Braess's Paradox - How Making Roads Could Slow Up Traffic

Param Rathour

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What is Braess's Paradox

This is an example of a Veridical Paradox.

Adding capacity to a transportation network can sometimes actually slow down the traffic!

Modelling a Transportation Network

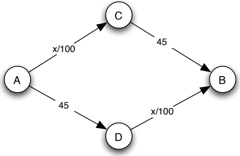

Figure 1: A highway network

- Directed Graph

  Edges Highways

  Nodes Exits to get on or off a particular Highway.

- Each edge has a designated travel time that depends on the amount of traffic it contains.

Strategic Form Games

Definition (Strategic Form Game)

A Strategic Form Game Γ is a tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where

- $N = \{1, 2, \ldots, n\}$ is a set of players
- $S_1, S_2, \ldots, S_n$ are sets called the strategy sets of the players $1, 2, \ldots, n$ respectively
- $u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$ for $i = 1, 2, \ldots, n$ are mappings called the utility functions or payoff functions.

Representation into a Strategic Form Game

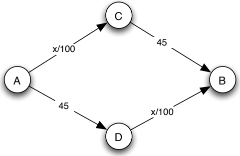

Figure 2: A highway network

- Assume $n = 4000$ cars, then $N = \{1, 2, \ldots, 4000\}$
- Strategy Sets are $S_1 = S_2 = \cdots = S_{4000} = \{C, D\}$
- Assume n_C (n_D) cars travel along C (D), Note that $n_C + n_D = n$
  So, the utility functions are

$$\begin{aligned} u_i(s_1, \ldots, s_n) &= -45 - \frac{n_C}{100} \quad \text{if} \quad s_i = C \\ &= -45 - \frac{n_D}{100} \quad \text{if} \quad s_i = D \end{aligned}$$

The notion of Nash Equilibrium

Definition (Pure Strategy Nash Equilibrium)

Given a strategic form game $\Gamma = \langle N, (S_i), (u_i) \rangle$, the strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is called a pure strategy Nash equilibrium of Γ if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \; \forall s_i \in S_i \; \forall i = 1, 2, \ldots, n$$

That is, each player's Nash equilibrium strategy is a best response to the Nash equilibrium strategies of the other players

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Definition (Best Response Correspondence)

Given a strategic form game $\Gamma = \langle N, (S_i), (u_i) \rangle$,the best response correspondence for player i is the mapping $b_i : S_{-i} \rightarrow 2^{S_i}$ defined by

$$b_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s{-}i) \geq u_i(s_i', s{-}i) \; \forall \; s_i' \in S_i\}$$

It can be seen that the strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ is a pure strategy Nash equilibrium iff

$$s_i^* \in b_i(s_{-i}^*), \forall i = 1, \ldots, n$$

Interpretations of Nash Equilibrium

- Prescription
- Prediction
- Self-Enforcing Agreement
- Evolution and Steady-State

Equilibrium Traffic

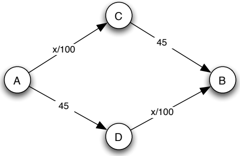


Figure 3: A highway network

- First consider case when $n_C \neq n_D$, then the two routes will have unequal travel times, and any driver on the slower route would have an incentive to switch to the faster one.
- Hence any list of strategies in which n_C is not equal to 2000 cannot be a Nash equilibrium; and any list of strategies in which $n_C = n_D = 2000$ is a Nash equilibrium.
- Time delay $= 45 + \frac{2000}{100} = 65$ minutes

Adding a Route from C to D

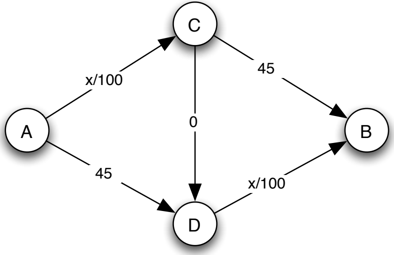


Figure 4: A highway network

- Now, a fast link from C to D to ease the congestion in the network is introduced
- We will assume the travel time from C to D to be zero as a degenerate case

Representation into a Strategic Form Game

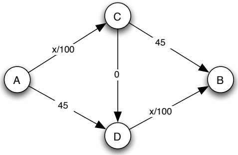


Figure 5: A highway network

- Again, assume $n = 4000$ cars, then $N = \{1, 2, \ldots, 4000\}$
- Strategy Sets are $S_1 = S_2 = \cdots = S_{4000} = \{C, D, CD\}$
- Assume n_C (n_D) (n_{CD}) cars travel along C (D) (CD), Note that $n_C + n_D + n_{CD} = n$

  So, the utility functions are

$$
\begin{aligned}
u_i(s_1, \ldots, s_n) &= -45 - \frac{n_C + n_{CD}}{100} \quad \text{if} \quad s_i = C \\
&= -45 - \frac{n_D + n_{CD}}{100} \quad \text{if} \quad s_i = D \\
&= -\frac{n_C + n_{CD}}{100} - \frac{n_D + n_{CD}}{100} \quad \text{if} \quad s_i = CD
\end{aligned}
$$

Equilibrium Traffic

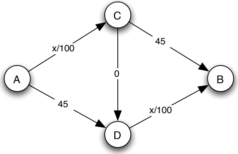
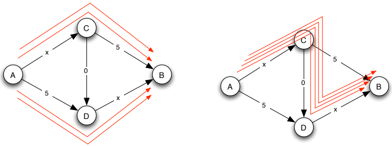


Figure 6: A highway network

- A surprising result is that now there is a unique Nash equilibrium (every driver uses the route CD).
- Why is it an equilibrium?
- Why is it unique?
- Time delay $= \frac{4000}{100} + \frac{4000}{100} = 80$ minutes
- This, time is clearly worse than 65 minutes we can get if half the people choose C and other the half choose D

Big Questions



(a) *The social optimum.* (b) *The Nash equilibrium.*

- Does an equilibrium traffic pattern always exists?
- How bad Braess's Paradox can be for networks in general?
- How much larger can the equilibrium travel time be after the addition of an edge, relative to what it was before?
- How to design networks to prevent bad equilibria from arising?

References

- Chapter 8 *Networks, Crowds, and Markets: Reasoning about a Highly Connected World* by David Easley and Jon Kleinberg, Cambridge University Press, 2010
- **How Bad is Selfish Routing?** by Tim Roughgarden and Eva Tardos