Bayes Nets

Chapter 14

Overview

* Bayes Nets (Graphical Models)
  * Syntax, Semantics
  * How to compactly represent Joint Distributions
  * How to efficiently do inference
* Dynamic Bayes Nets
  * How to adapt BN to reason over time
  * Markov Models, Hidden MM, Particle Filters
* Project 3 will use these concepts!

Conditional Independence

- Conditional Independence is our most basic and robust knowledge about uncertain environments

$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

$$X \perp\!\!\!\perp Y | Z$$

$$\forall x, y, z : P(x|z, y) = P(x|z)$$

- What about this domain:
  - Traffic
  - Need an Umbrella
  - Raining

Bayes Nets

* Two problems with full joint distribution tables for prob. models
  * gets WAY too big
  * awkward to specify joint prob for more than a few vars
* Bayes Nets are a technique for describing complex joint distributions with simple local distributions (Conditional Probabilities)

The Chain Rule

Remember....the definition of conditional probability, also called the Product Rule:

$$P(a \wedge b) = P(a \mid b)\, P(a)$$

$P(x_1,\ldots, x_n) = P(x_n \mid x_{n-1},\ldots, x_1)\, P(x_{n-1},\ldots, x_1)$

$P(x_1,\ldots, x_n) = P(x_n \mid x_{n-1},\ldots,x_1)\, P(x_{n-1} \mid x_{n-2},\ldots,x_1)\, P(x_{n-2},\ldots,x_1)$

$P(x_1,\ldots, x_n) = P(x_n \mid x_{n-1},\ldots, x_1)\, P(x_{n-1} \mid x_{n-2},\ldots, x_1) \ldots P(x_2 \mid x_1)\, P(x_1)$

$$P(x_1,\ldots, x_n) = \prod_{i=1}^{n} P(x_i \mid x_{i-1},\ldots, x_1)$$

Chain rule is the product rule applied multiple times, turning a joint probability into conditional probabilities

Traffic,Rain,Umbrella

* Trivial decomposition

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) =$$
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain}, \text{Traffic})$$

* Conditional Independence assumptions

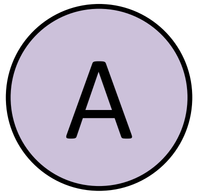
$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) =$$
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain})$$

* Bayes Nets let us compactly express conditional independence

Example: Factorization and Graph

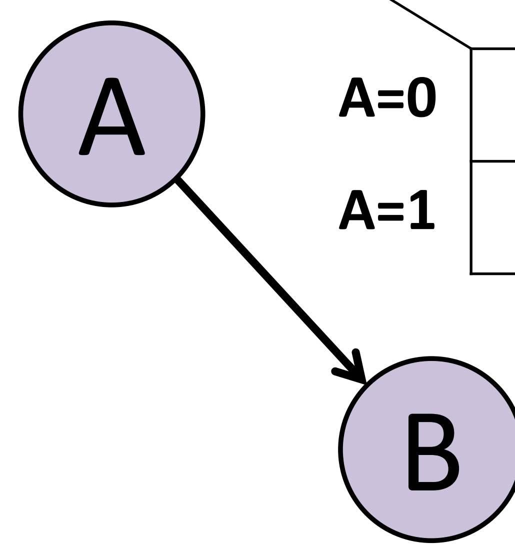
(chain rule of probability)

$$P(A,B,C,D) = P(D \mid C,B,A)P(C \mid B,A)P(B \mid A)\boxed{P(A)}$$



Example: Factorization and Graph

$$P(A,B,C,D) = P(D \mid C,B,A)P(C \mid B,A)\boxed{P(B \mid A)}P(A)$$

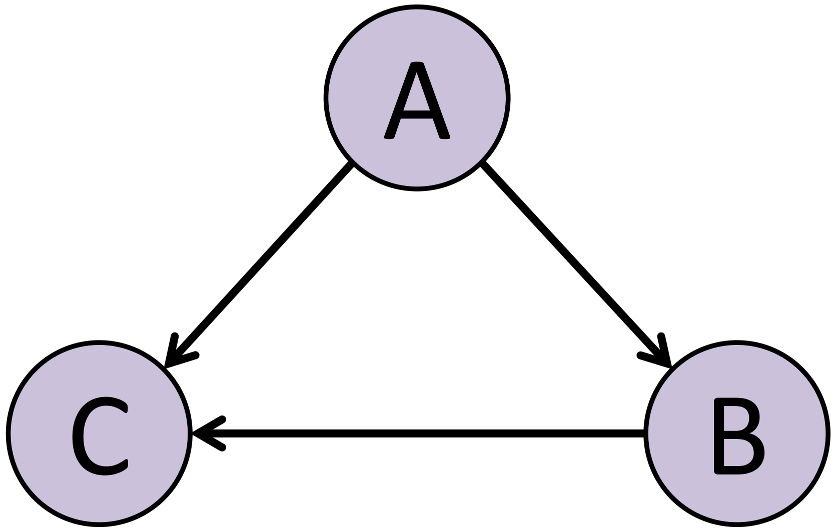


	B=0	B=1
A=0	0.4	0.6
A=1	0.7	0.3

Conditional Probability Table (CPT)

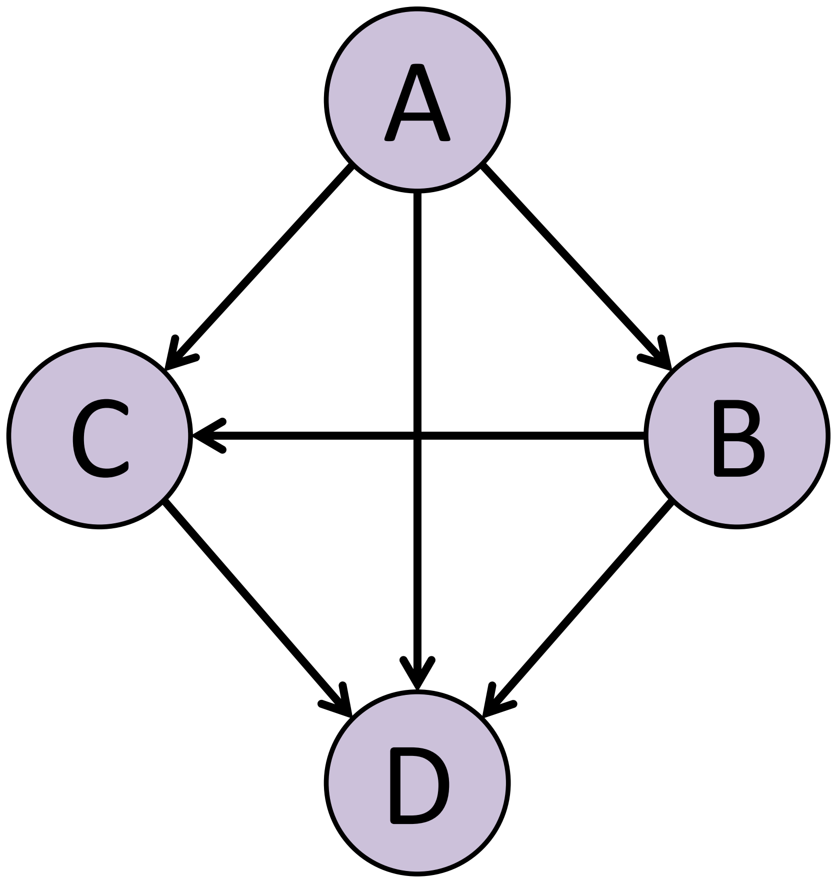
Example: Factorization and Graph

$$P(A, B, C, D) = P(D \mid C, B, A)\boxed{P(C \mid B, A)}P(B \mid A)P(A)$$



Example: Factorization and Graph

$$P(A,B,C,D) = \boxed{P(D \mid C,B,A)}P(C \mid B,A)P(B \mid A)P(A)$$



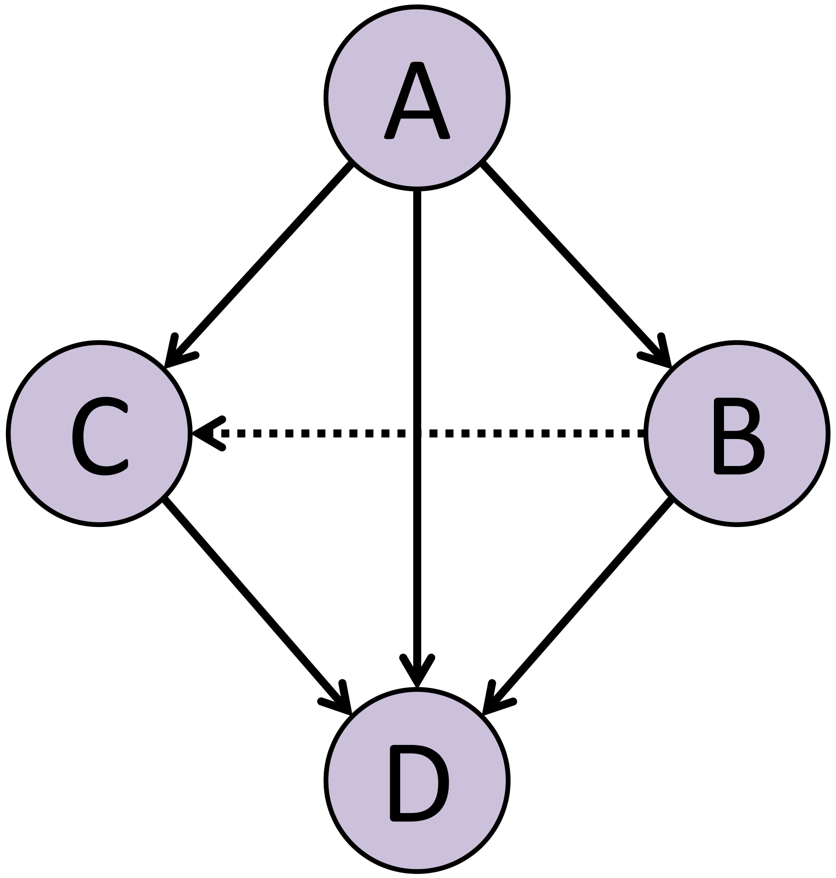
Complete graph

Example: Conditional Independencies

$$P(A,B,C,D) = P(D \mid C,B,A)\cancel{P(C \mid B,A)}P(B \mid A)P(A)$$

$$P(C \mid A)$$

$$\frac{I_\ell(P)}{C \perp B \mid A}$$



Example: Conditional Independencies

$$P(A, B, C, D) = P(D \mid C, B, A)\cancel{P(C \mid B, A)}P(B \mid A)P(A)$$

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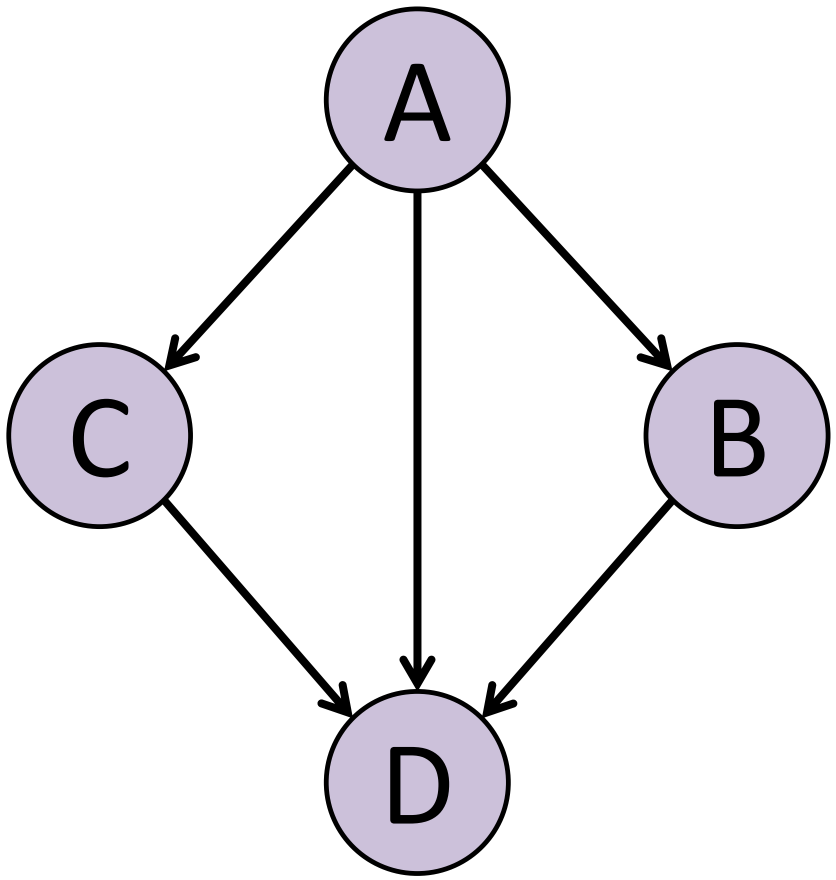
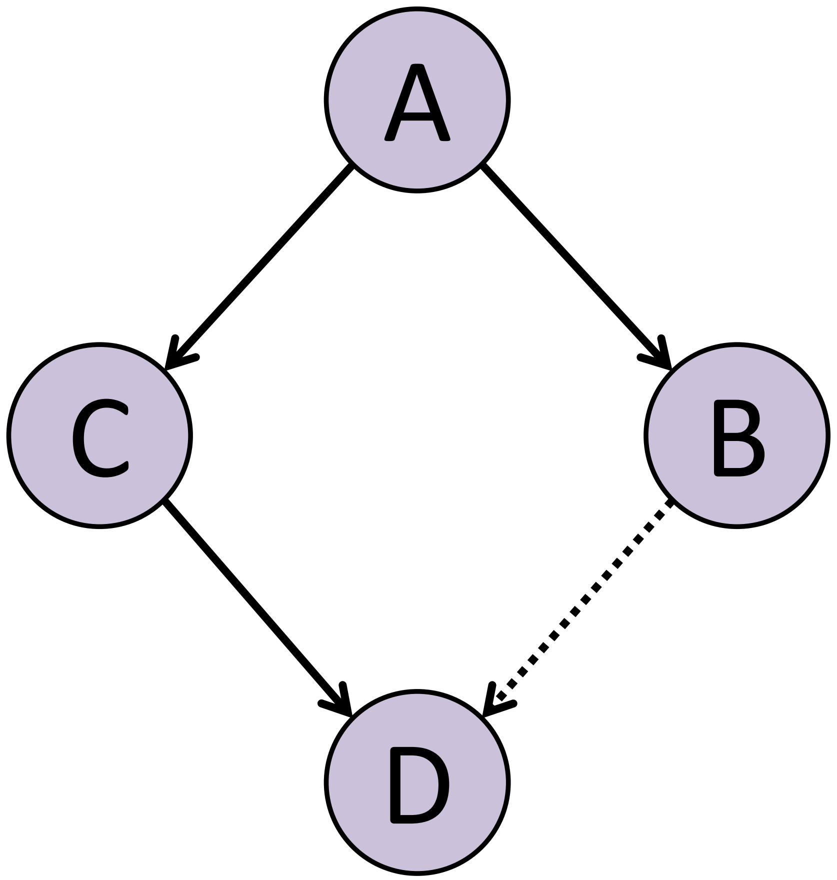


Example: Conditional Independencies

$$P(A,B,C,D) = \cancel{P(D \mid C,B,A)}\,\cancel{P(C \mid B,A)}\,P(B \mid A)P(A)$$

$$P(D \mid C) \qquad P(C \mid A)$$

$$\frac{I_\ell(P)}{\begin{array}{l} C \perp B \mid A \\ D \perp \{A,B\} \mid C \end{array}}$$



Example: Conditional Independencies

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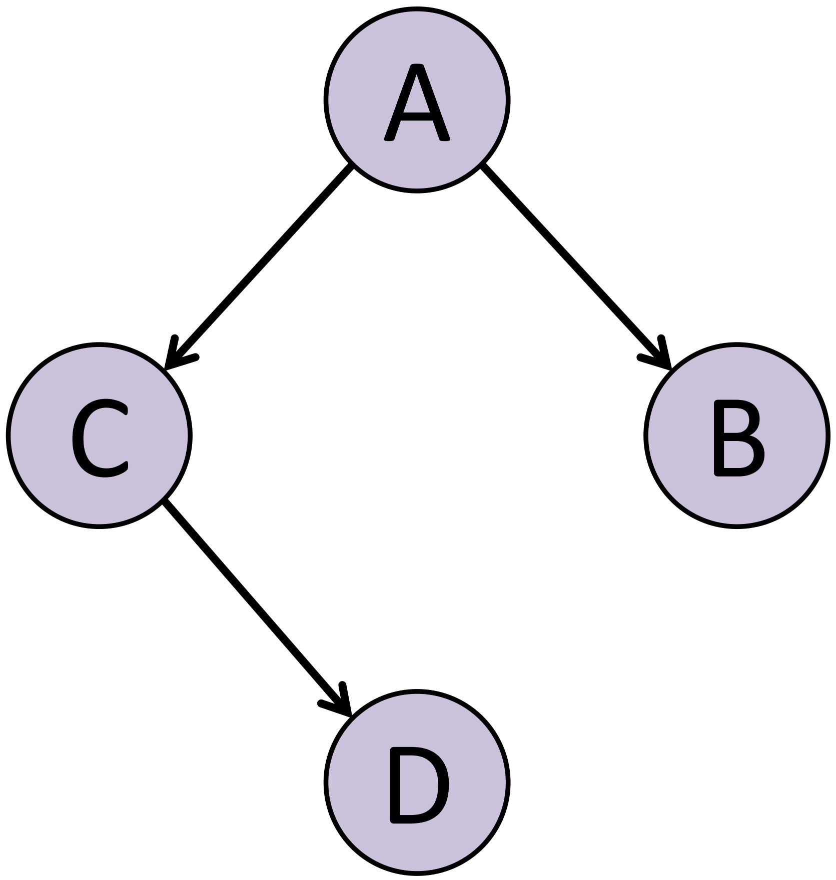
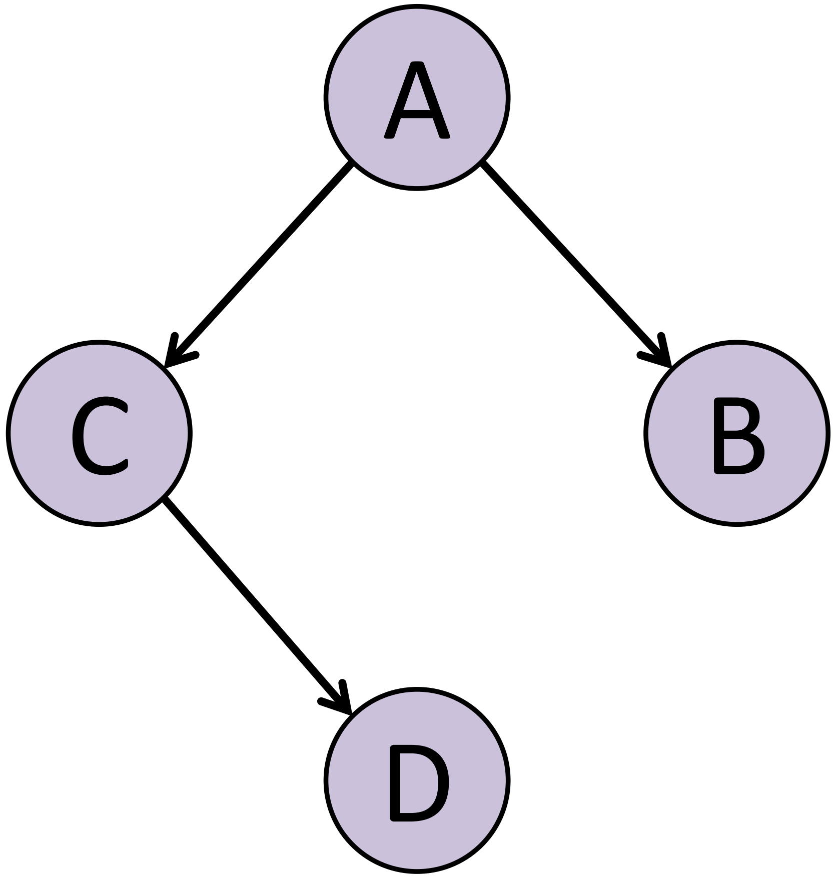


Chain Rule of Bayesian Networks

$$P(A,B,C,D) = P(D \mid C)P(C \mid A)P(B \mid A)P(A)$$

$$\frac{I_\ell(P)}{\begin{array}{l} C \perp B \mid A \\ D \perp \{A,B\} \mid C \end{array}}$$

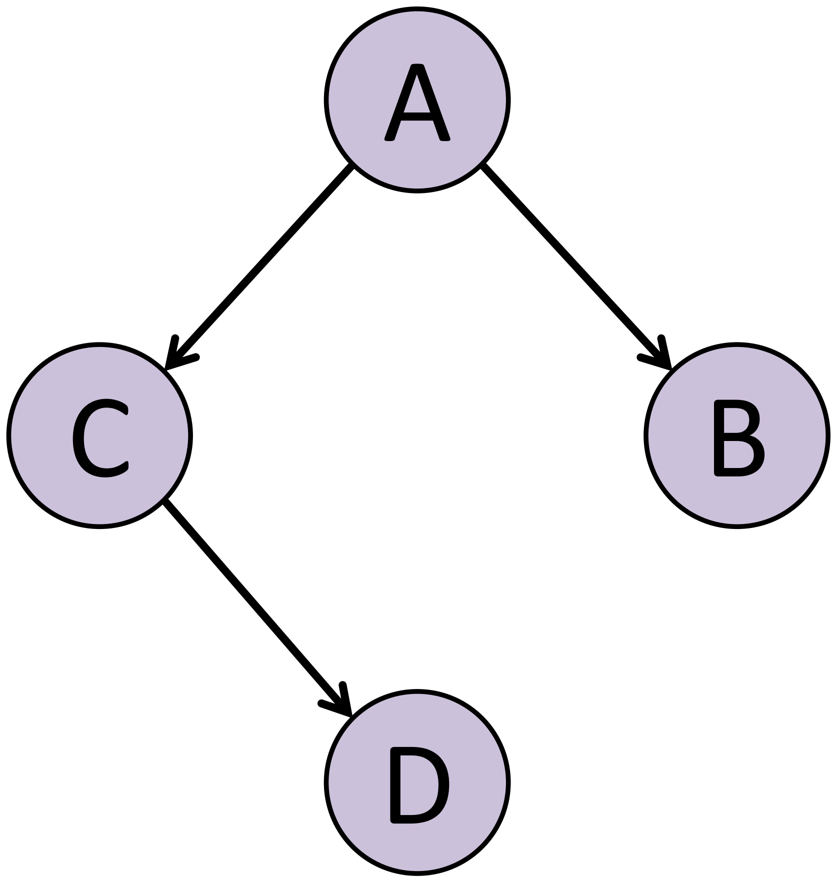


Chain Rule of Bayesian Networks

$$P(A, B, C, D) = P(D \mid C) P(C \mid A) P(B \mid A) P(A)$$

In general: $$P(X) = \prod_{i=1}^{N} P(x_i \mid \mathrm{pa}(x_i))$$ *Chain rule of Bayes nets*

$I_\ell(P)$
$C \perp B \mid A$
$D \perp \{A, B\} \mid C$



Chain Rule of Bayesian Networks

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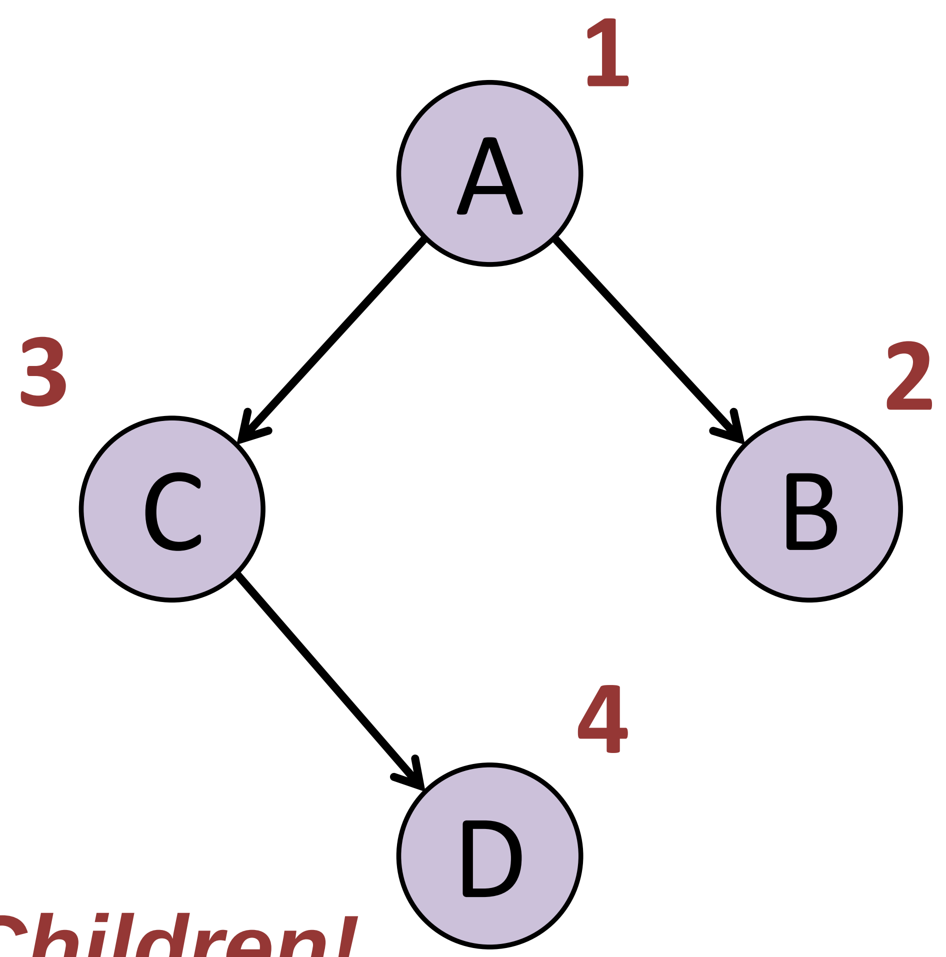
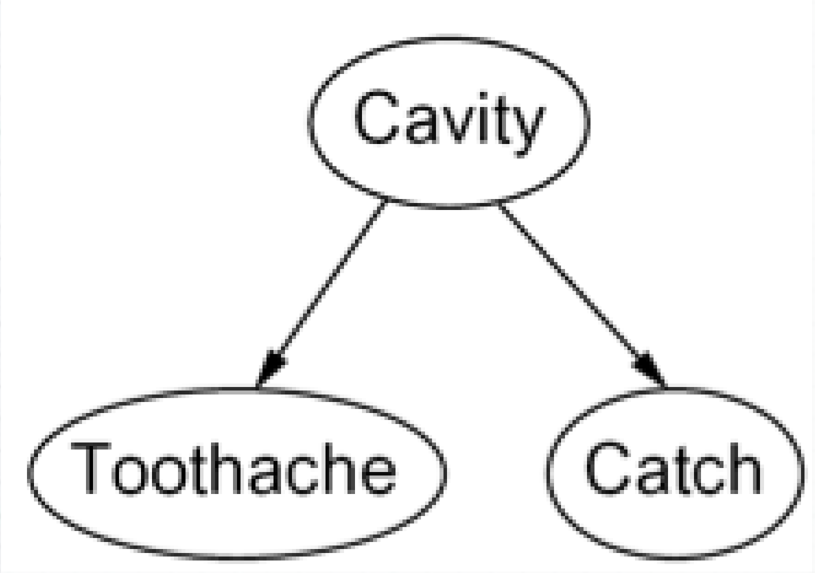


Topological Order:
A, B, C, D
Parents come before Children!

Variable Elimination Example

Bayes Net Notation

- Nodes: variables (with domains)
- Arcs: interactions
  - Directional
  - “Direct Influence” between vars
  - Formally: conditional indep.



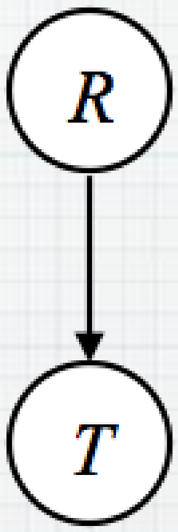
Example: Coin Flips

- N independent flips

- No interactions between variables

Example: Traffic

* Variables:
  * R: It rains
  * T: There’s traffic



* Model 1: independence
* Model 2: rain causes traffic
* Which is better for an agent to use?

Example: Traffic 2

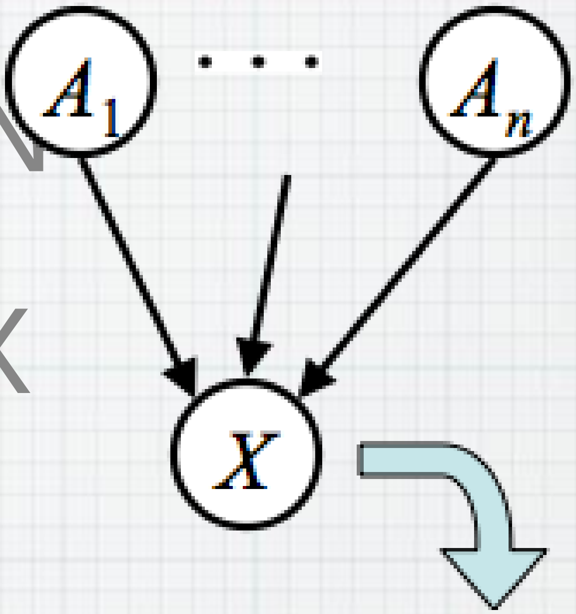
* Let’s build a causal graphical model
* Variables:
  * T: Traffic
  * R: It rains
  * L: Low air pressure
  * D: Roof drips
  * B: Ballgame
  * C: Cavity

Example: Burglar Alarm

- Variables:
  - B: Burglary
  - A: Alarm goes off
  - M: Mary calls
  - J: John calls
  - E: Earthquake

Bayes' Net Semantics

* Formalizing the semantics of a BN
* A set of nodes, one per variable X
* A directed, acyclic graph
* A conditional distribution for each node
  * Local conditional prob tables (CPT)
  * P(X | parent nodes)

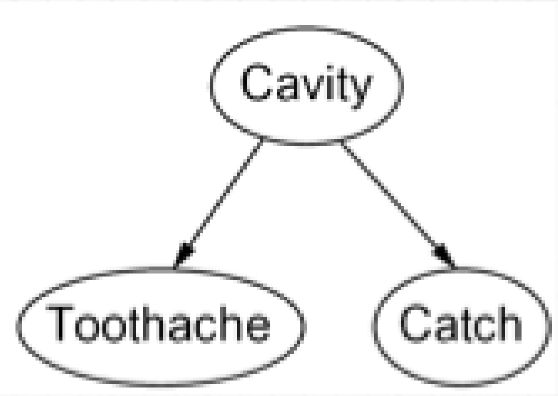


$P(X|A_1 \ldots A_n)$

Bayes' Net = Graph Topology + CPTs

Probabilities in BNs

* Bayes' nets implicitly encode joint distributions
  * As a product of local cond. distrib



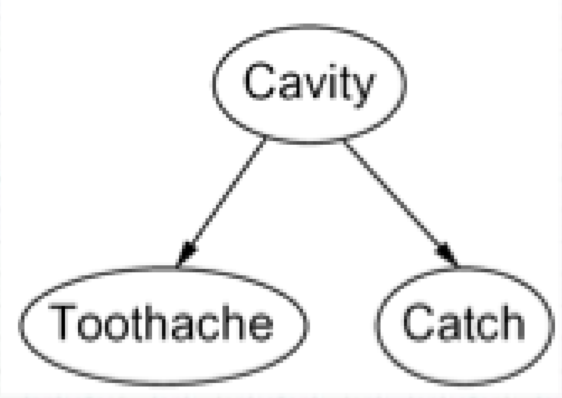
  * Can multiply all relevant conditionals to get any full joint

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

* Example: $P(+cavity, +catch, \neg toothache)$
* This let's us construct any entry in the full joint distribution table!

Probabilities in BNs

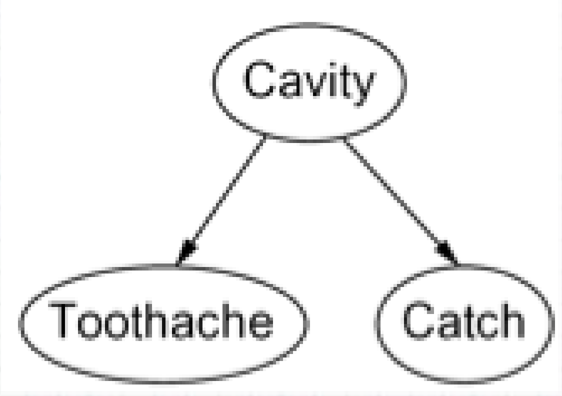
Bayes Net:
Structure + CPTs



P(Cavity)
P(Toothache | Cavity)
P(Catch | Cavity)

Probabilities in BNs

Bayes Net:
Structure + CPTs



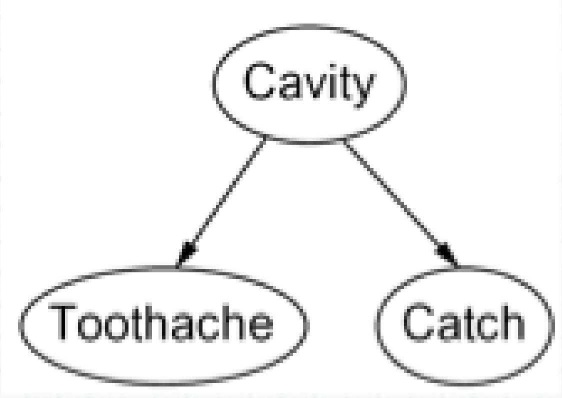
P(Cavity)
P(Toothache | Cavity)
P(Catch | Cavity)

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

P(catch, cavity, ¬toothache) =
P(cavity)

Probabilities in BNs

Bayes Net:
Structure + CPTs



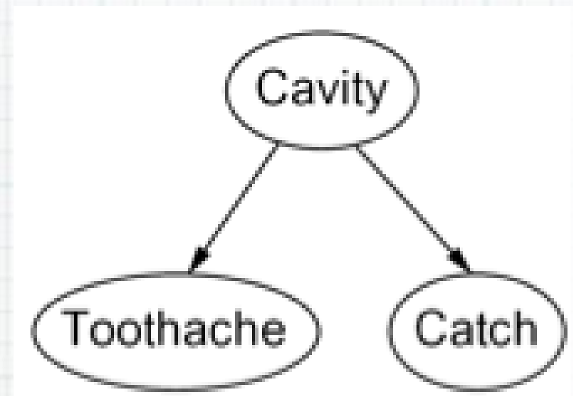
P(Cavity)
P(Toothache | Cavity)
P(Catch | Cavity)

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

P(catch, cavity, ¬toothache) =
P(cavity) P(¬toothache | cavity)

Probabilities in BNs

Bayes Net:
Structure + CPTs



P(Cavity)
P(Toothache | Cavity)
P(Catch | Cavity)

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

P(catch, cavity, ¬toothache) =
P(cavity) P(¬toothache | cavity) P(catch | cavity)

Example: Coin Flips

- N independent flips

$P(X_1)$

h	0.5
t	0.5

$P(X_2)$

h	0.5
t	0.5

$P(X_n)$

h	0.5
t	0.5

- P(h,h,t,h) = .5 * .5 * .5 * .5

Example: Traffic

* $P(r, \neg t) = P(r) * P(\neg t \mid r)$

  $1/4 * 1/4$

P(R)

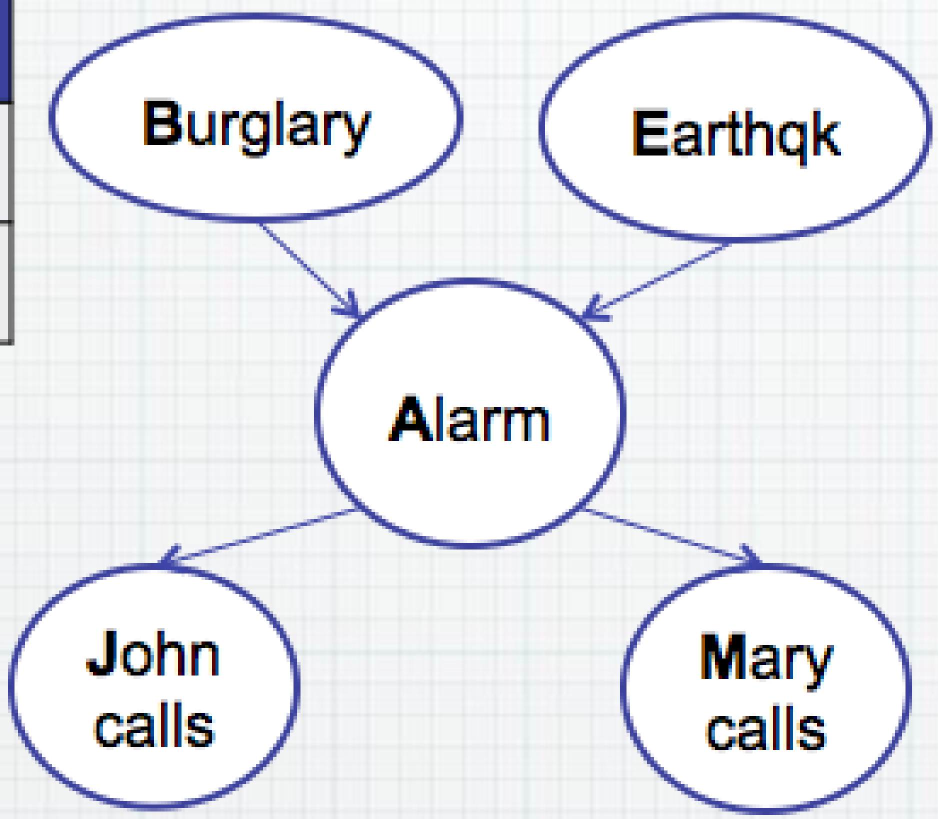
r	1/4
¬r	3/4

P(T | R)

r	t	3/4
	¬t	1/4
¬r	t	1/2
	¬t	1/2

Example CTPs: Alarm

B	P(B)
+b	0.001
¬b	0.999



E	P(E)
+e	0.002
¬e	0.998

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	¬a	0.05
+b	¬e	+a	0.94
+b	¬e	¬a	0.06
¬b	+e	+a	0.29
¬b	+e	¬a	0.71
¬b	¬e	+a	0.001
¬b	¬e	¬a	0.999

A	J	P(J|A)
+a	+j	0.9
+a	¬j	0.1
¬a	+j	0.05
¬a	¬j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	¬m	0.3
¬a	+m	0.01
¬a	¬m	0.99

Building (Entire) Joint

* We can use the Bayes' Net to build any entry from full distribution it encodes

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

* Typically no reason to build everything, just calc what we need on the fly

* Every BN over a domain of variables Implicitly Defines a Joint Distribution

Size of a Bayes' Net

- Size of joint dist table of N boolean vars
  - 2^N
- Size of N-node net where each node has up to k parents
  - $O(N*2^{k+1})$
- BN can be huge savings if $k << N$
- Easier to find local CPTs vs global joints

Bayes' Nets So Far...

- What we know:
  - Syntax and Semantics of BNs
- Next: properties of the joint distribution
  - Formalizing the notion of conditional independence and causality
  - Goal: answer queries about conditional independence and influence
  - Need to calc posterior probabilities quickly!