Data-Driven Analysis and Control using Informativity

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by

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Dissertation Approval

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Abstract

Control systems model the complex interactions between the constantly changing quantities and offer a mathematical framework for explaining the world around us. The recent increase in the amount of data and computational power have given rise to Data-Driven techniques, significantly reshaping the theory of control systems by taking over the classical geometrical approaches. This study investigates the newly introduced informativity approach to control problems, tracing its roots in behavioral theory and exploring its applications in analysing various control properties. We delve into the property of dissipativity, an important link between physics and control systems, and extend the results of informativity for dissipativity using the Matrix S-Lemma to systems with only input-output data. Lastly, the problem of robustness of such approaches in the presence of adversarial attacks is discussed.

The dissertation consists of a total of six chapters – the first chapter provides an *Introduction* to the research topic, the next chapter is a comprehensive *Literature Review* followed by the necessary pre-requisites of *Behavioral Theory* which views the dynamical systems as a family of trajectories and *Discrete Systems* theory which is very relevant in the development of Data-Driven Analysis and Control. Then, we come to the Informativity approach, covering its fundamentals and applications in Dissipativity analysis. Further, we discuss the limitations of this approach by assessing its robustness against Adversarial Attacks. Finally, the Conclusion chapter wraps up the topic and discusses future work in the area.

Index Terms — Behavioral Theory, Data-Driven analysis, Data-Driven control, Informativity approach, Adversarial Attacks.

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Abbreviations

ODE	**O**rdinary **D**ifferential **E**quation
LTI	**L**inear **T**ime **I**nvariant
LMI	**L**inear **M**atrix **I**nequality
QMI	**Q**uadratic **M**atrix **I**nequality
QDF	**Q**uadratic **D**ifferential **F**orms
SVD	**S**ingular **V**alue **D**ecomposition
DMD	**D**ynamic **M**ode **D**ecomposition
DMDc	**D**ynamic **M**ode **D**ecomposition with **c**ontrol
EDMD	**E**xtended **D**ynamic **M**ode **D**ecomposition
mrDMD	**m**ulti-**r**esolution**D**ynamic **M**ode **D**ecomposition
SINDy	**S**parse **I**dentification of **N**onlinear **Dy**namics
LANDO	**L**inear **A**nd **N**onlinear **D**isambiguation **O**ptimization
QDF	**Q**uadratic **D**ifferential **F**orm
FDLTI	**F**inite **D**imensional **L**inear **T**ime **I**nvariant
LQR	**L**inear–**Q**uadratic **R**egulator
FGSM	**F**ast **G**radient **S**ign **M**ethod
DGSM	**D**irected **G**radient **S**ign **M**ethod

Symbols

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{R}^{m\times n}$	Set of all $m \times n$ matrices with real entries
$\mathbb{S}^m$	Set of all symmetric $m \times m$ matrices with real entries
$\mathrm{tr}(M)$	Trace of the matrix M
$\|M\|$	Norm of the matrix M
$\mathrm{sign}(\cdot)$	Signum function
$M^\dagger$	Moore-Penrose pseudo-inverse of a real matrix M
$M^\sharp$	Right-inverse of a full row rank matrix M
$\mathbf{v}$	A general signal which maps each time instant to a vector
	Below are few data matrices of the concatenated vectors of $\mathbf{v}$
V	$\begin{bmatrix} v(0) & v(1) & \dots & v(T) \end{bmatrix}$
V_-	$\begin{bmatrix} v(0) & v(1) & \dots & v(T-1) \end{bmatrix}$
V_+	$\begin{bmatrix} v(1) & v(2) & \dots & v(T) \end{bmatrix}$
$V_{k,t,N}$	Hankel matrix associated with $\mathbf{v}$ with t rows and N columns

Chapter 1

Introduction

Dynamical systems model the complex interactions between the constantly changing quantities and offer a mathematical framework for explaining the world around us. It involves the analysis and prediction of the behaviour of systems. The focus of Systems and Control theory is to design *controllers* that meet desired specifications. These controllers are physical devices which are interconnected with the system.

The evolution of continuous systems is usually described by differential equations whereas iterative maps are used for discrete systems. From the classical mechanics of Newton and Leibniz to the control theory of Kalman and Willems, scientists have engaged in the analytical study of these mathematical models to derive laws of the system. But nowadays, the rising complexity of system models and cheap availability of data is giving way to *data-driven approaches*. The arrival of machine learning and big data has also helped the development of such data-driven frameworks. These approaches focus on finding the laws of the system directly, using measured data without losing out on theoretical guarantees. They can be applied directly to solve many critical problems in the fields of epidemiology, climate change.

This thesis presents a modern data-driven outlook on systems.

1.1 Problem Statement

To identify various control properties of the system and to design controllers to satisfy various control objectives of the system from its data.

1.2 Need of the Study

To resolve the following challenges in modern dynamical systems:

Nonlinearity Working with linear systems is simple and desirable, we can decouple the system with a similarity transform. Sadly, no such linear transformation exist for nonlinear systems, in general. A way is to perform local linearisations around fixed points, periodic orbits, etc. But predicting global behaviour in a locally linear model remains difficult.

Noise in the modelled data Noise changes the entire dynamical system and in most cases it will result in losing linearity of the system which leads to above mentioned problems

Unknown dynamics In many fields such as neuroscience, epidemiology, and ecology which tackle complex realistic systems, the physical laws governing these systems are unknown. Even when the dynamics are known, uncovering the higher dimension dominant behaviour is tough for phenomenon of turbulence, protein folding, etc.

1.3 Study Objectives

To study possible solutions for the above challenges

Informativity approach With this framework, we convert the control problem in our hand to a Linear Matrix Inequality (LMI). For some control problems, specific noise models can also be accommodated in its analysis making it a robust solution.

Operator-theoretic representations With this framework, we represent nonlinear systems in terms of infinite-dimensional linear operators using the Koopman operator.

Data-driven regression and machine learning With this framework, we directly discover dynamical systems from data using SINDy.

Chapter 2

Literature Review

2.1 Data-Driven Control

The foundation of data-driven control can be traced to the influential work in [1] in the behavioral framework. Assuming that the given input is persistently exciting, suitable data-dependent matrices can describe the behavior of any LTI system. The papers [2] and proven in [3] provided rigorous guarantees for system analysis and controller design using trajectories.

Data-Driven analysis methods can be categorised into three setups

- [4, 5, 6, 7, 8] focusses on estimating properties using online sampling, by repeatedly choosing the input and measuring the output.
- [9, 10, 11, 12, 13] focusses on analysing nonlinear systems using large amounts of input-output trajectories.
- [14, 15, 16, 17, 18] focusses on analysing LTI systems using a single input-state or input-output trajectory

A limitation of all the methods is the requirement of direct access to plants and the need for more time due to the complexity of the approach.

Recent data-driven developments include state-feedback design [19], controller synthesis [20], dissipativity analysis [14, 15, 16], model predictive control [21, 22] and data informativity [23].

2.1.1 Dissipativity Analysis

Willems' seminal papers [24, 25] on dissipative dynamical systems introduced the concept of dissipativity. He reformulated the dissipation inequality as a linear matrix inequality.

Dissipativity properties facilitate system analysis and application of feedback theorems ensuring closed-loop stability. [26, 27] includes the standard feedback theorems and controller design based on dissipativity properties which has led to numerous approaches for determining such properties from data.

Transition to the Discrete Framework Willems recognised the limitations of the input-output framework, leading to the development of the behavioral theory. Willems introduced the concept of quadratic differential forms (QDF) which are used to represent supply rates and storage functions for any dissipative system [28]. [14] introduced another notion of L-dissipativity for finite-horizon cases.

2.1.2 Informativity Approach

Recently, there has been significant interest in inferring dissipativity properties from data. The informativity approach bounds the noise by a quadratic matrix inequality (QMI). It determines whether *all* systems satisfying such QMI are dissipative. This approach has its similarities with robust-control [29, 30, 31, 32]. But here, data is directly mapped to storage functions.

2.2 Other Data-Driven Approaches

Though we will be focussing on the informativity approach, let us also discuss other data-driven approaches in some detail.

2.2.1 Dynamic Mode Decomposition

Dynamic Mode Decomposition (DMD) offers an operator-theoretic perspective based on system measurement evolution, complementing classical geometric and statistical perspectives on control systems. An example is Koopman operator theory [33], which represents nonlinear dynamics using intrinsic coordinate systems of a linear framework, exploiting the low-dimensional behavior of many real-life infinite-dimensional dynamical systems.

It decomposes the system into various modes where each mode consists of spatially correlated structures exhibiting the same linear behavior in time, i.e. the same characteristic value of $\lambda = a + ib$, where a represents the growth rate and b is the oscillation frequency. DMD also models the evolution of each mode over time, utilizing the computationally efficient singular value decomposition (SVD) to calculate these modes. DMD is fundamentally data-driven, as no knowledge of governing equations is required and the results are applicable to both experimental and numerical data. It has various variants applicable to existing system identification and modal extraction techniques. An example of DMD applied to fluid flow is shown in Figure 2.1.

2.2.1.1 Extensions, Applications and Limitations

Including inputs and control Proctor [35] introduced Dynamic Mode Decomposition with Control (DMDc) for the design of effective controllers using data.

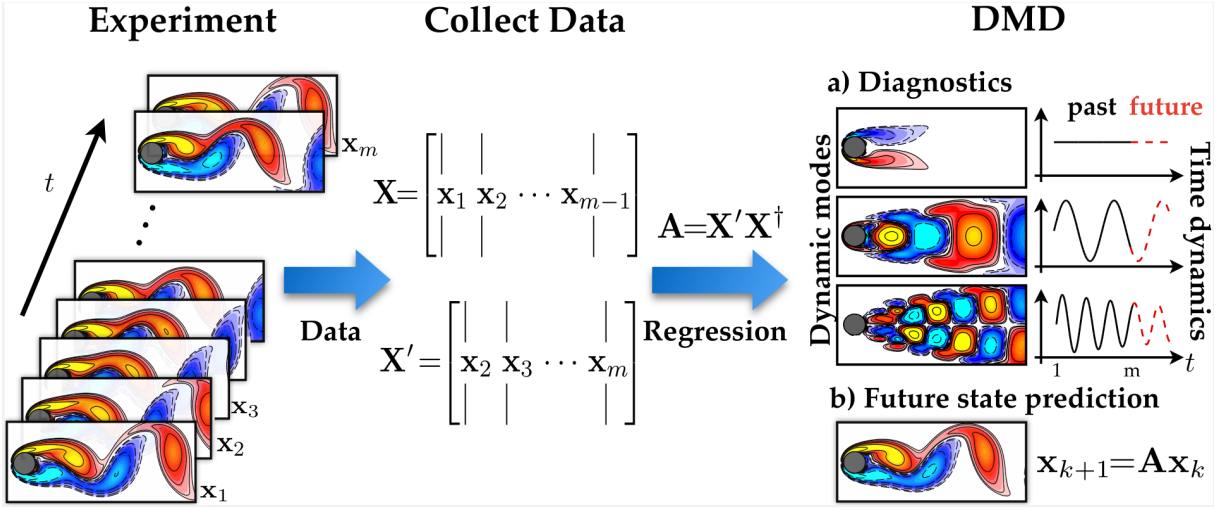


FIGURE 2.1: [34], Overview of DMD, using fluid flow past a circular cylinder

Compression and randomized linear algebra Randomized DMD algorithms [36] work more efficiently by projecting data onto lower-dimensional subspaces. *Sparsity* can also be utilised for efficient measurements [33].

Multiresolution multi-resolution Dynamic Mode Decomposition (mrDMD) [37] accurately captures dynamics with different timescales such as El Niño.

Epidemiology Proctor and Eckhoff [38] provided interpretable decompositions for data consisting of high-dimensional spatiotemporal time series measurements, such as the number of infections in a given neighbourhood or city.

Strong Nonlinearity DMD struggles to capture systems with nonlinear features such as chaos and multiple fixed points. Sparse Identification of Nonlinear Dynamics (SINDy) [39] is better at identifying nonlinear systems but it has it own limitations.

2.2.2 Sparse Identification of Nonlinear Dynamics (SINDy)

Discovering dynamical systems models from data has always been a central challenge in modelling systems, going back to the time when Kepler and Newton discovered the laws of

planetary motion. The automated discovery of governing equations is a new and exciting paradigm with increasing computational power and vast data.

The sparse identification of nonlinear dynamics (SINDy) algorithm [39] bypasses the intractable combinatorial search through all possible model structures, and uses the fact that many dynamical systems have dynamics with only a few active terms in the space of possible right-hand side functions as shown in 2.2.

This has been possible due to recent advances in compressed sensing and sparse regression. It also allows the dynamics to vary concerning bifurcation parameters $\boldsymbol{\mu} \in \mathbb{R}^q$.

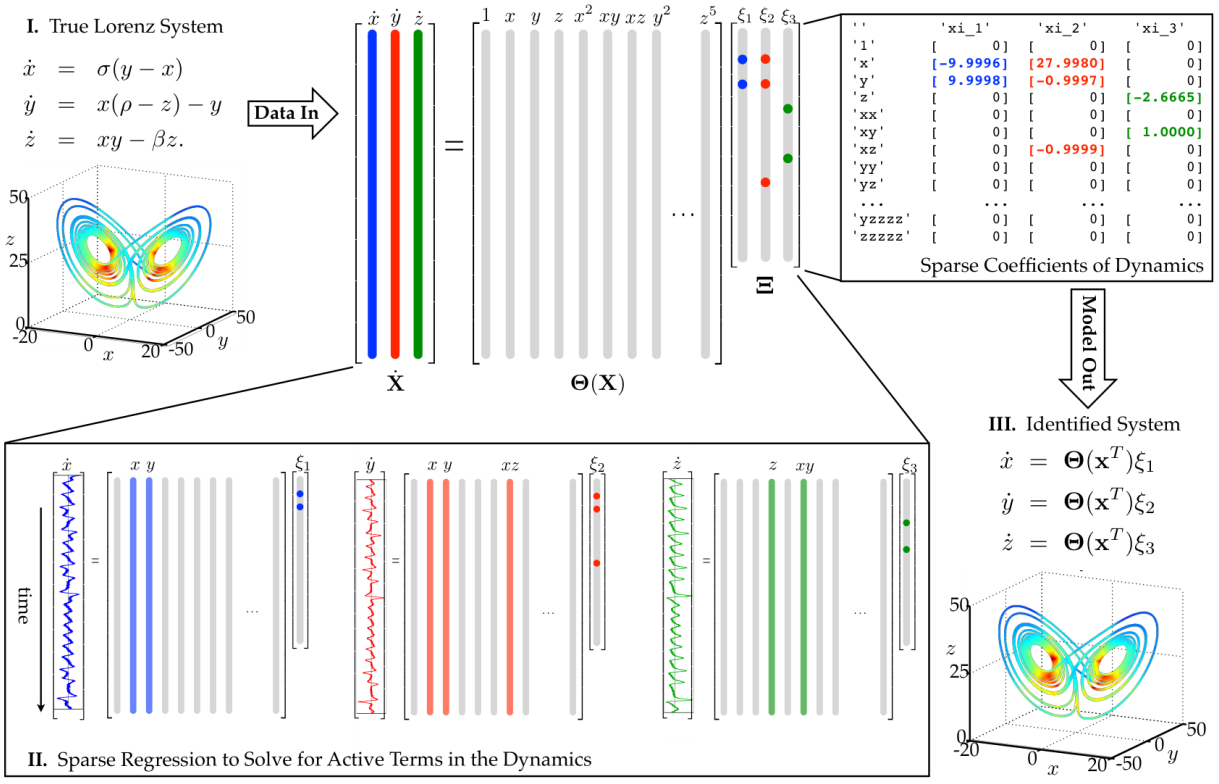


FIGURE 2.2: [39], Overview of SINDy, using the Lorenz equations

2.2.2.1 Extensions, Applications and Limitations

Constrained sparse Galerkin regression Loiseau and Brunton [40] generalised the SINDy framework to incorporate known physical constraints and symmetries in the equations by implementing a constrained sequentially thresholded least-squares

optimization by imposing energy-preserving constraints on the quadratic nonlinearities in the Navier–Stokes equations.

Rational Function Nonlinearities Kaheman [41] reformulated the dynamics in an implicit ordinary differential equation (ODE) and modified the optimization procedure to work for rational functions which are not a sparse linear combination of few basis functions. Such rational function nonlinearities occur in metabolic and regulatory networks in biology.

Implicit ODEs The optimization procedure above may be generalized to include a larger class of implicit ordinary differential equations in addition to those containing rational function nonlinearities. This can be achieved by updating our library of functions.

Limitations Linear and Nonlinear Disambiguation Optimization (LANDO) [42] resolves scaling issues for systems not inhibiting low-dimensional subspace by using kernels for learning representations [33].

2.3 Attacks on Data-Driven Control

A popular security measure is to focus on knowledge of explicit models against attacks such as zero-dynamics attacks [43], observer-based attacks [44], and moving target defence [45]. Nowadays, approaches using data-driven attack detection [48] and stealthy design [46, 47] are also being utilised.

2.4 Conclusion

Data-driven approaches represent a paradigm shift in the analysis and control of complex dynamical systems. By leveraging data to directly infer system properties and design controllers, these techniques offer a powerful alternative to traditional model-based approaches. However, the vulnerability of data-driven control algorithm has received less attention, and there is a need for dedicated techniques to address this issue which will be addressed in this report.

Chapter 3

Preliminaries

3.1 Behavioral Theory

The philosophy of the behavioral approach is that an input-state-output or input-output viewpoint of the dynamical system is conceptually limiting. Instead, the approach studies dynamical systems as a collection of trajectories, where these trajectories are vector-valued functions of time, consistent with the physical laws of the system.

A general form for representing Finite Dimensional Linear Time Invariant (FDLTI) systems are the following differential equations

$$R_n \frac{\mathrm{d}^n w}{\mathrm{d}t^n} + R_{n-1} \frac{\mathrm{d}^{n-1} w}{\mathrm{d}t^{n-1}} + \ldots + R_1 \frac{\mathrm{d}w}{\mathrm{d}t} + R_0 w = 0 \tag{3.1}$$

where $R_0, R_1, \ldots, R_n$ are real-valued matrices and w is a solution belonging to a function space of vector-valued functions which are locally square integrable. The set of all such trajectories w represents the *behavior* $\mathfrak{B}$ *of the system.* Equation 3.1 can be succinctly

represented as

$$R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \quad \rightarrow \quad \mathfrak{B} = \ker\left(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right) \tag{3.2}$$

$$\text{where } R(\xi) = R_n\xi^n + R_{n-1}\xi^{n-1} + \ldots + R_1\xi + R_0 \tag{3.3}$$

Now, $R(\xi)$ can be algebraically studied to infer properties of the behavior.

3.1.1 Controllability and Stabilizability

Behavioral theory uses trajectory-level properties to define the concepts of controllability and stabilizability instead of input-output-state representation. It still utilises the notion of state but instead, uses trajectories. At a time instant t_0, two trajectories belong to same state if they can be concatenated at that instant, i.e., $w_1, w_2 \in \mathfrak{B}$ at $t = t_0$ are in same state if the trajectory $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < t_0$ and $w(t) = w_2(t)$ for $t \geqslant t_0$.

Definition 3.1 ([49], Controllability). A behavior is controllable if for all trajectories $w_1, w_2 \in \mathfrak{B}$ and $t_0 \in \mathbb{R}$, there exists a $t_1 \geqslant t_0$ and a trajectory $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < t_0$ and $w(t) = w_2(t)$ for $t \geqslant t_1$.

Definition 3.2 ([49], Stabilizability). A behavior is stabilizable if for all trajectories $w_1 \in \mathfrak{B}$ and $t_0 \in \mathbb{R}$, there exists a trajectory $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < t_0$ and $\lim_{t\to\infty} w(t) = 0$.

3.1.1.1 Autonomous Systems

Autonomous Systems generalises the notion of an autonomous state-space system $\frac{\mathrm{d}x}{\mathrm{d}t} = \mathbf{A}x$

Definition 3.3 ([49], Autonomous Behavior). A behavior is autonomous if the future trajectories from $t_0 \in \mathbb{R}$ are fully determined by the state of the system at t_0

Equivalently, $\det(R(\xi)) \neq 0$.

The behavior of such systems can be divided into two parts: *controllable* and *autonomous*. That is, any trajectory w is uniquely given by the addition of two trajectories, where one belongs to the controllable sub-behavior while the other belongs to the autonomous sub-behavior.

3.1.2 Dissipative Dynamical Systems

[24, 25] formalises the concept of dissipativity of a system using the *storage function* of that system with respect to a *supply rate*.

3.1.2.1 State Transition Maps

Originally, dissipativity was developed for input-state-output systems defined using state transition maps ϕ, r where u, x, y represent input, state, output, respectively

$$x(t) = \phi(t, t_0, x_0, u) \text{ for } t \geqslant t_0 \tag{3.4}$$

$$y(t) = r(x(t), u(t)) \text{ for } t \geqslant t_0. \tag{3.5}$$

Assumption 3.4 ([49]). *It is assumed that ϕ satisfies the condition such that the system is time-invariant and the state at any time (t_0) can be uniquely determined by using any previous state (at time $t_1 < t_0$) and inputs to the system within that time (from $[t_1, t_0]$). Formally, for any state x_0, input u and $t_0 \leqslant t_1 \leqslant t_2$*

1. $\phi(t_0, t_0, x_0, u) = x_0$,
2. $\phi(t_1, t_0, x_0, u_1) = \phi(t_1, t_0, x_0, u_2)$ *whenever* $u_1(t) = u_2(t)$ *for all* $t_0 \leqslant t \leqslant t_1$,
3. $\phi(t_2, t_0, x_0, u) = \phi(t_2, t_1, \phi(t_1, t_0, x_0, u), u)$,
4. $\hat{u}(t) = u(t + T)\ \forall t \in \mathbb{R} \rightarrow \phi(t_1 + T, t_0 + T, x_0, \hat{u}) = \phi(t_1, t_0, x_0, u)$.

Definition 3.5 ([49], Dissipativity). A system is dissipative with respect to supply rate w, if there exists a non-negative function (the storage function) S of the state that satisfies

$$S(x_0) + \int_{t_0}^{t_1} w(u(t), y(t))\mathrm{dt} \geqslant S(x(t_1)) \tag{3.6}$$

Intuitively, the change in storage between a time interval must be always less than or equal to the total supply in that interval.

3.1.2.2 Trajectories

This notion of dissipativity was formalised for the state space systems by using the Kalman Yakubovich Popov (KYP) lemma. Consider the system in the following form

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u} \end{aligned} \tag{3.7}$$

where $\mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^p$.

Definition 3.6 ([49], Dissipativity). Assuming the system in controllable and observable, the system is dissipative with respect to the supply rate $w = u^T y$ iff the transfer function $G(s) = D + C(sI - A)^{-1}B$ is *positive-real* for all λ in open right half plane.

Alternatively, the system is dissipative iff there exists an $X \geqslant 0$ satisfying

$$\begin{bmatrix} -A^T X - XA & C^T - XB \\ C - B^T X & D + D^T \end{bmatrix} \geqslant 0 \tag{3.8}$$

Then, the available storage takes the form $S_a(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T X^- \mathbf{x}$ and required supply is $S_r(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T X^+ \mathbf{x}$ where $X^-, X^+ > 0$ are also solutions to 3.8 such that $X^+ \geqslant X \geqslant X^-$.

3.2 Transition to the Discrete Framework

Notation

Let's introduce some important notation for the next topics

- $\mathbb{S}^n$ is the set of all symmetric matrices in $\mathbb{R}^{n\times n}$
- $M^\dagger$ is the Moore-Penrose pseudo-inverse of a real matrix M
- $M^\sharp$ is the right-inverse of a full row rank matrix M
- For a general signal $\mathbf{v}$ and $k \in \mathbb{Z}$, $t, T \in \mathbb{N}$,

  $V_{k,k+T}$ is its restriction to the interval $\{k, k+1, \ldots, k+T\}$ as $V_{k,k+T} = \begin{bmatrix} v(k) \\ \vdots \\ v(k+T) \end{bmatrix}$
- $V_{k,t,N}$ is the Hankel matrix associated to $\mathbf{v}$ given by

$$V_{k,t,N} = \begin{bmatrix} v(k) & v(k+1) & \cdots & v(k+N-1) \\ v(k+1) & v(k+2) & \cdots & v(k+N) \\ \vdots & \vdots & \ddots & \vdots \\ v(k+t-1) & v(k+t) & \cdots & v(k+t+N-2) \end{bmatrix}$$

3.2.1 Inferring the System from the Data

We have seen analysis of continuous systems using the geometrical and behavioural approach, let us now study discrete systems which will help us to analyse data-driven techniques. A discrete-time linear system is given by

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{3.9a}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t) \tag{3.9b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n, \mathbf{u}(t) \in \mathbb{R}^m, \mathbf{y}(t) \in \mathbb{R}^p$ and A, B, C, D are real matrices. We assume that the system is controllable and observable.

Over the time $[0, t-1]$, the input-output response of the system can be expressed as

$$\left[\begin{array}{c} U_{0,t-1} \\ \hline Y_{0,t-1} \end{array}\right] = \left[\begin{array}{cc} I_t & 0_{tm\times n} \\ \hline \mathcal{T}_t & \mathcal{O}_t \end{array}\right] \left[\begin{array}{c} U_{0,t-1} \\ \hline X_{0,0} \end{array}\right] \tag{3.10}$$

where $\mathcal{T}_t$ is the order t Toeplitz matrix and $\mathcal{O}_t$ is the order t observability matrix,

$$\mathcal{T}_t := \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{t-2}B & CA^{t-3}B & CA^{t-4}B & \cdots & D \end{bmatrix} \qquad \mathcal{O}_t := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t-1} \end{bmatrix}$$

Now, the same response can be written using Hankel matrices by considering differents states of $\mathbf{x}$ as initial states and applying (3.10)

$$\left[\begin{array}{c} U_{0,t,T-t+1} \\ \hline Y_{0,t,T-t+1} \end{array}\right] = \left[\begin{array}{cc} I_{tm} & 0_{tm\times n} \\ \hline \mathcal{T}_t & \mathcal{O}_t \end{array}\right] \left[\begin{array}{c} U_{0,t,T-t+1} \\ \hline X_{0,1,T-t+1} \end{array}\right] \tag{3.11}$$

Now, when the actual values are unavailable, we replace those values of vectors by data to give us the following

$$\left[\begin{array}{c} (U_d)_{0,t,T-t+1} \\ \hline (Y_d)_{0,t,T-t+1} \end{array}\right] = \left[\begin{array}{cc} I_{tm} & 0_{tm\times n} \\ \hline \mathcal{T}_t & \mathcal{O}_t \end{array}\right] \left[\begin{array}{c} (U_d)_{0,t,T-t+1} \\ \hline (X_d)_{0,1,T-t+1} \end{array}\right] \tag{3.12}$$

This condition on data is the basis of control law design using data as we will see in the next section.

3.2.2 Persistently Exciting Data

Definition 3.7. [1] A signal $\mathbf{v}_{[0,T-1]} \in \mathbb{R}^n$ is persistently exciting of order L if the matrix $V_{0,L,T-L+1}$ has full rank nL.

Lemma 3.8. [1], Willem's Fundamental Lemma *For the system* (3.9a), *if the input* $u_{d,[0,T-1]}$ *is persistently exciting of order* $n+t$, *then the following condition holds*

$$\operatorname{rank} \left[\begin{array}{c} U_{0,t,T-t+1} \\ \hline X_{0,1,T-t+1} \end{array}\right] = n + tm \tag{3.13}$$

Using this lemma, a system can be uniquely determined from data when the input is persistently exciting. We also say, that such data is *informative* for system identification. The topic for informativity will be our next discussion.

Chapter 4

Informativity Approach

Notation

For the signals $\mathbf{x}, \mathbf{u}, \mathbf{y}$ and $T \in \mathbb{N}$, consider following data matrices

- $X := \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix}$
- $X_- := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}$
- $X_+ := \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}$
- $U_- := \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}$
- $Y_- := \begin{bmatrix} y(0) & y(1) & \cdots & y(T-1) \end{bmatrix}$

4.1 Data informativity framework

Let's start with few definitions that related to the informativity approach in data-driven control.

Definition 4.1 ([50], True System). Denoted by $\mathcal{S}$, is a mathematical model of the underlying unknown physical system.

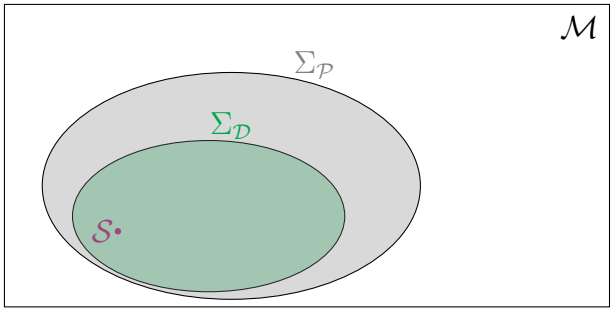
Definition 4.2 ([50], Model Class). Denoted by $\mathcal{M}$, is the set of systems that is assumed to contain the 'true' system.

Definition 4.3 ([50], Data). Denoted by $\mathcal{D}$, is the data generated by the true system.

Definition 4.4 ([50], Data Consistent Systems). Denoted by $\Sigma_{\mathcal{D}} \subseteq \mathcal{M}$, is the set of all systems in $\mathcal{M}$ that are consistent with the data $\mathcal{D}$.

Definition 4.5 ([50], Systems with property $\mathcal{P}$). Denoted by $\Sigma_{\mathcal{P}} \subseteq \mathcal{M}$, is the set of all systems in $\mathcal{M}$ having the property $\mathcal{P}$.

The key idea of the informativity approach is that the true system satisfies a property $\mathcal{P}$ if *all* the systems that are consistent with the data satisfy that property (4.1 and 4.2).



$\mathcal{M}$: model class $\Sigma_{\mathcal{D}}$: data consistent systems
$\mathcal{S}$: unknown system $\mathcal{P}$: system property
$\mathcal{D}$: given data set $\Sigma_{\mathcal{P}}$: systems with property $\mathcal{P}$

FIGURE 4.1: [50], Data is informative for property

Definition 4.6 ([50], Informativity for analysis). The data $\mathcal{D}$ is *informative* for property $\mathcal{P}$ if $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}}$.

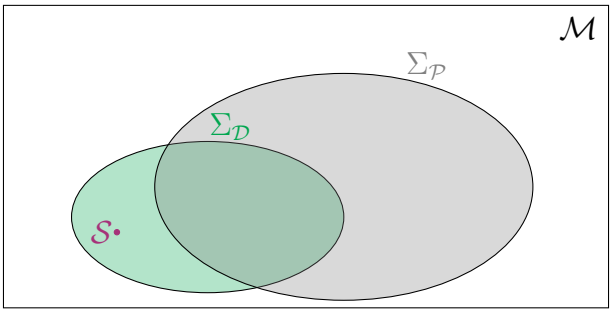


FIGURE 4.2: [50], Data is not informative for property

Problem 4.7 ([50], Informativity problem for analysis). Provide necessary and sufficient conditions on the data $\mathcal{D}$ under which these data are informative for property $\mathcal{P}$.

4.1.1 Incorporating Control problems

Consider the problem of designing a controller $\mathcal{K}$ meeting a desired control objective $\mathcal{O}$ for a data-driven control problem. Let $\Sigma_{\mathcal{O}}$ be the set of all systems satisfying the control objective $\mathcal{O}$ and $\Sigma_{\mathcal{D}}(\mathcal{K})$ are all systems obtained by interconnecting the systems in $\Sigma_{\mathcal{D}}$ with the controller. This give rise to the following informativity problem.

Definition 4.8 ([50], Informativity for control). The data $\mathcal{D}$ is *informative* for the control objective $\mathcal{O}$ if there exists a controller $\mathcal{K}$ such that $\Sigma_{\mathcal{D}}(\mathcal{K}) \subseteq \Sigma_{\mathcal{O}}$.

And the two problems associated with this informativity is to first determine if such controller exists and then to design the controller.

Problem 4.9 ([50], Informativity problem for control). Provide necessary and sufficient conditions on $\mathcal{D}$ under which the data are informative for the control objective $\mathcal{O}$.

Problem 4.10 ([50], Control design problem). Assuming the data $\mathcal{D}$ is informative for the control objective $\mathcal{O}$, find a controller $\mathcal{K}$ such that $\Sigma_{\mathcal{D}}(\mathcal{K}) \subseteq \Sigma_{\mathcal{O}}$.

This framework is already used for various control problems taking two main directions, based on exact data ('E') and noisy ('N') data and based on data requirements such as

the need of input ('I'), output ('O') and state ('S') data. Table 4.1 provides an summary of the available results and their data requirements.

TABLE 4.1: [50], Summary of results

Problem	Data
controllability	E-IS
observability	E-S
stabilizability	E-IS, N-IS
stability	E-S, N-S, N-IO
LQR	E-IS
dissipativity	E-ISO, N-ISO
tracking and regulation	E-IS

Initially we will review select analysis and design problems and then proceed to describe a new approach based on current literature.

4.2 Informativity for analysis (Controllability and Stabilizability)

The informativity notion is straightforward from the definition 4.6. Given data is *informative for controllability (stabilizability)* if all systems in $\Sigma_{\mathcal{D}}$ are controllable (stabilizable).

So, let's focus on the informativity problem for analysis which can be solved using the following theorem.

Theorem 4.11 ([50], Data-driven Hautus tests). *The data (U_-, X) is informative for controllability (stabilizability) if and only if*

$$\operatorname{rank}(X_+ - \lambda X_-) = n \quad \forall \lambda \in \mathbb{C} \quad (\textit{with } |\lambda| \geqslant 1). \tag{4.1}$$

This also shows that the rank condition (3.13) is not necessary for data-driven analysis.

4.3 Informativity for control (Stabilization)

Usingn 4.8. We define our sets as follows,

$$\Sigma_{\mathcal{O}} := \{A \in \mathbb{R}^{n\times n} \mid A \text{ is stable}\} \qquad \text{and} \qquad \Sigma_{\mathcal{D}}(K) = \{A + BK \mid (A,B) \in \Sigma_{\mathcal{D}}\}.$$

As per 4.8, data (U_-, X) is *informative for stabilization by state feedback* if there exists a $K \in \mathbb{R}^{m\times n}$ such that $\Sigma_{\mathcal{D}}(K) \subseteq \Sigma_{\mathcal{O}}$.

To solve the problem, consider the solution set of the homogeneous equation which will allow the to deduce the membership of A, B in $\Sigma_{\mathcal{D}}$

$$\Sigma_{\mathcal{D}}^0 := \left\{ (A_0, B_0) \mid 0 = \begin{bmatrix} A_0 & B_0 \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{4.2}$$

We first state a useful lemma using the fact that $(A,B) \in \Sigma_{\mathcal{D}}$ if and only if it is a solution of the the corresponding affine equation. Now let $\Sigma_{\mathcal{D}}^0$ denote the solution set of the corresponding homogeneous equation. That is,

Lemma 4.12 ([50], A necessary condition). *If the data (U_-, X) is informative for stabilization by state feedback then $A_0 + B_0K = 0$ for all $(A_0, B_0) \in \Sigma_{\mathcal{D}}^0$. Equivalently,*

$$\operatorname{im} \begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

This observations helps us to solve the problem of informativity for stabilization.

Theorem 4.13 ([50], Conditions for stabilization). *The data (U_-, X) is informative for stabilization by state feedback if and only if the matrix X_- has full row rank and there exists a right inverse $X_-^\sharp$ of X_- such that $X_+X_-^\sharp$ is stable.*

Moreover, K is such that $A+BK$ is stable for all $(A,B) \in \Sigma_{\mathcal{D}}$ if and only if $K = U_-X_-^\sharp$.

The above theorem though is not constructive as it does not provide the means to calculate such right inverse of X_-. To resolve this, the following LMI-based approach is utilised.

Theorem 4.14 ([50], LMI conditions for stabilization). *The data (U_-, X) are informative for stabilization by state feedback if and only if there exists a matrix $\Theta \in \mathbb{R}^{T\times n}$ satisfying*

$$X_-\Theta = (X_-\Theta)^\top \quad \text{and} \quad \begin{bmatrix} X_-\Theta & X_+\Theta \\ \Theta^\top X_+^\top & X_-\Theta \end{bmatrix} > 0. \tag{4.3}$$

Moreover, K is such that $A+BK$ is stable for all $(A,B) \in \Sigma_{\mathcal{D}}$ if and only if $K = U_-\Theta(X_-\Theta)^{-1}$ for some matrix Θ satisfying (4.3).

4.4 Dissipativity analysis

4.4.1 Dissipativity

Now, we will analyse the informativity problems for dissipativity of FDLTI systems using both exact and noisy data. Consider the discrete system state-space system 3.9 discussed before.

Theorem 4.15 ([50], Dissipativity LMI). *The system* (3.9) *is said to be* dissipative *with respect to the* supply rate

$$s(u,y) = \begin{bmatrix} u \\ y \end{bmatrix}^\top S \begin{bmatrix} u \\ y \end{bmatrix} \quad \text{where } S \in \mathbb{S}^{m+p} \tag{4.4}$$

if there exists $P \in \mathbb{S}^n$ *with* $P \succcurlyeq 0$ *such that the following holds*

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \succcurlyeq 0. \tag{4.5}$$

Or equivalently,

$$\left[\begin{array}{c|cc} I & & \\ \hline & A & B \\ & C & D \end{array}\right]^\top \underbrace{\begin{bmatrix} P & 0 & 0 & 0 \\ 0 & F & 0 & G \\ 0 & 0 & -P & 0 \\ 0 & G^\top & 0 & H \end{bmatrix}}_{M} \left[\begin{array}{c|cc} I & & \\ \hline & A & B \\ & C & D \end{array}\right] \succcurlyeq 0 \tag{4.6}$$

where we partition S *as follows*

$$S = \begin{bmatrix} F & G \\ G^\top & H \end{bmatrix} \quad \text{where } F \in \mathbb{R}^{m\times m},\ G \in \mathbb{R}^{m\times p},\ H \in \mathbb{R}^{p\times p} \tag{4.7}$$

4.4.2 Dissipativity from input-state-output trajectories

To solve the informativity problem now, we use the notion of persistently exciting input data. We will first consider the exact case and then the noisy case. As the system matrices are unknown, we use a subscript s to denote the same.

4.4.2.1 Exact case

Consider the unknown input-state-output system

$$\mathbf{x}(t+1) = A_s\mathbf{x}(t) + B_s\mathbf{u}(t), \tag{4.8a}$$

$$\mathbf{y}(t) = C_s\mathbf{x}(t) + D_s\mathbf{u}(t), \tag{4.8b}$$

with $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{y}(t) \in \mathbb{R}^p$ the input, state and output with the assumption that the dimensions m, n and p are known. The combined data is given by $\mathcal{D} = (U_-, X, Y_-)$ which satisfies

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}. \tag{4.9}$$

The set of all systems that are consistent with these data is then given by:

$$\Sigma_{(U_-,X,Y_-)} := \left\{ (A,B,C,D) \mid \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{4.10}$$

And hence, $(A_s, B_s, C_s, D_s) \in \Sigma_{(U_-,X,Y_-)}$.

Definition 4.16 ([50], Informativity for dissipativity of noiseless data). The data (U_-, X, Y_-) is *informative for dissipativity* with respect to the supply rate (4.4) if there exists a matrix $P \in \mathbb{S}^n$, $P \succcurlyeq 0$, such that the LMI (4.5) holds for every system $(A, B, C, D) \in \Sigma_{(U_-,X,Y_-)}$.

This definition requires all the systems in $\Sigma_{(U_-,X,Y_-)}$ to be dissipative with a *common* storage function which introduces additional assumptions on S such that the its inertia[1] is $\mathrm{In}(S) = (p, 0, m)$.

Theorem 4.17 ([50], Informativity for dissipativity of noiseless data). *Assuming that* $\mathrm{In}(S) = (p, 0, m)$ *and the data* (U_-, X, Y_-) *is informative for system identification,*

[1] number of negative, zero, and positive eigenvalues of S respectively, given by the tuple (ρ_-, ρ_0, ρ_+)

it is informative for dissipativity with respect to the supply rate (4.4) *if and only if there exists* $P = P^\top \succcurlyeq 0$ *such that*

$$\begin{bmatrix} X_- \\ X_+ \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} X_- \\ X_+ \end{bmatrix} + \begin{bmatrix} U_- \\ Y_- \end{bmatrix}^\top S \begin{bmatrix} U_- \\ Y_- \end{bmatrix} \succcurlyeq 0. \tag{4.11}$$

4.4.2.2 Noisy Data

Now, we add the unknown process noise and measurement noise to the system (4.8) resulting in the below system

$$\mathbf{x}(t+1) = A_s\mathbf{x}(t) + B_s\mathbf{u}(t) + \mathbf{w}(t), \tag{4.12a}$$

$$\mathbf{y}(t) = C_s\mathbf{x}(t) + D_s\mathbf{u}(t) + \mathbf{z}(t), \tag{4.12b}$$

where $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{y}(t) \in \mathbb{R}^p$ are the input, state and output with the assumption that the dimensions m, n and p are known. The unknown noise terms are $\mathbf{w}(t) \in \mathbb{R}^n$ and $\mathbf{z}(t) \in \mathbb{R}^p$ representing process and measurement noise, respectively. Again, we assume the assumptions on supply rate such that $S \in \mathbb{S}^{m+p}, \mathrm{In}(S) = (p, 0, m)$.

As this is the noisy case, we will need to make assumptions about noise to proceed any further.

Assumption 4.18 ([50], Noise model). *The noise samples satisfy the quadratic matrix inequality*

$$\begin{bmatrix} I \\ V_-^\top \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ V_-^\top \end{bmatrix} \succcurlyeq 0 \tag{4.13}$$

where

$$\underbrace{V_-}_{\in\mathbb{R}^{(n+p)\times T}} := \begin{bmatrix} w(0) & w(1) & \cdots & w(T-1) \\ z(0) & z(1) & \cdots & z(T-1) \end{bmatrix} \quad \textit{and} \quad \underbrace{\Phi}_{\in\mathbf{\Pi}_{n+p,T}} = \begin{bmatrix} \overbrace{\Phi_{11}}^{\in\mathbb{S}^{n+p}} & \overbrace{\Phi_{12}}^{\in\mathbb{R}^{(n+p)\times T}} \\ \underbrace{\Phi_{21}}_{\Phi_{12}^\top} & \underbrace{\Phi_{22}}_{\in\mathbb{S}^T} \end{bmatrix} \tag{4.14}$$

where $\mathbf{\Pi}_{m,n}$ *denotes following block partitioned matrices* $E \in \mathbf{\Pi}_{m,n} = \begin{bmatrix} \overbrace{E_{11}}^{\in\mathbb{S}^m} & \overbrace{E_{12}}^{\in\mathbb{R}^{m\times n}} \\ \underbrace{E_{21}}_{E_{12}^\top} & \underbrace{E_{22}}_{\in\mathbb{S}^n} \end{bmatrix}$

Also, let

$$\mathcal{Z}_T(\Phi) := \left\{ Z \in \mathbb{R}^{T\times(n+p)} \mid \begin{bmatrix} I_{n+p} \\ Z \end{bmatrix}^\top \Phi \begin{bmatrix} I_{n+p} \\ Z \end{bmatrix} \geqslant 0 \right\}, \tag{4.15}$$

As $\Phi \in \mathbf{\Pi}_{n+p,T}$, $\mathcal{Z}_T(\Phi)$ *is non-empty and convex.*

Now, V_- *satisfies* (4.34) *if and only if* $V_-^\top \in \mathcal{Z}_T(\Phi)$.

Similar to our exact case, we define $\Sigma_{\mathcal{D}}$ as follows

$$\Sigma_{\mathcal{D}} = \left\{ (A,B,C,D) \mid \left(\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right)^\top \in \mathcal{Z}_T(\Phi) \right\}. \tag{4.16}$$

We also define another matrix N

$$N := \left[\begin{array}{c:c} N_{11} & N_{12} \\ \hdashline N_{12}^\top & N_{22} \end{array}\right] = \left[\begin{array}{c|c} I & X_+ \\ & Y_- \\ \hline 0 & -X_- \\ & -U_- \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c|c} I & X_+ \\ & Y_- \\ \hline 0 & -X_- \\ & -U_- \end{array}\right]^\top \tag{4.17}$$

such that $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ if and only if

$$\left[\begin{array}{c|cc} I & & \\ \hline & A^\top & C^\top \\ & B^\top & D^\top \end{array}\right]^\top N \left[\begin{array}{c|cc} I & & \\ \hline & A^\top & C^\top \\ & B^\top & D^\top \end{array}\right] \geqslant 0. \equiv \quad \begin{bmatrix} A^\top & C^\top \\ B^\top & D^\top \end{bmatrix} \in \mathcal{Z}_{n+m}(N). \tag{4.18}$$

We can now do a similar analysis to solve the problem.

Definition 4.19 ([50], Informativity for dissipativity of noisy data). The noisy input-state-output data (U_-, X, Y_-) are *informative for dissipativity* with respect to the supply rate (4.4) if there exists a matrix $P \succcurlyeq 0$ such that the LMI (4.5) holds for all systems $(A, B, C, D) \in \Sigma_{\mathcal{D}}$.

Lemma 4.20 ([51], Necessity of positive definite storage). *In addition to the assumption on the inertia of S, we also need the Schur complement $N \,|\, N_{22}$ to be positive definite, then the common storage function P is necessarily positive definite.*

We invoke the non-strict matrix S-lemma to solve such problem,

Theorem 4.21 ([51], Matrix S-lemma). *For $M, N \in \mathbb{S}^{q+r}$, if there exists a real $\alpha \geqslant 0$ such that $M - \alpha N \succcurlyeq 0$ then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$. Also, assuming $N \in \mathbf{\Pi}_{q,r}$ and N has at least one positive eigenvalue, then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$ if and only if there exists a real $\alpha \geqslant 0$ such that $M - \alpha N \succcurlyeq 0$.*

To apply this lemma, an additional dualization result is needed as terms of (4.6) and (4.18) are *transpose* of each other.

Lemma 4.22 ([51], Dualization of dissipation inequality). *For $P > 0$, define*

$$\hat{S} := \begin{bmatrix} 0 & -I_p \\ I_m & 0 \end{bmatrix} S^{-1} \begin{bmatrix} 0 & -I_m \\ I_p & 0 \end{bmatrix}. \tag{4.19}$$

Then we have

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \geqslant 0 \tag{4.20}$$

if and only if

$$\begin{bmatrix} I & 0 \\ A^\top & C^\top \end{bmatrix}^\top \begin{bmatrix} P^{-1} & 0 \\ 0 & -P^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ A^\top & C^\top \end{bmatrix} + \begin{bmatrix} 0 & I \\ B^\top & D^\top \end{bmatrix}^\top \hat{S} \begin{bmatrix} 0 & I \\ B^\top & D^\top \end{bmatrix} \geqslant 0. \tag{4.21}$$

Intuitively, it connects the storage functions of the dual systems using an inverse relationship.

Now, we will do similar partitioning procedure for this dual system,

$$-S^{-1} = \begin{bmatrix} \hat{F} & \hat{G} \\ \hat{G}^\top & \hat{H} \end{bmatrix} \qquad \text{where } \hat{F} = \hat{F}^\top \in \mathbb{S}^m,\ \hat{G} \in \mathbb{R}^{m\times p}, \text{ and } \hat{H} = \hat{H}^\top \in \mathbb{S}^p$$

and define

$$\hat{M} := \begin{bmatrix} P^{-1} & 0 & 0 & 0 \\ 0 & \hat{H} & 0 & -\hat{G}^\top \\ 0 & 0 & -P^{-1} & 0 \\ 0 & -\hat{G} & 0 & \hat{F} \end{bmatrix}. \tag{4.22}$$

Then it is easily seen that $(A^\top, C^\top, B^\top, D^\top)$ satisfies the inequality (4.21) if and only if

$$\left[\begin{array}{c} I \\ \hline \begin{matrix} A^\top & C^\top \\ B^\top & D^\top \end{matrix} \end{array}\right]^\top \hat{M} \left[\begin{array}{c} I \\ \hline \begin{matrix} A^\top & C^\top \\ B^\top & D^\top \end{matrix} \end{array}\right] \geqslant 0. \tag{4.23}$$

Theorem 4.23 ([50], Informativity for dissipativity of noisy data). *The data* (U_-, X, Y_-) *with noise as stated in 4.18. Under the assumption of,* $\mathrm{In}(S) = (p, 0, m)$ *and* $N \,|\, N_{22} > 0$*, the data is informative for dissipativity with respect to the supply rate* (4.4) *if and only if there exist a real matrix* $Q \in \mathbb{S}^n$*,* $Q > 0$ *and a scalar* $\alpha \geqslant 0$ *such that*

$$\begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & \hat{H} & 0 & -\hat{G}^\top \\ 0 & 0 & -Q & 0 \\ 0 & -\hat{G} & 0 & \hat{F} \end{bmatrix} - \alpha \left[\begin{array}{c|c} I & X_+ \\ & Y_- \\ \hline 0 & -X_- \\ & -U_- \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c|c} I & X_+ \\ & Y_- \\ \hline 0 & -X_- \\ & -U_- \end{array}\right]^\top \geqslant 0. \tag{4.24}$$

The common storage function is given by $P := Q^{-1}$*.*

This case requires the knowledge of state data which is expensive and might not be readily available. Hence, the next section tackles the informativity for dissipativity problem using only input-output trajectories.

4.4.3 Dissipativity from input-output trajectories

In this case, we don't have state data, the combined data is given by $\mathcal{D} = (U_-, Y_-)$. This absence adds another complexity to the problem and hence we introduce another term for our next assumption.

Definition 4.24 ([52]). The lag $\underline{l}$ of system (4.8) is the smallest integer $l \in \mathbb{N}_+$ such that the observability matrix $\mathcal{O}_l$ has rank n

$$\mathcal{O}_l := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{l-1} \end{bmatrix} \tag{4.25}$$

4.4.3.1 Noiseless data

Now, we extend the original system (4.8) into another system where the state is given by only the known inputs and outputs under the assumption that the upper bound on the lag of the system is known as $l \geqslant \underline{l}$

Lemma 4.25 ([52]). *There exists a system $\widetilde{G}$ with system matrices $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ with the same input-output behavior as the original system, i.e., there exists ξ_0 such that for $k = 0, \ldots, N-1$,*

$$\xi_{k+1} = \widetilde{A}\xi_k + \widetilde{B}u_k, \quad y_k = \widetilde{C}\xi_k + \widetilde{D}u_k, \tag{4.26}$$

where the extended state is defined by

$$\xi_k = \begin{bmatrix} u_{k-l}^\top & u_{k-l+1}^\top & \cdots & u_{k-1}^\top & y_{k-l}^\top & y_{k-l+1}^\top & \cdots & y_{k-1}^\top \end{bmatrix}^\top.$$

Proof. The state equations can be rewritten as (4.27),

$$\begin{bmatrix} y_{k-l} \\ y_{k-l+1} \\ \vdots \\ y_{k-1} \end{bmatrix} = \underbrace{\begin{bmatrix} C_s \\ C_sA_s \\ \vdots \\ C_sA_s^{l-1} \end{bmatrix}}_{\mathcal{O}_l} x_{k-l} + \underbrace{\begin{bmatrix} D_s & 0 & & \ldots & 0 & 0 \\ C_sB_s & D_s & & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ C_sA_s^{l-2}B_s & C_sA_s^{l-3}B_s & \ldots & C_sA_sB_s & C_sB_s & D_s \end{bmatrix}}_{R} \begin{bmatrix} u_{k-l} \\ u_{k-l+1} \\ \vdots \\ u_{k-1} \end{bmatrix} \tag{4.27}$$

which can be simplified to

$$\begin{bmatrix} -R & I \end{bmatrix} \xi_k = \mathcal{O}_l x_k.$$

Now, as the lag $\underline{l} \leqslant l$, $\mathcal{O}_l$ has full column rank, and hence, there exists a left-inverse $\mathcal{O}_l^{-1}$ (which has full row rank) such that

$$\underbrace{\mathcal{O}_l^{-1}\begin{bmatrix} -R & I \end{bmatrix}}_{K}\xi_k = x_k.$$

This gives us

$$y_k = C_s A_s^l x_{k-l} + \begin{bmatrix} C_s A_s^{l-1} B_s & \dots & C_s B_s \end{bmatrix}\begin{bmatrix} u_{k-l} \\ \vdots \\ u_{k-1} \end{bmatrix}.$$

which can be expanded as,

$$\underbrace{\begin{bmatrix} u_{k-l+1} \\ \vdots \\ u_{k-1} \\ u_k \\ y_{k-l+1} \\ \vdots \\ y_{k-1} \\ y_k \end{bmatrix}}_{\xi_{k+1}} = \underbrace{\begin{bmatrix} 0 & I & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & I & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & I & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & I \\ C_s A_s^{l-1} B_s & \dots & \dots & C_s B_s & 0 & 0 & \dots & 0 \end{bmatrix}}_{\widetilde{A}} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ C_s A_s^l T \end{bmatrix}\underbrace{\begin{bmatrix} u_{k-l} \\ u_{k-l+1} \\ \vdots \\ u_{k-1} \\ y_{k-l} \\ y_{k-l+1} \\ \vdots \\ y_{k-1} \end{bmatrix}}_{\xi_k} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \\ D_s \end{bmatrix}}_{\widetilde{B}} u_k$$

$$y_k = \underbrace{\begin{bmatrix} 0 & \dots & 0 & I \end{bmatrix}}_{\widetilde{C}}\widetilde{A}\xi_k + \underbrace{D}_{\widetilde{D}} u_k$$

(4.28)

■

Now, we can get the informativity results by a analysis similar to the input-state-output case 4.17. The data matrices will have to be replaced appropriately with the new state ξ

as follows

$$\begin{aligned}
\Xi_- &\coloneqq \begin{bmatrix} \xi_l & \xi_{l+1} & \cdots & \xi_{T-1} \end{bmatrix}, \\
\Xi_+ &\coloneqq \begin{bmatrix} \xi_{l+1} & \xi_{l+2} & \cdots & \xi_T \end{bmatrix}, \\
Y_{\Xi_-} &\coloneqq \begin{bmatrix} y_l & y_{l+1} & \cdots & y_{T-1} \end{bmatrix}, \\
U_{\Xi_-} &\coloneqq \begin{bmatrix} u_l & u_{l+1} & \cdots & u_{T-1} \end{bmatrix},
\end{aligned}$$

Theorem 4.26 (Informativity for dissipativity of noiseless data). *Assuming that* $\mathrm{In}(S) = (p, 0, m)$*, the lag of the system* $\underline{l} \leqslant l$ *and the data* (U_-, Y_-) *is informative for system identification, then the data* (U_-, Y_-) *is informative for dissipativity with respect to the supply rate* (4.4) *if and only if there exists* $P = P^\top \geqslant 0$ *such that*

$$\begin{bmatrix} \Xi_- \\ \Xi_+ \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} \Xi_- \\ \Xi_+ \end{bmatrix} + \begin{bmatrix} U_{\Xi_-} \\ Y_{\Xi_-} \end{bmatrix}^\top S \begin{bmatrix} U_{\Xi_-} \\ Y_{\Xi_-} \end{bmatrix} \geqslant 0. \tag{4.29}$$

4.4.3.2 Noisy data

Now, consider the case of noisy input-output data. For this, we can rewrite the system (4.8) in the difference operator form

$$y_k = -a_l y_{k-1} - \cdots - a_2 y_{k-l+1} - a_1 y_{k-l} + d u_k + b_l u_{k-1} + \cdots + b_2 u_{k-l+1} + b_1 u_{k-l}, \tag{4.30}$$

with $a_i \in \mathbb{R}^{p \times p}$, $b_i \in \mathbb{R}^{p \times m}$, $i = 1, \ldots, l$, and l is an upper bound on the lag $\underline{l} \leqslant l$.

Now, the input-output behavior is corrupted by process noise of the following form

$$y_k = -a_l y_{k-1} - \cdots - a_2 y_{k-l+1} - a_1 y_{k-l} + d u_k + b_l u_{k-1} + \cdots + b_2 u_{k-l+1} + b_1 u_{k-l} + b_v v_k, \tag{4.31}$$

where $v_k \in \mathbb{R}^{m_v}$ denotes the noise and $b_v \in \mathbb{R}^{p \times m_v}$ is an extra term which can be used to utilise prior knowledge of noise.

Now, this behavior (4.31) can also be represented in state-space as shown below

$$\begin{bmatrix} u_{k-l+1} \\ \vdots \\ u_{k-1} \\ u_k \\ y_{k-l+1} \\ \vdots \\ y_{k-1} \\ y_k \end{bmatrix} = \begin{bmatrix} 0 & I & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & I & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & I & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & I \\ b_1 & b_2 & \dots & b_l & -a_1 & -a_2 & \dots & -a_l \end{bmatrix} \begin{bmatrix} u_{k-l} \\ u_{k-l+1} \\ \vdots \\ u_{k-1} \\ y_{k-l} \\ y_{k-l+1} \\ \vdots \\ y_{k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \\ D \end{bmatrix} u_k + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_v \end{bmatrix} v_k \tag{4.32}$$

This can be simplified as

$$\begin{aligned} \xi_{k+1} &= \underbrace{\begin{bmatrix} \widetilde{A}_1 \\ \widetilde{C} \end{bmatrix}}_{\widetilde{A}} \xi_k + \underbrace{\begin{bmatrix} \widetilde{B}_1 \\ \widetilde{D} \end{bmatrix}}_{\widetilde{B}} u_k + \begin{bmatrix} 0 \\ b_v \end{bmatrix} v_k, \\ y_k &= \widetilde{C}\xi_k + \widetilde{D}u_k + b_v v_k, \end{aligned} \tag{4.33}$$

where $\widetilde{A} \in \mathbb{R}^{(p+m)l \times (p+m)l}$, $\widetilde{B} \in \mathbb{R}^{(p+m)l \times m}$, $\widetilde{C} \in \mathbb{R}^{p \times (p+m)l}$, and $\widetilde{D} \in \mathbb{R}^{p \times m}$,

Here, $\widetilde{A}_1 \in \mathbb{R}^{((p+m)l-p) \times (p+m)l}$, and $\widetilde{B}_1 \in \mathbb{R}^{((p+m)l-p) \times m}$ are fixed.

Now, to incorporate noise, we can consider the following noise model

Assumption 4.27 (Noise model). *The noise samples satisfy the quadratic matrix inequality*

$$\begin{bmatrix} I \\ V_-^\top \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ V_-^\top \end{bmatrix} \geqslant 0 \tag{4.34}$$

where

$$\underbrace{V_-}_{\in \mathbb{R}^{m_v \times T-l}} := \begin{bmatrix} v(l) & v(l+1) & \cdots & v(T-1) \end{bmatrix} \quad \text{and} \quad \underbrace{\Phi}_{\in \mathbf{\Pi}_{m_v,T-l}} = \begin{bmatrix} \overbrace{\Phi_{11}}^{\in \mathbb{S}^{m_v}} & \overbrace{\Phi_{12}}^{\in \mathbb{R}^{m_v \times (T-l)}} \\ \underbrace{\Phi_{21}}_{\Phi_{12}^\top} & \underbrace{\Phi_{22}}_{\in \mathbb{S}^{T-l}} \end{bmatrix} \tag{4.35}$$

Also, let

$$\mathcal{Z}_{T-l}(\Phi) := \left\{ Z \in \mathbb{R}^{(T-l)\times m_v} \mid \begin{bmatrix} I_{m_v} \\ Z \end{bmatrix}^\top \Phi \begin{bmatrix} I_{m_v} \\ Z \end{bmatrix} \geqslant 0 \right\}, \tag{4.36}$$

As $\Phi \in \mathbf{\Pi}_{m_v,T-l}$, $\mathcal{Z}_{T-l}(\Phi)$ is non-empty and convex.

Now, V_- satisfies (4.34) if and only if $V_-^\top \in \mathcal{Z}_{T-l}(\Phi)$.

Similar to our previous case, we define $\Sigma_{\mathcal{D}}$ as follows after noting $\widetilde{A}_1, \widetilde{B}_1$ are known, hence we can reduce the size of our noise model

$$\begin{aligned} \Sigma_{\mathcal{D}} = \Bigg\{ & (A,B,C,D) \mid \\ & A = \begin{bmatrix} \widetilde{A}_1 \\ C \end{bmatrix}, B = \begin{bmatrix} \widetilde{B}_1 \\ D \end{bmatrix}, \left(\begin{bmatrix} Y_{\Xi_-} \end{bmatrix} - \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} \Xi_- \\ U_{\Xi_-} \end{bmatrix} \right)^\top = b_v V \text{ and } V \in \mathcal{Z}_{T-l}(\Phi) \Bigg\} \end{aligned} \tag{4.37}$$

Now, the only thing that needs to be taken care of is the matrix S, which needs to be extended suitably to get the equivalent $\widetilde{S} \in \mathbb{S}^{(m+p)l+pl}$ for the extended system. Then partition $\widetilde{S}$ as shown below, and construct a matrix M similar to the input-state-output

case (4.5) and then use the Matrix S-lemma 4.21 to get the final result.

$$-\widetilde{S}^{-1} = \begin{bmatrix} \hat{F} & \hat{G} \\ \hat{G}^\top & \hat{H} \end{bmatrix} \qquad \text{where } \hat{F} = \hat{F}^\top \in \mathbb{S}^{(m+p)l},\ \hat{G} \in \mathbb{R}^{(m+p)l \times pl},\ \text{and } \hat{H} = \hat{H}^\top \in \mathbb{S}^{pl} \tag{4.38}$$

Theorem 4.28 ([50], Informativity for dissipativity of noisy data). *The data* (U_-, Y_-) *with noise as stated in 4.30. Under the assumption of,* $\mathrm{In}(S) = (p, 0, m)$ *and* $N \,|\, N_{22} > 0$*, the data is informative for dissipativity with respect to the supply rate* (4.4) *if and only if there exist a real matrix* $Q \in \mathbb{S}^{(m+p)l}$*,* $Q > 0$ *and a scalar* $\alpha \geqslant 0$ *such that*

$$\begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & \hat{H} & 0 & -\hat{G}^\top \\ 0 & 0 & -Q & 0 \\ 0 & -\hat{G} & 0 & \hat{F} \end{bmatrix} - \alpha \left[\begin{array}{c|c} I & \Xi_+ \\ & Y_{\Xi_-} \\ \hline 0 & -\Xi_- \\ & -U_{\Xi_-} \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c|c} I & \Xi_+ \\ & Y_{\Xi_-} \\ \hline 0 & -\Xi_- \\ & -U_{\Xi_-} \end{array}\right]^\top \geqslant 0. \tag{4.39}$$

The common storage function is given by $P := Q^{-1}$*.*

Now, let us discuss some vulnerabilities of data-driven control.

Chapter 5

Adversarial Attacks to Data-Driven Control

This chapter evaluates the robustness of direct data-driven control methods against adversarial attacks. The objective of the attacker is to disrupt the stability of the system by making small perturbations to the data. It is assumed that the attacked has complete knowledge of the system.

Definition 5.1 (Transferability Property). Transferability property is the effectiveness of adversarial perturbations without partial knowledge.

Definition 5.2 (Transferability across data). Effectiveness of the attack without the knowledge of data.

Definition 5.3 (Transferability across parameters). Effectiveness of the attack without the knowledge of parameters.

Before delving into security, let us discuss the standard algorithms for optimal feedback.

5.1 Linear Quadratic Regulator

We consider the standard model as discussed in 4.3.

Consider the linear quadratic regulator (LQR) problem, where the pair (A, B) is unknown to the designer but it is known to be stabilizable. Now, the objective is to design as state-feedback control that minimizes the cost function

$$J(K) = \sum_{i=1}^{n} \sum_{t=0}^{\infty} \{x(t)^\top Q x(t) + u(t)^\top R u(t)\}|_{x(0)=e_i} \tag{5.1}$$

where $Q \geqslant 0$, $R > 0$ and e_i is the ith standard basis vector.

This cost function can be rewritten as

$$J(K) = \text{tr}(QP) + \text{tr}(K^\top RKP) \tag{5.2}$$

where $P \geqslant I$.

We will follow the previous notation for data matrices containing signals values of $\mathbf{u}, \mathbf{x}$. Let the disturbance signal be $\mathbf{d}$ and its data matrix be $D_0 := [d(0)\ d(1)\ \cdots\ d(T-1)] \in \mathbb{R}^{m \times T}$.

We will assume that Willem's fundamental lemma is satisfied which enables us to carry out data-driven control.

Lemma 5.4. [19, 53], LQR controller design *The noise free algorithm is given by*

$$\begin{aligned} &\min_{P,K,G} \quad \text{tr}(QP) + \text{tr}(K^\top RKP) \\ &\textit{such that} \quad X_+ G P G^\top X_+^\top - P + I \preceq 0,\ P \succeq I \quad \textit{and} \quad \begin{bmatrix} K \\ I^\top \end{bmatrix} = \begin{bmatrix} U_- \\ X_- \end{bmatrix} G \end{aligned} \tag{5.3}$$

For a robust approach, a regularized optimiser is proposed.

$$\begin{aligned} \min_{P,K,G} \quad & \operatorname{tr}(QP) + \operatorname{tr}(K^\top RKP) + \gamma \|\Pi G\| \\ \text{such that} \quad & X_+ GPG^\top X_+^\top - P + I \preceq 0, \; P \succeq I \;\; \text{and} \;\; \begin{bmatrix} K \\ I^\top \end{bmatrix} = \begin{bmatrix} U_- \\ X_- \end{bmatrix} G \end{aligned} \tag{5.4}$$

where $\gamma \geqslant 0$, $\Pi := I - W_0^\dagger W_0$ and $\|\cdot\|$ is any matrix norm.

5.2 Threat Model

The description of threat model is as follows (and shown in 5.1)

- A perturbation $(\Delta U, \Delta X)$ can be added to the input and output data
- The system model (A, B) and the design algorithms are known to the attacker

The controller designed using perturbed data $\hat{K}$ leads to a different system matrix $A+B\hat{K}$. Now, if the eigenvalues of $A + B\hat{K}$ perturb outside the unit circle then the system is destabilised.

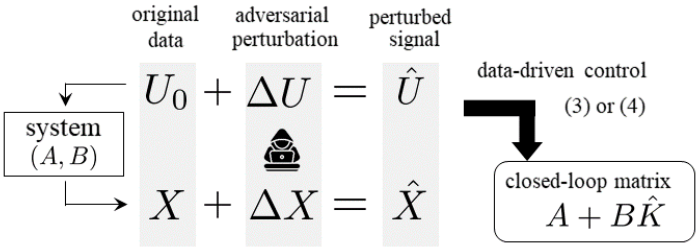


FIGURE 5.1: [54], Threat model

5.3 Fast Gradient Sign Method

[55] developed the fast gradient sign method (FGSM) to compute adversarial perturbations for given images efficiently. For a loss function of the neural network given

by $L(X, Y; \theta)$, $X \in \mathcal{X}$ is the input, $Y \in \mathcal{Y}$ is the label, and θ is the trained parameter. And $f : \mathcal{X} \to \mathcal{Y}$ is the trained classification model.

The aim of the attacker is to add small perturbations to cause misclassification in the model. This perturbation are designed in such a way that they locally maximises the loss function. This can be done by choosing a perturbation parallel to the direction of gradient of the loss function which is approximated using a linear approximation

$$L(X + \Delta, Y; \theta) \approx L(X, Y; \theta) + \sum_{i,j} (\nabla_X L(X, Y; \theta))_{i,j} \Delta_{i,j} \to \Delta = \epsilon \operatorname{sign}(\nabla_X L(X, Y; \theta)) \tag{5.5}$$

where the subscript $(\cdot)_{i,j}$ denotes the $(i, j)^{th}$ component.

5.4 Directed Gradient Sign Method

[54] developed the directed gradient sign method (DGSM) using similar principles as FGSM. It designs a perturbation such that the eigenvalues are shifted in less stable direction. It is calculated using linear approximations of eigenvalues as follows

$$\lambda_i(U_0, X, \Delta) \simeq \lambda_i(U_0, X, 0) + \sum_{i,j} \nabla_\Delta \lambda_i(U_0, X, \Delta)) \Delta_{i,j}. \tag{5.6}$$

$\Delta_{i,j}$ is chosen such that the right-hand side of (5.6) moves closer to the unit circle. Hence,

$$\Delta = \epsilon \operatorname{sign}(\Pi_{\lambda_i}(\nabla_\Delta \lambda_i(U_0, X, \Delta)))$$

where

$$\Pi_{\lambda_i}(Z) := \operatorname{Re} \lambda_i \operatorname{Re} Z + \operatorname{Im} \lambda_i \operatorname{Im} Z \quad \text{and} \quad Z := \nabla_\Delta \lambda_i(U_0, X, \Delta) \tag{5.7}$$

Figure 5.2 demonstrates the working of DGSM

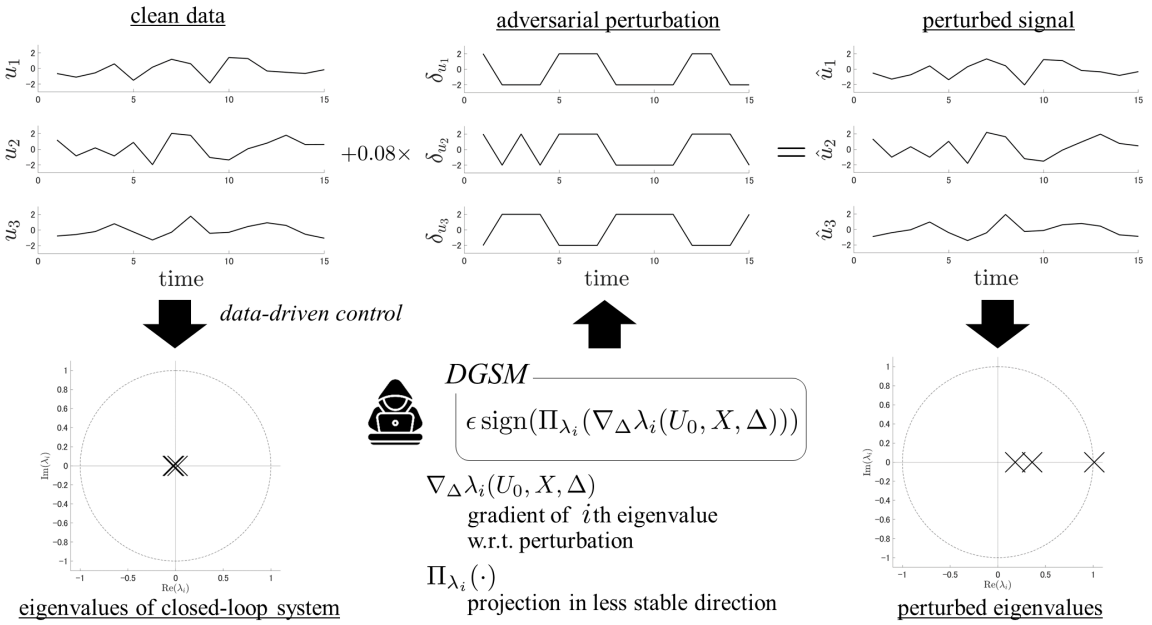


FIGURE 5.2: [54], Overview of DGSM, using a discrete-time linear system

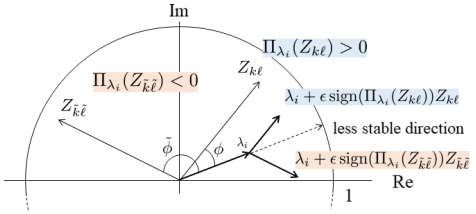


FIGURE 5.3: [54], Role of the function Π_{λ_i} in (5.7)

As shown in 5.3, the perturbed eigenvalue moves closer to the exterior irrespective of its direction. The perturbed eigenvalue $\hat{\lambda}_i$ can be approximated by

$$\hat{\lambda}_i \simeq \lambda_i + \epsilon \sum_{i,j} \mathrm{sign}(\Pi_{\lambda_i}(Z_{i,j}))Z_{i,j},$$

and as ϵ increases, this eigenvalue moves outside of the circle

DGSM uses this idea in an iterative fashion to calculate the smallest ϵ which destabilises the system from a given set.

Chapter 6

Conclusion

An introduction to the informativity approach was discussed and a new result for dissipativity using the Matrix S-lemma was developed. Though, there are certain limitations and possibilities of future research.

An obvious limitation is that the dissipativity approach works with only controllable and observable systems due to their need in the Willem's fundamental lemma. There are few works extending the result to such domains such as [52].

This approach deals with identifying the necessary and sufficient conditions for informativity. Such conditions are strong, in the sense that they assume the data and informativity can be interchanged. Instead, another situation can be analysed where data contains more information, giving rise to only sufficient conditions. These conditions can computationally perform better albeit with the loss of strong theoretical guarantees.

The informativity approach has three parts: the model class, the control objective, and noise model. While we have discussed on varying the control objective, the possibilities of analysis using different model class is endless due to the richer nature of nonlinear

systems such as bilinear systems [56, 57], polynomial systems [58, 59] or rational systems [60]. Different noise models can also be considered by incorporating measurement noise [19] or sample-bounded noise [61].

6.1 Attacks

This study primarily explored new avenues in the vulnerability of direct data-driven control. Further research can be done on different types of data, systems, attacks or adversaries. Investigating defence mechanisms such as detection of adversarial perturbations [62] is also crucial. Finally, a more comprehensive understanding of the approach and its vulnerabilities are needed for reliable data-driven control.

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