano Period

You probably heard about Fibonacci Numbers!

The Fibonacci numbers are the numbers in the integer sequence: (defined by the recurrence relation)

$$\begin{aligned} &F_0 = 0 \\ &F_1 = 1 \\ &F_n = F_{n-1} + F_{n-2} \quad n \in \mathbb{Z} \quad \text{(They can be extended to negative numbers)} \end{aligned} \tag{1}$$

For any integer n, the sequence of Fibonacci numbers F_i taken modulo n is periodic.

The Pisano period, denoted $\pi(n)$, is the length of the period of this sequence.

For example, the sequence of Fibonacci numbers modulo 3 begins:

$$0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, 0, 2, 2, 1, 0, \ldots \text{(A082115)}$$

This sequence has period 8, so $\pi(3) = 8$. (Basically, the remainder when these numbers are divided by n is a repeating sequence. You have to find the length of sequence)

Problem Statement:

(a) Find Pisano period of n numbers $k_1, k_2, \ldots, k_n$

Input Format
n
$k_1, k_2, \ldots, k_n$
Output Format
$\pi(k_i)$ (each on a newline)
Sample Input
```
3
3 10 25
```
Sample Output
```
8
60
100
```

(b) For n numbers $k_1, k_2, \ldots, k_n$, find $\max(\pi(i))$ for $i = 1, 2, \ldots, k$ and corresponding i
If there are 2 (or more) such i's, output smallest of them

Input Format
n
$k_1, k_2, \ldots, k_n$
Output Format
k $\pi(k)$ (each pair on a newline)
Here k is smallest possible integer satisfying $\pi(k) = \max(\pi(i))$ for possible i
Sample Input
```
5
20 40 60 80 100
```
Sample Output
```
10 60
30 120
50 300
50 300
98 336
```

2.2. π

$$\frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!} \tag{2}$$

Note. $n!!$ *is called double factorial.* $n!! \neq (n!)!$.

Problem Statement:
Calculate π using first $k_i + 1$ terms [1] of Equation (2) for n different natural numbers $k_1, k_2, \ldots, k_n$.
Give your answers correct to 10 decimal places

Input Format
n
$k_1, k_2, \ldots, k_n$
Output Format
Calculated π for k_i (each on a newline)
Sample Input
```
3
10 20 30
```
Sample Output
```
3.1411060216
3.1415922987
3.1415926533
```

2.3. Simpson's Rule

A method for numerical integration

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\right) \tag{3}$$

Note. *Simpson's rule can only be applied when an odd number of ordinates is chosen.*

Problem Statement:
Solve Equation (4) giving the answers correct to 7 decimal places (Use 101 ordinates)

$$\int_{0.5}^{1} \frac{\sin\theta}{\theta}\,\mathrm{d}\theta \tag{4}$$

Correct Answer $=$ 0.4529756

[1] i.e calculate π till $\frac{k_i!}{(2k_i+1)!!}$ term

§3. Practice Problems 3 - Functions

3.1. Calendar

Write a function that calculates the day of the week for any particular date in the past or future.
Consider Gregorian calendar (AD)

Task 1: As a programming exercise, try the naive approach:
Starting from 1 Jan 0001 (Saturday) and calculate day after day till you reach the given date
Use `switch-case` statement

Task 2: Try to make more efficent algorithm (reduce completion time) than Task 1
Implement it, and discuss your approach with me.

Also check for invalid dates (Write another function for this)
If dates are invalid, output -**1**

Input Format
n
Followed by n dates in **Date Month Year** format
Output Format
Day of the Week
Sample Input
5
19 2 1627
29 2 1700
15 4 1707
22 12 1887
23 6 1912
Sample Output
Monday
-1
Friday
Thursday
Sunday

Note. *Use your Task 1 program to check Task 2 implementation.*

3.2. Farey Sequence

This sequence has all rational numbers in range $[0/1 \text{ to } 1/1]$ sorted *in increasing order* such that the denominators are less than or equal to n and all numbers are in *reduced forms* i.e., 2/4 does not belong to this sequence as it can be reduced to 1/2.

Input Format
n
Output Format
Corresponding numbers in sequence in p/q format
Sample Input
7
Sample Output
0/1 1/7 1/6 1/5 1/4 2/7 1/3 2/5 3/7 1/2 4/7 3/5 2/3 5/7 3/4 4/5 5/6 6/7 1/1

Can you find efficient solution?

Fun Video

Funny Fractions and Ford Circles

3.3. Thue-Morse Sequence aka Fair Share Sequence

Thue-Morse Sequence is the infinite binary sequence obtained by starting with 0 and successively appending the Boolean complement of the sequence obtained thus far (called prefixes of the sequence).
First few steps :

- Start with 0
- Append complement of 0, we get 01
- Append complement of 01, we get 0110
- Append complement of 0110, we get 01101001

Problem Statement:
Consider appending complement of a prefix to itself as one iteration
Define a function to take a positive integer n as input then iterate n times to print the first 2^n digits

Input Format
n
Output Format
Corresponding digits in sequence
Sample Input
6
Sample Output
0110100110010110100101100110100110010110011010010110100110010110

Again, can you find better solution?

Fun Video
The Fairest Sharing Sequence Ever

3.4. Collatz Conjecture

Consider the following operation on an arbitrary positive integer:

- If the number is even, divide it by two.
- If the number is odd, triple it and add one.

Collatz Conjecture states that no matter which positive integer we start with; we always end up with 1.

Problem Statement:
Define a function which performs this operation repeatedly on the result at each step; beginning with a given input n ($n < 10^6$), returns the number of operations required to reach 1[2]

Input Format
Arbitrary number of testcases (each space separated)   Stop when input is negative
Output Format
Count of operations for each number (each on a newline)
Sample Input
1 3 7 9 27 871 77031 -1
Sample Output
0
7
16
19
111
178
350

[2]As of 2020, the conjecture has been checked by computer for all starting values up to $2^{68} \approx 2.95 \times 10^{20}$, so sequence from n will reach 1

§4. Practice Problems 4 - Recursion

Practice Problem 4 are inspired from the following video do watch till 6:10 to get clear understanding of recursion
5 Simple Steps for Solving Any Recursive Problem

- What's the simplest possible input?
- Play around with examples and visualize!
- Relate hard cases to simpler cases
- Generalize the pattern
- Write code by combining recursive pattern with base case

4.1. Ackermann function

Write a recursive function $\mathrm{A}()$ that takes two inputs n and m and outputs the number $\mathrm{A}(m,n)$ where $\mathrm{A}(m,n)$ is defined as

$$\mathrm{A}(0,n) = n+1 \tag{5}$$
$$\mathrm{A}(m+1,0) = \mathrm{A}(m,1) \tag{6}$$
$$\mathrm{A}(m+1,n+1) = \mathrm{A}(m,\mathrm{A}(m+1,n)) \tag{7}$$

Problem Statement:
For k pairs $(m_1,n_1),(m_2,n_2),\ldots,(m_k,n_k)$, find $\mathrm{A}(m,n)$ for such m,n

Input Format
k
m_i n_i (each pair on a newline)
Output Format
$\mathrm{A}(m_i,n_i)$ (each result on a newline)
Sample Input

```
7
0 0
0 4
1 3
2 2
3 4
4 0
4 1
```

Sample Output

```
1
5
5
7
125
13
65533
```

4.2. GridPaths

Write a recursive function $\text{NumberOfGridPaths}()$ that takes two inputs n and m and outputs the number of unique paths from the top left corner to bottom right corner of a $n \times m$ grid.

Constraints: $n, m \geq 1$ and you can only move down or right 1 unit at a time.
Examples given in Figure 2

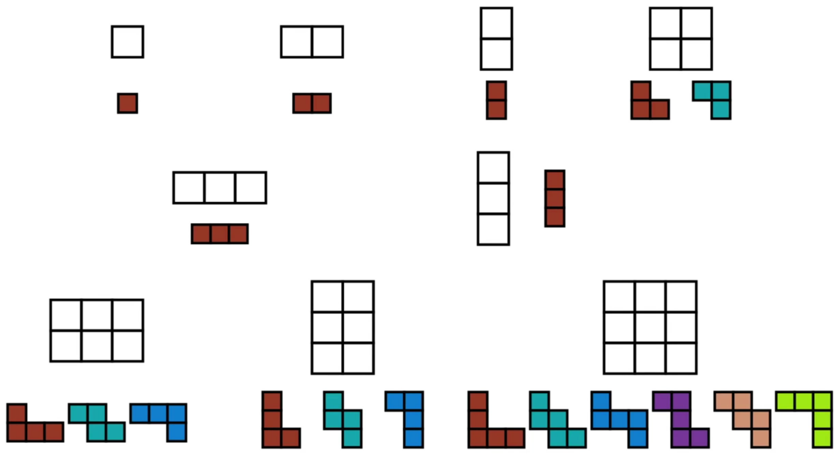


Figure 2: Number of Grid Paths for $m, n \in \{1, 2, 3\}$

Problem Statement:
For k pairs $(n_1, m_1), (n_2, m_2), \dots, (n_k, m_k)$, find $\text{NumberOfGridPaths}(n, m)$ for such n, m

Input Format
k
n_i m_i (each pair on a newline)
Output Format
$\text{NumberOfGridPaths}(n_i, m_i)$ (each result on a newline)
Sample Input

```
6
1 1
2 5
3 3
6 3
7 10
17 8
```

Sample Output

```
1
5
6
21
5005
245157
```

Can you find efficient solution?

4.3. Delannoy Numbers

Delannoy numbers describes the number of paths from the southwest corner (0, 0) of a rectangular grid to the northeast corner (m, n), using only single steps north, northeast, or east.
Write a recursive function $\text{DelannoyNumber}()$ to count number of these paths Constraints: $n, m \geq 0$
Examples given in Figure 3

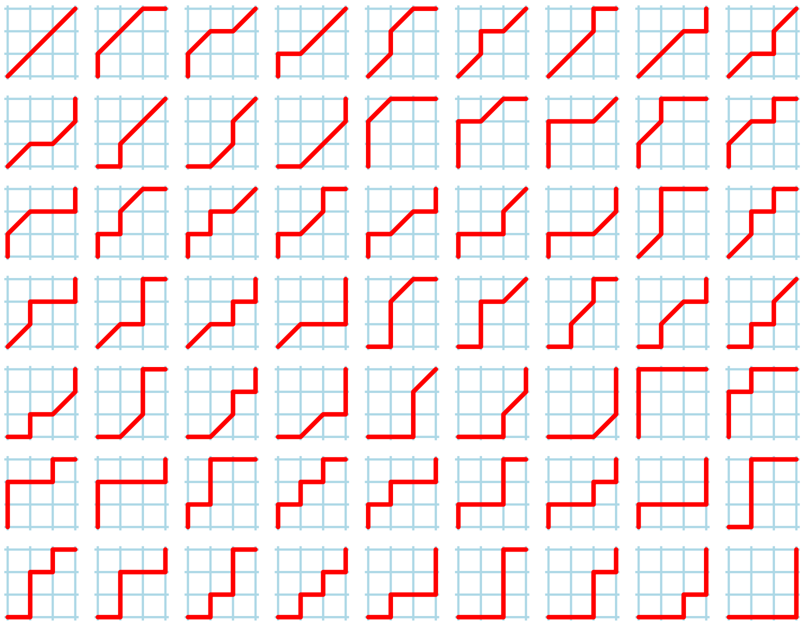


Figure 3: DelannoyNumber(3,3) = 63

Problem Statement:
For k pairs $(n_1, m_1), (n_2, m_2), \ldots, (n_k, m_k)$, find $\text{DelannoyNumbers}(n, m)$ for such n, m

Input Format
k
n_i m_i (each pair on a newline)
Output Format
$\text{DelannoyNumber}(n_i, m_i)$ (each result on a newline)
Sample Input
```
6
1 1
2 5
3 3
6 3
7 10
17 8
```
Sample Output
```
3
61
63
377
433905
245157
62390545
```

4.4. Partitions

(a) Write a recursive function $\mathrm{NumberOfPartitions}()$ that counts the number of ways you can partition n objects using parts up to m

Constraints: $n, m > 0$
Examples given in Figure 4

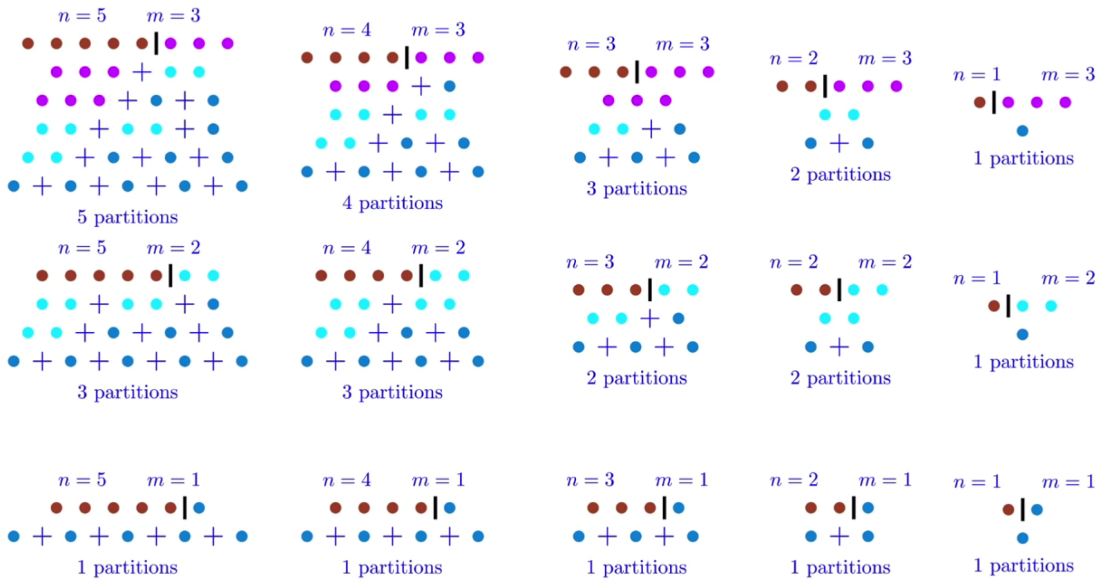


Figure 4: Partition n objects using parts up to m

Can you find efficient solution?
Problem Statement:
For k pairs $(n_1, m_1), (n_2, m_2), \ldots, (n_k, m_k)$, find $\mathrm{NumberOfPartitions}(n, m)$ for such n, m

Input Format
k
n_i m_i (each pair on a newline)
Output Format
$\mathrm{NumberOfPartitions}(n_i, m_i)$ (each result on a newline)
Sample Input
```
7
1 1
2 5
3 3
6 3
7 10
17 8
20 12
```
Sample Output
```
1
2
3
7
15
230
582
```

(b) Number of partitions $\mathrm{P}(n)$ of an integer n is same as $\mathrm{NumberOfPartitions}(n, n)$

Theorem 1 (Pentagonal Number Theorem)**.** *This theorem relates the product and series representations of the Euler function*

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k\, x^{k(3k-1)/2} = 1 + \sum_{k=1}^{\infty}(-1)^k \left(x^{k(3k+1)/2} + x^{k(3k-1)/2}\right) \tag{8}$$

In other words,

$$(1-x)(1-x^2)(1-x^3)\cdots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots \tag{9}$$

The exponents $1, 2, 5, 7, 12, \ldots$ *on the right hand side are called (generalized) pentagonal numbers (A001318) and are given by the formula* $g_k = k(3k-1)/2$ *for* $k = 1, -1, 2, -2, 3, -3, \ldots$

The equation (9) implies a recurrence for calculating $\mathrm{P}(n)$, the number of partitions of n:

$$\mathrm{P}(n) = \mathrm{P}(n-1) + \mathrm{P}(n-2) - \mathrm{P}(n-5) - \mathrm{P}(n-7) + \cdots \tag{10}$$

or more formally,

$$\mathrm{P}(n) = \sum_{k\neq 0}(-1)^{k-1}\,\mathrm{P}(n-g_k) \tag{11}$$

Problem Statement:
Write a function $\mathrm{P}()$ which calculates number of partitions using equation (10)

Input Format
Arbitrary number of testcases (each space separated)
Stop when input is negative
Output Format
Number of partitions for each testcase (each on a newline)
Sample Input

```
1 2 4 8 16 32 64 128 -1
```

Sample Output

```
1
2
5
22
231
8349
1741630
4351078600
```

Crazy Video

The hardest What comes next (Euler's pentagonal formula)

Exciting Puzzle

Towers of Hanoi: A Complete Recursive Visualization

§5. Practice Problems 5 - Arrays

5.1. Horner's method

An algorithm for polynomial evaluation

$$\begin{aligned} \mathrm{P}(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \\ &= a_0 + x\Big(a_1 + x\Big(a_2 + x\big(a_3 + \cdots + x(a_{n-1} + x\,a_n)\cdots\big)\Big)\Big) \end{aligned} \tag{12}$$

This allows the evaluation of a polynomial of degree n with only n multiplications and n additions. This is optimal, since there are polynomials of degree n that cannot be evaluated with fewer arithmetic operations

Problem Statement:
Write a function Horner() takes three inputs an array of coefficients P, degree of polynomial n & input x and returns $\mathrm{P}(x)$
For k testcases with coefficients $P_1[\ ], P_2[\ ], \ldots, P_n[\ ]$, degree $n_1, n_2, \ldots, n_k$ and variable $x_1, x_2, \ldots, x_n$ find $P_i(x_i)$ for each i in $\{1, 2, \ldots, n\}$

Input Format
k (number of testcases)
n_i (degree of polynomial), x and $P_i[\ ]$ ($n_i + 1$ coefficients with i^{th} index coefficient of i^{th} power)
Output Format
$\mathrm{Horner}(\mathrm{P}, n, x)$ (each result on a newline)
Sample Input

```
5
0   1   1
1   2   -3 2
2   2   15 -8 7
3   3   2 -1 -3 4
6   5   21 10 19 47 48 9 27
```

Sample Output

```
1
1
27
80
486421
```

5.2. Large Factorials

Problem Statement:
Compute factorial of large numbers $n > 20$

Note. `long long int` *can not store more than 20 digits. Maximum digits in testcases* $= 150$

Input Format
n (number of testcases)
k_i for $i = 1, 2, \ldots, n$
Output Format
$\mathrm{factorial}(k_i)$ (each result on a newline)
Sample Input

```
6
21 33 57 69 77 93
```

Sample Output

```
51090942171709440000
8683317618811886495518194401280000000
40526919504877216755680601905432322134980384796226602145184481280000000000000
171122452428141311372468338881272839092270544893520369393648040923257279754140647424000000000000000
145183092028285869634070784086308284983740379224208358846781574688061991349156420080065207861248000000000000000000
115677250708164157475920516230624043621475322957641353518614228121324680712146731521520328951684484530383899628938707809075200000000000000000000000
```

5.3. Remove Duplicates

Problem Statement:
Input an integer array and output all unique elements of that array (for repeated elements keep only first occurence)

Input Format
k (number of testcases)
n_i (size of array) followed by n_i elements ($<= 20$) of array (each testcase on a newline)
Output Format
Each result on a newline
Sample Input

```
2
15   1 4 9 10 18 1 4 9 16 17 6 10 11 13 17
20   1 3 4 8 16 1 11 15 17 20 1 3 14 15 18 4 5 17 19 20
```

Sample Output

```
1 4 9 10 18 16 17 6 11 13
1 3 4 8 16 11 15 17 20 14 18 5 19
```

5.4. Maximum Element

Problem Statement:
Given an array of integers which is initially increasing and then decreasing, find the maximum value in the array.
A naive approach is of $\Theta(n)$ time complexity, try to think an algorithm which runs in $\Theta(\log n)$ time (assume strictly increasing and strictly decreasing for this case)

Input Format
k (number of testcases)
n_i (size of array) followed by n_i distinct elements of array (each testcase on a newline)
Output Format
Max element of each array (each result on a newline)
Sample Input

```
2
8   1 4 9 10 18 17 13 11
9   8 13 25 87 167 235 454 512 32
```

Sample Output

```
18
512
```

5.5. Majority Element

Problem Statement:
Write a function which takes an array and prints the majority element (if it exists), otherwise prints -1.
A majority element in an array $A[\,]$ of size n is an element that appears more than $n/2$ times
A naive approach is of $\Theta(n^2)$ time complexity, try to think an algorithm which runs in $\Theta(n \log n)$ time.
Can you do it in $\Theta(n)$ time?

Input Format
k (number of testcases)
n_i (size of array) followed by n_i elements of array (each testcase on a newline)
Output Format
Max element of each array (each result on a newline)
Sample Input

```
2
9   3 3 4 2 4 4 2 4 4
8   3 3 4 2 4 4 2 4
```

Sample Output

```
4
-1
```

5.6. Matrix Inversion

Problem Statement:
Write a function which finds multiplicative inverse of a square matrix (if it exist) using Gaussian Elimination
Each element of your output should contain exactly 2 decimal places

Input Format
k (number of testcases)
n (size of matrix) followed by $n \times n = n^2$ elements of array
Output Format
Inverted Matrix
Sample Input

```
3
2
4 7
2 6
3
2 9 2
3 7 1
1 1 1
5
1 –4 3 –4 –4
–10 4 0 7 –3
–5 4 –5 0 –3
9 –2 –4 –7 1
8 –4 3 –7 5
```

Sample Output

```
0.60 –0.70
–0.20 0.40

–0.43 0.50 0.36
0.14 0.00 –0.29
0.29 –0.50 0.93

0.78 17.89 –5.56 11.78 5.67
1.69 39.93 –11.70 25.35 12.61
1.04 22.19 –6.74 14.37 7.22
–0.19 -2.93 0.70 –1.85 –1.11
–0.78 –14.89 4.56 –9.78 –4.67
```

§6. Practice Problems 6 - Give me name for this

6.1. Currency sums

India's currency consists of 1,2,5,10,20,50,100,200,500,2000.
Problem Statement:
Write a function to get number of different ways n can be made using any number of given coins/notes.

Input Format
n
$k_1, k_2, \ldots, k_n$
Output Format
Correct answer (each on a newline)
Sample Input
```
9
1 2 5 10 20 50 100 200 500 2000
```
Sample Output
```
1
4
11
41
451
4563
73682
6295435
27984272287
```

6.2. QuickSort

An efficient sorting algorithm.
It is a divide and conquer algorithm like merge sort discused in class.
It first divides the input array into two smaller sub-arrays: the low elements and the high elements. It then recursively sorts the sub-arrays. The steps are:

- Pick an element, called a pivot, from the array.
- Partitioning: reorder the array so that all elements with values less than the pivot come before the pivot, while all elements with values greater than the pivot come after it (equal values can go either way). After this partitioning, the pivot is in its final position. This is called the partition operation.
- Recursively apply the above steps to the sub-array of elements with smaller values and separately to the sub-array of elements with greater values.

The base case of the recursion is arrays of size zero or one, which are in order by definition, so they never need to be sorted.

The pivot selection and partitioning steps can be done in several different ways; the choice of specific implementation schemes greatly affects the algorithm's performance.
Problem Statement:
Your task is to implement QuickSort. For this problem take last element of array as pivot (Lomuto partition scheme). A complete runthrough is provided in Figure 5.

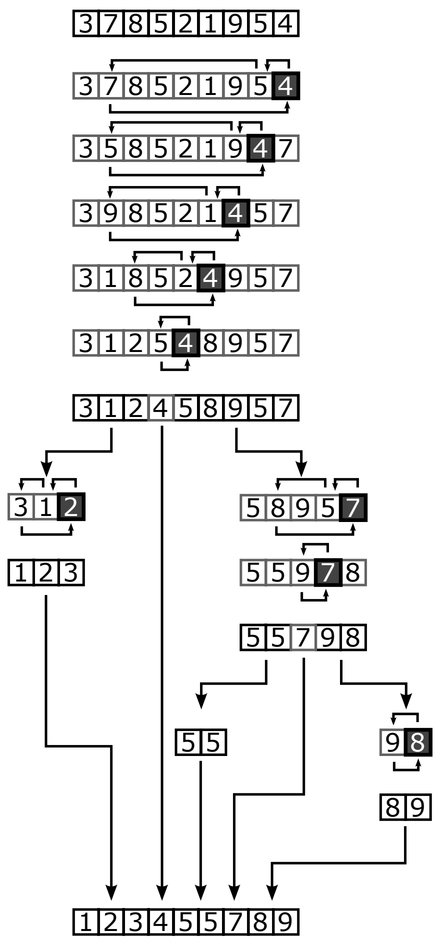


Figure 5: Quick Sort

Input Format
k (number of testcases)
n (size of array) n elements of array
Output Format
Sorted array for (each testcase on a newline)
Sample Input
2
30
86 56 24 26 55 73 77 100 53 20 52 59 74 43 19 21 74 51 44 79 76 15 54 62 6 43 42 5 28 84
27
17 9 10 6 6 12 5 16 18 1 14 11 6 12 14 12 13 10 12 3 2 16 16 14 11 12 7
Sample Output
5 6 15 19 20 21 24 26 28 42 43 43 44 51 52 53 54 55 56 59 62 73 74 74 76 77 79 84 86 100
1 2 3 5 6 6 6 7 9 10 10 11 11 12 12 12 12 12 13 14 14 14 16 16 16 17 18

6.3. Dyck words

A Dyck word is a string consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's.
Some examples for $n = 6$: XXXYYY XYXXYY XYXYXY XXYYXY XXYXYY
Problem Statement:
Find number of possible Dyck words for a given n modulo $10^9 + 7$. why this number?

Input Format
n (number of testcases)
n space separated integers
Output Format
Number of ways; don't forget to take modulo! (each result on a newline)
Sample Input

```
7
1 2 3 5 10 20 100
```

Sample Output

```
1
2
5
42
16796
564120378
558488487
```

6.4. Egyptian Fraction

A fraction is unit fraction if numerator is 1 and denominator is a positive integer, for example 1/3.
Every positive fraction can be represented as sum of unique unit fractions.
Such a representation is called Egyptian Fraction
Problem Statement:
For a given fraction, find its Egyptian Fraction Decomposition

Input Format
n (number of testcases)
N D numerator and denomination of given fraction
Output Format
corresponding decomposition, denominators is ascending order (each result on a newline)
Sample Input

```
5
2 5    3 8    7 12    5 23    14 19    5 31
```

Sample Output

```
1/3 1/15
1/3 1/24
1/2 1/12
1/5 1/58 1/6670
1/2 1/5 1/28 1/887 1/2359420
1/7 1/55 1/3979 1/23744683 1/2024035271
```

§7. Practice Problems 7 - Classes

Here are last set of practice problems. Hopefully you liked these problems and learnt something new in them. The last set are not standard programming problems, Before looking towards efficient techniques I have listed, think on yourself. Ofcourse these may not be "most efficient" (I hope you are comfortable with this by now :D).

Things as simple as polynomial multiplication, division, composition, root finding, factoring and many more are still under research, better/different approaches keep coming. I have linked some of them which you can go through and relatively simpler to implement. But do explore them yourself just a simple search will uncover many research papers and expositions. Take it as a exercise to go through the one you feel most interested in.
You will get a flavour of how things works in academia and also familiarity in reading published work.

El Psy Kongroo

7.1. Polynomials

In Lab-11 optional problem 2, you have made a simple polynomial class which does addition, subtraction, multiplication, differentiation, evaluation and printing the coefficients.

Here you have to make the same (+ some more operations) using efficient techniques.

You are free to decide input-output format but make a documentation for it. Some required features:

- Should work for any degree polynomial
- Member functions should return required polynomial(s) and operand polynomials should not get updated during operations
- Use operator overloading whenever possible

Operations required: (with efficient techniques resources also given for them) If you think there is better approach, feel free to discuss :)

- Evaluation (Horner's Algorithm)
- Polynomial Interpolation (Neville's algorithm)
- Addition
- Subtraction
- Multiplication (Using Fast Fourier Transform Video, FFT info, blog)
- Division (Remember synthetic division?, better approach)
- Composition (Based on Ranged Horner's Algorithm)
- Find all roots (Graeffe's Method, Durand–Kerner method, Aberth method, Real-root isolation helps)
  You may need to use complex number library developed in Lab-6 optional problem 2
- Differentiation
- Integration

Fast Fourier Transform Algorithms with Applications **Problem Statement:**
Try 5.2 after implementing this

7.2. Symbolic Computation aka Computer algebra

Symbolic Computation is evaluation of expressions without using numerical values directly, it directly operates on the symbols. The system which does this is called a Computer algebra system (CAS) (See Wolfram Alpha)

Example:
$3/9$ is $1/3$ for a CAS but $0.333\ldots$ to some finite number of decimals numerically. There is a loss of precision

Calculating $f = (x+y)^2$ gives $g = x^2 + 2xy + y^2$ but in C++ we need a value for x and y to calculate this expression. If we change values then we need to use f again but with CAS we can use simplified form g.
If this is still not clear try to calculate integral of a function using C++. We have to use numerical techniques but we can use CAS to calculate antiderivative (if it exists) and use that for further analysis.

Motivation

- Manipulate and calculate with symbols without needing to initially assign each symbol a numerical value.
- Answers are exact (infinite precision), as unlike floating point numbers, errors due to rounding are not introduced during calculations
- Solves for analytical expressions. By outputting an algebraic expression, the program can show relationships in a situation that are not apparent when the 'answer' is only a string of digits.
- The strength of most mathematical techniques lies in their generality which relies on the use of symbols rather than specific values

The greater computational power of computer algebra systems means they require more memory than most numerically based mathematics software. But it's benefits outclass this issue.

Problem Statement:
Look up examples of CAS systems and their working

Think how are they implemented?

Implement a class which supports $+, -, /, *$ and $roots, exponentiation$ over rational numbers, surds and user variables. (you may include more operations and π, e)

Calculate $\sum_{i=1}^{n} i^n$ (my dream :p) using Faulhaber polynomials symbolically using 5.6 or otherwise

References

Essay

§8. Advanced Concepts

8.1. Divide and Conquer

In computer science, divide and conquer is an algorithm design paradigm. A divide-and-conquer algorithm recursively breaks down a problem into two or more sub-problems of the same or related type, until these become simple enough to be solved directly. The solutions to the sub-problems are then combined to give a solution to the original problem.

Basically 3 steps:

- Divide the problem into sub-problems that are similar to the original but smaller in size
- Conquer the sub-problems by solving them recursively. If they are small enough, just solve them in a straightforward manner.
- Combine the solutions to create a solution to the original problem

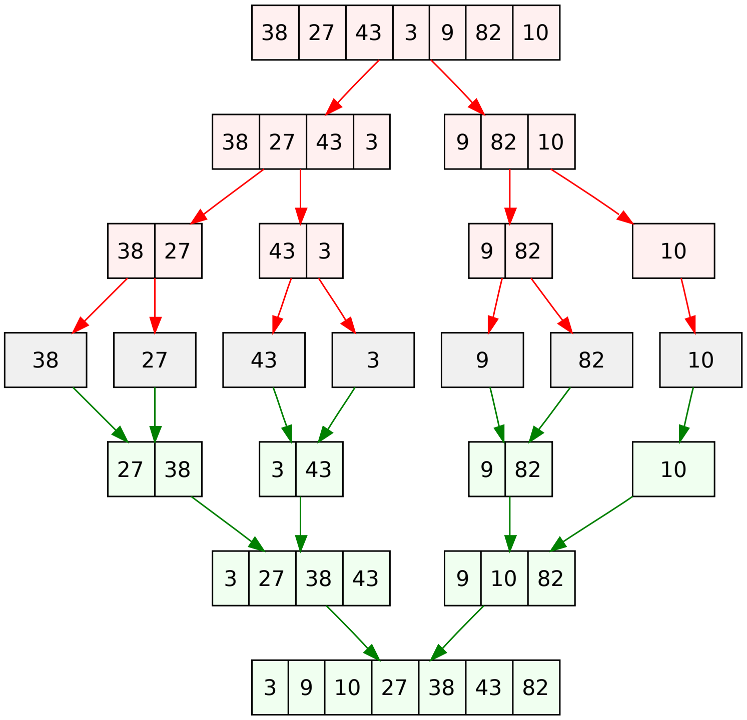


Figure 6: Merge Sort

The divide-and-conquer technique is the basis of efficient algorithms for many problems, such as sorting (e.g., quicksort, merge sort), multiplying large numbers (e.g., the Karatsuba algorithm), finding the closest pair of points, syntactic analysis (e.g., top-down parsers), and computing the discrete Fourier transform (FFT).

Solve questions 6.2,7.1 using divide and conquer approach

References

Wikipedia

8.2. Dynamic Programming

Dynamic Programming (DP) is an algorithmic technique for solving an optimization problem by breaking it down into simpler subproblems and utilizing the fact that the optimal solution to the overall problem depends upon the optimal solution to its subproblems.
Dynamic Programming Methods

- **Top-down with Memoization** - In this approach, we try to solve the bigger problem by recursively finding the solution to smaller sub-problems. Whenever we solve a sub-problem, we cache its result so that we don't end up solving it repeatedly if it's called multiple times. Instead, we can just return the saved result. This technique of storing the results of already solved subproblems is called Memoization.
- **Bottom-up with Tabulation** - Tabulation is the opposite of the top-down approach and avoids recursion. In this approach, we solve the problem "bottom-up" (i.e. by solving all the related sub-problems first). This is typically done by filling up an n-dimensional table. Based on the results in the table, the solution to the top/original problem is then computed.

Recursive Formula

```
int fibonacci(int n){
    if(n <= 1) {
        return n;
    }
    else {
        fibonacci(n-1)+fibonacci(n-2);
    }
}
```

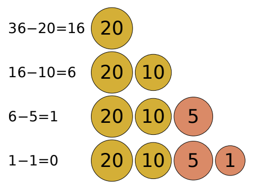
Bottom-up with Tabulation

```
int fibonacci[n];
fibonacci[0] = 0;
fibonacci[1] = 1;
for (int i = 2; i < n; ++i)
{
    fibonacci[i] = fibonacci[i-1] + fibonacci[i-2];
}
cout<<fibonacci[n];
```

Solve questions 4.2, 4.3, 4.4, 6.1 and 6.3 using dynamic programming

8.3. Greedy Algorithms

A greedy algorithm, as the name suggests, always makes the choice that seems to be the best at that moment. Means, it makes a locally-optimal choice in the hope that this choice will lead to a globally-optimal solution
In many problems, a greedy strategy does not usually produce an optimal solution, but nonetheless, a greedy heuristic may yield locally optimal solutions that approximate a globally optimal solution in a reasonable time.
An Example
Greedy algorithms to determine minimum number of coins to give while making change. These are the steps a human would take to emulate a greedy algorithm to represent 36 cents using only 1, 5, 10, 20 coins.



The coin of the highest value, less than the remaining change owed, is the local optimum
As a practice solve question 6.4 using greedy approach

References

5 Simple Steps for Solving Dynamic Programming Problems
What is Dynamic Programming
Coin sums: Project Euler

Greedy Algorithms: Wikipedia

8.4. Backtracking

Backtracking is a general algorithm for finding all (or some) solutions to some computational problems, notably constraint satisfaction problems, that incrementally builds candidates to the solutions, and abandons a candidate ("backtracks") as soon as it determines that the candidate cannot possibly be completed to a valid solution

Types

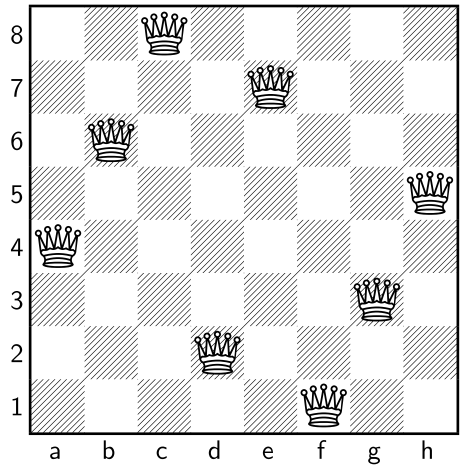
Decision Problem search for a feasible solution

Optimization Problem search for the best solution

Enumeration Problem find all feasible solutions

Backtracking can be applied only for problems which admit the concept of a "partial candidate solution" and a relatively quick test of whether it can possibly be completed to a valid solution. It is useless, for example, for locating a given value in an unordered table. When it is applicable, however, backtracking is often much faster than brute force enumeration of all complete candidates, since it can eliminate many candidates with a single test.

The classic textbook example of the use of backtracking is the eight queens puzzle, that asks for all arrangements of eight chess queens on a standard chessboard so that no queen attacks any other. In the common backtracking approach, the partial candidates are arrangements of k queens in the first k rows of the board, all in different rows and columns. Any partial solution that contains two mutually attacking queens can be abandoned.



Most of the problems, can be solved using other known algorithms like Dynamic Programming or Greedy Algorithms in logarithmic, linear, linear-logarithmic time complexity in order of input size, and therefore, outshine the backtracking algorithm in every respect (since backtracking algorithms are generally exponential in both time and space). However, a few problems still remain, that only have backtracking algorithms to solve them until now.

References

Wikipedia
Geeks for Geeks

§9. Hints

9.1. Practice Problems 1

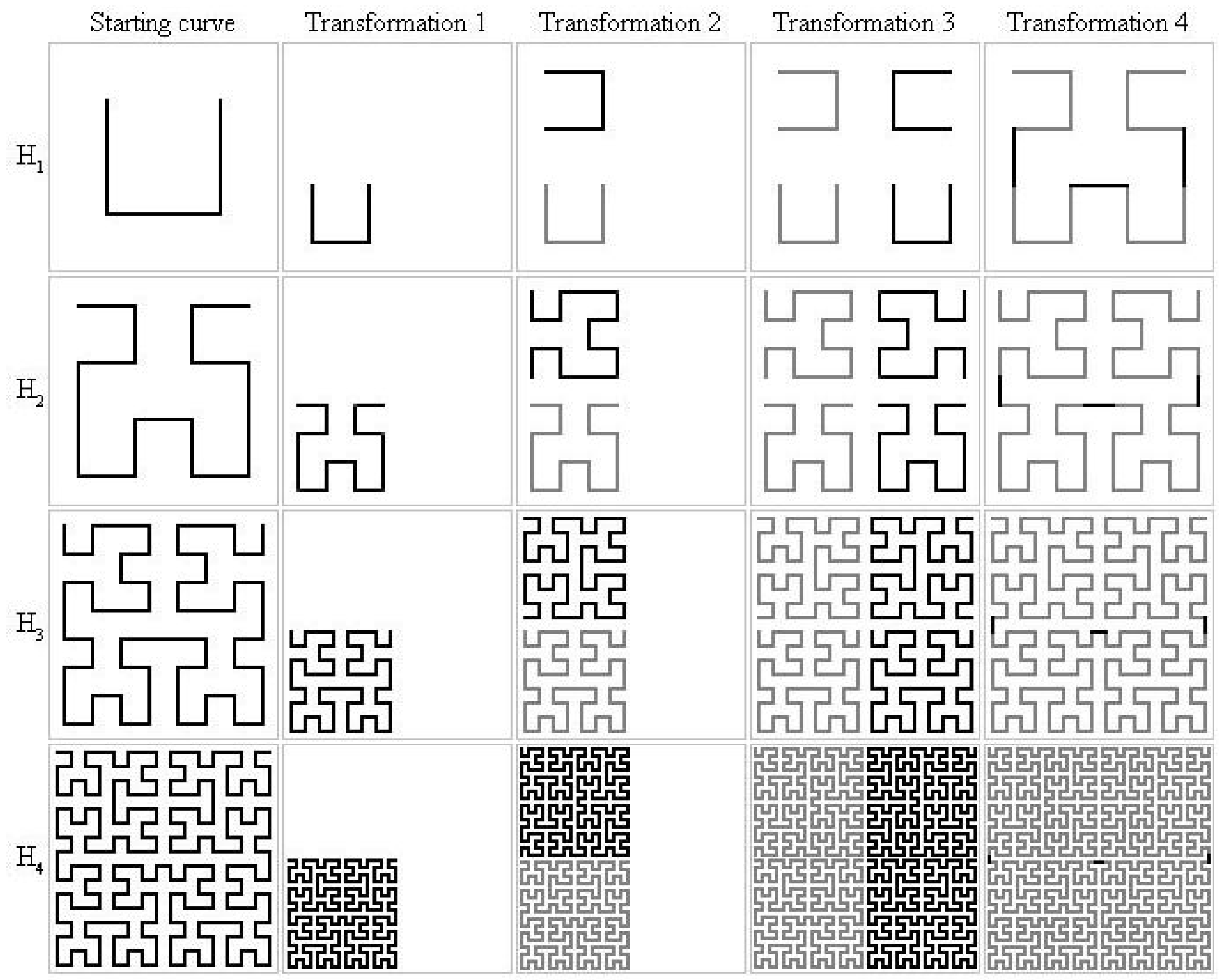


Figure 7: Hilbert Curve Transformations

9.2. Practice Problems 2

1. The sequence repeats when we encounter '0' then '1'
2. Similar to calculating e (problem given in slides)

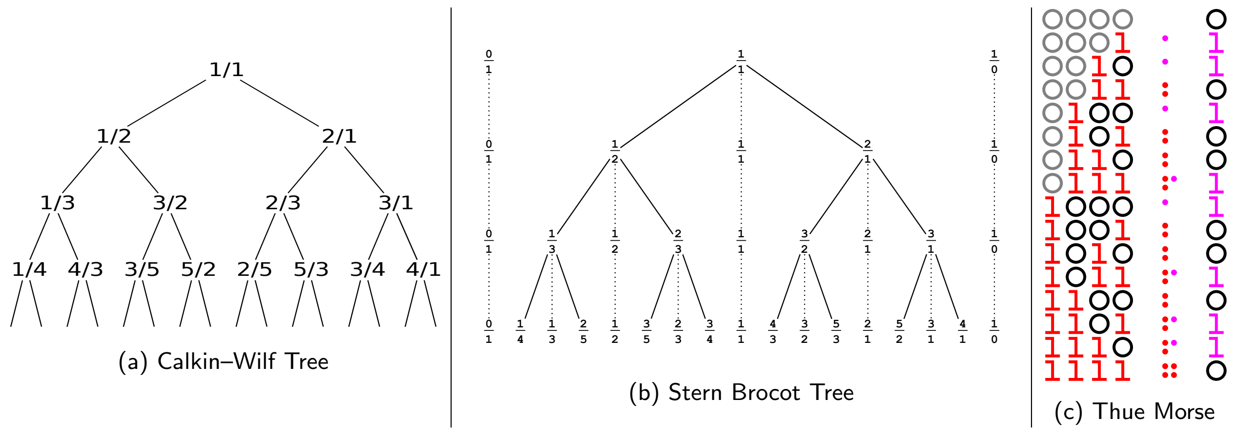
9.3. Practice Problems 3

1. Try skipping days, months, years and centuries :D

   Is there a formula?
2. Similar pattern as in Stern Brocot Tree. Also, exactly same pattern between 3 consecutive terms
3. How will you calculate n^{th} term?

9.4. Practice Problems 4

Straightforward implementation of given recursive definition in problems 4.1 & 4.4 b)



(a) Calkin–Wilf Tree (b) Stern Brocot Tree (c) Thue Morse

Figure 8

For 4.2, 4.3 & 4.4 a) try to find a recursive definition using previous terms
Observe carefully to find patterns in figures

9.5. Practice Problems 5

1. Straightforward
2. How will you store the answer? So, how can you arrive at final answer?
3. A duplicate element occurs more than one time. So, you can count occurence of each element
4. Do you really need to check all elements? Can you skip some of them?
5. Easy to solve in $\Theta(n^2)$ but there is a $\Theta(n)$ solution. You can traverse through all elements in $\Theta(n)$ time. So, how to find majority element while traversing?
6. Good Exercise. Straightforward if you know Gaussian Elimination, else you can also try other approach.

9.6. Practice Problems 6

1. Recursive approach like 4.4. How can you find answer for a n using answers for lower n?
2. Divide and Conquer Approach
3. Does it feel similar to 4.3?
4. How to calculate first element of decompositon? After subtracting first element what will you do?