Theorem Proving

Three important concepts for Logical Inference

Horn Clauses

* A disjunction in which at most one literal is positive
* Equivalent in form to an implication
* Example

$$\neg L_{1,1} \vee \neg Breeze \vee B_{1,1} \Longrightarrow (L_{1,1} \wedge Breeze) \Rightarrow B_{1,1}$$

$$P_{1,1} \vee \neg P_{2,1} \vee B_{1,2}$$

Inference Algorithms:
Forward and Backward Chaining

Forward and Backward Chaining

Horn Form (restricted)

KB = **conjunction** of **Horn clauses**

Horn clause =

◊ proposition symbol; or

◊ (conjunction of symbols) $\Rightarrow$ symbol

E.g., $C \wedge (B \Rightarrow A) \wedge (C \wedge D \Rightarrow B)$

- Restrict sentences to be written a particular way
- Only need one inference rule in your search for entailment

Forward and Backward Chaining

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Modus Ponens (for Horn Form): complete for Horn KBs

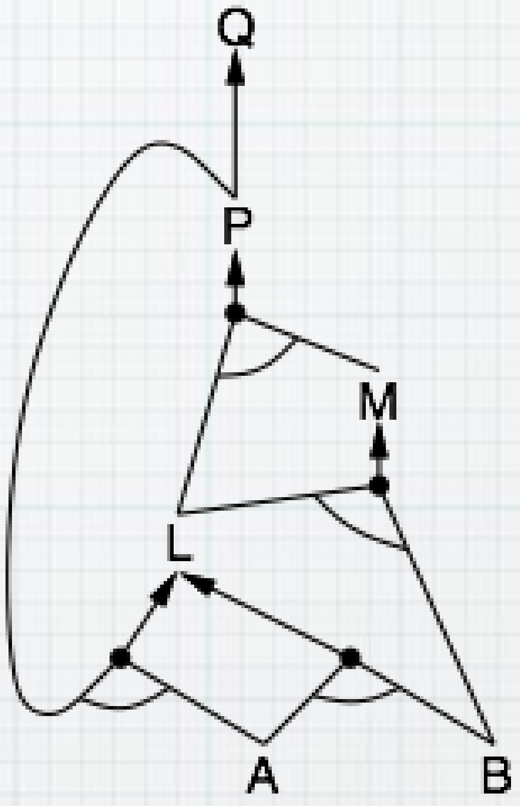
$$\frac{\alpha_1, \ldots, \alpha_n, \qquad \alpha_1 \wedge \cdots \wedge \alpha_n \Rightarrow \beta}{\beta}$$

Can be used with forward chaining or backward chaining.
These algorithms are very natural and run in **linear** time

Forward Chaining

Idea: fire any rule whose premises are satisfied in the KB,
add its conclusion to the KB, until query is found

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$

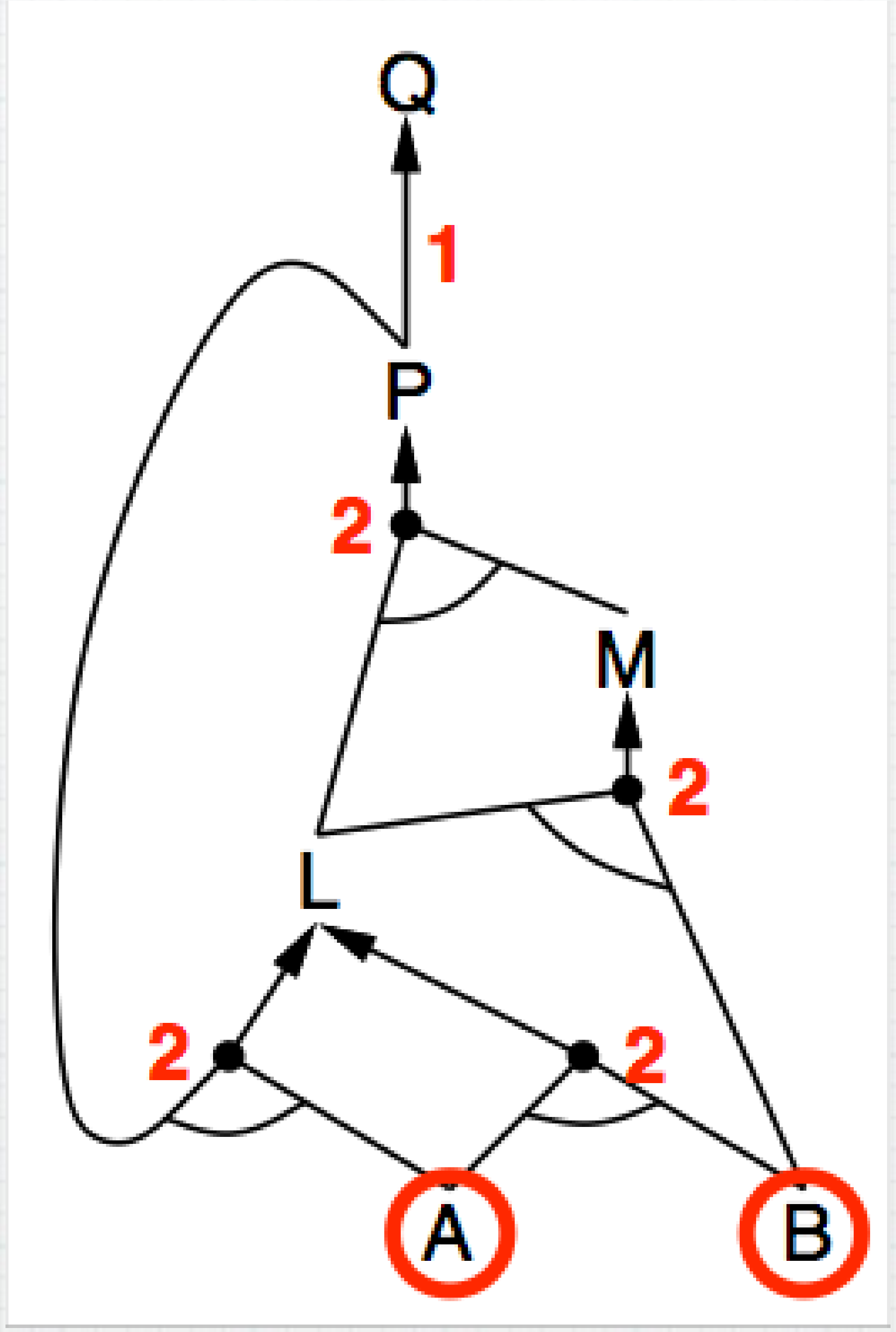


Forward Chaining

```
function PL-FC-ENTAILS?(KB, q) returns true or false
   inputs: KB, the knowledge base, a set of propositional Horn clauses
           q, the query, a proposition symbol
  local variables: count, a table, indexed by clause, initially the number of premises
                   inferred, a table, indexed by symbol, each entry initially false
                   agenda, a list of symbols, initially the symbols known in KB

   while agenda is not empty do
      p ← POP(agenda)
      unless inferred[p] do
         inferred[p] ← true
         for each Horn clause c in whose premise p appears do
            decrement count[c]
            if count[c] = 0 then do
               if HEAD[c] = q then return true
               PUSH(HEAD[c], agenda)
   return false
```

Forward chaining example



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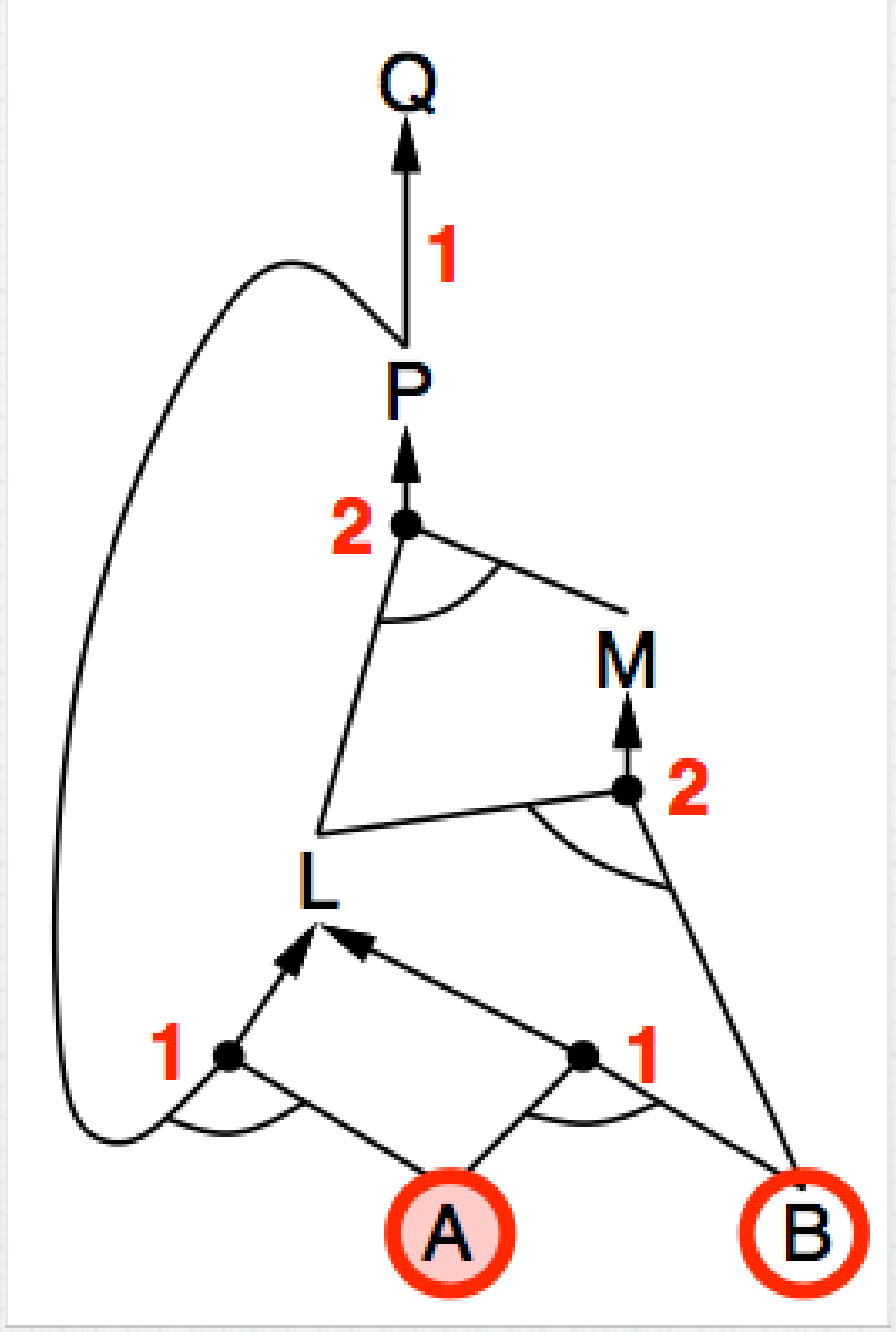

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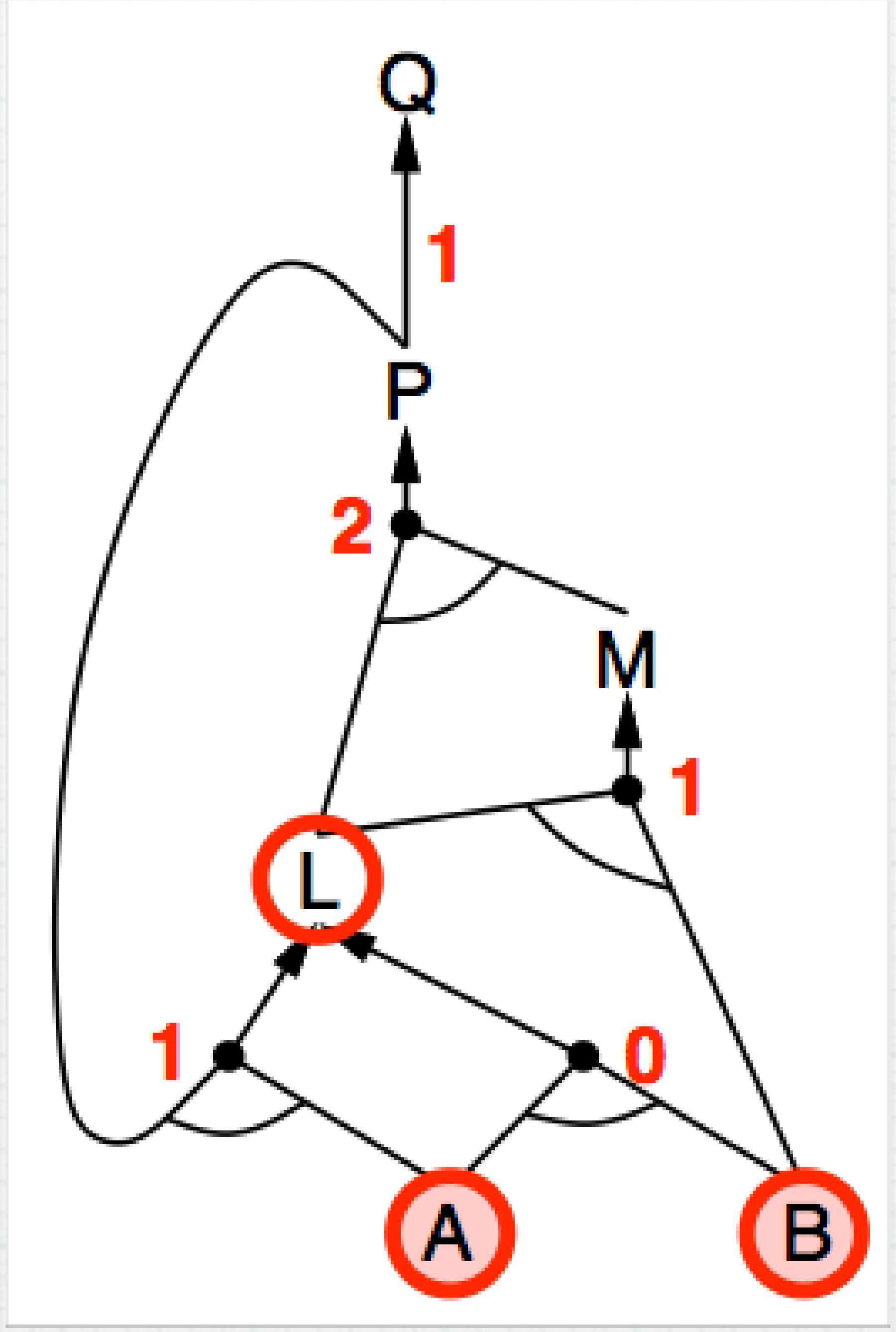

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B

L

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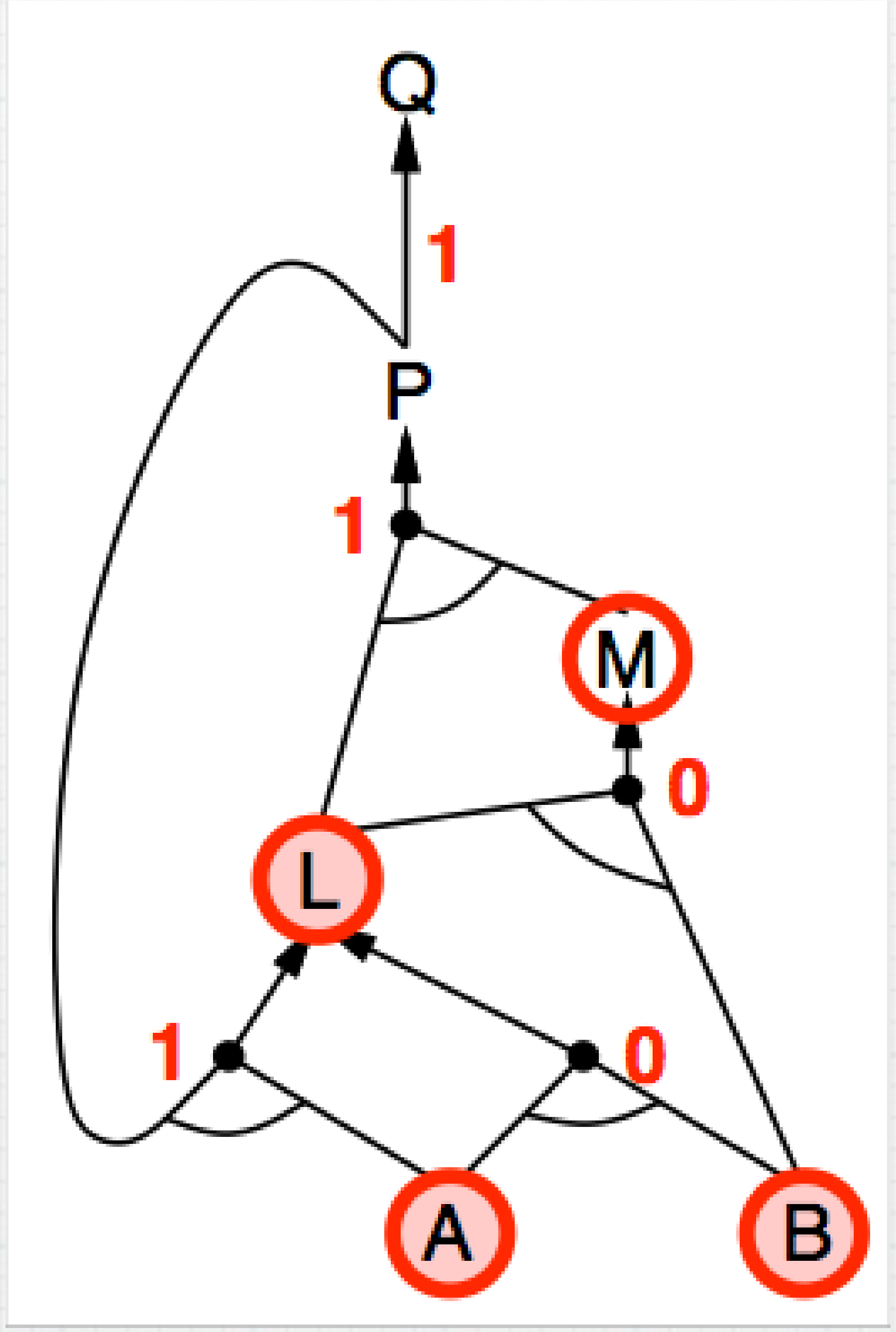

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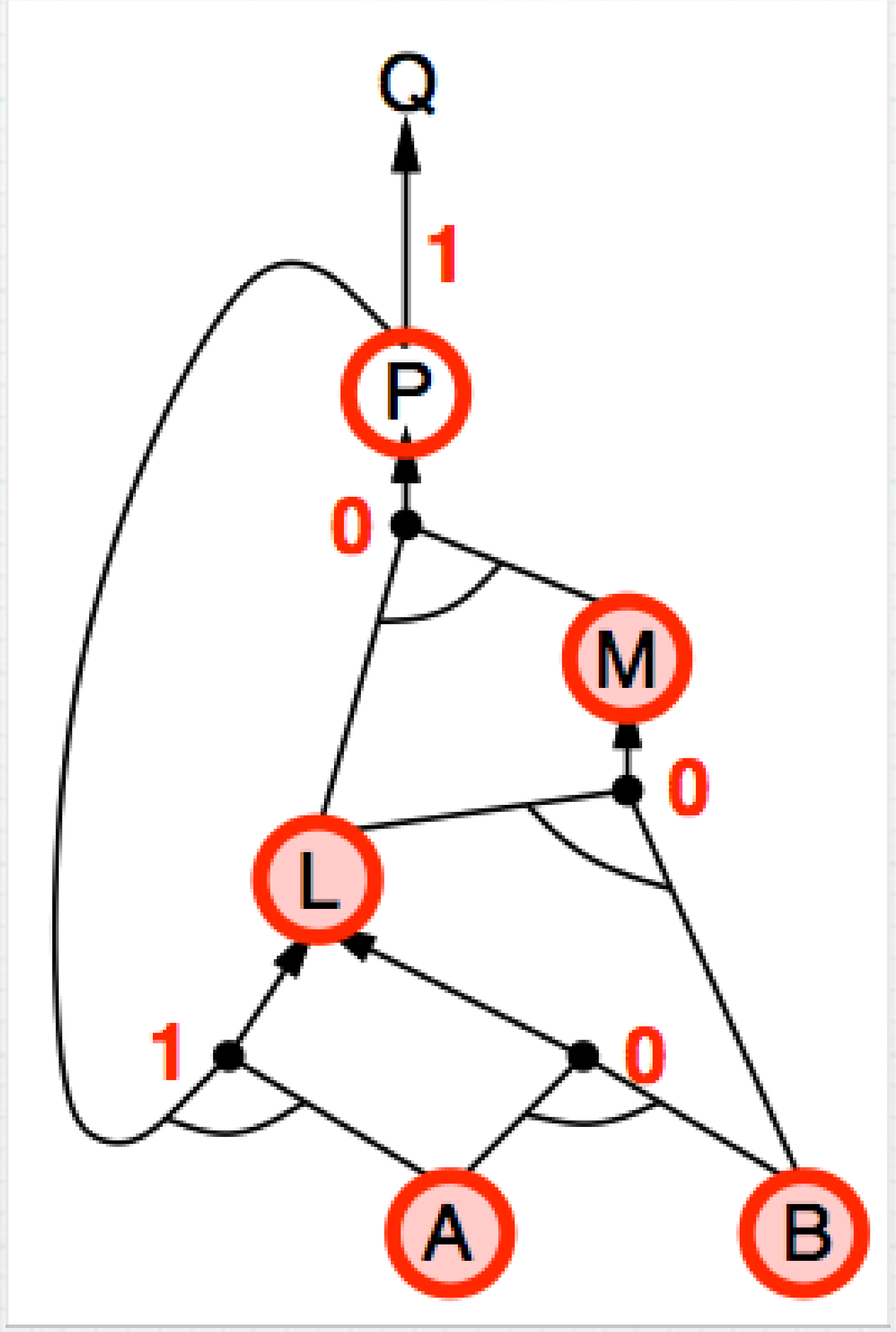

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M

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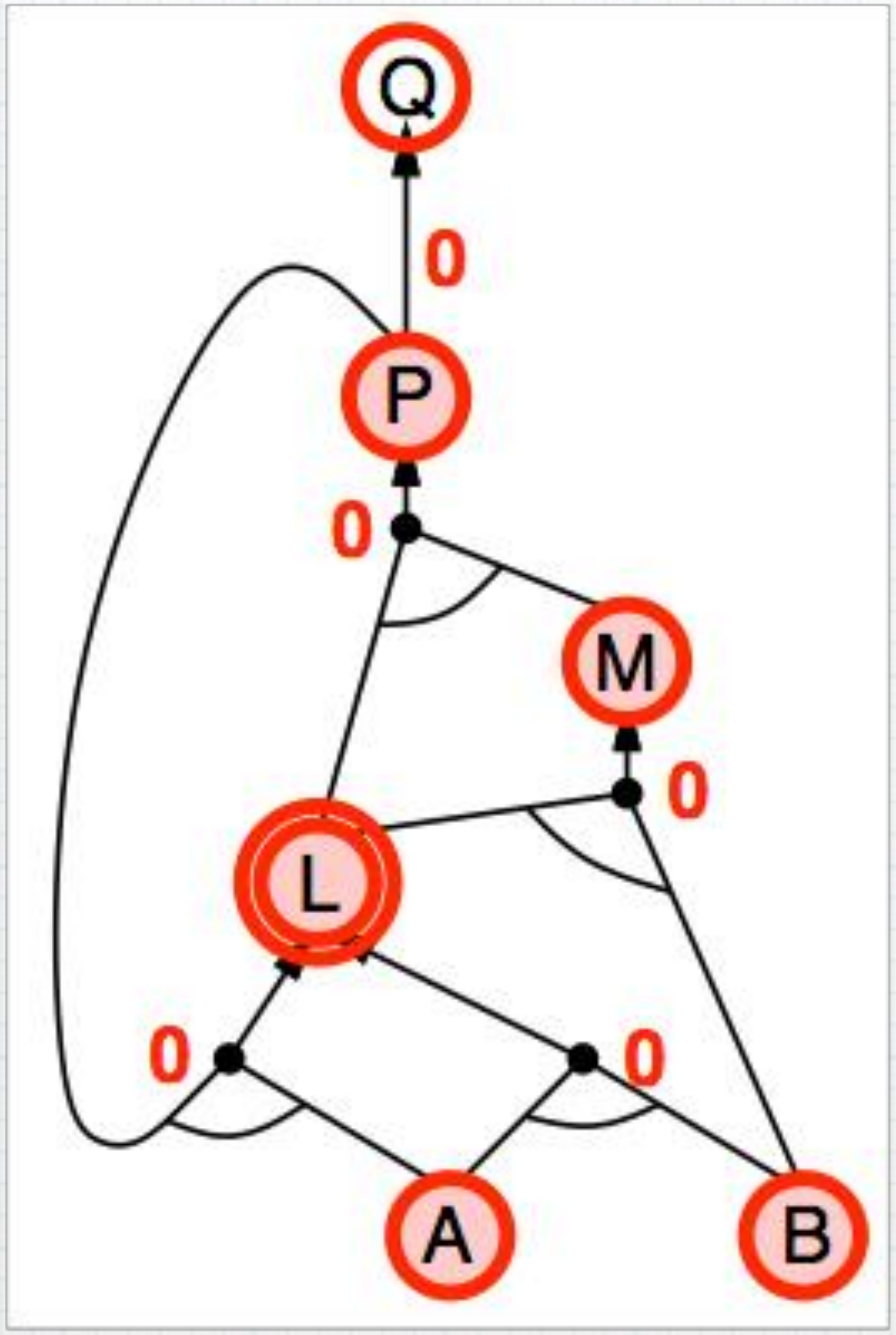

$$
\begin{aligned}
P &\Rightarrow Q \\
L \wedge M &\Rightarrow P \\
B \wedge L &\Rightarrow M \\
A \wedge P &\Rightarrow L \\
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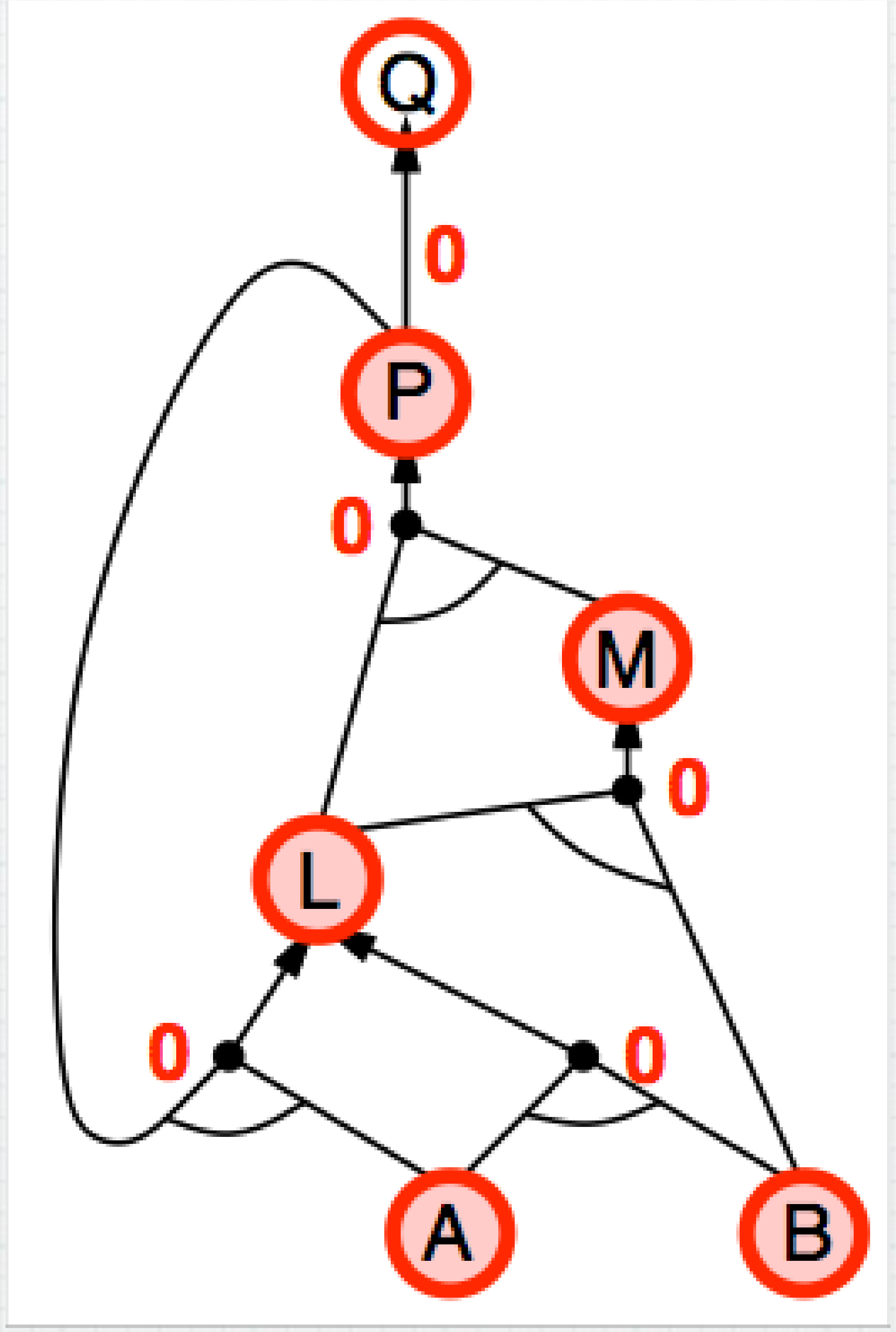
A & \\
B &
\end{aligned}
$$

L

M

P

Forward chaining example



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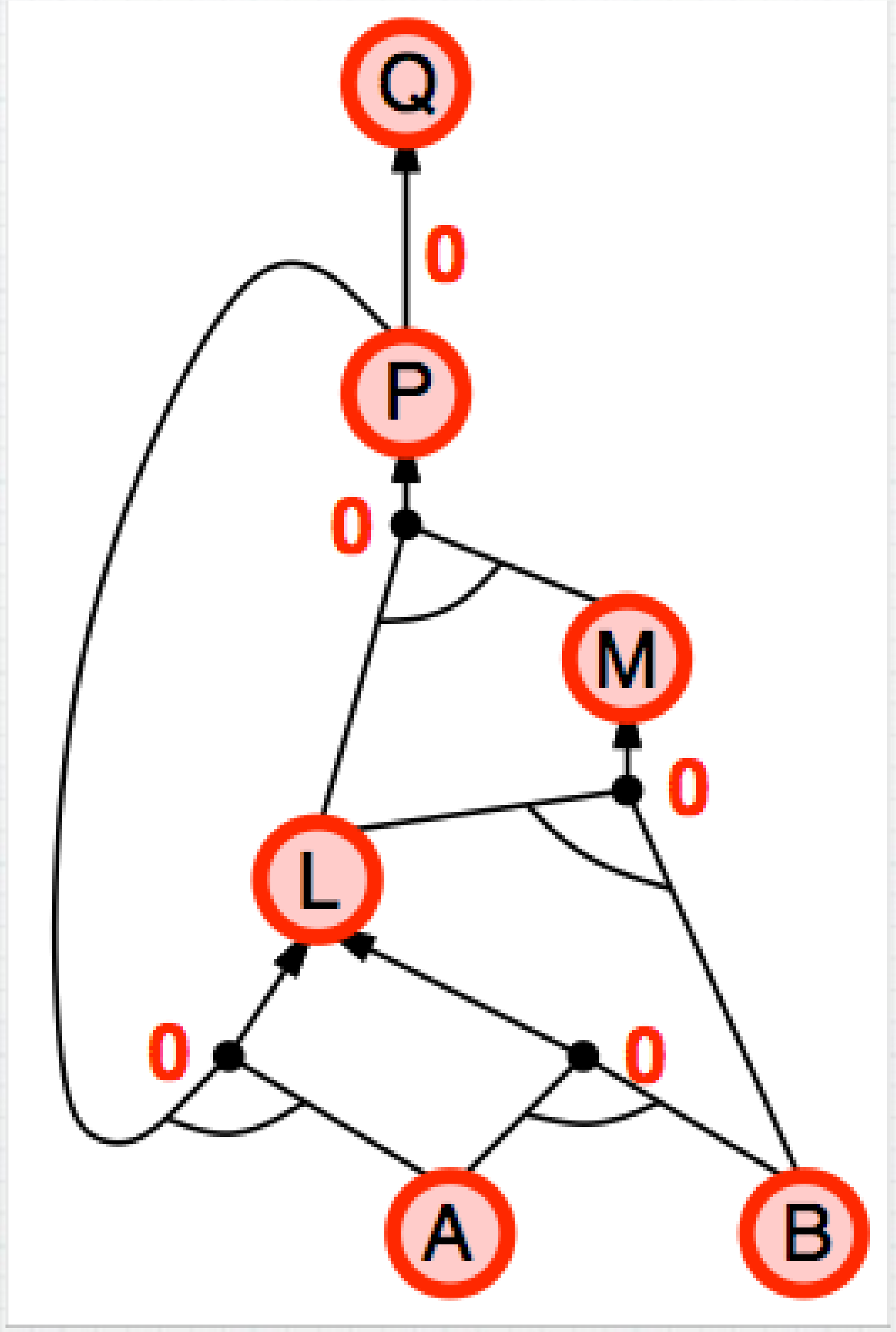

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Proof of Completeness

FC derives every atomic sentence that is entailed by KB

1. FC reaches a fixed point where no new atomic sentences are derived
2. Consider the final state as a model m, assigning true/false to symbols
3. Every clause in the original KB is true in m
   Proof: Suppose a clause $a_1 \wedge \ldots \wedge a_k \Rightarrow b$ is false in m
   Then $a_1 \wedge \ldots \wedge a_k$ is true in m and b is false in m
   Therefore the algorithm has not reached a fixed point!
4. Hence m is a model of KB
5. If $KB \models q$, q is true in **every** model of KB, including m

General idea: construct any model of KB by sound inference, check α

Backward Chaining

Idea: work backwards from the query q:

to prove q by BC,

check if q is known already, or

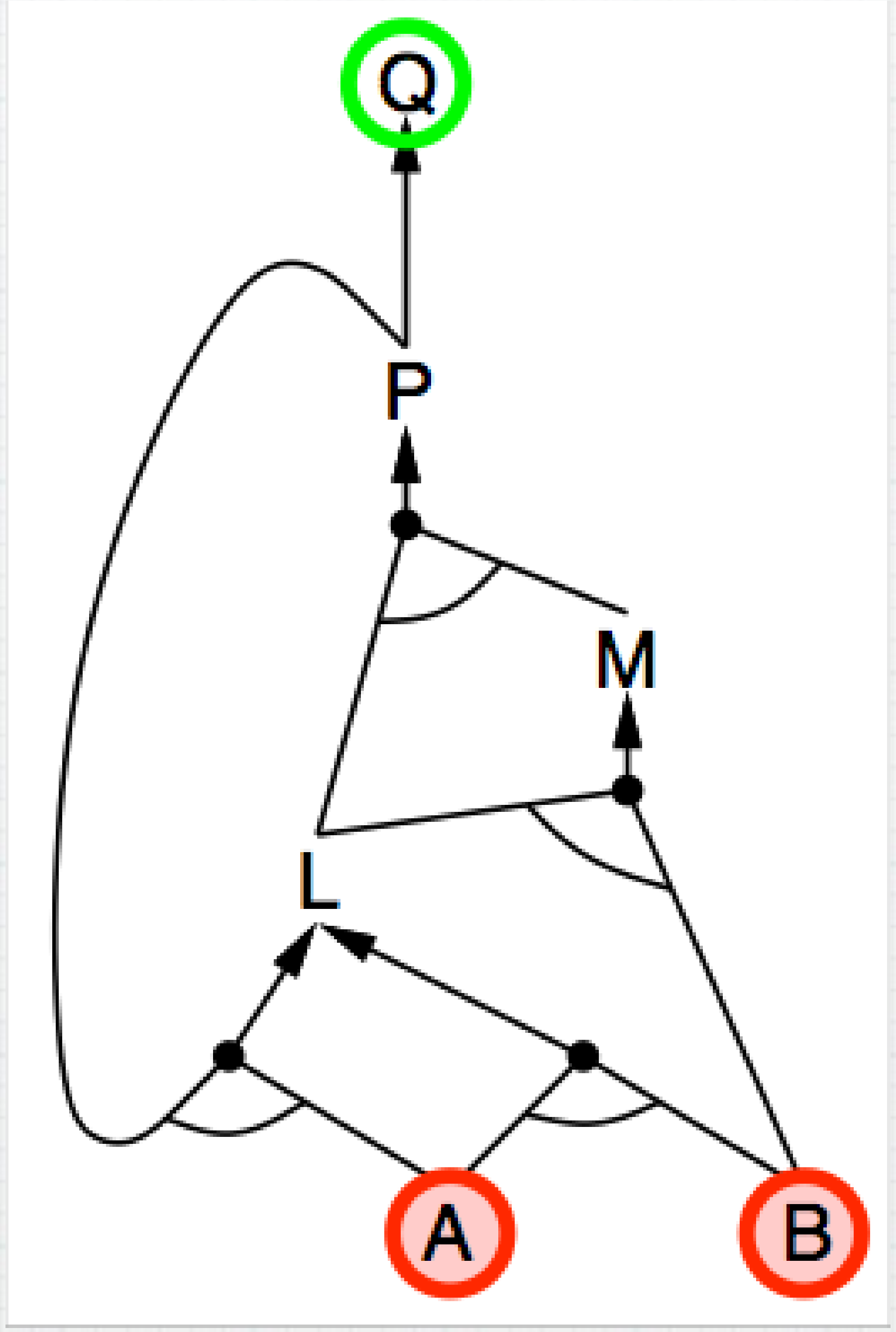
prove by BC all premises of some rule concluding q

Avoid loops: check if new subgoal is already on the goal stack

Avoid repeated work: check if new subgoal

1) has already been proved true, or
2) has already failed

Backward chaining example



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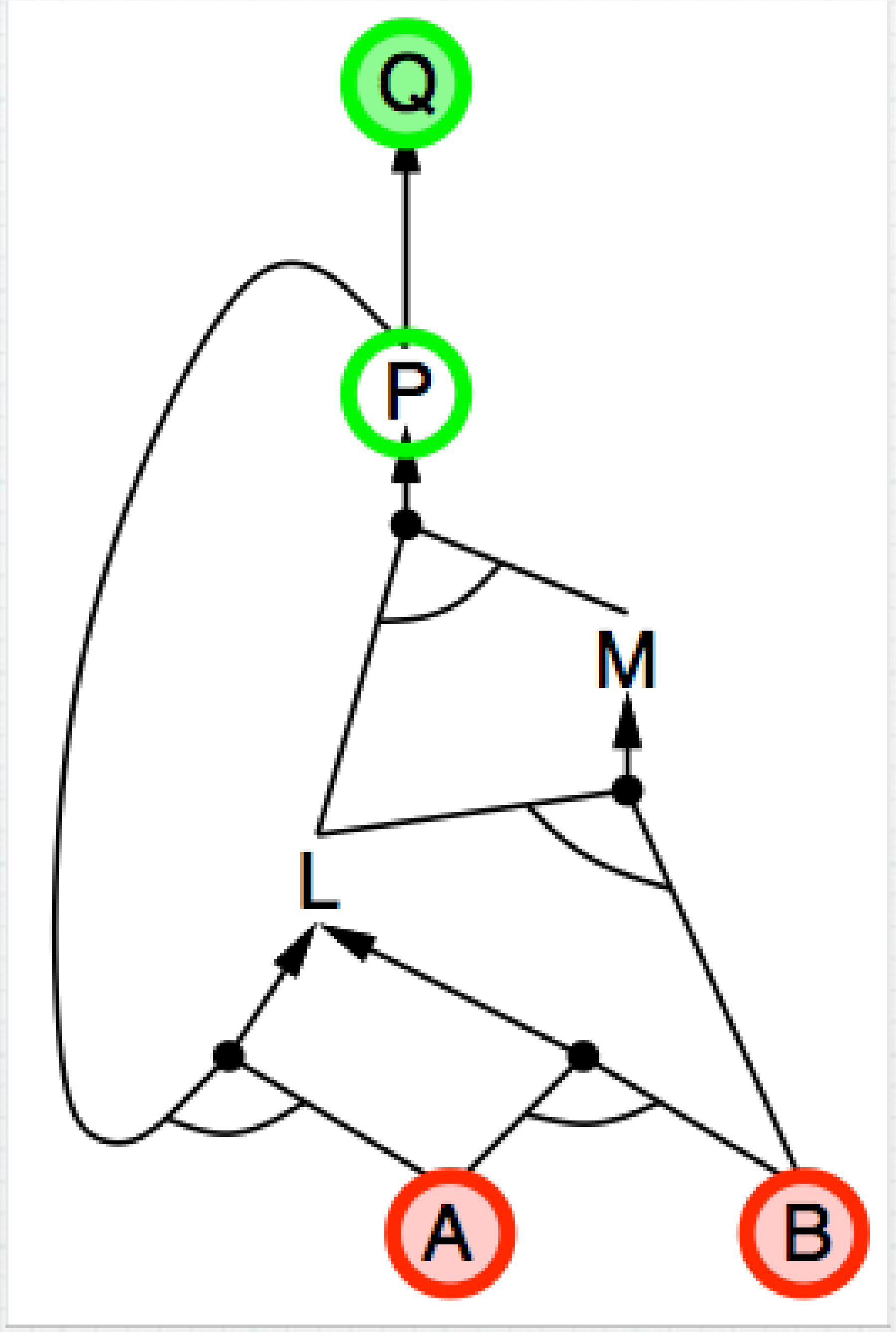

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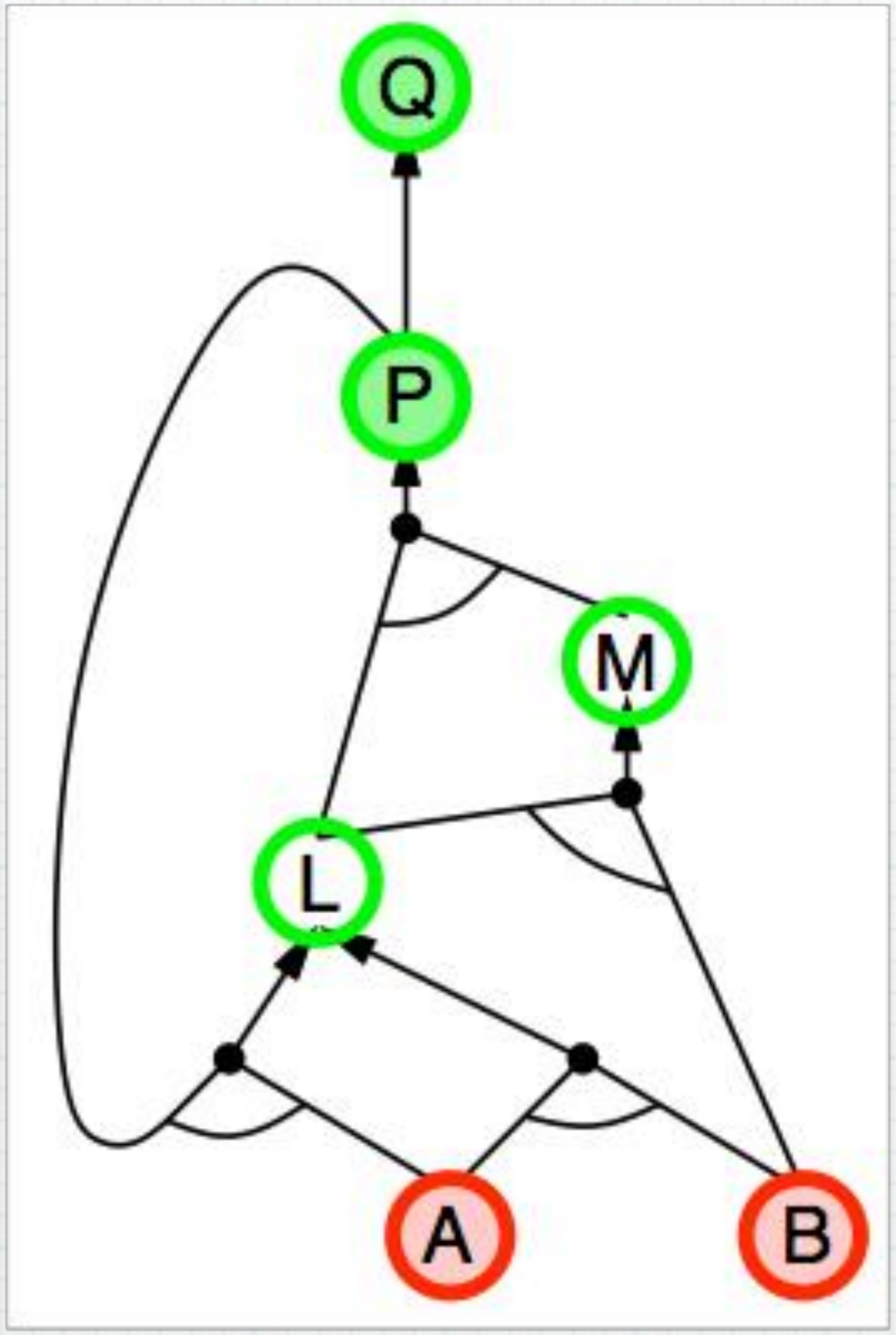

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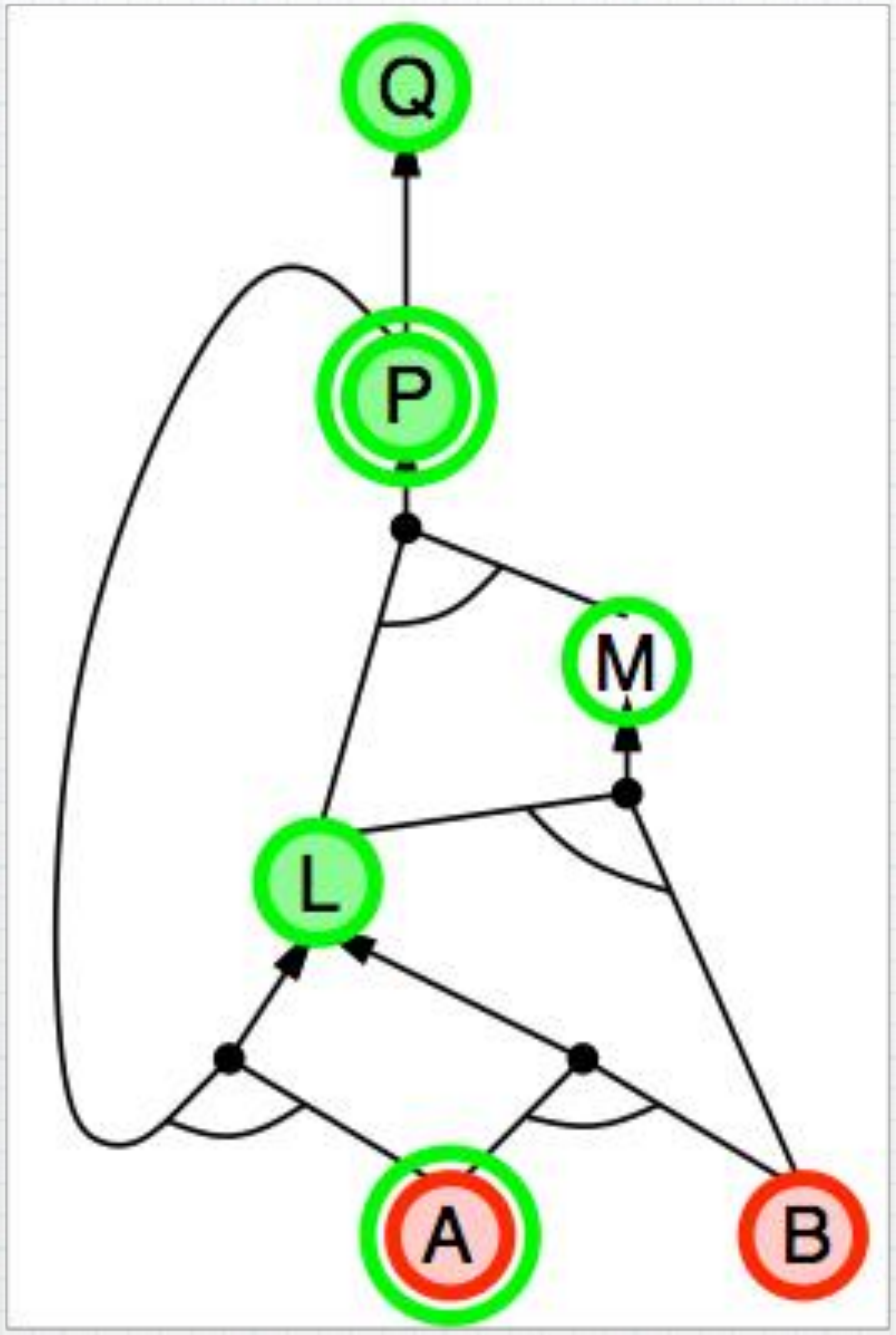

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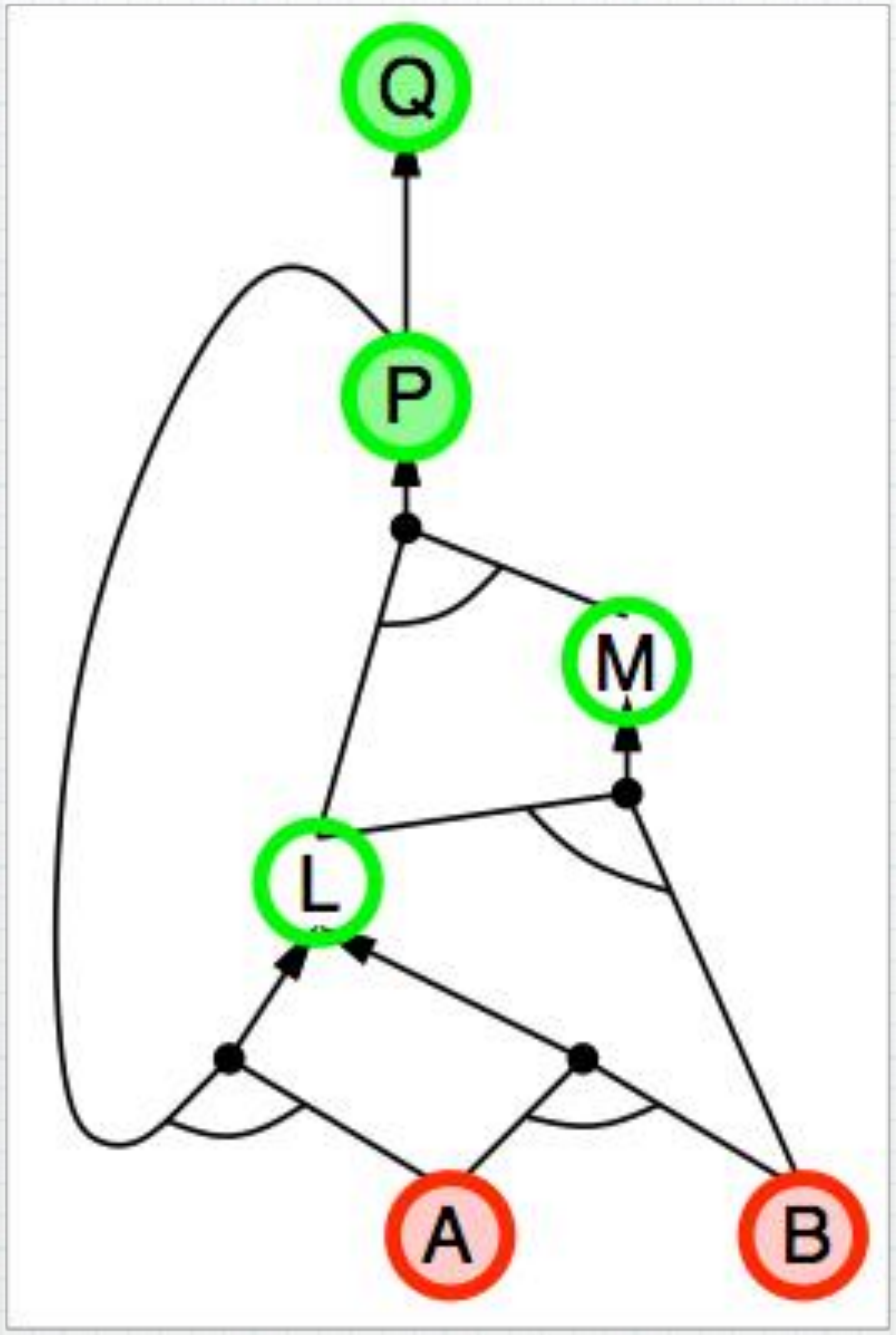

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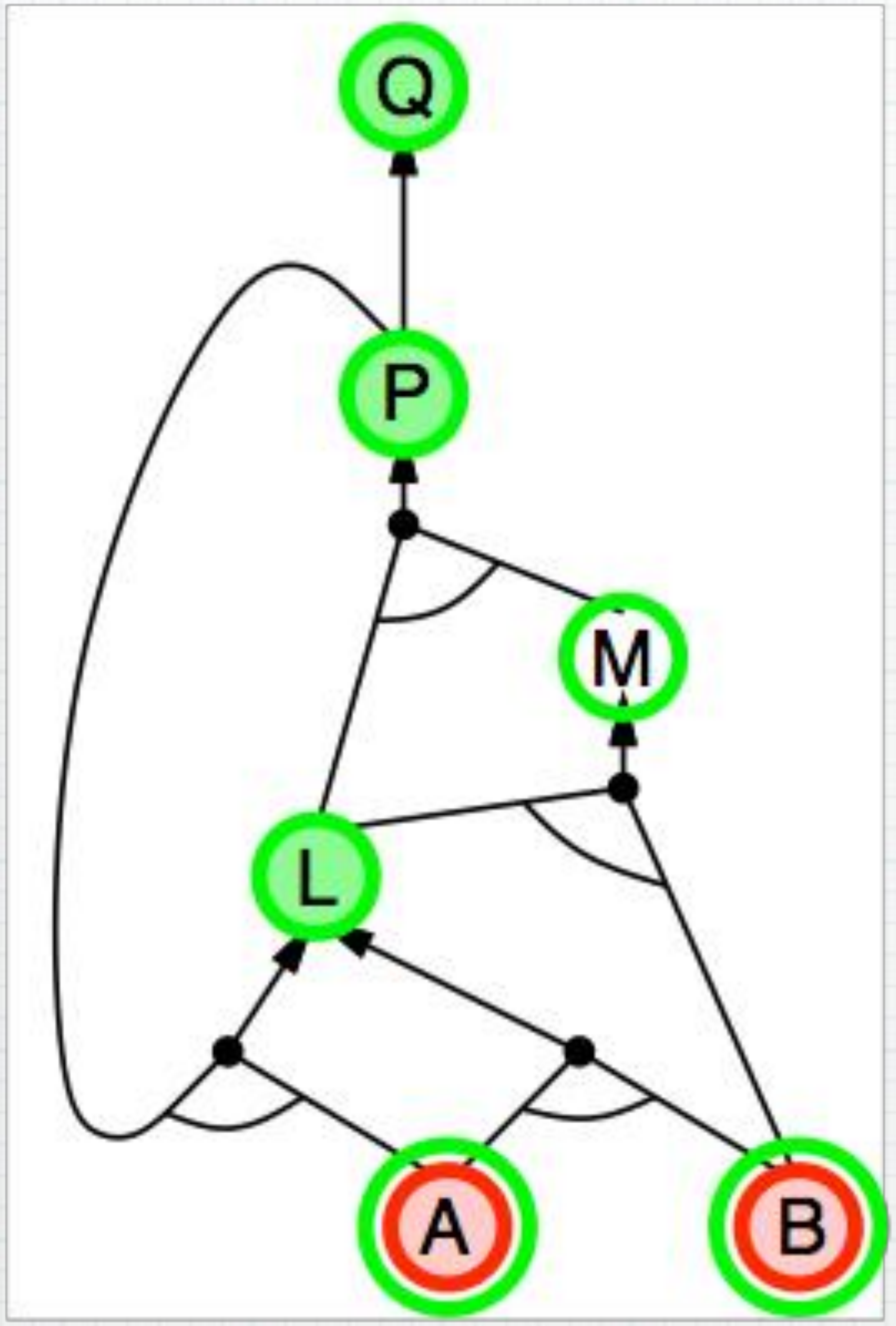

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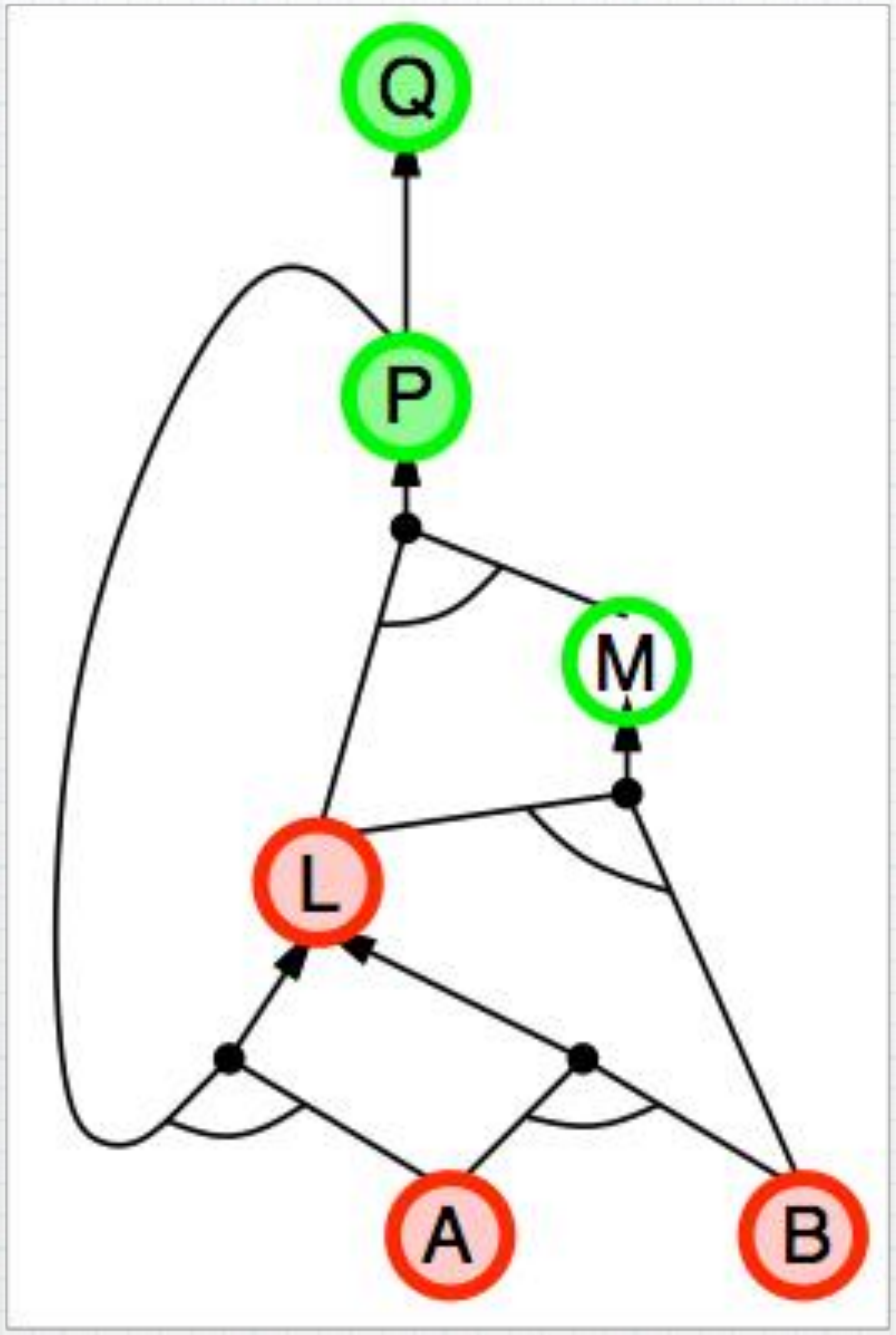

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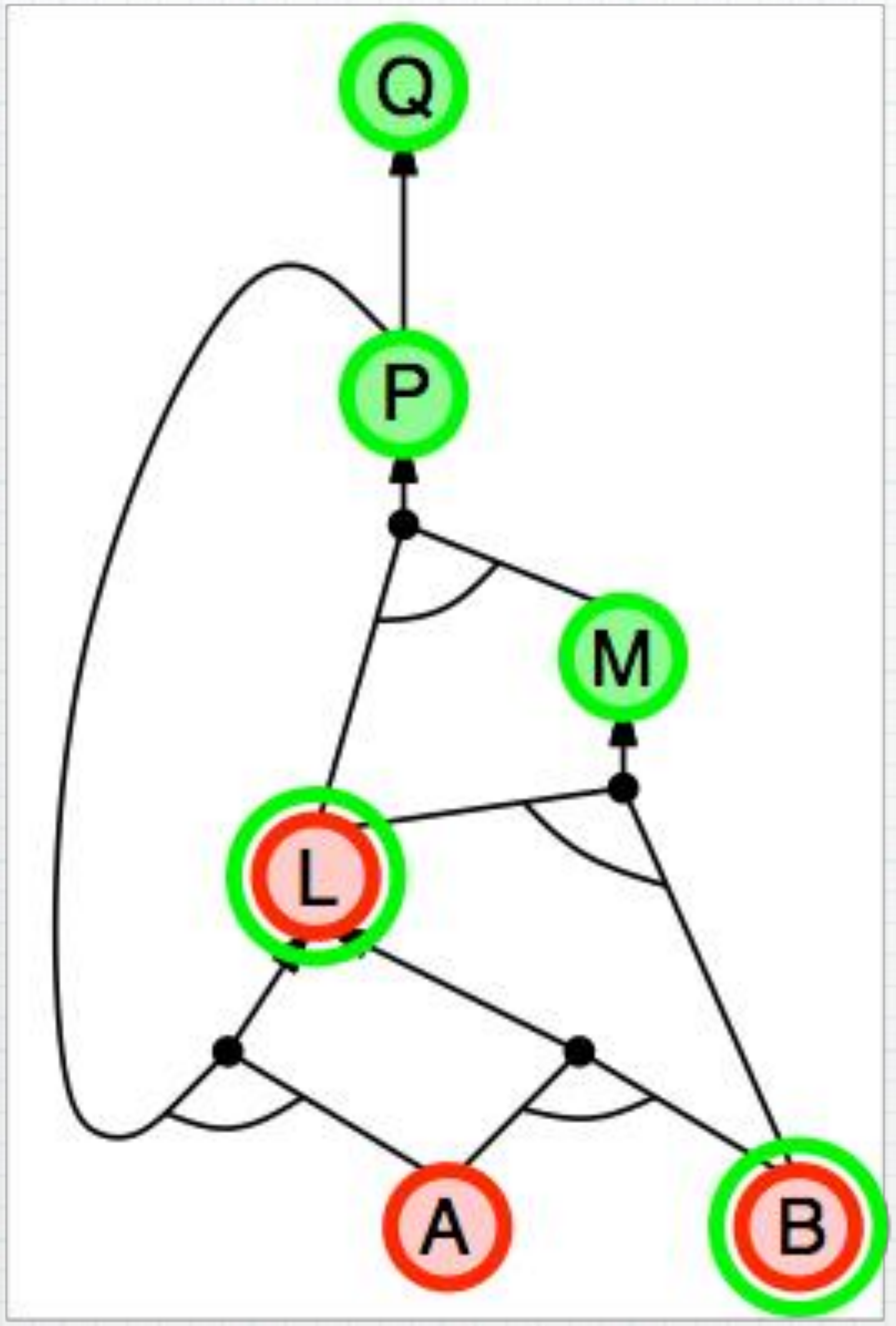

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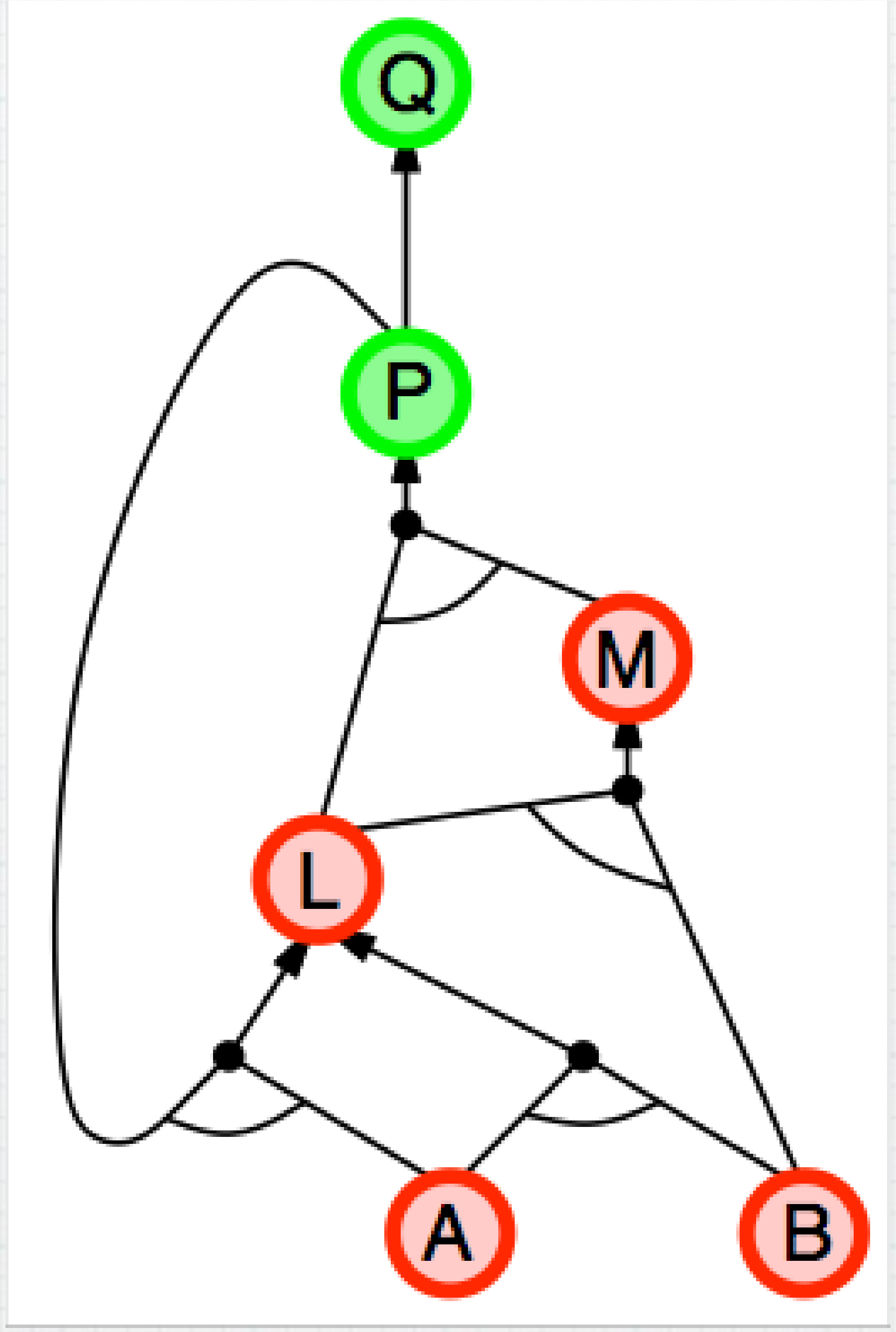

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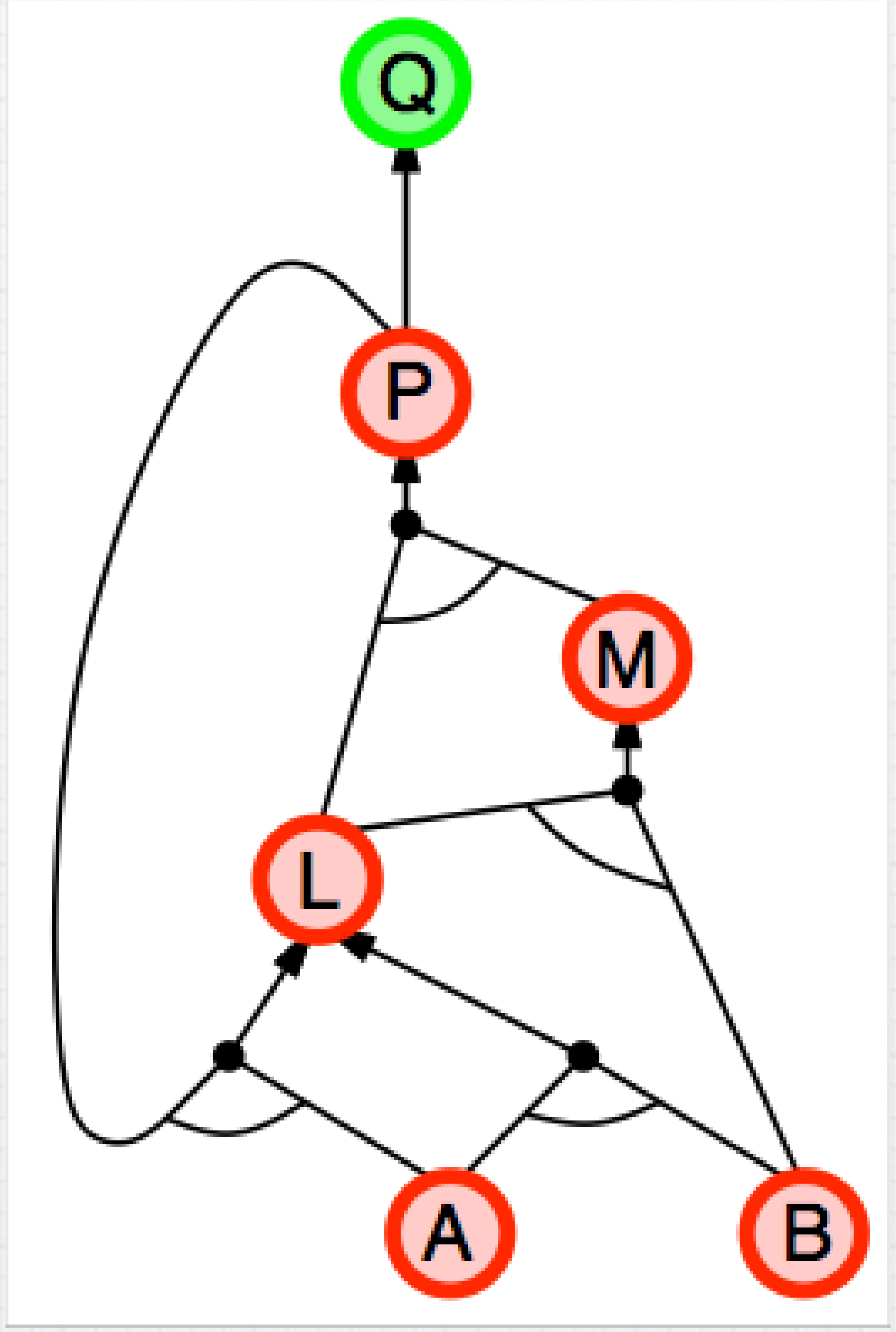

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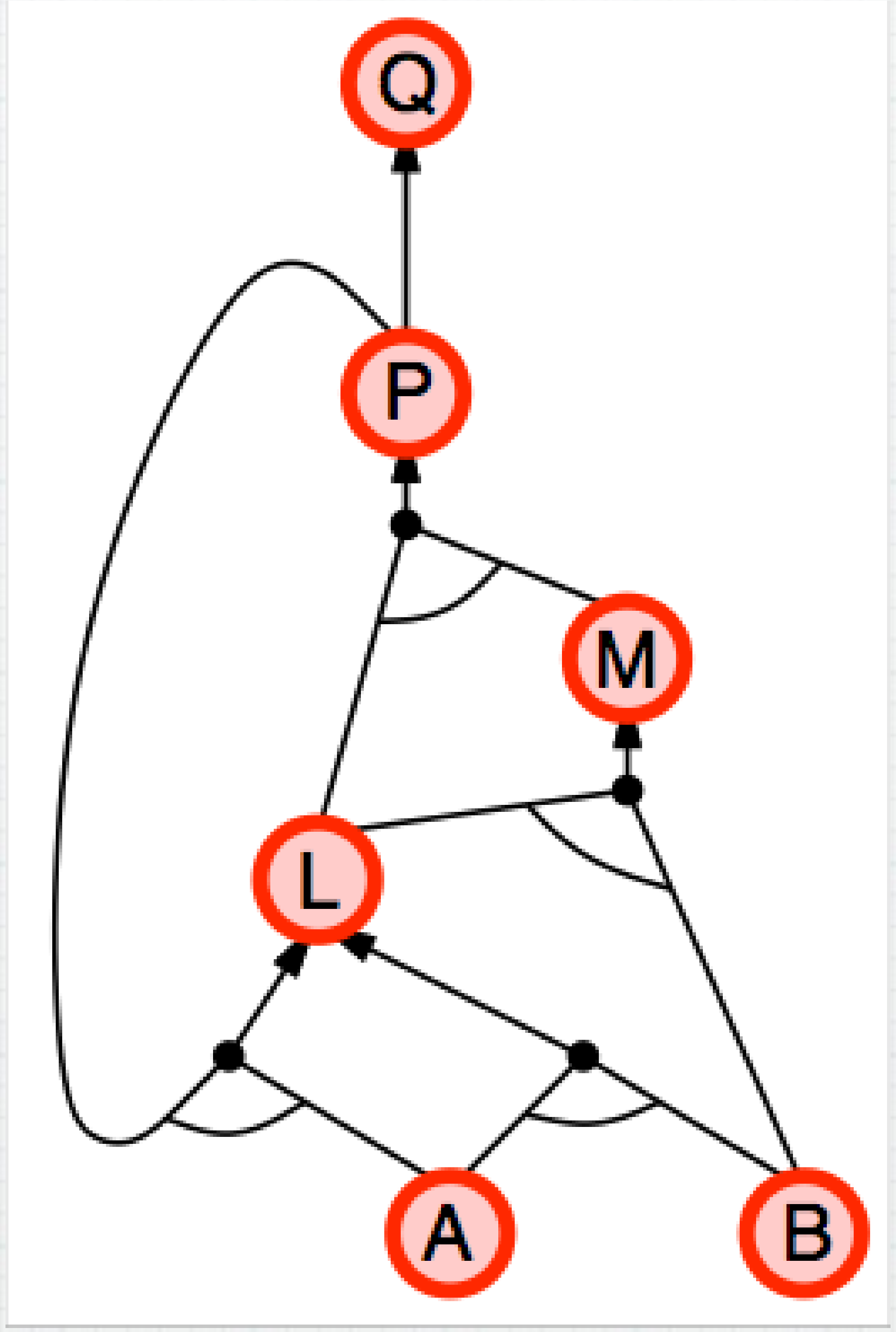


Forward vs. Backward Chaining

FC is data-driven, cf. automatic, unconscious processing,
e.g., object recognition, routine decisions

May do lots of work that is irrelevant to the goal

BC is goal-driven, appropriate for problem-solving,
e.g., Where are my keys? How do I get into a PhD program?

Complexity of BC can be **much less** than linear in size of KB

Inference Algorithms: Resolution

Resolution

Conjunctive Normal Form (CNF—universal)

$\textbf{conjunction}$ of $\underbrace{\textbf{disjunctions} \text{ of } \textbf{literals}}_{\textbf{clauses}}$

E.g., $(A \lor \neg B) \land (B \lor \neg C \lor \neg D)$

Resolution

Conjunctive Normal Form (CNF—universal)

conjunction of $\underbrace{\textbf{disjunctions} \text{ of } \textbf{literals}}_{\textbf{clauses}}$

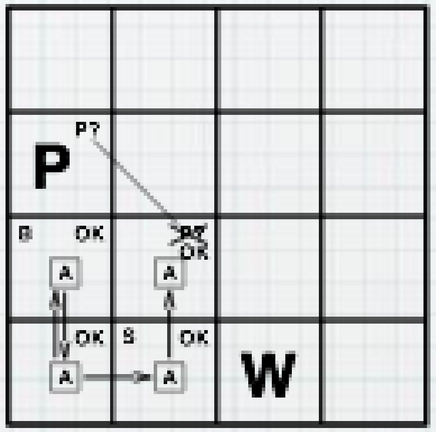
E.g., $(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$

Resolution inference rule (for CNF): complete for propositional logic

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n}$$

where ℓ_i and m_j are complementary literals. E.g.,

$$\frac{P_{1,3} \vee P_{2,2}, \qquad \neg P_{2,2}}{P_{1,3}}$$



Resolution is sound and complete for propositional logic

Resolution Algorithm

Proof by contradiction, i.e., show $KB \wedge \neg\alpha$ unsatisfiable

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```
function PL-RESOLUTION(KB, α) returns true or false
    inputs: KB, the knowledge base, a sentence in propositional logic
            α, the query, a sentence in propositional logic
    clauses ← the set of clauses in the CNF representation of KB ∧ ¬α
    new ← { }
    loop do
        for each Ci, Cj in clauses do
            resolvents ← PL-RESOLVE(Ci, Cj)
            if resolvents contains the empty clause then return true
            new ← new ∪ resolvents
        if new ⊆ clauses then return false
        clauses ← clauses ∪ new
```

Resolution Example

$KB = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$ $\alpha = \neg P_{1,2}$

$\boxed{\neg P_{2,1} \vee B_{1,1}}$ $\boxed{\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}}$ $\boxed{\neg P_{1,2} \vee B_{1,1}}$ $\boxed{\neg B_{1,1}}$ $\boxed{P_{1,2}}$

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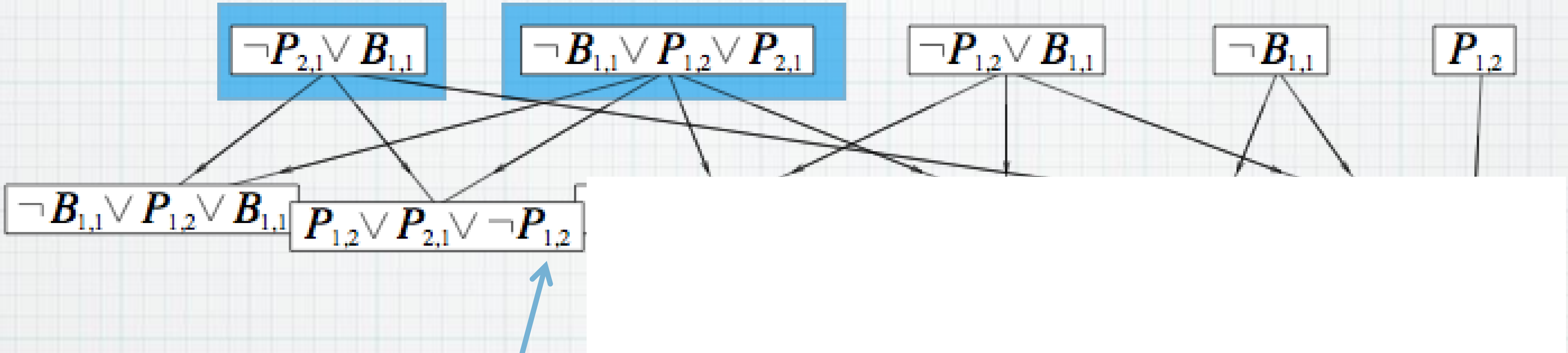

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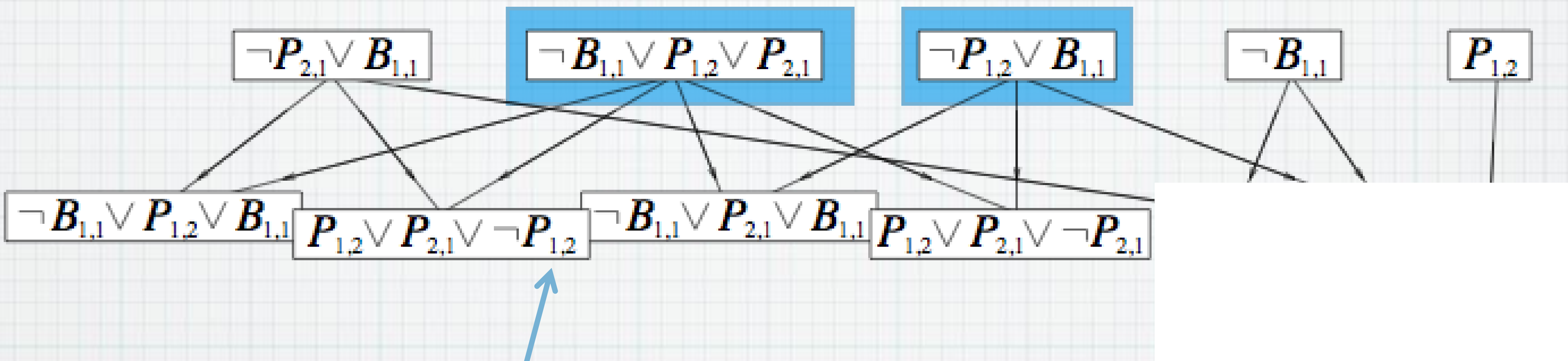

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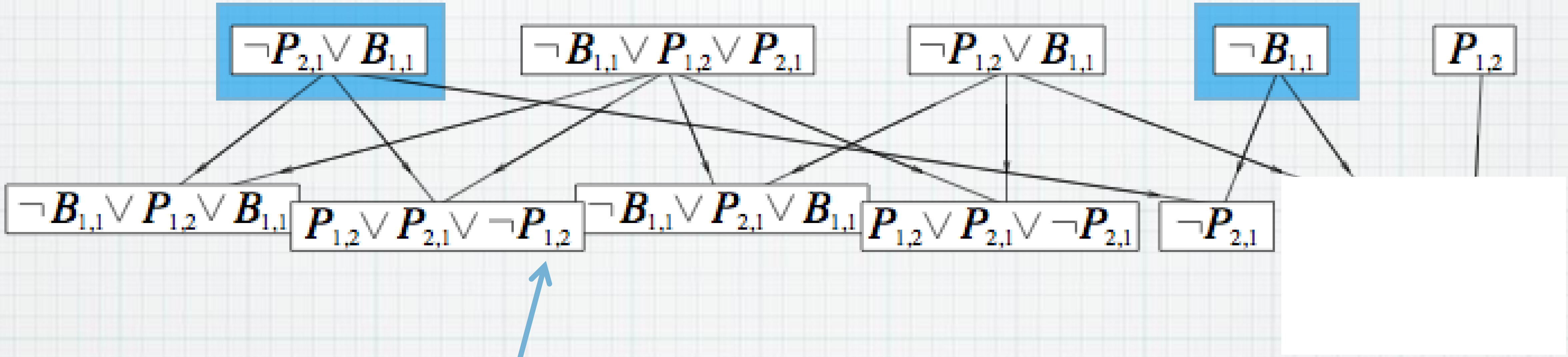

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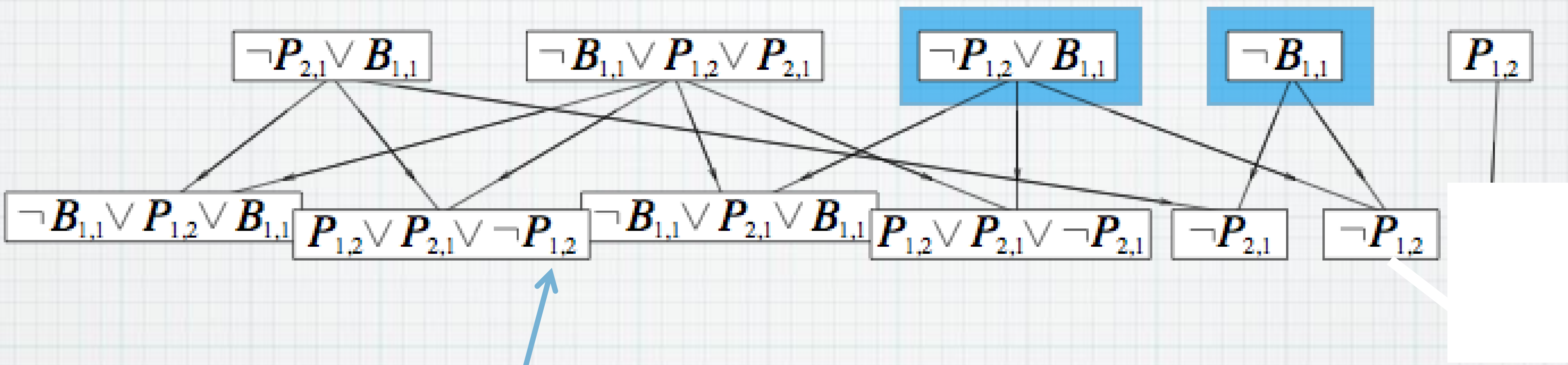

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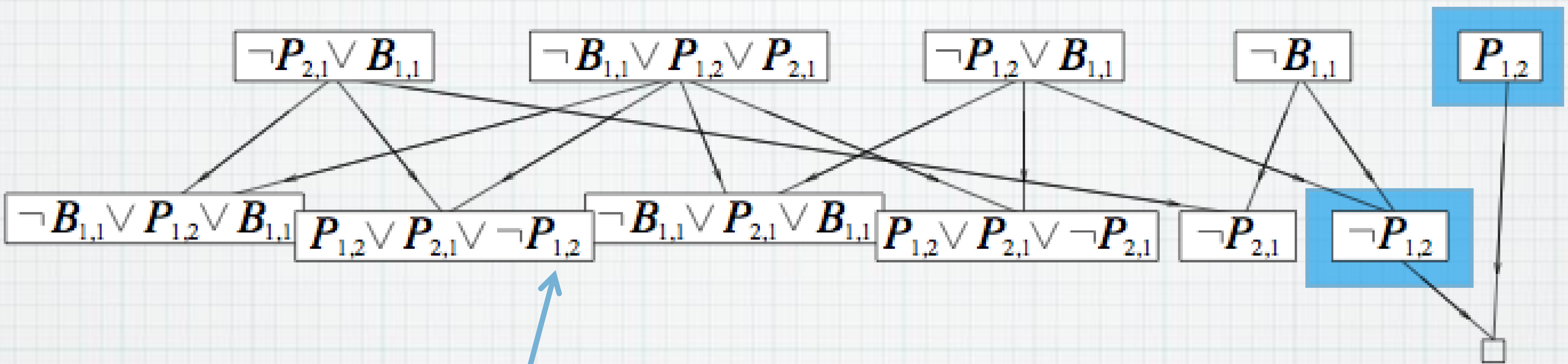

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Contradiction, therefore our sentence is entailed.