Social Choice Theory (Part 2)


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BORDA

Review: Group choosing between 2 alternatives

- Majority voting may be a good rule for aggregating *information* when interests are perfectly aligned. (Condorcet Jury Theorem)
  - Different opinions are due to different information.
  - Each vote provides evidence in favor of one alternative.
  - Alternative receiving most votes is (more) likely to be true.
  - In large groups (and given certain assumptions) it's almost certainly true.
- Majority voting may be a good way to aggregate *preferences* when they (genuinely) differ. (May's Theorem)
  - Satisfies democratic intuitions (responsiveness, equality, neutrality)
  - Is the *only rule* that does so

Three or more alternatives

- So far, we have considered only choices between two alternatives.
- In reality, groups face choices among many alternatives.

Example: Three alternatives $\{a, b, c\}$

- (How) can we apply majority rule to such a choice?
- One possibility: conduct majority vote between each pair of alternatives
- Question: Will this procedure produce a reasonable choice?

Example:

	a	b	c
	b	c	a
	c	a	b
N	3	4	5

Pairwise majority voting

- $a \succ_M b$ (8 to 4)
- $b \succ_M c$ (7 to 5)
- $c \succ_M a$ (9 to 3)

This is '**Condorcet's Paradox**'.

Three or more alternatives

- Pairwise majority comparisons
  - may not produce consistent ranking
  - may not identify a ‘winner’
- Possible ‘solution’: *Plurality* rule
  - One vote among all alternatives
  - Alternative with most votes wins
- One of several ‘scoring rules’
  - Rules that assign *points* to every alternative.
  - All scoring rules produce a complete and transitive ranking over the set of alternatives. (Why?)

Plurality rule

- Suppose individuals vote *sincerely* (for their favorite)
- Points = number of times an alternative is placed *first*

Example (Young 1997):

	b	a	c	a
	c	c	b	b
	a	b	a	c
N	7	7	6	1

- Plurality rule ranks the alternatives $a \succ_P b \succ_P c$

Discuss: Is *a* really the best choice “for the group”?

Jean-Charles de Borda (1744-1799)

- Criticized plurality rule for not using enough information
- First place votes do not accurately reflect ‘overall support’

Example (continued):

	b	a	c	a
	c	c	b	b
	a	b	a	c
N	7	7	6	1

- Plurality favors *a*, but *c* might be a better ‘compromise’
- Note that *c* receives the *fewest* first place votes!

Borda's rule (the 'Borda Count'):

- Each voter assigns points to every option
    - worst: 0 points
    - next best: 1 point
    - next best: 2 points
    - etc.
- Borda score: *sum* of points from all voters

Example:

	b	a	c	a
	c	c	b	b
	a	b	a	c
N	7	7	6	1

Exercise: What Borda score does option *b* achieve?

- ranked last by 1 voter $\Rightarrow 0 \cdot 1 = 0$ points
- ranked 2nd by 7 voters $\Rightarrow 1 \cdot 7 = 7$ points
- ranked 1st by 7 voters $\Rightarrow 2 \cdot 7 = 14$ points
- Borda score: $0 + 7 + 14 = 21$ points.

	b	a	c	a
	c	c	b	b
	a	b	a	c
N	7	7	6	1

Easy way to calculate Borda scores: *vote matrix*:

	a	b	c	score
a	-			
b		-		
c			-	

Exercise:

- Fill the matrix by entering the number of votes for the option in the row over the one in the column.
- Add the numbers in each row to find the Borda score for each option.
- (At home:) Verify that this method gives the same answer as counting points, and explain why.

Conclusion:

- Borda's rule ranks the options $c \succ_B b \succ_B a$
- This is exactly the *opposite* of plurality!

Point methods ('scoring rules')

- Plurality and Borda's rule both work by assigning *points* to each option.
- **Note:** *Any* system that assigns points to alternatives will produce a single consistent ranking.
- Borda's system uses more information than plurality rule (voters submit a complete ranking, not just their favorite).
- Borda's rule is arguably more *complex* than plurality voting.
- Another important rule in this class is **approval voting:** Each voter can give exactly 1 point to as many candidates as she likes.

Condorcet's critique of Borda's rule

	Peter	Peter	Paul	Paul	Jack	Jack
	Paul	Jack	Peter	Jack	Peter	Paul
	Jack	Paul	Jack	Peter	Paul	Peter
N	30	1	29	10	10	1

Exercise:

- Who would win using plurality rule?
- Which candidate attains the largest Borda score?

	Peter	Paul	Jack	score
Peter	-	41	60	101
Paul	40	-	69	109
Jack	21	12	-	33

- Does *pairwise majority voting* produce a rational (transitive) ranking?

$$\text{Peter} \succ_M \text{Paul} \succ_M \text{Jack}$$

Condorcet's critique of Borda's rule

	Peter	Paul	Jack	score
Peter	-	41	60	101
Paul	40	-	69	109
Jack	21	12	-	33

Exercise: What would happen if Jack dropped out of the race?

> *"The points method confuses votes comparing Peter and Paul with those comparing either Peter or Paul to Jack (...). As long as it relies on irrelevant factors to form its judgements, it is bound to lead to error, and that is the real reason why this method is defective (...)." (Condorcet 1788)*

- Borda's method is sensitive to an 'irrelevant' option being added or removed.
  - Condorcet considers Jack 'irrelevant' because he is ranked below the others anyway. (Watch out: Later on, the term 'irrelevant' will be used differently!!)
- Borda's method may not choose an option that defeats all others in a pairwise majority vote (a 'Condorcet winner')
  - In example: Peter is majority preferred to *both* Jack and Paul

Condorcet's rule (for three or more alternatives)

- As in the case of two options, Condorcet looked for a *rational* way to aggregate opinions.

Assumptions:

- There exists a single *correct* ranking of the alternatives.
- Each voter's judgment about a given pair of alternatives is independent of other voters' judgments and also independent of her own judgment about other pairs.*
- For *any two* alternatives, she ranks them correctly with probability $p > \frac{1}{2}$.

*Technical:

- With these assumptions it is possible that a voter ranks $a \succ b$, $b \succ c$ and $c \succ a$.
- However they simplify calculations and will lead to similar results as more elaborate systems.

Example

- 60 voters rank the alternatives as follows (as revealed by voting)

	a	b	b	c	c
	b	c	a	a	b
	c	a	c	b	a
N	23	17	2	10	8

Thought experiment:

- Suppose that the *correct* ranking is in fact $a \succ b \succ c$
- Under that assumption, how likely would it be to observe the particular *profile* of individual rankings summarized above?

$$p^{n_{ab}+n_{bc}+n_{ac}} \cdot (1-p)^{n_{ba}+n_{cb}+n_{ca}} = p^{100}(1-p)^{80},$$

  where n_{xy} denotes the number of persons preferring x over y.

- This is increasing in the *exponent* on p, that is $n_{ab} + n_{bc} + n_{ac}$
  - The number of pairwise votes *consistent* with the ranking $a \succ b \succ c$
  - This number is called the **support** for the ranking $a \succ b \succ c$.
- The support is proportional to the *likelihood of observing the votes that we see* if the true ranking were in fact $a \succ b \succ c$.

	a	b	b	c	c
	b	c	a	a	b
	c	a	c	b	a
N	23	17	2	10	8

Exercise

- Construct a *vote matrix* for this example

	a	b	c
a	-		
b		-	
c			-

- Find the *support* for ranking $b \succ a \succ c$

	a	a	b	b	c	c
	b	c	a	c	a	b
	c	b	c	a	b	a
Support	100	76	94	104	86	80

Interpretation

- Each of these numbers reflects the likelihood of the *votes actually observed* under alternative assumptions regarding the *true* ranking.

Condorcet's rule

- 'Social ranking' is the one with maximum 'support'
  - In our example, this is $b \succ_C c \succ_C a$
- We may call this the **'maximum likelihood ranking'**

Condorcet winner

- *Definition:* A **Condorcet winner** is an alternative that *does not lose any* pairwise majority vote against any other alternative. (In other words, it *wins or ties* against every other alternative).
- Condorcet argued that, when such an alternative exists, it is the best *choice* for the group.

EXERCISE (Home):

- Show that whenever a Condorcet winner exists, Condorcet's ranking places it first.
- (This problem is hard but doable! Start with three alternatives and then generalize.)

Finally, let's go back to Peter and Paul...

	Peter	Peter	Paul	Paul	Jack	Jack
	Paul	Jack	Peter	Jack	Peter	Paul
	Jack	Paul	Jack	Peter	Paul	Peter
N	30	1	29	10	10	1

EXERCISE (Home):

- How does Condorcet rank the candidates? What happens if Jack drops out?

Condorcet vs. Borda

- Borda's method is sensitive to 'irrelevant alternatives'.
  - Ranking of two options *x* and *y* may change if an option *z* which is ranked either below or above *both* *x* and *y* is removed.
- Condorcet's ranking is not sensitive to 'irrelevant alternatives' *in this sense*.
  - However, ranking of two option *x* and *y* may be affected if an option ranked *in between* is added or removed!
- If a Condorcet winner exists, Condorcet ranks it first.
- Borda's method *may not* rank a Condorcet winner first.
- If all rankings are equally likely to be 'true,'
  - Condorcet's *ranking* is more likely to be true than Borda's.
  - Option *ranked highest* by Borda is most likely to be truly best.

Discussion

- With three or more alternatives, pairwise majority rule may give rise to *cycles*.
- Various decision rules produce a consistent ranking by assigning *points*
    - Plurality rule
    - Borda's rule
    - Condorcet's rule
    - Others (approval voting, single transferable vote, instant runoff,...)

  ('scoring rules' or 'preferential voting systems')
- If a 'true' ranking exists, we can compare rules in terms of their ability to identify it or to chose the 'truly' best alternative.
- **What if individual rankings reflect different *preferences*?**
    - Can we identify 'good' rules to rank 3 or more alternatives?

The problem of social choice (Arrow 1951)

$I = \{1, 2, ..., I\}$ = set of individuals
$X =$ set of alternatives ("social states")
$\succsim_i =$ Mr. i's rational (complete, transitive) preference over X
$\mathcal{R} =$ set of rational preference relations over X
$\mathcal{A} = \mathcal{R}^I$ = set of preference *profiles*

Example:

- $I =$\{Alice, Ben, Charlie\}
- X=\{art museum, bar, cinema\}
- $\succsim_A = \left\{a \succ b, b \succ c, a \succ c\right\}$
- $\mathcal{R} = \left\{\{a \succ b, b \succ c, a \succ c\}, \{b \succ a, b \succ c, a \succ c\}, ...\right\}$
- Preference *profile*: $(\succsim_A, \succsim_B, \succsim_C)$ with each $\succsim_i \in \mathcal{R}$
- $\mathcal{A} = \left\{(\succsim_A, \succsim_B, \succsim_C), (\succsim'_A, \succsim'_B, \succsim'_C), ...\right\}$

Definition: A *social welfare functional* is a *rule* that produces a 'social' preference relation $\succsim_S$ from a given profile of individual preference relations $(\succsim_1, \succsim_2, ..., \succsim_I)$.

We will write:

- $x \succsim_S y$ if the social preference ranks x *at least as good as* y
- $x \succ_S y$ if the social preference ranks x *strictly better than* y

Question: Is there an "acceptable" social welfare functional? That is, is there a good way to "aggregate" any combination of individual preferences to yield a single "social preference"?

What is "acceptable"?

Arrow (1951) proposed the following axioms:

- **U** (Unrestricted domain) The rule must be applicable to all elements of A.
- **R** (Rationality) The rule must produce a *complete and transitive* ranking over all alternatives.
- **P** (Pareto principle) If $x \succ_i y$ for all i, then $x \succ_S y$
- **N** (Nondictatorship) There is no individual i such that for all x and y in X, $x \succ_i y$ implies $x \succ_S y$ regardless of the preferences of other individuals.
- **I** (Independence of irrelevant alternatives) The relative social ranking of any two options x and y depends only on the individual rankings of x and y, not on the relative ranking of any alternatives $z \neq x, y$.

Does **pairwise majority voting** satisfy...

- P Pareto principle?
- N Non-dictatorship?
- I Independence of Irrelevant Alternatives?
- R Rationality?

Does the **Borda count** satisfy...

- P Pareto principle?
- N Non-dictatorship?
- R Rationality?
- I Independence of Irrelevant Alternatives?

What about **Condorcet's rule**?

Example

	a	a	b	c
	b	c	c	a
	c	b	a	b
N	3	1	3	2

- Show that the maximum likelihood ranking is $a \succ_C b \succ_C c$.
- Change the third column by placing b last and leaving the preference between a and c unchanged. What happens?

Conclusion

- None of the rules we have considered are 'acceptable' in Arrow's sense.
- Are there other SWFLs that do satisfy Arrow's axioms?

Arrow's Impossibility Theorem (Arrow 1951): Suppose the number of alternatives is at least three. Then, every social welfare functional that satisfies axioms I,P,R, and U is dictatorial.

Proof (Geanakoplos 1996, Jehle and Reny 2011)

- Consider any option z and suppose it is ranked *last* by all individuals
- The **Pareto Principle** implies that the social ranking also places z last.

$\succ_1$	$\succ_2$	...	$\succ_N$	$\succ_S$
a	a'	...	a"	a"'
b	b'	...	b"	b"'
c	c'	...	c"	c"'
.	.	...	.	
.	.	...	.	
.	.	...	.	
z	**z**	...	**z**	**z**

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Proof (Geanakoplos 1996, Jehle and Reny 2011)

- Starting at the left, sequentially move *z* to the top of the individual rankings, one by one. (Leaving all other positions unchanged.)
- At *some* point in this process, *z*’s *social* rank must change (why?)

$\succ_1$	$\succ_2$	...	$\succ_N$	$\succ_S$
z	**z**			
a	a’	...	a”	a”’
b	b’	...	b”	
c	c’	...	c”	b”’
.	.	...	.	
.	.	...	.	c”’
.	.	...	.	
		...	**z**	

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Proof (Geanakoplos 1996, Jehle and Reny 2011)

- Let k be the **first** person such that when z is moved to the top of her ranking, its social rank changes.
- **Claim:** When z is moved to the top of k's ranking, it *immediately* moves to the *top* of the social ranking.

$\succ_1$	...	$\succ_k$	...	$\succ_N$	$\succ_S$
z	...	**z**			**z**
a	...	a'	...	a"	a"'
b	...	b'	...	b"	b"'
c	...	c'	...	c"	c"'
.	.	.	...	.	
.	.	.	...	.	
.	.	.	...	.	
			...	**z**	

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- Let k be the **first** person such that when z is moved to the top of her ranking, its social rank changes.
- **Claim:** When z is moved to the top of k's ranking, it *immediately* moves to the *top* of the social ranking.

Proof (of the claim):

- Suppose instead that $x \succ_S z \succ_S y$ for some $x, y \neq z$.
- Reshuffle all preferences such that $y \succ_i x$ for all i, *leaving* z at bottom or top.

  Note that we can do this without affecting the individual rankings of x or y relative to z.
- Pareto principle implies that now $y \succ_S x$.
- But since individual rankings of x and y relative to z are unchanged, IIA implies that $x \succ_S z$ and $z \succ_S y$.
- By transitivity, $x \succ_S y$. This is a contradiction.

Arrow's Impossibility Theorem (Arrow 1951): Suppose the number of alternatives is at least three. Then, every social welfare functional that satisfies axioms I,P,R, and U is dictatorial.

Proof (Geanakoplos 1996, Jehle and Reny 2011)

- For any two options $x, y \neq z$ reshuffle preferences such that $x \succ_k z \succ_k y$.
- For all others, rank x and y in *any way*, but leave the position of z unchanged.
- **Claim:** It then follows that $x \succ_S y$.

$\succ_1$	...	$\succ_k$	...	$\succ_N$	$\succ_S$
z	...	.			
.	...	.	...	.	
.	...	x	...	.	
.	...	**z**	...	.	
.	.	y	...	.	
.	.	.	...	.	
.	.	.	...	.	
			...	**z**	

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- For any two options $x, y \neq z$ reshuffle preferences such that $x \succ_k z \succ_k y$.
- For all others, rank x and y in *any way*, but leave the position of z unchanged.
- **Claim:** It then follows that $x \succ_S y$.

Proof (of the claim):

- x and z are individually ranked exactly as they were *before* z was moved to the top of k's ranking. $\Rightarrow$ By IIA, we have $x \succ_S z$
- y and z are individually ranked exactly as they were *after* z was moved to the top of k's ranking. $\Rightarrow$ By IIA, we have $z \succ_S y$
- By transitivity (rationality), it follows that $x \succ_S y$
- By IIA, the same will be true *whenever* $x \succ_k y$.

Arrow's Impossibility Theorem (Arrow 1951): Suppose the number of alternatives is at least three. Then, every social welfare functional that satisfies axioms I,P,R, and U is dictatorial.

Proof (Geanakoplos 1996, Jehle and Reny 2011)

- For any two options $x, y \neq z$ reshuffle preferences such that $x \succ_k z \succ_k y$.
- For all others, rank x and y in *any way*, but leave the position of z unchanged.
- **Claim:** It then follows that $x \succ_S y$.

It follows:

- By IIA, the social ranking of any two alternatives $x, y \neq z$ always agrees with Mr. k's ranking, no matter how the other individuals rank x and y.
- Thus, Mr. k is a dictator on all pairs *not involving* z.

Arrow's Impossibility Theorem (Arrow 1951): Suppose the number of alternatives is at least three. Then, every social welfare functional that satisfies axioms I,P,R, and U is dictatorial.

Proof (Geanakoplos 1996, Jehle and Reny 2011)

- We have seen that Mr. k is a dictator on all pairs not involving option z.
- We can repeat the argument with any other alternative $w \neq z$ to see that some individual k' is a dictator on all pairs *not including* w.
- But, since Mr. k's ranking of $z \neq w$ affected it's social ranking before, it must be that $k' = k$.
- I.e., **Mr. k is a dictator on all pairs**.

Arrow's Impossibility Theorem (Arrow 1951): Suppose the number of alternatives is at least three. Then, every social welfare functional that satisfies axioms I,P,R, and U is dictatorial.

$\Rightarrow$ An 'acceptable' SWFL as Arrow defined it does not exist.

Interpretation (Normative)

- There is no 'acceptable' (democratic) way to attribute *rational preferences* to a group of individuals whose preferences differ.
  - Majority rule may give rise to 'irrational' (non-transitive) judgements (Condorcet Paradox).
  - No other (acceptable) voting system can get around this problem.
- Decisions arrived at using democratic procedures cannot be interpreted as expressions of a single 'collective will'.
- Any concept of 'Social Welfare' as a measure of 'group utility' necessarily violates basic democratic principles.

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Interpretation (Positive)

- Group choices are not unambiguously *determined* by individual preferences alone.
- Collective decisions (also) depend on the rules and procedures used.
- 'Democratic' procedures may not produce rational preferences over all policies (or 'social states').
- Indecision and instability
    - Perhaps *no* decision is made one way or the other.
        - Debates may cycle without agreement
        - Status quo option winds up 'chosen'
    - Perhaps decisions taken are later overturned.

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$\Rightarrow$ An 'acceptable' SWFL as Arrow defined it does not exist.

Questions for discussion

- Despite Arrow's Theorem, policy makers *must* aggregate preferences somehow, right?
- Does Arrow's Theorem imply that there is no acceptable way to make collective decisions?

  (You should come back to these questions after we have talked about the constitutional economics perspective.)

Critique (Buchanan 1954)

- Social Choice Theorists take it for granted that group **choices** must reflect collective **preferences**.
    - This perspective is inherited from economists' theory of *individual choice*.
- Extending the preference-based theory of individual choice to a group is inconsistent with methodological and normative *individualism*
    - The idea of "rationality (...) as an attribute of the social group" (...) is incompatible with the principle that "the individual is the only entity possessing ends or values."
- In their analysis of *market exchange*, economists do not ask whether outcomes are 'socially preferred,' but only if they are *efficient*.
    - The *outcome* (allocation) produced by market exchange is **obviously not a 'choice'** and therefore no attempt is made to explain or justify it with reference to a collective preference.
    - The emphasis is on describing how voluntary transactions can be organized in such a way as to produce *mutual advantages* (Pareto improvements), not whether they increase 'social utility'.
- A consistent application of the 'economic perspective' to politics should follow the same principle: Investigate the conditions under which *non-market exchange* can be organized in a way that facilitates *additional* Pareto improvements beyond those achieved through voluntary private transactions.

Literature

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