Theorems in InterEcon

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Theorem: Stolper and Samuelson

In a two-good, two-factor Heckscher–Ohlin model under perfect competition, an **increase in the relative price** of one good **raises the real return to the factor used intensively** in its production and lowers the real return to the other factor.

Assumptions Two goods (**A and B**) and two factors (**capital K and labor L**). PF $F(K, L)$ is C^2, **linearly homogeneous (CRS)**, strictly concave, and has positive marginal products ($\mathbf{F_K > 0}$, $\mathbf{F_L > 0}$, $\mathbf{F_{KK} < 0}$, $\mathbf{F_{LL} < 0}$). Factors are mobile domestically but immobile internationally. **Good A is capital-intensive** ($\mathbf{K_A/L_A > K_B/L_B}$), and good B is labor-intensive.

Proof(Sketch) Suppose P_B/P_A increases. Under perfect competition, unit costs satisfy:

$$P_A = c_A(r,\, w)$$
$$P_B = c_B(r,\, w),$$

where $c_i(r,\, w)$ is the unit-cost function in sector i. Since B is more labor-intensive, $c_B(r,\, w)$ increases with w/r more rapidly than $c_A(r,\, w)$. Hence there is a one-to-one relation

$$\frac{P_B}{P_A} \uparrow \iff \frac{w}{r} \uparrow .$$

An increase in P_B/P_A raises w/r, inducing firms in both sectors to substitute capital for labor and thus to increase $k = K/L$. Expressing $F(K, L) = L\, f(k)$ ($\because F$: CRS). Then

$$\mathrm{MPL}(K, L) = g(k) \equiv f(k) - k\, f'(k),$$
$$\mathrm{MPK}(K, L) = f'(k).$$

From concavity, $f''(k) < 0$ implies:

$$g'(k) = -k\, f''(k) > 0, \quad \text{so}\, \mathrm{MPL} = w/P\, \text{rises with } k,$$
$$f''(k) < 0, \quad \text{so}\, \mathrm{MPK} = r/P\, \text{falls with } k.$$

Thus, the real wage (w/P) increases and the real rental rate (r/P) falls in both sectors. □

Role in the Heckscher–Ohlin Model

- In a country, links relative goods prices to factor returns
- Serves as a stepping stone to the *Factor Price Equalization Theorem.*

Difference between Theorems

- *Stolper–Samuelson theorem*: Price fluctuations of goods → Explains **fluctuations** in **domestic** factor prices (real wages & interest rates)
- *Factor price equalization theorem*: When the relative prices of two goods are unified across countries through **international trade**, then the **real returns to factors** are equalized across countries.

Theorem: Rybczynski

Assume the **relative price** of two goods is **fixed**. In a two-good, two-factor Heckscher–Ohlin model under perfect competition, an **increase in the endowment of one factor** (holding the other factor and goods prices constant) **raises the output** of the good which uses that **factor intensively** and **lowers the output** of the other good.

Assumptions Same as above, except that P_A, P_B are taken as given (**small open economy**).

Proof (Sketch) Let total factor endowments be K, L with

$$K = K_A + K_B, \quad L = L_A + L_B.$$

Define the economy-wide capital-labor ratio

$$\frac{K}{L} = \underbrace{\frac{L_A}{L}}_{\alpha_A} \frac{K_A}{L_A} + \underbrace{\frac{L_B}{L}}_{\alpha_B} \frac{K_B}{L_B}, \quad \alpha_A + \alpha_B = 1.$$

Since there is a one-to-one relation as before

$$\frac{P_B}{P_A} \text{ const.} \iff \frac{w}{r} \text{ const..}$$

meaning goods prices fix factor price ratio w/r. Thus the sectoral intensities K_i/L_i are determined by cost minimization and remain constant when endowments change. Now increase K. Because each K_i/L_i is constant, the weighted average must satisfy

$$\frac{K}{L} = \alpha_A \frac{K_A}{L_A} + \alpha_B \frac{K_B}{L_B} \quad \text{where } K_A/L_A > K_B/L_B.$$

This implies that K/L is written as increasing linear function of α_A (decreasing in α_B)

$$\frac{K}{L} = \underbrace{(\frac{K_A}{L_A} - \frac{K_B}{L_B})}_{>0,\text{ const.}} \alpha_A + \frac{K_B}{L_B} = \underbrace{(\frac{K_B}{L_B} - \frac{K_A}{L_A})}_{<0,\text{ const.}} \alpha_B + \frac{K_A}{L_A}$$

An increase in K/L implies α_A must rise and α_B fall. Hence

$$K \uparrow \implies L_A/L = \alpha_A \uparrow \implies K_A = \frac{K_A}{L_A} \text{ const.} \implies L_A \uparrow, K_A \uparrow$$
$$K \uparrow \implies L_B/L = \alpha_B \downarrow \implies K_B = \frac{K_B}{L_B} \text{ const.} \implies L_B \downarrow, K_B \downarrow.$$

Thus both capital and labor shift from the labor-intensive sector B to the capital-intensive sector A, raising output of A and reducing output of B. □

Role in the Heckscher–Ohlin Model

- In a **small open economy**, showing how changes in factor endowments affect outputs
- The result can be seen as a bench-mark for a larger model with endogenous price ratio
- To prove the *Heckscher-Ohlin-Theorem*, the result is used like:

  For a **given relative price**, the **relative supply** of the (labor-) capital-intensive good is **higher** in the (labor-) capital-**abundant** country than in the (capital-) labour-abundant one

Theorem: Heckscher and Ohlin

The labor-abundant country exports the labor intensive good, the capital abundant country exports the capital-intensive good; i.e. **each country exports the good which uses the country's abundant factor intensively**. By Stolper and Samuelson Theorem, those price changes with the international trade express **distribution effects of starting trade**!

Theorem: Factor Price Equalization

In a Standard model under perfect competition and free trade, if both countries face the same relative goods prices, then the **real returns to factors** are equalized across countries.

- 2 countries (Home & Foreign), 2 goods (X & Y), and 2 factors (capital K & labor L).
- Identical and constant-returns-to-scale (CRS) production functions in both countries:

$$F^X(K,L),\ F^Y(K,L) \quad (C^2,\ \text{strictly concave},\ F_K > 0,\ F_L > 0,\ F_{KK} < 0,\ F_{LL} < 0).$$

- Factors are perfectly mobile within each country but immobile between countries.
- X is capital-intensive, Y is labor-intensive in both countries: $K_X/L_X > K_Y/L_Y$.
- Free trade implies that world relative prices are equal:

$$\frac{P_X^H}{P_Y^H} = \frac{P_X^F}{P_Y^F} = \frac{P_X^W}{P_Y^W}.$$

Proof Since both countries share identical technology and face the same world prices (P_X^W, P_Y^W), under perfect competition, unit-cost functions in Home and Foreign satisfy:

$$\frac{c_X(w,\, r)}{c_Y(w,\, r)} = \frac{P_X^W}{P_Y^W}, \qquad \frac{c_X^*(w^*,\, r^*)}{c_Y^*(w^*,\, r^*)} = \frac{P_X^W}{P_Y^W}$$

where (w, r) are Home factor prices and (w^*, r^*) are Foreign factor prices, and $c_i(r,\, w)$ is the unit-cost function in sector i. Since Y is more labor-intensive, $c_Y(r,\, w)$ increases with w/r more rapidly than $c_X(r,\, w)$ in both countries. Hence there are one-to-one relations:

$$\frac{P_Y^W}{P_X^W} = a \iff \frac{w}{r} = b, \qquad \frac{P_Y^W}{P_X^W} = c \iff \frac{w^*}{r^*} = d\ .$$

By the one-to-one relationships, each system pins down a unique ratio w/r in Home and w^*/r^* in Foreign. But because both systems share the same relative price (P_X^W/P_Y^W) and identical unit-cost functions, it must be that

$$\frac{w}{r} = \frac{w^*}{r^*}\ .$$

Furthermore, given CRS, zero-profit conditions imply that

$$w = \mathrm{MPL}_X \cdot P_X^H = \mathrm{MPL}_Y \cdot P_Y^H, \quad r = \mathrm{MPK}_X \cdot P_X^H = \mathrm{MPK}_Y \cdot P_Y^H \quad .$$

$$w^* = \mathrm{MPL}_X^* \cdot P_X^F = \mathrm{MPL}_Y^* \cdot P_Y^F, \quad r^* = \mathrm{MPK}_X^* \cdot P_X^F = \mathrm{MPK}_Y^* \cdot P_Y^F \quad .$$

By identity of production functions and CRS PF's property,

$$\frac{w}{r} = \frac{w^*}{r^*} \implies \frac{K}{L}\,|_{\mathrm{OPT}} = \frac{K^*}{L^*}\,|_{\mathrm{OPT}}\ .$$

It's because there also exists 1-1 relation between K/L and w/r comming from the company's PMP. Rigorously, we have to think it too. From identity of PF and CRS PF's property:

$$\mathrm{MPL}_X = \mathrm{MPL}_X^*,\ \mathrm{MPL}_Y = \mathrm{MPL}_Y^*,\ \mathrm{MPK}_X = \mathrm{MPK}_X^*,\ \mathrm{MPK}_Y = \mathrm{MPK}_Y^*\ .$$

We checked the fact graphically, but if you want to check it more rigorously, we should check that the gradient vector of CRS PF ∇F only depends on $k = K/L$.
Rearranging those relationships:

$$w/P_X^H = \mathrm{MPL}_X = w^*/P_X^F,\ w/P_Y^H = \mathrm{MPL}_Y = w^*/P_Y^F\ ,$$
$$r/P_X^H = \mathrm{MPK}_X = r^*/P_X^F,\ r/P_Y^H = \mathrm{MPK}_Y = r^*/P_Y^F\ .$$

$$\therefore \quad r/P_X^H = r^*/P_X^F,\ r/P_Y^H = r^*/P_Y^F\ , w/P_X^H = w^*/P_X^F,\ w/P_Y^H = w^*/P_Y^F\ .$$

Therefore, both the real return to factors are equalized internationally. □