Theorem Proving

Three important concepts for Logical Inference

Inference Algorithms:
Forward and Backward Chaining

Forward and Backward Chaining

Horn Form (restricted)

KB = **conjunction** of **Horn clauses**

Horn clause =

◇ proposition symbol; or

◇ (conjunction of symbols) $\Rightarrow$ symbol

E.g., $C \wedge (B \Rightarrow A) \wedge (C \wedge D \Rightarrow B)$

- Restrict sentences to be written a particular way
- Only need one inference rule in your search for entailment

Forward and Backward Chaining

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- ◊ (conjunction of symbols) $\Rightarrow$ symbol

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Modus Ponens (for Horn Form): complete for Horn KBs

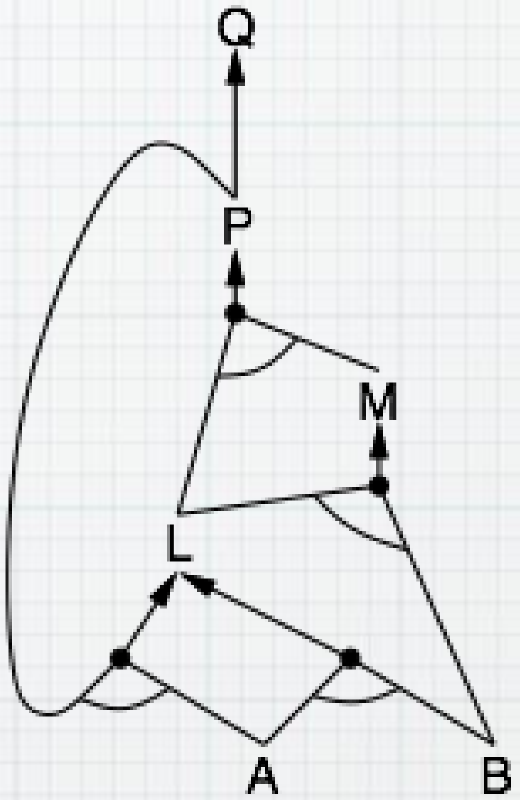
$$\frac{\alpha_1, \ldots, \alpha_n, \qquad \alpha_1 \wedge \cdots \wedge \alpha_n \Rightarrow \beta}{\beta}$$

Can be used with forward chaining or backward chaining.
These algorithms are very natural and run in **linear** time

Forward Chaining

Idea: fire any rule whose premises are satisfied in the KB,
add its conclusion to the KB, until query is found

$$
\begin{aligned}
P &\Rightarrow Q \\
L \wedge M &\Rightarrow P \\
B \wedge L &\Rightarrow M \\
A \wedge P &\Rightarrow L \\
A \wedge B &\Rightarrow L \\
A & \\
B &
\end{aligned}
$$

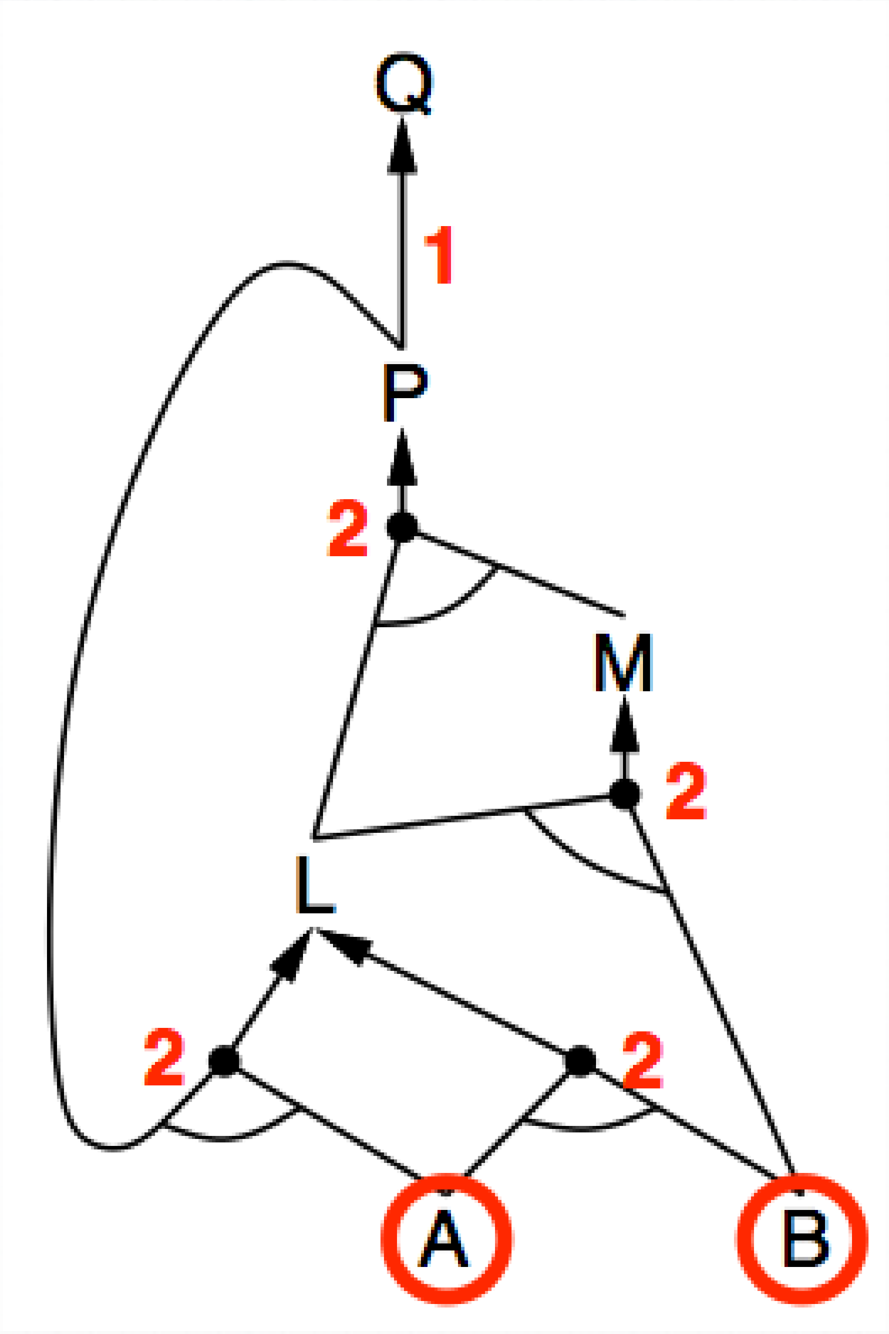


Forward Chaining

```
function PL-FC-ENTAILS?(KB, q) returns true or false
   inputs: KB, the knowledge base, a set of propositional Horn clauses
           q, the query, a proposition symbol
   local variables: count, a table, indexed by clause, initially the number of premises
                    inferred, a table, indexed by symbol, each entry initially false
                    agenda, a list of symbols, initially the symbols known in KB

   while agenda is not empty do
       p ← POP(agenda)
       unless inferred[p] do
           inferred[p] ← true
           for each Horn clause c in whose premise p appears do
               decrement count[c]
               if count[c] = 0 then do
                   if HEAD[c] = q then return true
                   PUSH(HEAD[c], agenda)
   return false
```

Forward chaining example



$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
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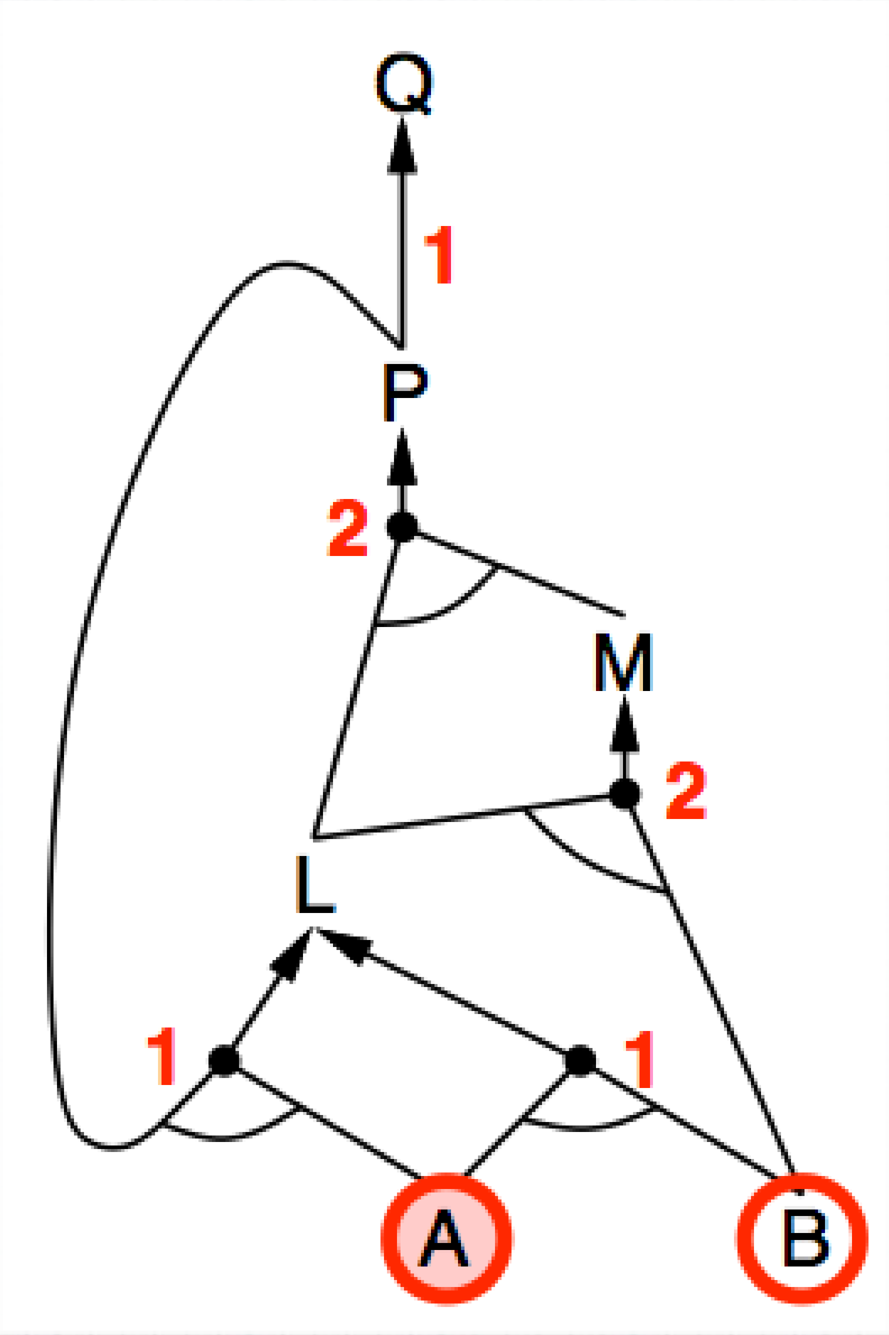
$$A$$
$$B$$

Forward chaining example



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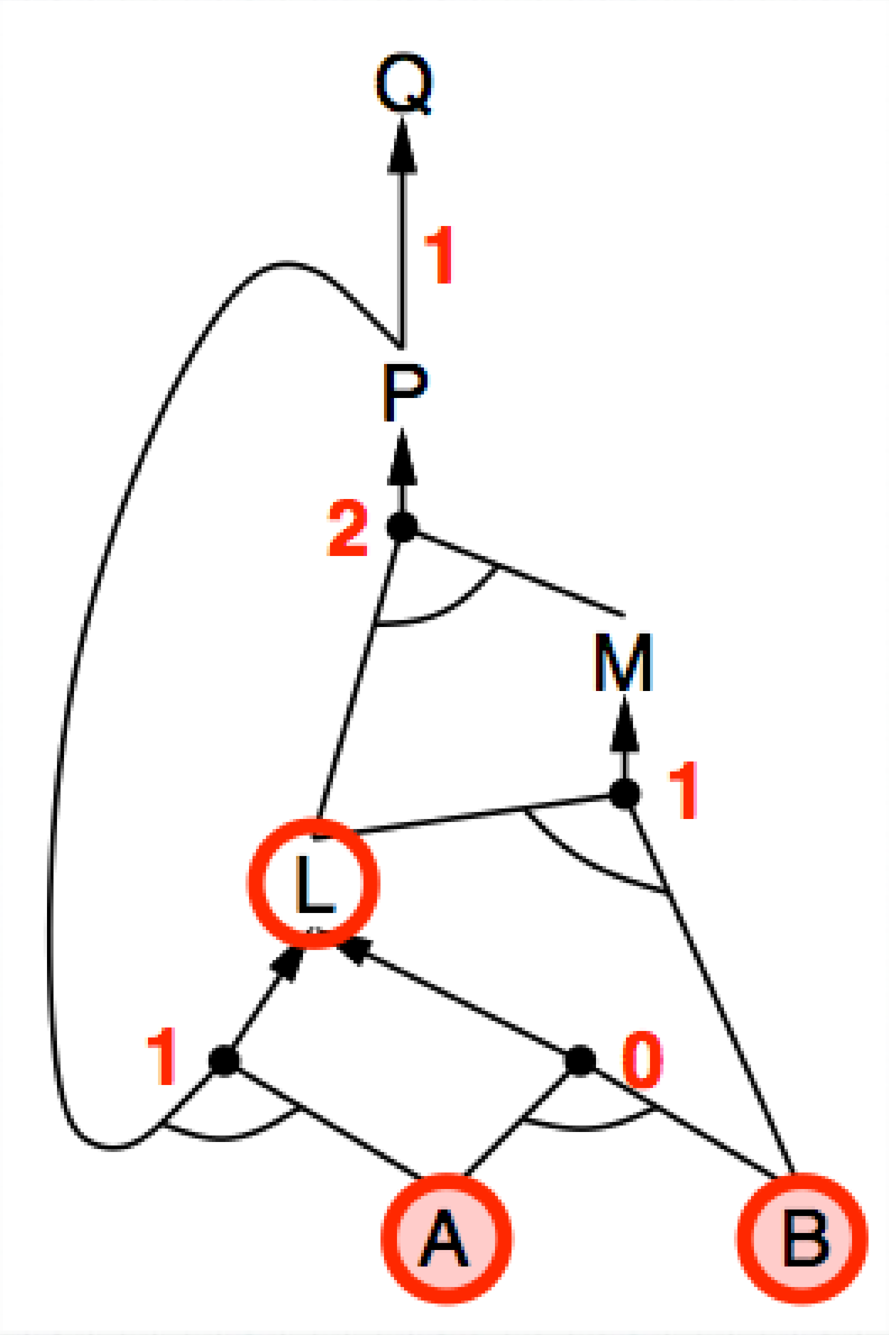

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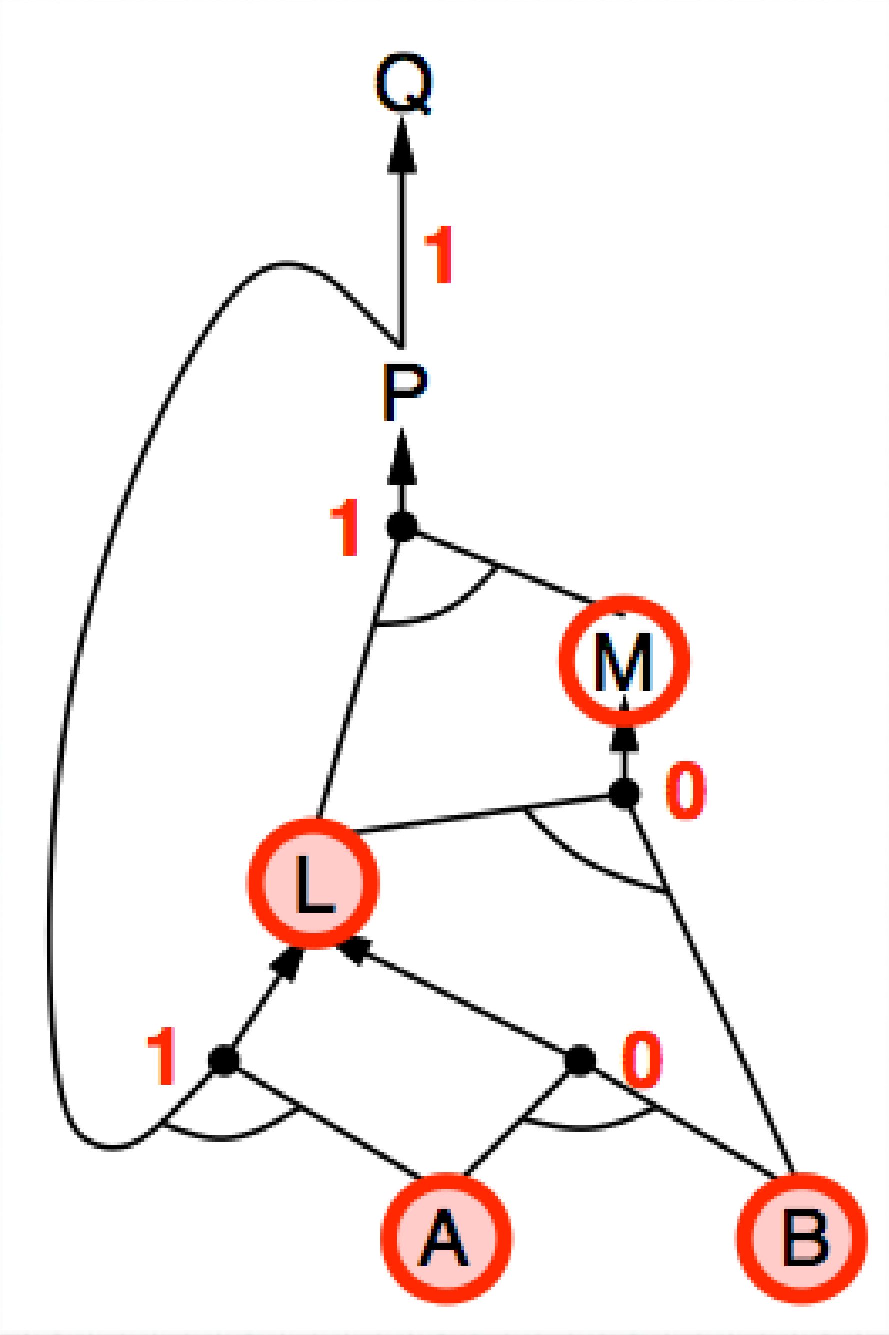

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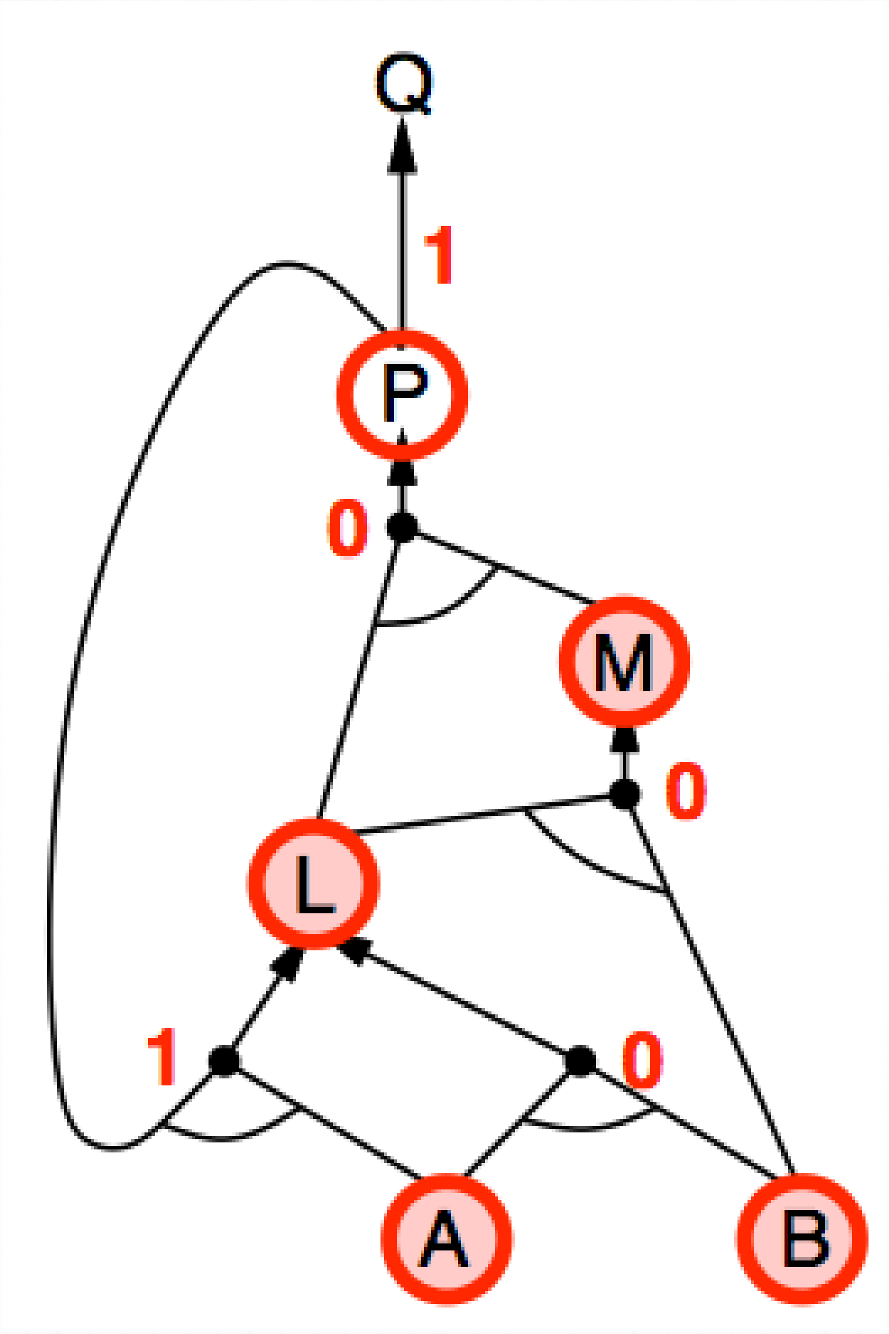

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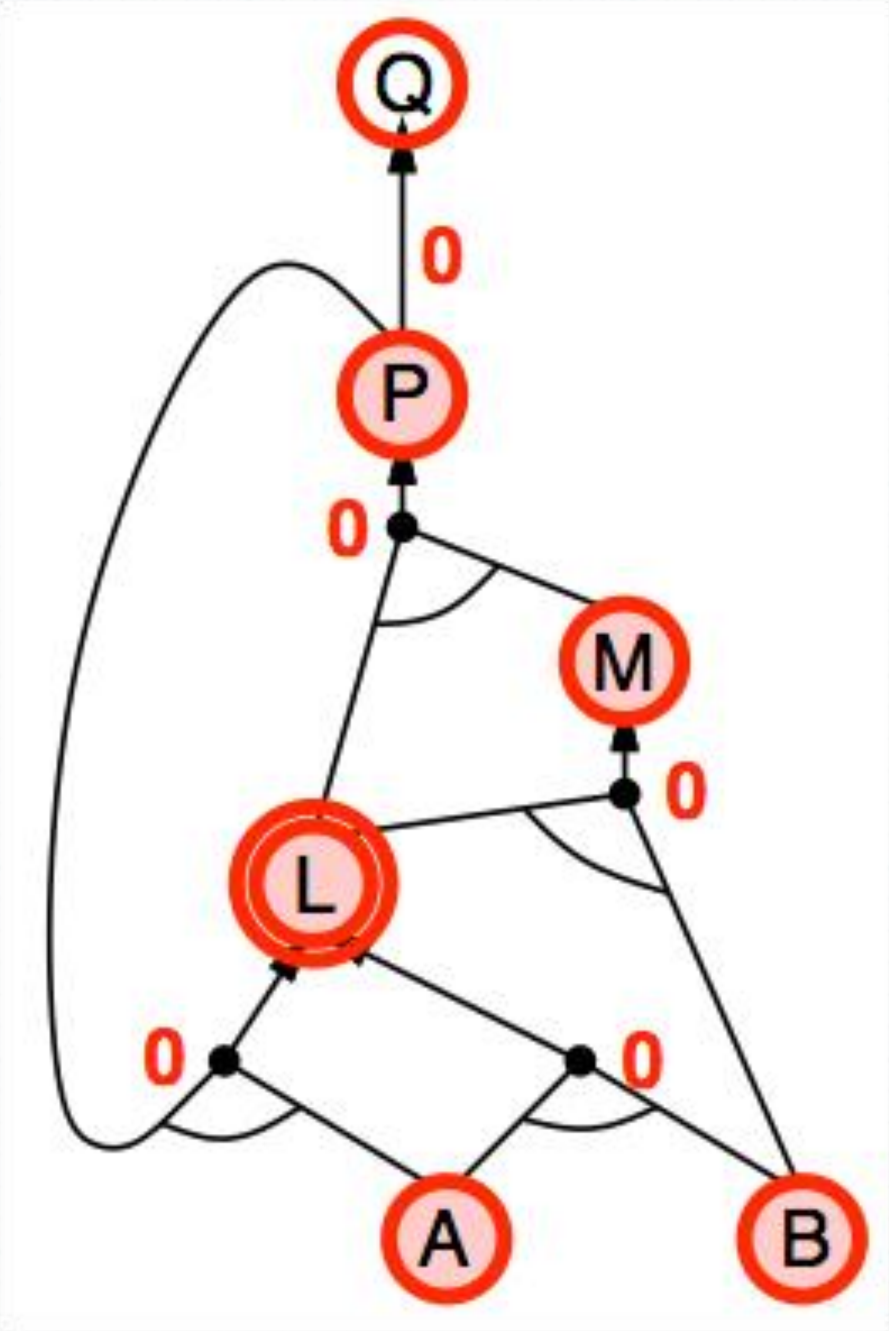
L
M

Forward chaining example



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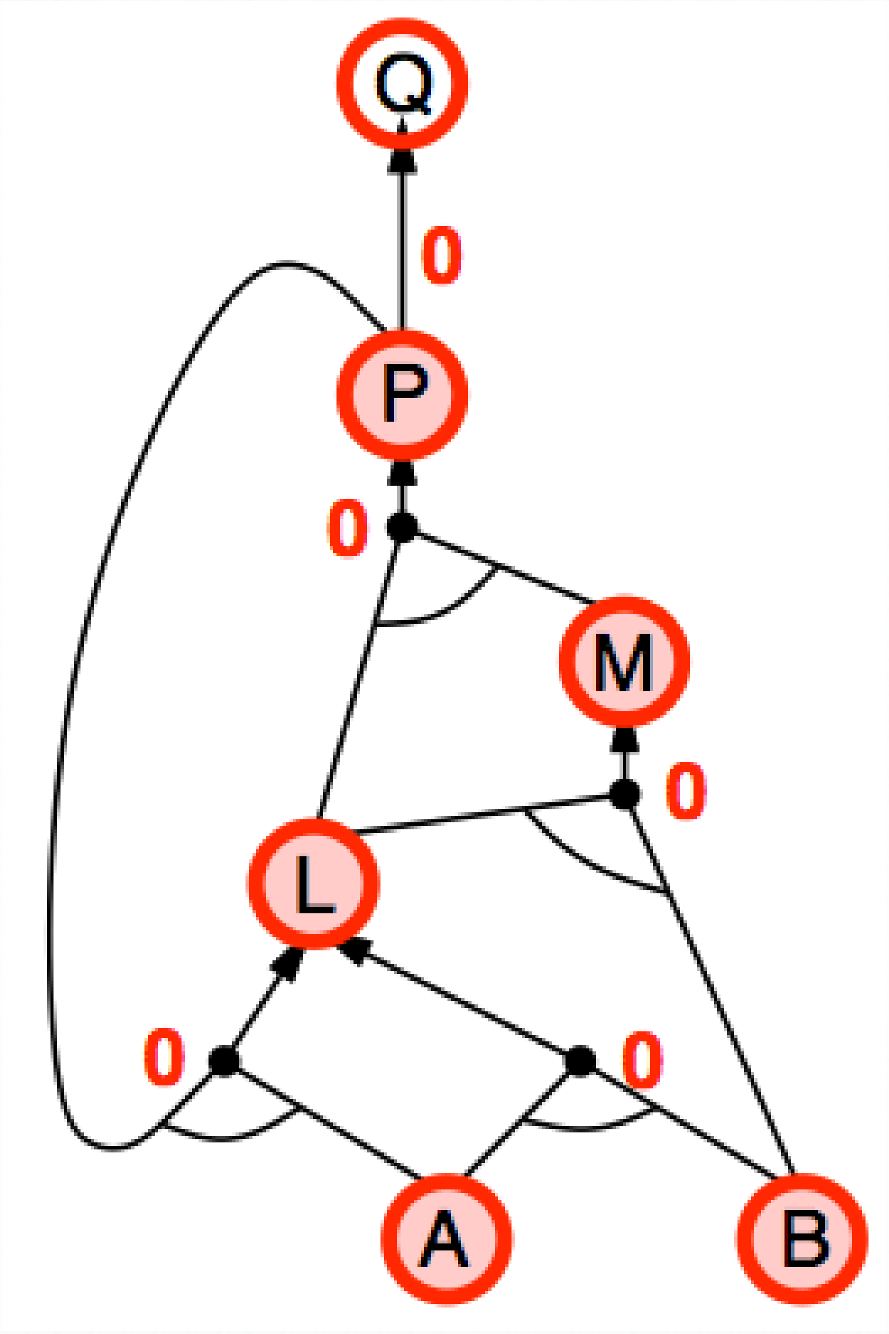

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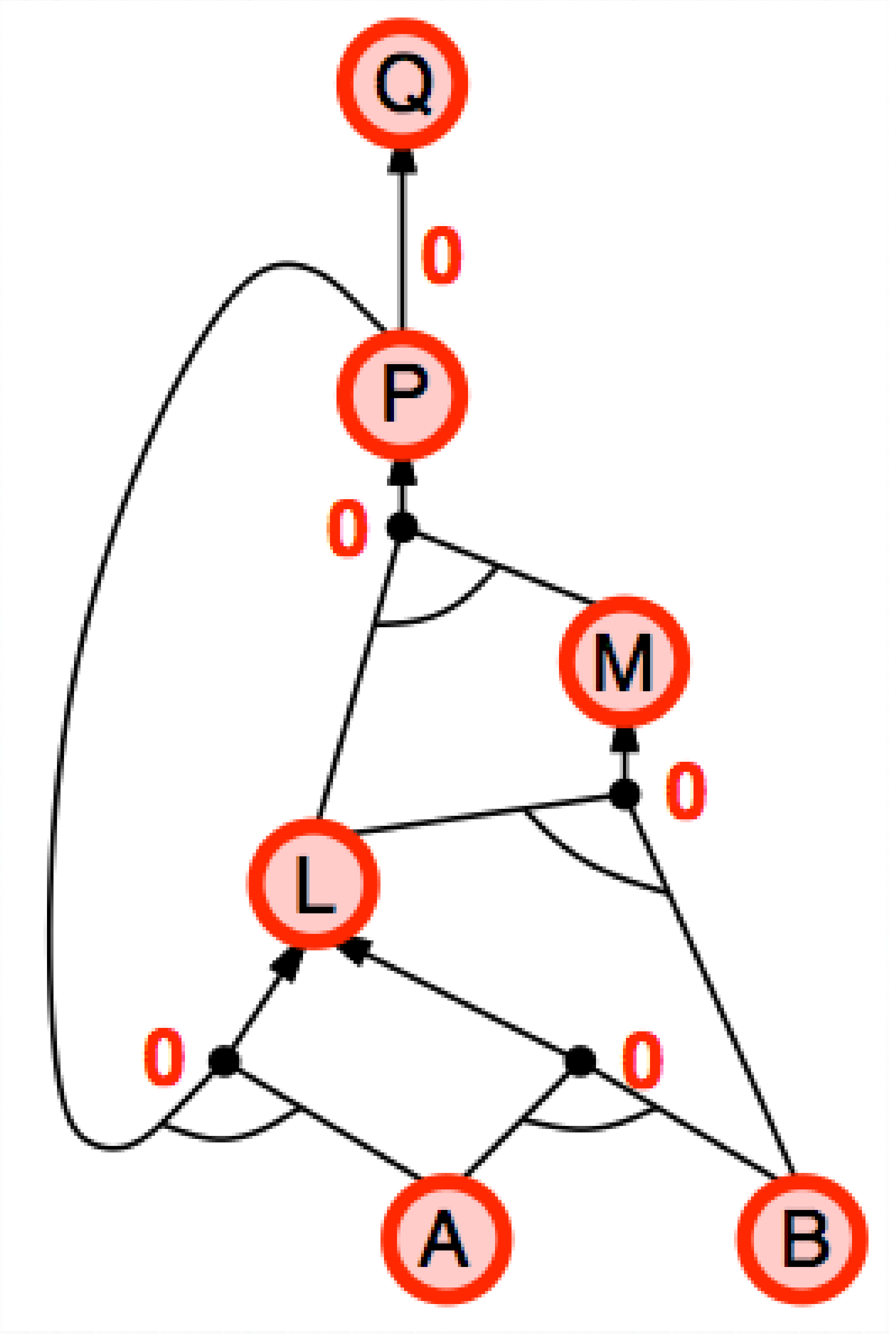

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M

P

Q

Proof of Completeness

FC derives every atomic sentence that is entailed by KB

1. FC reaches a fixed point where no new atomic sentences are derived
2. Consider the final state as a model m, assigning true/false to symbols
3. Every clause in the original KB is true in m
   Proof: Suppose a clause $a_1 \wedge \ldots \wedge a_k \Rightarrow b$ is false in m
   Then $a_1 \wedge \ldots \wedge a_k$ is true in m and b is false in m
   Therefore the algorithm has not reached a fixed point!
4. Hence m is a model of KB
5. If $KB \models q$, q is true in **every** model of KB, including m

General idea: construct any model of KB by sound inference, check α

Backward Chaining

Idea: work backwards from the query q:

to prove q by BC,

check if q is known already, or

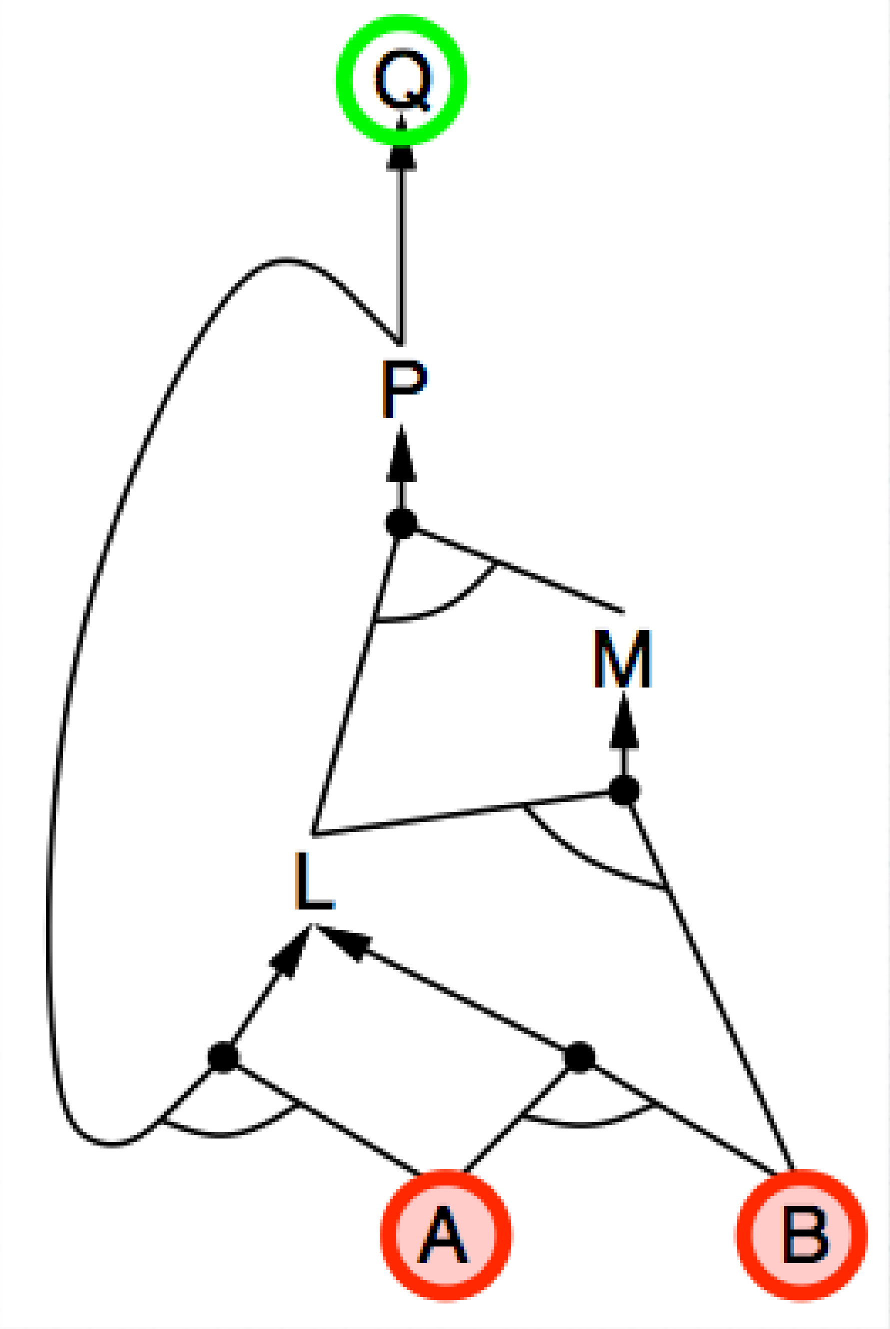
prove by BC all premises of some rule concluding q

Avoid loops: check if new subgoal is already on the goal stack

Avoid repeated work: check if new subgoal

1) has already been proved true, or
2) has already failed

Backward chaining example



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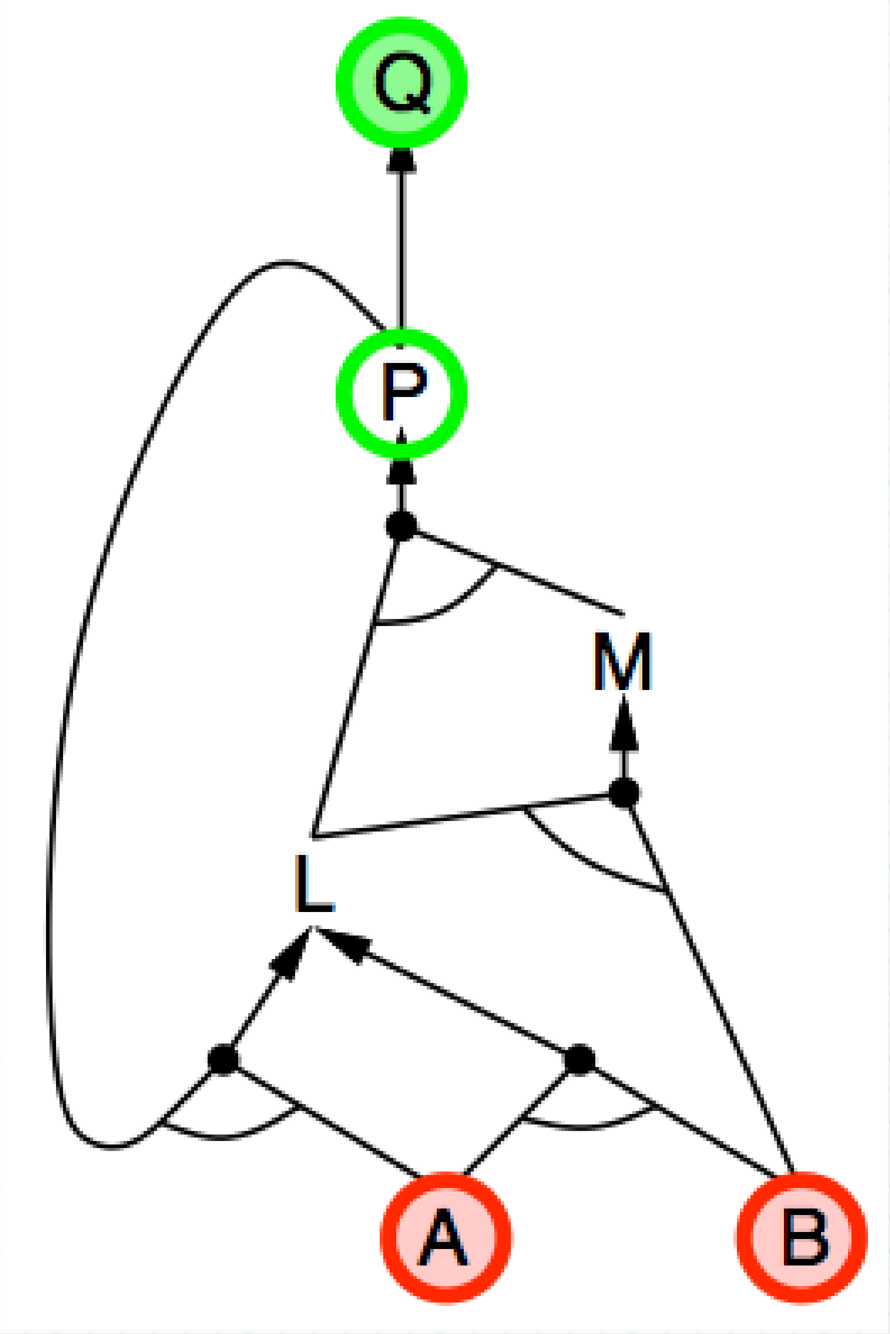

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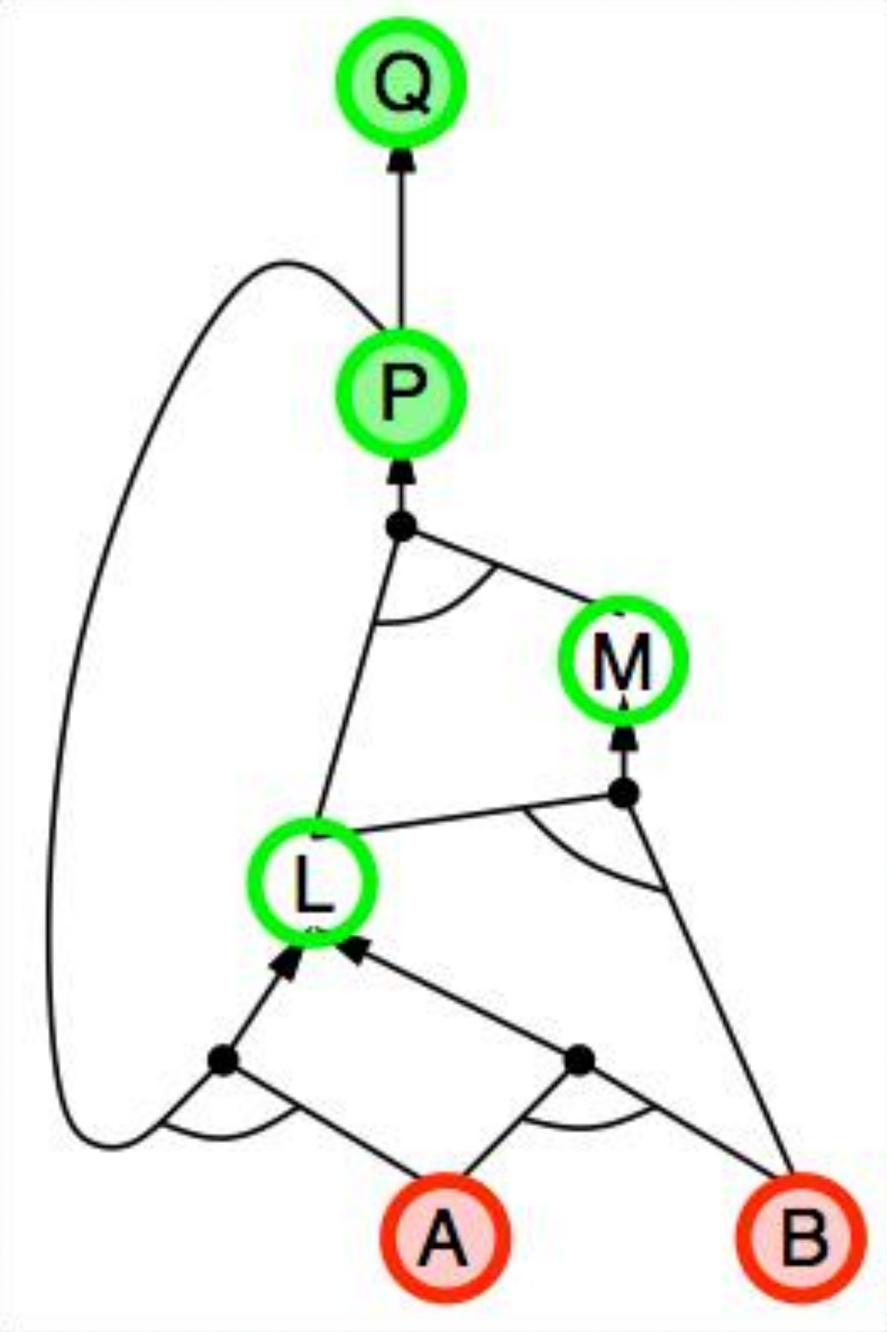

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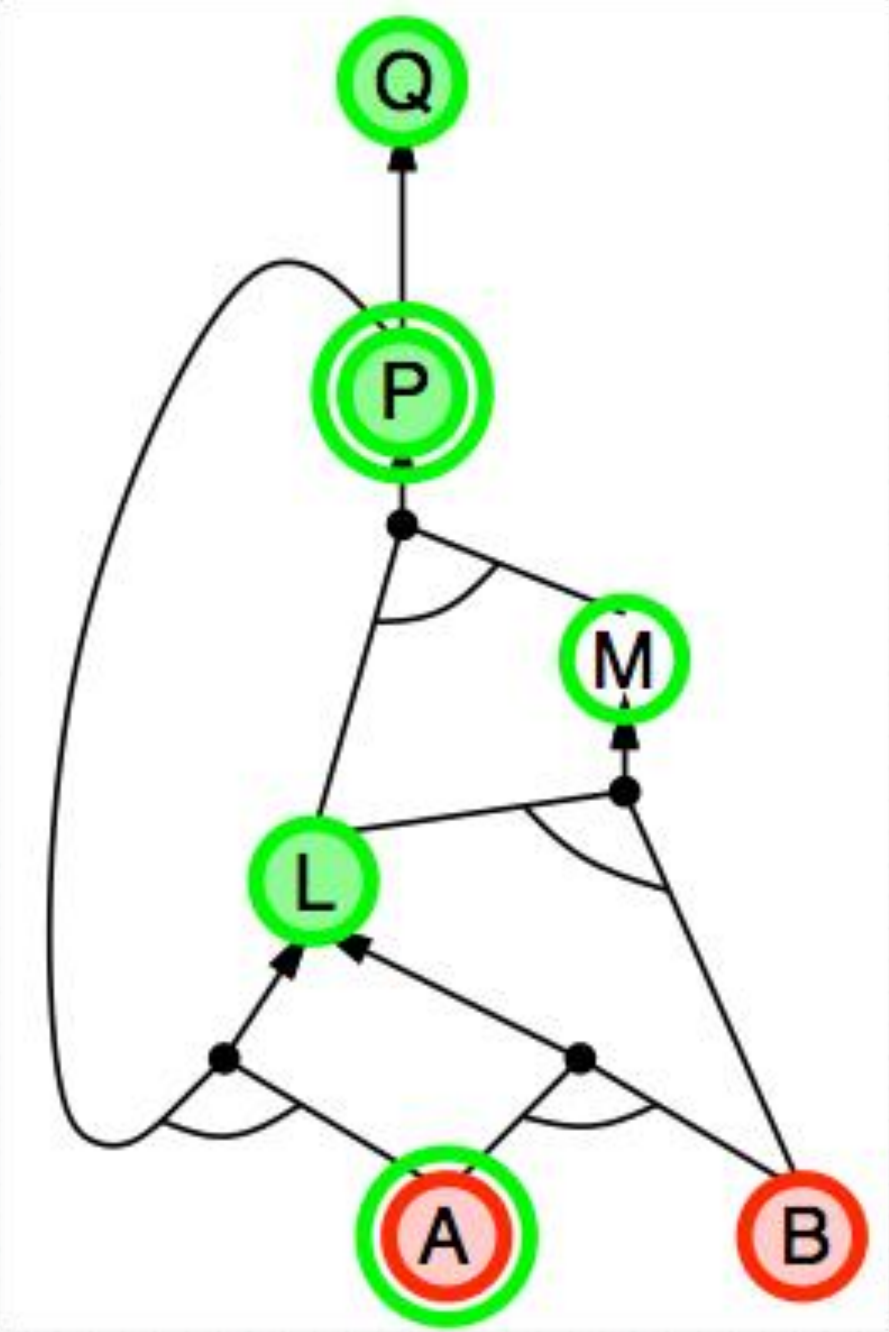

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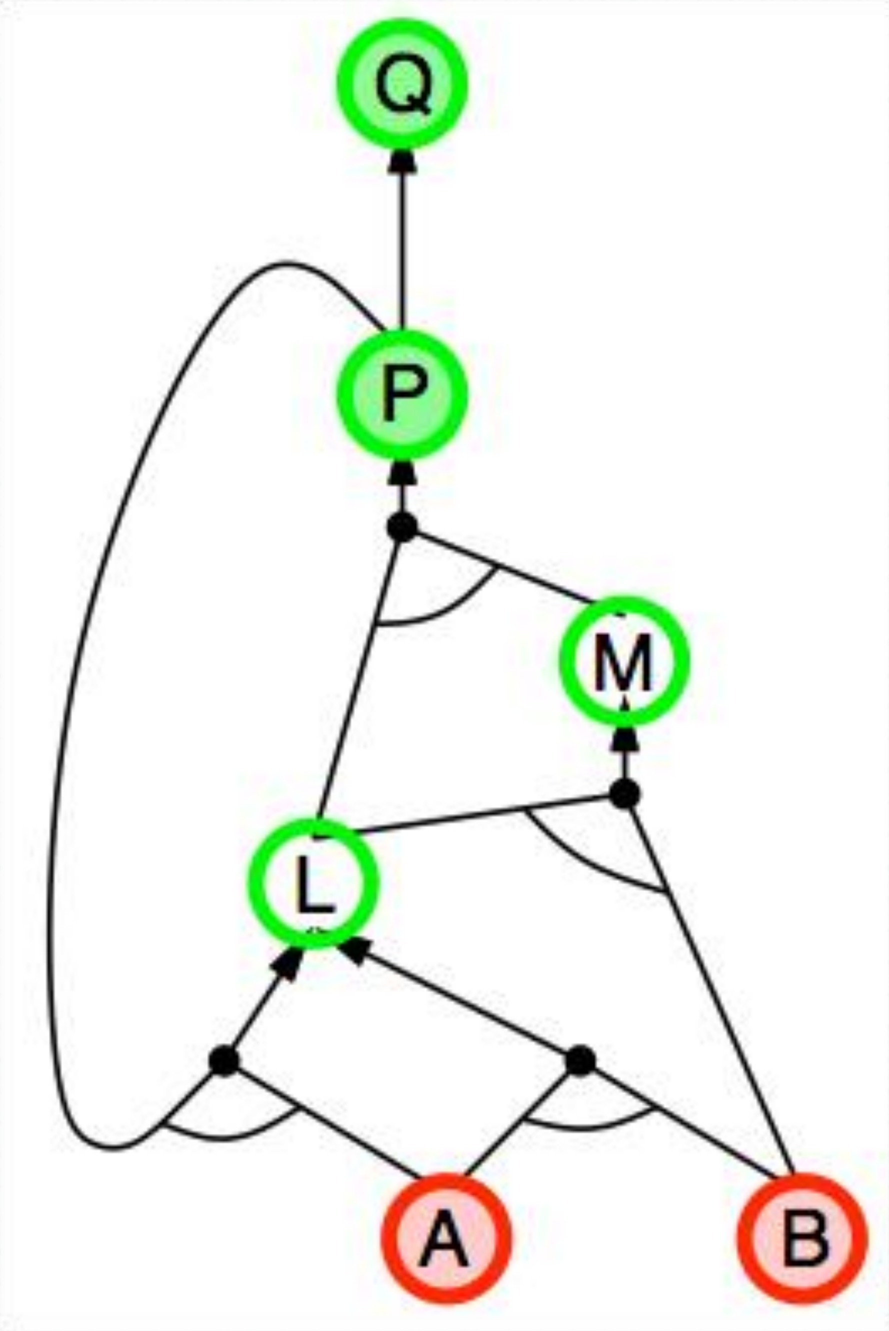

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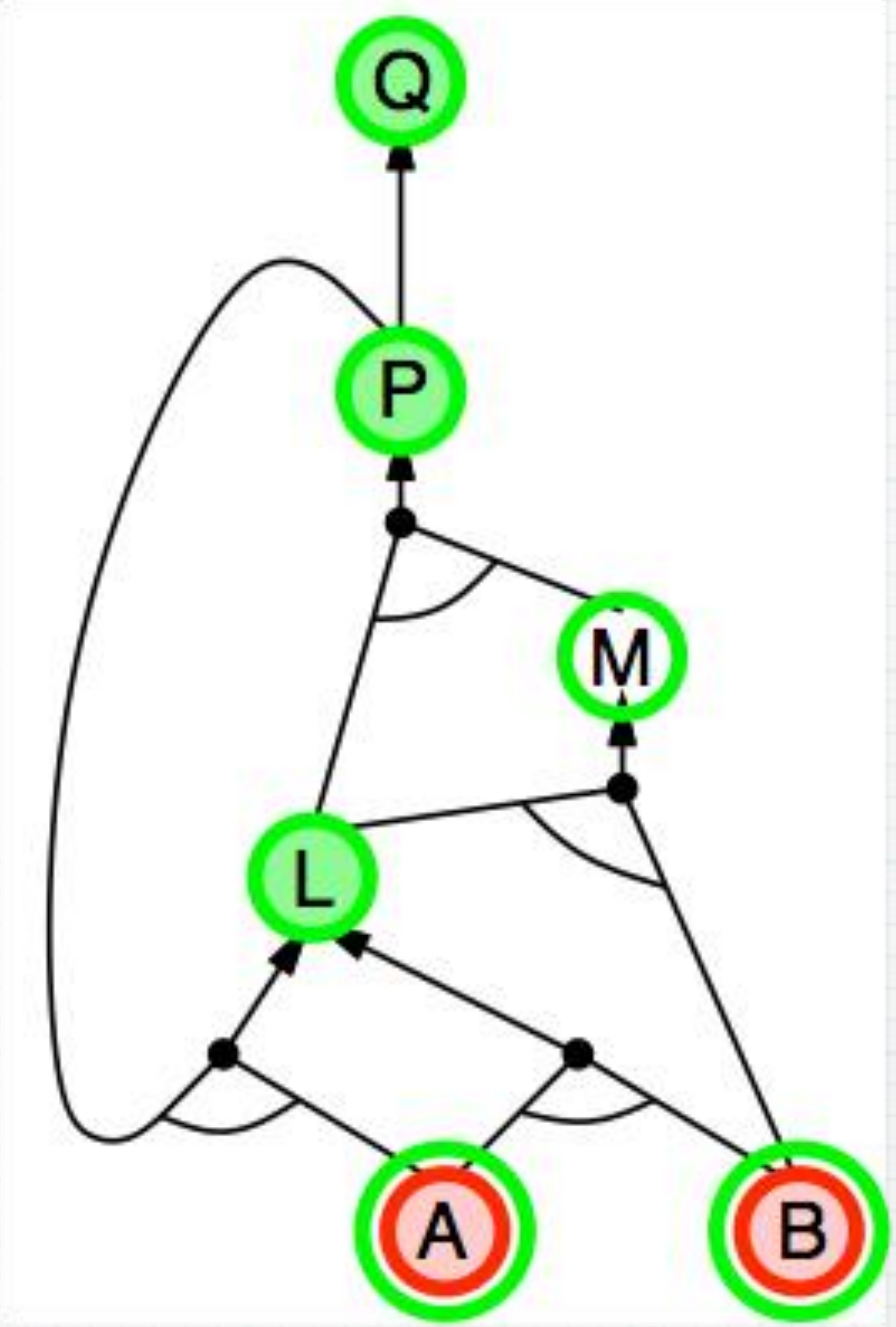

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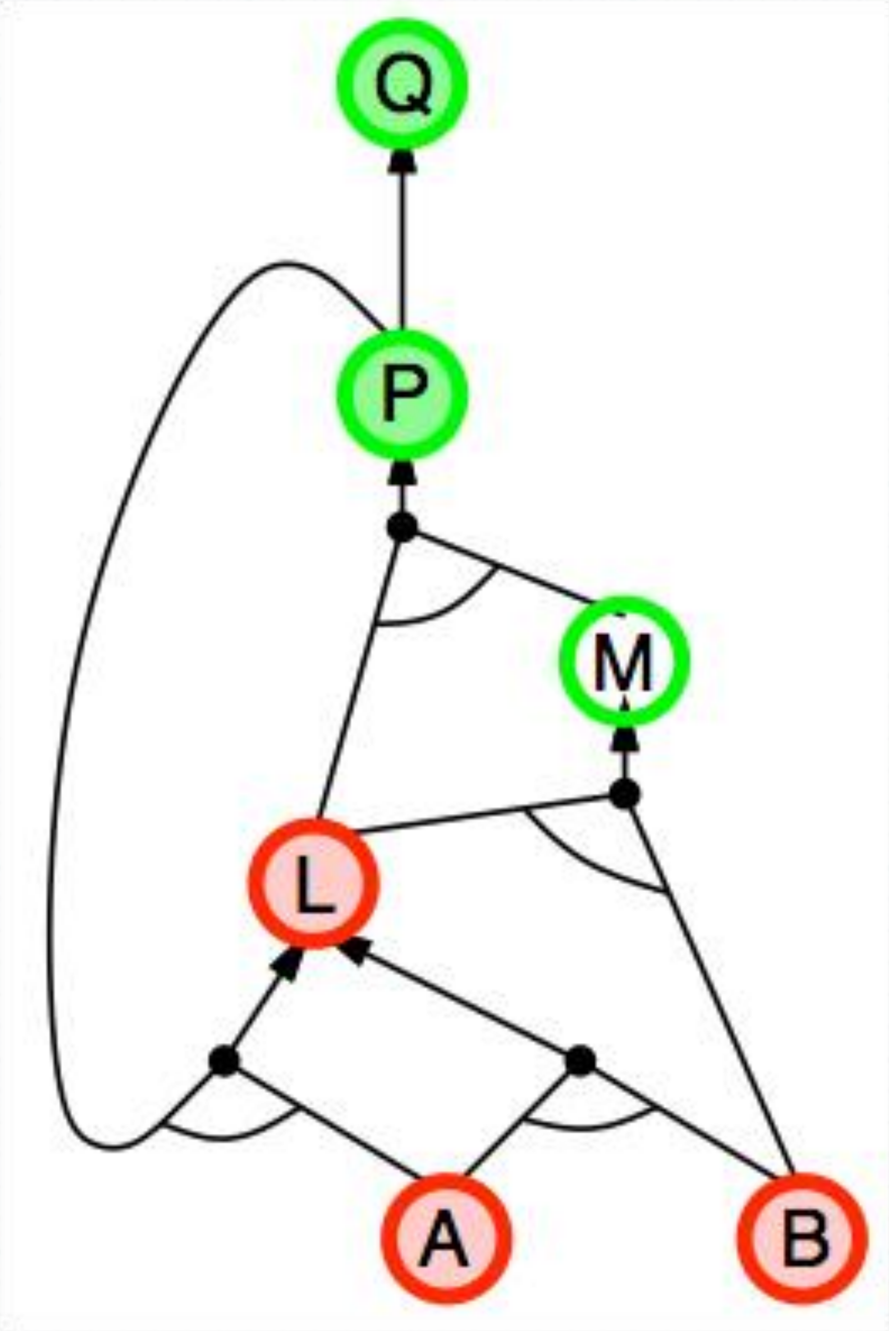

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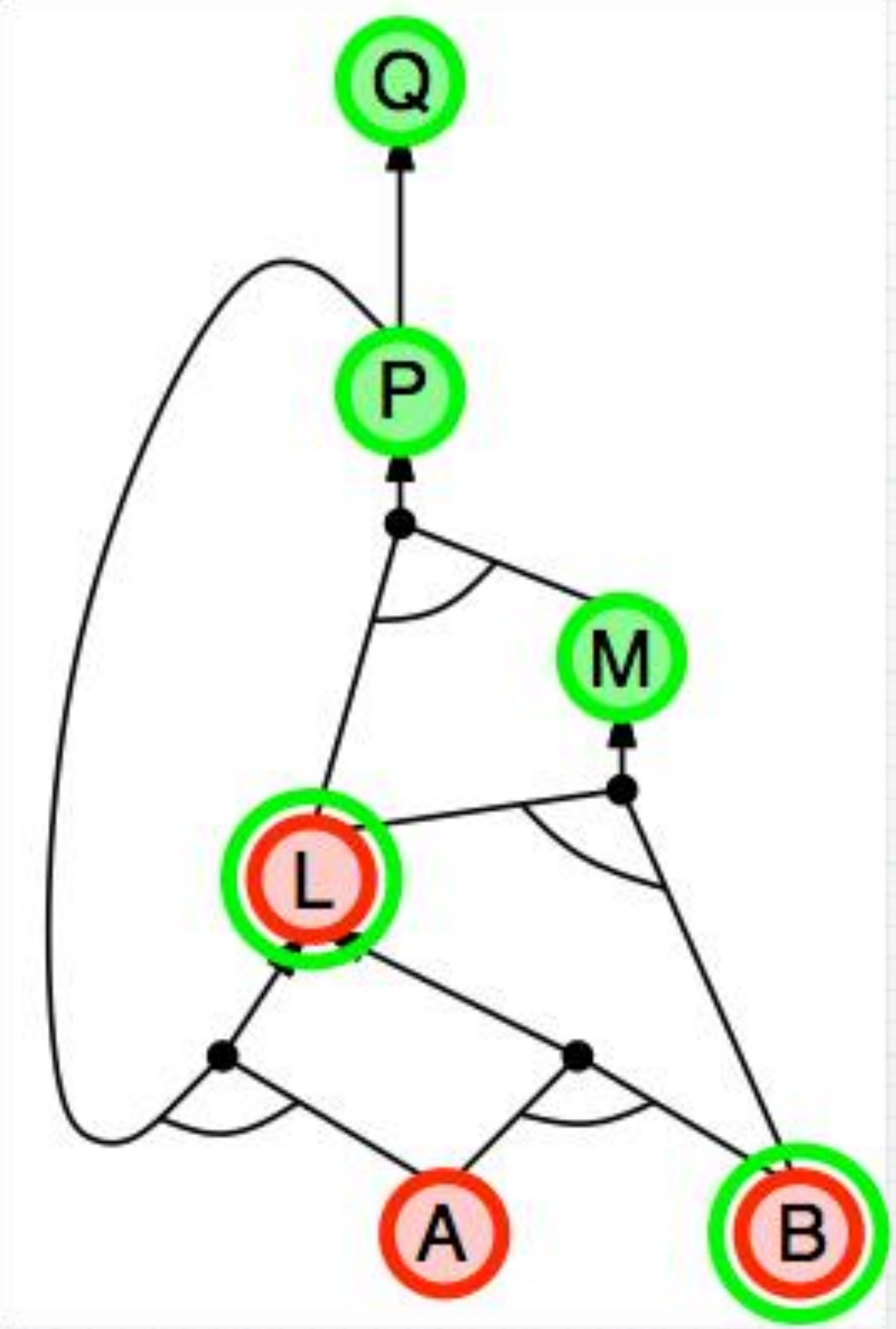

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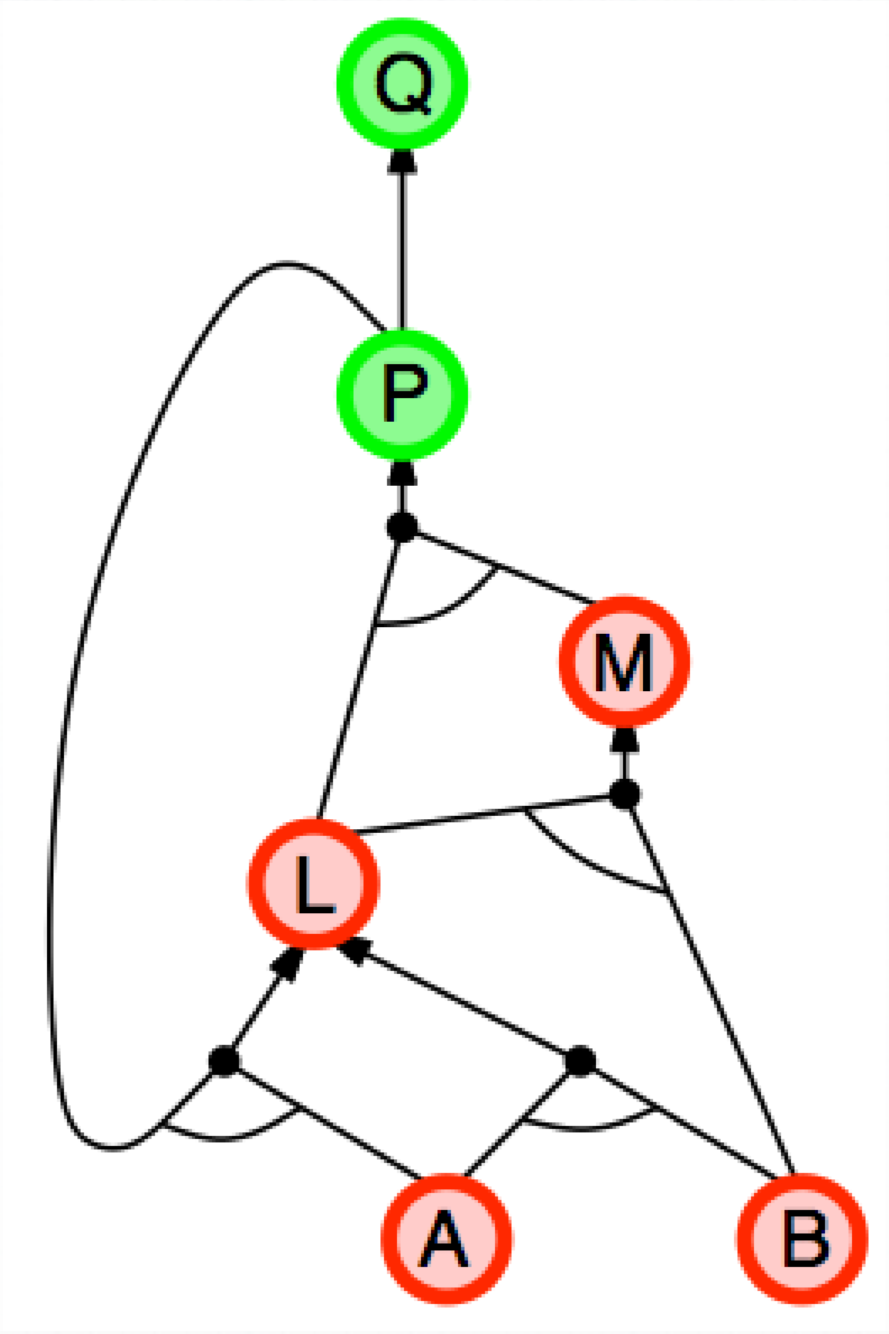

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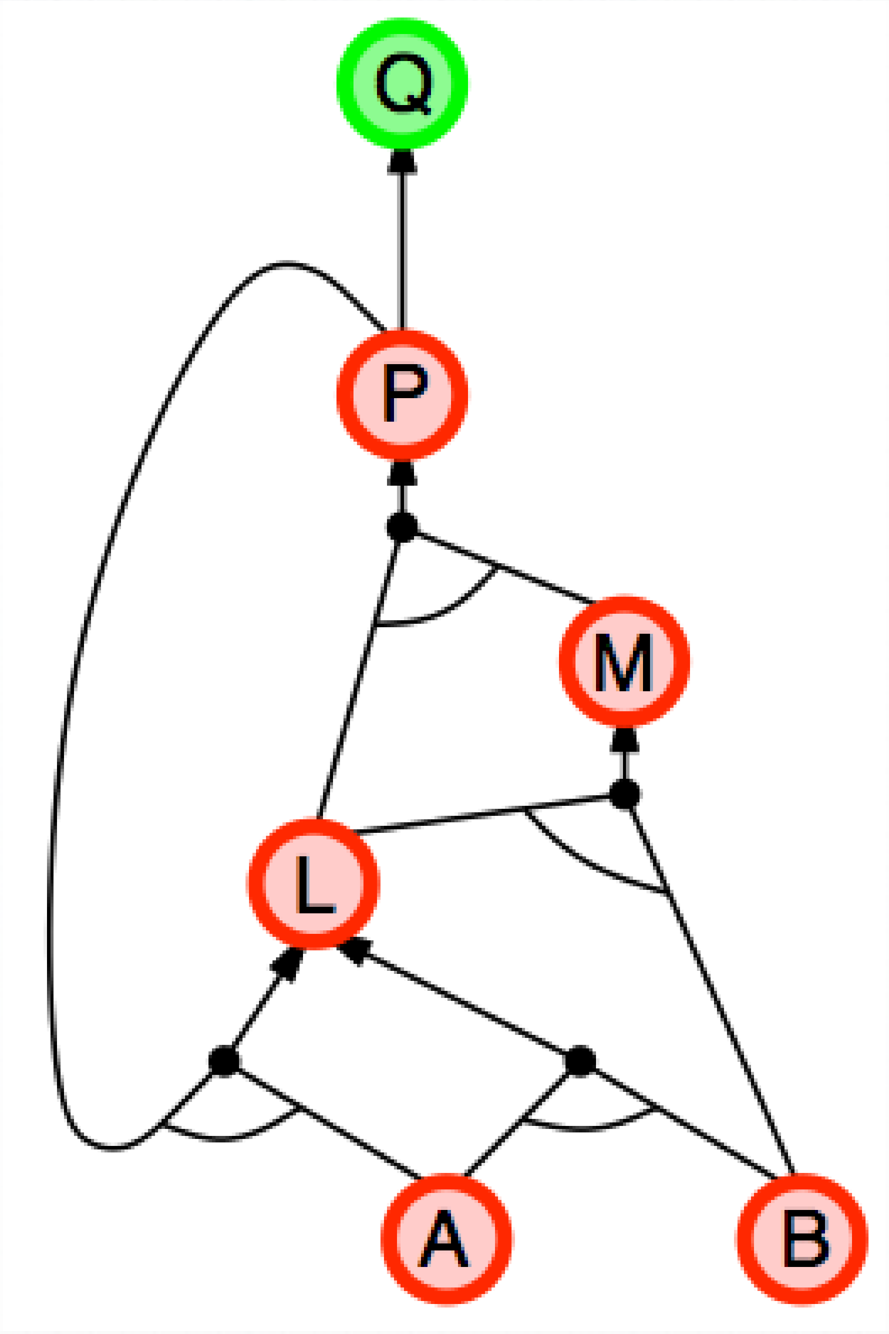

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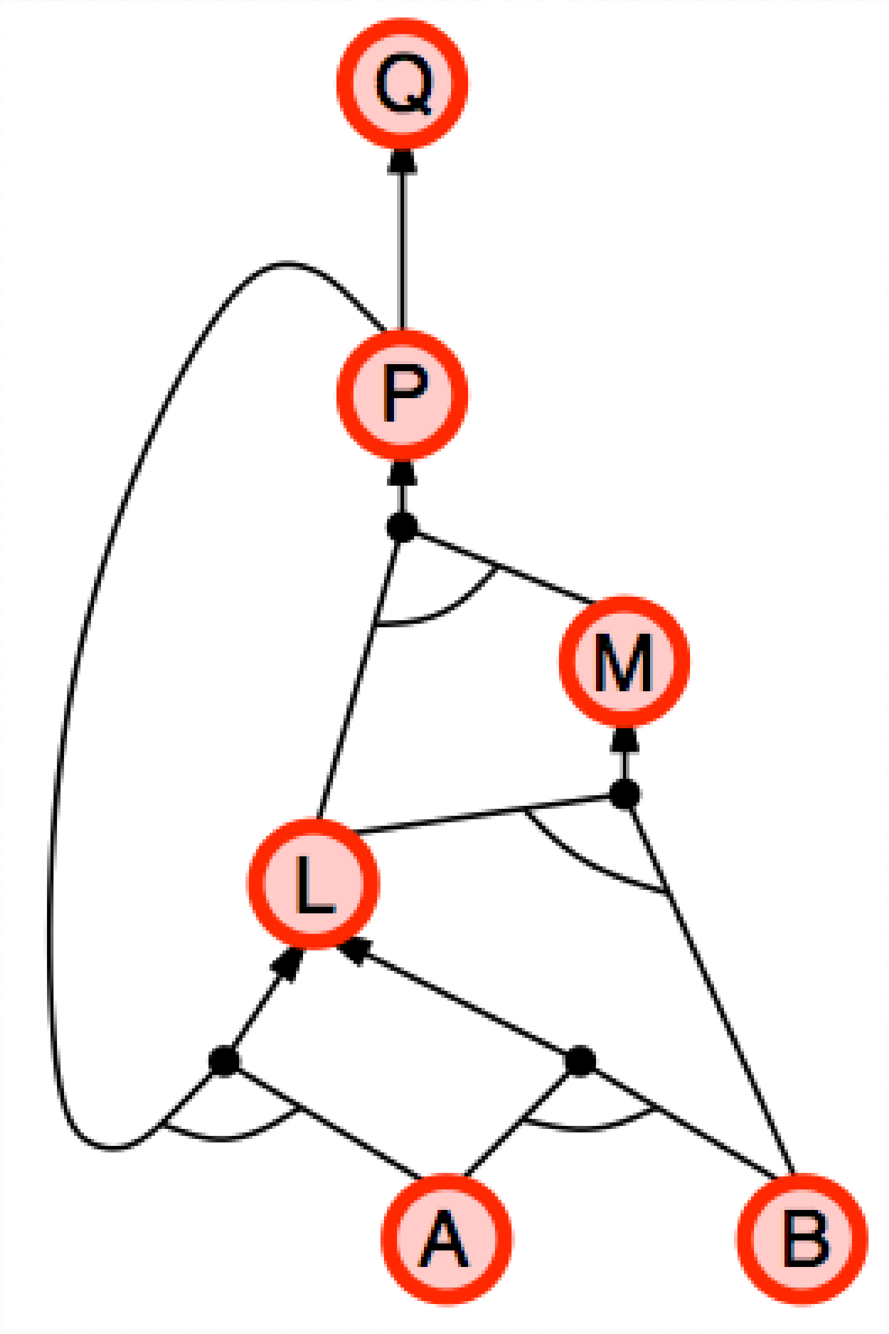

Forward vs. Backward Chaining

FC is data-driven, cf. automatic, unconscious processing,
e.g., object recognition, routine decisions

May do lots of work that is irrelevant to the goal

BC is goal-driven, appropriate for problem-solving,
e.g., Where are my keys? How do I get into a PhD program?

Complexity of BC can be much less than linear in size of KB

Inference Algorithms: Resolution

Resolution

Conjunctive Normal Form (CNF—universal)

$$\textbf{conjunction} \text{ of } \underbrace{\textbf{disjunctions} \text{ of } \textbf{literals}}_{\textbf{clauses}}$$

E.g., $(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$

Resolution

Conjunctive Normal Form (CNF—universal)

conjunction of $\underbrace{\textbf{disjunctions} \text{ of } \textbf{literals}}_{\textbf{clauses}}$

E.g., $(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$

Resolution inference rule (for CNF): complete for propositional logic

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n}$$

where ℓ_i and m_j are complementary literals. E.g.,

Resolution

Conjunctive Normal Form (CNF—universal)

$$\underbrace{\textbf{conjunction of } \textbf{disjunctions of literals}}$$

$\textbf{conjunction}$ of $\underbrace{\textbf{disjunctions} \text{ of } \textbf{literals}}_{\textbf{clauses}}$

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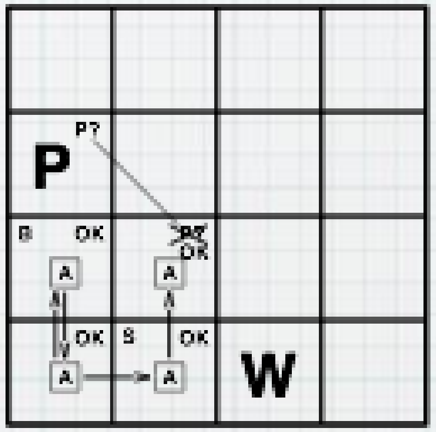
where ℓ_i and m_j are complementary literals. E.g.,

$$\frac{P_{1,3} \vee P_{2,2}, \qquad \neg P_{2,2}}{P_{1,3}}$$



Resolution is sound and complete for propositional logic

Example CNF Convert

- Great! Can we make everything CNF?
- Convert sentences to Conjunctive Normal Form with our rules of logical equivalence

$(\alpha \wedge \beta)$	$\equiv$	$(\beta \wedge \alpha)$	commutativity of $\wedge$
$(\alpha \vee \beta)$	$\equiv$	$(\beta \vee \alpha)$	commutativity of $\vee$
$((\alpha \wedge \beta) \wedge \gamma)$	$\equiv$	$(\alpha \wedge (\beta \wedge \gamma))$	associativity of $\wedge$
$((\alpha \vee \beta) \vee \gamma)$	$\equiv$	$(\alpha \vee (\beta \vee \gamma))$	associativity of $\vee$
$\neg(\neg\alpha)$	$\equiv$	α	double-negation elimination
$(\alpha \Rightarrow \beta)$	$\equiv$	$(\neg\beta \Rightarrow \neg\alpha)$	contraposition
$(\alpha \Rightarrow \beta)$	$\equiv$	$(\neg\alpha \vee \beta)$	implication elimination
$(\alpha \Leftrightarrow \beta)$	$\equiv$	$((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha))$	biconditional elimination
$\neg(\alpha \wedge \beta)$	$\equiv$	$(\neg\alpha \vee \neg\beta)$	De Morgan
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$(\alpha \wedge (\beta \vee \gamma))$	$\equiv$	$((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$	distributivity of $\wedge$ over $\vee$
$(\alpha \vee (\beta \wedge \gamma))$	$\equiv$	$((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$	distributivity of $\vee$ over $\wedge$

Example CNF Convert

$$B_{11} \Leftrightarrow (P_{12} \vee P_{21})$$

Biconditional Elimination

$$B_{11} \Rightarrow (P_{12} \vee P_{21}) \wedge (P_{12} \vee P_{21}) \Rightarrow B_{11}$$

Implication Elimination

$$(\neg B_{11} \vee P_{12} \vee P_{21}) \wedge (\neg(P_{12} \vee P_{21}) \vee B_{11})$$

Move neg. in, DeMorgans

$$(\neg B_{11} \vee P_{12} \vee P_{21}) \wedge ((\neg P_{12} \wedge \neg P_{21}) \vee B_{11})$$

Distribute over and/or

$$(\neg B_{11} \vee P_{12} \vee P_{21}) \wedge (\neg P_{12} \vee B_{11}) \wedge (\neg P_{21} \vee B_{11})$$

Resolution Algorithm

Proof by contradiction, i.e., show $KB \wedge \neg\alpha$ unsatisfiable

Resolution Algorithm

Proof by contradiction, i.e., show $KB \wedge \neg\alpha$ unsatisfiable

```
function PL-RESOLUTION(KB, α) returns true or false
    inputs: KB, the knowledge base, a sentence in propositional logic
            α, the query, a sentence in propositional logic

    clauses ← the set of clauses in the CNF representation of KB ∧ ¬α
    new ← { }
    loop do
        for each Ci, Cj in clauses do
            resolvents ← PL-RESOLVE(Ci, Cj)
            if resolvents contains the empty clause then return true
            new ← new ∪ resolvents
        if new ⊆ clauses then return false
        clauses ← clauses ∪ new
```

Resolution Example

$KB = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$ $\alpha = \neg P_{1,2}$

$\boxed{\neg P_{2,1} \vee B_{1,1}}$ $\boxed{\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}}$ $\boxed{\neg P_{1,2} \vee B_{1,1}}$ $\boxed{\neg B_{1,1}}$ $\boxed{P_{1,2}}$

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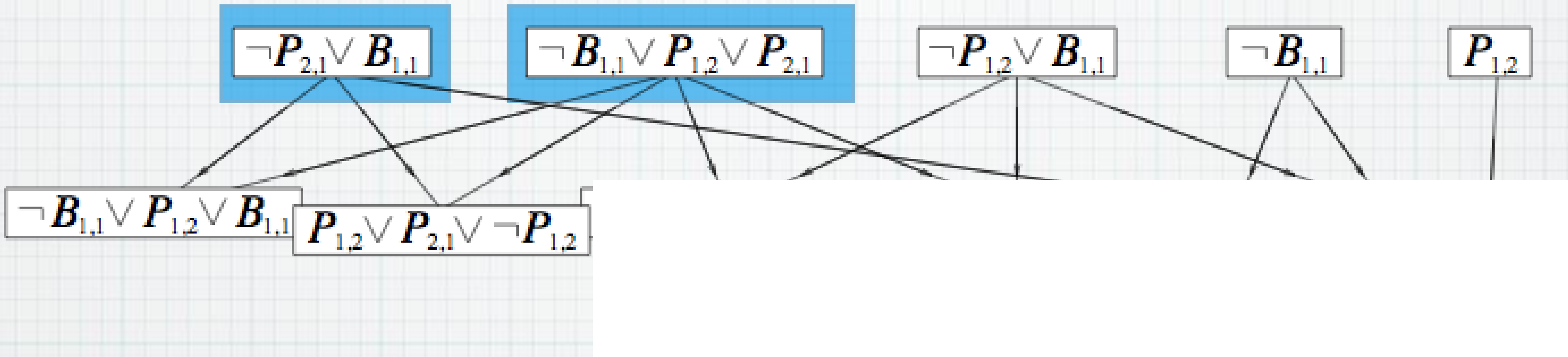
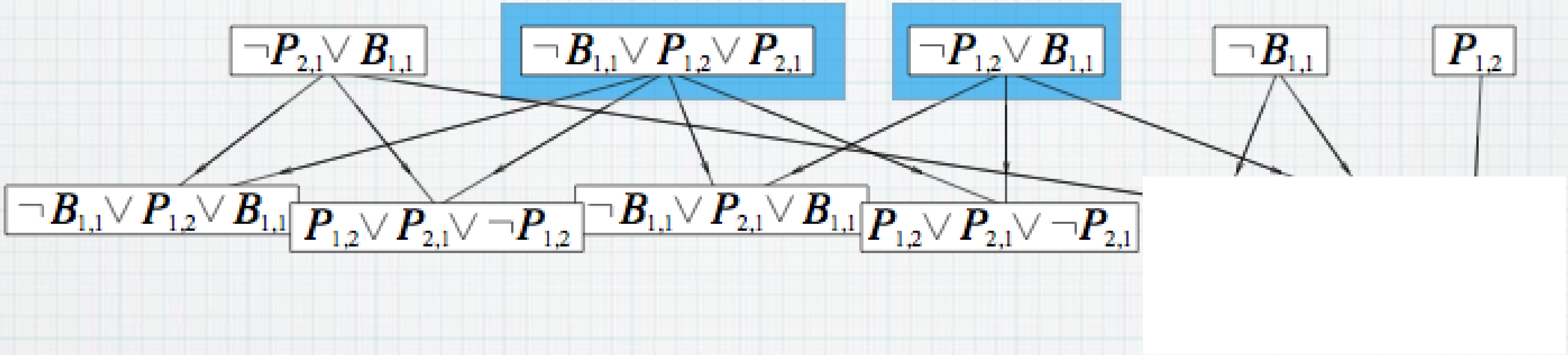


Resolution Example

$KB = (B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})) \land \neg B_{1,1}$ $\alpha = \neg P_{1,2}$



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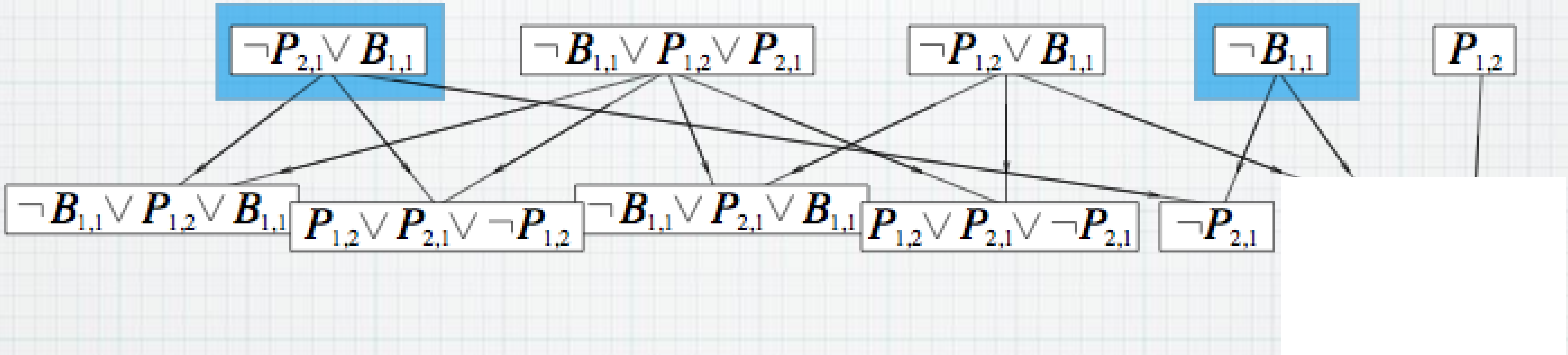

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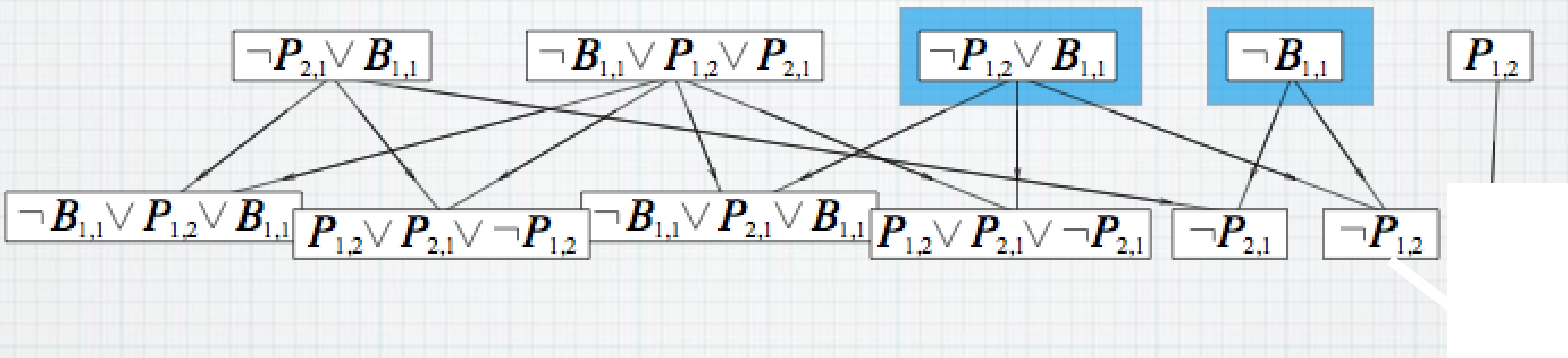

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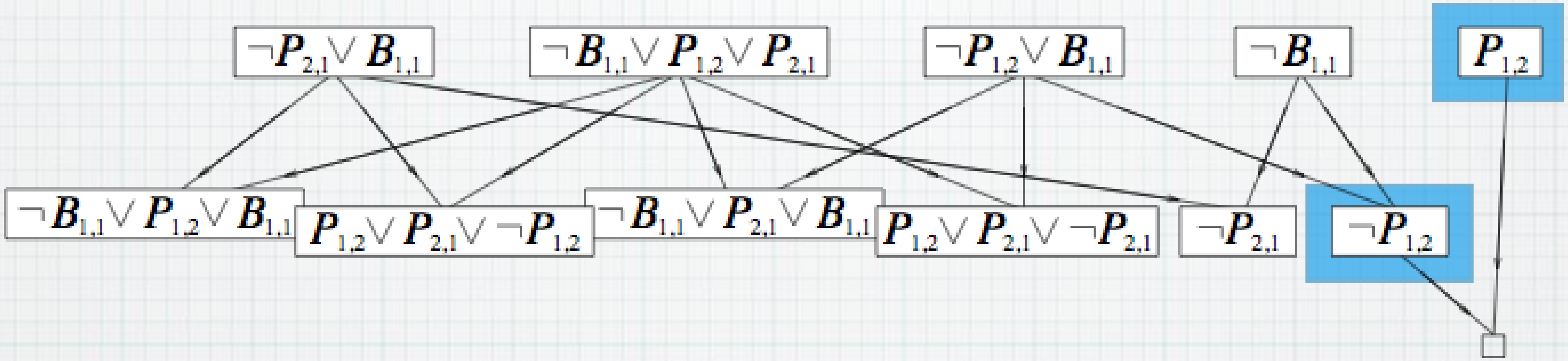

Contradiction, therefore our sentence is entailed.

Agent based on Prop. Logic

Inference-based agents in the Wumpus World

A wumpus-world agent using propositional logic:

$\leftarrow P_{1,1}$

$\leftarrow W_{1,1}$

$B_{x,y}$ ™$(P_{x,y+1}$ ($P_{x,y-1}$ ($P_{x+1,y}$ ($P_{x-1,y})$

$S_{x,y}$ ™$(W_{x,y+1}$ ($W_{x,y-1}$ ($W_{x+1,y}$ ($W_{x-1,y})$

$W_{1,1}$ ($W_{1,2}$ (… ($W_{4,4}$

$\leftarrow W_{1,1}$ ($\leftarrow W_{1,2}$

$\leftarrow W_{1,1}$ ($\leftarrow W_{1,3}$

…

64 distinct proposition symbols, 155 sentences

function PL-WUMPUS-AGENT(*percept*) **returns** an *action*
  inputs: *percept*, a list, [*stench*, *breeze*, *glitter*]
  static: *KB*, initially containing the "physics" of the wumpus world
    x, *y*, *orientation*, the agent's position (init. [1,1]) and orient. (init. *right*)
    visited, an array indicating which squares have been visited, initially *false*
    action, the agent's most recent action, initially null
    plan, an action sequence, initially empty

  update *x*,*y*,*orientation*, *visited* based on *action*
  if *stench* **then** TELL(KB, $S_{x,y}$) **else** TELL(KB, $\neg S_{x,y}$)
  if *breeze* **then** TELL(KB, $B_{x,y}$) **else** TELL(KB, $\neg B_{x,y}$)
  if *glitter* **then** *action* ← *grab*
  else if *plan* is nonempty then *action* ← POP(*plan*)
  else if for some fringe square [*i*,*j*], ASK(KB, $(\neg P_{i,j} \wedge \neg W_{i,j})$) is *true* **or**
      for some fringe square [*i*,*j*], ASK(KB, $(P_{i,j} \vee W_{i,j})$) is *false* **then do**
    plan ← A*-GRAPH-SEARCH(ROUTE-PB([*x*,*y*], *orientation*, [*i*,*j*], *visited*))
    action ← POP(*plan*)
  else *action* ← a randomly chosen move
  return *action*

Propositional Logic Practice

Clauses:

- If it rains, the aquaphobes will not vote — $R \Rightarrow \neg A$
- John will win only if the aquaphobes and vegetarians vote — $J \Rightarrow (A \wedge V)$
- Either John or Peter will win but not both — $J \Leftrightarrow \neg P$
- Want to conclude: If it rains, Peter will win — $\neg(R \Rightarrow P)$

Propositional Logic Practice

Clauses:

$R \Rightarrow \neg A$

$J \Rightarrow (A \wedge V)$

$J \Leftrightarrow \neg P$

$\neg(R \Rightarrow P)$

Convert to CNF:

$(\alpha \wedge \beta)$	$\equiv$	$(\beta \wedge \alpha)$	commutativity of $\wedge$
$(\alpha \vee \beta)$	$\equiv$	$(\beta \vee \alpha)$	commutativity of $\vee$
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Propositional Logic Practice

$R \Rightarrow \neg A = \neg R \vee \neg A$	(Implication Elimination)
$J \Rightarrow (A \wedge V) = \neg J \vee (A \wedge V)$	(Implication Elimination)
$= (\neg J \vee A) \wedge (\neg J \vee V)$	(Distribute OR over AND)
$J \Leftrightarrow \neg P = (J \Rightarrow \neg P) \wedge (\neg P \Rightarrow J)$	(Bi-cond. Elimination)
$= (\neg J \vee \neg P) \wedge (P \vee J)$	(Implication Elimination)
$\neg(R \Rightarrow P) = \neg(\neg R \vee P)$	(Implication Elimination)
$= R \wedge \neg P$	(De Morgan)

Propositional Logic Practice

Proof by Resolution:

Start with KB and ¬α, look for contradiction