Forecasts of Inflation and Interest Rates in No-Arbitrage Affine Models*

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Abstract

In this paper we examine the forecasting ability of affine term structure framework that jointly models the markets for Treasuries, inflation protected securities (TIPS), and inflation derivatives, based on no-arbitrage restrictions across these markets. Focusing on the Treasury market only, affine term structure models can produce better forecasts of nominal Treasury yields than those produced by simply assuming yields follow random walks. However, we show that once extended to the markets for TIPS and inflation derivatives, affine term structure models can no longer beat forecasts of nominal yields produced by the random-walk model. Intuitively, the no-arbitrage restrictions that link these markets also constrain the flexibility in modeling risk compensation and interest rate volatility in these markets. Furthermore, we show that inflation derivatives help improve forecasts of inflaiton and (nominal and real) yields. However, the improvement in nominal yield forecasts is limited. This paper poses a challenge to existing affine term structures models in terms of their ability to simultaneously forecast inflation and interest rates.

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1 Introduction

Essentially affine term structure models can produce better forecasts of nominal Treasury yields than those produced by simply assuming yields follow random walks (Duffee (2002)). The success of essentially affine models stems from breaking up the tight link between risk compensation and interest rate volatility prevalent that exists in completely affine models. Therefore, essentially affine models are capable of simultaneously fitting the behavior of expected excess returns in time series and the term structure shapes in the cross section. More recently, researchers have developed affine term structure models to fruther study inflation and Treasury Inflatoin Protected Securities (TIPS) yields based on no-arbitrage restrictions across these markets.[1] However, relatively under-researched is the ability of the workhorse no-arbitrage framework to simultaneously forecast inflation and (nominal or real) interest rates. In this paper, we examine the performance of the essentially afine class of term structure models in simultatneously forecasting inflation and, both nominal and real, interest rates. A model that produces accurate forecasts of inflation and interest rates has very important applications for policy makers and investors. It can also shed light on the time variation in expected returns to several important asset classes, namely, Treasuries, TIPS, and inflation products.

As our first main result, we show that essentially affine models, once extended to model real yields and inflation, can no longer beat the forecast of random-walk models, both in-sample and out-of-sample. The key intuition is that the flexibility that allows the essentially affine term structure models to accurately forecast Treasury yields are no longer flexible enough in the multi-market context. The reason is that to rule out arbitrage, the cross-sectional and time-series characteristics of the term structure in both Treasury and TIPS markets are inherently linked. Focusing on one single market can produce better forecast for that market, but only for that market. Once both markets are included in an essentially affine term structure model, no arbitrage imposes a tight link between risk compensations demanded in both markets. As a result, we may be able to break up the link between risk compensation and interest rate volatility in one market, but it is quite challenging to break up the link in both markets. Our results show that the essentially affine term structure models that jointly model Treasury and TIPS yields fail to produce accurate forecasts for nominal yields, although they can still produce accurate forecasts for TIPS yields and inflation.

Our second main result is the finding that forecasts of inflation and interest rates are inherently linked via no-arbitrage restrictions. Affine term structure models that are better at identifying the prices of risk associated with risk factors driving both inflation and interest rates can usually forecast them better. Along this line of reasoning, we show that inflation derivative markets contain information not only for inflation forecasting for also for interest rate forecasting. The intuition is the following. Under no arbitrage, there exist nominal and real stochastic discount factors, labeled as M_t^N and M_t^R, respectively, satisfying the restriction $M_t^N = M_t^R Q_t$ where Q_t denotes the price level (e.g., CPI) at time t. It immediately follows that break-even inflation rate, defined as the

[1] See, for example, Kim and Wright (2005), Christensen-Lopez-Rudebusch (2010), Haubrich, Pennacchi, and Ritchken (2012), and D'Amico, Kim, and Wei (2014).

difference between nominal and real yields of bonds with the same maturity, can be decomposed into *inflation expectation* and *inflation risk premium* as follows:

$$BEI_{t,\tau} = \underbrace{\left\{\frac{1}{\tau}\ln E_t\left[\frac{Q_{t+\tau}}{Q_t}\right]\right\}}_{\text{inflation expectation (IE)}} + \underbrace{\left\{-\frac{1}{\tau}\ln\left(\frac{E_t\left[\left(M^R_{t+\tau}/M^R_t\right)\left(Q_t/Q_{t+\tau}\right)\right]}{E_t\left[M^R_{t+\tau}/M^R_t\right]E_t\left[Q_t/Q_{t+\tau}\right]}\right)\right\}}_{\text{inflation risk premium (IRP)}}. \tag{1}$$

It is worthwhile to note that the above decomposition is model-independent. The no-arbitrage restriction $M^N_t = M^R_t Q_t$ further implies that if there exists a common risk factor in driving inflation as well as the nominal and the real economies, this factor would carry significant inflation risk premium as well as (nominal or real) bond risk premiums. In an affine term structure model, all these risk premiums are tied to this factor's price-of-risk parameter. Consequently, introducing data that are more informative on estimating inflation risk premium can help improve forecasts of both inflation and interest rates. The improvement in inflation forecast is a direct result of the decomposition of break-even rate because better identification of one component leads to better identification of the other component, while the improvement in interest rate forecsat comes from better identification of the prices of risk.

In this paper, we focus on the information content in inflation derivatives, particularly inflation options. Under the put-call parity, the values of an inflation cap and an inflation floor with the same tenor and strike price implies expected inflation under the *forward* measure. Under no arbitrage, the forward measure, the risk-neutral measure, and the physical measure are all connected through the prices of risk, and the connection has a further affine structure in an affine term structure model. Therefore, introducing inflation options provides further information about the risk premiums, and thus can help improve forecasts of inflation and interest rates for the aforementioned reasons.

To elicit some of the novelties and contributions of our approach, it is useful to position our paper in the growing literature on decomposing the break-even inflation into inflation expectations and risk premium that includes Abrahams, Adrian, Crump and Moench (2015), Ang, Bekaert and Wei (2008), D'Amico, Kim and Wei (2014), Chen, Liu and Cheng (2010), Christensen, Lopez and Rudebusch (2010), Grishchenko and Huang (2013), Haubrich, Pennacchi and Ritchken (2012), Hördahl and Tristani (2012), Joyce, Lildholdt and Sorensen (2010), among others.[2] While our objective is similar in spirit to the objective of these papers, our modeling and empirical strategy is different. First, we combine information in yields (nominal and real) and inflation derivative prices to pin down the prices of risk and the dynamics of a new *inflation risk factor*. We derive a closed-form solution of the inflation caps (floors) under the forward measure, which is a function of the structural parameters, the traditional state variables (interpreted as level, slope and curvature

[2] Our paper is most closely related to D'Amico, Kim and Wei (2014) and Kitsul and Wright (2013). D'Amico, Kim and Wei (2014) augment the traditional term structure model with a liquidity factor but they maintain the constant volatility assumption and use only information in nominal yields, real yields and inflation (as well as survey inflation expectations and forecasts of interest rates). On the other hand, Kitsul and Wright (2013) are primarily interested in the options-implied distribution of inflation and do not model explicitly a term structure model as a link to nominal and real yields. We capitalize on some ideas in both of these papers and develop a no-arbitrage, affine model of nominal yields, real yields, inflation and inflation derivatives.

of the term structure) and the inflation factor. Importantly, the information in inflation options is incorporated through option-implied inflation expectations which preserves the affine structure of the model.[3] The inclusion of the inflation factor increases the variability of inflation expectations which brings them in line with the level and dynamics of households' survey inflation expectations.

Second, the inclusion of the inflation factor generates a more realistic dynamics of inflation expectations and is critical for pricing inflation options. It increases the variability of inflation expectations which brings them in line with the level and dynamics of households' survey inflation expectations. We find that this factor is highly correlated with the oil price. Anchoring inflation expectations and inflation option prices around the oil price may seem somewhat surprising but it appears to be an effective way of exploiting the transparency and economic information in futures oil prices in the absence of major movements in the nominal yield curve. The relative importance of oil prices over time (relative to the term structure state variables) can also explain the time-varying correlation between inflation expectations and oil prices.

Finally, we decompose the risk premium into two components: (i) inflation risk premium which is demanded to compensate for the time-varying level of inflation and (ii) inflation variance risk premium which is demanded to compensate for the time-varying uncertainty of inflation. Higher inflation uncertainty tends to push real yields (and term premia; see Wright, 2011) higher as a compensation for the higher risk. As a by-product, we also obtain the option-implied distribution of inflation over time which allows us to conduct various policy experiments and explore different inflation scenarios.

The rest of the paper is organized as follows. Section 2 introduces our new affine model of nominal yields, TIPS and inflation option prices and derives its implications for inflation expectations, uncertainty, and risk premium. Section 3 describes the data and contains the main empirical results. Section 4 concludes. The proofs of the main results are relagated to Appendix A. The state-space representation of the model and its estimation are discussed in the Appendix B.

2 A Canonical Model of Yields and Inflation

In this section we describe in detail the main no-arbitrage framework that we use to jointly model nominal Treasury and TIPS yields, inflation. Following Duffee (2002), we focus on N-factor "essentially affine" models. Let $X_t \equiv (v_t', x_t')'$ denote the vector of N state variables. The state vector is ordered so that the first m elements, denoted by $v_t \equiv (v_{1,t}, \cdots, v_{m,t})'$, affect the instantaneous variance of X_t, and the last $n \equiv N - m$ elements, denoted by $x_t \equiv (x_{1,t}, \cdots, x_{n,t})'$, do not, where $0 \leq m, n \leq N$.

[3] Incorporating additional independent information from surveys or other asset markets also help the potential identification of hidden or unspanned factors (Fisher and Gilles, 2000; Duffee, 2011; Chernov and Mueller, 2012; Joslin, Priebsch and Singleton, 2014) that pass undetected through the term structure of interest rates. For example, as Chernov and Mueller (2012) point out, a factor can remain hidden from the nominal and real term structure of interest rates if it has an equal but opposite effect on inflation expectations and inflation risk premium.

Real Economy. Consider the real economy first. Following the normalization in Dai and Singleton (2000) and Duffee (2002), we assume the real risk-neutral dynamics of X_t is the following:

$$dX_t = K_X^{R*} \left(\mu_X^{R*} - X_t \right) dt + \mathcal{D}_t dW_t^{R*}, \tag{2}$$

where $\mathcal{D}_t$ is a diagonal matrix with its i_{th} diagonal element given by

$$\mathcal{D}_{t(ii)} = \begin{cases} \sqrt{v_{i,t}}, \text{ if } i = 1, \cdots, m \\ \sqrt{1 + \beta_i' X_t}, \text{ if } i = m+1, \cdots, N \end{cases}$$

and $\beta_i \equiv \left(\beta_{i(1)}, \cdots, \beta_{i(m)}, 0, \cdots, 0 \right)'$. The price of risk factor in the essentially affine model is

$$\Lambda_t^R = \mathcal{D}_t \lambda_0^R + \mathcal{D}_t^- \lambda_X^R X_t$$

where $\mathcal{D}_t^-$ is a diagonal matrix with its diagonal element $\mathcal{D}_{t(ii)}^-$ being zero for $i \in \{1, \cdots, m\}$ and being $\left(1 + \beta_i' X_t\right)^{-1/2}$ for $i \in \{m+1, \cdots, N\}$. Note that $\mathcal{D}_t \Lambda_t^R$ is still affine. Therefore, the real physical dynamics of X_t takes a similar affine form:

$$\begin{aligned} dX_t &= K_X^{R*} \left(\mu_X^{R*} - X_t \right) dt + \mathcal{D}_t \left[\mathcal{D}_t \lambda_0^R + \mathcal{D}_t^- \lambda_X^R X_t \right] dt + \mathcal{D}_t dW_t \\ &\equiv K_X \left(\mu_X - X_t \right) dt + \mathcal{D}_t dW_t, \end{aligned} \tag{3}$$

where

$$\begin{aligned} K_X &= K_X^{R*} - \lambda_0^R \beta' + I^- \lambda_X^R, \\ K_X \mu_X &= K_X^{R*} \mu_X^{R*} + I^- \lambda_0^R, \end{aligned}$$

and β is a matrix with its row i being β_i', and I^- is a diagonal matrix with its diagonal element being zero for $i \leq m$ and being 1 otherwise.

Lastly, the real pricing kernel is given by

$$dM_t^R / M_t^R = -r_t^R dt - \Lambda_t^{R\prime} dW_t,$$

where the real short rate takes the following affine form:

$$r_t^R = \rho_0^R + \rho_X^{R\prime} X_t.$$

Inflation and Nominal Economy. To capture the possibility that some shcoks to inflation are neutral and have no pricing impacts in the real economy, we assume that the the following price level process:

$$\begin{aligned} dq_t &= \pi_t dt + \textstyle\sum_{i=1}^m \sigma_{q,i} \sqrt{v_{i,t}} dW_{(i),t} + \sum_{i=m+1}^N \sigma_{q,i} \sqrt{1 + \beta_i' X_t} dW_{i,t} + \sigma_q^\perp dW_{\perp,t} \\ &= \pi_t dt + \left[\mathcal{D}_t \sigma_q \right]' dW_t + \sigma_q^\perp dW_{\perp,t} \end{aligned} \tag{4}$$

where $q_t \equiv \log Q_t$ denotes the logarithm of the price level Q_t, π_t is the instantaneous expected inflation rate, and $W_t = (W_{(1),t}, \cdots, W_{(N),t})'$ and $W_{\perp,t}$ are independent standard Brownian motions. The instantaneous expected inflation rate is assumed to be affine in the state vector X_t:

$$\pi_t = \rho_0^\pi + \rho_X^{\pi\prime} X_t. \tag{5}$$

The real and the nominal pricing kernels are linked by the no-arbitrage condition: $M_t^R = M_t^N Q_t$. By Ito's lemma, it is straightforward to show that the nominal pricing kernel M_t^N follows

$$\begin{aligned} dM_t^N/M_t^N &= dM_t^R/M_t^R + dQ_t^{-1}/Q_t^{-1} + \left(dM_t^R/M_t^R\right)\left(dQ_t^{-1}/Q_t^{-1}\right) \\ &= -r_t^R dt - \Lambda_t^{R\prime} dW_t - \pi_t dt - \left[\mathcal{D}_t \sigma_q\right]' dW_t - \sigma_q^\perp dW_{\perp,t} \\ &\quad + \frac{1}{2}\left[\textstyle\sum_{i=1}^m \sigma_{q,i}^2 v_{i,t} + \sum_{i=m+1}^N \sigma_{q,i}^2 \left(1 + \beta_i' X_t\right) + \left(\sigma_q^\perp\right)^2\right] dt + \left[\mathcal{D}_t \sigma_q\right]' \Lambda_t^R dt \\ &\equiv -r_t^N dt - \Lambda_t^{N\prime} dW_t - \sigma_q^\perp dW_{\perp,t} \end{aligned}$$

where

$$\begin{aligned} r_t^N &= r_t^R + \pi_t - \left[\mathcal{D}_t \sigma_q\right]' \Lambda_t^R \\ &\quad - \frac{1}{2}\left[\textstyle\sum_{i=1}^m \sigma_{q,i}^2 v_{i,t} + \sum_{i=m+1}^N \sigma_{q,i}^2 \left(1 + \beta_i' X_t\right) + \left(\sigma_q^\perp\right)^2\right] \\ &\equiv \rho_0^N + \rho_X^{N\prime} X_t, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \Lambda_t^N &= \Lambda_t^R + \mathcal{D}_t \sigma_q = \mathcal{D}_t\left(\lambda_0^R + \sigma_q\right) + \mathcal{D}_t^- \lambda_X^R X_t \\ &\equiv \mathcal{D}_t \lambda_0^N + \mathcal{D}_t^- \lambda_X^N X_t. \end{aligned} \tag{7}$$

Eq. (6) is the generalized Fisher equation in which the nominal short rate is decomposed into the real short rate, expected inflation rate, instantaneous inflation risk premium, and a convexity term due to Jensen's inequality.

The existing models are special cases of the above canonical framework (up to slight modifications), such as the models in Haubrich, Pennacchi, and Ritchken (201) and D'Amico, Kim, and Wei (2014, hereafter DKW), etc. In examing the forecasting ability of this class of affine models, we specialize to the model in DKW and its extensions, to which we turn next.

The DKW model falls into the category of $EA_0\,(3)$ and is very tractable. We further extend it by introducing stochastic volatility in inflation, and the extended model is essentially $EA_1\,(4)$. In particular, we assume the following price level process:

$$dq_t = \pi_t dt + \sigma_q' dW_{x,t} + \sqrt{v_t} dW_{\perp,t}, \tag{8}$$

where $q_t \equiv \log Q_t$ denotes the logarithm of the price level Q_t, π_t is the instantaneous expected inflation rate, and $W_{x,t} = (W_{1,t}, W_{2,t}, W_{3,t})'$ and $W_{\perp,t}$ are independent standard Brownian motions. The instantaneous expected inflation rate is assumed to be affine in the latent factors $x_t = (x_{1t}, x_{2t}, x_{3t})'$ and v_t:

$$\pi_t = \rho_0^\pi + \rho_x^{\pi\prime} x_t + \rho_v^\pi v_t. \tag{9}$$

The state variables x_t capture the drivers in the yield curve movements that are orthogonal to the volatility process v_t. The shock $dW_{\perp,t}$ is introduced to capture inflation-specific dynamics that is not reflected in x_t. Our choice of allowing for stochastic volatility of the inflation-specific shock and treating it as a risk factor, that affects directly the term structure of nominal and real yields and inflation expectations, is motivated by two empirical regularities. First, there is abundant evidence of time-varying volatility in inflation (for instance, Engle (1982), among many others). Second, Wright (2011) argues that inflation uncertainty appears to be an important component of bond term premia. This new inflation factor plays a key role in our subsequent analysis.

The dynamics of the state variables under the physical measure $\mathbb{P}$ is given by

$$\begin{aligned} dx_t &= \mathcal{K}(\mu - x_t)\,dt + \Sigma dW_{x,t}, && (10) \\ dv_t &= \kappa_v(\mu_v - v_t)\,dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t} + \rho dW_{\perp,t}\right), && (11) \end{aligned}$$

where $W_{v,t}$ is an independent standard Brownian motion, and the parameters κ_v, μ_v and σ_v govern the behavior (persistence, long-run mean, volatility of volatility) of v_t. To account for the strong postive relationship between inflation uncertainty and level of inflation (see Golob, 1994), the volatility shocks are allowed to be correlated with $dW_{\perp,t}$ through the correlation coefficient ρ. The structure of the 3×3 matrices $\mathcal{K}$ and Σ and the 3×1 vector μ will be later restricted for identification purposes.

2.1 Nominal and Real Bond Prices

The nominal bond prices are determined by the following nominal pricing kernel

$$dM_t^N/M_t^N = -r_t^N dt - \Lambda_{x,t}^{N\prime} dW_{x,t} - \Lambda_{v,t}^N dW_{v,t} - \Lambda_{\perp,t}^N dW_{\perp,t}, \tag{12}$$

where r_t^N is the nominal short rate, specified as an affine function of the latent variables

$$r_t^N = \rho_0^N + \rho_x^{N\prime} x_t + \rho_v^N v_t, \tag{13}$$

and the vector of prices of risk is assumed to be

$$\begin{aligned} \Lambda_{x,t}^N &= \lambda_0^N + \lambda_x^N x_t, \\ \Lambda_{v,t}^N &= \gamma_v\sqrt{v_t}, \\ \Lambda_{\perp,t}^N &= \gamma_\perp\sqrt{v_t}. \end{aligned}$$

The real and the nominal pricing kernels are linked by the no-arbitrage condition: $M_t^R = M_t^N Q_t$. By Ito's lemma, it is straightforward to show that the real pricing kernel M_t^R follows

$$\begin{aligned} dM_t^R/M_t^R &= dM_t^N/M_t^N + dQ_t/Q_t + \left(dM_t^N/M_t^N\right)(dQ_t/Q_t) \\ &= -r_t^R dt - \left(\Lambda_{x,t}^N - \sigma_q\right)' dW_{x,t} - \Lambda_{v,t}^N dW_{v,t} - \left(\Lambda_{\perp,t}^N - \sqrt{v_t}\right) dW_{\perp,t}, \end{aligned}$$

where the real short rate r_t^R is given by

$$r_t^R = r_t^N - \pi_t + \left(\sigma_q' \Lambda_{x,t}^N + \sqrt{v_t} \Lambda_{\perp,t}^N\right) - \frac{1}{2}\left(\sigma_q' \sigma_q + v_t\right). \tag{14}$$

Eq. (14) is the generalized Fisher equation in which the nominal short rate is decomposed into the real short rate, expected inflation rate, instantaneous inflation risk premium, and a convexity term due to Jensen's inequality.

Let $P_{t,\tau}^N$ (or $P_{t,\tau}^R$) denote the time-t price of a nominal (or real) τ-year zero-coupon bond that pays \$1 at maturity. Under the model, nominal or real bond prices can then be determined under the risk neutral measure $\mathbb{P}^*$:

$$P_{t,\tau}^i = E_t^{\mathbb{P}^*}\left[\exp\left(-\int_t^{t+\tau} r_s^i ds\right)\right], \text{ for } i = N, R.$$

where $E_t^{\mathbb{Q}}[\cdot]$ (resp., $Var_t^{\mathbb{Q}}[\cdot]$ and $Cov_t^{\mathbb{Q}}[\cdot]$) is generic notation for the expectation (resp., variance and covariance) operator under a particular $\mathbb{Q}$ measure, conditional on information at time t. As is standard for affine term structure models, bond prices can be shown to be exponential affine in the latent variables. In Proposition 1 below, we derive the closed-form expressions of bond prices and yields in term of the underlying parameters.

Proposition 1 *Under this model, nominal and real bond prices take the exponential-affine form*

$$P_{t,\tau}^i = \exp\left(A_\tau^i + B_\tau^{i\prime} x_t + C_\tau^i v_t\right), \ i = N, R \tag{15}$$

and nominal and real yields take the affine form

$$y_{t,\tau}^i = a_\tau^i + b_\tau^{i\prime} x_t + c_\tau^i v_t, \ i = N, R, \tag{16}$$

where $a_\tau^i \equiv -A_\tau^i/\tau$, $b_\tau^i \equiv -B_\tau^i/\tau$, *and* $c_\tau^i \equiv -C_\tau^i/\tau$, *and* A_τ^i, B_τ^i, C_τ^i $(i = N, R)$ *satisfy the following system of ODEs:*

$$\begin{aligned}
\frac{dA_\tau^i}{d\tau} &= -\rho_0^i + \left(\mathcal{K}\mu - \Sigma\lambda_0^i\right)' B_\tau^i + \kappa_v \mu_v C_\tau^i + \frac{1}{2} B_\tau^{i\prime} \Sigma\Sigma' B_\tau^i, \text{ with } A_0^i = 0 \\
\frac{dB_\tau^i}{d\tau} &= -\rho_x^i - \left(\mathcal{K} + \Sigma\lambda_x^i\right)' B_\tau^i, \text{ with } B_0^i = 0 \\
\frac{dC_\tau^i}{d\tau} &= -\rho_v^i - \left[\kappa_v + \sqrt{1-\rho^2}\sigma_v \gamma_v + \rho\sigma_v\gamma_\perp\right] C_\tau^i + \frac{1}{2}\sigma_v^2 \left(C_\tau^i\right)^2, \text{ with } C_0^i = 0.
\end{aligned}$$

The coefficients $\rho_0^R, \rho_x^R, \rho_v^R, \lambda_0^R, \lambda_x^R$ *are given in Appendix A.*

Proof of Proposition 1. See Appendix A. ■

It follows from Proposition 1 that (for $i = N, R$)

$$\frac{dP_{t,\tau}^i}{P_{t,\tau}^i} = r_t^i dt + B_\tau^{i\prime} \Sigma dW_{x,t} + C_\tau^i \sigma_v \sqrt{v_t}\left(\sqrt{1-\rho^2} dW_{v,t} + \rho dW_{\perp,t}\right),$$

i.e., we allow for substantial generality as the dynamics of nominal and real bond prices can be affected by $dW_{x,t}$, $dW_{v,t}$ and $dW_{\perp,t}$. In the case of constant volatility ($\sigma_v = 0$), the last term drops out. Another special case arises when $\rho = 0$ and the orthogonal shock to unexpected inflation $dW_{\perp,t}$ has no effect on the bond price dynamics.

2.2 Inflation Option Prices

Next we turn our attention to pricing of inflation options under the forward measure $\tilde{\mathbb{P}}^{\tau}$. The Radon-Nikodym derivative of the forward measure $\tilde{\mathbb{P}}^{\tau}$ with respect to the risk neutral measure $\mathbb{P}^*$ is given by

$$\left(\frac{d\tilde{\mathbb{P}}^{\tau}}{d\mathbb{P}^*}\right)_{t,t+\tau} = \frac{1}{P^N_{t,\tau}} \exp\left(-\int_t^{t+\tau} r^N_s ds\right).$$

In the rest of the paper, we omit the superscript and simply denote the forward measure by $\tilde{\mathbb{P}}$ wherever there is no confusion. The forward measure $\tilde{\mathbb{P}}$ is used for analytical tractability. In particular, the value of an inflation option is simply the expected payoff at maturity under the forward measure discounted by the nominal yield $y^N_{t,\tau}$; that is, denote by by , respectively, then

$$\begin{aligned}
P^{CAP}_{t,\tau,K} &= \exp\left(-\tau y^N_{t,\tau}\right) E^{\tilde{\mathbb{P}}}_t\left[\left(\frac{Q_{t+\tau}}{Q_t} - (1+K)^{\tau}\right)^+\right], \\
P^{FLO}_{t,\tau,K} &= \exp\left(-\tau y^N_{t,\tau}\right) E^{\tilde{\mathbb{P}}}_t\left[\left((1+K)^{\tau} - \frac{Q_{t+\tau}}{Q_t}\right)^+\right],
\end{aligned}$$

where $P^{CAP}_{t,\tau,K}$ ($P^{FLO}_{t,\tau,K}$) denotes the price of a τ-maturity inflation cap (floor) with the strike price K. If the put-call parity further holds, then one can extract option-implied inflation expectation from option prices as follows:

$$E^{\tilde{\mathbb{P}}}_t\left[\frac{Q_{t+\tau}}{Q_t}\right] = \frac{P^{CAP}_{t,\tau,K} - P^{FLO}_{t,\tau,K}}{P^N_{t,\tau}} + (1+K)^{\tau}. \tag{17}$$

Two observations are worth mentioning here. First, the expression for the option-implied inflation expectation in Eq. (17) holds in general and is not model-specific. The only assumption behind this result is the put-call parity. Second, although we can extract inflation expectation from option data following Eq. (17), the expectation is taken under the forward measure. As a result, it cannot be directly compared against the breakeven inflation rate which concerns inflation expectation under the physical measure. The affine term structure model in this paper allows us to further translate the option-implied inflation expectation based on Girsanov's theorem which implies the following relationship between Brownian motions under the forward and risk-neutral measures:

$$\begin{aligned}
d\tilde{W}_{x,t} &= dW^*_t - \Sigma' B^N_{\tau} dt, \\
d\tilde{W}_{v,t} &= dW^*_{v,t} - \sqrt{1-\rho^2}\sigma_v C^N_{\tau}\sqrt{v_t}dt, \\
d\tilde{W}_{\perp,t} &= dW^*_{\perp,t} - \rho\sigma_v C^N_{\tau}\sqrt{v_t}dt.
\end{aligned}$$

Denote

$$\begin{aligned}
\mathcal{IE}_{t,\tau} &\equiv \frac{1}{\tau} E^{\tilde{\mathbb{P}}}_t\left[q_{t+\tau} - q_t\right], \\
\mathcal{IU}_{t,\tau} &\equiv \frac{1}{\tau} Var^{\tilde{\mathbb{P}}}_t\left[q_{t+\tau} - q_t\right].
\end{aligned}$$

$\mathcal{IE}_{t,\tau}$ is an alternative definition of inflation expectation[4], while $\mathcal{IU}_{t,\tau}$ measures inflation uncertainty. Under the assumption that the change in log price levels approximately follows a normal distribution, i.e., $q_{t+\tau} - q_t|_{\mathcal{F}_t} \sim N\left(\tau \cdot \mathcal{IE}_{t,\tau}, \tau \cdot \mathcal{IU}_{t,\tau}\right)$, then the option-implied inflation expectation in Eq. (17) has the following (approximate) exponential-affine form:

$$E_t^{\tilde{\mathbb{P}}}\left[\frac{Q_{t+\tau}}{Q_t}\right] \approx \exp\left[\tau\left(\mathcal{IE}_{t,\tau}\right) + \frac{1}{2}\tau\left(\mathcal{IU}_{t,\tau}\right)\right], \tag{18}$$

where

$$\begin{aligned} \mathcal{IE}_{t,\tau} &= \left(\widetilde{\rho}_0^\pi + \widetilde{\rho}_x^{\pi\prime}\widetilde{\mu} + \widetilde{\rho}_v^{\pi\prime}\widetilde{\mu}_v\right) + \widetilde{\rho}_x^{\pi\prime}\left(\widetilde{\mathcal{K}}\tau\right)^{-1}\left(I - e^{-\widetilde{\mathcal{K}}\tau}\right)\left(x_t - \widetilde{\mu}\right) \\ &\quad + \widetilde{\rho}_v^\pi\left(\widetilde{\kappa}_v\tau\right)^{-1}\left(1 - \exp\left(-\widetilde{\kappa}_v\tau\right)\right)\left(v_t - \widetilde{\mu}_v\right), \end{aligned}$$

and

$$\mathcal{IU}_{t,\tau} = \widetilde{H}_0 + \widetilde{H}_1\widetilde{\mu}_v + \widetilde{H}_2\left(v_t - \widetilde{\mu}_v\right),$$

and the notations such as $\widetilde{\rho}_0^\pi$, etc. denote corresponding parameters under the forward measure (see the Appendix for the expressions of these parameters as well as those for $\widetilde{H}_i$, $i = 0, 1, 2$). The above approximation result holds exactly in the simplified models without inflation uncertainty factor v_t (e.g., $\mathcal{M}_x^{N,R}$ or $\mathcal{M}_x^{N,R,O}$). Moreover, we can also derive approximate closed-form pricing formula for inflation caps and floors. The results are presented in Proposition 2 below.

Proposition 2 *In our affine term structure model, inflation expectations the forward measure $\tilde{\mathbb{P}}$ implied from an inflation cap and floor with the same maturity τ and strike K is given in Eq. (17) and has the exponential-affine form in Eq. (18). Furthermore, the price of a τ-maturity inflation cap with strike K is approximately given by*

$$P_{t,\tau,K}^{CAP} \approx P_{t,\tau}^N \left[\begin{array}{c} e^{\tau\left(\mathcal{IE}_{t,\tau} + \frac{1}{2}\mathcal{IU}_{t,\tau}\right)}\Phi\left(\frac{-\log(1+K) + \left(\mathcal{IE}_{t,\tau} + \mathcal{IU}_{t,\tau}\right)}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}}\right) \\ -\left(1+K\right)^\tau \Phi\left(\frac{-\log(1+K) + \mathcal{IE}_{t,\tau}}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}}\right) \end{array}\right], \tag{19}$$

and the price of a τ-maturity inflation floor with strike K is approximately given by

$$P_{t,\tau,K}^{FLO} \approx P_{t,\tau}^N \left[\begin{array}{c} -e^{\tau\left(\mathcal{IE}_{t,\tau} + \frac{1}{2}\mathcal{IU}_{t,\tau}\right)}\Phi\left(-\frac{-\log(1+K) + \left(\mathcal{IE}_{t,\tau} + \mathcal{IU}_{t,\tau}\right)}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}}\right) \\ +\left(1+K\right)^\tau \Phi\left(-\frac{-\log(1+K) + \mathcal{IE}_{t,\tau}}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}}\right) \end{array}\right], \tag{20}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable. The approximation becomes exact in the Gaussian models without stochastic inflation uncertainty v_t.

Proof of Proposition 2. See Appendix A. ■

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[4] We define inflation expectations as in Christensen, Lopez and Rudebusch (2010)

$$IE_{t,\tau} \equiv -\frac{1}{\tau}\ln\left(E_t\left[\frac{Q_t}{Q_{t+\tau}}\right]\right).$$

This differs from D'Amico, Kim and Wei (2014) who define it as $IE_{t,\tau} \equiv \frac{1}{\tau}E_t\left[\ln\left(\frac{Q_{t+\tau}}{Q_t}\right)\right]$. The difference between these two slightly different measures is due to the Jensen's inequality term.

2.3 Model Implications for Inflation Expectation and Risk Premium

Under the model, inflation expectations, inflation uncertainty, and risk premium all take affine forms. Proposition 1 below decomposes nominal yields into real yields, inflation expectations, risk premium associated with time-varying level of future inflation and risk premium associated with time-varying uncertainty of future inflation.

Proposition 3 *(i) Under no-arbitrage, the nominal yield of a τ-year zero-coupon bond can be decomposed into inflation expectations ($IE_{t,\tau}$) and inflation risk premium ($IRP_{t,\tau}$) as follows:*

$$y^N_{t,\tau} = y^R_{t,\tau} + IE_{t,\tau} + IRP_{t,\tau}, \tag{21}$$

where $IE_{t,\tau} = \frac{1}{\tau} \ln E^{\mathbb{P}}_t \left[\frac{Q_{t+\tau}}{Q_t} \right]$ *and* $IRP_{t,\tau} = -\frac{1}{\tau} \ln \left(\frac{E^{\mathbb{P}}_t \left[\left(M^R_{t+\tau}/M^R_t \right) (Q_t/Q_{t+\tau}) \right]}{E^{\mathbb{P}}_t \left[\left(M^R_{t+\tau}/M^R_t \right) \right] E^{\mathbb{P}}_t [(Q_t/Q_{t+\tau})]} \right)$.

(ii) In our affine term structure model, inflation risk premium $IRP_{t,\tau}$ can be approximated by

$$IRP_{t,\tau} \approx \frac{1}{\tau} E^{\mathbb{P}}_t \left[\int_t^{t+\tau} \left(\sigma_q' \Lambda^R_{x,s} - v_s \right) ds \right] \tag{22}$$

$$= IRP^x_{t,\tau} + IRP^v_{t,\tau}, \tag{23}$$

where can further be decomposed into the following components $IRP^x_{t,\tau} = \frac{1}{\tau} E^{\mathbb{P}}_t \left[\int_t^{t+\tau} \sigma_q' \Lambda^R_{x,s} ds \right]$ *and* $IRP^v_{t,\tau} = \frac{1}{\tau} E^{\mathbb{P}}_t \left[\int_t^{t+\tau} \Lambda^R_{\perp,s} \sqrt{v_s} ds \right]$.

Proof of Proposition 3. See Appendix. ■

As mentioned in Eq. (1) in the Introduction, the decomposition in Proposition 3 (i) holds as long as there is no arbitrage. An affine term structure model is needed in order to disengle the component of inflation expectation from the component of inflation risk premium. Proposition 3 (ii) provides an approximate closed-form expression for the inflation risk premium. It shows that we can further decompose the inflation risk premium into different components that are driven by state variables x_t and v_t, respectively.

3 Estimation Methodology and Data

3.1 Estimation Methodology

The set of variables that enter our model consists of observables ($y^N_{t,\tau}$, $y^R_{t,\tau}$ and $y^O_{t,\tau,K}$) as well as latent (x_t and v_t) or partially observed (q_t) state variables, where $y^O_{t,\tau,K} \equiv \frac{1}{\tau} \ln E^{\tilde{\mathbb{P}}}_t \left[\frac{Q_{t+\tau}}{Q_t} \right] \approx \mathcal{IE}_{t,\tau} + \frac{1}{2} \mathcal{IU}_{t,\tau}$ (see Eq. (17). The superscript "O" refers to the use of options data in deriving the inflation expectations under the forward measure. Since the dimension of the observables is typically larger than the dimension of the state vector, the term structure models are inherently stochastically singular (Piazzesi, 2010, p.726). There are two approaches to dealing with this singularity. One approach is to "invert" the state variables from a small subset (of the same dimension as the state vector) of observables, add measurement errors to the rest of the observable vector and proceed with quasi-maximum likelihood estimation. The drawback of this method is that the choice of the observables from which the state variables are extracted is arbitrary and naturally affects the quality and the dynamics of the derived state variables. An additional problem that arises in our setup is that the volatility process should satisfy some properties (positivity, for example) which may not be guaranteed by this inversion.

For these reasons, we pursue the second approach in which all yields and option-implied inflation expectations are assumed to be observed with a measurement error. This specification arises naturally in our framework since our yield data is obtained from an interpolated zero-coupon yield curve (see also Piazzesi, 2010, for the plausibility of this assumption). This approach requires the use of a filtering method and, given the assumptions and the structure of our model, we employ the extended Kalman filter which is discussed later. The nominal yields, real yields and cap prices are at weekly frequency while the CPI inflation is based on monthly data. Let $Y_t = (q_t, y^{N\prime}_{t,\tau}, y^{R\prime}_{t,\tau}, p^{CAP\prime}_{t,\tau,K})'$ denote the $m \times 1$ vector of observables, where $\{q_t\}$, $t \in \{1, \cdots, T_q\}$, $\{y^N_{t,\tau}\}$ with $t \in \{1, \cdots, T_N\}$ and $\tau \in \{\tau_1, \cdots, \tau_{m^N}\}$, $\{y^R_{t,\tau}\}$ with $t \in \{1, \cdots, T_R\}$ and $\tau \in \{\tau_1, \cdots, \tau_{m^R}\}$, $\left\{y^O_{t,\tau,K}\right\}$ with $t \in \{1, \cdots, T_{IE}\}$, $\tau \in \{\tau_1, \cdots, \tau_{m^{IE}}\}$ and $K \in \{K_1, \cdots, K_{m^K}\}$, and $m = 1 + m^N + m^R + m^{IE} \cdot m^K$. The state-space representation of the model is characterized by the measurement equation for Y_t and the transition equation for the augmented state vector $X_t = (q_t, x_t', v_t)'$ whose construction is described in Appendix B. The discretized state-space system is given by

$$\begin{aligned} X_t &= \mathcal{A} + \mathcal{B} X_{t-1} + \eta_t \\ Y_t &= a + b X_t + e_t, \end{aligned}$$

where $\mathcal{A}$, $\mathcal{B}$, a, and b are functions of the underlying parameters of interest $\theta = (vec(\mathcal{K})'$, $vech(\Sigma\Sigma')'$, ρ, μ', $\rho_0^{N\prime}$, $\rho_x^{N\prime}$, ρ_v^N, $\lambda_0^{N\prime}$, $vec(\lambda_x^N)'$, ρ_0^π, $\rho_x^{\pi\prime}$, ρ_v^π, σ_q', κ_v, σ_v, μ_v, γ_v, $\gamma_\perp)'$. For identification purposes, we follow D'Amico, Kim and Wei (2014) and impose the restrictions that μ is a zero vector, $\mathcal{K}$ is a diagonal matrix and Σ is a lower triangular matrix with diagonal matrix set equal to 0.01. The parameter vector θ is then estimated by Kalman filter (see Appendix B for details).

3.2 Data

All data variables are converted to weekly frequency and end in the last week of February 2015 (but have different start date). Continuously-compounded, zero-coupon yields on U.S. Treasury notes with 1-, 2-, 4-, 7- and 10-year maturities are obtained from the U.S. Treasury yield curve of Gürkaynak, Sack and Wright (2007), maintained by the Federal Reserve Board (available at `http://www.federalreserve.gov/Pubs/feds/2006/200628/200628abs.html`). The 3- and 6-month rates are obtained from the 3- and 6-month T-bill rates with constant maturity from the Federal Reserve Board's H.15 statistical release by converting them from discount basis to continuously-compounded rates. The sample period for the nominal yields starts in the first week of January 1990. Figure 1 plots the nominal yields across maturities (yield curve) over time. Over this period, the level of nominal yields dropped dramatically (and stayed low for the last five years of the sample) with the shape (slope and curvature) of the yield curve also exhibiting substantial time variability.

For the TIPS yields, we use data for 5-, 7- and 10-year continuously-compounded, zero-coupon yields from the TIPS yield curve[5] of Gürkaynak, Sack and Wright (2010), maintained by the Federal Reserve Board (`http://www.federalreserve.gov/pubs/feds/2008/200805/200805abs.html`). The sample period for TIPS yields starts in the first week of January 1999. Figure 2 presents the real yield curves over the period January 1999 - February 2015. In Figure 3, we also plot the 5-year nominal and real yields along with the break-even inflation rate. For most of the period, the break-even rate varies between 1% and 3% except for a sharp decrease in the wake of the recent financial crisis.

Data for inflation cap and floor prices, starting in the first week of October 2009, with strike prices from 1% to 6% in increments of 0.5% (for caps) and -2% to 3% in increments of 0.5% (for floors) with 1-, 3-, 5- 7- and 10-year maturities, were provided by BGC Partners. BGC options data have few missing values for 1- and 3-year maturities. In the empirical analysis, we consider 1- and 3-year inflation caps and floors with 1%, 2% and 3% strike prices. Since one of the main input variable that summarizes information from the options market is the option-implied inflation expectations described above, Figure 4 plots these series at 1- and 3-year maturity, along with the break-even inflation. With very few exceptions, the option-implied inflation expectations lie above the break-even inflation rate inferred from Treasury and TIPS bond prices.

Weekly series for nominal yields, TIPS yields and inflation option prices are constructed by using the Wednesday observation of each week (if the market is closed on Wednesday, we take the Tuesday observation or Thursday's observation if the Tuesday's is not available).

We use the CPI for all urban consumers (all items, seasonally adjusted) from the U.S. Bureau of Labor Statistics, covering the period January 1990 – February 2015. Following D'Amico, Kim and Wei (2014), the monthly CPI is assumed to be observed on the last Wednesday of each month. The remaining weeks are treated as missing observations which are filled in via the Kalman filter. Similarly, the missing weekly observations up to January 1999 for TIPS and October 2009 for inflation options are also estimated using the Kalman filter.

We also use forecasts of the 3-month T-bill yield from Blue Chip Financial Forecasts (BCFF). In particular, we use the 6- and 12-month-ahead forecasts (available at the monthly frequency) and the long-range forecast over the next 5 to 10 years (available twice a year). Following DKW, we fix the standard deviation of the latter's measurement error at 75 basis points in order to prevent the long-range forecast to have too strong influence on the estimation.

[5] For some specific aspects of the U.S. TIPS market, see Fleckenstein, Longstaff and Lustig (2014), Fleming and Krishnan (2012), Gürkaynak, Sack and Wright (2010), and Sack and Elsasser (2004).

4 Empirical Results

We estimate five versions of the model that include different state variables and different input variables. The different models are denoted by $\mathcal{M}$ with superscripts N, R, O (for nominal yields, real yields, option-implied inflation expectations) for the input variables and subscripts x and v for the state variables. The benchmark model, $\mathcal{M}_x^{N,R}$, is the model used in D'Amico, Kim and Wei (2014) without a liquidity factor. This model uses nominal (Treasury) and real (TIPS) yields (as well as inflation and survey nominal yield forecasts) as input variables and estimates only the vector x as state variables. In the evaluation of the in-sample and out-of-sample predictions of nominal yields, we also consider the more restricted model, $\mathcal{M}_x^{N,R}$, which uses only nominal yields as input variables. As discussed below, this model is expected to fit better the nominal yield curve at the expense of poor fit for the real yield curve and inflation options.

The other three models are new. The first one, $\mathcal{M}_x^{N,R,O}$, is designed to evaluate the information content of options (option-implied inflation expectations, to be more precise) for identifying and estimating the same set of model parameters as in $\mathcal{M}_x^{N}$ and $\mathcal{M}_x^{N,R}$. The models $\mathcal{M}_x^{N}$, $\mathcal{M}_x^{N,R}$ and $\mathcal{M}_x^{N,R,O}$ maintain the assumption of constant inflation volatility by replacing $\sqrt{v_t}$ with the constant parameter $\sigma_q^{\perp}$. The next two models allow for stochastic volatility in the inflation equation. $\mathcal{M}_{xv}^{N,R,O}$ is the main model proposed in this paper which, in addition to x, estimates the inflation volatility factor v employing information from nominal yields, real yields and option-implied inflation expectations. Finally, to show that the inflation option prices can be fitted better by introducing the price information more directly, i.e., using individual caps and floors prices of different maturities and different strikes. The forecast performance of all these models for inflation, nominal and real yields is evaluated relative to a random walk (RW) model for the respective variable.

Table 1 presents the estimation results for different models. All model specifications estimate similar dynamics for the state variables in x. Figure 1 plots the estimated state variables x. There is little difference in the estimates of these state variables which roughly correspond to the level (x_2), slope (x_1) and curvature (x_3) of the nominal and real term structure. The loadings of the nominal spot rate on these state variables as well as the estimated prices of risk are also similar across the different models.

The fourth state variable is new and represents our inflation volatility (uncertainty) factor. Several interesting observations emerge from examining the dynamics of this state variable. As documented in previous research (see Golob, 1994), inflation uncertainty appears to be strongly positively correlated with the level of 1-year CPI inflation. This positive co-movement between the inflation level and inflation uncertainty explains the sharp decrease in inflation uncertainty that coincides with the disinflation in the first half of 2009. By contrast, estimating a stochastic volatility or a GARCH model for inflation suggests an increase in volatility during this period. Finally, our measure of inflation uncertainty captures the recent drop in the headline inflation although the level of uncertainty is still higher than the levels of inflation uncertainty in the 1990s which were characterized with fairly stable inflation rates.

Table 1 also reveals that inflation uncertainty is highly persistent and the model inflation expectations load heavily on this state variable. The loadings of the model inflation expectations on the yield curve variables x exhibit variation across the model specifications. Combined with the large estimates of ρ_v^π, the increased loadings on x in models $\mathcal{M}_{xv}^{N,R,O}$ produce more volatile model-implied inflation expectations. The estimates of the price of risk parameters for the inflation and volatility risk factors, $\gamma_\perp$ and γ_v, are also large. Importantly, the explicit introduction of the state variable v_t is crucial for fitting the inflation option prices. Figures 6 and 7 plot the actual and model-implied option prices for caps and floors, respectively. It is clear that the model $\mathcal{M}_{xv}^{N,R,O}$, the only model in this graph that includes v_t as a state variable, produces the best fit.

While inflation uncertainty has a large direct effect on expected inflation, it is less clear if inflation uncertainty affects nominal and real yields. In other words, it is instructive to see if imposing the restrictions $\rho_v^N = 0$ and $\rho_v^R = 0$ would be a more appropriate model choice. The implied parameter restrictions, $\rho_v^N = 0$ and $\rho_v^\pi = (\gamma_\perp - \frac{1}{2})$, are tested using a likelihood ratio (LR) test. The value of the LR test statistic of these restrictions is 8.8554 with a p-value of 0.012. Therefore, we reject this restricted specification at the 5% level. This result is mainly driven by the importance of inflation uncertainty for the dynamics of real yields. This is confirmed by the large improvement in the fit to real yields in Table 1. The fit of inflation option prices is also very good as illustrated in Figure 2.

Given the parameter estimates, we can construct the model-implied inflation expectations and inflation risk premium as discussed in Section 2.3. We then compare these inflation expectations (and risk premium) from the two main models, $\mathcal{M}_x^{N,R}$ and $\mathcal{M}_{xv}^{N,R,O}$, as well as the University of Michigan household survey and the Blue Chip inflation expectations. Figure 8 presents the 1- and 5-year inflation expectations and inflation risk premium for models $\mathcal{M}_x^{N,R}$ and $\mathcal{M}_{xv}^{N,R,O}$. One noticeable feature of the inflation expectations from $\mathcal{M}_{xv}^{N,R,O}$ is that they are much more volatile and shifted upwards compared to the $\mathcal{M}_x^{N,R}$-based inflation expectations. Counterfactually, the DKW inflation expectations are going up (1-year) or stay constant (5-year) in the last year while our model-implied inflation expectations have drifted downwards, consistent with the subdued inflation in the US (as well as the rest of the world) and falling oil prices. In unreported results, and similarly to DKW, the explicit introduction of a liquidity factor, makes the inflation risk premium positive. From our decomposition of inflation expectations and risk premium into components associated with x and v, it appears that the contribution of v to the level and the dynamics of inflation expectations and risk premium is quite distinct and important.

Finally, Figure 9 shows that our (weekly) inflation expectations trace fairly closely the dynamics of the (monthly) inflation household expectations from the University of Michigan survey and, to a smaller extent, the less volatile (quarterly) Blue Chip inflation expectations. Our inflation risk factor, identified from market option prices, appears to be crucial for matching the volatility of the household inflation expectations. Also, our implied inflation uncertainty measure tends to be in line with the inflation uncertainty measured from the Livingston survey of inflation expectations.

In Tables 2 to 7, we report the in-sample and out-of-sample fit, measured by the root mean squared error (RMSE), of the models for forecasting inflation, nominal yields, real yields, and inflation option prices. The in-sample period is January 1990 - December 2012 and the out-of-sample period is January 2013 - February 2015. For CPI inflation, the RMSE is computed as $\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{\pi}_{i+h}-\hat{\pi}_{i+h}\right)^{2}}$ for $h=1,6,12$, where $\hat{\pi}_{i+h}=\frac{Q_{i+h}-Q_i}{Q_i}$ is the actual inflation rate at horizon h and $\tilde{\pi}_{i+h}$ is the model forecast of inflation. The RMSE for the other variables is computed in a similar manner.

The results presented in Table 2 show that incorporating information from the options market reduces the RMSEs and the inclusion of v_t further improves the in-sample fit. Tables 3 and 4 report the results for the nominal bonds with the reminder that the level and the dynamics of short-term nominal yields have been quite stale. Since none of the models considered have any built-in-features to handle this type of behavior, it is not surprising that they are substantially out-performed by the RW model. Nevertheless, model $\mathcal{M}_{x}^{N,R,O}$ offers improvements at longer horizons for longer maturity bonds. While this is also true for real yields (Tables 5 and 6), $\mathcal{M}_{xv}^{N,R,O}$ performs best in the in-sample evaluation. Finally, the models that use option price information and incorporate v_t dominate by a large margin the remaining models in fitting inflation option prices.

4.1 Why affine models fail to simultaneously forecast inflation and interest rates?

Our results demonstrate that essentially affine models, once extended to model real yields and inflation, can no longer beat the forecast of random-walk models, both in-sample and out-of-sample. Here we use a one-factor model to illustrate the intuition behind the failure. The key intuition is that the flexibility that allows the essentially affine term structure models to accurately forecast Treasury yields are no longer flexible enough in the multi-market context.

Consider the following one-factor model:

$$\begin{aligned} dq_t &= \pi_t dt + \sigma_q \sqrt{v_t} dW_t + \sigma_q^{\perp} dW_{\perp,t} \\ dv_t &= \kappa\left(\mu - v_t\right) dt + \sqrt{v_t} dW_t \end{aligned}$$

where the instantaneous expected inflation rate and the real short rate are both affine:

$$\begin{aligned} r_t^R &= \rho_0^R + \rho_1^R v_t \\ \pi_t &= \rho_0^{\pi} + \rho_1^{\pi} v_t \\ \Lambda_t^R &= \lambda^R \sqrt{v_t} \end{aligned}$$

Following the same argument in Section 2, we can show that no arbitrage implies the following risk-neutral dynamcis of the state variable, in either nominal or real terms,

$$\begin{aligned} &= \kappa\left(\mu - v_t\right) dt + \sqrt{v_t}\left(dW_t^{*R} - \Lambda_t^R dt\right) \\ dv_t &= \kappa^{*i}\left(\mu^{*i} - v_t\right) + \sqrt{v_t} dW_t^{*i}, i = N \text{ or } R \end{aligned}$$

where

$$
\begin{aligned}
\kappa^{*R} &= \kappa + \lambda^{R} \\
\kappa^{*N} &= \kappa + \lambda^{N} \\
\kappa^{*R}\mu^{*R} &= \kappa^{*N}\mu^{*N} = \kappa\mu
\end{aligned}
$$

The nominal price of risk λ^N is determined under no arbitrage as $\lambda^N = \lambda^R + \sigma_q$. As a result, to rule out arbitrage, the cross-sectional and time-series characteristics of the term structure in both Treasury and TIPS markets are inherently linked. The instantaneous expected excess return to a τ-maturity bond, in nominal and real terms, is given by

$$e^i_{t,\tau} = B^i_\tau \lambda^i x_t,\ i = N, R$$

Therefore, the inverse of its coefficient of variation is given by

$$\frac{E\left[e^i_{t,\tau}\right]}{\sqrt{Var\left(e^i_{t,\tau}\right)}} = \frac{\mu}{\sqrt{Var\left(x\right)}},\ i = N, R.$$

Note that first, in this one-factor model, the inverse of coefficient of variation is identical for nominal or real bonds; and second, this model is completely affine. Although extending it to multi-factor essentially affine models helps to break up the link between risk compensation and interest rate volatility. However, such flexibility can be achieved if we only focus on one of the markets. That is, focusing on one single market can produce better forecast for that market, but only for that market.

However ,once markets for both nominal and real yields are included in an essentially affine term structure model, no arbitrage imposes a tight link between the nominal and real prices of risk. For example, the link is similar to the result $\lambda^N = \lambda^R + \sigma_q$ in this completely affine one-factor model. The parameter σ_q governs the inflation process, and, once estimated, imposes a tight link between λ^N and λ^R. Therefore, it is challenging to freely break up risk compensations demanded in both markets. That is, we may be able to break up the link between risk compensation and interest rate volatility in one market, but it is quite challenging to break up the link in both markets. This is the key reason why the essentially affine term structure models that jointly model Treasury and TIPS yields fail to produce accurate forecasts for nominal yields, although they can still produce accurate forecasts for TIPS yields and inflation.

5 Conclusions

In this paper we examine the forecasting ability of affine term structure framework that jointly models the markets for Treasuries, inflation protected securities (TIPS), and inflation derivatives, based on no-arbitrage restrictions across these markets. In particular, we develop and estimate an affine term-structure model that allows for stochastic volatility in inflation and incorporates information from nominal bond yields, TIPS and inflation options. In addition to fitting all these three asset markets simultaneously, the model provides measures of inflation expectations, uncertainty and risk premium with plausible estimated dynamics and wide policy implications.

As our first main result, we show that affine term structure models , once extended to the markets for TIPS and inflation derivatives, can no longer produce better forecasts of nominal Treasury yields than those produced by simply assuming yields follow random walks, although it is the case if the models are built for the Treasury market only. Intuitively, the no-arbitrage restrictions that link these markets also constrain the flexibility in modeling risk compensation and interest rate volatility in these markets. Furthermore, we show that inflation derivatives help improve forecasts of inflaiton and (nominal and real) yields. However, the improvement in nominal yield forecasts is limited. This paper poses a challenge to existing affine term structures models in the ability to simultaneously forecast inflation and interest rates.

We contribute to the literature along several dimensions. First, we demonstrate the importance of no-arbitrage restrictions across different markets for improved forecasting of inflation and bond yields. Second, we use a novel way to introduce information from the options market into an affine framework. More specifically, we establish that option-implied inflation expectations, under the forward measure, are affine in the state variables. We link the different (physical, risk-neutral and forward) measure through common prices of risk and show that inflation options help to identify the price of risk parameters. Finally, we introduce a new inflation volatility factor that proves to be of key importance for pricing inflation options.

The version of the model that we present in this paper does not incorporate explicitly a liquidity factor, as in D'Amico, Kim and Wei (2014), that captures the relative illiquidity of the TIPS and inflation derivative markets. While this liquidity factor could be important, we wanted to keep our model simple so we can emphasize the novelties of our approach. Our preliminary results with an additional liquidity factor suggest that the main message and results of this paper are largely unchanged but the introduction of a liquidity factor helps in isolating this component from the composite risk premium and ensuring more intuitive sign and dynamics of the resulting inflation risk premium.

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Appendix A: Proofs of Propositions

A.1 Proof of Proposition 1

We first derive the pricing formula for nominal bonds. Let

$$\Lambda_t^N \equiv \begin{pmatrix} \Lambda_{x,t}^N \\ \Lambda_{v,t}^N \\ \Lambda_{\perp,t}^N \end{pmatrix}, W_t \equiv \begin{pmatrix} W_{x,t} \\ W_{v,t} \\ W_{\perp,t} \end{pmatrix}.$$

Then, the Radon-Nikodym derivative of the risk neutral measure $\mathbb{P}^*$ with respect to the physical measure $\mathbb{P}$ is given by

$$\left(\frac{d\mathbb{P}^*}{d\mathbb{P}}\right)_{t,T} = \exp\left[-\frac{1}{2}\int_t^T \Lambda_s^{N\prime}\Lambda_s^N ds - \int_t^T \Lambda_s^{N\prime} dW_s\right] \tag{A1}$$

By the Girsanov theorem, $dW_t^* = dW_t + \Lambda_t^N dt$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. It implies that under the risk neutral measure,

$$\begin{aligned} dx_t &= \mathcal{K}(\mu - x_t)\,dt + \Sigma\left(dW_{x,t}^* - \Lambda_{x,t}^N dt\right) \\ &\equiv \mathcal{K}^*(\mu^* - x_t) + \Sigma dW_{x,t}^*, \end{aligned}$$

$$\begin{aligned} dv_t &= \kappa_v(\mu_v - v_t)\,dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}\left(dW_{v,t}^* - \Lambda_{v,t}^N dt\right) + \rho\left(dW_{\perp,t}^* - \Lambda_{\perp,t}^N dt\right)\right) \\ &\equiv \kappa_v^*(\mu_v^* - v_t)\,dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t}^* + \rho dW_{\perp,t}^*\right) \end{aligned}$$

and

$$\begin{aligned} dq_t &= \left(\rho_0^\pi + \rho_x^{\pi\prime}x_t + \rho_v^\pi v_t\right)dt + \sigma_q'\left(dW_{x,t}^* - \Lambda_{x,t}^N dt\right) + \sqrt{v_t}\left(dW_{\perp,t}^* - \Lambda_{\perp,t}^N dt\right) \\ &\equiv \left(\rho_0^{\pi*} + \rho_x^{\pi*\prime}x_t + \rho_v^{\pi*}v_t\right)dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t}dW_{\perp,t}^*, \end{aligned}$$

where $\mathcal{K}^* = \mathcal{K} + \Sigma\lambda_x^N$, $\mathcal{K}^*\mu^* = \mathcal{K}\mu - \Sigma\lambda_0^N$, $\kappa_v^* = \kappa_v + \sqrt{1-\rho^2}\sigma_v\gamma_v + \rho\sigma_v\gamma_\perp$, $\kappa_v^*\mu_v^* = \kappa_v\mu_v$, $\pi_t^* = \rho_0^{\pi*} + \rho_x^{\pi*\prime}x_t + \rho_v^{\pi*}v_t$, $\rho_0^{\pi*} = \rho_0^\pi - \sigma_q'\lambda_0^N$, $\rho_x^{\pi*} = \rho_x^\pi - \lambda_x^{N\prime}\sigma_q$ and $\rho_v^{\pi*} = \rho_v^\pi - \gamma_\perp$. From the fact that $\exp\left(-\int_0^t r_s^N ds\right)P_{t,\tau}^N$ is a martingale under the risk neutral measure, we can derive the ODE system for the nominal yields.

To price real bonds, we need to derive the dynamics of the real pricing kernel. Based on the fact that $M_t^R = M_t^N Q_t$ and using Ito's lemma, it is straightforward to show that the real pricing kernel M_t^R follows the process

$$\begin{aligned} dM_t^R/M_t^R &= dM_t^N/M_t^N + dQ_t/Q_t + \left(dM_t^N/M_t^N\right)\cdot(dQ_t/Q_t) \\ &= -r_t^R dt - \Lambda_{x,t}^{R\prime}dW_{x,t} - \Lambda_{v,t}^R dW_{v,t} - \Lambda_{\perp,t}^R dW_{q,t}, \end{aligned}$$

where the real short rate is

$$r_t^R = \rho_0^R + \rho_x^{R\prime}x_t + \rho_v^R v_t$$

with coefficients $\rho_0^R = \rho_0^N - \rho_0^\pi - \frac{1}{2}\sigma_q'\sigma_q + \lambda_0'\sigma_q$, $\rho_x^R = \rho_x^N - \rho_x^\pi + \lambda_x^{N\prime}\sigma_q$, $\rho_v^R = \rho_v^N - \rho_v^\pi + \gamma_\perp - \frac{1}{2}$. The real prices of risk can be derived as

$$\begin{aligned}
\Lambda_{x,t}^R &= \Lambda_{x,t}^N - \sigma_q \equiv \lambda_0^R + \lambda_x^{R\prime} x_t, \\
\Lambda_{v,t}^R &= \Lambda_{v,t}^N = \gamma_v \sqrt{v_t},
\end{aligned}$$

where $\lambda_0^R = \lambda_0^N - \sigma_q$ and $\lambda_x^R = \lambda_x^N$. Based on the dynamics for the real pricing kernel, the derivation of the pricing formula for real bonds is similar and thus omitted.

A.2 Proof of Proposition 2

Under the forward measure, we have

$$\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right)_{t,T} = \frac{\exp\left(-\int_t^T r_s^N ds\right)}{P_{t,\tau}^N}$$

and

$$\Psi_t \equiv E_t^{\mathbb{P}^*}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right)_{0,T}\right] = E_t^{\mathbb{P}^*}\left[\frac{\exp\left(-\int_0^T r_s^N ds\right)}{P_{0,T}^N}\right] = \frac{P_{t,\tau}^N}{P_{0,T}^N}\exp\left(-\int_0^t r_s^N ds\right)$$

We have

$$\begin{aligned}
d\Psi_t &= \frac{\exp\left(-\int_0^t r_s^N ds\right)}{P_{0,T}^N}\left[dP_{t,\tau}^N - r_t^N P_{t,\tau}^N dt\right] \\
&= \Psi_t\left[B_\tau^{N\prime}\Sigma dW_{x,t} + C_\tau^N \sigma_v \sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t} + \rho dW_{\perp,t}\right)\right]
\end{aligned}$$

By Girsanov's Theorem, we have

$$d\tilde{W}_t = dW_t^* - \frac{d\Psi_t}{\Psi_t}\cdot dW_t^*$$

or

$$\begin{aligned}
d\tilde{W}_{x,t} &= dW_t^* - \Sigma' B_\tau^N dt \\
d\tilde{W}_{v,t} &= dW_{v,t}^* - \sqrt{1-\rho^2}\sigma_v C_\tau^N \sqrt{v_t}dt \\
d\tilde{W}_{\perp,t} &= dW_{\perp,t}^* - \rho\sigma_v C_\tau^N \sqrt{v_t}dt.
\end{aligned}$$

The dynamics of the state variables under the risk neutral measure is given by

$$\begin{aligned}
dx_t &= \mathcal{K}^*\left(\mu^* - x_t\right) + \Sigma dW_{x,t}^* \\
dv_t &= \kappa_v^*\left(\mu_v^* - v_t\right)dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t}^* + \rho dW_{\perp,t}^*\right) \\
dq_t &= \left(\rho_0^{\pi*} + \rho_x^{\pi*\prime}x_t + \rho_v^{\pi*}v_t\right)dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t}dW_{\perp,t}^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
dx_t &= \mathcal{K}^* (\mu^* - x_t) dt + \Sigma \left(d\tilde{W}_{x,t} + \Sigma' B_\tau^N dt \right) \\
&= \left(\mathcal{K}^* \mu^* + \Sigma\Sigma' B_\tau^N - \mathcal{K}^* x_t \right) dt + \Sigma d\tilde{W}_{x,t} \\
&\equiv \tilde{\mathcal{K}} (\tilde{\mu} - x_t) dt + \Sigma d\tilde{W}_{x,t},
\end{aligned}$$

$$\begin{aligned}
dv_t &= \kappa_v^* (\mu_v^* - v_t) dt + \sigma_v \sqrt{v_t} \left(\sqrt{1-\rho^2} dW_{v,t}^* + \rho dW_{\perp,t}^* \right) \\
&= \left[\kappa_v^* \mu_v^* - \left(\kappa_v^* - \sigma_v^2 C_\tau^N \right) v_t \right] dt + \sigma_v \sqrt{v_t} \left(\sqrt{1-\rho^2} d\tilde{W}_{v,t} + \rho d\tilde{W}_{\perp,t} \right) \\
&\equiv \tilde{\kappa}_v (\tilde{\mu}_v - v_t) dt + \sigma_v \sqrt{v_t} \left(\sqrt{1-\rho^2} d\tilde{W}_{v,t} + \rho d\tilde{W}_{\perp,t} \right)
\end{aligned}$$

and

$$\begin{aligned}
dq_t &= \left(\rho_0^{\pi *} + \rho_x^{\pi *\prime} x_t + \rho_v^{\pi *} v_t \right) dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t} dW_{\perp,t}^* \\
&= \left(\left[\rho_0^{\pi *} + \sigma_q' \Sigma' B_\tau^N \right] + \rho_x^{\pi *\prime} x_t + \left[\rho_v^{\pi *} + \rho \sigma_v C_\tau^N \right] v_t \right) dt + \sigma_q' d\tilde{W}_{x,t} + \sqrt{v_t} d\tilde{W}_{\perp,t} \\
&\equiv \left(\widetilde{\rho}_0^{\pi} + \widetilde{\rho}_x^{\pi\prime} x_t + \widetilde{\rho}_v^{\pi} v_t \right) dt + \sigma_q' d\tilde{W}_{x,t} + \sqrt{v_t} d\tilde{W}_{\perp,t}
\end{aligned}$$

where $\tilde{\mathcal{K}} = \mathcal{K}^* = \mathcal{K} + \Sigma \lambda_1^N$, $\tilde{\mathcal{K}} \tilde{\mu} = \mathcal{K}^* \mu^* + \Sigma\Sigma' B_\tau^N = \mathcal{K}\mu - \Sigma \lambda_0^N + \Sigma\Sigma' B_\tau^N$, $\tilde{\kappa}_v = \kappa_v^* - \sigma_v^2 C_\tau^N = \kappa_v + \sqrt{1-\rho^2} \sigma_v \gamma_v + \rho \sigma_v \gamma_\perp - \sigma_v^2 C_\tau^N$, $\tilde{\kappa}_v \tilde{\mu}_v = \kappa_v^* \mu_v^* = \kappa_v \mu_v$, $\tilde{\pi}_t = \widetilde{\rho}_0^{\pi} + \widetilde{\rho}_x^{\pi\prime} x_t + \widetilde{\rho}_v^{\pi} v_t$, $\widetilde{\rho}_0^{\pi} = \rho_0^{\pi *} + \sigma_q' \Sigma' B_\tau^N = \rho_0^{\pi} - \sigma_q' \lambda_0^N + \sigma_q' \Sigma' B_\tau^N$, $\widetilde{\rho}_x^{\pi} = \rho_x^{\pi *} = \rho_x^{\pi} - \lambda_x^{N\prime} \sigma_q$ and $\widetilde{\rho}_v^{\pi} = \rho_v^{\pi *} + \rho \sigma_v C_\tau^N = \rho_v^{\pi} - \gamma_\perp + \rho \sigma_v C_\tau^N$.

Next, we compute the expected values of the state variables over the period t to $t+\tau$ under the forward measure. First, using the stochastic differential equation for v_t, it follows that

$$E_t^{\tilde{\mathbb{P}}} (v_s) = \tilde{\mu}_v + \exp(-\tilde{\kappa}_v (s-t)) (v_t - \tilde{\mu}_v)$$

and

$$\frac{1}{\tau} E_t^{\tilde{\mathbb{P}}} \left[\int_t^{t+\tau} v_s ds \right] = \tilde{\mu}_v + (\tilde{\kappa}_v \tau)^{-1} (1 - \exp(-\tilde{\kappa}_v \tau)) (v_t - \tilde{\mu}_v) \equiv \tilde{a}_\tau^v + \tilde{b}_\tau^v v_t$$

where $\tilde{a}_\tau^v = \tilde{\mu}_v \left[1 - (\tilde{\kappa}_v \tau)^{-1} (1 - \exp(-\tilde{\kappa}_v \tau)) \right]$ and $\tilde{b}_\tau^v = (\tilde{\kappa}_v \tau)^{-1} (1 - \exp(-\tilde{\kappa}_v \tau))$. Second, because $x_s = e^{-\tilde{\mathcal{K}}(s-t)} x_t + \int_t^s \left[e^{\tilde{\mathcal{K}}(u-s)} \tilde{\mathcal{K}} \tilde{\mu} du + e^{\tilde{\mathcal{K}}(u-s)} \Sigma d\tilde{W}_{x,u} \right]$, we have

$$E_t^{\tilde{\mathbb{P}}} (x_s) = \tilde{\mu} + \exp\left(-\tilde{\mathcal{K}} (s-t) \right) (x_t - \tilde{\mu})$$

and

$$\begin{aligned}
\frac{1}{\tau} E_t^{\tilde{\mathbb{P}}} \left[\int_t^{t+\tau} x_s ds \right] &= \frac{1}{\tau} \left[\int_t^{t+\tau} \left(\tilde{\mu} + \exp\left(-\tilde{\mathcal{K}} (s-t) \right) (x_t - \tilde{\mu}) \right) ds \right] \\
&= \tilde{\mu} + \left(\tilde{\mathcal{K}} \tau \right)^{-1} \left(I - \exp\left(-\tilde{\mathcal{K}} \tau \right) \right) (x_t - \tilde{\mu}) \equiv \tilde{a}_\tau^x + \tilde{b}_\tau^x x_t.
\end{aligned}$$

Third,

$$E_t^{\tilde{\mathbb{P}}} \left[\ln\left(\frac{Q_{t+\tau}}{Q_t} \right) \right] = E_t^{\tilde{\mathbb{P}}} \left[\int_t^{t+\tau} \tilde{\pi}_s ds \right] = E_t^{\tilde{\mathbb{P}}} \left[\int_t^{t+\tau} \left(\rho_0^{\pi} + \rho_x^{\pi\prime} x_s + \rho_v^{\pi} v_s \right) ds \right]$$

$$= \tau \left(\rho_0^{\pi} + \rho_x^{\pi\prime} \left(\tilde{a}_\tau^x + \tilde{b}_\tau^x x_t \right) + \rho_v^{\pi} \left(\tilde{a}_\tau^v + \tilde{b}_\tau^v v_t \right) \right) \equiv \tau \left(\tilde{a}_\tau^{\pi} + \tilde{b}_\tau^{\pi\prime} x_t + \tilde{c}_\tau^{\pi} v_t \right)$$

and

$$Var_t^{\tilde{\mathbb{P}}}\left[\ln\left(\frac{Q_{t+\tau}}{Q_t}\right)\right] = E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau}\left(\sigma_q'\sigma_q + v_s\right)ds\right] = \tau\left(\sigma_q'\sigma_q + \tilde{a}_\tau^v + \tilde{b}_\tau^v v_t\right) \equiv \tau\left(\tilde{d}_\tau^\pi + \tilde{e}_\tau^\pi v_t\right),$$

where $\tilde{a}_\tau^\pi = \rho_0^\pi + \rho_x^{\pi\prime}\tilde{a}_\tau^x + \rho_v^\pi \tilde{a}_\tau^v$, $\tilde{b}_\tau^\pi = \tilde{b}_\tau^{x\prime}\rho_x^\pi$, $\tilde{c}_\tau^\pi = \rho_v^\pi \tilde{b}_\tau^v$, $\tilde{d}_\tau^\pi = \sigma_q'\sigma_q + \tilde{a}_\tau^v$ and $\tilde{e}_\tau^\pi = \tilde{b}_\tau^v$.

Finally, we turn to inflation option pricing. The price of a τ-maturity inflation cap with strike K is given by

$$\begin{aligned}
P_{t,\tau,K}^{CAP} &= \exp\left(-\tau y_{t,\tau}^N\right) E_t^{\tilde{\mathbb{P}}}\left[\left(\frac{Q_{t+\tau}}{Q_t} - (1+K)^\tau\right)^+\right] \\
&= \exp\left(-\tau y_{t,\tau}^N\right) E_t^{\tilde{\mathbb{P}}}\left[\left(\exp\left(\ln\left(\frac{Q_{t+\tau}}{Q_t}\right)\right) - (1+K)^\tau\right)^+\right] \\
&= \exp\left(-\tau y_{t,\tau}^N\right)\left[\exp\left(\tau\left(\left(\tilde{a}_\tau^\pi + \frac{\tilde{d}_\tau^\pi}{2}\right) + \tilde{b}_\tau^{\pi\prime}x_t + \left(\tilde{c}_\tau^\pi + \frac{\tilde{e}_\tau^\pi}{2}\right)v_t\right)\right)\right. \\
&\quad \times\Phi\left(\frac{\tau}{\sigma}\left[-\ln(1+K) + \left(\tilde{a}_\tau^\pi + \tilde{d}_\tau^\pi\right) + \tilde{b}_\tau^{\pi\prime}x_t + \left(\tilde{c}_\tau^\pi + \tilde{e}_\tau^\pi\right)v_t\right]\right) \\
&\quad \left. -(1+K)^\tau\,\Phi\left(\frac{\tau}{\sigma}\left[-\ln(1+K) + \tilde{a}_\tau^\pi + \tilde{b}_\tau^{\pi\prime}x_t + \tilde{c}_\tau^\pi v_t\right]\right)\right],
\end{aligned}$$

and the price of a τ-maturity inflation cap with strike K is given by

$$\begin{aligned}
P_{t,\tau,K}^{FLO} &= \exp\left(-\tau y_{t,\tau}^N\right) E_t^{\tilde{\mathbb{P}}}\left[\left((1+K)^\tau - \frac{Q_{t+\tau}}{Q_t}\right)^+\right] \\
&= \exp\left(-\tau y_{t,\tau}^N\right) E_t^{\tilde{\mathbb{P}}}\left[\left((1+K)^\tau - \exp\left(\ln\left(\frac{Q_{t+\tau}}{Q_t}\right)\right)\right)^+\right] \\
&= \exp\left(-\tau y_{t,\tau}^N\right)\left[-\exp\left(\tau\left(\left(\tilde{a}_\tau^\pi + \frac{\tilde{d}_\tau^\pi}{2}\right) + \tilde{b}_\tau^{\pi\prime}x_t + \left(\tilde{c}_\tau^\pi + \frac{\tilde{e}_\tau^\pi}{2}\right)v_t\right)\right)\right. \\
&\quad \times\Phi\left(-\frac{\tau}{\sigma}\left[-\ln(1+K) + \left(\tilde{a}_\tau^\pi + \tilde{d}_\tau^\pi\right) + \tilde{b}_\tau^{\pi\prime}x_t + \left(\tilde{c}_\tau^\pi + \tilde{e}_\tau^\pi\right)v_t\right]\right) \\
&\quad \left. +(1+K)^\tau\,\Phi\left(-\frac{\tau}{\sigma}\left[-\ln(1+K) + \tilde{a}_\tau^\pi + \tilde{b}_\tau^{\pi\prime}x_t + \tilde{c}_\tau^\pi v_t\right]\right)\right]
\end{aligned}$$

where we used the fact that for a normal random variable $\tilde{z} \sim N\left(\mu, \sigma^2\right)$,

$$\begin{aligned}
E\left[\left(ae^{\tilde{z}} - b\right)^+\right] &= a\exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(\frac{\ln(a/b) + \left(\mu + \sigma^2\right)}{\sigma}\right) - b\Phi\left(\frac{\ln(a/b) + \mu}{\sigma}\right), \\
E\left[\left(b - ae^{\tilde{z}}\right)^+\right] &= -a\exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(-\frac{\ln(a/b) + \left(\mu + \sigma^2\right)}{\sigma}\right) + b\Phi\left(-\frac{\ln(a/b) + \mu}{\sigma}\right).
\end{aligned}$$

A.3 Proof of Proposition 3

First, we calculate expected average future state variables x_t and v_t:

$$\begin{aligned}
E_t^{\mathbb{P}}\left[\frac{1}{\tau}\int_t^{t+\tau} x_s ds\right] &= \frac{1}{\tau}\int_t^{t+\tau}\left(\mu + e^{-\mathcal{K}(s-t)}\left(x_t-\mu\right)\right)ds \equiv a_\tau^x + b_\tau^x x_t,\\
E_t^{\mathbb{P}}\left[\frac{1}{\tau}\int_t^{t+\tau} v_s ds\right] &= \frac{1}{\tau}\int_t^{t+\tau}\left(\mu_v + e^{-\kappa_v(s-t)}\left(v_t-\mu_v\right)\right)ds \equiv a_\tau^v + b_\tau^v v_t,
\end{aligned}$$

where $a_\tau^x \equiv \left(I - \left(\mathcal{K}\tau\right)^{-1}\left(I-\exp\left(-\mathcal{K}\tau\right)\right)\right)\mu$, $a_\tau^v \equiv \left(1-\left(\kappa_v\tau\right)^{-1}\left(1-\exp\left(-\kappa_v\tau\right)\right)\right)\mu_v$, $b_\tau^x \equiv \left(\mathcal{K}\tau\right)^{-1}$ $\left(1-\exp\left(-\mathcal{K}\tau\right)\right)$ and $b_\tau^v \equiv \left(\kappa_v\tau\right)^{-1}\left(1-\exp\left(-\kappa_v\tau\right)\right)$.

Next, we calculate the time-t expected mean and variance of $\ln\left(Q_{t+\tau}/Q_t\right)$ as follows:

$$\begin{aligned}
\frac{1}{\tau}E_t^{\mathbb{P}}\left[\ln\left(\frac{Q_{t+\tau}}{Q_t}\right)\right] &= \frac{1}{\tau}E_t^{\mathbb{P}}\left[\int_t^{t+\tau}\pi_s ds\right] = \frac{1}{\tau}E_t^{\mathbb{P}}\left[\int_t^{t+\tau}\left(\rho_0^\pi + \rho_1^{\pi\prime}x_s + \rho_v^\pi v_s\right)ds\right]\\
&= \rho_0^\pi + \rho_x^{\pi\prime}\left(a_\tau^x + b_\tau^x x_t\right) + \rho_v^\pi\left(a_\tau^v + b_\tau^v v_t\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\tau}Var_t^{\mathbb{P}}\left[\ln\left(\frac{Q_{t+\tau}}{Q_t}\right)\right] &= \frac{1}{\tau}E_t^{\mathbb{P}}\left[\left(\int_t^{t+\tau}\sigma_q' dW_{x,s} + \sqrt{v_s}dW_{\perp,s}\right)^2\right] = \sigma_q'\sigma_q + \frac{1}{\tau}E_t^{\mathbb{P}}\left[\int_t^{t+\tau}v_s ds\right]\\
&= \sigma_q'\sigma_q + a_\tau^v + b_\tau^v v_t.
\end{aligned}$$

Thus, the inflation expectations, defined as in Christensen, Lopez and Rudebusch (2010), are given by

$$\begin{aligned}
IE_{t,\tau} &\equiv -\frac{1}{\tau}\ln\left(E_t^{\mathbb{P}}\left[\frac{Q_t}{Q_{t+\tau}}\right]\right)\\
&= \frac{1}{\tau}\left(E_t^{\mathbb{P}}\left[\ln\left(Q_{t+\tau}/Q_t\right)\right] - \frac{1}{2}Var_t^{\mathbb{P}}\left[\ln\left(Q_{t+\tau}/Q_t\right)\right]\right)\\
&= \rho_0^\pi + \rho_x^{\pi\prime}\left(a_\tau^x + b_\tau^{x\prime}x_t\right) + \rho_v^\pi\left(a_\tau^v + b_\tau^v v_t\right) - \frac{1}{2}\left(\sigma_q'\sigma_q + a_\tau^v + b_\tau^v v_t\right).
\end{aligned} \tag{A2}$$

Furthermore, it can be shown that

$$\begin{aligned}
& y_{t,\tau}^N - y_{t,\tau}^R + \frac{1}{\tau}\ln\left(E_t^{\mathbb{P}}\left[Q_t/Q_{t+\tau}\right]\right)\\
=\ & -\frac{1}{\tau}\ln\left(\frac{E_t^{\mathbb{P}}\left[\left(M_{t+\tau}^R/M_t^R\right)\left(Q_t/Q_{t+\tau}\right)\right]}{E_t^{\mathbb{P}}\left[\left(M_{t+\tau}^R/M_t^R\right)\right]E_t^{\mathbb{P}}\left[\left(Q_t/Q_{t+\tau}\right)\right]}\right)
\end{aligned}$$

Then, note that for normal random variables $(\widetilde{x},\widetilde{y})$, we have

$$\frac{E\left[\exp\left(\widetilde{x}+\widetilde{y}\right)\right]}{E\left[\exp\left(\widetilde{x}\right)\right]E\left[\exp\left(\widetilde{y}\right)\right]} = \exp\left(Cov\left[\widetilde{x},\widetilde{y}\right]\right).$$

Therefore,

$$
\begin{aligned}
& y^N_{t,\tau} - y^R_{t,\tau} + \frac{1}{\tau}\ln\left(E^{\mathbb{P}}_t\left[Q_t/Q_{t+\tau}\right]\right) = -\frac{1}{\tau}Cov^{\mathbb{P}}_t\left[\ln\left(M^R_{t+\tau}/M^R_t\right)\ln\left(Q_{t+\tau}/Q_t\right)\right] \\
= \;& -\frac{1}{\tau}E^{\mathbb{P}}_t\left[\int_t^{t+\tau}\left(-\sigma_q'\Lambda^R_{x,s} - \Lambda^R_{\perp,s}\sqrt{v_s}\right)ds\right] \\
= \;& \sigma_q'\left(\lambda^R_0 + \lambda^{R\prime}_x\left(a^x_\tau + b^{x\prime}_\tau x_t\right)\right) + \left(\gamma_\perp - 1\right)\left(a^v_\tau + b^{v\prime}_\tau v_t\right) \\
\equiv \;& IRP^x_{t,\tau} + IRP^v_{t,\tau}.
\end{aligned}
$$

Appendix B: Estimation Methodology

B.1 Transition and Measurement Equations

We first consider the discrete-time dynamics of the state variable q_t between time $t-\Delta t$ and t. From $dq_t = \pi_t dt + \sigma_q' dW_{x,t} + \sqrt{v_t}dW_{\perp,t}$, we have

$$
q_t = q_{t-\Delta t} + \left(\rho^\pi_0\Delta t + \left(\rho^{\pi\prime}_x\Delta t\right)x_t + \left(\rho^\pi_v\Delta t\right)v_t\right) + \eta^q_t
$$

where $\eta^q_t \equiv \int_{t-\Delta t}^t\left(\sigma_q' dW_{x,s} + \sqrt{v_s}dW_{\perp,s}\right) \sim N\left(0, \Omega^q_{t-\Delta t}\right)$ and

$$
\begin{aligned}
\Omega^q_{t-\Delta t} &\equiv Var_{t-\Delta t}\left(\eta^q_t\right) = \sigma_q'\sigma_q\Delta t + E_{t-\Delta t}\left[\int_{t-\Delta t}^t v_s ds\right] \\
&= \sigma_q'\sigma_q\Delta t + \left[\int_0^{\Delta t}\left(\mu_v\left(1-\exp(-\kappa_v s)\right) + \exp(-\kappa_v s)v_{t-\Delta t}\right)ds\right] \\
&= \sigma_q'\sigma_q\Delta t + \mu_v\Delta t + \left(v_{t-\Delta t} - \mu_v\right)\frac{1-\exp\left(-\kappa_v\Delta t\right)}{\kappa_v}.
\end{aligned}
$$

Similarly, for the state variable x_t we have

$$
x_t = \exp\left(-\mathcal{K}\Delta t\right)x_{t-\Delta t} + \left(I - \exp\left(-\mathcal{K}\Delta t\right)\right)\mu + \eta^x_t,
$$

where $\eta^x_t = \int_0^{\Delta t}\exp\left(-\kappa s\right)\Sigma dW_{x,s} \sim N\left(0, \Omega^x_{t-\Delta t}\right)$ and

$$
\Omega^x_{t-\Delta t} \equiv Var_{t-\Delta t}\left(\eta^x_t\right) = \int_0^{\Delta t}\exp\left(-\mathcal{K}s\right)\Sigma\Sigma'\exp\left(-\mathcal{K}'s\right)ds = N\Xi N',
$$

with $\mathcal{K} = NDN^{-1}$, $D = diag\left([d_1, ..., d_N]\right)$, and $\Xi_{i,j} = \left[(N^{-1}\Sigma)(N^{-1}\Sigma)'\right]_{i,j}\frac{1-\exp[-(d_i+d_j)\cdot t]}{(d_i+d_j)}$. The covariance matrix between η^x_t and η^q_t is given by

$$
\Omega^{xq}_{t-\Delta t} = Cov_{t-\Delta t}\left[\eta^x_t, \eta^q_t\right] = \int_0^{\Delta t}\exp\left(-\mathcal{K}s\right)\Sigma\sigma_q ds = \mathcal{K}^{-1}\left(I - \exp\left(-\mathcal{K}s\Delta t\right)\right)\Sigma\sigma_q.
$$

where I signifies the identity matrix.

Next, we consider the state variable v_t. From $dv_t = \kappa_v\left(\mu_v - v_t\right)dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t} + \rho dW_{\perp,t}\right)$, we have

$$
v_t = \exp(-\kappa_v\Delta t)v_{t-\Delta t} + \mu_v\left(1 - \exp(-\kappa_v\Delta t)\right) + \eta^v_t
$$

where $\eta_t^v = \sigma_v \int_{t-\Delta t}^t \exp(\kappa_v (s-t))\sqrt{v_s}\left(\sqrt{1-\rho^2}dB_s^v + \rho dB_s^\perp\right) \sim N\left(0, \Omega_{t-1}^v\right)$ and

$$\begin{aligned}
\Omega_{t-\Delta t}^v &= E_{t-\Delta t}\left[\sigma_v^2 \int_{t-\Delta t}^t \exp(2\kappa_v (s-t)) v_s ds\right] \\
&= \mu_v \sigma_v^2 \frac{(1-\exp(-\kappa_v \Delta t))^2}{2\kappa_v} + v_{t-1}\sigma_v^2 \frac{\exp(-\kappa_v \Delta t) - \exp(-2\kappa_v \Delta t)}{\kappa_v}
\end{aligned}$$

The covariance between η_t^v and η_t^x is zero by construction while the covariance between the inflation and volatility innovation terms is

$$\begin{aligned}
\Omega_{t-\Delta}^{vq} &= Cov_{t-\Delta t}\left[\eta_t^v, \eta_t^q\right] = \rho\sigma_v E_{t-\Delta t}\left[\int_{t-\Delta t}^t \exp(\kappa_v (s-t)) v_s ds\right] \\
&= \rho\sigma_v \left[\mu_v \frac{\left(1-\exp(-\frac{\kappa_v}{2}\Delta t)\right)^2}{\kappa_v} + v_{t-\Delta t}\frac{\exp(-\frac{\kappa_v}{2}\Delta t) - \exp(-\kappa_v \Delta t)}{\kappa_v/2}\right].
\end{aligned}$$

In summary, the dynamics of the state vector $X_t = (q_t, x_t', v_t)'$ follows the VAR process

$$X_t = \mathcal{A} + \mathcal{B} X_{t-\Delta t} + \eta_t$$

where $\mathcal{A} = \begin{bmatrix} \rho_0^\pi \Delta t \\ (I - \exp(-\mathcal{K}\Delta t))\,\mu \\ (1-\exp(-\kappa_v \Delta t))\,\mu_v \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 1 & \rho_x^{\pi\prime}\Delta t & \rho_v^\pi \Delta t \\ 0 & \exp(-\mathcal{K}\Delta t) & 0_{3\times 1} \\ 0 & 0_{1\times 3} & \exp(-\kappa_v \Delta t) \end{bmatrix}$, $\eta_t = \begin{pmatrix} \eta_t^q \\ \eta_t^x \\ \eta_t^v \end{pmatrix} \sim$ $N(0, \Omega_{t-\Delta t})$ and $\Omega_{t-\Delta t} = \begin{bmatrix} \Omega_{t-\Delta t}^q & \Omega_{t-\Delta t}^{xq\prime} & \Omega_{t-\Delta t}^{vq} \\ \Omega_{t-\Delta t}^{xq} & \Omega_{t-\Delta t}^x & 0_{3\times 1} \\ \Omega_{t-\Delta t}^{vq} & 0_{1\times 3} & \Omega_{t-\Delta t}^v \end{bmatrix}$.

Combining the expressions for nominal and real yields in Proposition 1 and for option-implied inflation expectations in Proposition 2, the measurement equation[6] takes the form

$$\begin{aligned}
Y_t &= \begin{bmatrix} 0 \\ a_\tau^N \\ a_\tau^R \\ a_{\tau,K}^{IE} \end{bmatrix} + \begin{bmatrix} 1 & 0_{1\times 3} & 0 \\ 0 & b_\tau^{N\prime} & c_\tau^N \\ 0 & b_\tau^{R\prime} & c_\tau^R \\ 0 & b_{\tau,K}^{IE\prime} & c_{\tau,K}^{IE} \end{bmatrix} X_t + \begin{bmatrix} 0 \\ e_t^N \\ e_t^R \\ e_t^{IE} \end{bmatrix} \\
&\equiv a + bX_t + e_t,
\end{aligned}$$

where e_t is an $m \times 1$ vector of iid measurement errors with $e_{t,\tau_i}^N \sim N\left(0, (\delta_{\tau_i}^N)^2\right)$, $e_{t,\tau_i}^R \sim N\left(0, (\delta_{\tau_i}^R)^2\right)$ and $e_{t,\tau_i}^{IE} \sim N\left(0, (\delta_{\tau_i}^{IE})^2\right)$ with $\delta_{\tau_i}^{IE} = \delta^{IE}$ for all $\tau \in \left\{\tau_1, \cdots, \tau_m^{IE}\right\}$.

B.2 Kalman Filter

With the derived state and observation equations, the state-space system is given by

$$\begin{aligned}
X_t &= \mathcal{A} + \mathcal{B} X_{t-1} + \eta_t \\
Y_t &= a + bX_t + e_t
\end{aligned}$$

[6] The vector of observables can be further augmented with inflation survey expectations as well as short-term (up to a year) and long-term (5 to 10 years) forecasts of the spot rate as in D'Amico, Kim and Wei (2014).

where $\mathcal{A}$, $\mathcal{B}$, a, and b are functions of the underlying parameters of interest $\theta = (vec(\mathcal{K})', vech(\Sigma\Sigma')'$, ρ, μ', $\rho_0^{N\prime}$, $\rho_x^{N\prime}$, ρ_v^N, $\lambda_0^{N\prime}$, $vec(\lambda_x^N)'$, ρ_0^π, $\rho_x^{\pi\prime}$, ρ_v^π, σ_q', κ_v, σ_v, μ_v, γ_v, $\gamma_\perp)'$. The estimation procedure, based on the Kalman filter, that we adopt in this paper is the following. Denote the one-period-ahead prediction and variance of X_t (or Y_t) as $X_{t|t-1}$ and $P_{t|t-1}$ (or $Y_{t|t-1}$ and $V_{t|t-1}$), respectively. That is,

$$\begin{aligned} X_{t|t-1} &\equiv E_{t-1}[X_t] \text{ and } P_{t|t-1} \equiv Var_{t-1}[X_t] \\ Y_{t|t-1} &\equiv E_{t-1}[Y_t] \text{ and } V_{t|t-1} \equiv Var_{t-1}[Y_t]. \end{aligned}$$

1. Calculate the unconditional mean and variance of X_t as the prediction and variance of X_0, denoted by $X_{0|0}$ and $P_{0|0}$.

2. Compute the one-period-ahead prediction and variance of X_t, given $X_{t-1|t-1}$ and $P_{t-1|t-1}$

$$\begin{aligned} X_{t|t-1} &= \mathcal{A} + \mathcal{B}X_{t-1|t-1} \\ P_{t|t-1} &= \mathcal{B}P_{t-1|t-1}\mathcal{B}' + \Omega_{t-1} \end{aligned}$$

3. Compute the one-period-ahead prediction and variance of Y_t, given $X_{t-1|t-1}$ and $P_{t-1|t-1}$

$$\begin{aligned} Y_{t|t-1} &= a + bX_{t|t-1} \\ V_{t|t-1} &= bP_{t-1|t-1}b' + \Gamma \end{aligned}$$

4. Compute the forecast error in Y_t as

$$\epsilon_{t|t-1} = Y_t - Y_{t|t-1} \sim N\left(0, V_{t|t-1}\right)$$

5. Update the prediction of X_t as

$$\begin{aligned} X_{t|t} &= X_{t|t-1} + P_{t|t-1}b'V_{t|t-1}^{-1}\epsilon_{t|t-1} \\ P_{t|t} &= P_{t|t-1} - P_{t|t-1}b'V_{t|t-1}^{-1}bP_{t|t-1} \end{aligned}$$

6. Repeat steps 2-5 for $t = 1, \cdots, T$, the parameter vector θ maximizes the following pseudo log-likelihood function

$$\mathcal{L}(Y_t; X_t, \theta) = \max \textstyle\sum_{t=1}^T \left[-\frac{1}{2}\left(m \ln(2\pi) + \ln\left|V_{t|t-1}\right| + \epsilon_{t|t-1}' V_{t|t-1}^{-1} \epsilon_{t|t-1}\right)\right].$$

Table 1: Parameter Estimates (January 1990 - December 2012)

Parameter	$\mathcal{M}_x^{N,R}$	$\mathcal{M}_x^{N}$	$\mathcal{M}_x^{N,R,O}$	$\mathcal{M}_{x,v}^{N,R,O}$
$\mathcal{K}_{11}$	0.8798	0.8716	0.8804	0.8896
$\mathcal{K}_{22}$	0.1367	0.2277	0.1338	0.1373
$\mathcal{K}_{33}$	1.4473	1.3629	1.4501	1.4467
$100 \times \Sigma_{21}$	-0.6935	1.6443	-0.6880	-0.6804
$100 \times \Sigma_{31}$	-4.5187	-5.9575	-4.5035	-4.8217
$100 \times \Sigma_{32}$	-0.9687	-0.9512	-0.9606	-0.9773
ρ_0^N	0.0473	0.0345	0.0474	0.0391
$\rho_x^N(1)$	3.7723	2.5158	3.7580	3.9000
$\rho_x^N(2)$	0.8952	0.8262	0.8896	0.8914
$\rho_x^N(3)$	0.7398	0.7082	0.7408	0.7234
ρ_v^N	NaN	NaN	NaN	78.9460
$\lambda_0^N(1)$	0.3073	0.2973	0.3099	0.3640
$\lambda_0^N(2)$	-0.4085	-0.5386	-0.4095	-0.4188
$\lambda_0^N(3)$	-1.2332	-1.3232	-1.2383	-1.2283
$\left(\Sigma\lambda_x^N\right)_{11}$	-0.6940	-0.5834	-0.6945	-0.7793
$\left(\Sigma\lambda_x^N\right)_{21}$	2.1130	1.8509	2.1084	2.3929
$\left(\Sigma\lambda_x^N\right)_{31}$	3.1788	1.5766	3.1920	3.4719
$\left(\Sigma\lambda_x^N\right)_{12}$	0.0304	0.0606	0.0305	0.0460
$\left(\Sigma\lambda_x^N\right)_{22}$	-0.1440	-0.1715	-0.1445	-0.1546
$\left(\Sigma\lambda_x^N\right)_{32}$	-0.3505	-0.7089	-0.3566	-0.4359
$\left(\Sigma\lambda_x^N\right)_{13}$	-0.0664	-0.1010	-0.0660	-0.0945
$\left(\Sigma\lambda_x^N\right)_{23}$	0.5703	0.1080	0.5723	0.6154
$\left(\Sigma\lambda_x^N\right)_{33}$	0.1281	0.0574	0.1292	0.2052
ρ_0^π	0.0268	0.0268	0.0257	-0.1067
$\rho_x^\pi(1)$	-0.0988	0.6105	-0.0957	1.3881
$\rho_x^\pi(2)$	0.1182	0.1981	0.1302	0.3433
$\rho_x^\pi(3)$	-0.2299	0.1318	-0.1728	0.1493
ρ_v^π	NaN	NaN	NaN	1437.5308
$100 \times \sigma_q(1)$	-0.0657	-0.0557	-0.0595	-0.0169
$100 \times \sigma_q(2)$	-0.0056	0.1295	-0.0064	-0.0235
$100 \times \sigma_q(3)$	-0.0615	-0.0241	-0.0686	-0.0878
$100 \times \sigma_q^\perp$	0.9139	0.8923	0.9099	NaN
κ_v	NaN	NaN	NaN	0.1397
$100 \times \sigma_v$	NaN	NaN	NaN	0.0105
$100 \times \mu_v$	NaN	NaN	NaN	0.0783
γ_v	NaN	NaN	NaN	-69.6551
$\gamma_\perp$	NaN	NaN	NaN	116.2959
ρ	NaN	NaN	NaN	0.5188

Table 1 (continued): Parameter Estimates (January 1990 - December 2012)

Parameter	$\mathcal{M}_x^{N,R}$	$\mathcal{M}_x^{N}$	$\mathcal{M}_x^{N,R,O}$	$\mathcal{M}_{x,v}^{N,R,O}$
$100 \times \delta_{3m}^N$	0.1324	0.1321	0.1324	0.1322
$100 \times \delta_{6m}^N$	0.0183	0.0163	0.0181	0.0188
$100 \times \delta_{1y}^N$	0.0657	0.0656	0.0658	0.0657
$100 \times \delta_{2y}^N$	0.0000	0.0000	0.0000	0.0000
$100 \times \delta_{4y}^N$	0.0397	0.0394	0.0398	0.0401
$100 \times \delta_{7y}^N$	-0.0000	0.0000	-0.0000	0.0000
$100 \times \delta_{10y}^N$	0.0529	0.0536	0.0529	0.0528
$100 \times \delta_{5y}^R$	0.5987	NaN	0.6070	0.0907
$100 \times \delta_{7y}^R$	0.4703	NaN	0.4771	-0.0000
$100 \times \delta_{10y}^R$	0.4115	NaN	0.4093	0.0706
$100 \times \delta_{6m}^F$	0.1886	NaN	0.1885	0.1887
$100 \times \delta_{12m}^F$	0.2944	NaN	0.2942	0.2941
$100 \times \delta_{1y}^{IE}$	NaN	NaN	0.6706	0.5623
$100 \times \delta_{3y}^{IE}$	NaN	NaN	0.3851	0.2795

Notes: The table presents the estimated parameters and measurement error standard deviations (δ) for the models: (1) model $\mathcal{M}_x^{N,R}$ with state variables x that uses data from nominal and real yields; (2) model $\mathcal{M}_x^{N}$ with state variables x that uses data from nominal yields only; (3) model $\mathcal{M}_x^{N,R,O}$ with state variables x that uses data from nominal and real yields, and option-implied inflation expectations; (4) model $\mathcal{M}_{x,v}^{N,R,O}$ with state variables x and v that uses data from nominal and real yields, and option-implied inflation expectations.

Table 2: In-Sample and Out-of-Sample Inflation Forecast RMSEs

Panel A: In-Sample Forecasts				
horizon	RW	$\mathcal{M}_x^{N,R}$	$\mathcal{M}_x^{N,R,O}$	$\mathcal{M}_{x,v}^{N,R,O}$
1	0.0034	0.0027	0.0027	0.0025
6	0.0153	0.0088	0.0086	0.0069
12	0.0275	0.0123	0.0117	0.0084
Panel B: Out-of-Sample Forecasts				
horizon	RW	$\mathcal{M}_x^{N,R}$	$\mathcal{M}_x^{N,R,O}$	$\mathcal{M}_{x,v}^{N,R,O}$
1	0.0026	0.0029	0.0027	0.0030
6	0.0085	0.0080	0.0066	0.0094
12	0.0156	0.0107	0.0067	0.0133

Notes: The table reports the root mean squared errors (RMSEs) of CPI inflation, computed as $\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{\pi}_{i+h}-\hat{\pi}_{i+h}\right)^2}$ for $h = 1, 6, 12$, where $\hat{\pi}_{i+h} = \frac{Q_{i+h}-Q_i}{Q_i}$ is the actual inflation rate at horizon h and $\tilde{\pi}_{i+h}$ is the model forecast of inflation. The in-sample period is January 1990 - December 2012 and the out-of-sample period is January 2013 - February 2015. The models are: (1) random walk (RW); (2) model $\mathcal{M}_x^{N,R}$ with state variables x that uses data from nominal and real yields; (3) model $\mathcal{M}_x^{N,R,O}$ with state variables x that uses data from nominal and real yields, and option-implied inflation expectations; (4) model $\mathcal{M}_{x,v}^{N,R,O}$ with state variables x and v that uses data from nominal and real yields, and option-implied inflation expectations.

Table 3: In-Sample Nominal Yields Forecast RMSEs

maturity	horizon	RW	$\mathcal{M}_x^{N,R}$	$\mathcal{M}_x^{N}$	$\mathcal{M}_x^{N,R,O}$	$\mathcal{M}_{x,v}^{N,R,O}$
0.5	3	0.4970	0.4763	0.4280	0.4762	0.4759
1	3	0.5203	0.5316	0.4807	0.5319	0.5316
2	3	0.5519	0.5948	0.5374	0.5955	0.5960
3	3	0.5583	0.6054	0.5510	0.6062	0.6076
4	3	0.5499	0.5922	0.5439	0.5929	0.5951
5	3	0.5351	0.5702	0.5290	0.5708	0.5733
6	3	0.5183	0.5471	0.5129	0.5475	0.5502
7	3	0.5020	0.5264	0.4984	0.5268	0.5293
8	3	0.4871	0.5094	0.4862	0.5097	0.5120
9	3	0.4740	0.4962	0.4765	0.4965	0.4987
10	3	0.4626	0.4869	0.4691	0.4872	0.4892
0.5	6	0.8398	0.8193	0.7048	0.8195	0.8193
1	6	0.8447	0.8775	0.7555	0.8785	0.8782
2	6	0.8337	0.9221	0.7936	0.9238	0.9242
3	6	0.8128	0.9061	0.7889	0.9078	0.9091
4	6	0.7862	0.8682	0.7673	0.8697	0.8719
5	6	0.7585	0.8253	0.7413	0.8264	0.8292
6	6	0.7320	0.7846	0.7164	0.7855	0.7885
7	6	0.7077	0.7493	0.6945	0.7500	0.7530
8	6	0.6858	0.7198	0.6760	0.7204	0.7234
9	6	0.6664	0.6961	0.6605	0.6966	0.6994
10	6	0.6493	0.6776	0.6477	0.6779	0.6807
0.5	12	1.4754	1.4493	1.1984	1.4507	1.4517
1	12	1.4380	1.4785	1.2279	1.4810	1.4818
2	12	1.3196	1.4446	1.1993	1.4476	1.4483
3	12	1.2134	1.3535	1.1344	1.3564	1.3571
4	12	1.1256	1.2534	1.0662	1.2556	1.2567
5	12	1.0551	1.1604	1.0051	1.1621	1.1634
6	12	0.9989	1.0800	0.9539	1.0812	1.0827
7	12	0.9538	1.0133	0.9124	1.0141	1.0156
8	12	0.9171	0.9591	0.8792	0.9596	0.9611
9	12	0.8865	0.9159	0.8525	0.9161	0.9176
10	12	0.8605	0.8821	0.8313	0.8821	0.8836

Notes: The table reports the in-sample root mean squared errors (RMSEs), multiplied by 100, of τ-maturity nominal yields for the period January 1990 - December 2012, computed as $100 \times \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{y}^N_{\tau,i+h} - \hat{y}^N_{\tau,i+h}\right)^2}$ for $h = 1, 6, 12$, where $\hat{y}^N_{\tau,i+h}$ is the actual nominal yield and $\tilde{y}^N_{\tau,i+h}$ is the model forecast of nominal yield. The models are described in the notes to Table 2.

Table 4: **Out–Of-Sample Nominal Yields Forecast RMSEs**

maturity	horizon	RW	$\mathcal{M}_x^{N,R}$	$\mathcal{M}_x^{N}$	$\mathcal{M}_x^{N,R,O}$	$\mathcal{M}_{x,v}^{N,R,O}$
0.5	3	0.0297	0.1198	0.1615	0.1197	0.1202
1	3	0.0487	0.1694	0.1407	0.1695	0.1680
2	3	0.0934	0.2345	0.1192	0.2347	0.2358
3	3	0.1662	0.2439	0.1514	0.2440	0.2488
4	3	0.2305	0.2591	0.1946	0.2591	0.2652
5	3	0.2809	0.2890	0.2312	0.2890	0.2951
6	3	0.3182	0.3320	0.2655	0.3320	0.3377
7	3	0.3447	0.3842	0.3031	0.3840	0.3893
8	3	0.3623	0.4418	0.3463	0.4415	0.4462
9	3	0.3733	0.5022	0.3950	0.5017	0.5058
10	3	0.3791	0.5640	0.4484	0.5633	0.5665
0.5	6	0.0350	0.2666	0.2496	0.2666	0.2671
1	6	0.0589	0.3574	0.2169	0.3575	0.3561
2	6	0.1249	0.4233	0.1929	0.4232	0.4253
3	6	0.2248	0.3900	0.1746	0.3896	0.3969
4	6	0.3124	0.3751	0.1872	0.3746	0.3850
5	6	0.3844	0.4062	0.2262	0.4059	0.4169
6	6	0.4424	0.4686	0.2785	0.4684	0.4791
7	6	0.4876	0.5429	0.3356	0.5427	0.5531
8	6	0.5211	0.6182	0.3939	0.6179	0.6282
9	6	0.5445	0.6904	0.4519	0.6898	0.7000
10	6	0.5597	0.7584	0.5095	0.7576	0.7674
0.5	12	0.0300	0.6178	0.4549	0.6179	0.6184
1	12	0.0637	0.7182	0.4664	0.7175	0.7171
2	12	0.1955	0.7203	0.3877	0.7182	0.7221
3	12	0.3603	0.6343	0.2476	0.6315	0.6412
4	12	0.4807	0.6036	0.1538	0.6010	0.6143
5	12	0.5624	0.6348	0.1518	0.6326	0.6473
6	12	0.6192	0.6966	0.2047	0.6947	0.7100
7	12	0.6605	0.7670	0.2690	0.7650	0.7808
8	12	0.6920	0.8371	0.3355	0.8347	0.8511
9	12	0.7164	0.9043	0.4036	0.9015	0.9182
10	12	0.7358	0.9686	0.4736	0.9652	0.9820

Notes: The table reports the out-of-sample root mean squared errors (RMSEs), multiplied by 100, of τ-maturity nominal yields for the period January 2013 - February 2015, computed as $100 \times \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{y}_{\tau,i+h}^{N} - \hat{y}_{\tau,i+h}^{N}\right)^2}$ for $h = 1, 6, 12$, where $\hat{y}_{\tau,i+h}^{N}$ is the actual nominal yield and $\tilde{y}_{\tau,i+h}^{N}$ is the model forecast of nominal yield. The models are described in the notes to Table 2.

Table 5: In-Sample Real Yields Forecast RMSEs

maturity	horizon	RW	$\mathcal{M}_x^{N,R}$	$\mathcal{M}_x^{N,R,O}$	$\mathcal{M}_{x,v}^{N,R,O}$
2	3	0.9800	0.9870	0.9778	0.8269
3	3	0.7472	0.7840	0.7783	0.6723
4	3	0.6254	0.6893	0.6861	0.5979
5	3	0.5542	0.6410	0.6391	0.5563
6	3	0.5072	0.6169	0.6155	0.5284
7	3	0.4725	0.6088	0.6073	0.5067
8	3	0.4446	0.6131	0.6113	0.4886
9	3	0.4212	0.6281	0.6258	0.4733
10	3	0.4008	0.6523	0.6496	0.4609
2	6	1.3347	1.0009	0.9907	1.1034
3	6	1.0284	0.8542	0.8497	0.9462
4	6	0.8627	0.8071	0.8054	0.8587
5	6	0.7622	0.7923	0.7915	0.7997
6	6	0.6937	0.7894	0.7887	0.7539
7	6	0.6424	0.7934	0.7924	0.7155
8	6	0.6013	0.8033	0.8017	0.6824
9	6	0.5672	0.8192	0.8169	0.6542
10	6	0.5384	0.8410	0.8381	0.6309
2	12	1.5568	1.1098	1.1155	1.3515
3	12	1.2643	1.0276	1.0390	1.2311
4	12	1.0992	1.0093	1.0206	1.1493
5	12	0.9905	1.0064	1.0160	1.0815
6	12	0.9107	1.0084	1.0157	1.0221
7	12	0.8477	1.0136	1.0186	0.9694
8	12	0.7958	1.0224	1.0254	0.9232
9	12	0.7521	1.0358	1.0368	0.8834
10	12	0.7149	1.0540	1.0534	0.8501

Notes: The table reports the in-sample root mean squared errors (RMSEs), multiplied by 100, of τ-maturity real (TIPS) yields for the period January 1990 - December 2012, computed as $100 \times \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{y}^R_{\tau,i+h} - \hat{y}^R_{\tau,i+h}\right)^2}$ for $h = 1, 6, 12$, where $\hat{y}^R_{\tau,i+h}$ is the actual real (TIPS) yield and $\tilde{y}^R_{\tau,i+h}$ is the model forecast of real yield. The models are described in the notes to Table 2.

Table 6: Out-of-Sample Real Yields Forecast RMSEs

maturity	horizon	RW	$\mathcal{M}_x^{N,R}$	$\mathcal{M}_x^{N,R,O}$	$\mathcal{M}_{x,v}^{N,R,O}$
2	3	2.0493	1.9572	1.8975	1.9115
3	3	1.8509	1.6637	1.6045	1.6784
4	3	1.6209	1.3765	1.3197	1.4531
5	3	1.3968	1.1114	1.0575	1.2446
6	3	1.1912	0.8766	0.8259	1.0558
7	3	1.0083	0.6815	0.6348	0.8881
8	3	0.8493	0.5416	0.5016	0.7434
9	3	0.7140	0.4789	0.4509	0.6245
10	3	0.6018	0.5007	0.4869	0.5363
2	6	2.2112	1.9170	1.8402	1.9673
3	6	2.0296	1.6280	1.5559	1.7360
4	6	1.8114	1.3622	1.2953	1.5216
5	6	1.5940	1.1271	1.0654	1.3273
6	6	1.3924	0.9288	0.8726	1.1541
7	6	1.2123	0.7757	0.7260	1.0026
8	6	1.0553	0.6781	0.6370	0.8738
9	6	0.9211	0.6434	0.6131	0.7692
10	6	0.8087	0.6670	0.6475	0.6911
2	12	2.1786	1.6471	1.5497	1.7894
3	12	2.0646	1.3926	1.3069	1.5830
4	12	1.8827	1.1701	1.0951	1.3963
5	12	1.6830	0.9810	0.9163	1.2289
6	12	1.4895	0.8314	0.7773	1.0809
7	12	1.3128	0.7301	0.6880	0.9535
8	12	1.1574	0.6848	0.6555	0.8484
9	12	1.0247	0.6944	0.6770	0.7676
10	12	0.9146	0.7474	0.7392	0.7126

Notes: The table reports the out-of-sample root mean squared errors (RMSEs), multiplied by 100, of τ-maturity real (TIPS) yields for the period January 1990 - December 2012, computed as $100 \times \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{y}_{\tau,i+h}^{R} - \hat{y}_{\tau,i+h}^{R}\right)^2}$ for $h = 1, 6, 12$, where $\hat{y}_{\tau,i+h}^{R}$ is the actual real (TIPS) yield and $\tilde{y}_{\tau,i+h}^{R}$ is the model forecast of real yield. The models are described in the notes to Table 2.

Table 7: In-Sample and Out-of-Sample Option Prices Fitting RMSEs

Panel A: In-Sample RMSEs					
option	maturity	strike	$\mathcal{M}_x^{N,R}$	$\mathcal{M}_x^{N,R,O}$	$\mathcal{M}_{x,v}^{N,R,O}$
caps	1	1%	0.5610	0.4806	0.4205
	1	2%	0.8817	0.7886	0.5886
	1	3%	1.5133	1.5049	0.8084
	3	1%	0.1461	0.1372	0.0857
	3	2%	0.3667	0.4080	0.1202
	3	3%	0.9503	1.1627	0.2145
floors	1	1%	1.5237	1.4073	0.8059
	1	2%	0.8729	0.7670	0.5461
	1	3%	0.5253	0.4551	0.3680
	3	1%	0.7552	0.8058	0.0976
	3	2%	0.2776	0.2490	0.0824
	3	3%	0.1183	0.0959	0.0580
Panel B: Out–of-Sample RMSEs					
option	maturity	strike	$\mathcal{M}_x^{N,R}$	$\mathcal{M}_x^{N,R,O}$	$\mathcal{M}_{x,v}^{N,R,O}$
caps	1	1%	0.9547	0.8926	0.6801
	1	2%	1.3039	1.1991	1.0724
	1	3%	1.4063	1.2677	1.5925
	3	1%	0.1367	0.1323	0.0556
	3	2%	0.2639	0.2986	0.1369
	3	3%	0.6172	0.8398	0.2324
floors	1	1%	1.2355	1.0903	0.9387
	1	2%	0.7891	0.6806	0.5358
	1	3%	0.4838	0.4119	0.3006
	3	1%	0.4995	0.5341	0.3640
	3	2%	0.2037	0.1738	0.2003
	3	3%	0.0962	0.0807	0.0769

Notes: The table reports the root mean squared errors (RMSEs) of inflation options (caps and floors) with maturity τ and strike price K, computed as $\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{y}_{i,\tau,K}^{O}-\hat{y}_{i,\tau,K}^{O}\right)^2}$, where $\hat{y}_{i,\tau}^{O}=-\frac{\ln P_{i,\tau,K}^{CAP}}{\tau}$ or $\hat{y}_{i,\tau}^{O}=-\frac{\ln P_{i,\tau,K}^{FLO}}{\tau}$ is the actual option "yield" and $\tilde{y}_{i,\tau,K}^{O}$ is the model fitted option "yield". The in-sample period is January 1990 - December 2012 and the out-of-sample period is January 2013 - February 2015. The models are described in the notes to Table 2.

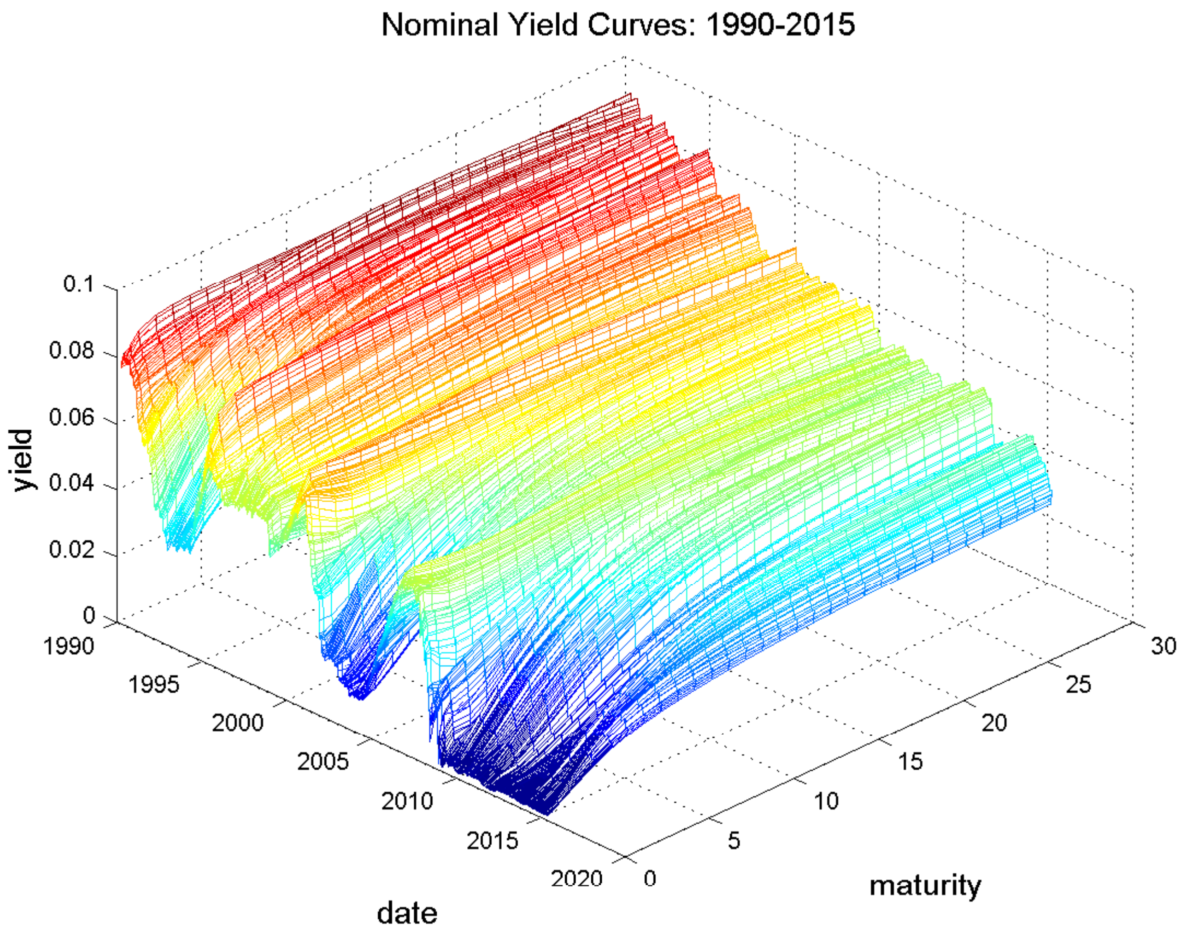

Figure 1. Nominal yield curves, January 1990 - February 2015 (weekly data).

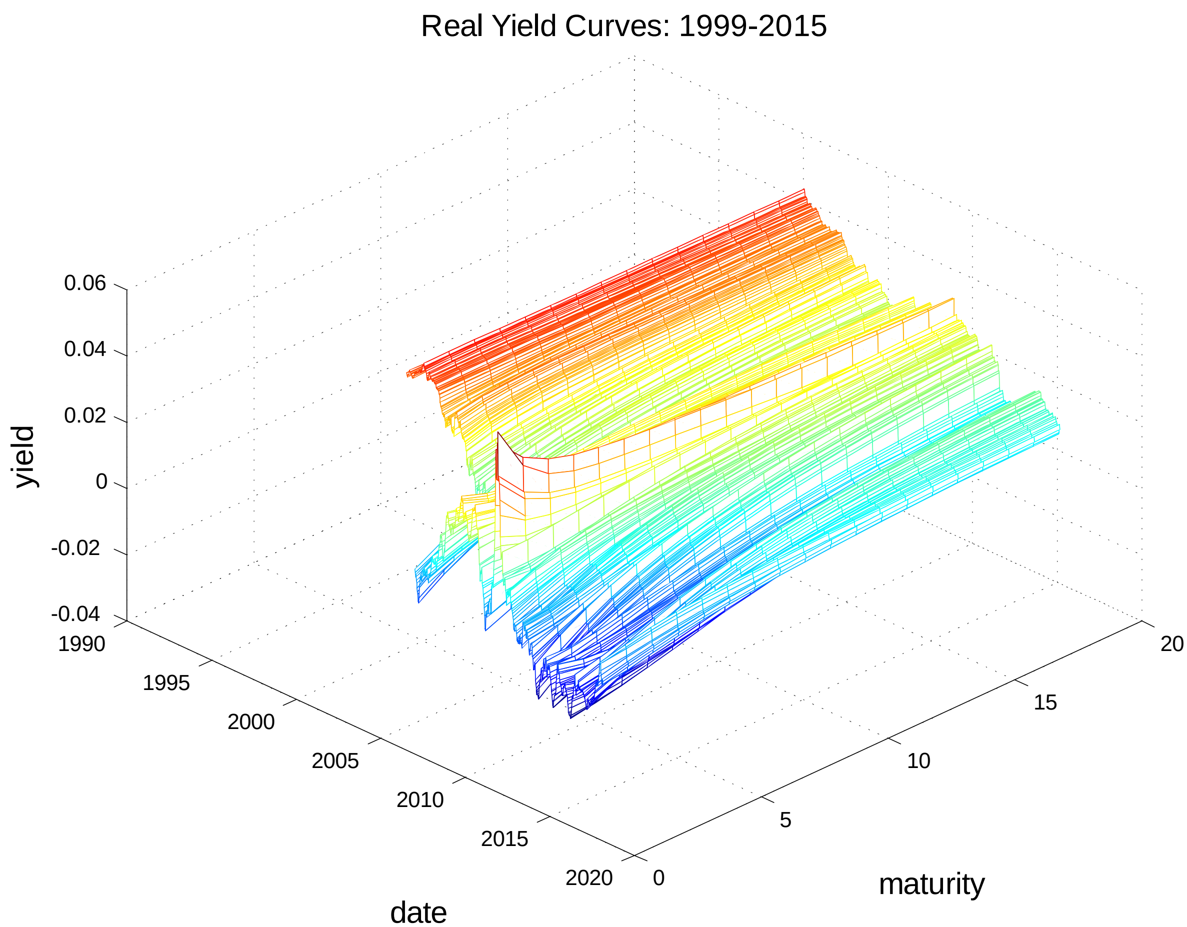

Figure 2. Real yield curves, January 1999 - February 2015 (weekly data).

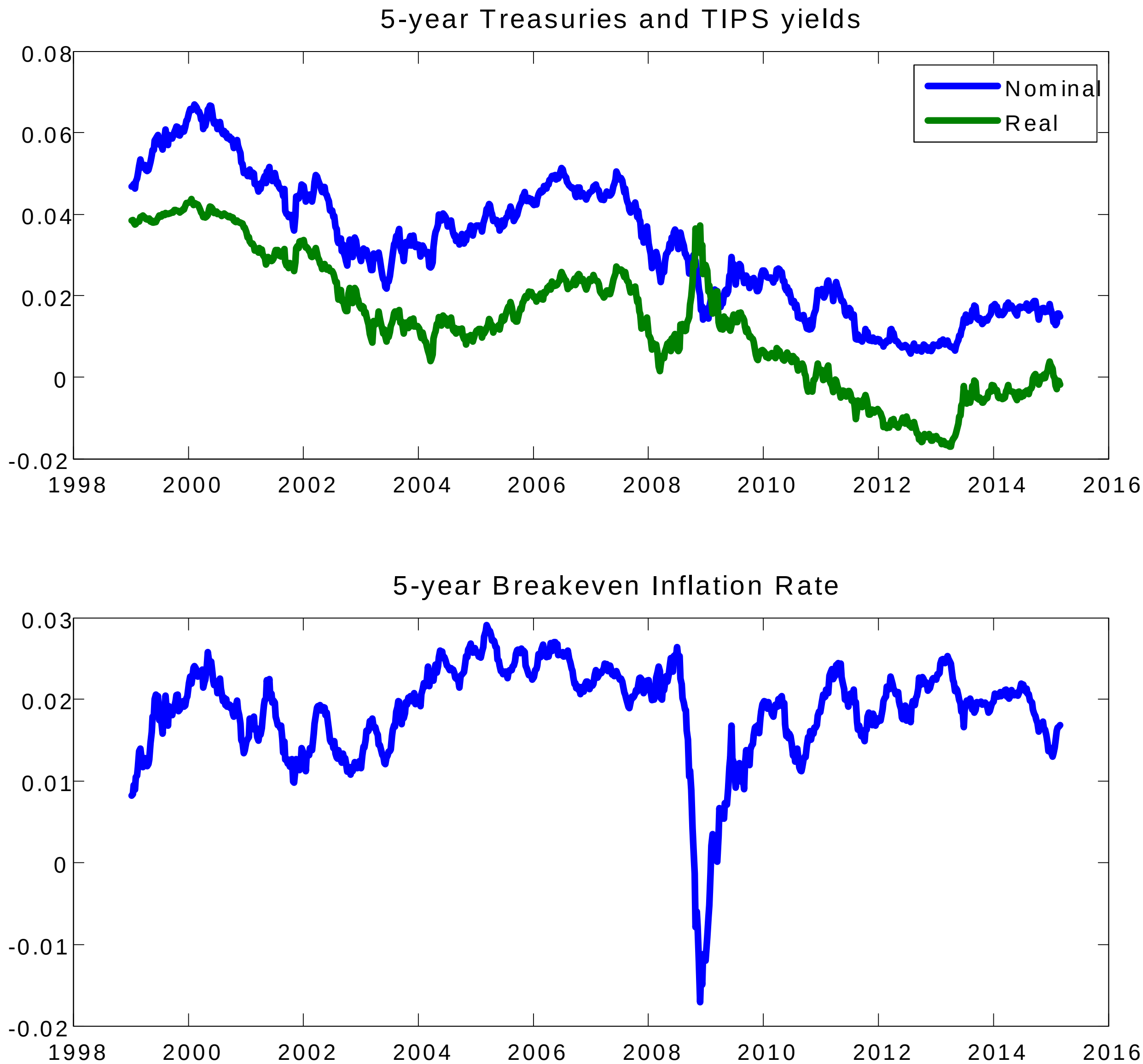

Figure 3. 5-year Treasury and TIPS yields, and break-even inflation.

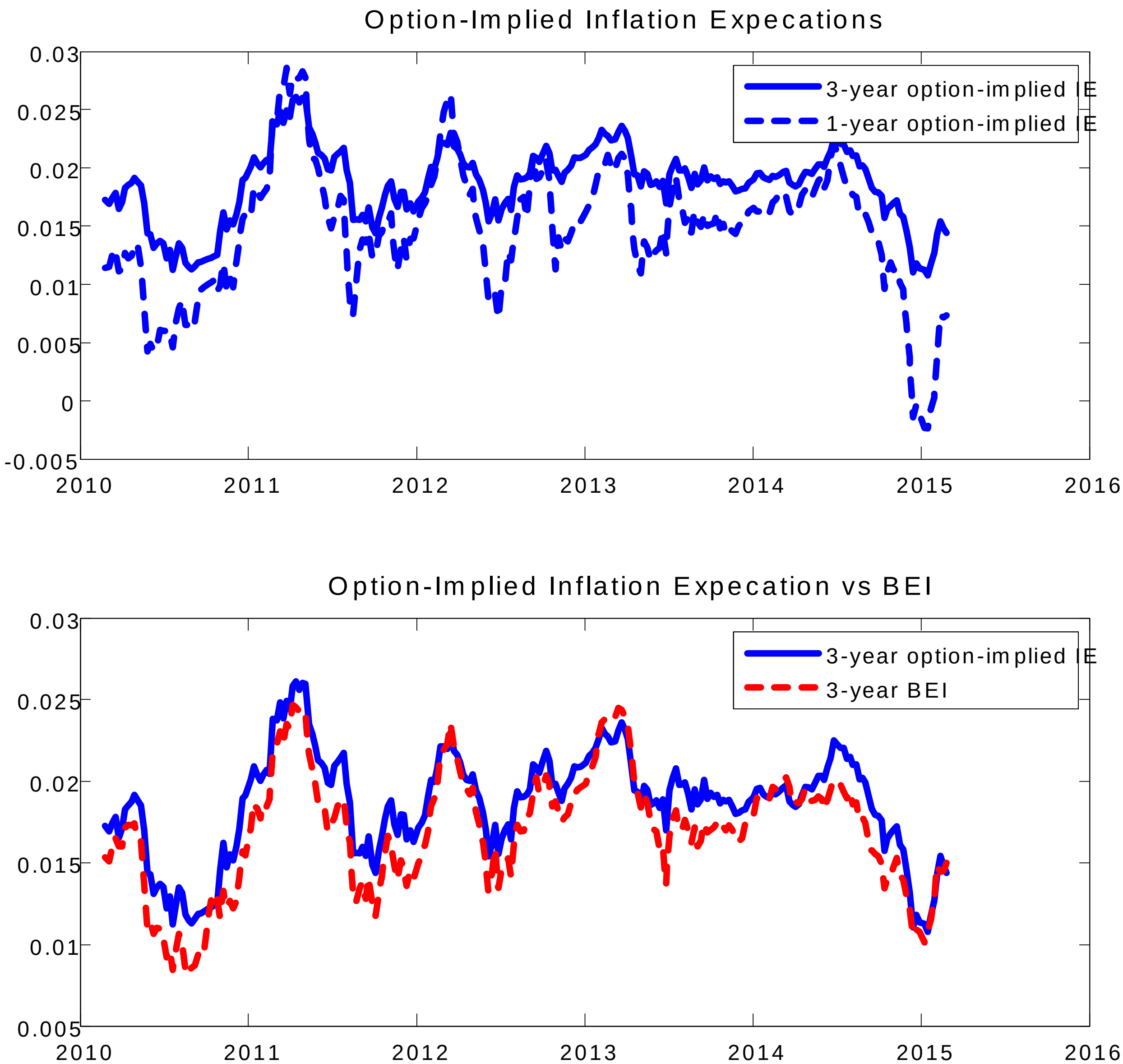

Figure 4. Option-implied inflation expectations.

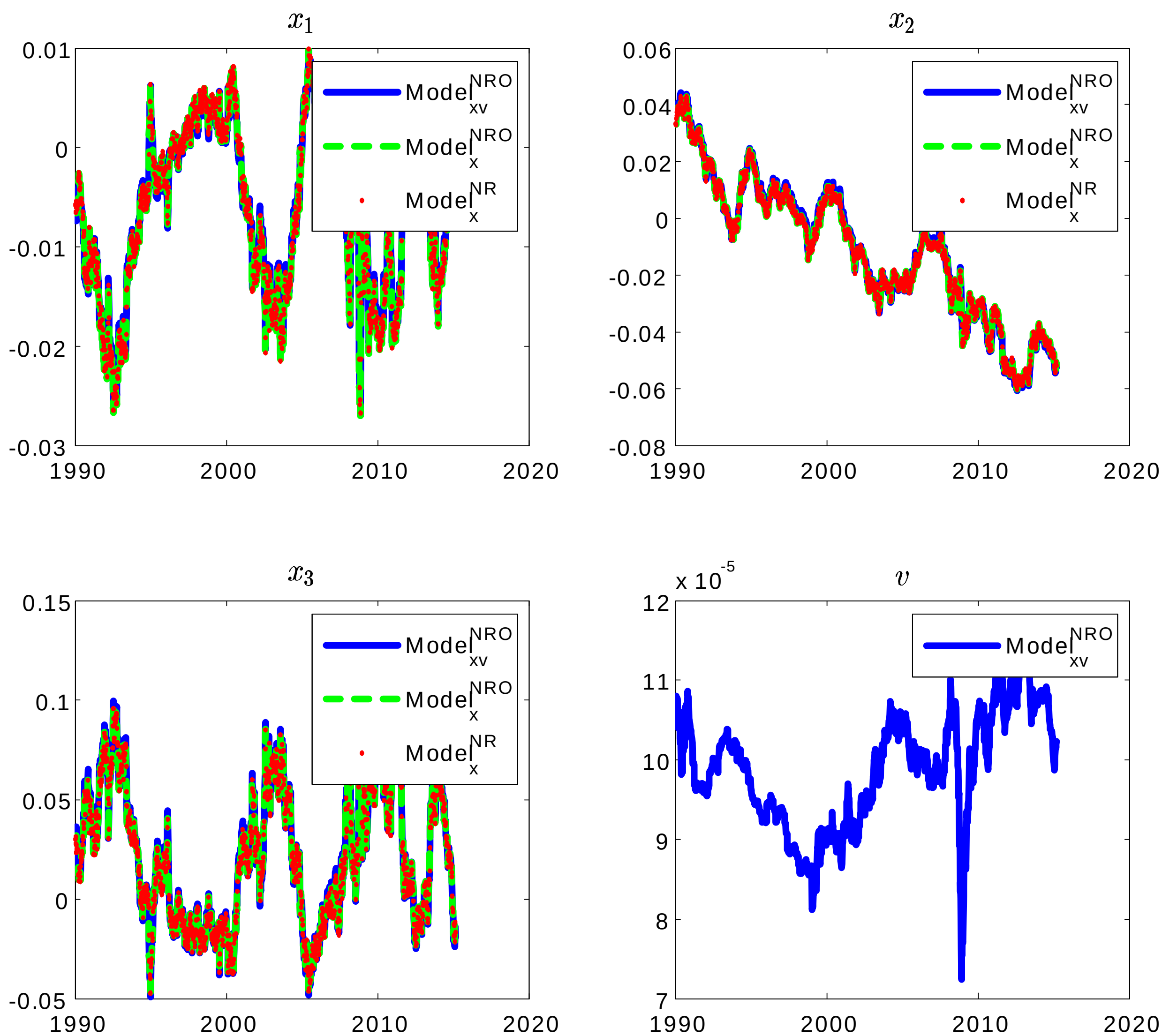

Figure 5. State variables.

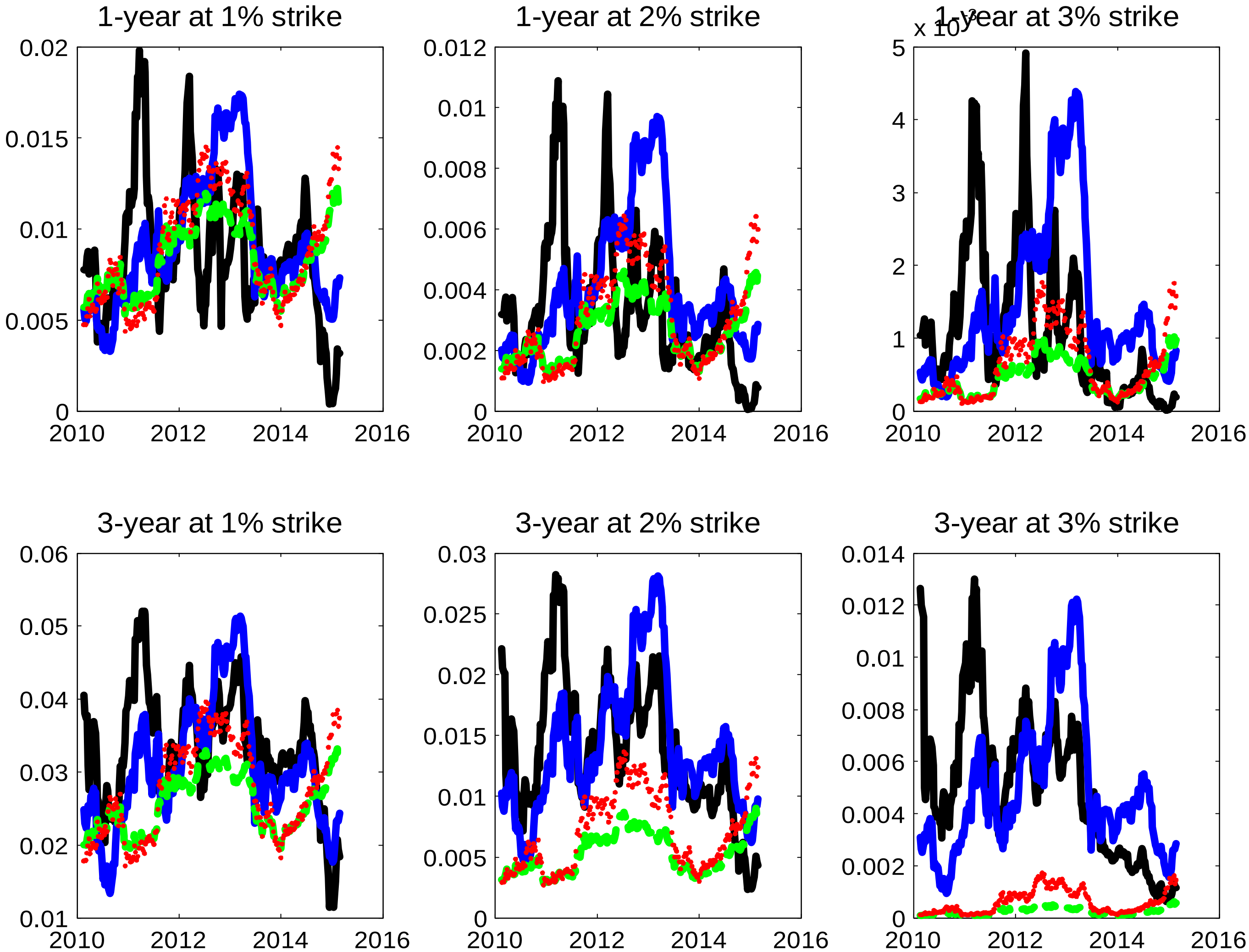

Figure 6. Inflation caps and model fitted prices. The black solid line denotes the data; the blue solid line denotes the fitted prices from model $\mathcal{M}_{xv}^{N,R,O}$; the green dashed line denotes the fitted prices from model $\mathcal{M}_{x}^{N,R,O}$; and the red dotted line denotes the fitted prices from model $\mathcal{M}_{x}^{N,R}$.

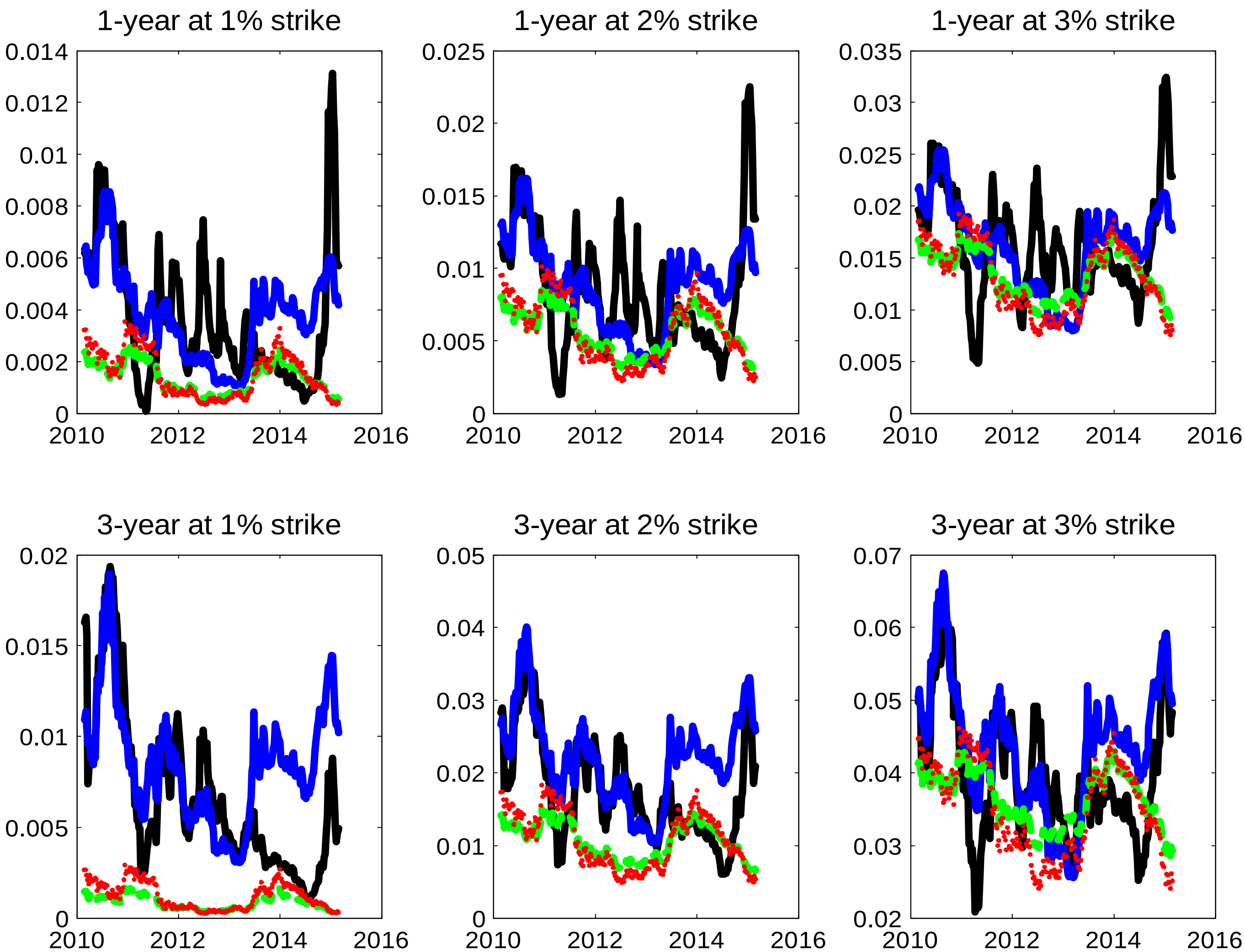

Figure 7. Inflation floors and model fitted prices. The black solid line denotes the data; the blue solid line denotes the fitted prices from model $\mathcal{M}_{xv}^{N,R,O}$; the green dashed line denotes the fitted prices from model $\mathcal{M}_{x}^{N,R,O}$; and the red dotted line denotes the fitted prices from model $\mathcal{M}_{x}^{N,R}$.

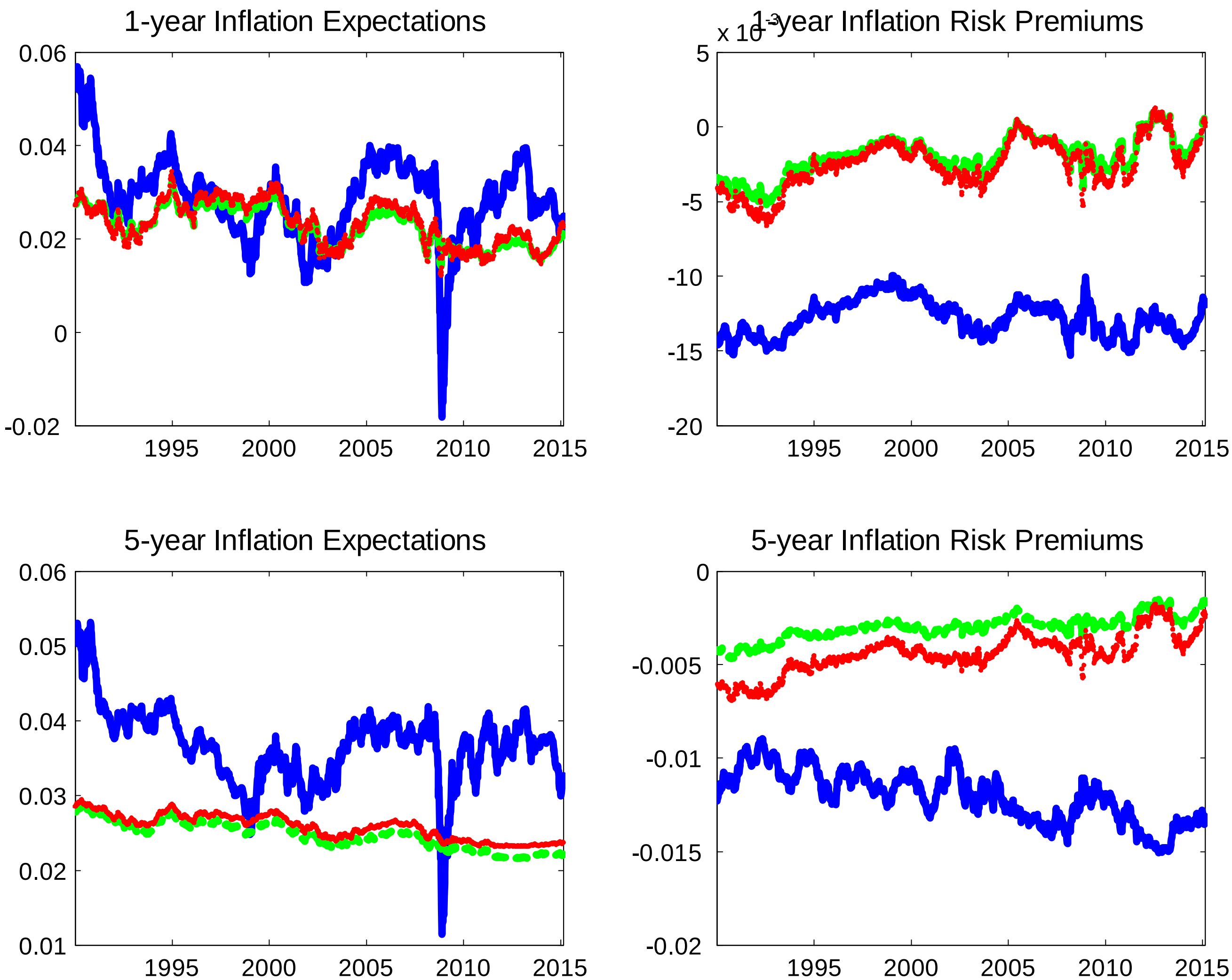

Figure 8. 1- and 5-year inflation expectations and inflation risk premiums. The blue solid line denotes the fitted prices from model $\mathcal{M}_{xv}^{N,R,O}$; the green dashed line denotes the fitted prices from model $\mathcal{M}_{x}^{N,R,O}$; and the red dotted line denotes the fitted prices from model $\mathcal{M}_{x}^{N,R}$.

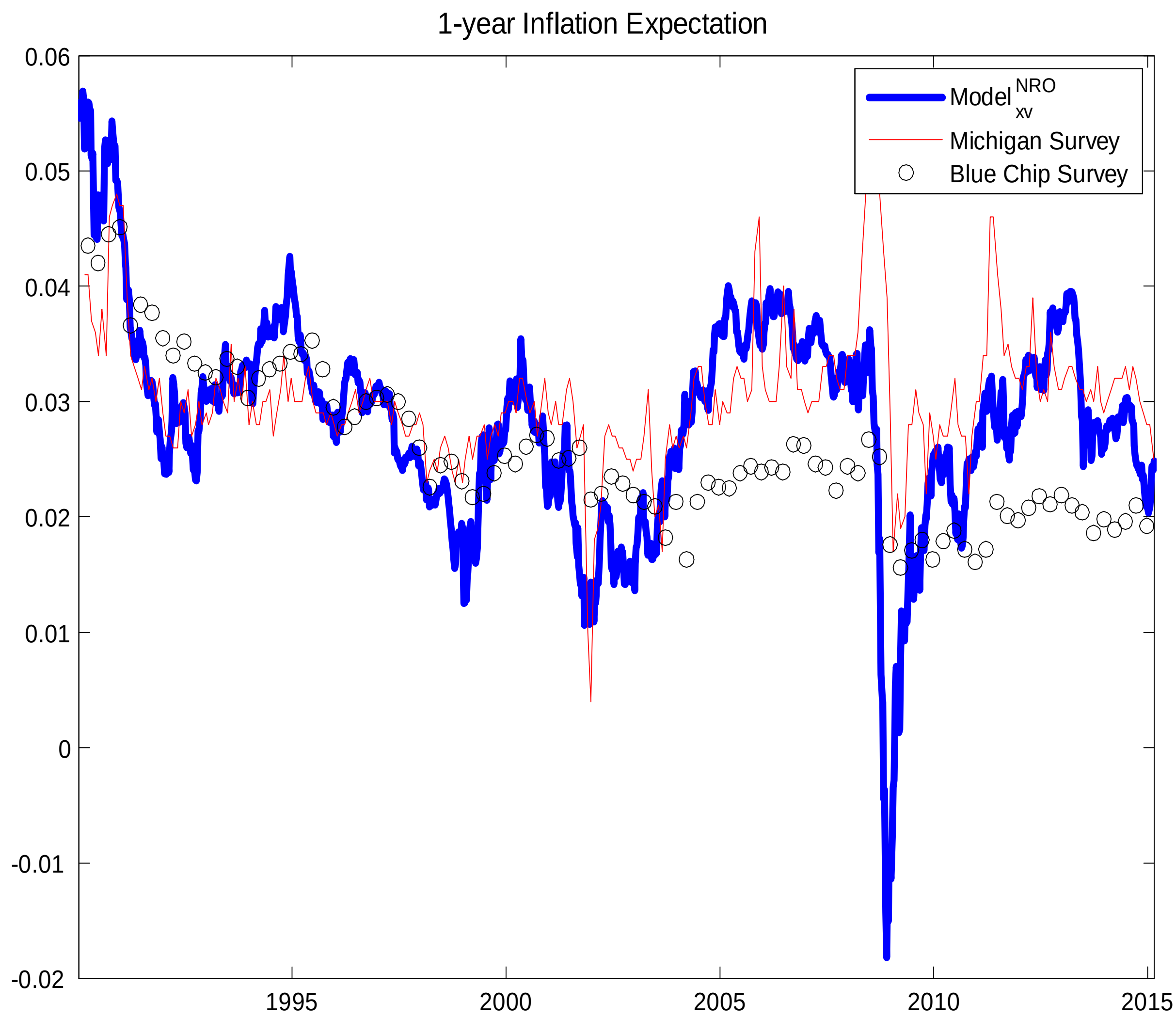

Figure 9. 1-year inflation expectations from model $\mathcal{M}_{xv}^{N,R,O}$, University of Michigan survey and Blue Chip survey.