Sant'Anna
Scuola Universitaria Superiore Pisa

NETWORK SIMULATION

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YAMAN PARASHER

PHOTONICS INTEGRATED CIRCUITS, SENSORS & NETOWORKS
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Assignment No.1
Random Variate Generation

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1 Objective of the Assignment

1. Generate random varietes for the following continuous random variables density functions
   - Uniform (0,1]
   - Uniform [4,10]
   - Negative exponential (β=2)
   - Pareto (α=3, k=2)
2. Verify that the set of generated random variates approximate the considered density functions (plot the random variates results and the pdf in the same plot as a function of x)

2 Introduction

The given assignment provides a comprehensive analysis of the inverse transform method which is widely regarded as a very useful approach to generate random variate corresponding to a given distribution function. This whole process is widely known as Pseudo-random number sampling or non-uniform pseudo-random variate generation which is the numerical practice of generating pseudo-random numbers that are distributed according to a given probability distribution.

In the assignment, I observed that the distribution of these random variate inside the given range of the functions are highly influenced by the different pseudo-random number generator algorithm one of which is the LCG (Linear congruential generators (LCGs) are a class of pseudorandom number generator (PRNG) algorithms). Depending upon the values of parameters m, the modulus, a, the multiplier, and c, the increment in LCG there are a list of LCGs in common use, including built-in rand() functions in runtime libraries of various compilers.

Therefore, before going directly to the objective of the assignment, I provide a brief & concise introduction to the whole concept of probability distribution and related things from the basics.Starting from what a probabliity distribution function is all about I dig deeper to define its classification and the functions which I have to taken into account to fulfil the given objective of the assignment.Apart from all this, this whole assignment has been implemented in R which is a programming language and free software environment for statistical computing and graphics employed generally to solve various real world problems.

2.1 Probability Distribution its Classification

In probability theory and statistics, a probability distribution is a mathematical function that provides the probabilities of occurrence of different possible outcomes in an experiment. In more technical terms, the probability distribution is a description of a random phenomenon in terms of the probabilities of events. For instance, if the random variable X is used to denote the outcome of a coin toss ("the experiment"), then the probability distribution of X would take the value 0.5 for X = heads, and 0.5 for X = tails (assuming the coin is fair). Examples of random phenomena can include the results of an experiment or survey.

A probability distribution is specified in terms of an underlying sample space, which is the set of all possible outcomes of the random phenomenon being observed. The sample space may be the set of real numbers or a set of vectors, or it may be a list of non-numerical values; for example, the sample space of a coin flip would be heads, tails .

They are generally divided into two classes.

1. A discrete probability distribution (applicable to the scenarios where the set of possible outcomes is discrete, such as a coin toss or a roll of dice) can be encoded by a discrete list of the probabilities of the outcomes, known as a probability mass function.
2. On the other hand, a continuous probability distribution (applicable to the scenarios where the set of possible outcomes can take on values in a continuous range (e.g. real numbers), such as the temperature on a given day) is typically described by probability density functions (with the probability of any individual outcome actually being 0).

2.2 Random Variate

In the mathematical fields of probability and statistics, a random variate is a particular outcome of a random variable: the random variates which are other outcomes of the same random variable might have different values. The concept of Random variates are generally employed when we simulate processes which are driven by random influences (stochastic processes). In modern applications, such simulations would derive random variates corresponding to any given probability distribution. Procedures to generate random variates corresponding to a given distribution are known as procedures for random variate generation or pseudo-random number sampling,which is the core theme of this assignment for a given set of distribution functions.

2.3 Inverse Transform : An approach to generate Random Variate

For a given random variable X which is continuous and has distribution function F that is continuous and strictly increasing when $0 < F(x) < 1$. Which ultimately means that if $x_1 < x_2$ and $0 < F(x_1) \leq F(x_2) < 1$, then $F(x_1) < F(x_2)$. Let F^{-1} denotes the inverse of function F.Then an algorithm to generate random variate X having distribution function F is given as follows,

1. Generate U $\equiv$ U(0,1)
2. Return x=$F^{-1}(U)$

Note that $F^{-1}(U)$ will always be defined.
If F is not continuous or increasing, one must use the generalized inverse function given by $F^{-1}(u) = min\ (x : F(x) \geq \mu)$

2.4 Linear congruential generators (LCGs)

Linear congruential generators (LCGs) are a class of pseudorandom number generator (PRNG) algorithms used for generating sequences of random-like numbers. The generation of random numbers plays a large role in many applications ranging from cryptography to Monte Carlo methods. Linear congruential generators are one of the oldest and most well-known methods for generating random numbers primarily due to their comparative ease of implementation and speed and their need for little memory. Other methods such as the Mersenne Twister are much more common in practical use today.

Linear congruential generators are defined by a recurrence relation:

$$x_{i+1} = (aX_i + c)\ mod\ m \tag{1}$$

There are many choices for the parameters m, the modulus, a, the multiplier, and c the increment.

2.4.1 Selecting a Seed Number

Random number generators such as LCGs are known as 'pseudorandom' as they require a seed number to generate the random sequence. Due to this requirement, random number generators today are not truly 'random.'

3 Pseudo-random number sampling of Uniform Function(0,1]

The continuous uniform distribution is the probability distribution of random number selection from the continuous interval between a and b. Its density function is defined by the following.

$$f(x) = \begin{cases} \frac{1}{b-a} & for\ a \leq x \geq b \\ 0 & x < a\ or\ x > b \end{cases}$$

Fig. 1 shows a group of random variates corresponding to the given distribution, that is the Uniform(0,1]. To plot this, I have used predefined functions in R for the given function which take into consideration the LCG as the argument to generate the given graph. Here, the histograms represents set of random variates that are plotted corresponding to the Uniform function (0,1] shown using the green line.

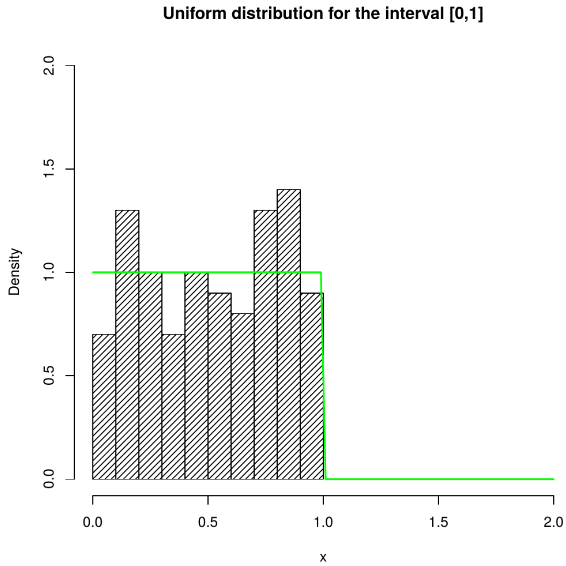

Figura 1: Random Variates corresponding to Uniform (0,1]

- **Uniform (0,1**]

The Uniform distribution function for the given case can be expressed as

$$F(x) = \begin{cases} 0 & for x < 0 \\ x & for\ 0 \leq x \leq 1 \\ 1 & for\ x > 1 \end{cases}$$

Random Variate can be generated from $x = F^{-1}(u) = a + (b - a)u$ where $a = 0$ & $b = 1$. Therefore,x=u.

4 Pseudo-random number sampling of Uniform Function[4,10]

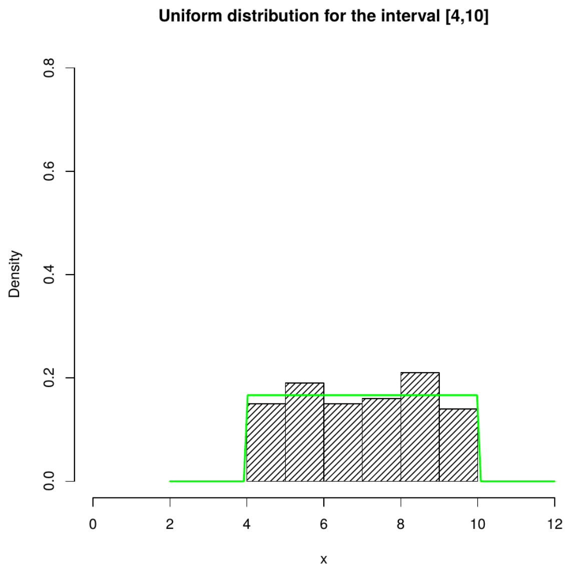

Figura 2: Random Variates corresponding to Uniform [4,10]

Similarly for this section, I have obtained the plot shown in Fig. 2. It can be clearly seen from the graph that the computed random variates properly populate the given function distribution represented by the green line.

- **Uniform [4,10]**

The Uniform distribution function for the given case can be expressed as

$$F(x) = \begin{cases} 0 & for x < a \\ \frac{x-a}{x-b} & for\ a \leq x \leq b \\ 1 & for\ x > b \end{cases}$$

Random Variate can be generated from $x = F^{-1}(u) = a + (b - a)u$ where $a = 4$ & $b = 10$. Therefore,x=4+6u.

5 Pseudo-random number sampling of Negative Exponential Function (β=2)

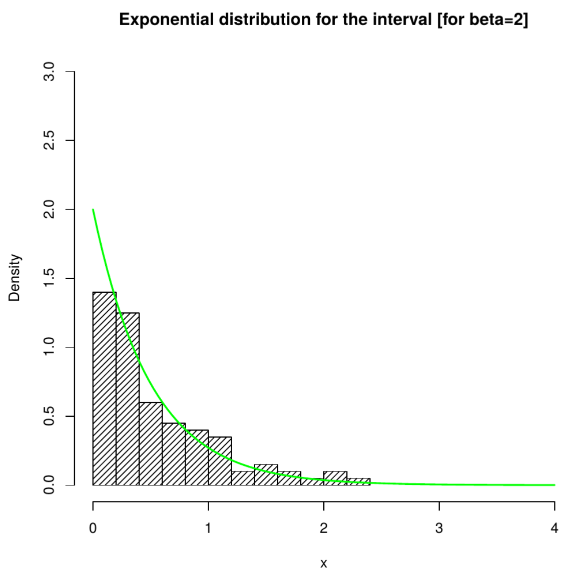

Figura 3: Random Variates corresponding to Negative Exponential Function (β=2)

In probability theory and statistics, the exponential distributions (a.k.a. negative exponential distributions) are a class of continuous probability distributions. They describe the times between events in a Poisson process, i.e. a process in which events occur continuously and independently at a constant average rate. The probability density function of an exponential distribution is:

$$f(x; \mu) = \mu e^{-\mu x} \tag{2}$$

If a random variable X has this distribution, we say that the random variable is exponentially distributed with parameter μ. The cumulative distribution function is:

$$F(x; \mu) = P(X \leq x) = 1 - e^{-\mu x} \tag{3}$$

Mean (β), is given by $\frac{1}{\mu}$

- **Negative Exponential Function (β=2)**

The distribution function for the given case can be expressed as

$$F(x) = \begin{cases} 1 - e^{-(\frac{x}{\beta})} & if\ x \geq 0 \\ 0 & otherwise \end{cases}$$

Random Variate can be generated from $x = F^{-1}(u) = -\beta ln(u)$
Fig. 3 shows a group of random variates corresponding to the given distribution, that is the Negative Exponential Function (β=2).

6 Pseudo-random number sampling of Pareto Function (α=3, k=2)

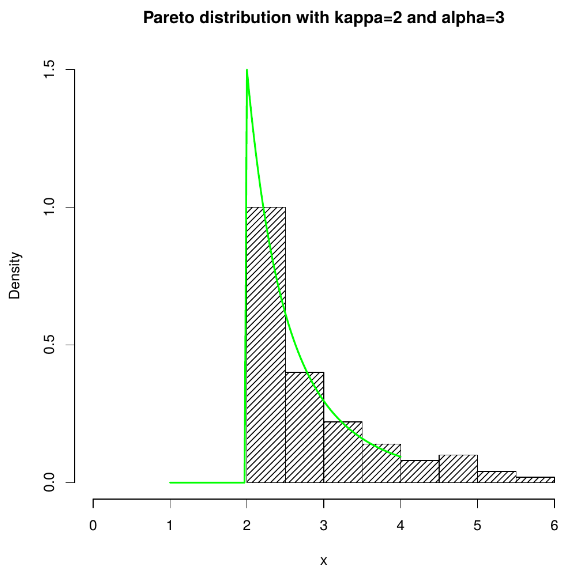

Figura 4: Random Variates corresponding to Pareto Function (α=3, k=2)

The Pareto distribution, named after the Italian civil engineer, economist, and sociologist Vilfredo Pareto, is a power-law probability distribution that is used in description of social, scientific, geophysical, actuarial, and many other types of observable phenomena.
Let X be a pareto random variable with parameters α=3 and k=2. The density function of X is generally given by,

$$f(x; k, \alpha) = \frac{\alpha k^{\alpha}}{x^{\alpha+1}}, \alpha > 0, k > 0, x \geq k \tag{4}$$

Similarly, the cumulative distribution function of X is given by

$$F(x; k, \alpha) = \int_{x}^{k} f(u)du = 1 - (\frac{k}{x})^{\alpha} \tag{5}$$

- **Pareto Function (α=3, k=2)**

The distribution function for the given case can be expressed as $F(x; k, \alpha) = \int_{x}^{k} f(u)du = 1 - (\frac{k}{x})^{\alpha}$
Random Variate can be generated from $x = \frac{k}{(1-u)^{\frac{1}{\alpha}}}$
Fig. 4 shows a group of random variates corresponding to the given distribution, that is the Pareto Function (α=3, k=2).

7 Conclusion

For a given set of parameters of different distribution functions,I was able to generate random variates using Inverse transform method using Linear Congurential generator algorithm in R, which approximate the considered density functions in a proper way.