<u>CS201c Quiz 1 solutions</u>

Question 1.

First note that, sin(pi/2)=1,sin((3*pi)/2)=-1, and sin(2*pi+x)=sin(x).

f(1)=f(3)=f(5)= … = 0. Thus, f(2k+1)=0 for all k>=0.
f(2)=2.2^2, f(4)=2.4^2, f(6)=2.6^2, …. Thus, f(2k)=2.(2k)^2 for all k>=1.

First proof:

(a) No.

Assume g(n) = O(f(n)).
=> There exists a constant c>0, such that g(n) <= c.f(n) for all n=1,2,3,...
=> g(1) <= c.f(1)
=> 1 <= 0 (a contradiction)

(b) Yes.
For k>=0:
f(2k+1) = 0 < 2.(2k+1)^2 = 2.g(2k+1)
For k>=1:
f(2k) = 2.(2k)^2 = 2.g(2k)

Thus, f(n) <= 2.g(n) for all n=1,2,3,...
Hence, f(n)=O(g(n)).

Second proof.

Consider the set S={ f(n)/g(n) | n=1,2,3,...}

For n odd, f(n)=0 and g(n)=n^2, and hence f(n)/g(n)=0.
For n even, f(n)=2n^2 and g(n)=n^2, and hence f(n)/g(n)=2.

Thus, f(n)/g(n) for n=1,2,3,... is the sequence 0,2,0,2,0,2,....

Thus, l.u.b(S)=2 and g.l.b(S)=0.

Thus, f(n)=O(g(n)), but g(n) is not O(f(n)).

Question 2.

Let $M(h)$ denote the minimum possible length of a root-to-leaf path among all AVL trees of height exactly h.

Our inductive hypothesis (for h'=0,1,...) is the following:

P(h'): M(h) >= floor(h/2) for all h <= h'.

Base case:

P(0) is true. The only AVL tree of height h=0 consists of a single node, and in this case M(h) = 0 = floor(h/2).

P(1) is true. Further, there are exactly two AVL trees of height h=1. For both of them, M(h)=1 > floor(h/2).

Suppose P(h') is true for some h' >=1.

We now derive P(h'+1). Let T be an AVL-tree of height exactly (h'+1). Let T1 and T2 denote the left and right subtrees of the root of T. Let h1 and h2 be their respective heights. By property of AVL tree (balance factor at root is -1, 0, or +1), we get that:

min(h1,h2) >= (h'-1)
max(h1,h2) = h'

Thus, since P(h') is true:

- Any root-to-leaf path in T1 has length at least floor((h1)/2), which is at least floor((h'-1)/2). [floor(x) is an non-decreasing function of x.]
- Similarly, any root-to-leaf path in T2 has length at least floor((h2)/2), which is at least floor((h'-1)/2).

Therefore, any root to leaf path in T has length at least:

1 + floor((h'-1)/2) = floor((h'-1)/2 + 1) (Using the equality, floor(x)+1 = floor(x+1))
= floor((h'+1)/2).

Since T was arbitrary, we obtain that M(h'+1) >= floor((h'+1)/2). Hence proved.

Question 3.

Level	Number of keys in each node at this level	Number of nodes at this level	Total number of keys at this level
0	2	1	2
1	1	3	3
2	2	3.2	3.2^2
3	1	3^2.2	3^2.2
4	2	3^2.2^2	3^2.2^3
5	1	3^3.2^2	3^3.2^2

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Levels 0,2,...,2h ---> level 2k, 0<=k<=h.
Levels 1,3,5,...,2h-1 ---> level 2k+1, 0<=k<=h-1

Total number of keys
= sum_{k=0}^{h} [3^k.2^{k+1}] + sum_{k=0}^{h-1} [3^{k+1}2^k]
= 2 . sum_{k=0}^{h} 6^k + 3 . sum_{k=0}^{h-1} 6^k
= 2.6^h + 5 . sum_{k=0}^{h-1} 6^k
= 2.6^h + 5. (6^h-1)/(6-1) (By formula for sum of geometric series)
= 2.6^h + 6^h - 1
= 3.6^h - 1