Voting and Social Choice Theory (1)

Recap

- The ‘standard’ economic model of market exchange assumes a *completely decentralized* form of social organization.
  - All individuals decide independently, taking prices as given.
- This system may be imperfect in the sense that not all potential Pareto improvements are realized.
  - Prices do not reflect ‘external effects’
  - Individual incentives to ‘free ride’
- *Privately* organized ‘collective’ action may solve some problems.
  - Coasian bargains between sovereign individuals.
  - Based on *mutual (unanimous) agreement*
- Private solutions may work well if transactions costs are low.
- In large groups, ‘efficiency’ might require ‘the state’ to ‘choose’ (in some sense) and to use *coercion* to enforce its decisions.

Social Choice Theory

- Question: How should a group of individuals make ‘collective’ choices (from some set of alternatives) when there is disagreement among its members as to which is best?
- Different opinions can arise for (at least) two reasons:
  - Different (factual) *beliefs* about the properties of the options.
  - Different *preferences* over the true characteristics of options.
- These reasons lead to two different ‘functions’ of collective choice mechanisms (decision rules / voting procedures)
  - *information aggregation*.
  - *preference aggregation*.
- We will begin our analysis by assuming that the group is choosing between *two* alternatives.

Choosing between two alternatives

Marquis de Condorcet (1743-1794)

- Condorcet assumed that the purpose of voting is to identify which option is ‘truly best’ for the group.
- Differences of opinion are due to different *beliefs* about which is the *objectively correct* answer.

Examples

- Board members deciding whether to undertake an investment.
- Jury deciding whether the defendant is guilty of murder.

In these cases,

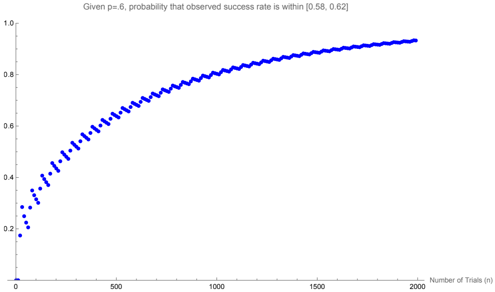
- There is an “objectively” correct answer
- Different opinions due to different information
- Voting is a method of *information aggregation*

Condorcet's Model (with 2 alternatives)

- 2 alternatives, a and b.
  - One alternative is *'truly'* better
  - Both alternatives are equally likely to be better
- n individuals, each casts one vote (no abstention)
- Voters decide *independently*
  - probability of voting for the *correct* alternative: p
- Assumption: Voters are more likely to be right than to be wrong

$$p > \frac{1}{2}$$

The Law of Large Numbers: (Bernoulli 1713) Consider an 'experiment' that can result in exactly two outcomes ('success' or 'failure'). Let the *probability* of success be p, and suppose you repeat the experiment n times. Let $\hat{p}$ be the *proportion* of those experiments that result in success. Then, the probability that $\hat{p}$ is (arbitrarily) close to p approaches 1 as n becomes large.

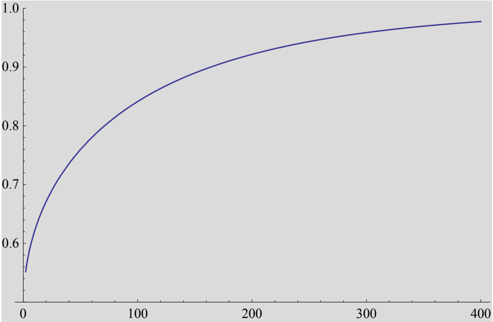


Applied to Condorcet's model: As the number of voters, n, becomes large, it becomes increasingly likely that the fraction voting for the correct option is very close to p. When n approaches infinity, this probability approaches 1.

Implication (1)

- As the group gets larger, we can be more and more certain that the fraction of voters supporting the 'correct' option is close to p.
- Since $p > \frac{1}{2}$, this means that we can be increasingly certain that a *majority* will vote for the 'correct' option.

Example: $p = 0.55$

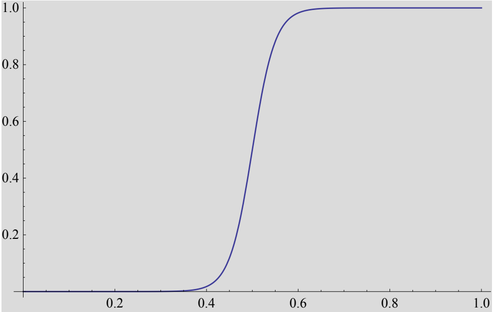


Probability that majority is correct given n

Implication (2)

- If the group is *very large*, the majority (however slight!) is almost certainly correct.

Example: $p = 0.55$, $n = 100$



Probability that option **A** is correct, given share voting for **A**.

Condorcet's 'Jury Theorem': (Condorcet 1785)

Let n voters (n odd) choose between two alternatives that have equal likelihood of being correct a priori. Assume that voters make their judgments independently and have the same probability of being correct, $p > .5$. Then the probability that the group makes a correct judgement using majority rule approaches 1 as n becomes large.

What about other q-majority rules?

- Consider rules of the form 'Option A is chosen if at least q people vote for A. Else B is chosen.'
- If A and B are equally likely ex ante, what's the best value for q?

Theorem: (Nitzan and Paroush 1982; Shapley and Grofman 1984): Let n voters (n odd) choose between two alternatives (... as above). Simple majority rule (i.e. $q = \frac{n+1}{2}$) *maximizes* the probability that the group judgement is correct.

Discussion

(1) What assumptions are responsible for the main result?

(2) (When) do these assumptions seem reasonable?

(3) How does this approach differ from the perspective taken by Buchanan and Tullock?

Possible answers:

(1) There is an objective truth, and all voters (no matter how many) receive *independent* signals. (Lots of information is being gathered!)

(2) When "true" interests are aligned (e.g. electing a manager) and when each voter does his own original "research" (e.g. interviews the manager in a different way).

(3) B&T assume that each binary decision concerns a change relative to a status quo, and that "true" preferences with respect to these changes differ.

When differences of opinion reflect different *interests*

- What if differences of opinion reflect genuinely different *preferences*?
  - Some like cheese, others prefer pudding.
  - Which do “we” prefer (as a group)?
- Is there a good way to derive a ‘**social preference**’ from individual tastes?
- *For now*, we continue to restrict attention to a choice between *two* alternatives.

Social preferences over *two* alternatives

- Set of alternatives: $X = \{x, y\}$
- I individuals, each with individual preference denoted $\alpha_i \in \{-1, 0, +1\}$

  $\alpha_i = +1$ means that Mr. i prefers x to y
  $\alpha_i = -1$ means that Mr. i prefers y to x
  $\alpha_i = 0$ means that Mr. i is indifferent

- A *profile* of preferences is $\alpha = (\alpha_1, ..., \alpha_I) \in \{-1, 0, 1\}^I$, i.e. it is a list summarizing the individual preferences in the group.

Example: Movie night

- $x =$ "Die Hard" $y =$ "When Harry Met Sally"
- $I = \{$Andrew, Jacob, Nicole$\}$
- Andrew prefers "Die Hard": $\alpha_{Andrew} = +1$
- Jacob prefers "When Harry Met Sally": $\alpha_{Jacob} = -1$
- Nicole is indifferent: $\alpha_{Nicole} = 0$
- The preference profile for the group is $\alpha = (\alpha_{Andrew}, \alpha_{Jacob}, \alpha_{Nicole}) = (1, -1, 0)$

Definition: A **social welfare functional (SWF)** is a rule F that assigns a "*social preference*" $F(\alpha) \in \{-1, 0, 1\}$ to any possible profile of individual preferences $\alpha = (\alpha_1, ..., \alpha_I) \in \{-1, 0, 1\}^I$.

Exercise:

- In our example, what does it mean to say that a SWF assigns a social preference to *any possible profile* of individual preferences?
- Would it be appropriate to view a *SWF* as a 'decision rule'?

Discuss:

- Suppose you had to define a 'good' SWF for N=3 people.
- What 'social preference' should the SWF produce for each of the cases below?

α	$F(\alpha)$	α	$F(\alpha)$	α	$F(\alpha)$
(0,0,0)		(0,0,1)		(0,0,-1)	
(0,1,0)		(0,1,1)		(0,1,-1)	
(0,-1,0)		(0,-1,1)		(0,-1,-1)	
(1,0,0)		(1,0,1)		(1,0,-1)	
(1,1,0)		(1,1,1)		(1,1,-1)	
(1,-1,0)		(1,-1,1)		(1,-1,-1)	
(-1,0,0)		(-1,0,1)		(-1,0,-1)	
(-1,1,0)		(-1,1,1)		(-1,1,-1)	
(-1,-1,0)		(-1,-1,1)		(-1,-1,-1)	

Discuss:

- What *reasons* did you have in mind as you filled out the table?
- Were you applying basic *principles*? What were they?
- Can you provide a justification for those principles?

Commonly cited principles:

- All individuals should be treated equally.
- Neither option should be favored by rule.
- Output should depend on preferences in a reasonable way, e.g.
  - If all $\alpha_i = 1$ then $F(\alpha) = 1$
  - ...

Axiomatic analysis:

- Basic principles (properties) are formulated as axioms.
- Each axiom divides the set of SWFs into two classes:
  - Those that satisfy the axiom.
  - Those that do not.
- Thus, each axiom *restricts* the set of SWFs
- Then we can ask: Which SWFs (if any) satisfy a given *set* of axioms?

Definition: A social welfare functional $F(\alpha)$ is **symmetric among agents** if, whenever α' is a permutation of $\alpha = (\alpha_1, ..., \alpha_I)$, then

$$F(\alpha') = F(\alpha).$$

Explanation: If F treats all agents the same, then reshuffling their opinions (or numbering them differently) should not affect the output.

Implications for the $n = 3$ case:

α	$F(\alpha)$	α	$F(\alpha)$	α	$F(\alpha)$
(0,0,0)		(0,0,1)	X	(0,0,-1)	Z
(0,1,0)	X	(0,1,1)	W	(0,1,-1)	Y
(0,-1,0)	Z	(0,-1,1)	Y	(0,-1,-1)	U
(1,0,0)	X	(1,0,1)	W	(1,0,-1)	Y
(1,1,0)	W	(1,1,1)		(1,1,-1)	V
(1,-1,0)	Y	(1,-1,1)	V	(1,-1,-1)	T
(-1,0,0)	Z	(-1,0,1)	Y	(-1,0,-1)	U
(-1,1,0)	Y	(-1,1,1)	V	(-1,1,-1)	T
(-1,-1,0)	U	(-1,-1,1)	T	(-1,-1,-1)	

Definition: A social welfare functional F is **neutral between alternatives** if, for any profile $\alpha = (\alpha_1, ..., \alpha_I)$,

$$F(-\alpha) = -F(\alpha).$$

Explanation: If neither option is favored by rule, then "flipping" their labels (x becomnes y and y becomes x) should "flip" the social preference too.

EXERCISE: Fill out the table to show what this axiom implies in our example.

α	$F(\alpha)$	α	$F(\alpha)$	α	$F(\alpha)$
(0,0,0)		(0,0,1)		(0,0,-1)	
(0,1,0)		(0,1,1)		(0,1,-1)	
(0,-1,0)		(0,-1,1)		(0,-1,-1)	
(1,0,0)		(1,0,1)		(1,0,-1)	
(1,1,0)		(1,1,1)		(1,1,-1)	
(1,-1,0)		(1,-1,1)		(1,-1,-1)	
(-1,0,0)		(-1,0,1)		(-1,0,-1)	
(-1,1,0)		(-1,1,1)		(-1,1,-1)	
(-1,-1,0)		(-1,-1,1)		(-1,-1,-1)	

Definition: A social welfare functional F is **positively responsive** if whenever $F(\alpha) \geq 0$ then for any $\alpha' \neq \alpha$ such that $(\alpha'_1, ..., \alpha'_I) \geq (\alpha_1, ...\alpha_I)$, we have $F(\alpha') = +1$.

Explanation: If "society" is indifferent according to the rule F, then after one or more people 'change their mind' to favor x, society should favor x as well.

Example: If F is positively responsive and $F(1, -1, -1) = 0$, then $F(1, -1, 0) = 1$.

EXERCISE (NOW): Identify all implications of this axiom in the table below. (Note that one social preference is given.)

α	$F(\alpha)$	α	$F(\alpha)$	α	$F(\alpha)$
(0,0,0)		(0,0,1)		(0,0,-1)	
(0,1,0)		(0,1,1)		(0,1,-1)	
(0,-1,0)		(0,-1,1)		(0,-1,-1)	
(1,0,0)		(1,0,1)		(1,0,-1)	
(1,1,0)		(1,1,1)		(1,1,-1)	
(1,-1,0)	1	(1,-1,1)		(1,-1,-1)	
(-1,0,0)		(-1,0,1)		(-1,0,-1)	
(-1,1,0)		(-1,1,1)		(-1,1,-1)	
(-1,-1,0)		(-1,-1,1)		(-1,-1,-1)	

Definition: The *majority voting social welfare functional* is

$$F(\alpha_1, ..., \alpha_I) = sign\left(\sum_{i=1}^{I} \alpha_i\right),$$

where

$$sign(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ +1 & x > 0 \end{cases}$$

EXERCISE (NOW): Verify that majority voting satisfies symmetry among agents, neutrality between alternatives, and positive responsiveness.

α	$F(\alpha)$	α	$F(\alpha)$	α	$F(\alpha)$
(0,0,0)		(0,0,1)		(0,0,-1)	
(0,1,0)		(0,1,1)		(0,1,-1)	
(0,-1,0)		(0,-1,1)		(0,-1,-1)	
(1,0,0)		(1,0,1)		(1,0,-1)	
(1,1,0)		(1,1,1)		(1,1,-1)	
(1,-1,0)		(1,-1,1)		(1,-1,-1)	
(-1,0,0)		(-1,0,1)		(-1,0,-1)	
(-1,1,0)		(-1,1,1)		(-1,1,-1)	
(-1,-1,0)		(-1,-1,1)		(-1,-1,-1)	

Theorem (May's Theorem): Majority voting is the *only* social welfare functional which satisfies *symmetry among agents*, *neutrality between alternatives*, and *positive responsiveness*.

Proof logic:

- We have already seen that majority voting satisfies the 3 axioms. Thus the following statement is true: *'If F is the majority voting SWF, then F satisfies the axioms.'*
- A mathematician would say that satisfaction of the axioms is *necessary* for the SWF to be majority voting.
- Now we want to show the converse, i.e. we want to prove the following statement:
  'If F satisfies the axioms, then F is the majority voting SWF.'
- A mathematician would say that we are trying to show that satisfaction of the axioms is *sufficient* for the SWF to be majority voting.

Proof (sufficiency): Suppose F satisfies *symmetry among agents*, *neutrality between alternatives*, and *positive responsiveness*. Then,

(1) *Symmetry among agents* implies that only the *number* of agents who prefer x to y or y to x matter.

Therefore it must be possible to write F as a function only of those numbers:

$$F(\alpha) = G(n^{+}(\alpha), n^{-}(\alpha))$$

where $n^{+}(\alpha)$ is the number of 1's and $n^{-}(\alpha)$ the number of -1's in α.

(2) *Neutrality between alternatives* implies that whenever $n^{+}(\alpha) = n^{-}(\alpha)$, we have

$$F(\alpha) = G(n^{+}(\alpha), n^{-}(\alpha)) = G(n^{-}(\alpha), n^{+}(\alpha))$$

$$= G(n^{+}(-\alpha), n^{-}(-\alpha)) = F(-\alpha) = -F(\alpha)$$

Thus $F(\alpha) = -F(\alpha)$, implying $F(\alpha) = 0$.

$$\Rightarrow \textbf{If } n^{+}(\alpha) = n^{-}(\alpha)\textbf{, then } F(\alpha) = 0\textbf{.}$$

(3) Consider any α such that $n^{+}(\alpha) > n^{-}(\alpha)$, and apply *positive responsiveness*:

Imagine replacing $+1$ entries by 0's until you get to α' such that $n^{+}(\alpha') = n^{-}(\alpha')$.
Then $F(\alpha') = 0$, $\alpha \geq \alpha'$, and $\alpha \neq \alpha'$, hence $F(\alpha) = +1$.

$$\Rightarrow \textbf{If } n^{+}(\alpha) > n^{-}(\alpha)\textbf{, then } F(\alpha) = +1$$

(4) Consider any α such that $n^{+}(\alpha) < n^{-}(\alpha)$, and apply *neutrality between alternatives*:

Clearly $n^{+}(-\alpha) > n^{-}(-\alpha)$ and hence $F(-\alpha) = 1$.
Therefore $F(\alpha) = -F(-\alpha) = -1$.

$$\Rightarrow \textbf{If } n^{+}(\alpha) < n^{-}(\alpha)\textbf{, then } F(\alpha) = -1$$

Together, properties (2-4) exactly characterize *simple majority rule*.

Theorem (May's Theorem): Majority voting is the *only* social welfare functional which satisfies *symmetry among agents*, *neutrality between alternatives*, and *positive responsiveness*.

Comments

- This is a 'powerful' result: A set of only three axioms completely 'characterizes' a single SWF.
  - Any other way of socially ranking two alternatives will violate at least one of the three axioms.
  - What would happen if we *add* additional axioms? What could such an axiom be?

Summary (2 alternatives)

- Assuming that the group is choosing between **two alternatives**...
- Majority voting may be a good rule for *aggregating information* **(Condorcet Jury Theorem)**
  - Different opinions are due to different information.
  - Each vote provides evidence in favor of one alternative.
  - Alternative receiving most votes is (most) likely to be best.
- Majority voting may be a good rule for *aggregating preferences* **(May's Theorem)**
  - It satisfies intuitions about democratic principles.
  - It is the *only rule* that does so!

Discussion (2 alternatives)

- Recall Buchanan and Tullock's (1965) arguments concerning optimal decision rules.
- They also look at *binary* decisions (whether or not to undertake some collective action)
- However they *do not* conclude that simple majority rule is necessarily best.
- What explains this difference in their conclusions?