Spatial Models of Majority Rule

Recap

- For group choices between two alternatives, we have two Theorems that seem to support the use of *majority rule* (Condorcet's Jury Theorem and May's Theorem).
- With three or more alternatives, Condorcet's Paradox shows that pairwise majority voting may produce cycles / intransitivities.
- Other rules (e.g. Borda, Condorcet) can produce transitive rankings, but have other "flaws" (e.g. dependence on "irrelevant alternatives")
- Arrow's "Impossibility Theorem" suggests that there is no "good solution" to this.
- According to Buchanan, the source of this impossibility is a fundamental conflict between the collectivist concept of "group rationality" and an individualist perspective on democracy.

Ways to achieve ‘possibility’ results

- Relaxing axioms (e.g. IIA)
  - Various ‘scoring rules’ (e.g. Borda) satisfy remaining axioms.
  - These rules are often susceptible to strategic manipulation.
- Restricting the domain
  - Arrow requires that one rule must handle all possible constellations of preferences.
  - In practice, some preference constellations may seem more plausible than others.
  - In particular, this will be true if the options under consideration are related to one another in a way that allows us to imagine them occupying points in a “space” of policy alternatives.

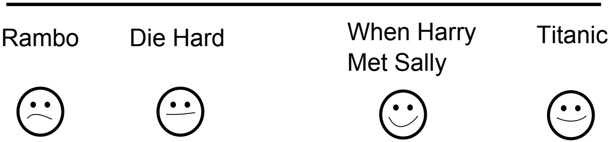
One dimensional policy space

- Many issues are such that options can be *ordered* in a natural[†] way (small - big, left - right,...)

Rambo | Die Hard | When Harry Met Sally | Titanic

Single peaked preferences

- Often, each individual will have a favorite option, and other options are less preferred the farther away they are.



Framework:

N voters $I = 1, ..., N$ (N odd)
Set of alternatives $X \subseteq \mathbb{R}$
Individual preferences $\succ_i$ (strict)

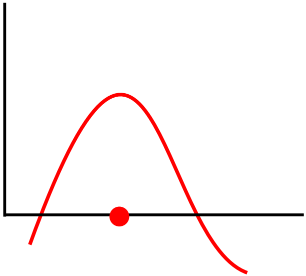
Definition: Voter i has single peaked preferences if there exists an option x_i^* such that for any two options y and z,

$$\text{If } x_i^* \geq z > y \text{ then } z \succ_i y$$

and

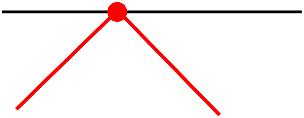
$$\text{If } y > z \geq x_i^* \text{ then } z \succ_i y$$

We call x_i^* i's 'ideal point'.

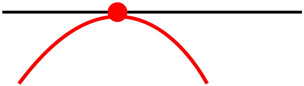


Example: Distance preferences

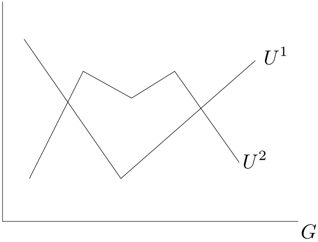
- $u_i(x) = -|x - x_i^*|$

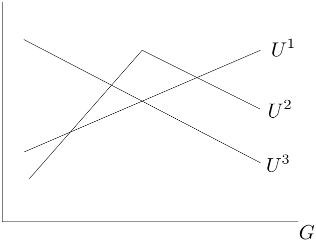


- $u_i(x) = -(x - x_i^*)^2$



Examples: Single peaked?





Lemma: Let $a < b < c$. If voter i has single peaked preferences, then either $b \succ_i a$ or $b \succ_i c$ (or both).
(I.e. the 'middle' option cannot be the worst - for any voter who has single peaked preferences.)

Proof: At home! (Hint: Suppose $a \succ_i b$ then where can x_i^* (not) lie and what does this imply about the statement $b \succ_i c$?)

Definition: Suppose that all N voters have single peaked preferences over $X \subseteq \mathbb{R}$. Denote their ideal points by $(x_1^*, ..., x_N^*)$. Mr. m is a median voter if

$$\#\{i \in I : x_i^* \geq x_m^*\} \geq \frac{N}{2} \text{ and } \#\{i \in I : x_i^* \leq x_m^*\} \geq \frac{N}{2}$$

Black's Median Voter Theorem: Suppose that all voters have single peaked preferences over $X \subseteq \mathbb{R}$. Let m be a median voter. Then, option x_m^* is not defeated by any other option in a pairwise majority vote. That is, the median voter's ideal point is a *Condorcet winner*.

Proof: At home!

Black's Single-Peakedness Theorem: Suppose that all voters have single peaked preferences over $X \subseteq \mathbb{R}$. Then, pairwise majority voting over all alternatives gives rise to a rational preference relation.

Proof: Suppose that all voters have single peaked preferences over $X \subseteq \mathbb{R}$. Consider 3 options a, b, and c, and let $x \succ_M y$ stand for 'a majority prefers x over y'. Suppose that $a \succ_M b$, $b \succ_M c$. We need to show that $a \succ_M c$.

- Since all preferences are single peaked, there exists $\hat{x} \in \{a, b, c\}$ which no individual ranks last among the three. (See Lemma above)
- Suppose $\hat{x} = a$.
  - Since $b \succ_M c$, then a majority of voters have individual preference relations $a \succ_i b \succ_i c$ or $b \succ_i a \succ_i c$. In *both* cases, $a \succ_i c$, and so $a \succ_M c$.
- Suppose $\hat{x} = b$.
  - Since $a \succ_M b$, then for a majority of voters $a \succ_i b \succ_i c$, and so $a \succ_M c$.
- Suppose $\hat{x} = c$.
  - Since $b \succ_M c$, then for a majority of voters $b \succ_i c \succ_i a$, and so $b \succ_M a$. This contradics $a \succ_M b$, thus $\hat{x} \neq c$.

$\Rightarrow$ If all voters have single peaked preferences, then whenever $a \succ_M b$ and $b \succ_M c$, we have $a \succ_M c$.

(Note: The proof suggests that the 'social preference' generated by pairwise majority voting will never place the 'middle' option last. I.e. above we saw that c cannot be the 'middle' option.)

Example: Voting on redistribution (Romer 1975)

- Consider an economy inhabited by multiple households, each with preferences represented by

$$U(x, \ell) = x - \frac{1}{2}\ell^2$$

  where x is consumption and ℓ is labor.
- The price of consumption is 1, and each household earns some wage ω, where ω *differs* among households. It is distributed according to a cdf $F(\omega)$.
- Consider a *linear* tax system consisting of a single tax rate t applied to all income, as well as a transfer b paid to each household.
- Then a household who works for ℓ hours and earns wage ω achieves utility

$$U(b + (1-t)\omega\ell, \ell) = b + (1-t)\omega\ell - \frac{1}{2}\ell^2$$

- Questions to analyse:
  - What tax system (combination b and t) would be chosen using majority rule?
  - **In particular:** How does the chosen tax system depend on the distribution of wages?

Example: Voting on redistribution (Romer 1975)

- Steps in the analysis
  - (1) Derive labor supply for any given combination (t, b)
  - (2) Restrict attention to feasible systems, where b is fully funded $\Rightarrow$ one dimensional space of alternatives

$$t \in [0, 1]$$

  - (3) Derive a consumer's *indirect* utility from a given tax system.
  - (4) Check that preferences (indirect utility) are single peaked
  - (5) Apply median voter theorem $\Rightarrow$ median voter's preferred policy

Example: Voting on redistribution (Romer 1975)

- Labor supply: $\ell(t, \omega) = (1 - t)\omega$
- Feasibility: $b = t(1 - t)E(\omega^2)$
- Indirect utility: $v_\omega(t) = t(1 - t)E(\omega^2) + \frac{1}{2}(1 - t)^2\omega^2$
- Is this single peaked? (Unique maximum + increasing before, decreasing after?)

$$v'_\omega(t) = (1 - t)\left(E(\omega^2) - \omega^2\right) - tE(\omega^2)$$

If $\omega^2 > E(\omega^2)$, $v_\omega(t)$ is strictly decreasing on $[0, 1]$ and the unique max occurs at $t = 0$.

$$v''_\omega(t) = -2E(\omega^2) + \omega^2$$

If $\omega^2 < E(\omega^2)$, $v_\omega(t)$ is strictly concave on $[0, 1]$. and the unique max occurs at $t = \frac{E(\omega^2) - \omega^2}{2E(\omega^2) - \omega^2}$.
Thus, **preferences over t are single peaked**.

- By the median voter theorem, the unique stable outcome of majority voting is the *median* voter's ideal t:

$$t^* = \max\left\{0, \frac{E(\omega^2) - \omega_m^2}{2E(\omega^2) - \omega_m^2}\right\}$$

Example: Voting on redistribution (Romer 1975)

$$t^* = \max\left\{0, \frac{E(\omega^2) - \omega_m^2}{2E(\omega^2) - \omega_m^2}\right\}$$

Interpretation

- The degree or redistribution depende on the relationship between the (squared) *median* wage level ω_m^2 and the *average* (squared) wage level $E(\omega^2)$.
- Thus it depends on how *skewed* the wage distribution is.
- In most countries, median wages are significantly below mean wages, such that t^* would be positive.

- The model predicts that the degree of redistribution in a democracy depends on the difference between the median and mean wage.

Single peaked preferences in multiple dimensions

Black's Theorems assume that options can be aligned in a single dimension. Complications arise if we make the (more realistic) assumption that policies are multi-dimensional.

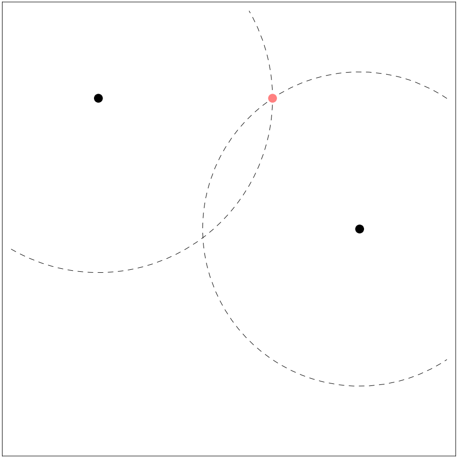
Assume: Options are $x_j \in \mathbb{R}^L$ and voters have preferences $v_i(x) : \mathbb{R}^L \to \mathbb{R}$.

Example: Distance preferences in $\mathbb{R}^2$:

$$u(x_1, x_2, x_{1i}, x_{2i}) = -\sqrt{(x_1 - x_{1i})^2 + (x_2 - x_{2i})^2}$$

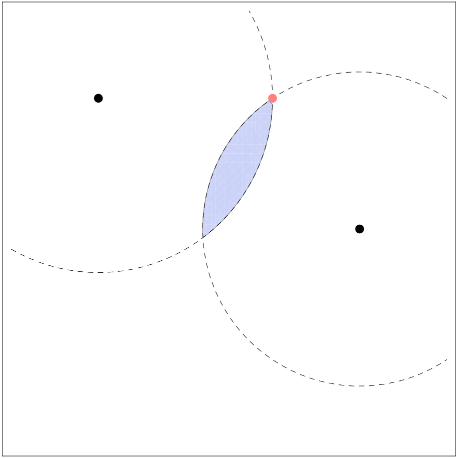
Single peaked (distance) preferences in 2 dimensions

- Circular indifference curves
- Preferred-to sets: areas inside IC through given point



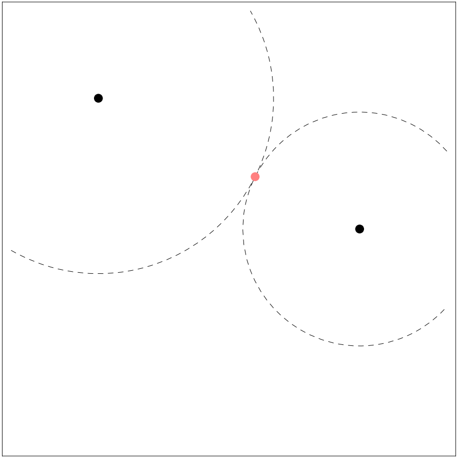
Pareto improvements (2 voters)

- Intersection of preferred-to sets
- Mutually agreeable alternatives



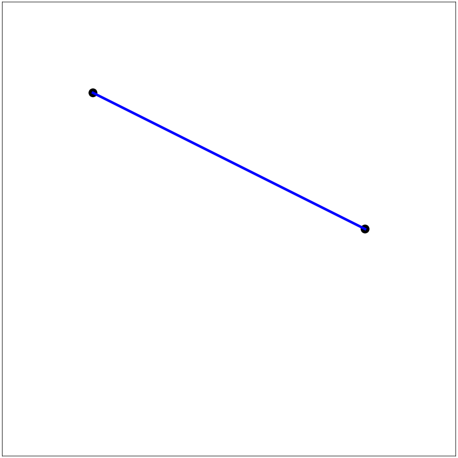
Pareto optimal points (2 voters)

- Intersecion of preferred-to sets is empty
- Indifference curves are tangent



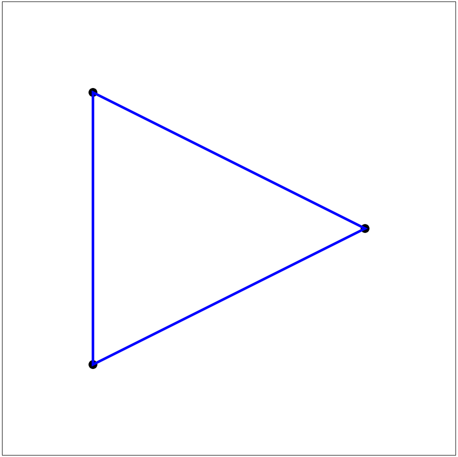
Pareto set (2 voters)

- Set of points such that 2 individuals would not mutually prefer another point.
- For any point *not* on this line, there exist other points that both prefer.



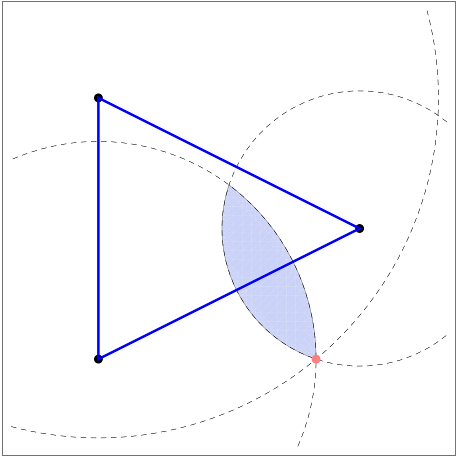
Three individuals and majority rule

- Note that *no point* is in all three Pareto sets.
- $\Rightarrow$ Therefore *any point* can be defeated under majority rule.



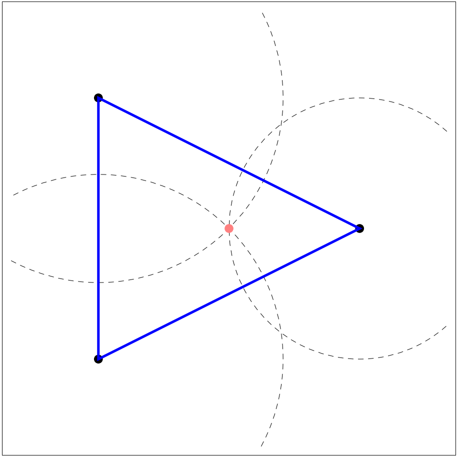
Three individuals and majority rule

- Points outside triangle are not Pareto efficient (Why?)
- $\Rightarrow$ These points can be defeated even under *unanimity rule*



Three individuals and majority rule

- Points inside triangle are Pareto efficient (Why?)
- They are stable under *unanimity rule*, but unstable under majority rule.



Conclusions from our example

- No point is ‘stable’ under majority rule.
- For *any* point, there exists a majority coalition which prefers some other point.
- Graphically: no single point is located on all three 2-person ‘Pareto sets’

Exercise (now)

- Find an arrangement of three voter ideal points such that there *does* exist a stable point under majority rule!

Generalization

Theorem (McKelvey 1979): Suppose voters have single peaked preferences in more than two dimensions. Then, the existence of a Condorcet winner is *extremely* unlikely (i.e. occurs only under very special arrangements of ideal points).

Instability, agenda setting, and the importance of institutions

- McKelvey (1979) shows that, in general, one can move from *any* point in the space to *any* other point by some sequence of pairwise votes.
- An *agenda setter* can therefore construct a sequence of votes such that her ideal point is reached (provided others vote 'sincerely' at each stage).
- More generally, the outcomes produced by majoritarian institutions strongly depend on the *procedures* employed (rules governing the sequence of alternatives, opportunities to introduce amendments, etc.).

Literature

(* = required, !=highly recommended)

(*) SHEPLE, K. *Analyzing Politics*, Chapter 5 (Semesterapparat)

MCKELVEY, R. 1979. Intransitivities in multidimensional voting models and some implications for agenda control. *Journal of Economic Theory*