Data-Driven Analysis and Control using Informativity

EE593 Dual Degree Project - II

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Outline I

Challenges in Modern Dynamical Systems

Introduction

Nonlinearity Linear systems are simple and desirable. But nonlinear aren't.
Can perform local linearizations around fixed points, periodic orbits, etc. but predicting global phenomena is tough

Noise in the modelled data Noise changes the entire dynamical system and in most cases it will result in losing linearity of the system which leads to the above mentioned problems

Unknown dynamics Basic lack of physical laws governing systems in fields such as neuroscience, epidemiology, and ecology which tackle complex realistic systems.
Even for known dynamics are like in turbulence, and protein folding, with higher dimensions uncovering dominant behviour is difficult.

Possible Solutions

Informativity approach With this framework, we convert the control problem in our hand to a Linear Matrix Inequality (LMI). For some control problems, specific noise models can also be accomodated in its analysis making it a robust solution.

Operator-theoretic representations Using Koopman operator, we can represent nonlinear dynamical systems in terms of infinite-dimensional linear operators

Data-driven regression and machine learning is becoming a critical tool to discover discover dynamical systems from data and it forms the basis of DMD, SINDy.

Dynamic Mode Decomposition (DMD)

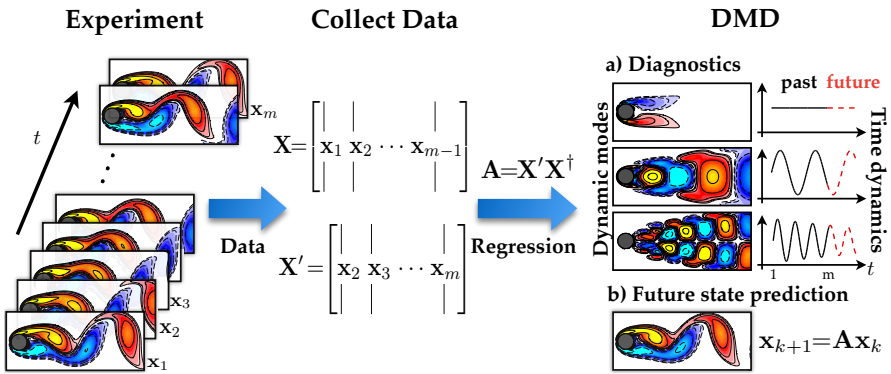


Figure: [1]

Modern koopman theory for dynamical systems, Steven L. Brunton, 2021

Sparse Identification of Nonlinear Dynamics (SINDy)

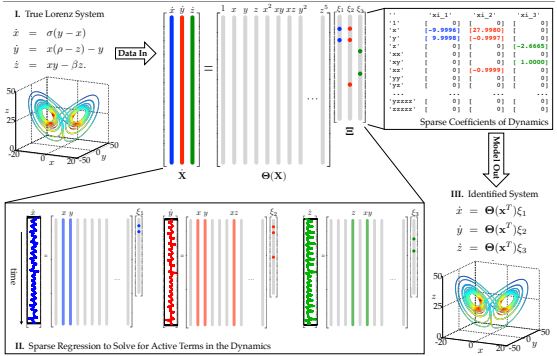


Figure: [1]

Modern koopman theory for dynamical systems, Steven L. Brunton, 2021

Notation I

Informativity

- Model class $\mathcal{M}$
- True system $\mathcal{S}$
- Set of data $\mathcal{D}$
- $\Sigma_{\mathcal{D}}$ is the set of all systems in $\mathcal{M}$ that are consistent with the data $\mathcal{D}$
- Property $\mathcal{P}$
- $\Sigma_{\mathcal{P}}$ is the set of all systems in $\mathcal{M}$ satisfying $\mathcal{P}$

Definition (Informativity for analysis)

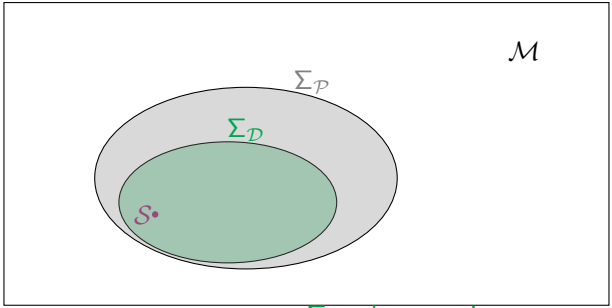
We say that the data $\mathcal{D}$ are *informative* for property $\mathcal{P}$ if $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}}$, i.e., all systems that are consistent with the data have property $\mathcal{P}$.

Problem (Informativity problem for analysis)

Provide necessary and sufficient conditions on the data $\mathcal{D}$ under which these data are informative for property $\mathcal{P}$.

Notation II

Informativity



$\mathcal{M}$: model class $\Sigma_{\mathcal{D}}$: data consistent systems
$\mathcal{S}$: unknown system $\mathcal{P}$: system property
$\mathcal{D}$: given data set $\Sigma_{\mathcal{P}}$: systems with property $\mathcal{P}$

Figure: [5], The data are informative for property $\mathcal{P}$ as $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}}$.

The informativity approach to data-driven analysis and control, Henk J. van Waarde, Jaap Eising, M. Kanat Camlibel, and Harry L. Trentelman, 2023

Notation III

Informativity

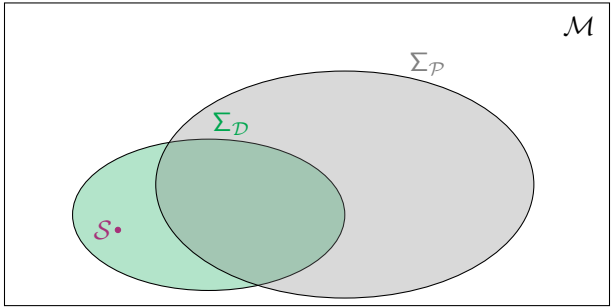


Figure: [5], The data are not informative for property $\mathcal{P}$.

The informativity approach to data-driven analysis and control, Henk J. van Waarde, Jaap Eising, M. Kanat Camlibel, and Harry L. Trentelman, 2023

Notation IV

Informativity

Definition (Informativity for control)

We say that the data $\mathcal{D}$ are *informative* for the control objective $\mathcal{O}$ if there exists a controller $\mathcal{K}$ such that $\Sigma_{\mathcal{D}}(\mathcal{K}) \subseteq \Sigma_{\mathcal{O}}$.

Problem (Informativity problem for control)

Provide necessary and sufficient conditions on $\mathcal{D}$ under which the data are informative for the control objective $\mathcal{O}$.

Problem (Control design problem)

Under the assumption that the data $\mathcal{D}$ are informative for the control objective $\mathcal{O}$, find a controller $\mathcal{K}$ such that $\Sigma_{\mathcal{D}}(\mathcal{K}) \subseteq \Sigma_{\mathcal{O}}$.

Results within Informativity approach

Informativity

- Discrete-time systems
- 'E' refers to exact data, and 'N' to noisy data
- State, input-state, input-state-output and input-output are denoted by 'S', 'IS', 'ISO' and 'IO', respectively

Problem	Data
controllability	E-IS
observability	E-S
stabilizability	E-IS, N-IS
stability	E-S, N-S, N-IO
LQR	E-IS
dissipativity	E-ISO, N-ISO
tracking and regulation	E-IS

Table: Summary of results within the informativity approach to data-driven analysis and control.

The informativity approach to data-driven analysis and control, Henk J. van Waarde, Jaap Eising, M. Kanat Camlibel, and Harry L. Trentelman, 2023

Discrete System Framework

$$\mathbf{x}(t+1) = A_s\mathbf{x}(t) + B_s\mathbf{u}(t), \tag{1}$$

where $\mathbf{x}$ denotes the n-dimensional state and $\mathbf{u}$ the m-dimensional input.
Collect input-state data from the true system on a set of time instances $\{0, 1, \ldots, T\}$ and obtain measurements

$$U_- := \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}, \tag{2a}$$
$$X := \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix}. \tag{2b}$$

If, additionally, we define the matrices

$$X_- := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}, \tag{3a}$$
$$X_+ := \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}, \tag{3b}$$

we have $X_+ = A_sX_- + B_sU_-$
Data $\mathcal{D} = (U_-, X)$ and $\Sigma_{\mathcal{D}} = \{(A_s, B_s)\}$ then

$$\Sigma_{\mathcal{D}} = \left\{ (A, B) \in \mathcal{M} \mid X_+ = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{4}$$

Willems' Fundamental Lemma I

Lemma

The data are informative for system identification if and only if the full rank condition

$$\text{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m. \tag{5}$$

Consider Hankel Matrix

$$H_k(u_{[0,T-1]}) := \begin{bmatrix} u(0) & u(1) & \cdots & u(T-k) \\ u(1) & u(2) & \cdots & u(T-k+1) \\ \vdots & \vdots & & \vdots \\ u(k-1) & u(k) & \cdots & u(T-1) \end{bmatrix}.$$

The input sequence $u_{[0,T-1]}$ is called *persistently exciting* of order k if $H_k(u_{[0,T-1]})$ has full row rank.

Willems' Fundamental Lemma II

Suppose that the system is controllable and observable, and that the input sequence $u_{[0,T-1]}$ is persistently exciting of order $n+L$. Denote $X_L = \begin{bmatrix} x(0) & \cdots & x(T-L) \end{bmatrix}$. Then a consequence of Willems' fundamental lemma is that the following matrix is full rank

$$\begin{bmatrix} X_L \\ H_L(u_{[0,T-1]}) \end{bmatrix}, \tag{6}$$

More generally, full row rank of (6) enables the identification of the system matrices (A, B, C, D) up to similarity transformation if L is larger than the so-called *lag* of the system.

Dissipativity when state functions are known I

Dissipativity analysis

Problem

$$\begin{aligned} \mathbf{x}(t+1) &= A\mathbf{x}(t) + B\mathbf{u}(t), \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t), \end{aligned} \tag{7}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$ are given matrices.

Let $S \in \mathbb{S}^{m+p}$. The system (7) is said to be *dissipative* with respect to the *supply rate*

$$s(u,y) = \begin{bmatrix} u \\ y \end{bmatrix}^\top S \begin{bmatrix} u \\ y \end{bmatrix} \tag{8}$$

if there exists $P \in \mathbb{S}^n$ with $P \geqslant 0$ such that the *dissipation inequality*

$$\mathbf{x}(t+1)^\top P \mathbf{x}(t+1) - \mathbf{x}(t)^\top P \mathbf{x}(t) \leqslant s\big(\mathbf{u}(t), \mathbf{y}(t)\big) \tag{9}$$

holds for all $t \geqslant 0$ and for all trajectories $(\mathbf{u}, \mathbf{x}, \mathbf{y}) : \mathbb{Z}_+ \to \mathbb{R}^{m+n+p}$ of (7).

Dissipativity when state functions are known II

Dissipativity analysis

dissipativity with respect to the supply rate (8) is equivalent with the feasibility of the linear matrix inequalities $P \geqslant 0$ and

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \geqslant 0. \tag{10}$$

Dissipativity when state functions are unknown I

Dissipativity from input-state-output trajectories

Problem

$$\begin{aligned} \mathbf{x}(t+1) &= A_s\mathbf{x}(t) + B_s\mathbf{u}(t), \\ \mathbf{y}(t) &= C_s\mathbf{x}(t) + D_s\mathbf{u}(t), \end{aligned} \tag{11}$$

with $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{x}(t) \in \mathbb{R}^n$ *and* $\mathbf{y}(t) \in \mathbb{R}^p$ *the input, state and output.*
We assume that the dimensions m, n *and* p *are known, but the true system matrices* (A_s, B_s, C_s, D_s) *are unknown. Instead, we know finite number of input-state-output measurements of* (11).

Let U_-, X, X_-, and X_+ be defined as the previous section and let Y_- be defined in a similar way as U_-. Our data are now given by $\mathcal{D} = (U_-, X, Y_-)$.

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}. \tag{12}$$

Dissipativity when state functions are unknown II

Dissipativity from input-state-output trajectories

The set of all systems that are consistent with these data is then given by:

$$\Sigma_{\mathcal{D}} = \Sigma_{(U_-,X,Y_-)} := \left\{ (A,B,C,D) \mid \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{13}$$

Noiseless Data I

Dissipativity from input-state-output trajectories

We now define the property of *informativity for dissipativity* for the case of noiseless data.

Definition (Informativity of noiseless data)

The data (U_-, X, Y_-) are *informative for dissipativity* with respect to the supply rate (8) if there exists a matrix $P \in \mathbb{S}^n$, $P \succcurlyeq 0$, such that the LMI (10) holds for every system $(A, B, C, D) \in \Sigma_{(U_-, X, Y_-)}$.

Note that this definition of informativity for dissipativity requires the systems in $\Sigma_{(U_-, X, Y_-)}$ to be dissipative with a *common* storage function.

We will impose the following assumption on the inertia of $S = (\rho_-, \rho_0, \rho_+) = (p, 0, m)$.

For positive-real case, $S = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$, so that $\mathrm{In}(S) = (m, 0, m)$.

For bounded-real case, $S = \begin{bmatrix} \gamma^2 I_m & 0 \\ 0 & -I_p \end{bmatrix}$ for some $\gamma > 0$, which implies that $\mathrm{In}(S) = (p, 0, m)$.

Noiseless Data II

Dissipativity from input-state-output trajectories

Theorem (Informativity of noiseless data)

Assuming that $\mathrm{In}(S) = (p, 0, m)$. *Then the data* (U_-, X, Y_-) *are informative for dissipativity with respect to the supply rate* (8) *if and only if they are informative for system identification and there exists* $P = P^\top \succcurlyeq 0$ *such that*

$$\begin{bmatrix} X_- \\ X_+ \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} X_- \\ X_+ \end{bmatrix} + \begin{bmatrix} U_- \\ Y_- \end{bmatrix}^\top S \begin{bmatrix} U_- \\ Y_- \end{bmatrix} \succcurlyeq 0. \tag{14}$$

Noisy Data I

Dissipativity from input-state-output trajectories

Problem

$$\begin{aligned} \mathbf{x}(t+1) &= A_s\mathbf{x}(t) + B_s\mathbf{u}(t) + \mathbf{w}(t), \\ \mathbf{y}(t) &= C_s\mathbf{x}(t) + D_s\mathbf{u}(t) + \mathbf{z}(t), \end{aligned} \tag{15}$$

where $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{x}(t) \in \mathbb{R}^n$ *and* $\mathbf{y}(t) \in \mathbb{R}^p$ *are the input, state and output.*
The dimensions m, n *and* p *are assumed to be known.*
The noise terms $\mathbf{w}$ *and* $\mathbf{z}$ *are unknown, so* $w(0), w(1), \ldots, w(T-1)$ *and* $z(0), z(1), \ldots, z(T-1)$ *are not measured, and are therefore not part of the data.*

Noisy Data II

Dissipativity from input-state-output trajectories

Definition (Noise model)

The noise samples, collected in the real $(n+p) \times T$ matrix

$$V_- := \begin{bmatrix} w(0) & w(1) & \cdots & w(T-1) \\ z(0) & z(1) & \cdots & z(T-1) \end{bmatrix}$$

satisfy the quadratic matrix inequality

$$\begin{bmatrix} I \\ V_-^\top \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ V_-^\top \end{bmatrix} \geqslant 0, \tag{16}$$

Noisy Data III

Dissipativity from input-state-output trajectories

Definition

where $\Phi \in \mathbb{S}^{n+p+T}$ is a given partitioned matrix

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{17}$$

with $\Phi_{11} \in \mathbb{S}^{n+p}$, $\Phi_{12} \in \mathbb{R}^{(n+p)\times T}$, $\Phi_{21} = \Phi_{12}^\top$ and $\Phi_{22} \in \mathbb{S}^T$.
We assume that $\mathcal{Z}_T(\Phi)$ is nonempty and convex.[a]
We have that V_- satisfies (49) if and only if $V_-^\top \in \mathcal{Z}_T(\Phi)$ where

$$\mathcal{Z}_T(\Phi) := \left\{ Z \in \mathbb{R}^{T\times q} \mid \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \Phi \begin{bmatrix} I_q \\ Z \end{bmatrix} \geqslant 0 \right\}, \tag{18}$$

[a] This assumption comes from Quadratic matrix inequalities results.

Noisy Data IV

Dissipativity from input-state-output trajectories

$$\Sigma_{\mathcal{D}} = \left\{ (A,B,C,D) \,|\, \left(\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right)^\top \in \mathcal{Z}_T(\Phi) \right\}. \tag{19}$$

We assume that the data have been obtained from the unknown system, i.e., $(A_s, B_s, C_s, D_s) \in \Sigma_{\mathcal{D}}$. Therefore, $\Sigma_{\mathcal{D}}$ is nonempty. Define

$$N := \left[\begin{array}{c:c} N_{11} & N_{12} \\ \hdashline N_{12}^\top & N_{22} \end{array}\right] = \left[\begin{array}{c|c} I & X_+ \\ \hline 0 & \begin{matrix} Y_- \\ \hline -X_- \\ -U_- \end{matrix} \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c|c} I & X_+ \\ \hline 0 & \begin{matrix} Y_- \\ \hline -X_- \\ -U_- \end{matrix} \end{array}\right]^\top. \tag{20}$$

Note that $(A,B,C,D) \in \Sigma_{\mathcal{D}}$ if and only if

$$\left[\begin{array}{c} I \\ \hline \begin{matrix} A^\top & C^\top \\ B^\top & D^\top \end{matrix} \end{array}\right]^\top N \left[\begin{array}{c} I \\ \hline \begin{matrix} A^\top & C^\top \\ B^\top & D^\top \end{matrix} \end{array}\right] \geqslant 0. \qquad \equiv \qquad \begin{bmatrix} A^\top & C^\top \\ B^\top & D^\top \end{bmatrix} \in \mathcal{Z}_{n+m}(N). \tag{21}$$

Informativity of noisy data I

Dissipativity from input-state-output trajectories

Definition (Informativity of noisy data)

The noisy input-state-output data (U_-, X, Y_-) are *informative for dissipativity* with respect to the supply rate (8) if there exists a matrix $P \succcurlyeq 0$ such that the LMI (10) holds for all systems $(A, B, C, D) \in \Sigma_{\mathcal{D}}$.

Lemma (Necessity of full row rank condition)

Assuming that $\mathrm{In}(S) = (p, 0, m)$. *If the data* (U_-, X, Y_-) *are informative for dissipativity with respect to the supply rate* (8) *then* 5 *holds.*

Lemma (Necessity of positive definite storage)

Suppose that $\mathrm{In}(S) = (p, 0, m)$ *and that* $N \mid N_{22} > 0$. *If* $P \succcurlyeq 0$ *satisfies the dissipation inequality* (10) *for all* $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ *then* $P > 0$.

Informativity of noisy data II

Dissipativity from input-state-output trajectories

Our next step is to partition

$$S = \begin{bmatrix} F & G \\ G^\top & H \end{bmatrix}, \tag{22}$$

where $F \in \mathbb{R}^{m \times m}$, $G \in \mathbb{R}^{m \times p}$, $H \in \mathbb{R}^{p \times p}$. For any $P \geqslant 0$ define

$$M := \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & F & 0 & G \\ 0 & 0 & -P & 0 \\ 0 & G^\top & 0 & H \end{bmatrix}. \tag{23}$$

Then the system (A, B, C, D) can be seen to satisfy the dissipation inequality (10) if and only if

$$\left[\begin{array}{c} I \\ \hline A \quad B \\ C \quad D \end{array}\right]^\top M \left[\begin{array}{c} I \\ \hline A \quad B \\ C \quad D \end{array}\right] \geqslant 0 \tag{24}$$

Informativity of noisy data III

Dissipativity from input-state-output trajectories

Theorem (Matrix S-lemma [4])

Let $M, N \in \mathbb{S}^{q+r}$. If there exists a real $\alpha \geqslant 0$ such that $M - \alpha N \geqslant 0$, then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$. Next, assume that $N \in \mathbf{\Pi}_{q,r}$ and N has at least one positive eigenvalue. Then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$ if and only if there exists a real $\alpha \geqslant 0$ such that $M - \alpha N \geqslant 0$.

Informativity of noisy data IV

Dissipativity from input-state-output trajectories

Lemma (Dualization of dissipation inequality [5])

Let $P > 0$ and let (A, B, C, D) be any system with input dimension m, state space dimension n and output dimension p. Assume that $\mathsf{In}(S) = (p, 0, m)$. Define

$$\hat{S} := \begin{bmatrix} 0 & -I_p \\ I_m & 0 \end{bmatrix} S^{-1} \begin{bmatrix} 0 & -I_m \\ I_p & 0 \end{bmatrix}. \tag{25}$$

Then we have

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \geqslant 0 \tag{26}$$

if and only if

$$\begin{bmatrix} I & 0 \\ A^\top & C^\top \end{bmatrix}^\top \begin{bmatrix} P^{-1} & 0 \\ 0 & -P^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ A^\top & C^\top \end{bmatrix} + \begin{bmatrix} 0 & I \\ B^\top & D^\top \end{bmatrix}^\top \hat{S} \begin{bmatrix} 0 & I \\ B^\top & D^\top \end{bmatrix} \geqslant 0. \tag{27}$$

Informativity of noisy data V

Dissipativity from input-state-output trajectories

Theorem (Informativity of noisy data)

Suppose that the data (U_-, X, Y_-) are collected from system 4 with noise as in Assumption 23. In addition, assume that $\ln(S) = (p, 0, m)$ and that the data (U_-, X, Y_-) are such that $N \mid N_{22} > 0$. Partition

$$-S^{-1} = \begin{bmatrix} \hat{F} & \hat{G} \\ \hat{G}^\top & \hat{H} \end{bmatrix}, \tag{28}$$

where $\hat{F} = \hat{F}^\top \in \mathbb{R}^{m\times m}$, $\hat{G} \in \mathbb{R}^{m\times p}$, and $\hat{H} = \hat{H}^\top \in \mathbb{R}^{p\times p}$.
Then the data are informative for dissipativity with respect to the supply rate (8) if and only if there exist a real $n \times n$ matrix $Q \in \mathbb{S}^n$, $Q > 0$ and a scalar $\alpha \geqslant 0$ such that

$$\begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & \hat{H} & 0 & -\hat{G}^\top \\ 0 & 0 & -Q & 0 \\ 0 & -\hat{G} & 0 & \hat{F} \end{bmatrix} - \alpha \left[\begin{array}{c|c} I & X_+ \\ Y_- & \\ \hline 0 & -X_- \\ & -U_- \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c|c} I & X_+ \\ Y_- & \\ \hline 0 & -X_- \\ & -U_- \end{array}\right]^\top \geqslant 0. \tag{29}$$

Dissipativity when state functions are unknown I

Dissipativity from input-output trajectories

Problem

$$\begin{aligned} \mathbf{x}(t+1) &= A_s\mathbf{x}(t) + B_s\mathbf{u}(t), \\ \mathbf{y}(t) &= C_s\mathbf{x}(t) + D_s\mathbf{u}(t), \end{aligned} \tag{30}$$

with $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{x}(t) \in \mathbb{R}^n$ *and* $\mathbf{y}(t) \in \mathbb{R}^p$ *the input, state and output.*
We assume that the dimensions m, n *and* p *are known, but the true system matrices* (A_s, B_s, C_s, D_s) *are unknown. Instead, we know finite number of input-state-output measurements of* (11).

Let U_-, be defined as the previous section and let Y_- be defined in a similar way as U_-. Our data are now given by $\mathcal{D} = (U_-, Y_-)$.

Assumptions

Dissipativity from input-output trajectories

Definition

The lag $\underline{l}$ of system (11) is the smallest integer $l \in \mathbb{N}_+$ such that the observability matrix given by

$$\mathcal{O}_l := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{l-1} \end{bmatrix}$$

has rank n.

The upper bound on the lag of the system

$$\underline{l} \leqslant l$$

Noiseless Data I

Dissipativity from input-output trajectories

Lemma

Let $l \geqslant \underline{l}$. Then there exists a system $\widetilde{G}$ with matrices $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ which can explain the data $\{u_k\}_{k=0}^{N-1}$, $\{y_k\}_{k=0}^{N-1}$, i.e., there exists ξ_0 such that for $k = 0, \ldots, N-1$,

$$\xi_{k+1} = \widetilde{A}\xi_k + \widetilde{B}u_k, \quad y_k = \widetilde{C}\xi_k + \widetilde{D}u_k, \tag{31}$$

where the extended state is defined by

$$\xi_k = \begin{bmatrix} u_{k-l}^\top & u_{k-l+1}^\top & \cdots & u_{k-1}^\top & y_{k-l}^\top & y_{k-l+1}^\top & \cdots & y_{k-1}^\top \end{bmatrix}^\top.$$

Proof.

The input-output behavior of the system G in (11) over l steps can be written as (32),

$$\begin{bmatrix} y_{k-l} \\ y_{k-l+1} \\ \vdots \\ y_{k-1} \end{bmatrix} = \underbrace{\begin{bmatrix} C_s \\ C_s A_s \\ \vdots \\ C_s A_s^{l-1} \end{bmatrix}}_{\mathcal{O}_l} x_{k-l} + \underbrace{\begin{bmatrix} D_s & 0 & \dots & 0 & 0 \\ C_s B_s & D_s & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ C_s A_s^{l-2} B_s & C_s A_s^{l-3} B_s & \dots & C_s A_s B_s & C_s B_s & D_s \end{bmatrix}}_{R} \begin{bmatrix} u_{k-l} \\ u_{k-l+1} \\ \vdots \\ u_{k-1} \end{bmatrix} \tag{32}$$

Noiseless Data III

Dissipativity from input-output trajectories

Proof.

which yields with the introduced matrix notation

$$\begin{bmatrix} -R & I \end{bmatrix} \xi_k = \mathcal{O}_l x_k.$$

Using the definition of the lag, we know that $\mathcal{O}_l$ has full column rank, and hence, there exists a left-inverse $\mathcal{O}_l^{-1}$ (which has full row rank) such that

$$\underbrace{\mathcal{O}_l^{-1} \begin{bmatrix} -R & I \end{bmatrix}}_{K} \xi_k = x_k. \tag{33}$$

$$y_k = C_s A_s^l x_{k-l} + \begin{bmatrix} C_s A_s^{l-1} B_s & \dots & C_s B_s \end{bmatrix} \begin{bmatrix} u_{k-l} \\ \vdots \\ u_{k-1} \end{bmatrix}.$$

Noiseless Data IV

Dissipativity from input-output trajectories

Proof.

$$\underbrace{\begin{bmatrix} u_{k-l+1} \\ \vdots \\ u_{k-1} \\ u_k \\ y_{k-l+1} \\ \vdots \\ y_{k-1} \\ y_k \end{bmatrix}}_{\xi_{k+1}} = \underbrace{\begin{bmatrix} 0 & I & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & I & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & I & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & I \\ C_s A_s^{l-1} B_s & \dots & \dots & C_s B_s & 0 & 0 & \dots & 0 \end{bmatrix}}_{\tilde{A}} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ C_s A_s^l T \end{bmatrix} \underbrace{\begin{bmatrix} u_{k-l} \\ u_{k-l+1} \\ \vdots \\ u_{k-1} \\ y_{k-l} \\ y_{k-l+1} \\ \vdots \\ y_{k-1} \end{bmatrix}}_{\xi_k} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \\ D_s \end{bmatrix}}_{\tilde{B}} u_k$$

$$y_k = \underbrace{\begin{bmatrix} 0 & \dots & 0 & I \end{bmatrix}}_{\tilde{C}} \tilde{A} \xi_k + \underbrace{D}_{\tilde{D}} u_k \tag{34}$$

□

Noiseless Data

Dissipativity from input-output trajectories

Theorem (Informativity of noiseless data)

Assuming that $\mathsf{ln}(S) = (p, 0, m)$ *and the lag of the system* $\underline{l} \leqslant l$. *Then the data* (U_-, Y_-) *are informative for dissipativity with respect to the supply rate* (8) *if and only if they are informative for system identification and there exists* $P = P^\top \succcurlyeq 0$ *such that*

$$\begin{bmatrix} \Xi_- \\ \Xi_+ \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} \Xi_- \\ \Xi_+ \end{bmatrix} + \begin{bmatrix} U_{\Xi_-} \\ Y_{\Xi_-} \end{bmatrix}^\top S \begin{bmatrix} U_{\Xi_-} \\ Y_{\Xi_-} \end{bmatrix} \succcurlyeq 0. \tag{35}$$

where $\xi_k = \begin{bmatrix} u_{k-l}^\top & u_{k-l+1}^\top & \cdots & u_{k-1}^\top & y_{k-l}^\top & y_{k-l+1}^\top & \cdots & y_{k-1}^\top \end{bmatrix}^\top$ *and,*

$$\begin{aligned} \Xi_- &:= \begin{bmatrix} \xi_l & \xi_{l+1} & \cdots & \xi_{T-1} \end{bmatrix}, \\ \Xi_+ &:= \begin{bmatrix} \xi_{l+1} & \xi_{l+2} & \cdots & \xi_T \end{bmatrix}, \\ Y_{\Xi_-} &:= \begin{bmatrix} y_l & y_{l+1} & \cdots & y_{T-1} \end{bmatrix}, \\ U_{\Xi_-} &:= \begin{bmatrix} u_l & u_{l+1} & \cdots & u_{T-1} \end{bmatrix}, \end{aligned}$$

Noisy Data I

Dissipativity from input-output trajectories

Problem

$$\begin{aligned} \mathbf{x}(t+1) &= A_s\mathbf{x}(t) + B_s\mathbf{u}(t) + \mathbf{w}(t), \\ \mathbf{y}(t) &= C_s\mathbf{x}(t) + D_s\mathbf{u}(t) + \mathbf{z}(t), \end{aligned} \tag{36}$$

where $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{x}(t) \in \mathbb{R}^n$ *and* $\mathbf{y}(t) \in \mathbb{R}^p$ *are the input, state and output.*
The dimensions m, n *and* p *are assumed to be known.*
The noise terms $\mathbf{w}$ *and* $\mathbf{z}$ *are unknown, so* $w(0), w(1), \ldots, w(T-1)$ *and* $z(0), z(1), \ldots, z(T-1)$ *are not measured, and are therefore not part of the data.*

Informativity of noisy data using the complete system I

Dissipativity from input-output trajectories

Using 34, we define

Definition (Noise model)

The noise samples, collected in the real $(n+p) \times (T-l)$ matrix

$$V_- := \begin{bmatrix} \tilde{w}(l) & \tilde{w}(l+1) & \cdots & \tilde{w}(T-1) \\ \tilde{z}(l) & \tilde{z}(l+1) & \cdots & \tilde{z}(l-1) \end{bmatrix}$$

satisfy the quadratic matrix inequality

$$\begin{bmatrix} I \\ V_-^\top \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ V_-^\top \end{bmatrix} \geqslant 0, \tag{37}$$

Informativity of noisy data using the complete system II

Dissipativity from input-output trajectories

Definition

where $\Phi \in \mathbb{S}^{(n+p)+(T-l)}$ is a given partitioned matrix

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{38}$$

with $\Phi_{11} \in \mathbb{S}^{n+p}$, $\Phi_{12} \in \mathbb{R}^{(n+p)\times(T-l)}$, $\Phi_{21} = \Phi_{12}^\top$ and $\Phi_{22} \in \mathbb{S}^{(T-l)}$.
We assume that $\mathcal{Z}_T(\Phi)$ is nonempty and convex.[a]
We have that V_- satisfies (49) if and only if $V_-^\top \in \mathcal{Z}_T(\Phi)$ where

$$\mathcal{Z}_T(\Phi) := \left\{ Z \in \mathbb{R}^{T\times q} \mid \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \Phi \begin{bmatrix} I_q \\ Z \end{bmatrix} \geqslant 0 \right\}, \tag{39}$$

[a] This assumption comes from Quadratic matrix inequalities results.

Informativity of noisy data using the complete system III

Dissipativity from input-output trajectories

$$\Sigma_{\mathcal{D}} = \left\{ (A,B,C,D) \,|\, \left(\begin{bmatrix} \Xi_+ \\ Y_{\Xi_-} \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Xi_- \\ U_{\Xi_-} \end{bmatrix} \right)^\top \in \mathcal{Z}_T(\Phi) \right\}. \tag{40}$$

We assume that the data have been obtained from the unknown system, i.e., $(A_s, B_s, C_s, D_s) \in \Sigma_{\mathcal{D}}$. Therefore, $\Sigma_{\mathcal{D}}$ is nonempty. Define

$$N := \left[\begin{array}{c:c} N_{11} & N_{12} \\ \hdashline N_{12}^\top & N_{22} \end{array}\right] = \left[\begin{array}{c|c} I & \Xi_+ \\ & Y_{\Xi_-} \\ \hline 0 & -\Xi_- \\ & -U_{\Xi_-} \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c|c} I & \Xi_+ \\ & Y_{\Xi_-} \\ \hline 0 & -\Xi_- \\ & -U_{\Xi_-} \end{array}\right]^\top. \tag{41}$$

Note that $(A,B,C,D) \in \Sigma_{\mathcal{D}}$ if and only if

$$\left[\begin{array}{c} I \\ \hline \begin{matrix} A^\top & C^\top \\ B^\top & D^\top \end{matrix} \end{array}\right]^\top N \left[\begin{array}{c} I \\ \hline \begin{matrix} A^\top & C^\top \\ B^\top & D^\top \end{matrix} \end{array}\right] \geqslant 0. \qquad \equiv \qquad \begin{bmatrix} A^\top & C^\top \\ B^\top & D^\top \end{bmatrix} \in \mathcal{Z}_{n+m}(N). \tag{42}$$

Informativity of noisy data using the complete system IV

Dissipativity from input-output trajectories

Theorem (Informativity of noisy data)

Suppose that the data (U_-, Y_-) are collected from system 4 with noiseIn addition, assume that $\mathrm{In}(S) = (p, 0, m)$ and that the data (U_-, X, Y_-) are such that $N \,|\, N_{22} > 0$. Partition the equivalent supply rate $\tilde{S}$,

$$-\tilde{S}^{-1} = \begin{bmatrix} \hat{F} & \hat{G} \\ \hat{G}^\top & \hat{H} \end{bmatrix}, \tag{43}$$

Then the data are informative for dissipativity with respect to the supply rate (8) if and only if there exist a real $n \times n$ matrix $Q \in \mathbb{S}^n$, $Q > 0$ and a scalar $\alpha \geqslant 0$ such that

$$\begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & \hat{H} & 0 & -\hat{G}^\top \\ 0 & 0 & -Q & 0 \\ 0 & -\hat{G} & 0 & \hat{F} \end{bmatrix} - \alpha \left[\begin{array}{c|c} I & \Xi_+ \\ & Y_{\Xi_-} \\ \hline 0 & -\Xi_- \\ & -U_{\Xi_-} \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c|c} I & \Xi_+ \\ & Y_{\Xi_-} \\ \hline 0 & -\Xi_- \\ & -U_{\Xi_-} \end{array}\right]^\top \geqslant 0. \tag{44}$$

Difference Operator I

Proposition ([2])

Consider the system in the difference operator form

$$\begin{aligned} y_k = &- a_l y_{k-1} - \cdots - a_2 y_{k-l+1} - a_1 y_{k-l} \\ &+ d u_k + b_l u_{k-1} + \cdots + b_2 u_{k-l+1} + b_1 u_{k-l} + \underbrace{b_v v_k}_{\hat{v}_k} \end{aligned} \tag{45}$$

with $a_i \in \mathbb{R}^{p\times p}$, $b_i \in \mathbb{R}^{p\times m}$, $i = 1, \ldots, l$
$v_k \in \mathbb{R}^{m_v}$ *denotes the noise and, as before, the choice of* $b_v \in \mathbb{R}^{p\times m_v}$

This can also be represented in state space form

$$\begin{aligned} \xi_{k+1} &= \begin{bmatrix} \widetilde{A}_1 \\ \widetilde{A}_2 \end{bmatrix} \xi_k + \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{D} \end{bmatrix} u_k + \begin{bmatrix} 0 \\ b_v \end{bmatrix} v_k, \\ y_k &= \widetilde{A}_2 \xi_k + \widetilde{D} u_k + b_v v_k, \end{aligned} \tag{46}$$

Difference Operator II

$$\begin{bmatrix} u_{k-l+1} \\ \vdots \\ u_{k-1} \\ u_k \\ y_{k-l+1} \\ \vdots \\ y_{k-1} \\ y_k \end{bmatrix} = \begin{bmatrix} 0 & I & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & I & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & I & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & I \\ b_1 & b_2 & \dots & b_l & -a_1 & -a_2 & \dots & -a_l \end{bmatrix} \begin{bmatrix} u_{k-l} \\ u_{k-l+1} \\ \vdots \\ u_{k-1} \\ y_{k-l} \\ y_{k-l+1} \\ \vdots \\ y_{k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \\ D \end{bmatrix} u_k + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_v \end{bmatrix} v_k \quad (47)$$

Here $\tilde{A}_1 \in \mathbb{R}^{((p+m)l-p)\times(p+m)l}$ and $\widetilde{B}_1 \in \mathbb{R}^{((p+m)l-p)\times m}$ are known, and $\tilde{A}_2 \in \mathbb{R}^{p\times(p+m)l}$ and $\widetilde{D} \in \mathbb{R}^{p\times m}$ are unknown.

Informativity of noisy data, optimally I

Dissipativity from input-output trajectories

Definition (Noise model)

The noise samples, collected in the real $p \times (T-l)$ matrix

$$V = \begin{bmatrix} \hat{v}_l & \hat{v}_{l+1} & \dots & \hat{v}_{T-1} \end{bmatrix} \tag{48}$$

satisfy the quadratic matrix inequality

$$\begin{bmatrix} I \\ V_-^\top \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ V_-^\top \end{bmatrix} \geqslant 0, \tag{49}$$

Informativity of noisy data, optimally II

Dissipativity from input-output trajectories

Definition

where $\Phi \in \mathbb{S}^{m_v+T-l}$ is a given partitioned matrix

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{50}$$

with $\Phi_{11} \in \mathbb{S}^{m_v}$, $\Phi_{12} \in \mathbb{R}^{m_v \times (T-l)}$, $\Phi_{21} = \Phi_{12}^\top$ and $\Phi_{22} \in \mathbb{S}^{T-l}$.
We assume that $\mathcal{Z}_T(\Phi)$ is nonempty and convex.[a]
We have that V_- satisfies (49) if and only if $V_-^\top \in \mathcal{Z}_T(\Phi)$ where

$$\mathcal{Z}_T(\Phi) := \left\{ Z \in \mathbb{R}^{T \times q} \mid \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \Phi \begin{bmatrix} I_q \\ Z \end{bmatrix} \geqslant 0 \right\}, \tag{51}$$

[a] This assumption comes from Quadratic matrix inequalities results.

Informativity of noisy data, optimally III

Dissipativity from input-output trajectories

$$\Sigma_{\mathcal{D}} = \{(A_2, D) | Y_\Xi = A_2 \Xi + D U_\Xi + b_v V, V \in \mathcal{Z}_T(\Phi)\}.$$

We assume that the data have been obtained from the unknown system, i.e., $(A_s, B_s, C_s, D_s) \in \Sigma_{\mathcal{D}}$. Therefore, $\Sigma_{\mathcal{D}}$ is nonempty. Define

$$N := \left[\begin{array}{c:c} N_{11} & N_{12} \\ \hdashline N_{12}^\top & N_{22} \end{array}\right] = \left[\begin{array}{c|c} I & Y_{\Xi_-} \\ \hline 0 & -\Xi_- \\ & -U_{\Xi_-} \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c|c} I & Y_{\Xi_-} \\ \hline 0 & -\Xi_- \\ & -U_{\Xi_-} \end{array}\right]^\top. \tag{52}$$

Note that $(A_2, D) \in \Sigma_{\mathcal{D}}$ if and only if

$$\left[\begin{array}{c} I \\ \hline A_2^\top \\ D^\top \end{array}\right]^\top N \left[\begin{array}{c} I \\ \hline A_2^\top \\ D^\top \end{array}\right] \geqslant 0 \qquad \equiv \qquad \begin{bmatrix} A_2^\top \\ D^\top \end{bmatrix} \in \mathcal{Z}_{n+m}(N). \tag{53}$$

Informativity of noisy data, optimally IV

Dissipativity from input-output trajectories

Theorem (Informativity of noisy data [2])

Suppose that the data (U_-, Y_-) are collected from system 4 with noise as in Assumption 23. In addition, assume that $\mathsf{In}(S) = (p, 0, m)$ and that the data (U_-, Y_-) are such that $N \mid N_{22} > 0$. Partition

$$-S^{-1} = \begin{bmatrix} \hat{F} & \hat{G} \\ \hat{G}^\top & \hat{H} \end{bmatrix}, \tag{54}$$

where $\hat{F} = \hat{F}^\top \succeq 0 \in \mathbb{R}^{m\times m}$, $\hat{G} \in \mathbb{R}^{m\times p}$, and $\hat{H} = \hat{H}^\top \in \mathbb{R}^{p\times p}$.
If there exists a matrix $P = P^\top \succ 0$, $\alpha > 0$ such that (55) holds, then (45) is dissipative for all matrices consistent with the data $(A_2, D) \in \Sigma_{\mathcal{D}}$.

$$\begin{bmatrix} \begin{pmatrix} \widetilde{A}_1^\top & 0 \end{pmatrix} & 0 & \begin{pmatrix} I & 0 \end{pmatrix} \\ -I & 0 & 0 \\ \begin{pmatrix} \widetilde{B}_1^\top & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & I \end{pmatrix} \\ 0 & -I & 0 \\ 0 & 0 & I \\ \begin{pmatrix} 0 & I \end{pmatrix} & I & 0 \end{bmatrix}^\top \left[\begin{array}{cccccc} -P & 0 & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \hat{F} & \hat{G} & 0 & 0 \\ 0 & 0 & \hat{G}^\top & \hat{H} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\alpha\hat{\Phi}_{22} & -\alpha\hat{\Phi}_{12} \\ 0 & 0 & 0 & 0 & -\alpha\hat{\Phi}_{21} & -\alpha\hat{\Phi}_{11} \end{array}\right] \begin{bmatrix} \begin{bmatrix} \widetilde{A}_1^\top & 0 \end{bmatrix} & 0 & \begin{bmatrix} I & 0 \end{bmatrix} \\ -I & 0 & 0 \\ \begin{bmatrix} \widetilde{B}_1^\top & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 & I \end{bmatrix} \\ 0 & -I & 0 \\ 0 & 0 & I \\ \begin{bmatrix} 0 & I \end{bmatrix} & I & 0 \end{bmatrix} \succ 0 \tag{55}$$

Adversarial Attacks to Data-Driven Control I

Introduction to LQR

Consider the discrete time linear system

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{56}$$

where $\mathbf{x}$ is the n-dimensional state and $\mathbf{u}$ the m-dimensional input.

Definition (The quadratic cost functional)

$$J(x_0, \mathbf{u}) = \sum_{t=0}^{\infty} \mathbf{x}^\top(t) Q \mathbf{x}(t) + \mathbf{u}^\top(t) R \mathbf{u}(t), \tag{57}$$

where $Q \in \mathbb{S}^n$ is positive semidefinite and $R \in \mathbb{S}^m$ is positive definite.

Problem (The LQR problem)

Determine for every initial condition x_0 an input $\mathbf{u}^*$, such that $\lim_{t\to\infty} \mathbf{x}_{x_0,\mathbf{u}^*}(t) = 0$, and the cost functional $J(x_0, \mathbf{u})$ is minimized under this constraint.

Theorem (Conditions for LQR)

Let $Q \geqslant 0$ and $R > 0$. Then the following statements hold:

1. *If (A, B) is stabilizable, there exists a unique largest real symmetric solution P^+ to the discrete-time algebraic Riccati equation (DARE)*

$$P = A^\top PA - A^\top PB(R + B^\top PB)^{-1}B^\top PA + Q, \tag{58}$$

   in the sense that $P^+ \geqslant P$ for every real symmetric P satisfying (58). The matrix P^+ is positive semidefinite.

2. *If, in addition to stabilizability of (A, B), every eigenvalue of A on the unit circle is (Q, A)-observable then for every x_0 a unique optimal input $\mathbf{u}^*$ exists. Furthermore, this input sequence is generated by the feedback law $\mathbf{u} = K\mathbf{x}$, where*

$$K := -(R + B^\top P^+ B)^{-1}B^\top P^+ A. \tag{59}$$

3. *In fact, the optimal LQR problem is solvable for (A, B, Q, R) if and only if (A, B) is stabilizable and every eigenvalue of A on the unit circle is (Q, A)-observable.*

Informativity for LQR

Definition (Informativity for LQR)

Given matrices Q and R, we say that the data $\mathcal{D} = (U_-, X)$ are *informative for optimal linear quadratic regulation* if the optimal LQR problem is solvable for all $(A, B) \in \Sigma_{\mathcal{D}}$ and there exists K such that $\Sigma_{\mathcal{D}} \subseteq \Sigma_K^{Q,R}$.
where

$$\Sigma_K^{Q,R} := \{\, (A, B) \in \mathcal{M} \,\} \; K \text{ is optimal for } (A, B, Q, R).$$

Assume that the time series

$$U_- := [u(0)\ u(1)\ \cdots u(T-1)] \in \mathbb{R}^{m\times T} \tag{60}$$

$$X_: = [x(0)\ x(1)\ \cdots x(T-1)\ x(T)] \in \mathbb{R}^{n\times(T+1)} \tag{61}$$

$$X_- := [x(0)\ x(1)\ \cdots x(T-1)] \in \mathbb{R}^{n\times T} \tag{62}$$

$$X_+ := [x(1)\ x(2)\ \cdots x(T)] \in \mathbb{R}^{n\times T} \tag{63}$$

$$\tag{64}$$

Let the disturbance $D_0 := [d(0)\ d(1)\ \cdots\ d(T-1)] \in \mathbb{R}^{m\times T}$, then

$$X_+ - D_0 = [B\ A]W_0,$$

where $W_0 := \begin{bmatrix} U_- \\ X_- \end{bmatrix}$. We here assume that $\operatorname{rank} W_0 = n + m$

LQR Design I

The key idea is to parameterize the controller using the available data by introducing a new variable $G \in \mathbb{R}^{T\times n}$ with the relationship

$$[K^\mathsf{T}\ I]^\mathsf{T} = W_0 G. \tag{65}$$

Then the closed-loop matrix can be parameterized directly by data matrices as $A+BK = [B\ A]W_0 G = (X_+ - D_0)G$. The LQR controller design can be formulated as follwing by disregarding the noise term.

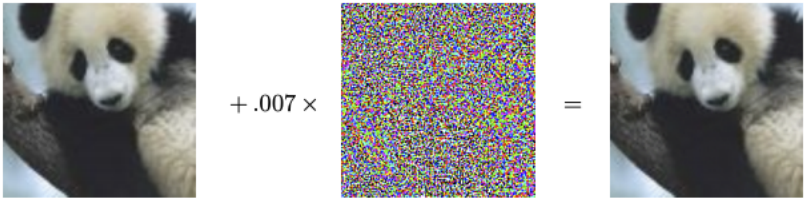
$$\begin{array}{ll} \min\limits_{P,K,G} & \mathrm{tr}(QP) + \mathrm{tr}(K^\mathsf{T} RKP) \\ \text{s.t.} & X_+ GPG^\mathsf{T} X_+^\mathsf{T} - P + I \preceq 0 \\ & P \succeq I \text{ and } (65) \end{array} \tag{66}$$

However, it has been revealed that the formulation (66) is not robust to disturbance.
To enhance robustness against disturbance, we can do regularization

$$\begin{array}{ll} \min\limits_{P,K,G} & \mathrm{tr}(QP) + \mathrm{tr}(K^\mathsf{T} RKP) + \gamma\|\Pi G\| \\ \text{s.t.} & X_+ GPG^\mathsf{T} X_+^\mathsf{T} - P + I \preceq 0 \\ & P \succeq I \text{ and } (65) \end{array} \tag{67}$$

with a constant $\gamma \geqslant 0$ where $\Pi := I - W_0^\dagger W_0$

Fast Gradient Sign Method I



Adversarial example using FGSM, Tensorflow, 2024

Fast Gradient Sign Method II

The core idea of FGSM is to choose a perturbation that locally maximizes the loss function.
The linear approximation of the loss function with respect to Δ is given by

$$L(X+\Delta, Y; \theta) \simeq L(X, Y; \theta) + \sum_{k,\ell} (\nabla_X L(X, Y; \theta))_{k\ell} \Delta_{k\ell} \tag{68}$$

where the subscript $(\cdot)_{k\ell}$ denotes the (k, ℓ) component. The right-hand side of (68) is maximized by choosing $\Delta_{k\ell} = \epsilon \operatorname{sign}(\nabla_X L(X, Y; \theta))_{k\ell}$, whose matrix form is given by

$$\Delta = \epsilon \operatorname{sign}(\nabla_X L(X, Y; \theta)).$$

Threat Model

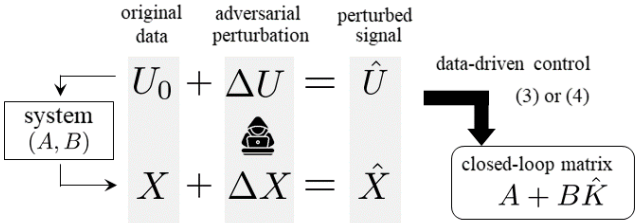


Figure: [3], Threat model.
The adversary is able to add a perturbation $(\Delta U, \Delta X)$ to the original input and output data (U_0, X) with knowledge of the system model, the signals, and the controller design algorithm. The controller $\hat{K}$ is designed using the perturbed data $(\hat{U}, \hat{X})$, which results in the closed-loop matrix $A + B\hat{K}$.

Definition (Transferability)

The effectiveness of the attack without knowledge of the data

Adversarial attacks to direct data-driven control for destabilization, Hampei Sasahara, 2023

Directed Gradient Sign Method I

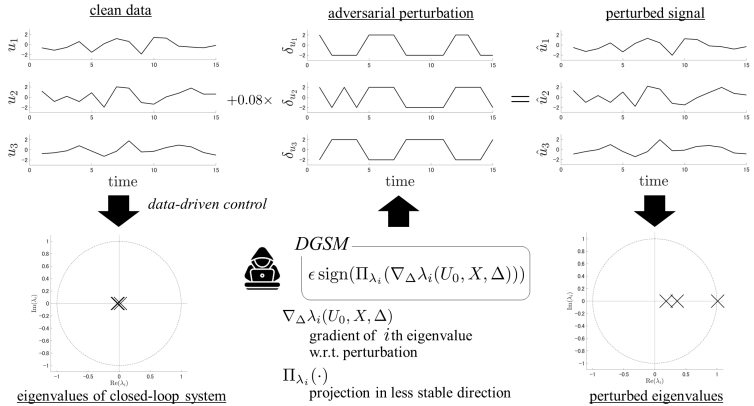


Figure: [3], Demonstration of DGSM applied to a discrete-time linear system with three-dimensional input.

Adversarial attacks to direct data-driven control for destabilization, Hampei Sasahara, 2023

Directed Gradient Sign Method II

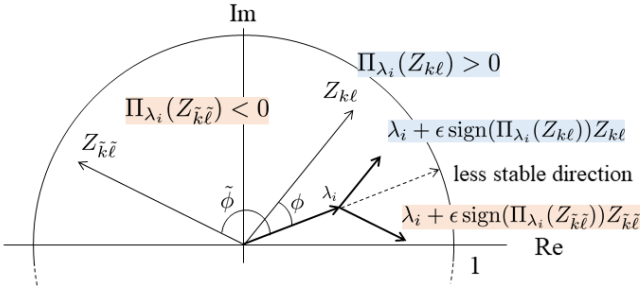


Figure: [3], Role of the function Π_{λ_i} in (70). Since $Z_{k\ell}$ faces the direction of λ_i, the angle ϕ between λ_i and $Z_{k\ell}$ is less than $\pi/2$, which leads to $\Pi_{\lambda_i}(Z_{k\ell}) > 0$. On the other hand, since $\tilde{\phi}$ between λ_i and $Z_{\tilde{k}\tilde{\ell}}$ is greater than $\pi/2$, $\pi_{\lambda_i}(Z_{\tilde{k}\tilde{\ell}}) < 0$. As a result, in both cases, the perturbed eigenvalue moves closer to the unit circle.

Adversarial attacks to direct data-driven control for destabilization, Hampei Sasahara, 2023

Directed Gradient Sign Method III

The core idea of DGSM is to choose a perturbation that locally shifts an eigenvalue in the less stable direction. We temporarily fix the eigenvalue of interest, denoted by $\lambda_i(U_0, X, \Delta)$, and denote its gradient with respect to Δ by $\nabla_\Delta \lambda_i(U_0, X, \Delta)$. The linear approximation of the eigenvalue with respect to Δ is given by

$$\lambda_i(U_0, X, \Delta) \simeq \lambda_i(U_0, X, 0) + \sum_{k,\ell} \nabla_\Delta \lambda_i(U_0, X, \Delta))\Delta_{k\ell}. \tag{69}$$

We choose $\Delta_{k\ell}$ such that the right-hand side of (69) moves closer to the unit circle. Specifically, DGSM crafts the perturbation

$$\Delta = \epsilon \operatorname{sign}(\Pi_{\lambda_i}(\nabla_\Delta \lambda_i(U_0, X, \Delta)))$$

where $\Pi_{\lambda_i} : \mathbb{C}^{(m+n)\times(2T+1)} \to \mathbb{R}^{(m+n)\times(2T+1)}$ is defined by

$$\Pi_{\lambda_i}(Z) := \operatorname{Re}\lambda_i \operatorname{Re} Z + \operatorname{Im}\lambda_i \operatorname{Im} Z \qquad \text{with} \qquad Z := \nabla_\Delta \lambda_i(U_0, X, \Delta). \tag{70}$$

Directed Gradient Sign Method IV

Algorithm 1 Directed Gradient Sign Method (DGSM)

Input: $\{\epsilon_k\}, A, B, U_0, X, \gamma, \rho$
Output: Δ

flag $\leftarrow 0$
$k \leftarrow 0$
while flag $= 0$ **do**
  $k \leftarrow k + 1$
  for $i = 1, \ldots, n$ **do**
    $\Delta \leftarrow \epsilon_k \text{sign}(\Pi_{\lambda_i}(\nabla_\Delta \lambda_i(U_0, X, \Delta)))$
    if $|\lambda_i(U_0, X, \Delta)| > 1$ **then**
      flag $\leftarrow 1$
      break
    end if
  end for
end while
return Δ

Figure: [3], The Algorithm.

Adversarial attacks to direct data-driven control for destabilization, Hampei Sasahara, 2023

Conclusions and Future Work

- Stabilisation algorithms are susceptible to adversarial attacks
- Performance of other control objectives against adversarial attacks
- Development of stronger attack models
- System simulation

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