# 上級マクロ Lec5 線形化

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**■Euler 方程式の厳密な線形化**　線形化の結果得られた $(*)$ を導出する．元の関数をまず与える．

$$G(k_t, k_{t+1}, k_{t+2}) = -\big(k_t^\alpha - k_{t+1}\big)^{-\sigma} + \beta\,\alpha k_{t+1}^{\alpha-1}\big(k_{t+1}^\alpha - k_{t+2}\big)^{-\sigma}.$$

Chain rule を厳密に用いて各偏導関数を導出する．

$$G_1 \equiv \frac{\partial G}{\partial k_t} = -\frac{d}{dk_t}\big(k_t^\alpha - k_{t+1}\big)^{-\sigma} = \sigma\alpha\,k_t^{\alpha-1}\big(k_t^\alpha - k_{t+1}\big)^{-\sigma-1}. \tag{1}$$

$$\begin{aligned} G_2 \equiv \frac{\partial G}{\partial k_{t+1}} &= -\frac{d}{dk_{t+1}}\big(k_t^\alpha - k_{t+1}\big)^{-\sigma} + \beta\frac{d}{dk_{t+1}}\Big[\alpha k_{t+1}^{\alpha-1}\big(k_{t+1}^\alpha - k_{t+2}\big)^{-\sigma}\Big], \\ &= -\sigma\big(k_t^\alpha - k_{t+1}\big)^{-\sigma-1} + \beta\Big\{A'B + AB'\Big\}, \quad \text{where } A = (k_{t+1}^\alpha - k_{t+2})^{-\sigma},\ B = \alpha k_{t+1}^{\alpha-1}. \end{aligned}$$

ここで,

$$A' = -\sigma\,(k_{t+1}^\alpha - k_{t+2})^{-\sigma-1}\cdot\alpha k_{t+1}^{\alpha-1}, \qquad B' = \alpha(\alpha-1)k_{t+1}^{\alpha-2},$$

であるから,

$$G_2 = -\sigma\big(k_t^\alpha - k_{t+1}\big)^{-\sigma-1} + \beta\Big[-\sigma\alpha^2 k_{t+1}^{2\alpha-2}\big(k_{t+1}^\alpha - k_{t+2}\big)^{-\sigma-1} + \alpha(\alpha-1)k_{t+1}^{\alpha-2}\big(k_{t+1}^\alpha - k_{t+2}\big)^{-\sigma}\Big]. \tag{2}$$

$$G_3 \equiv \frac{\partial G}{\partial k_{t+2}} = \beta\alpha k_{t+1}^{\alpha-1}\frac{d}{dk_{t+2}}\big(k_{t+1}^\alpha - k_{t+2}\big)^{-\sigma} = \beta\sigma\alpha\,k_{t+1}^{\alpha-1}\big(k_{t+1}^\alpha - k_{t+2}\big)^{-\sigma-1}. \tag{3}$$

定常状態を $k_t = k_{t+1} = k_{t+2} = \bar{k}$ とおき, $\bar{c} = \bar{k}^\alpha - \bar{k}$ と定義する．線形化で % 偏差 $\hat{k}_{t+i} = (k_{t+i} - \bar{k})/\bar{k}$ を使うので, 係数として $\tilde{G}_i \equiv G_i(\bar{k}, \bar{k}, \bar{k})\,\bar{k}$ を用いる．

$$\tilde{G}_1 = \sigma\alpha\,\bar{k}^\alpha\,\bar{c}^{-\sigma-1}, \tag{4}$$

$$\tilde{G}_3 = \beta\sigma\alpha\,\bar{k}^\alpha\,\bar{c}^{-\sigma-1}, \tag{5}$$

$$\tilde{G}_2 = -\sigma\bar{k}\,\bar{c}^{-\sigma-1} - \beta\sigma\alpha^2\bar{k}^{2\alpha-1}\bar{c}^{-\sigma-1} + \beta\alpha(\alpha-1)\bar{k}^{\alpha-1}\bar{c}^{-\sigma}. \tag{6}$$

このとき線形化は次の形になる：

$$\tilde{G}_1\,\hat{k}_t + \tilde{G}_2\,\hat{k}_{t+1} + \tilde{G}_3\,\hat{k}_{t+2} = 0. \tag{L}$$

定常状態の Euler 方程式は:

$$\bar{c}^{-\sigma} = \beta\bar{c}^{-\sigma}\alpha\,\bar{k}^{\alpha-1} \implies 1 = \beta\alpha\,\bar{k}^{\alpha-1}. \tag{E}$$

全体をある正の定数で割ると, 係数が単純化される．

$$D := \sigma\,\bar{k}^\alpha\,\bar{c}^{-\sigma-1}$$

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として, まず明らかに

$$\frac{\tilde{G}_1}{D} = \frac{\sigma\alpha\bar{k}^{\alpha}\bar{c}^{-\sigma-1}}{\sigma\bar{k}^{\alpha}\bar{c}^{-\sigma-1}} = \alpha, \tag{7}$$

$$\frac{\tilde{G}_3}{D} = \frac{\beta\sigma\alpha\bar{k}^{\alpha}\bar{c}^{-\sigma-1}}{\sigma\bar{k}^{\alpha}\bar{c}^{-\sigma-1}} = \alpha\beta. \tag{8}$$

次に $\tilde{G}_2/D$ を計算する：

$$\begin{aligned}\frac{\tilde{G}_2}{D} &= \frac{-\sigma\bar{k}\bar{c}^{-\sigma-1}}{\sigma\bar{k}^{\alpha}\bar{c}^{-\sigma-1}} + \frac{-\beta\sigma\alpha^2\bar{k}^{2\alpha-1}\bar{c}^{-\sigma-1}}{\sigma\bar{k}^{\alpha}\bar{c}^{-\sigma-1}} + \frac{\beta\alpha(\alpha-1)\bar{k}^{\alpha-1}\bar{c}^{-\sigma}}{\sigma\bar{k}^{\alpha}\bar{c}^{-\sigma-1}}\\ &= -\bar{k}^{1-\alpha} - \beta\alpha^2\bar{k}^{\alpha-1} + \frac{\beta\alpha(\alpha-1)}{\sigma}\,\bar{k}^{-1}\bar{c}.\end{aligned}$$

ここで $\bar{c} = \bar{k}^{\alpha} - \bar{k} = \bar{k}(\bar{k}^{\alpha-1}-1)$ と, (E) より $\beta\bar{k}^{\alpha-1} = 1/\alpha$ を用いると, 各項をさらに整理できる：

$$\begin{aligned}-\bar{k}^{1-\alpha} &= -(\beta\alpha),\\ -\beta\alpha^2\bar{k}^{\alpha-1} &= -\alpha^2\cdot(\beta\bar{k}^{\alpha-1}) = -\alpha^2\cdot\frac{1}{\alpha} = -\alpha,\\ \frac{\beta\alpha(\alpha-1)}{\sigma}\,\bar{k}^{-1}\bar{c} &= \frac{\beta\alpha(\alpha-1)}{\sigma}\,(\bar{k}^{\alpha-1}-1) = \frac{\alpha-1}{\sigma}\big(\beta\alpha\bar{k}^{\alpha-1}-\beta\alpha\big) = \frac{\alpha-1}{\sigma}\,(1-\beta\alpha).\end{aligned}$$

したがって

$$\frac{\tilde{G}_2}{D} = -(\beta\alpha) - \alpha + \frac{\alpha-1}{\sigma}(1-\beta\alpha) = \frac{\alpha-1}{\sigma}(1-\alpha\beta) - (\alpha\beta+\alpha). \tag{9}$$

(L) を $D$ で割ると, $\hat{k}$ に対する線形結合の係数は上で得た定数となる. すなわち

$$\frac{\sum_{i=0}^{2} G_{i+1}(\bar{k},\bar{k},\bar{k})\bar{k}\hat{k}_{t+i}}{D} = \frac{1}{D}[\tilde{G}_1\,\hat{k}_t + \tilde{G}_2\,\hat{k}_{t+1} + \tilde{G}_3\,\hat{k}_{t+2}] = 0.$$

$$\alpha\beta\,\hat{k}_{t+2} + \left[\frac{1}{\sigma}(\alpha-1)(1-\alpha\beta) - (\alpha\beta+\alpha)\right]\hat{k}_{t+1} + \alpha\,\hat{k}_t = 0. \tag{$*$}$$

これはスライドに示された $(*)$ と一致する. □