CS201c: Quiz 1
Total: 60 points, Time: 45 minutes
Instructor: Apurva Mudgal

Instructions. For full credit, you will have to support your answers with proofs.

Notation. Let V denote a AVL tree, or a 2-3-4 tree. A leaf of V is a node which has no children.

The length of a root-to-leaf path P in tree V is equal to the number of nodes on path P minus 1. The height of tree V is the maximum length of a root-to-leaf path in V.

Further, levels are assigned to the nodes of tree V as follows: (i) the root is the only node at level 0, and (ii) if a node v is a child of a node at level i, then the level of v is set to $i + 1$.

1. (20 points) Suppose n varies through positive integers $1, 2, 3, \ldots$.

   Consider the two functions:

$$f(n) = n^2\left[\sin\left(n\pi + \frac{\pi}{2}\right) + 1\right]$$

$$g(n) = n^2$$

   (a) (10 points) Is $g(n) = O(f(n))$? Support your answer with a proof.

   (b) (10 points) Is $f(n) = O(g(n))$? Support your answer with a proof.

2. (20 points) Suppose the height of a AVL tree T is h. Prove that the minimum length of a root-to-leaf path in T is at least $\lfloor \frac{h}{2} \rfloor$.

   Note. $\lfloor \frac{h}{2} \rfloor$ denotes the *largest* integer x such that $x \leq \frac{h}{2}$.

3. (20 points) Consider a 2-3-4 tree U of height $2h$ such that:

   (a) All nodes at even levels (levels 0, 2, ..., $2h$) have exactly 2 keys.

   (b) All nodes at odd levels (levels 1, 3, ..., $2h - 1$) have exactly 1 key.

   Derive an expression for the total number of keys in tree U. Simplify your answer as much as possible.