Theorems in InterEcon

Sora Maekawa

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Theorem: Stolper and Samuelson

In a two-good, two-factor Heckscher-Ohlin model under perfect competition, an increase in the relative price of one good raises the real return to the factor used intensively in its production and lowers the real return to the other factor.

Assumptions Two goods (A and B) and two factors (capital K and labor L). PF F(K, L) is C^2 , linearly homogeneous (CRS), strictly concave, and has positive marginal products ($\mathbf{F_K} > \mathbf{0}$, $\mathbf{F_L} > \mathbf{0}$, $\mathbf{F_{KK}} < \mathbf{0}$, $\mathbf{F_{LL}} < \mathbf{0}$). Factors are mobile domestically but immobile internationally. Good A is capital-intensive ($\mathbf{K_A}/\mathbf{L_A} > \mathbf{K_B}/\mathbf{L_B}$), and good B is labor-intensive.

Proof(Sketch) Suppose P_B/P_A increases. Under perfect competition, unit costs satisfy:

$$P_A = c_A(r, w)$$

$$P_B = c_B(r, w),$$

where $c_i(r, w)$ is the unit-cost function in sector i. Since B is more labor-intensive, $c_B(r, w)$ increases with w/r more rapidly than $c_A(r, w)$. Hence there is a one-to-one relation

$$\frac{P_B}{P_A} \uparrow \iff \frac{w}{r} \uparrow.$$

An increase in P_B/P_A raises w/r, inducing firms in both sectors to substitute capital for labor and thus to increase k = K/L. Expressing F(K, L) = L f(k) (: F: CRS). Then

$$MPL(K, L) = g(k) \equiv f(k) - k f'(k),$$

$$MPK(K, L) = f'(k).$$

From concavity, f''(k) < 0 implies:

$$g'(k) = -k f''(k) > 0$$
, so MPL = w/P rises with k , $f''(k) < 0$, so MPK = r/P falls with k .

Thus, the real wage (w/P) increases and the real rental rate (r/P) falls in both sectors. \square

Role in the Heckscher-Ohlin Model

- In a country, links relative goods prices to factor returns
- Serves as a stepping stone to the Factor Price Equalization Theorem.

Difference between Theorems

- Stolper-Samuelson theorem: Price fluctuations of goods → Explains fluctuations in domestic factor prices (real wages & interest rates)
- Factor price equalization theorem: When the relative prices of two goods are unified across countries through international trade, then the real returns to factors are equalized across countries.

Theorem: Rybczynski

Assume the **relative price** of two goods is **fixed**. In a two-good, two-factor Heckscher-Ohlin model under perfect competition, an **increase in the endowment of one factor** (holding the other factor and goods prices constant) **raises the output** of the good which uses that **factor intensively** and **lowers the output** of the other good.

Assumptions Same as above, except that P_A, P_B are taken as given (small open economy).

Proof (Sketch) Let total factor endowments be K, L with

$$K = K_A + K_B, \quad L = L_A + L_B.$$

Define the economy-wide capital-labor ratio

$$\frac{K}{L} = \underbrace{\frac{L_A}{L}}_{\alpha_A} \underbrace{\frac{K_A}{L_A}}_{L_A} + \underbrace{\frac{L_B}{L}}_{\alpha_B} \underbrace{\frac{K_B}{L_B}}_{L_B}, \quad \alpha_A + \alpha_B = 1.$$

Since there is a one-to-one relation as before

$$\frac{P_B}{P_A}$$
 const. $\iff \frac{w}{r}$ const..

meaning goods prices fix factor price ratio w/r. Thus the sectoral intensities K_i/L_i are determined by cost minimization and remain constant when endowments change. Now increase K. Because each K_i/L_i is constant, the weighted average must satisfy

$$\frac{K}{L} = \alpha_A \, \frac{K_A}{L_A} + \alpha_B \, \frac{K_B}{L_B} \quad \text{where } K_A/L_A > K_B/L_B.$$

This implies that K/L is written as increasing linear function of α_A (decreasing in α_B)

$$\frac{K}{L} = (\underbrace{\frac{K_A}{L_A} - \frac{K_B}{L_B}}_{>0, \text{ const.}}) \alpha_A + \underbrace{\frac{K_B}{L_B}}_{=0} = (\underbrace{\frac{K_B}{L_B} - \frac{K_A}{L_A}}_{<0, \text{ const.}}) \alpha_B + \underbrace{\frac{K_A}{L_A}}_{=0, \text{ const.}}$$

An increase in K/L implies α_A must rise and α_B fall. Hence

$$K \uparrow \Longrightarrow L_A/L = \alpha_A \uparrow \Longrightarrow K_A = \frac{K_A}{L_A} \text{ const.} \implies L_A \uparrow, K_A \uparrow$$

 $K \uparrow \Longrightarrow L_B/L = \alpha_B \downarrow \Longrightarrow K_B = \frac{K_B}{L_B} \text{ const.} \implies L_B \downarrow, K_B \downarrow.$

Thus both capital and labor shift from the labor-intensive sector B to the capital-intensive sector A, raising output of A and reducing output of B. \Box

Role in the Heckscher-Ohlin Model

- In a small open economy, showing how changes in factor endowments affect outputs
- The result can be seen as a bench-mark for a larger model with endogenous price ratio
- To prove the *Heckscher-Ohlin-Theorem*, the result is used like:

 For a **given relative price**, the **relative supply** of the (labor-) capital-intensive good is **higher** in the (labor-) capital-**abundant** country than in the (capital-) labour-abundant one

Theorem: Heckscher and Ohlin

The labor-abundant country exports the labor intensive good, the capital abundant country exports the capital-intensive good; i.e. **each country exports the good which uses the country's abundant factor intensively**. By Stolper and Samuelson Theorem, those price changes with the international trade express **distribution effects of starting trade!**

Theorem: Factor Price Equalization

In a Standard model under perfect competition and free trade, if both countries face the same relative goods prices, then the **real returns to factors** are equalized across countries.

- 2 countries (Home & Foreign), 2 goods (X & Y), and 2 factors (capital K & labor L).
- Identical and constant-returns-to-scale (CRS) production functions in both countries:

$$F^X(K,L), F^Y(K,L)$$
 (C², strictly concave, $F_K > 0, F_L > 0, F_{KK} < 0, F_{LL} < 0$).

- Factors are perfectly mobile within each country but immobile between countries.
- X is capital-intensive, Y is labor-intensive in both countries: $K_X/L_X > K_Y/L_Y$.
- Free trade implies that world relative prices are equal:

$$\frac{P_X^H}{P_Y^H} = \frac{P_X^F}{P_Y^F} = \frac{P_X^W}{P_Y^W}.$$

Proof Since both countries share identical technology and face the same world prices (P_X^W, P_Y^W) , under perfect competition, unit-cost functions in Home and Foreign satisfy:

$$\frac{c_X(w, r)}{c_Y(w, r)} = \frac{P_X^W}{P_Y^W}, \qquad \frac{c_X^*(w^*, r^*)}{c_Y^*(w^*, r^*)} = \frac{P_X^W}{P_Y^W}$$

where (w, r) are Home factor prices and (w^*, r^*) are Foreign factor prices, and $c_i(r, w)$ is the unit-cost function in sector i. Since Y is more labor-intensive, $c_Y(r, w)$ increases with w/r more rapidly than $c_X(r, w)$ in both countries. Hence there are one-to-one relations:

$$\frac{P_Y^W}{P_X^W} = a \iff \frac{w}{r} = b, \qquad \frac{P_Y^W}{P_X^W} = c \iff \frac{w^*}{r^*} = d \;.$$

By the one-to-one relationships, each system pins down a unique ratio w/r in Home and w^*/r^* in Foreign. But because both systems share the same relative price (P_X^W/P_Y^W) and identical unit-cost functions, it must be that

$$\frac{w}{r} = \frac{w^*}{r^*} .$$

Furthermore, given CRS, zero-profit conditions imply that

$$w = \mathrm{MPL}_X \cdot P_X^H = \mathrm{MPL}_Y \cdot P_Y^H, \quad r = \mathrm{MPK}_X \cdot P_X^H = \mathrm{MPK}_Y \cdot P_Y^H$$

$$w^* = \text{MPL}_X^* \cdot P_X^F = \text{MPL}_Y^* \cdot P_Y^F, \quad r^* = \text{MPK}_X^* \cdot P_X^F = \text{MPK}_Y^* \cdot P_Y^F$$
.

By identity of production functions and CRS PF's property,

$$\frac{w}{r} \, = \, \frac{w^*}{r^*} \, \implies \frac{K}{L} \mid_{\text{OPT}} \, = \, \frac{K^*}{L^*} \mid_{\text{OPT}} \, .$$

It's because there also exists 1-1 relation between K/L and w/r comming from the company's PMP. Rigorously, we have to think it too. From identity of PF and CRS PF's property:

$$\mathrm{MPL}_X = \mathrm{MPL}_X^*, \ \mathrm{MPL}_Y = \mathrm{MPL}_Y^*, \ \mathrm{MPK}_X = \mathrm{MPK}_X^*, \ \mathrm{MPK}_Y = \mathrm{MPK}_Y^* \ .$$

We checked the fact graphically, but if you want to check it more rigorously, we should check that the gradient vector of CRS PF ∇F only depends on k = K/L. Rearranging those relationships:

$$\begin{split} w/P_X^H &= \text{MPL}_X = w^*/P_X^F, \ w/P_Y^H = \text{MPL}_Y = w^*/P_Y^F, \\ r/P_X^H &= \text{MPK}_X = r^*/P_X^F, \ r/P_Y^H = \text{MPK}_Y = r^*/P_Y^F. \end{split}$$

$$\therefore r/P_X^H = r^*/P_X^F, \ r/P_Y^H = r^*/P_Y^F \ , w/P_X^H = w^*/P_X^F, \ w/P_Y^H = w^*/P_Y^F \ .$$

Therefore, both the real return to factors are equalized internationally.