Assignment 7

Due: March 3, 2022

Your solutions must be typed (preferably typeset in LaTeX) and submitted as a PDF through Canvas before the beginning of class on the day its due.

Problem 1: Lower Bounds [10 points] Prove that any comparison- based algorithm for constructing a binary search tree from an arbitrary list of n elements takes $\Omega(n \log n)$ time in the worst case. (Hint: Consider how to reduce the sorting problem to performing a set of operations on a binary search tree. In other words, show that if a faster algorithm existedfor constructing a binary search tree then you would violate the $\Omega(n \log n)$ comparison-based sorting lower bound.)

Answer: - The time complexity of lg n is for binary search tree for the n number of key (for every comparison)

Therefore, the total cost for the generation of binary search tree becomes as: - $\Omega(n \log n)$

The above point is valid only for unsorted keys.

If the keys are sorted, then the time complexity becomes as: - $O(n^2)$.

Now as we know that for quicksort method, the time complexity is O(n lg n) as it uses divide and conquer method, therefore,

Using a similar approach for a binary search tree, the worst case can be lg n. This is because for the binary search tree, there needs to be a comparison with respect to the height of the tree (for each compared parameter).

From this, we can say that, lower bound of the comparison sets when generating the tree is set at $\Omega(n \log n)$.

But this rule can easily be violated as proved in the above scenario. This is because there can exist a faster, more quicker and efficient algorithm for developing a binary search tree which can and will violate this lower bound .

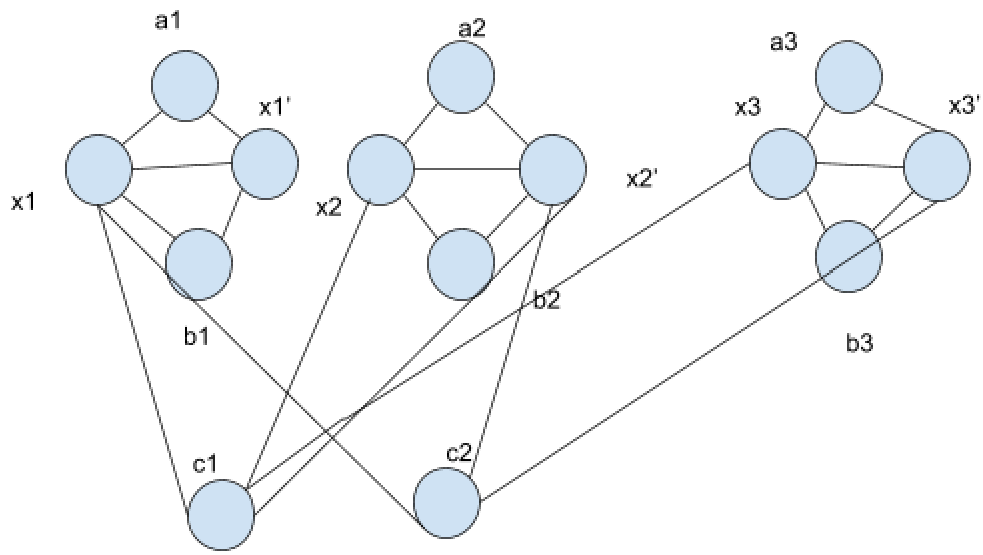
Problem 2: Fire Houses [10 points] Show that the following problem is NP-Complete (Hint: reduce from 3-SAT or Vertex Cover).

FIREHOUSES: Given an undirected graph G with positive integer distances on the edges, and two integers f and d, is there a way to select f vertices on G on which to locate firehouses, so that no vertex of G is at distance more than d from a firehouse?

Answer: - We can check in polynomial time, if there exists f (not more than f) vertices on G to locate firehouses, from the set consisting of all the vertices, and all vertices are at a particular distance "d" from the firehouse, we can use depth first search to calculate distance in the graph. Therefore it is in NP.
Now if we can prove that this problem is NP-HARD, we can say that it is NP-COMPLETE.
To prove it, let us reduce from 3-SAT problem, say there is an instance of 3-SAT, having X {x_1, x_2, x_3, …., x_i} as set of variables and C as set of clauses, and each clause will be having 3

literals l_{i1}, l_{i2}, l_{i3} and each literal is for x_i or x_i' for any x_i. Now, we can construct graph G with vertices V and edges E if and only if all the clauses are satisfiable.
Now, let us assume that d = 1 i.e. all the vertices are one edge away from the firehouse. Now let us consider the vertices x_i, x_i', a_1 and b_1 for a part of Graph G, with edges as shown below.

Now, we have to connect this part of the graph (x_i, x_i') to the rest of the graph such that a_1 and b_1 are connected and are at distance 1 to one of the firehouses. This means that this part should be assigned to at least one firehouse (to either x_i or x_i'), also we will have a vertex for each $C_{i.}$



These clauses c1 and c2 will have 3 literals each i.e. c1: {x1 ,x2, x3} and c2: {x1, x2', x3'}.
Now, let's say that we have satisfying values for the clauses, this means the firehouses need to be placed at each x_i or x_i' with total number equal to f. And these firehouses are within distance 1 from the firehouses in the subparts with the true assignment settings, also as the clauses are satisfied, therefore one out of three edges from each clause should go to the firehouse vertex, hence every clause is within distance 1, and therefore it satisfies the firehouse instance. Similarly, let's say that we have a set of firehouse vertices f, with all the vertices of G at distance 1 from firehouse. This means that we must have one firehouse (at either x_i or x_i') for each part of the graph. Now, as each clause is at distance 1, and there are no more firehouses (f = n), then one of the literals in the clause must be having a firehouse and therefore each clause is satisfied.

As from above, the construction of the firehouse has taken polynomial time, we can say that 3 - SAT reduces to FIREHOUSES and can be solved in polynomial time, hence FIREHOUSES is NP-HARD and thus NP-COMPLETE.

References: https://people.cs.clemson.edu/~goddard/texts/theoryOfComputation/19b.pdf
http://drona.csa.iisc.ernet.in/~chandan/courses/complexity15/notes/lec3.pdf
https://www.quora.com/Why-is-3-SAT-NP-complete