

Probabilistic Reasoning Over Time

Chapter 15

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Bayesian Reasoning

- Quick recap
- Bayes Nets over time (HMMs)
- Exact Inference for HMMs
- Approximate Inference for HMMs
- All the tasks for Project 3...

Probabilistic Models

- A probabilistic model is a joint distribution over a set of variables

$$P(X_1, X_2, \dots, X_n)$$

- Given a joint distribution, we can reason about unobserved variables given observations (evidence)
- General form of a query:

Stuff you care about $\xrightarrow{\quad} P(X_q | x_{e_1}, \dots, x_{e_k}) \xleftarrow{\quad} \text{*Stuff you already know*}$

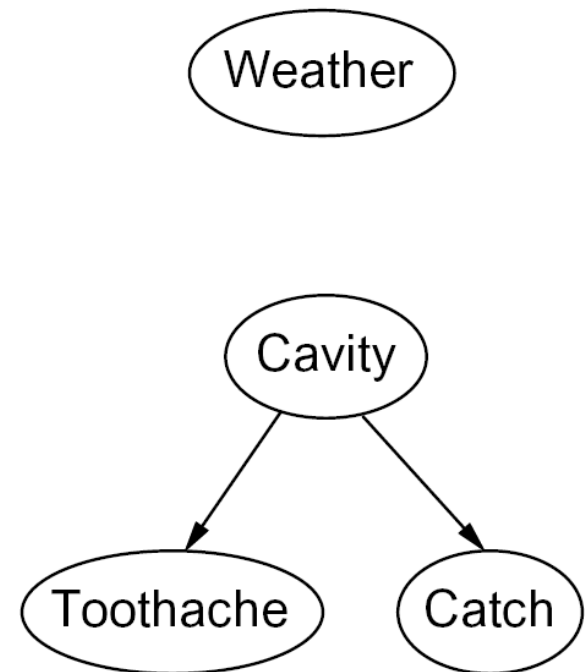
- This kind of **posterior distribution** is also called the **belief function** of an agent which uses this model

Bayes' Nets: Big Picture

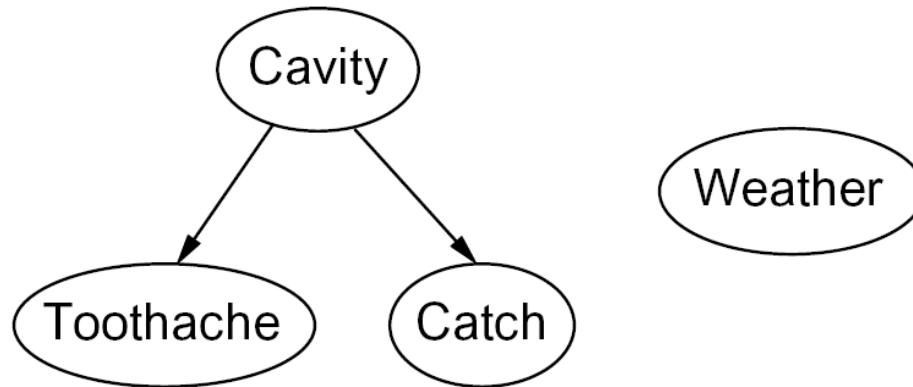
- Two problems with using full joint distribution tables as our probabilistic models:
 - Unless there are only a few variables, the joint is WAY too big to represent explicitly
 - Hard to learn (estimate) anything empirically about more than a few variables at a time
- **Bayes' nets:** a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
 - More properly called **graphical models**
 - We describe how variables locally interact
 - Local interactions chain together to give global, indirect interactions

Graphical Model Notation

- **Nodes: variables (with domains)**
 - Can be assigned (observed) or unassigned (unobserved)
- **Arcs: interactions**
 - Indicate “direct influence” between variables
 - Formally: encode conditional independence



Bayes Net



- Factor the joint density according to the parent-child relationships in the graph

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

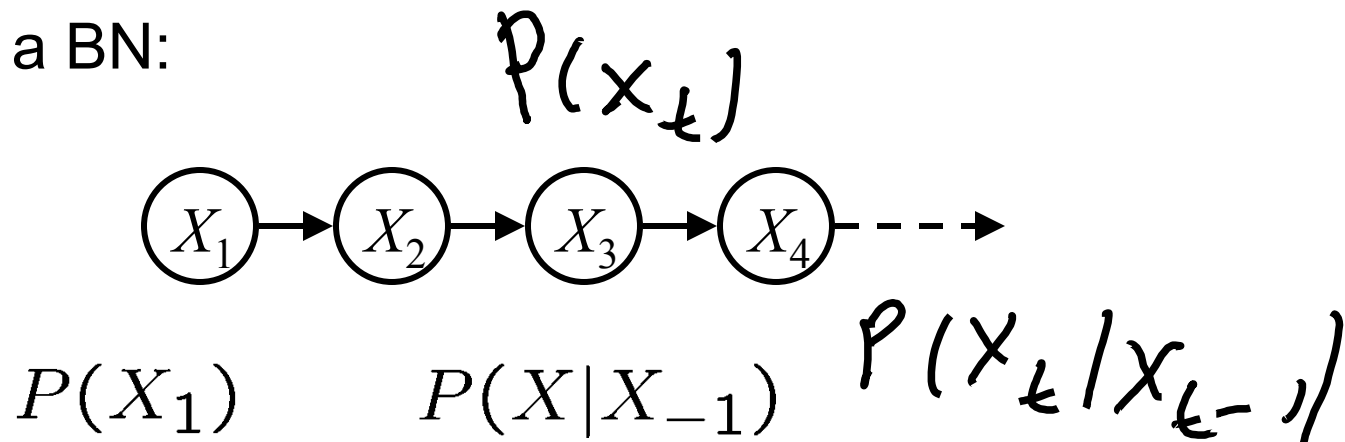
- Fundamental Theorem of Bayes Nets
- Requires directed acyclic graph (no cycles)

Reasoning over Time

- Often, we want to reason about a sequence of observations
 - Speech recognition
 - Robot localization
 - User attention
 - Medical monitoring
- Need to introduce time into our models
- Basic approach: hidden Markov models (HMMs)
- More general: dynamic Bayes' nets

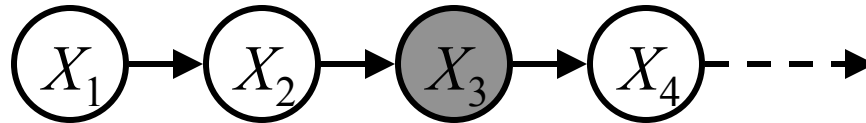
Markov Models

- A **Markov model** is a chain-structured BN
 - Each node is identically distributed (stationarity)
 - Value of X at a given time is called the **state**
 - As a BN:



- Parameters: called **transition probabilities** or dynamics, specify how the state evolves over time (also, initial probs)

Conditional Independence

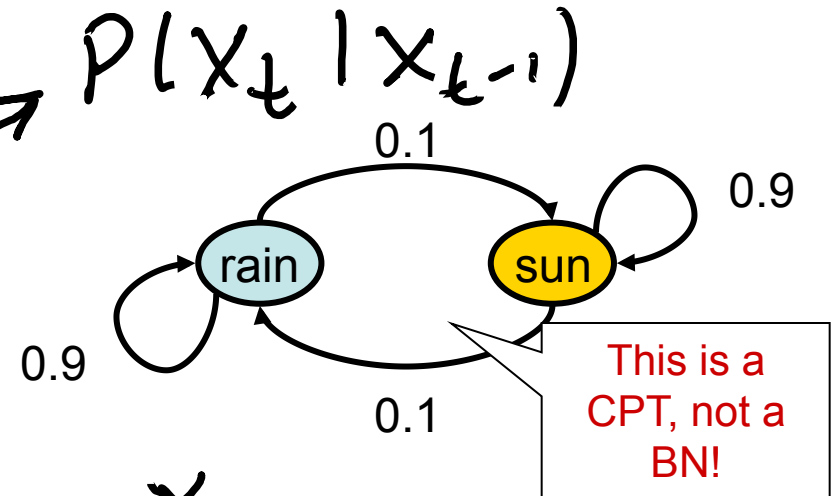


- **Basic conditional independence:**
 - Past and future independent of the present
 - Each time step only depends on the previous
 - This is called the (first order) Markov property
- **Note that the chain is just a (growing) BN**
 - We can always use generic BN reasoning on it if we truncate the chain at a fixed length

Example: Markov Chain

Weather:

- States: $X = \{\text{rain}, \text{sun}\}$
- Transitions: $P(X|X_{-1})$



$$P(X_t | X_{t-1}) =$$

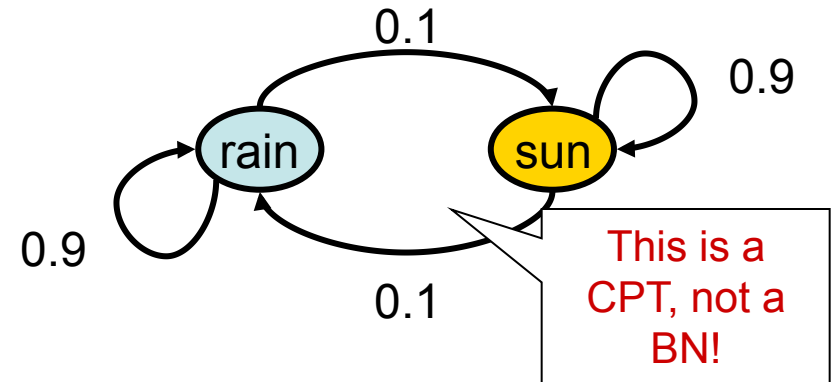
		X_{t-1}	
		R	S
X_t	R	0.9	0.1
	S	0.1	0.9

prob of going from sun to rain

Example: Markov Chain

- Weather:

- States: $X = \{\text{rain}, \text{sun}\}$
- Transitions: $P(X|X_{-1})$



- Initial distribution: $P(X_1 = \text{sun}) = 1.0$
- What's the probability distribution after one step?

$$\begin{aligned} P(X_2 = \text{sun}) &= P(X_2 = \text{sun} | X_1 = \text{sun})P(X_1 = \text{sun}) + \\ &\quad P(X_2 = \text{sun} | X_1 = \text{rain})P(X_1 = \text{rain}) \\ &= 0.9 \cdot 1.0 + 0.1 \cdot 0.0 = 0.9 \end{aligned}$$

Mini-Forward Algorithm

- Question: probability of being in state x at time t ?
- Slow answer:
 - Enumerate all sequences of length t which end in x
 - Add up their probabilities

$$P(X_t = \text{sun}) = \sum_{x_1 \dots x_{t-1}} P(x_1, \dots, x_{t-1}, \text{sun})$$

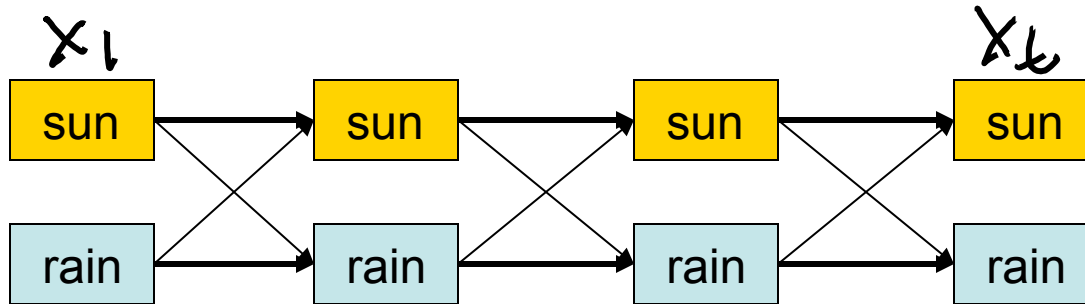
$$P(X_1 = \text{sun})P(X_2 = \text{sun}|X_1 = \text{sun})P(X_3 = \text{sun}|X_2 = \text{sun})P(X_4 = \text{sun}|X_3 = \text{sun})$$

$$P(X_1 = \text{sun})P(X_2 = \text{rain}|X_1 = \text{sun})P(X_3 = \text{sun}|X_2 = \text{rain})P(X_4 = \text{sun}|X_3 = \text{sun})$$

⋮

Mini-Forward Algorithm

- Question: What's $P(X)$ on some day t ?
 - An instance of variable elimination!



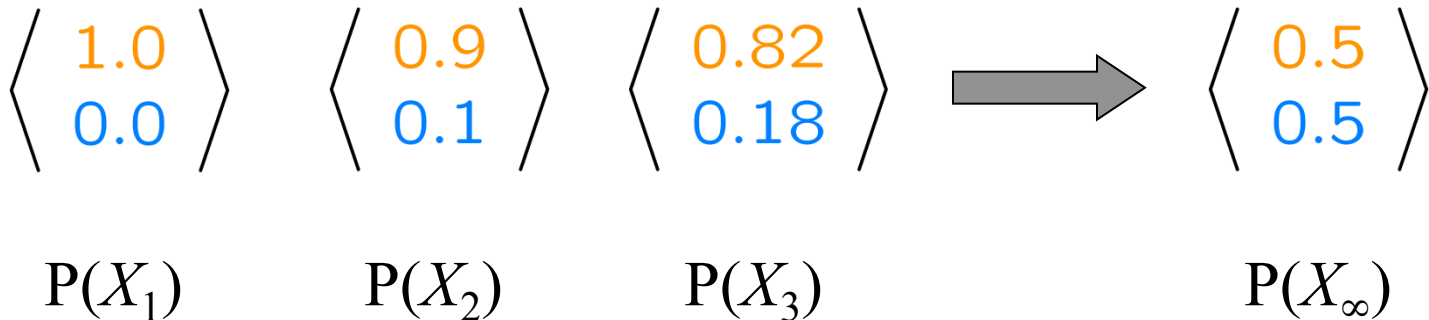
$$P(x_t) = \sum_{x_{t-1}} P(x_t | x_{t-1}) P(x_{t-1})$$

$$P(x_1) = \text{known}$$

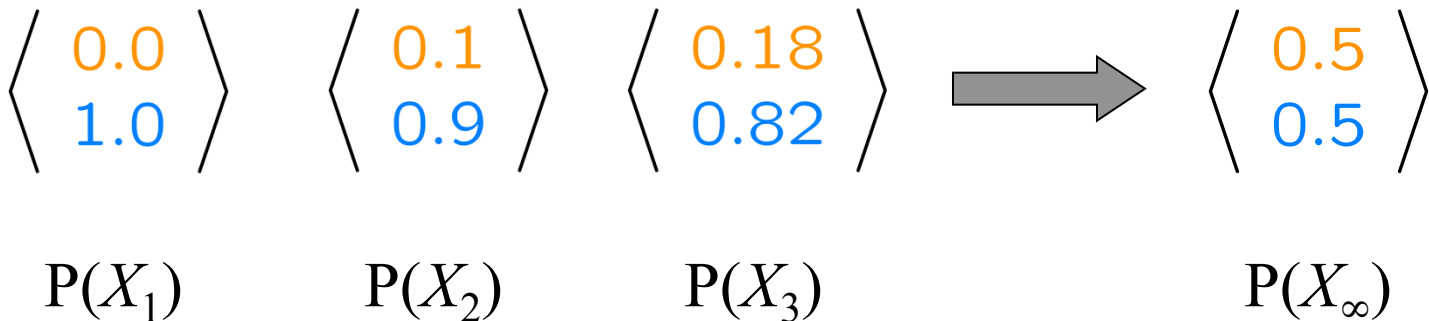
Forward simulation

Example

- From initial observation of sun



- From initial observation of rain



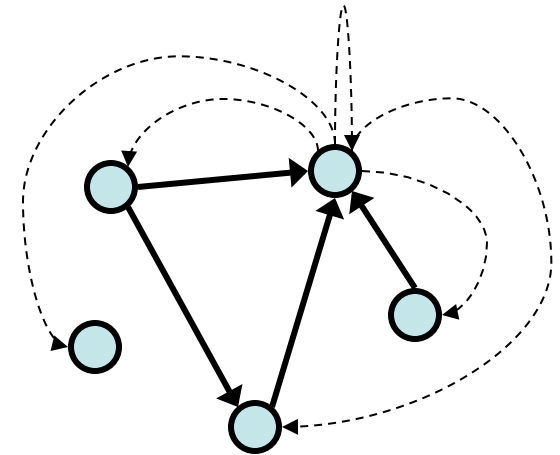
Stationary Distributions

- If we simulate the chain long enough:
 - What happens?
 - Uncertainty accumulates
 - Eventually, we have no idea what the state is!
- Stationary distributions:
 - For most chains, the distribution we end up in is independent of the initial distribution
 - Called the **stationary distribution** of the chain
 - Usually, can only predict a short time out

Web Link Analysis

- PageRank over a web graph

- Each web page is a state
- Initial distribution: uniform over pages
- Transitions:
 - With prob. c , uniform jump to a random page (dotted lines, not all shown)
 - With prob. $1-c$, follow a random outlink (solid lines)

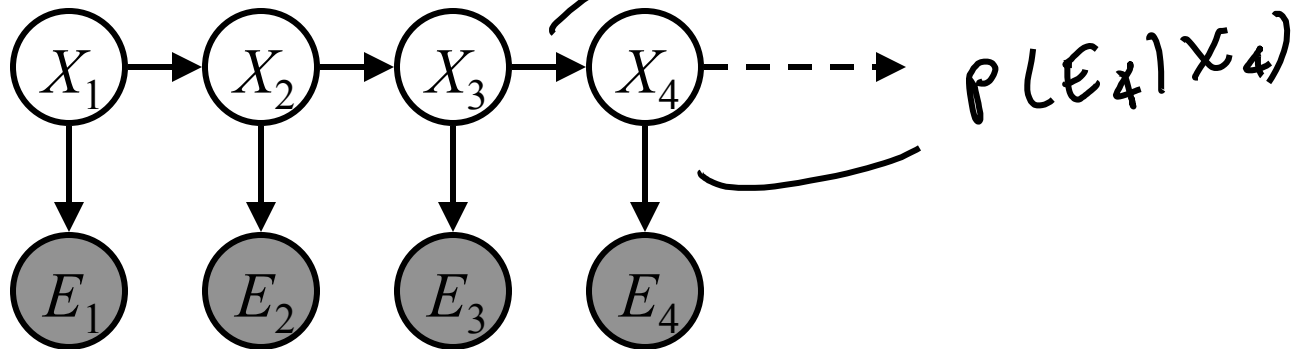


- Stationary distribution

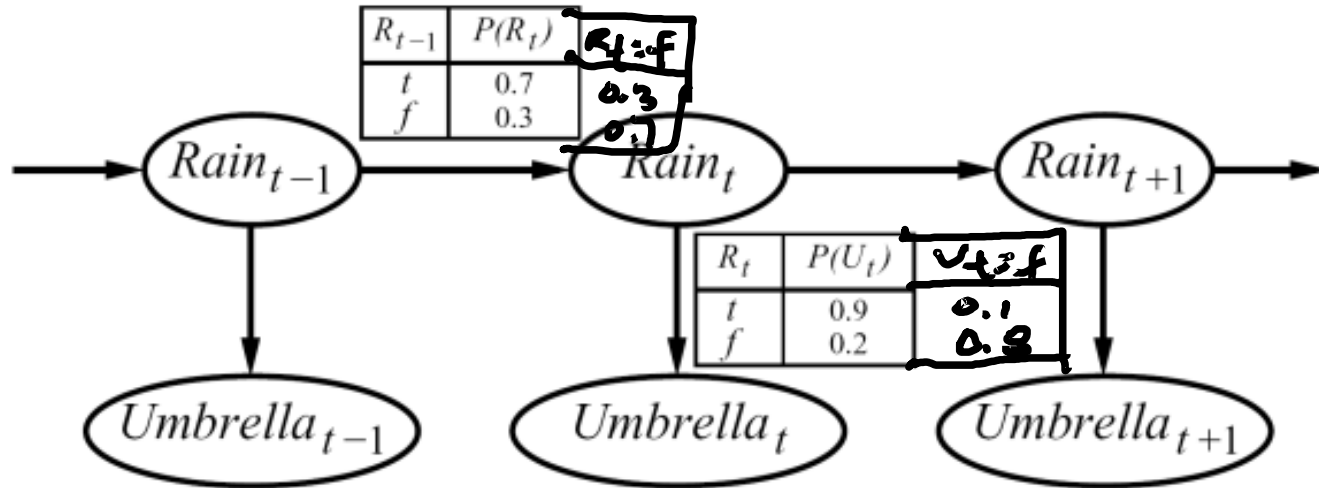
- Will spend more time on highly reachable pages
- E.g. many ways to get to the Acrobat Reader download page
- Somewhat robust to link spam
- Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)

Hidden Markov Models

- Markov chains not so useful for most agents
 - Eventually you don't know anything anymore
 - Need observations to update your beliefs
- Hidden Markov models (HMMs)
 - Underlying Markov chain over states X
 - You observe outputs (Effects) at each time step
 - As a Bayes' net this looks like:



Example



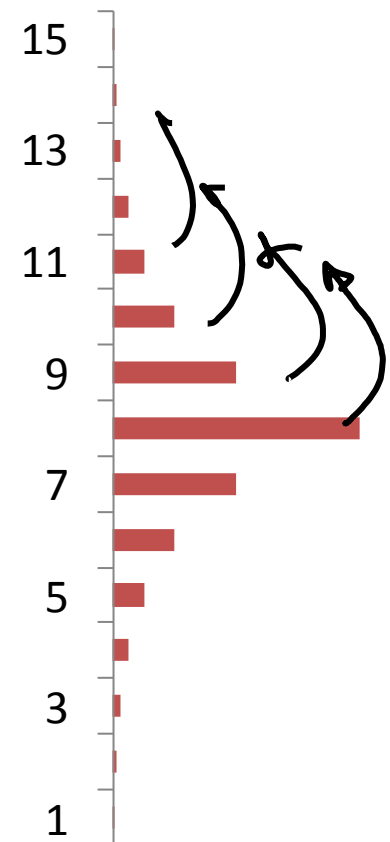
- An HMM is defined by:
 - Initial distribution: $P(X_1)$
 - Transition Model: $P(X|X_{-1})$
 - Emission Model: $P(E|X)$

P3: Ghostbusters

- **Plot:** Pacman's grandfather, Grandpac, learned to hunt ghosts for sport.
- He was blind, but could hear the ghosts 'banging and clanging.'
- **Transition Model:** All ghosts move
 - **Plot:** Pacman's grandfather, Grandpac, learned to hunt ghosts for sport.
 - He was blind, but could hear the

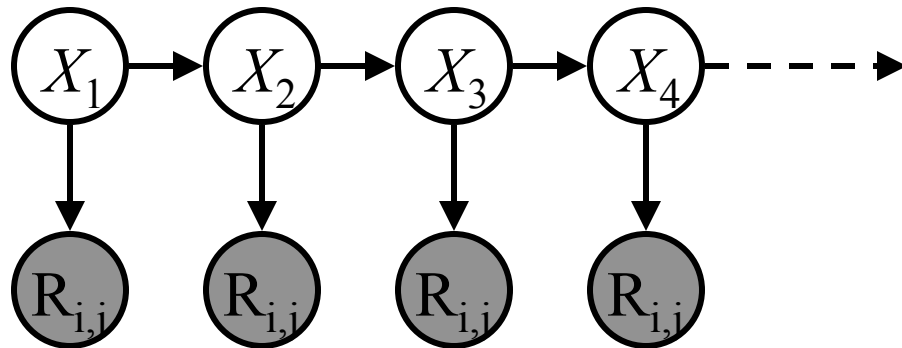
Sample $\tilde{E}_t \mid X_t = 8$

Noisy distance prob
True distance = 8



Ghostbusters HMM

- X = location of 1 ghost
- $P(X_1)$ = uniform
- $P(X_2|X_1)$ = biased to move clockwise, but sometimes move in a random direction or stay in place
- $P(R_{ij}|X) =$ sensor reading at $\langle i,j \rangle$
red means close, green means far away.



We stopped here on Wed

1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$P(X_1)$

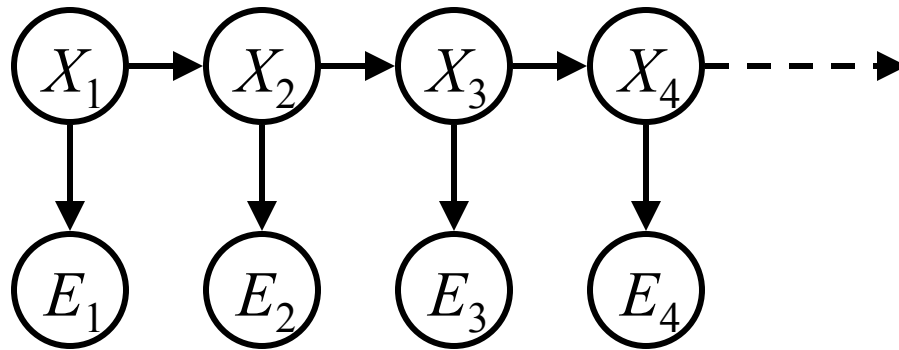
R_{1,2}

1/6	1/6	1/2
0	1/6	0
0	0	0

$P(X_2|X_1=\langle 1,2 \rangle)$

Conditional Independence

- HMMs have two important independence properties:
 - 1st order Markov process, future depends on 1 past state
 - Current Effects independent of all else given current state

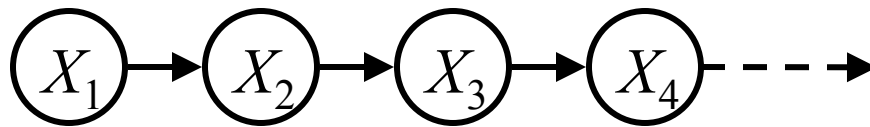


Real HMM Examples

- **Speech recognition HMMs:**
 - Observations are acoustic signals (continuous valued)
 - States are specific positions in specific words (so, tens of thousands)
- **Machine translation HMMs:**
 - Observations are words (tens of thousands)
 - States are translation options
- **Robot tracking:**
 - Observations are range readings (continuous)
 - States are positions on a map (continuous)

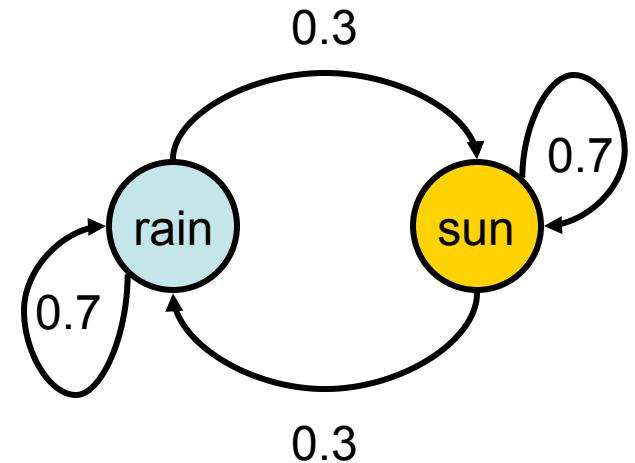
Recap: Reasoning Over Time

- Stationary Markov models



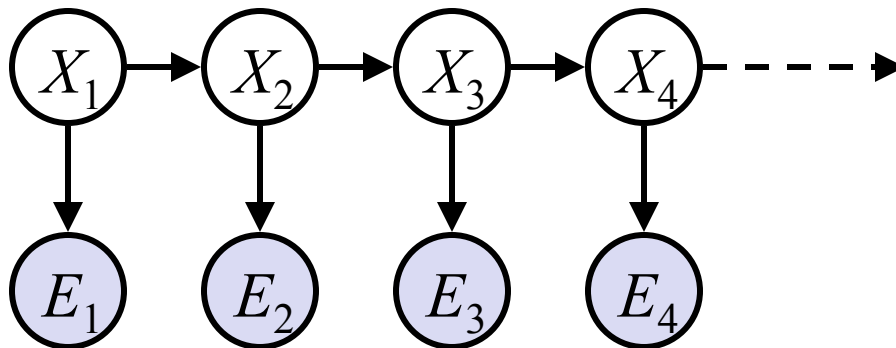
$$P(X_1)$$

$$P(X|X_{-1})$$



$$P(E|X)$$

- Hidden Markov models



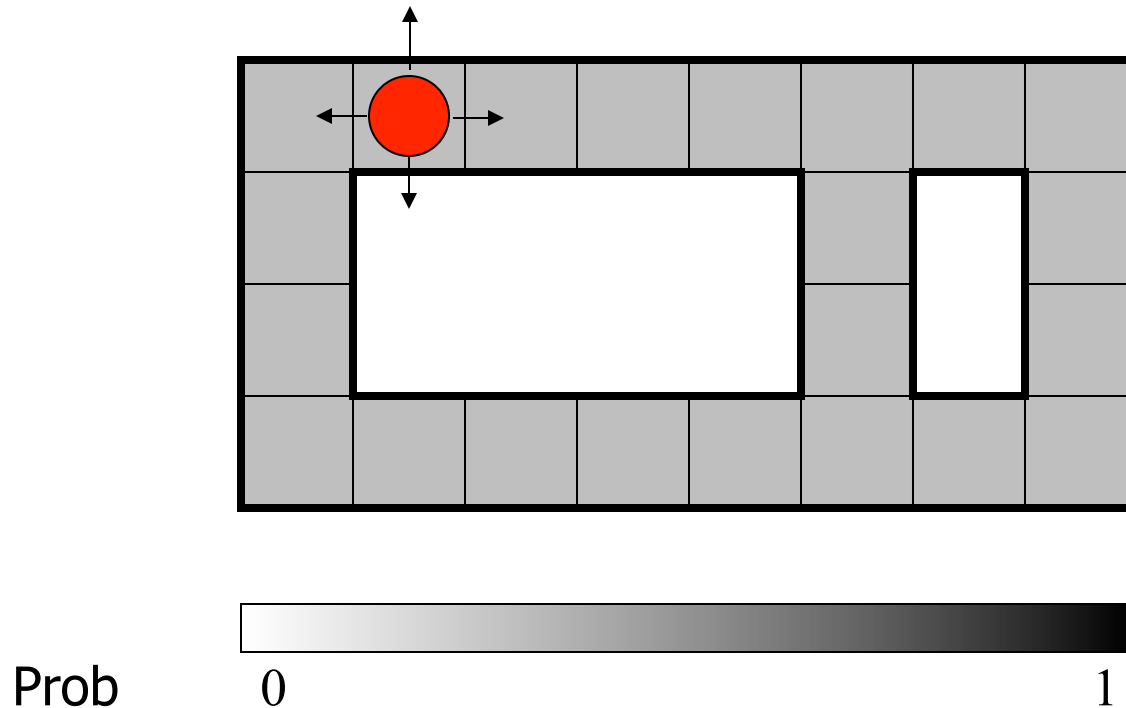
X	E	P
rain	umbrella	0.9
rain	no umbrella	0.1
sun	umbrella	0.2
sun	no umbrella	0.8

Filtering / Monitoring

- Filtering, or monitoring, is the task of tracking the distribution $B(X)$ (the belief state) over time
- We start with $B(X)$ in an initial setting, usually uniform
- As time passes, or we get observations, we update $B(X)$
- The Kalman filter was invented in the 60's and first implemented as a method of trajectory estimation for the Apollo program

Example: Robot Localization

*Example from
Michael Pfeiffer*

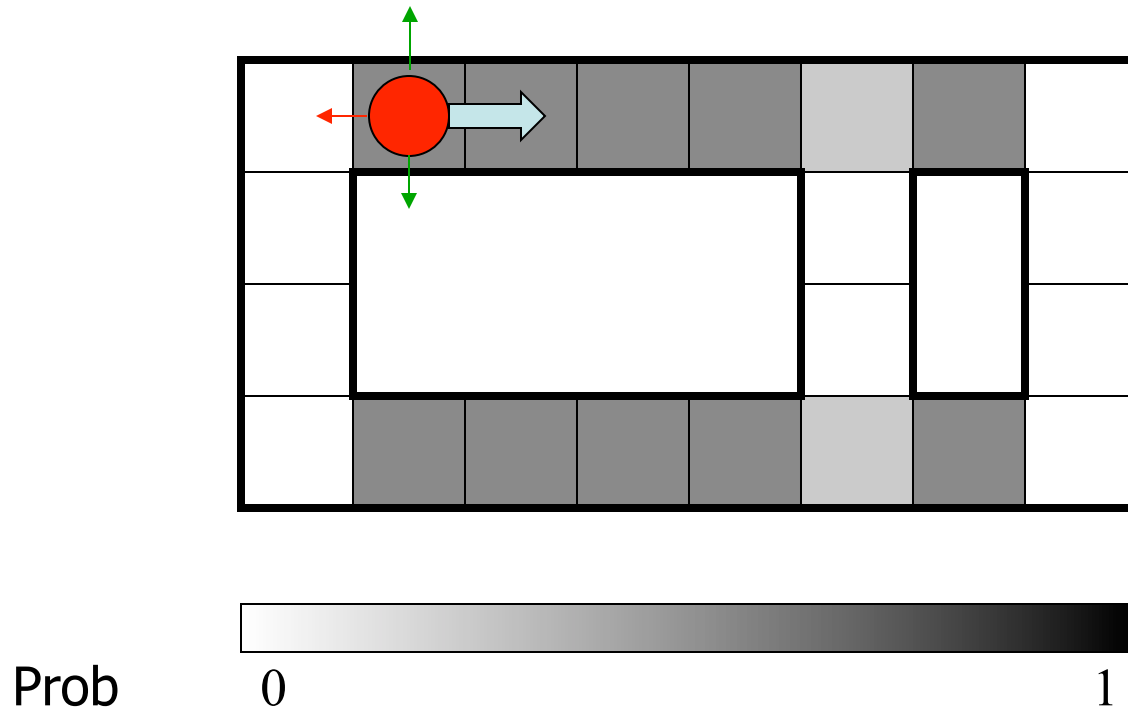


$t=0$

Sensor model: never more than 1 mistake

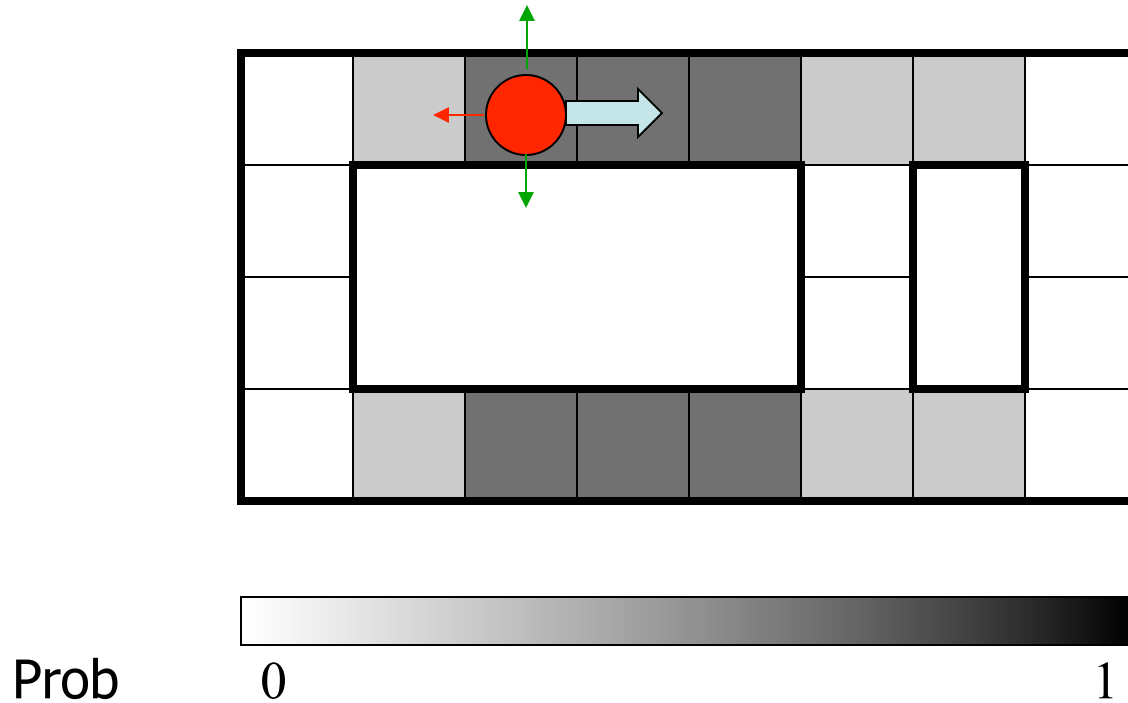
Motion model: may not execute action with small prob.

Example: Robot Localization



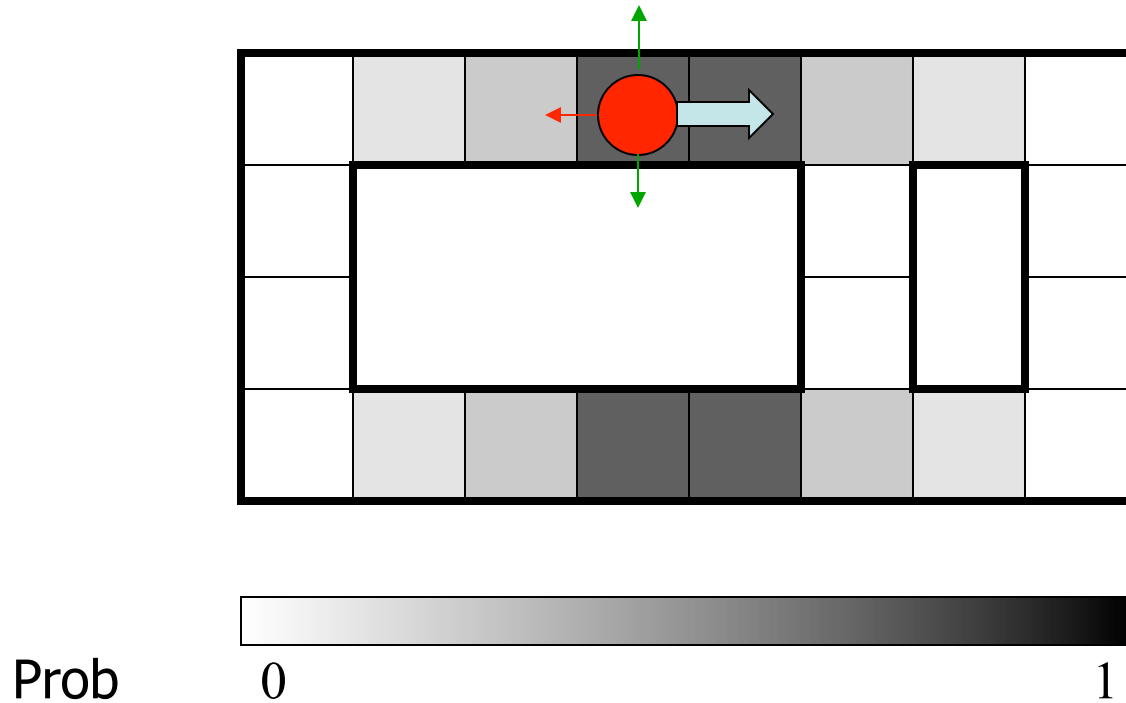
$t=1$

Example: Robot Localization



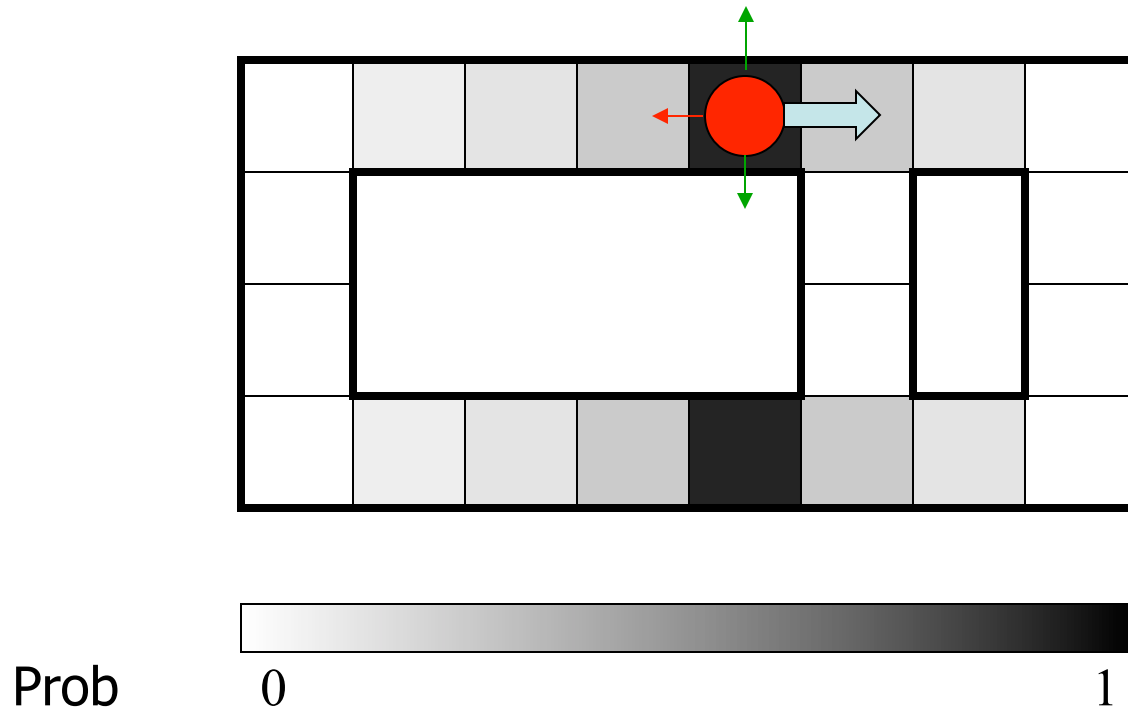
t=2

Example: Robot Localization



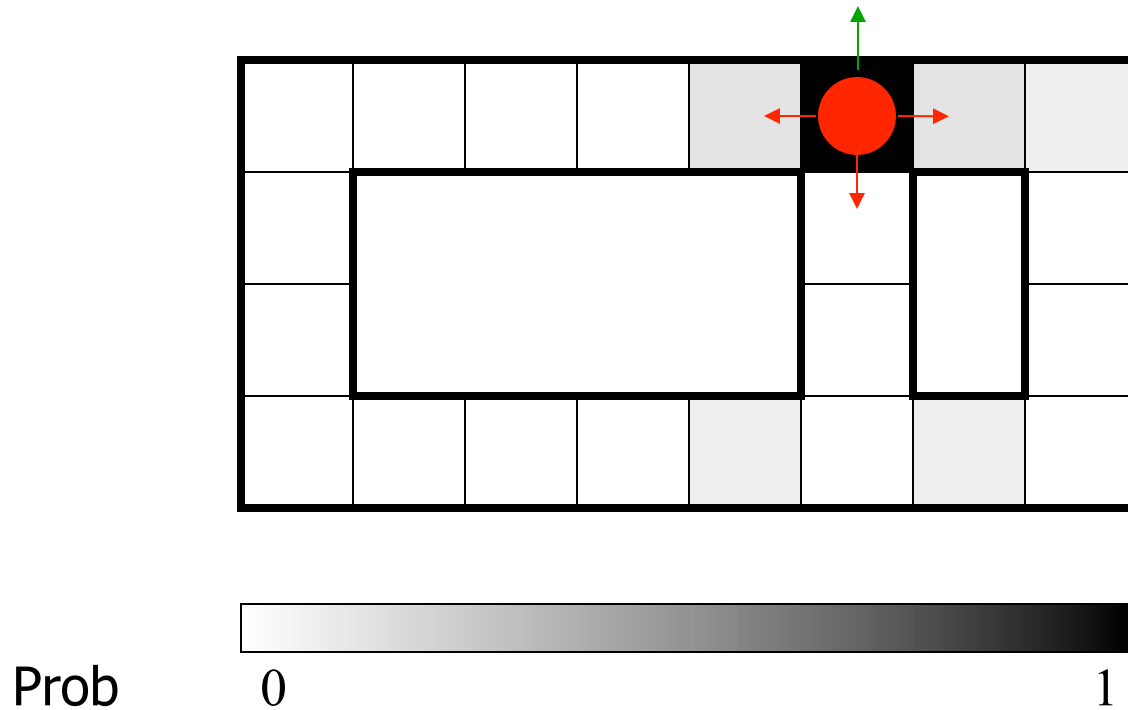
$t=3$

Example: Robot Localization



$t=4$

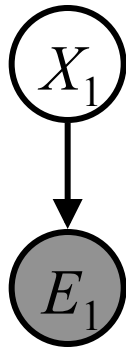
Example: Robot Localization



$t=5$

Inference Recap: Simple Cases

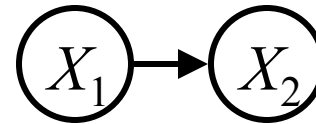
■ Observation



$$P(X_1|e_1)$$

$$\begin{aligned} P(x_1|e_1) &= P(x_1, e_1)/P(e_1) \\ &\propto_{X_1} P(x_1, e_1) \\ &= P(x_1)P(e_1|x_1) \end{aligned}$$

Passage of Time



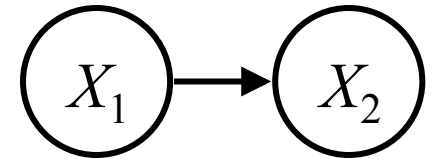
$$P(X_2)$$

$$\begin{aligned} P(x_2) &= \sum_{x_1} P(x_1, x_2) \\ &= \sum_{x_1} P(x_1)P(x_2|x_1) \end{aligned}$$

Passage of Time

- Assume we have current belief $P(X \mid \text{evidence to date})$

$$B(X_t) = P(X_t | e_{1:t})$$



- Then, after one time step passes:

$$P(X_{t+1} | e_{1:t}) = \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t})$$

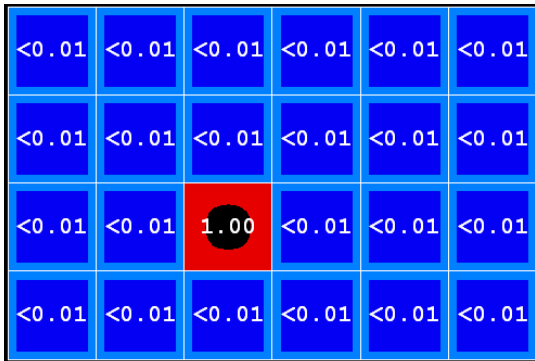
- Or, compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X' | x) B(x_t)$$

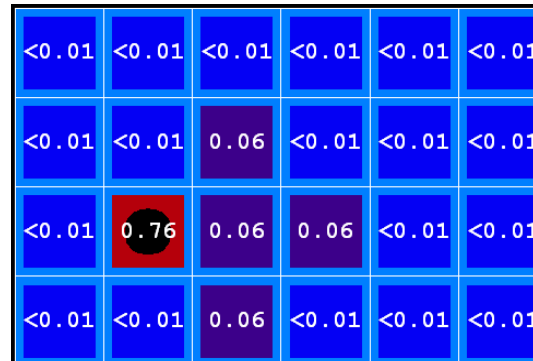
- Basic idea: beliefs get “pushed” through the transitions
 - With the “B” notation, we have to be careful about what time step t the belief is about, and what evidence it includes

Example: Passage of Time

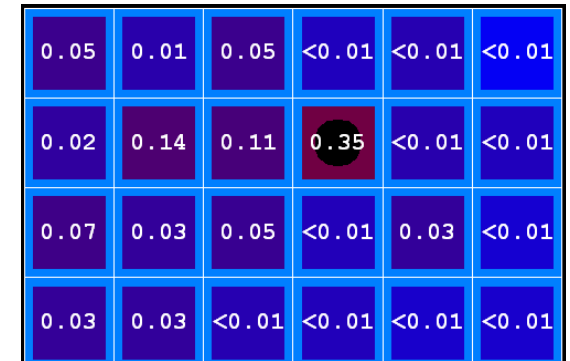
- As time passes, uncertainty “accumulates”



T = 1



T = 2



T = 5

$$B'(X') = \sum_x P(X'|x) B(x)$$

Transition model: ghosts usually go clockwise

Observation

- Assume we have current belief $P(X \mid \text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1} | e_{1:t})$$

- Then:

$$P(X_{t+1} | e_{1:t+1}) \propto P(e_{t+1} | X_{t+1}) P(X_{t+1} | e_{1:t})$$

- Or:

$$B(X_{t+1}) \propto P(e | X) B'(X_{t+1})$$

- Basic idea: beliefs reweighted by likelihood of evidence
- Unlike passage of time, we have to renormalize



Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

Before observation

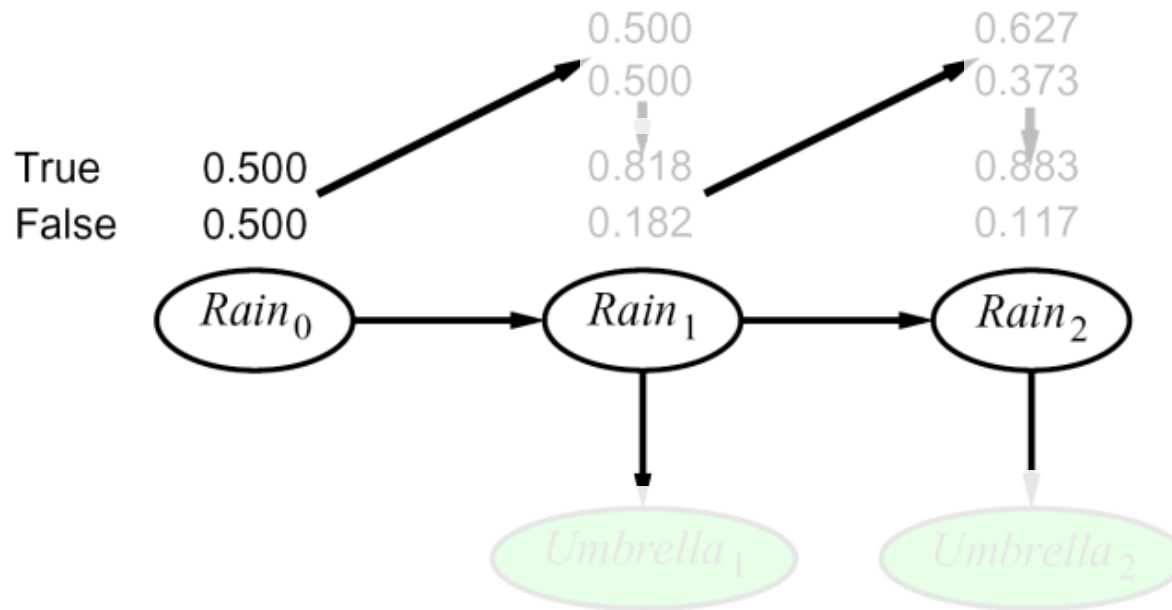
<0.01	<0.01	<0.01	<0.01	0.02	<0.01
<0.01	<0.01	<0.01	0.83	0.02	<0.01
<0.01	<0.01	0.11	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

After observation

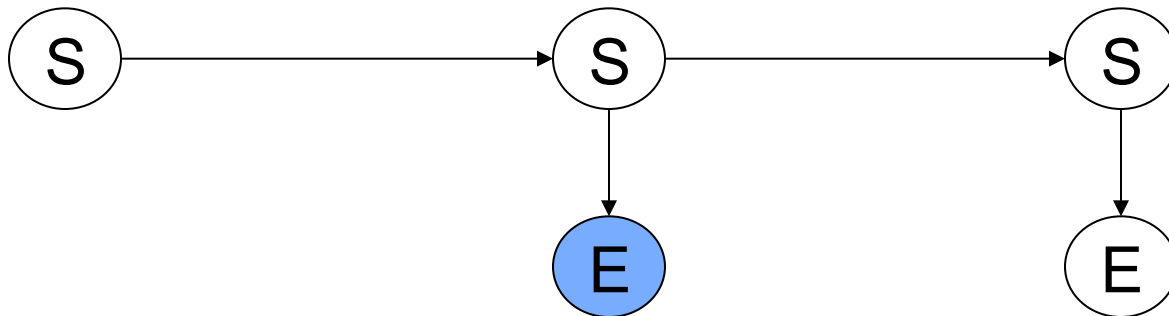
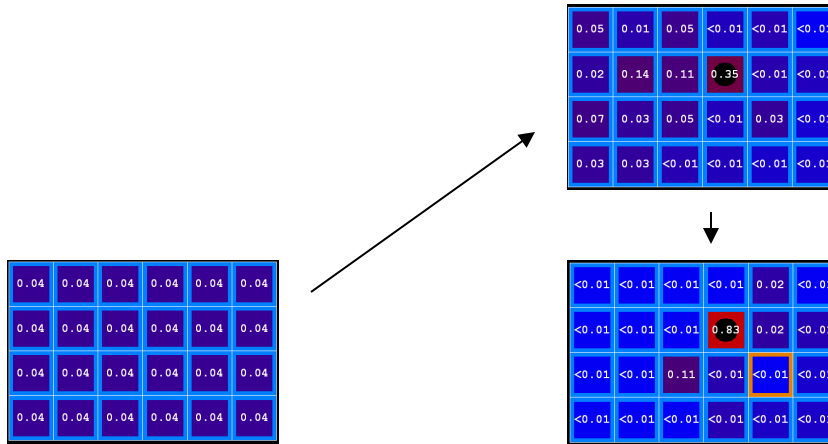
$$B(X) \propto P(e|X)B'(X)$$

Example HMM

- Iteratively Infer Time+Observation



Example HMM



The Forward Algorithm

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t|e_{1:t})$$

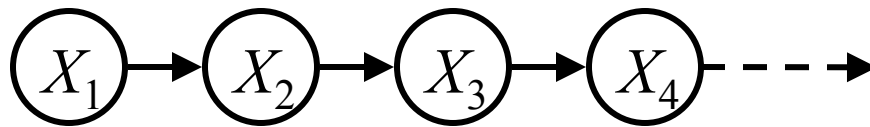
- We can derive the following updates

$$\begin{aligned} P(x_t|e_{1:t}) &\propto_X P(x_t, e_{1:t}) \\ &= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t}) \\ &= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t) \\ &= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) P(x_{t-1}, e_{1:t-1}) \end{aligned}$$

We can normalize as we go if we want to have $P(x|e)$ at each time step, or just once at the end...

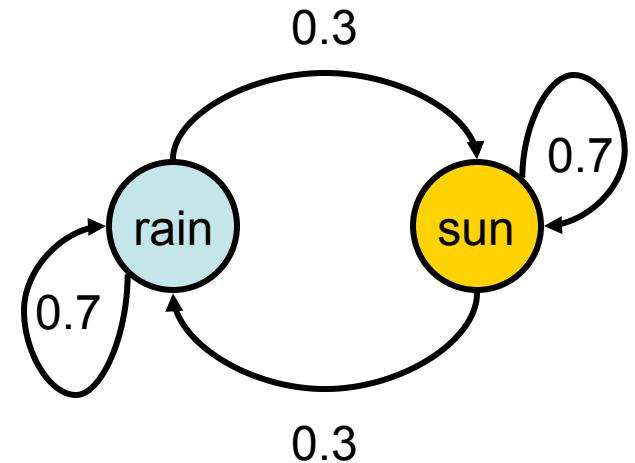
Recap: Reasoning Over Time

- Stationary Markov models



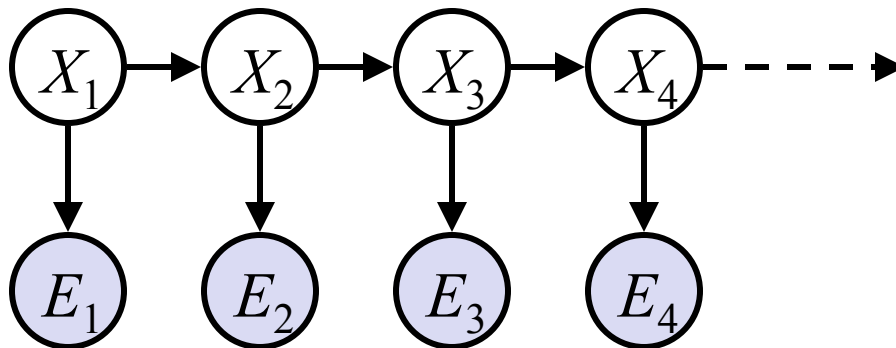
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$$P(X|X_{-1})$$



$$P(E|X)$$

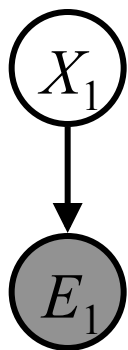
- Hidden Markov models



X	E	P
rain	umbrella	0.9
rain	no umbrella	0.1
sun	umbrella	0.2
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Exact Inference: Simple Cases

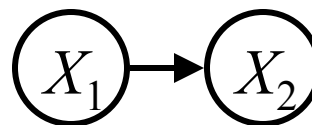
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Passage of Time



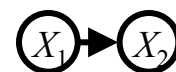
$$P(X_2)$$

$$\begin{aligned} P(x_2) &= \sum_{x_1} P(x_1, x_2) \\ &= \sum_{x_1} P(x_1)P(x_2|x_1) \end{aligned}$$

Online Belief Updates

- Every time step, we start with current $P(X \mid \text{evidence})$
- We update for time:

$$P(x_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} | e_{1:t-1}) \cdot P(x_t | x_{t-1})$$



- We update for evidence:

$$P(x_t | e_{1:t}) \propto_X P(x_t | e_{1:t-1}) \cdot P(e_t | x_t)$$



- The forward algorithm does both at once (and doesn't normalize)
- Problem: space is $|X|$ and time is $|X|^2$ per time step

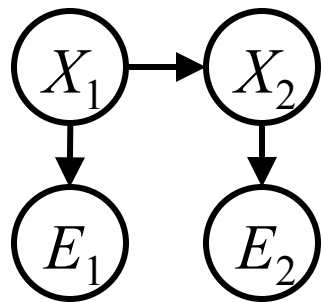
Recap: Filtering

Elapse time: compute $P(X_t | e_{1:t-1})$

$$P(x_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} | e_{1:t-1}) \cdot P(x_t | x_{t-1})$$

Observe: compute $P(X_t | e_{1:t})$

$$P(x_t | e_{1:t}) \propto P(x_t | e_{1:t-1}) \cdot P(e_t | x_t)$$



Belief: $\langle P(\text{rain}), P(\text{sun}) \rangle$

Prior on X_1 $P(X_1)$ $\langle 0.5, 0.5 \rangle$

Observe $P(X_1 | E_1 = \text{umbrella})$ $\langle .9 \ .2 \rangle \langle .5 \ .5 \rangle = \langle 0.82, 0.18 \rangle$

Elapse time $P(X_2 | E_1 = \text{umbrella})$ $\langle .7 \ .3 \rangle * .82 + \langle .3 \ .7 \rangle * .18$
 $= \langle 0.63, 0.37 \rangle$

Observe $P(X_2 | E_1 = \text{umb}, E_2 = \text{umb})$ $\langle .9 \ .2 \rangle \langle .63 \ .37 \rangle = \langle 0.88, 0.12 \rangle$

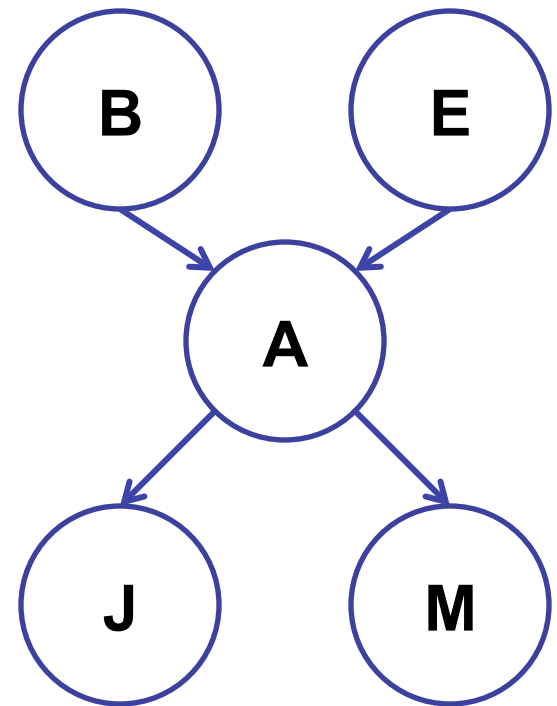
Approximate Filtering

-
- So far we've talked about Exact Inference for Filtering
 - The size of the state you want to track may be too big for exact inference
 - So we'll use a sample based approach – Particle Filters
- First let's talk about approximate Inference in Bayes Nets
 - Set evidence nodes
 - Sample the non-evidence nodes
 - Weight that sample by likelihood of the evidence

Recap: Exact Inference

- Inference: calculating some useful quantity from a joint probability distribution
- Examples:
 - Posterior probability:

$$P(Q|E_1 = e_1, \dots, E_k = e_k)$$



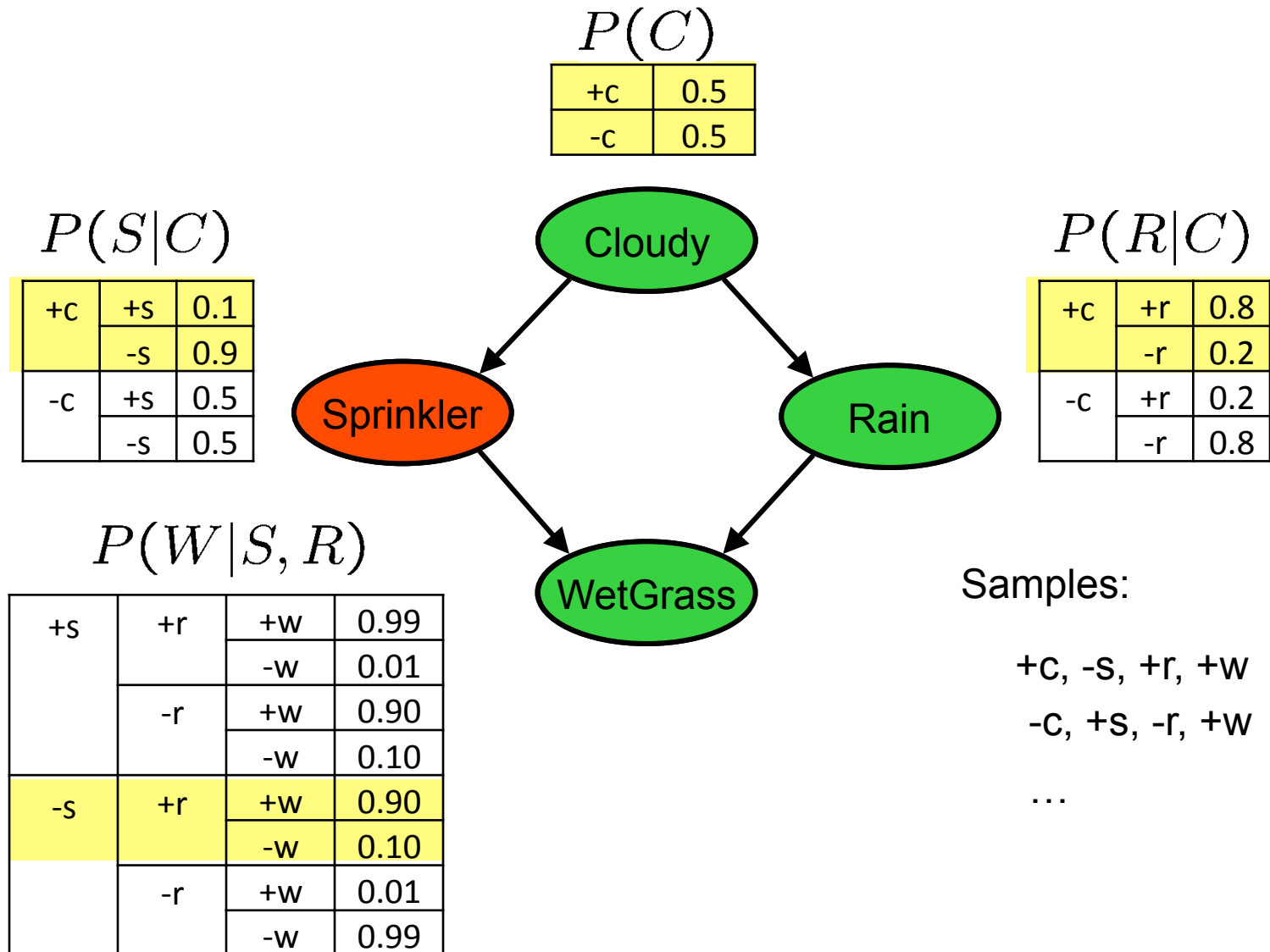
General Variable Elimination

- Query: $P(Q|E_1 = e_1, \dots, E_k = e_k)$
- Start with initial factors:
 - Local CPTs (but instantiated by evidence)
- While there are still hidden variables (not Q or evidence):
 - Pick a hidden variable H
 - Join all factors mentioning H
 - Eliminate (sum out) H
- Join all remaining factors and normalize

Approximate Inference

- Simulation has a name: sampling
- Sampling is a hot topic in machine learning, and it's really simple
- Basic idea:
 - Draw N samples from a sampling distribution S
 - Compute an approximate posterior probability
 - Show this converges to the true probability P
- Why sample?
 - Learning: get samples from a distribution you don't know
 - Inference: getting a sample is faster than computing the right answer (e.g. with variable elimination)

Prior Sampling



Prior Sampling

- This process generates samples with probability:

$$S_{PS}(x_1 \dots x_n) = \prod_{i=1}^n P(x_i | \text{Parents}(X_i)) = P(x_1 \dots x_n)$$

...i.e. the BN's joint probability

- Let the number of samples of an event be $N_{PS}(x_1 \dots x_n)$

- Then
$$\begin{aligned} \lim_{N \rightarrow \infty} \hat{P}(x_1, \dots, x_n) &= \lim_{N \rightarrow \infty} N_{PS}(x_1, \dots, x_n) / N \\ &= S_{PS}(x_1, \dots, x_n) \\ &= P(x_1 \dots x_n) \end{aligned}$$

- I.e., the sampling procedure is **consistent**

Example

- We'll get a bunch of samples from the BN:

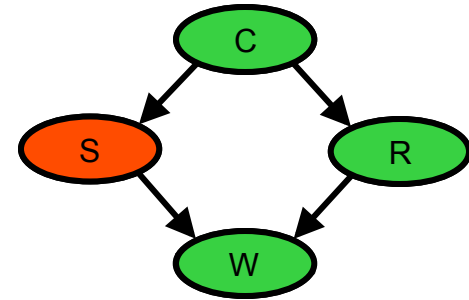
1: (+c, -s, +r, +w)

2: (+c, +s, +r, +w)

3: (-c, +s, +r, -w)

4: (+c, -s, +r, +w)

5: (-c, -s, -r, +w)

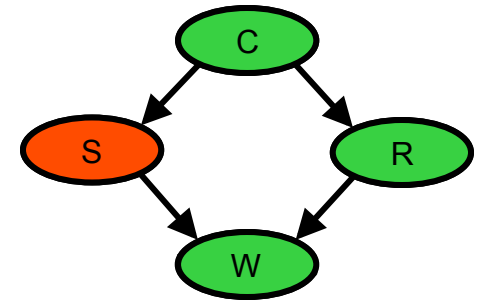


- If we want to know $P(W)$

- We have counts $\langle +w:4, -w:1 \rangle$
- Normalize to get $P(W) = \langle +w:0.8, -w:0.2 \rangle$
- This will get closer to the true distribution with more samples
- Can estimate anything else, too
- What about $P(C \mid +w)$? $P(C \mid +r, +w)$? $P(C \mid -r, -w)$?
- Fast: can use fewer samples if less time (what's the drawback?)

Rejection Sampling

- Let's say we want $P(C)$
 - No point keeping all samples around
 - Just tally counts of C as we go
- Let's say we want $P(C | +s)$
 - Same thing: tally C outcomes, but ignore (reject) samples which don't have $S=+s$
 - This is called rejection sampling
 - It is also consistent for conditional probabilities (i.e., correct in the limit)



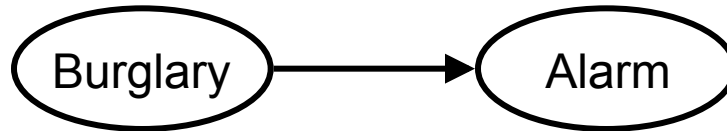
+c, -s, +r, +w
+c, +s, +r, +w
-c, +s, +r, -w
+c, -s, +r, +w
-c, -s, -r, +w

Likelihood Weighting

- Problem with rejection sampling:

- If evidence is unlikely, you reject a lot of samples
- You don't exploit your evidence as you sample
- Consider $P(B|+a)$

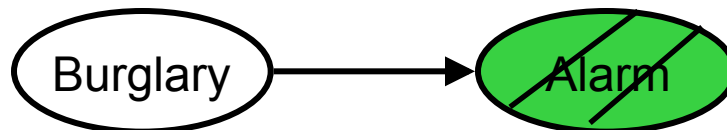
+b	.001
-b	.999



+b	+a	0.95
	-a	0.05
-b	+a	0.01
	-a	0.99

-b, -a
-b, -a
-b, -a
-b, -a
+b, +a

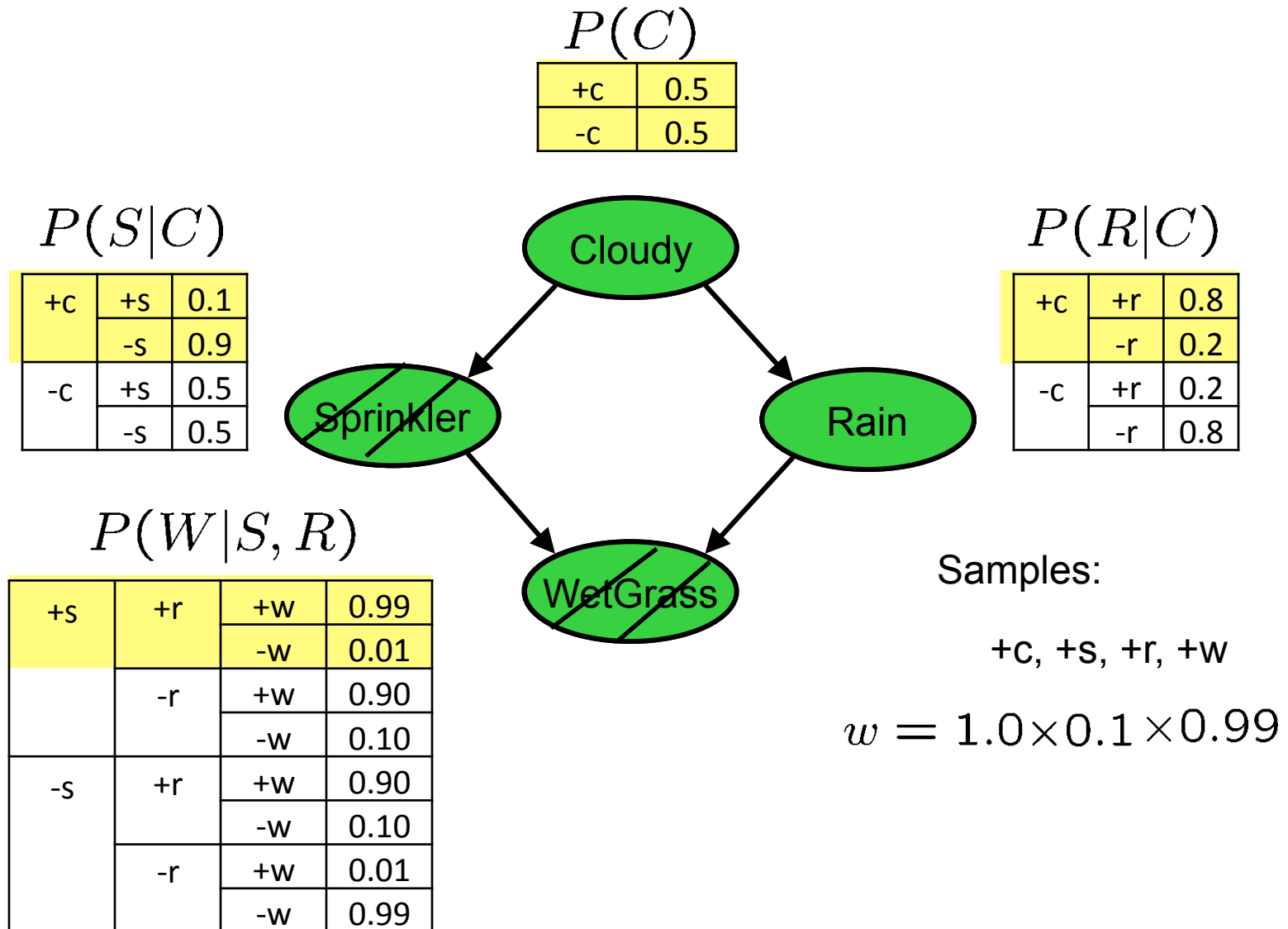
- Idea: fix evidence variables and sample the rest



-b +a
-b, +a
-b, +a
-b, +a
+b, +a

- Problem: sample distribution not consistent!
- Solution: weight by probability of evidence given parents

Likelihood Weighting



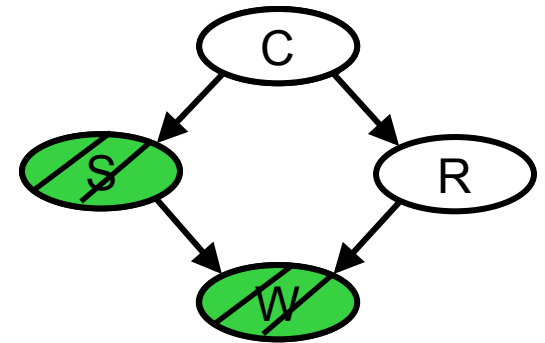
Likelihood Weighting

- Sampling distribution if z sampled and e fixed evidence

$$S_{WS}(z, e) = \prod_{i=1}^l P(z_i | \text{Parents}(Z_i))$$

- Now, samples have weights

$$w(z, e) = \prod_{i=1}^m P(e_i | \text{Parents}(E_i))$$

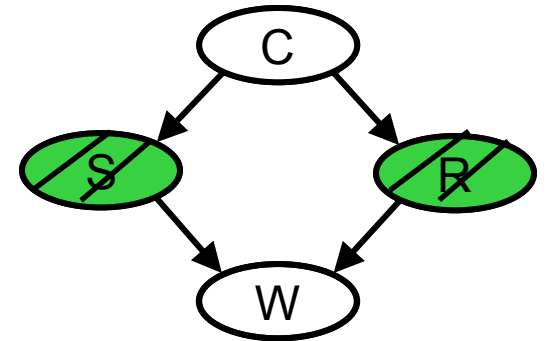


- Together, weighted sampling distribution is consistent

$$\begin{aligned} S_{WS}(z, e) \cdot w(z, e) &= \prod_{i=1}^l P(z_i | \text{Parents}(z_i)) \prod_{i=1}^m P(e_i | \text{Parents}(e_i)) \\ &= P(z, e) \end{aligned}$$

Likelihood Weighting

- Likelihood weighting is good
 - We have taken evidence into account as we generate the sample
 - E.g. here, W 's value will get picked based on the evidence values of S , R
 - More of our samples will reflect the state of the world suggested by the evidence



Approximate Filtering

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- Now let's talk about approximate Inference for our filtering problem – Particle Filters
 - Sample $P(X)$ instead of calculating exactly
 - Weight that sample by likelihood of the evidence

Particle Filtering

- Sometimes $|X|$ is too big to use exact inference
 - $|X|$ may be too big to even store $B(X)$
 - E.g. X is continuous
 - $|X|^2$ may be too big to do updates
- Solution: approximate inference
 - Track samples of X , not all values
 - Samples are called particles
 - Time per step is linear in the number of samples
 - But: number needed may be large
 - In memory: list of particles, not states
- This is how robot localization works in practice

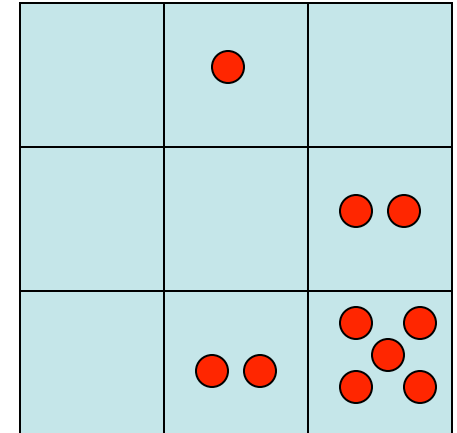
0.0	0.1	0.0
0.0	0.0	0.2
0.0	0.2	0.5



	●	
		● ●
	● ●	● ● ● ●

Representation: Particles

- Our representation of $P(X)$ is now a list of N particles (samples)
 - Generally, $N \ll |X|$
 - Storing map from X to counts would defeat the point
- $P(x)$ approximated by number of particles with value x
 - So, many x will have $P(x) = 0$!
 - More particles, more accuracy
- For now, all particles have a weight of 1



Particles:

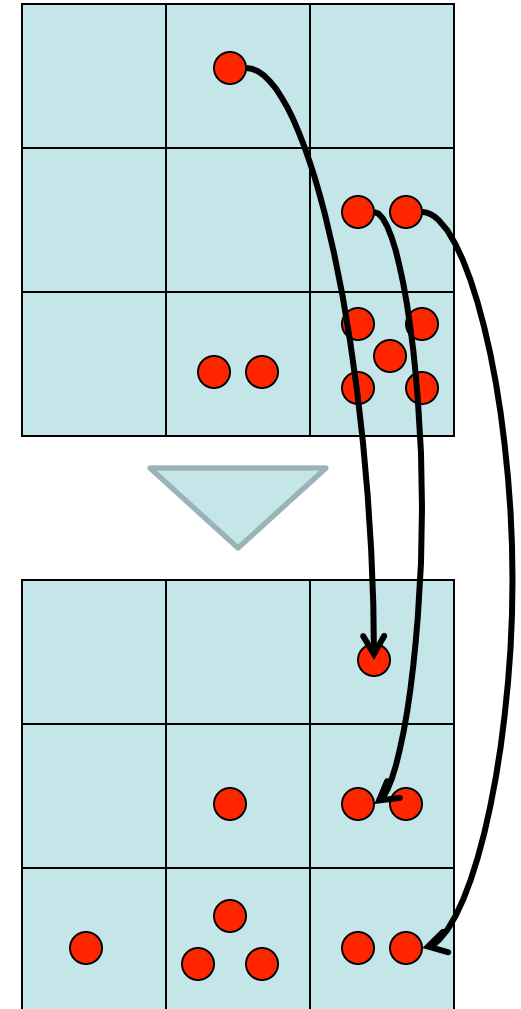
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(2,1)
(3,3)
(3,3)
(2,1)

Particle Filtering: Elapse Time

- Each particle is moved by sampling its next position from the transition model

$$x' = \text{sample}(P(X'|x))$$

- This is like prior sampling – samples' frequencies reflect the transition probs
 - Here, most samples move clockwise, but some move in another direction or stay in place
- This captures the passage of time
 - If we have enough samples, close to the exact values before and after (consistent)



Particle Filtering: Observe

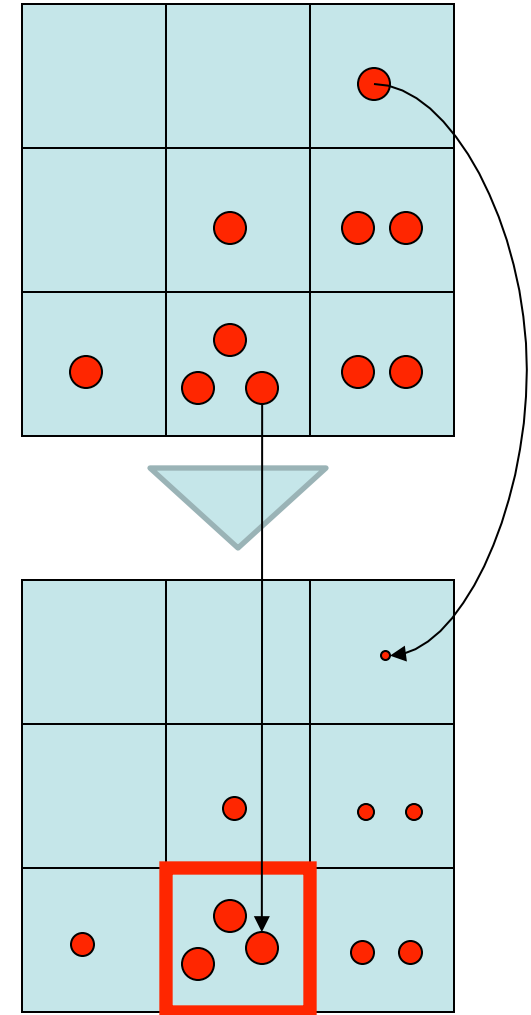
- Slightly trickier:

- Don't sample the observation, it's fixed...
- This is similar to likelihood weighting, so we downweight our samples based on the evidence

$$w(x) = P(e|x)$$

$$B(X) \propto P(e|X)B'(X)$$

- Note that, as before, the probabilities don't sum to one, since most have been downweighted (in fact they sum to an approximation of $P(e)$)

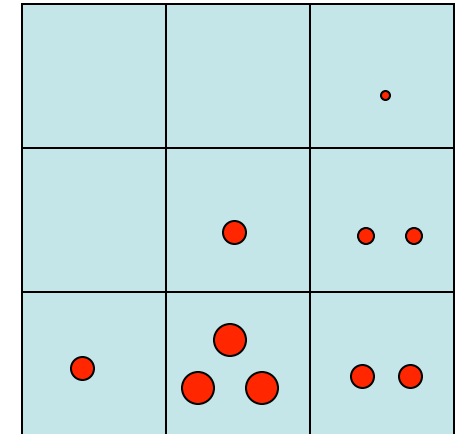


Particle Filtering: Resample

- Rather than tracking weighted samples, we resample
- N times, we choose from our weighted sample distribution (i.e. draw with replacement)
- This is equivalent to renormalizing the distribution
- Now the update is complete for this time step, continue with the next one

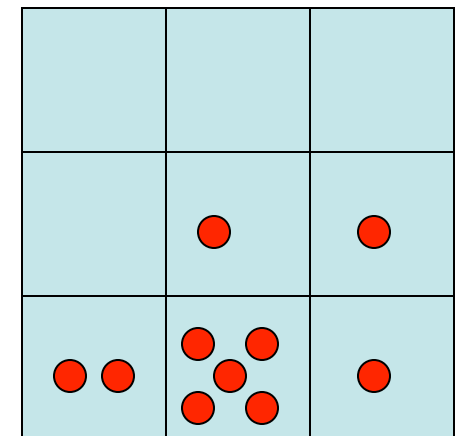
Old Particles:

(3,3) $w=0.1$
(2,1) $w=0.9$
(2,1) $w=0.9$
(3,1) $w=0.4$
(3,2) $w=0.3$
(2,2) $w=0.4$
(1,1) $w=0.4$
(3,1) $w=0.4$
(2,1) $w=0.9$
(3,2) $w=0.3$



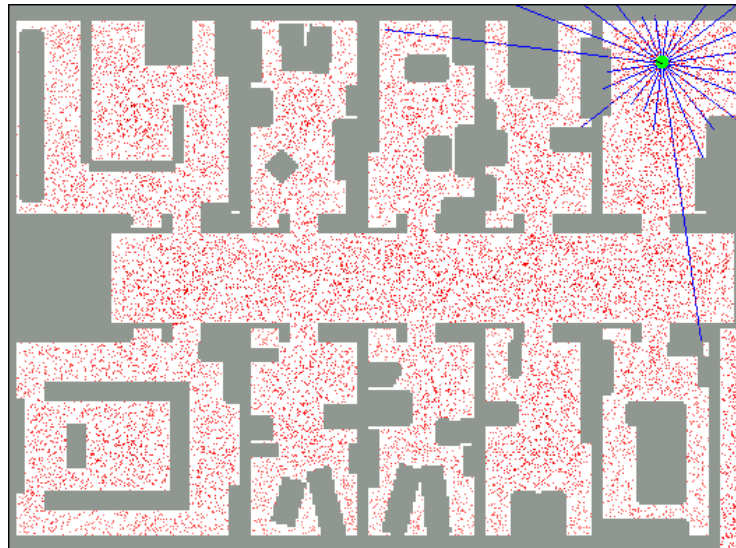
New Particles:

(2,1) $w=1$
(2,1) $w=1$
(2,1) $w=1$
(3,2) $w=1$
(2,2) $w=1$
(2,1) $w=1$
(1,1) $w=1$
(3,1) $w=1$
(2,1) $w=1$
(1,1) $w=1$



Robot Localization

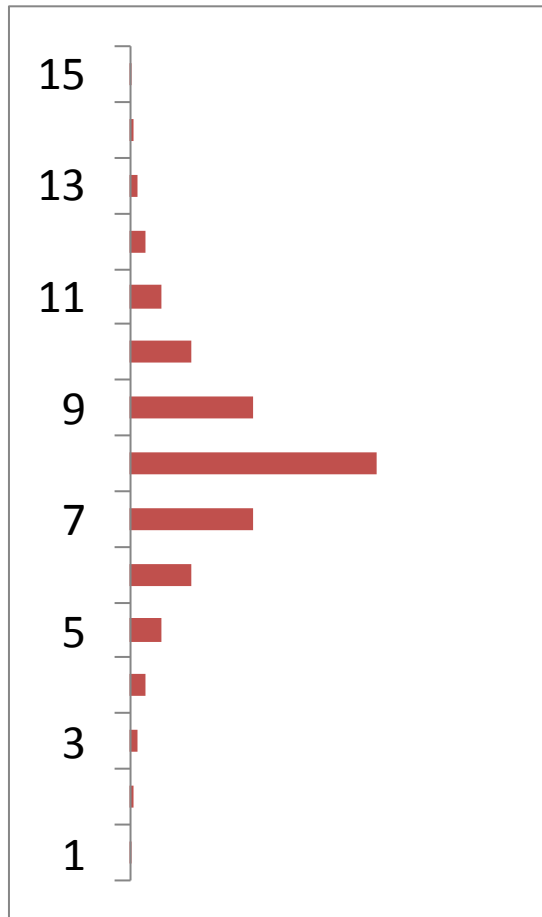
- In robot localization:
 - We know the map, but not the robot's position
 - Observations may be vectors of range finder readings
 - State space and readings are typically continuous (works basically like a very fine grid) and so we cannot store $B(X)$
 - Particle filtering is a main technique



P3: Ghostbusters

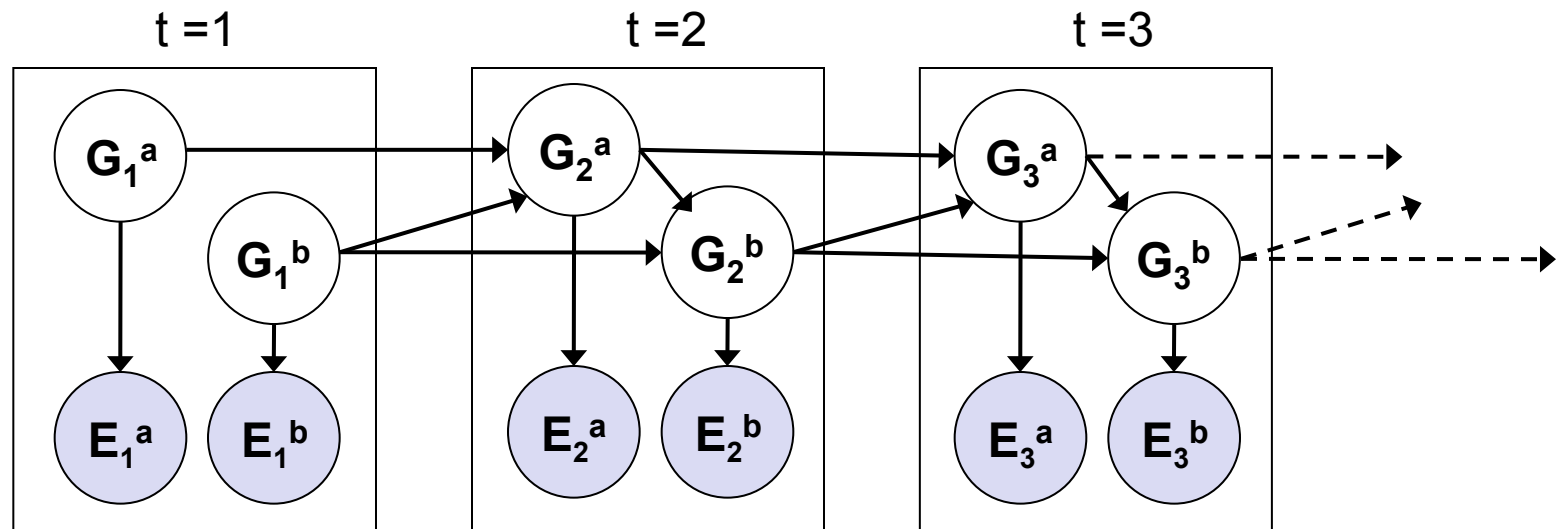
- **Plot:** Pacman's grandfather, Grandpac, learned to hunt ghosts for sport.
- He was blind, but could hear the ghosts 'banging and clanging.'
- **Transition Model:** All ghosts move randomly, but are sometimes biased
- **Emission Model:** Pacman knows a "noisy" distance to each ghost

Noisy distance prob
True distance = 8



Dynamic Bayes Nets (DBNs)

- We want to track multiple variables over time, using multiple sources of evidence
- Idea: Repeat a fixed Bayes net structure at each time
- Variables from time t can condition on those from $t-1$



DBN Particle Filters

- A particle is a complete sample for a time step
- **Initialize:** Generate prior samples for the $t=1$ Bayes net
 - Example particle: $\mathbf{G}_1^a = (3,3)$ $\mathbf{G}_1^b = (5,3)$
- **Elapse time:** Sample a successor for each particle
 - Example successor: $\mathbf{G}_2^a = (2,3)$ $\mathbf{G}_2^b = (6,3)$
- **Observe:** Weight each entire sample by the likelihood of the evidence conditioned on the sample
 - Likelihood: $P(\mathbf{E}_1^a | \mathbf{G}_1^a) * P(\mathbf{E}_1^b | \mathbf{G}_1^b)$
- **Resample:** Select prior samples (tuples of values) in proportion to their likelihood