

Classification of Sets by Cardinality

Today's Agenda

- Cardinality of Sets
- Cardinality of Natural Numbers
- Cardinality of Whole Numbers
- Cardinality of Even Numbers
- Cardinality of Odd Numbers
- Cardinality of Integers
- Cardinality of Rational Numbers
- Cardinality of Real Numbers

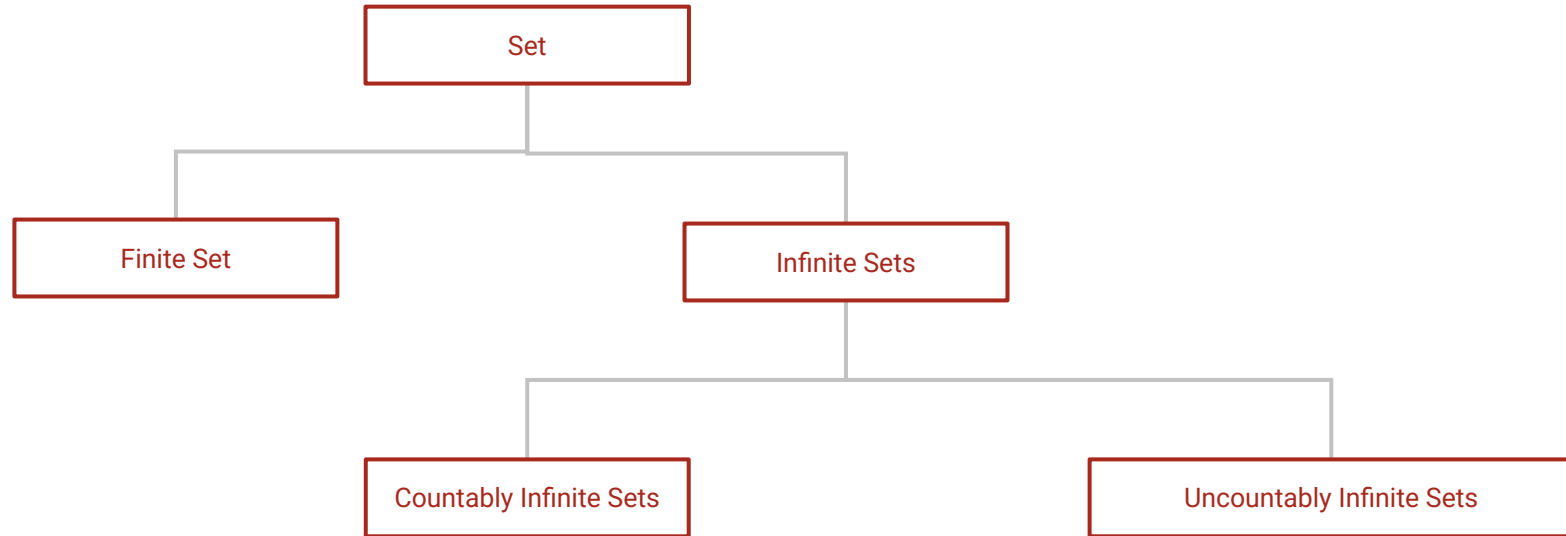
Why study set sizes

Most of the mathematical constructs are defined in terms of sets.

Previously we studied how sets can be constructed and de-structured in various ways. It includes union, intersection, subsets etc.

We will now focus more on the count of elements.

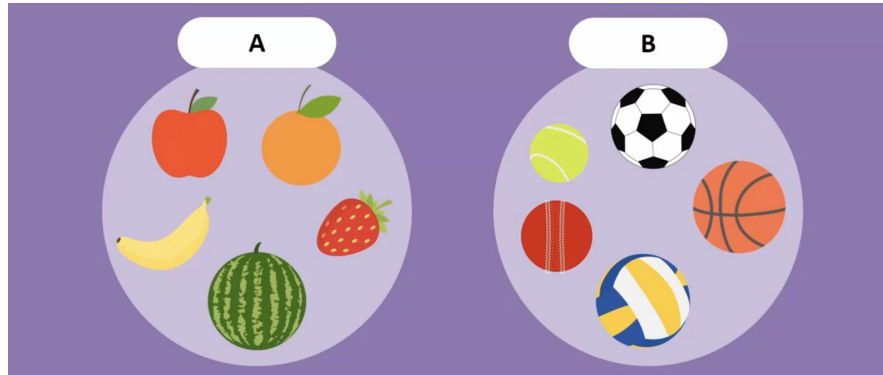
Classification of Sets by Sizes - Preview



Equivalent Sets

Equivalent Sets - Two finite sets A and B are equivalent if their element count is same. i.e. $n(A) = n(B)$

Example: {5, 10, 15, 20, 25}, & {a, e, i, o, u} are equivalent.



Cardinality

The **number of elements** in a set is called **cardinality** of that set. It is denoted by **|A|** for the *number of elements* in set A.

$$S = \{ 2, 3, 5, 7, 11, 16 \} \quad \Rightarrow \quad |S| = 6$$

The use of term **cardinality** instead of **size** is

- Being precise over idea of *count of the elements*
- Other similar terms like *measure, length, boundary* etc.

Rapid fire

What is the cardinality of set

$$A = \{ x \mid x \in \mathbb{N}, 3 < x \leq 10 \}?$$



Rapid fire

What is the cardinality of set

$$A = \{ x \mid x \in \mathbb{N}, 3 < x \leq 10 \}?$$

Ans: 7



Rapid fire

What is the cardinality of set

$E = \{ a, b, c, d, \dots, x, y, z \}$?

Ans:



Rapid fire

What is the cardinality of set

$E = \{ a, b, c, d, \dots, x, y, z \}$?

Ans: 26



Rapid fire

What is the cardinality of \emptyset

Ans:



Rapid fire

What is the cardinality of \emptyset

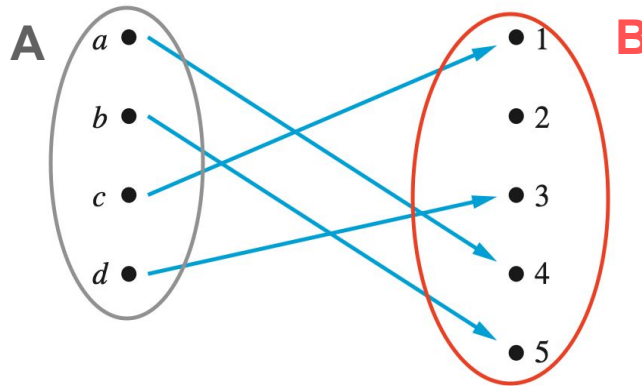
Ans: 0



Cardinality - Comparative

- If there exists an injective function from A to B,
We say the cardinality of A is less than or same as B.

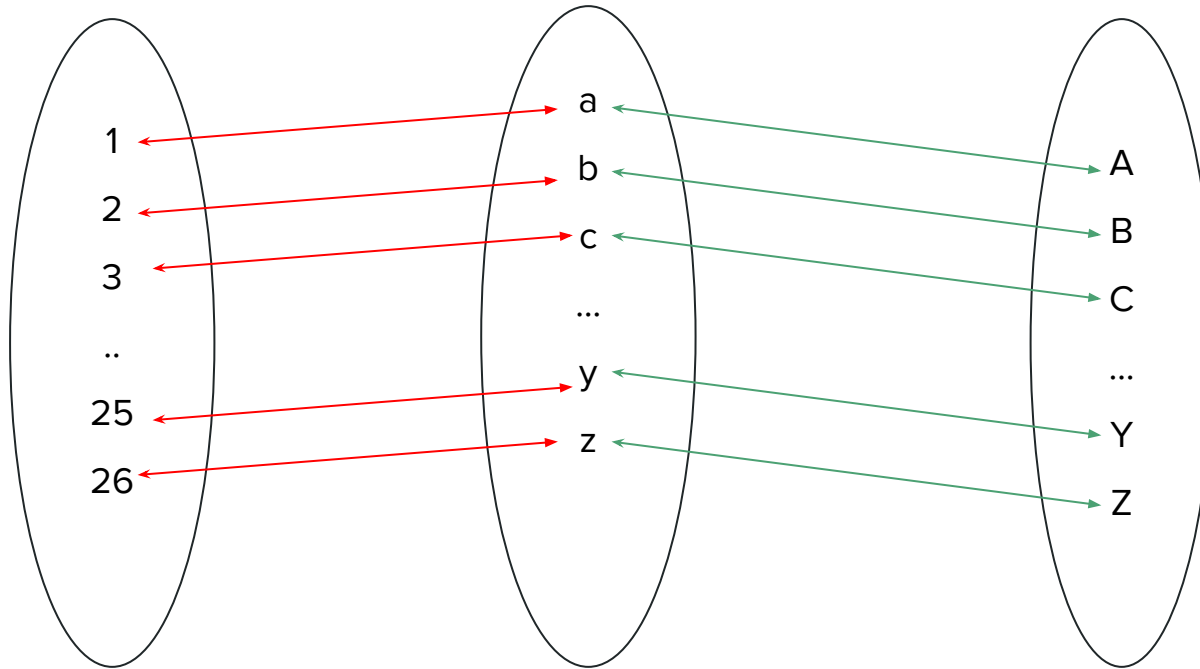
$$|A| \leq |B|$$



Cardinality - Equivalent

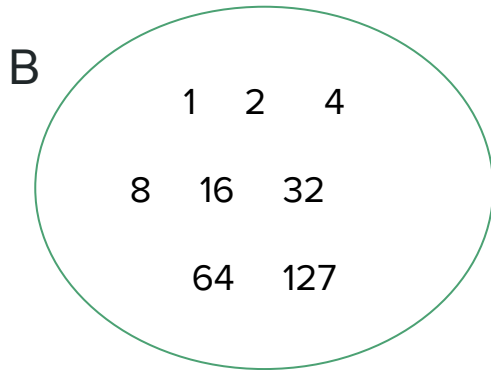
- If our injective function is also surjective, it becomes bijective, then A will have same number of elements as B , meaning same cardinality.
- The sets A and B have the **same cardinality** if and only if there is a **bijective function** (one-to-one correspondence) from A to B .
- When A and B have the same cardinality, we write $|A| = |B|$.

Cardinality - Equivalence Relation



Cardinality of Finite Sets

- A set with *no elements* is called a **null set**. i.e. \emptyset
- A **finite set** has a cardinality that can be represented by a nonnegative integer.



$$|B| = 8$$

Finite Vs Infinite Sets

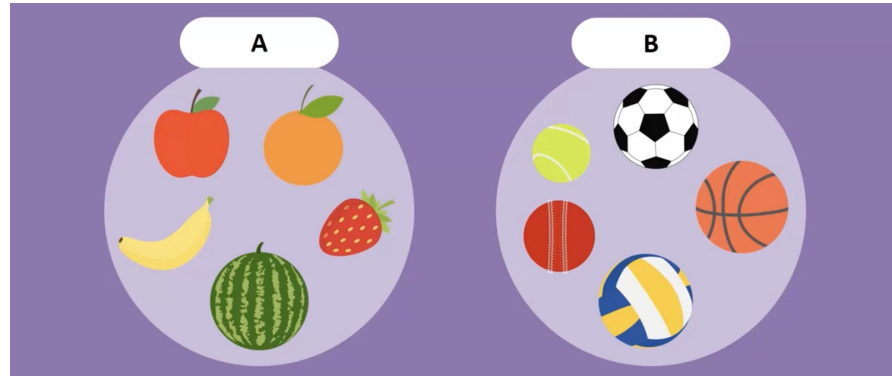
Equivalent Sets - Two finite sets A and B are equivalent if their element count is same. i.e. $|A| = |B|$

Example:

$P = \{5, 10, 15, 20, 25\}$ and

$Q = \{a, e, i, o, u\}$

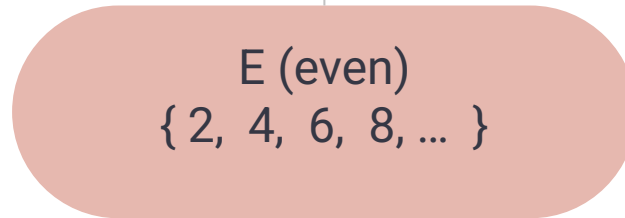
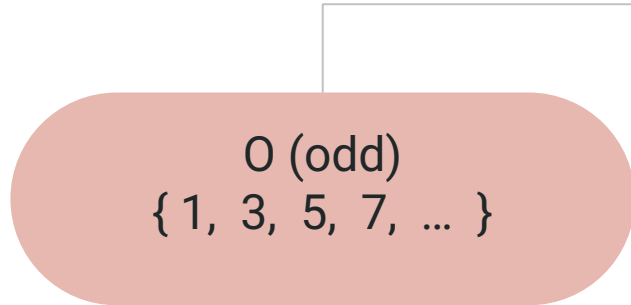
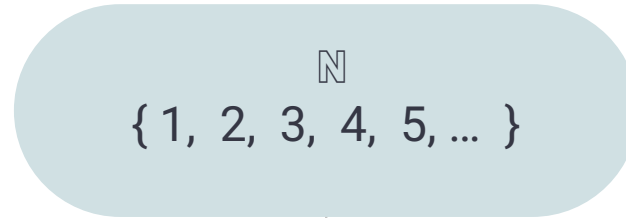
are equivalent.



We can easily compare two **finite sets** based on their cardinality.

But can we do the same for **infinite sets**?

Splitting the Naturals in two parts



Try Answering

1. $|\mathbb{N}| = |E| \quad ??$
2. $|\mathbb{N}| = |O| + |E| \quad ??$
3. $|E| = |O| \quad ??$

Cardinality of Infinite Set

- An Infinite set is a set that is **not finite**.
- An infinite set is one that can be put into bijection with one of its proper subsets.
- The above statement feels counter-intuitive, but it can be shown with the help of famous Hilbert Hotel.

Hilbert Hotel - A short Visit

Room No	1	2	3	4	5
Day 1	A1	A2	A3	A4	A5

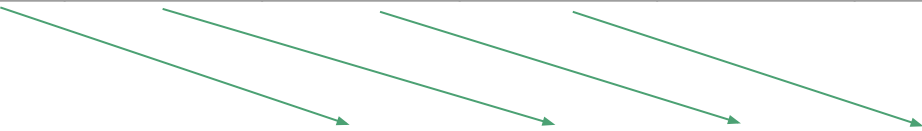
Hilbert Hotel - A short Visit

Room No	1	2	3	4	5
Day 1	A1	A2	A3	A4	A5

Day 2	B1	B2
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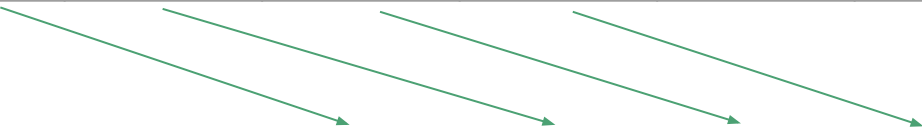
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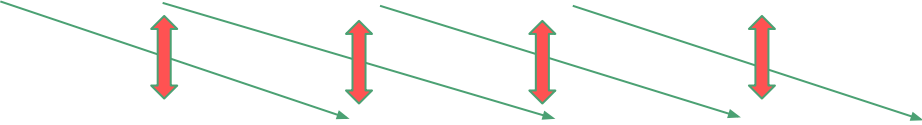
Hilbert Hotel - A short Visit

Room No	1	2	3	4	5
Day 1	A1	A2	A3	A4	A5
Day 2	B1	B2	A1	A2	A3



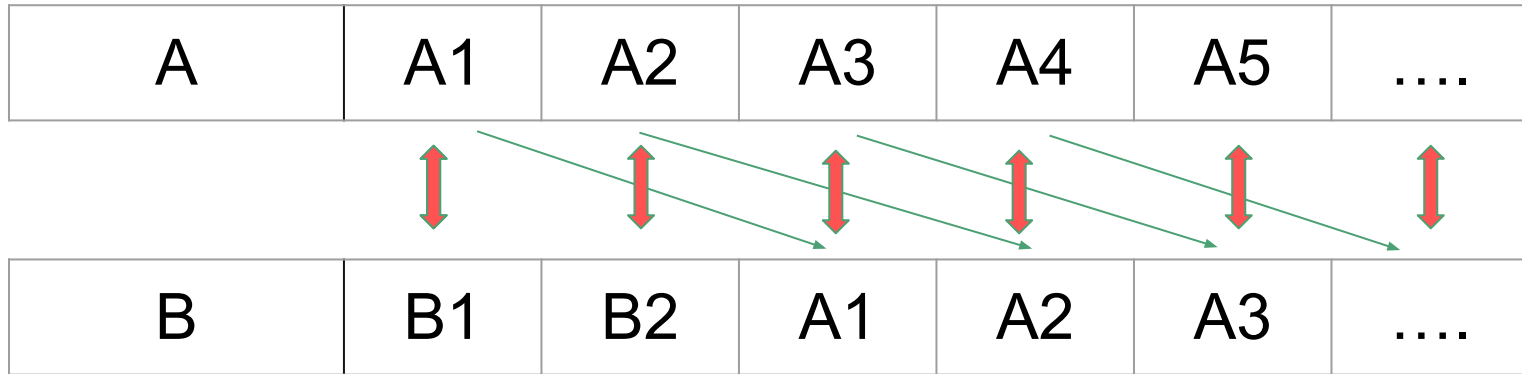
Hilbert Hotel - A short Visit

Room No	1	2	3	4	5
Day 1	A1	A2	A3	A4	A5
						
Day 2	B1	B2	A1	A2	A3



Hilbert Hotel - A short Visit

- Here set A is clearly a proper subset of set B



- Nonetheless, there is a bijection between A and B.
- This is the property of infinite set.
- Finite sets cannot have bijection if A is proper subset of B.

Cardinality of Natural Numbers

- We know that the size of \mathbb{N} is **infinite**.
- Cardinality is typically discussed **between two sets**, by checking for a bijection.
- Therefore, when discussing the cardinality of **infinite sets**, we try to find a bijection between \mathbb{N} and the set in question.

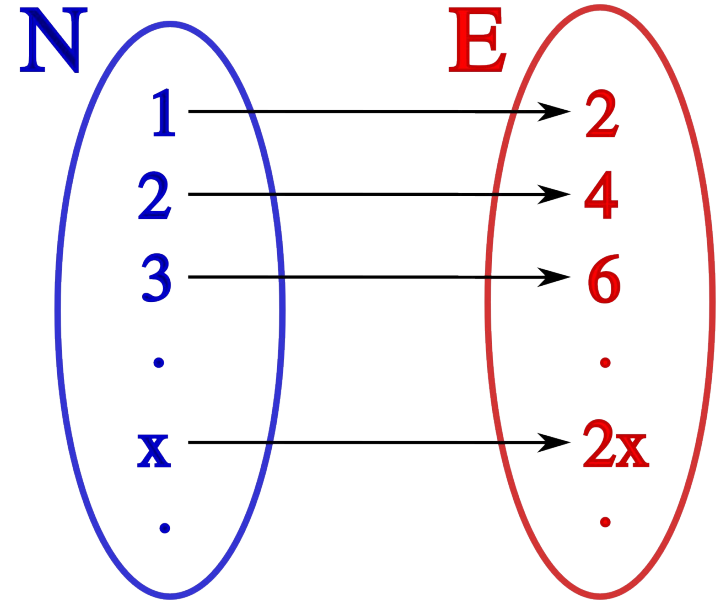
Now, let's **revisit the odd-even split** of \mathbb{N} to explore these ideas further.

Cardinality - Natural Vs Even Numbers

Lets try bijection from Natural numbers to Even numbers.

We can see that, with function $f(x) = 2x$,
For every natural number x (in set N),
there exists an even number $2x$ (in set E).

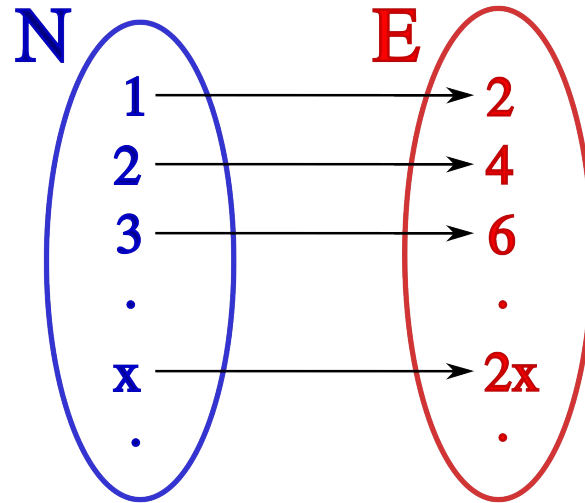
Also, since no even number is mapped twice,
this function is bijective.



Cardinality of Odd and Even Numbers

Since bijection exists between \mathbb{N} and E (or O),

The cardinality of even numbers is same as cardinality of \mathbb{N} .



Countably Infinite Sets - Definition

- We say we can “**count**” the natural numbers because we can list them in order:

$$\{ 1, 2, 3, \dots \}$$

- A set that has the **same cardinality** as \mathbb{N} is called **countably infinite**.
- The cardinality of the natural numbers is often denoted by: \aleph_0

Questions

Which of the following sets have same cardinality as of \mathbb{N} .

Extra: If they do, can you represent your bijection as an algebraic function?

1. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots\}$
2. The set of all square numbers less than a googol.
3. Set of all integers

Cardinality of Integers

To show that \mathbb{Z} is countable, we define following **mapping**. This mapping pairs each natural number with a unique integer, proving that \mathbb{Z} and \mathbb{N} have the same cardinality.

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n	1	2	3	4	5	6	7	8	9
$f(n)$		1		2		3		4	

$$f(n) = \frac{n}{2}, \text{ for } \mathbb{Z}^+$$

Cardinality of Integers

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n	1	2	3	4	5	6	7	8	9
$f(n)$	0		-1		-2		-3		-4

$$f(n) = -\frac{(n-1)}{2}, \text{ for } Z^- \text{ and } 0$$

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$$f(n) = \frac{n}{2}, \text{ for } \mathbb{Z}^+$$

$$f(n) = -\frac{(n-1)}{2}, \text{ for } \mathbb{Z}^- \text{ and } 0$$

Cardinality of $\mathbb{N} \times \mathbb{N}$ (Cartesian Product of \mathbb{N})

		y					
		0	1	2	3	4	5
x	0	0	1	3	6	10	15
	1	2	4	7	11	16	22
	2	5	8	12	17	23	30
	3	9	13	18	24	31	39
	4	14	19	25	32	40	49
	5	20	26	33	41	50	60

$\mathbb{N} \times \mathbb{N}$ denotes the cartesian product of natural numbers (ordered pair of natural number).

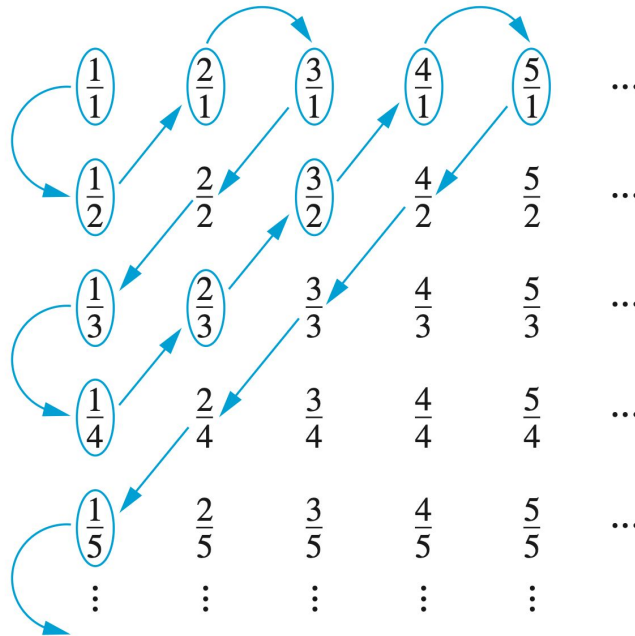
We show here a mapping from $\mathbb{N} \times \mathbb{N}$ to \mathbb{N} .

$$P(x, y) = \frac{(x + y)(x + y + 1)}{2} + x,$$

This shows that the cardinality of $\mathbb{N} \times \mathbb{N}$ to \mathbb{N} is same.

Cardinality of Rational Numbers

Terms not circled
are not listed
because they
repeat previously
listed terms



We previously saw rationals as

$$\mathbb{Q} = \{ a / b : a, b \in \mathbb{Z} \}$$

Here's a mapping of rational
numbers to natural numbers.

We skip the repeated pairs here.