# Lecture: Auctions (Part I)

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# Overview

This Lecture will cover (roughly) the following papers:

- Milgrom and Weber (1982)
- Laffont, Ossard, Vuong (1995)
- Guerre, Perrigne, Vuong (2000)
- *Athey and Haile (2005)

# Introduction

Auctions are (static) games of *incomplete information*. There is a literature that relates the seller's problem in the auction to that of the monopolist.

The principle difference is that in the auction context the *seller does not have perfect information* about the demand curve she faces.

Auctions are regarded to be one of the great empirical successes of the structural approach.

# Why Auctions?

- Introduces the ideas behind indentification in structural models in a formal yet digestible way
- Assymetric information is one of the most fertile areas for empirical work: auctions present an easily understandable environment for examining the impact of assym. info issue (ie we can see the rules of the game, understand the strategy set etc)
- Tools developed in the auction literature could be adapted to other environments (e.g. Laffont and Tirole)
- Auctions can be thought of as a kind of monopoly model with fixed capacity and imperfect info (see Bulow and Klemperer)
- Something like 10% of GDP is transacted through auction markets and their design and application has been, and continues to be, a topic of ongoing interest to both policy makers and business.

# Why Auctions?

- Validating basic assumptions: the theory makes a big deal of the role of assymetric informa- tion. To be confident that our models mean anything we should spend some time making sure that assymetric information does, indeed, matter. (Hendricks Porter)
- Testing theory: Theory makes some pretty specific predictions about how model primitives map to outcomes. These seem worth testing. Experimental work has been successful here.
- Evaluating policy: the optimality of design decisions usually depends on the properties on the underlying primitives we can divide these into two areas.

# Why Auctions?

Evaluating Policy

- Uncovering the specific distribution of private information: in a FPSB IPV single unit auction the reserve price depends on the specific distribution of private information
- Uncovering the properties of the structure of private information: in a single unit auction, the attractiveness of the FPSB auction depends on whether we are in the IPV or common value environment or some other private info structure.

# Introductory Theory of Auctions

Assume that there are $N$ bidders indexed by $i$ so that $i = 1, \ldots, N$. All auctions have the following two characteristics:

Private Signals $X_1, \ldots, X_N$. (generally scalar RV's).

Valuations/Utilities $V_i = u_i(X_i, X_{-i})$. These are random variables from the point of view of all bidders.

# Identification

- We are often interested in learning about the distribution of the bidders' private information $F(X_1, \ldots, X_N) \in \mathbf{F}$.
- the auction also needs some rules for allocating the winner and payments and saying what permitted bids are. Let $\mathbf{G}$ denote the set of all distributions over the space of permitted bids.
- Theory gives us a mapping from private information to bids so that $\gamma \in \mathbf{\Gamma}$ and $\gamma : \mathbf{F} \rightarrow \mathbf{G}$
- *Identification* tells us that from a given distribution of bids we can recover the most likely distribution of private information.

# Identification

## Identification

A model $(\mathbf{F}, \Gamma)$ is identified if for every $(F, \tilde{F}) \in \mathbf{F^2}$ and $(\gamma, \tilde{\gamma}) \in \Gamma^2$ that $\gamma(F) = \tilde{\gamma}(\tilde{F})$ implies that $(F, \gamma) = (\tilde{F}, \tilde{\gamma})$.

- This requires we take some kind of stand on $\gamma$ in order to recover $F$ (might be parametric or not).
- Identification is a property of the model (not of your data). You may still not recover $F$ in finite sample.
- Simulations and plots are helpful to diagnose whether your data has sufficient identifying variation.

# Informational Content

The form of $V_i = u_i(X_i, X_{-i})$ determines the *informational structure* of the auction.

Private Values $u_i(X_i, X_{-i}) = u_i(X_i) = X_i$. Your signal is your value.

(Pure) Common Values $u_i(X_i, X_{-i}) = V$. The value $V$ does not depend on your signal $X_i$ but each bidder's $X_i$ is believed to be a noisy estimate of the true random variable $V$.

We have *interdependent values* any time $u_i(\cdot)$ depends on $X_{-i}$.

# Informational Content

Another restriction that is often made is that of symmetry of bidders. Some examples include:

## Symmetric Independent Private Values

$X_i \sim F$ i.i.id. across all bidders $i$, and $V_i = X_i$. Which implies that $F(X_1, X_2, \ldots, X_N) = F(X_1) * F(X_2) \ldots * F(X_N)$ and $F(V_1, X_1, \ldots, V_N, X_N) = \prod_i [F(X_i)]^2$.

## Conditional Independent Values

Signals are independent conditional on a common $V$ so that $V_i = V \ \forall i$. Implying $F(V_1, X_1, \ldots, V_N, X_N) = F(V) \prod_i F(X_i)$.

# Auction Mechanisms

Auctions also differ in the mechanism (how price is determined, how bids are collected, and how the winner is chosen).

**first-price** high bidder wins they pay their bid.

**second-price** high bidder wins they pay second highest bid.

**English** auctioneer continuously raises price. last man standing wins.

**Dutch** price is continuously lowered. first bid wins.

Theory often uses *RET* to show when mechanisms are equivalent.

# Milgrom Weber (1982)

## Affiliation

For some $Z_1, \ldots, Z_M$ and $Z_1^*, \ldots, Z_M^*$ we say that $Z$ and $Z^*$ are affiliated IFF $F(\overline{Z})F(\underline{Z}) \geq F(Z_1, \ldots, Z_M)F(Z_1^*, \ldots, Z_M^*)$.
That is higher values of $Z^*$ make higher values of $Z$ more likely.

Assumptions:

- $V_i = u_i(X_i, X_{-i})$
- Symmetry: $F(V_1, X_1, \ldots, V_N, X_N)$ is invariant to permutations, for example: $F(V_1, X_1, \ldots, V_N, X_N) = F(V_N, X_N, \ldots, V_1, X_1)$.
- $V_1, \ldots, V_N, X_1, \ldots, X_N$ are *affiliated*
- $Y_i \equiv \max_{j \neq i} X_j$ the highest of the signals observed by bidder $i$'s rivals.
- Therefore $E[V_i | X_i, Y_i]$ is increasing in both arguments.

# Milgrom Weber (1982)

## Winners Curse

$$E[V_i|X_i] \geq E[V_i > Y_i]$$

Your expected utility is lower once we condition on winning.

$$\begin{aligned}
E[V_i|X_i] &= E_{X_{-i}} E[V_i|X_i, X_{-i}] \\
&= \underbrace{\int \cdots \int}_{N-1} E[V_i|X_i, X_{-i}] F(dX_1, \ldots, dX_N) \\
&\geq \underbrace{\int^{X_i} \cdots \int^{X_i}}_{N-1} E[V_i|X_i, X_{-i}] F(dX_1, \ldots, dX_N) \\
&= E[V_i|X_i > X_j, j \neq i] = E[V_i|X_i > Y_i]
\end{aligned}$$

Bidding your value gives you negative payoffs for every $X_i$.

# First Price Auctions

We can derive the symmetric monotone bidding strategy $b^*(\cdot)$ for the first price auction. In this case the winner pays $b^*(X_i)$ and expected profit is:

$$\begin{aligned} &= E[(V_i - b)\mathbb{I}[b^*(Y_i) < b]|X_i = x] \\ &= E_{Y_i} E[(V_i - b)\mathbb{I}[Y_i < b^{*-1}(b)]|X_i = x, Y_i] \\ &= \int_{-\infty}^{b^{*-1}(b)} (V(x, Y_i) - b) f(Y_i|x) dY_i \end{aligned}$$

First order conditions are:

$$\begin{aligned} 0 &= -\int_{-\infty}^{b^{*-1}(b)} f(Y_i|x) dY_i + \frac{1}{b^{*\prime}(x)}[(V(x,x) - b) * f_{Y_i|X_i}(x|x)] \\ 0 &= -F(Y_i|x)(x|x) + \frac{1}{b^{*\prime}(x)}[(V(x,x) - b) * f_{Y_i|X_i}(x|x)] \\ b^{*\prime}(x) &= (V(x,x) - b^*(x)) \left[\frac{f(x|x)}{F(x|x)}\right] \end{aligned}$$

Now solve the Diff Eq.

# First Price Auctions

Helps to write $L(\alpha|x) = \exp\left(-\int_\alpha^x \frac{f(s|s)}{F(s|s)}\right)$. The resulting equilibrium bidding function becomes

$$b^*(x) = \exp\left(-\int_{\underline{x}}^{x} \frac{f(s|s)}{F(s|s)} ds\right) b(\underline{x}) + \int_{\underline{x}}^{x} V(\alpha,\alpha) dL(\alpha|x)$$

With the initial condition $b(\underline{x}) = V(\underline{x},\underline{x})$ (Reserve Price ..)

## IPV Case

$$\begin{aligned} V(\alpha,\alpha) &= \alpha \\ F(s|s) &= F(s)^{N-1} \\ f(s|s) &= (n-1)F(s)^{N-2} f(s) \end{aligned}$$

# First Price Auctions: Example

## Example: Uniform Case

$V(\alpha,\alpha) = \alpha, f(s|s) = 1, F(s|s) = s$

$$\begin{aligned}
b^*(x) &= 0 + \int_0^x \alpha \exp\left( - \int_\alpha^x \frac{(n-1)f(s)}{F(s)} ds \right) \frac{(n-1)f(\alpha)}{F(\alpha)} d\alpha \\
&= \int_0^x \exp\left( -(n-1)(\log \frac{x}{\alpha}) \right) (n-1) d\alpha \\
&= \int_0^x \left(\frac{\alpha}{x}\right)^{N-1} (N-1) d\alpha \\
&= \alpha \left(\frac{N-1}{N}\right) \left(\frac{\alpha}{x}\right)^N |_0^x = \frac{N-1}{N} x
\end{aligned}$$

# Laffont Ossard Vuong (1995)

- Structural Estimation of parametric 1PA model in IPV context.
- Observe bids $b_1, \ldots, b_n$ and want to recover values $v_1, \ldots, v_n$.
- Want to evaluate *market power* of bidders $v - p$ and relation to $n$.
- Counterfactual: Optimal design of auctions
- Estimation via SNLLS

# Laffont Ossard Vuong (1995)

## Model

- $I$ bidders
- IPV information structure
- valuations are IID from $F(\cdot|z_l,\theta)$ for auction $l$
- $p^0$ is reserve price : reject $b < p^0$
- Dutch Auction: strategically identical to FPSB

## Bidding Strategy

$$b^i = e(v^i, I, p^0, F) = \begin{cases} v^i - \frac{\int_{p^0}^{v^i} F(x)^{I-1} dx}{F(v^i)^{I-1}} & \text{if } v^i > p^0, \\ 0 & o.w. \end{cases}$$

Show bid function is strictly increasing in $v_i$.

# Laffont Ossard Vuong (1995)

- Dataset: observe only winning bid $b_l^w$ for each auction $l$ (Dutch Auction).
- Monotonicity $b^w = e(v_{(I)}, I, p^0, F)$ where $v_{(I)} \equiv max_i v^i$
- Symm/Indep tells us that $v_{(I)} \sim F(\cdot|z_l, \theta)^I$ with density $I \cdot F^{I-1}(\cdot) f(\cdot)$

# Laffont Ossard Vuong (1995)

## Estimation

Find $\theta$ the parameters of $F$ by matching winning bid with its expectation

$$\begin{aligned}
E_{v(I)>p^0}(b^w) &= \int_{p^0}^{\infty} e(v_{(I)}, I, p^0, F) I \cdot F(v|\theta)^{I-1} f(v|\theta) dv \\
&= I \left( v^i - \frac{\int_{p^0}^{v^i} F(x)^{I-1} dx}{F(v^i)^{I-1}} \right) \int_{p^0}^{\infty} F(v|\theta)^{I-1} f(v|\theta) dv \\
(*) \quad &= I \int_{p^0}^{\infty} \left( v^i F(v^i)^{I-1} - \int_{p^0}^{v^i} F(x)^{I-1} dx \right) F(v|\theta)^{I-1} f(v|\theta) dv
\end{aligned}$$

# Laffont Ossard Vuong (1995)

## SNLLS: Naive Method

Find the $\theta$ that minimizes sum of squares between winning bids and predicted winning bids in $(*)$:

- Draw valuations $v^s, s = 1, \ldots, S$ i.i.d. according to $f(v|\theta)$. (How?)
- For each simulated $v_s$, compute $V_s = v_s F(v_s|\theta)^{I-1} - \int_{p^0}^{v_s} F(x|\theta)^{I-1} dx$ (note the second integral can make it tricky).
- Approximate expected bid as $\frac{1}{S} \sum_s V_s$

# Revenue Equivalence (Myerson (1981), Klemperer (1999))

## Revenue Equivalence Theorem

Assume each of $N$ risk-neutral bidders has a privately- known signal $X$ independently drawn from a common distribution $F$ that is strictly increasing and atomless on its support $[\underline{X}, \overline{X}]$. Any auction mechanism which is (i) efficient in awarding the object to the bidder with the highest signal with probability one; and (ii) leaves any bidder with the lowest signal $X$ with zero surplus; yields the same expected revenue for the seller, and results in a bidder with signal $x$ making the same expected payment.

- Independent signal case: no advantage to make agents payoff depend on rivals' reports
- Offer independent contract to each individual agent.
- In efficient auction the probability that agent with signal $x$ wins is the same (same expected payment schedule).

# Laffont Ossard Vuong (1995)

This is exploited for estimation purposes:

By RET

- expected revenue in 1PA same as expected revenue in 2PA
- expected revenue in 2PA is $Ev^{(I-1)}$
- with reserve price, expected revenue in 2PA is $E\max(v^{(I-1)}, p^0)$ (with IPV, reserve price $r$ screens out same subset of valuations $v \leq r$ in both formats).

# Laffont Ossard Vuong (1995)

## Improved SNLLS Procedure

$$Eb^*(v_{(I)}) = E[\max(v_{(I-1)}, p^0)]$$

For a given $\theta$ and each $s = 1, \dots S$:

- Draw $v_1^s, \dots, v_{I_l}^s$ simulated valuations for each auction $l$
- Sort the draws: $v_{1:I_l}^s < \cdots < v_{I_l:I_l}^s$.
- Compute winning bid $b_l^{w,s} = v_{I_l-1:I_l}$
- If $b_l^{w,s} < p_l^0$, set $b_l^{w,s} = p_l^0$.
- Approximate $E(b_l^w; \theta) = \frac{1}{S} \sum_s b_l^{w,s}$.
- Estimate via SNLLS: $\min_\theta \frac{1}{L} \sum_{l=1}^{L} (b_l^w - E(b_l^w; \theta))^2$.

# Laffont Ossard Vuong (1995)

## Challenges

- Problem: bias when number of simulation draws $S$ is fixed (as number of auctions $L \to \infty$). Propose bias correction estimator, which is consistent and asymptotic normal under these conditions (tradeoff with variance).
- This clever methodology is useful for independent value models: works for all cases where revenue equivalence theorem holds.
- Does not work for affiliated value models (including common value models)
- Parametric form for $F(\theta)$.

## Guerre Perrigne Vuong (2000)

The previous paper examined how to *parametrically* estimate an auction model. But suppose we do not want to impose a form for $F(\theta)$.

$$b'(x) = (V(x,x) - b(x)) \cdot \frac{f_{y_i|x_i}(x|x)}{F_{y_i|x_i}(x|x)}; y_i \equiv \max_{j \neq i} x_i$$

### IPV Case

$$\begin{aligned}
v(x,x) &= x \\
F_{y_i|x_i}(x|x) &= F(x)^{n-1} \\
f_{y_i|x_i}(x|x) &= \frac{\partial}{\partial x} F(x)^{n-1} = (n-1)F(x)^{n-2} f(x) \\
\Rightarrow b'(x) &= (x - b(x)) \cdot (n-1)\frac{f(x)}{F(x)}
\end{aligned}$$

# Guerre Perrigne Vuong (2000)

Bidding function is monotone $b_i \equiv b(x_i)$, simple change of variables

$$G(b_i) = F(x_i); \qquad g(b_i) = \frac{f(x_i)}{b'(x_i)}$$

$$\frac{1}{g(b_i)} = (n-1)\frac{x_i - b_i}{G(b_i)} \quad \Leftrightarrow \quad x_i = b_i + \frac{G(b_i)}{(n-1)g(b_i)}$$

The RHS of the equation is observed, and we can estimate the CDF $G$ and density $g$ of the bids *nonparametrically*.

# Guerre Perrigne Vuong (2000)

## Nonparametric Estimates

$$\begin{aligned}
\hat{g}(b) &\approx \frac{1}{T \cdot n} \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{1}{h} \mathcal{K}\left(\frac{b - b_{it}}{h}\right) \\
\hat{G}(b) &\approx \frac{1}{T \cdot n} \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{I}(b_{it} \leq b) \\
\hat{x}_i &= b_i + \frac{G(\hat{b}_i)}{(n-1) g(\hat{b}_i)}
\end{aligned}$$

# Guerre Perrigne Vuong (2000)

## Two Step Estimator

- Recover $\hat{G}(b)$ and $\hat{g}(b)$ from the bid data nonparametrically.
- Recover the "pseudo-values" from the bidding function
- Recover the distribution of private values
  $\hat{f}(x) \approx \frac{1}{T \cdot n} \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{1}{h} \mathcal{K}\left(\frac{b - b_{it}}{h}\right)$ via kernel estimator.

# Estimation Details

## Kernel Estimator Properties

- A valid density function $\mathcal{K}(u) \geq 0$ and $\int_{-\infty}^{\infty} \mathcal{K}(u) = 1$
- Symmetric about zero $\mathcal{K}(u) = \mathcal{K}(-u)$
- bandwidth $h$
- Examples :
    1. $\mathcal{K}(u) = \phi(u)$ (standard normal)
    2. $\mathcal{K}(u) = \frac{1}{2}\mathbb{I}(|u| \leq 1)$ (uniform kernel)
    3. $\mathcal{K}(u) = \frac{3}{4}(1 - u^2)\mathbb{I}(|u| \leq 1)$ (Epanechnikov kernel)

# Estimation Details

## Kernel Estimator Bandwidth

Bandwidth is an important property of the kernel estimator. Consider the histogram:

$$h(b) = \frac{1}{Tn} \sum_t \sum_i \mathbb{I}(b_{it} \in [b - \epsilon, b + \epsilon])$$

for small $\epsilon > 0$ then the histogram at $b$, $h(b)$ is the frequency of often the observed bid lands within an $\epsilon$-neighborhood of $b$

## Kernel Estimator Bandwidth

Consider kernel $\frac{1}{h}\mathcal{K}(\frac{b-b_{it}}{h})$

- Always positive, takes values in $\mathbb{R}^+$ larger than 1.
- Values of $b$ near $b_{it} \to$ large values, far $\to$ small values.
- $h$ and bias/variance tradeoff
- smoothed histogram/neighborhood size

# Parametric or Not?

- for the theory to work we need a compact support of $u_i$. Issues arise when the estimator is near the boundary of the support - consistency is no longer assured. Li, Perrigne and Vong (2002, p 180 on) sort this out
- while nonparametric estimation is very attractive ex ante it may be that the data set you are facing works better with the extra structure imposed by a parametric specification. That is more structure may give you more power.
- In considering whether to go parametric think about what you want to use the estimates for, where identification is coming from and whether any auxillary data can but used to justify the parametric assumption.
- there is still a lot of structure being imposed on the data here. Take some time to think about how much work the functional form assumptions are doing.
- lastly, and most importantly, note the big assumptions on auction heterogeniety, bidder heterogeniety etc.

# Identification Results

Athey Haile (2002) cover a number of identification results (FPSB, SPSB, Symm, Asymm) and show when we can use only a subset of bids to identify values.

## Example: Single Bid Identification

Identification continues to hold, even when only the highest-bid in each auction is observed. Specifically, if only $b_{n:n}$ is observed, we can estimate $G_{n:n}$, the CDF of the maximum bid, from the data. Note that the relationship between the CDF of the maximum bid and the marginal CDF of an equilibrium bid is $G_{n:n}(b) = G(b)^n$ implying that $G(b)$ can be recovered from knowledge of $G_{n:n}(b)$. Once $G(b)$ is recovered, the corresponding density g(b) can also be recovered.

# Non-Identification Results

## Laffont and Vuong (1996) nonidentification result

from observation of bids in n-bidder auctions, the affiliated private value model (ie. a PV model where valuations are dependent across bidders) is indistinguishable from a CV model.

## Explanation

Intuitively, all you identify from observed bid data is joint density of $b_1, \ldots, b_n$. In particular, can recover the correlation structure amongst the bids. But correlation of bids in an auction could be due to both affiliated PV, or to CV.