Documentation: MODEL_FRBA_full

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1 Model

Suppose that there are a vector of four latent variables $x_t = (x_{1t}, x_{2t}, x_{3t})'$ and v_t that drive nominal and real yields as well as inflation. Their dynamics under the physical measure is

$$
\begin{aligned}
dx_t &= \mathcal{K}(\mu - x_t)\,dt + \Sigma dW_{x,t} && (1)\\
dv_t &= \kappa_v(\mu_v - v_t)\,dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t} + \rho dW_{\perp,t}\right) && (2)
\end{aligned}
$$

and

$$
\begin{aligned}
d\delta_t &= \kappa_\delta(\mu_\delta - \delta_t)\,dt + \sigma_\delta dW_{\delta,t} && (3)\\
ds_t &= \left(r_t^N - \delta_t - \frac{1}{2}\sigma_s^2 + \sigma_s\Lambda_{\delta,t}^N\right)dt + \sigma_s dW_{\delta,t} && (4)
\end{aligned}
$$

The nominal pricing kernel takes the form

$$
dM_t^N/M_t^N = -r_t^N dt - \Lambda_{x,t}^{N\prime} dW_{x,t} - \Lambda_{v,t}^N dW_{v,t} - \Lambda_{\perp,t}^N dW_{\perp,t} - \Lambda_{\delta,t}^N dW_{\delta,t} \tag{5}
$$

where the nominal short rate is

$$
r_t^N = \rho_0^N + \rho_x^{N\prime} x_t + \rho_v^N v_t + \rho_\delta^N \delta_t + \rho_s^N s_t \tag{6}
$$

and the vector of prices of risk is given by

$$
\begin{aligned}
\Lambda_{x,t}^N &= \lambda_{0,x}^N + \lambda_x^N x_t\\
\Lambda_{v,t}^N &= \gamma_v\sqrt{v_t}\\
\Lambda_{\perp,t}^N &= \gamma_v^\perp\sqrt{v_t}\\
\Lambda_{\delta,t}^N &= \lambda_{0,\delta}^N + \lambda_\delta^N \delta_t
\end{aligned}
$$

Let $q_t \equiv \log Q_t$ denote the log price level. The price level evolves as follows:

$$
dq_t = \pi_t dt + \sigma_q' dW_{x,t} + \sqrt{v_t}dW_{\perp,t} \tag{7}
$$

where the instantaneous expected inflation rate is given by

$$
\pi_t = \rho_0^\pi + \rho_x^{\pi\prime} x_t + \rho_v^\pi v_t + \rho_\delta^\pi \delta_t + \rho_s^\pi s_t \tag{8}
$$

Under the risk-neutral measure,

$$
\begin{aligned}
d\delta_t &= \kappa_\delta(\mu_\delta - \delta_t)\,dt + \sigma_\delta dW_{\delta,t}\\
&= \kappa_\delta(\mu_\delta - \delta_t)\,dt + \sigma_\delta\left(dW_{\delta,t}^* - \Lambda_{\delta,t}^N dt\right)\\
&\equiv \kappa_\delta^*(\mu_\delta^* - \delta_t)\,dt + \sigma_\delta dW_{\delta,t}^*
\end{aligned}
$$

and

$$
\begin{aligned}
ds_t &= \left(r_t^N - \delta_t - \frac{1}{2}\sigma_s^2 + \sigma_s \Lambda_{\delta,t}^N\right) dt + \sigma_s dW_{\delta,t} \\
&\equiv \left(\phi_0 + \phi_x' x_t + \phi_v v_t + \phi_\delta \delta_t + \phi_s s_t\right) dt + \sigma_s dW_{\delta,t} \\
&\equiv \left(\phi_0^* + \phi_x^{*\prime} x_t + \phi_v^* v_t + \phi_\delta^* \delta_t + \phi_s^* s_t\right) dt + \sigma_s dW_{\delta,t}^*
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_\delta^* &= \kappa_\delta + \sigma_\delta \lambda_\delta^N \\
\kappa_\delta^* \mu_\delta^* &= \kappa_\delta \mu_\delta - \sigma_\delta \lambda_{0,\delta}^N
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_0 &= \rho_0^N - \frac{1}{2}\sigma_s^2 + \sigma_s \lambda_{0,\delta}^N \\
\phi_x &= \rho_x^N \\
\phi_v &= \rho_v^N \\
\phi_\delta &= \rho_\delta^N - 1 + \sigma_s \lambda_\delta^N \\
\phi_s &= \rho_s^N
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_0^* &= \rho_0^N - \frac{1}{2}\sigma_s^2 = \phi_0 - \sigma_s \lambda_{0,\delta}^N \\
\phi_x^* &= \rho_x^N \\
\phi_v^* &= \rho_v^N \\
\phi_\delta^* &= \rho_\delta^N - 1 = \phi_\delta - \sigma_s \lambda_\delta^N \\
\phi_s^* &= \rho_s^N
\end{aligned}
$$

Remark. We initialize λ_δ^N so that $\phi_\delta = 0 = \rho_\delta^N - 1 + \sigma_s \lambda_\delta^N$, or $\lambda_\delta^N = \left(1 - \rho_\delta^N\right)/\sigma_s$. Furthermore, as we show later, the long run mean of s_t is

$$
s_\infty \equiv -\frac{1}{\phi_s}\left(\phi_0 + \phi_x' \mu + \phi_v \mu_v + \phi_\delta \mu_\delta\right)
$$

Given the other parameters including ϕ_s, we can initialize $\lambda_{0,\delta}^N$ by matching sample mean of s_t, denoted by $\widehat{s}$, to its long run mean s_∞. Given that $\phi_\delta = 0$ via initialization and $\mu = 0$ by normalization and $\phi_v = \rho_v^N = 0$ by restriction, we have

$$
\begin{aligned}
\phi_0 &= -\left(\phi_v \mu_v + \phi_s \widehat{s}\right) = \rho_0^N - \frac{1}{2}\sigma_s^2 + \sigma_s \lambda_{0,\delta}^N \\
&\Rightarrow \quad \lambda_{0,\delta}^N = -\frac{\phi_v \mu_v + \phi_s \widehat{s} + \rho_0^N - \frac{1}{2}\sigma_s^2}{\sigma_s} \\
&\Rightarrow \quad \lambda_{0,\delta}^N = -\frac{\phi_s \widehat{s} + \rho_0^N - \frac{1}{2}\sigma_s^2}{\sigma_s}
\end{aligned}
$$

NOTE:

$$
\begin{aligned}
dx_t &= \mathcal{K}^*\left(\mu^* - x_t\right) dt + \Sigma dW_{x,t}^* && (9) \\
dv_t &= \kappa_v^*\left(\mu_v^* - v_t\right) dt + \sigma_v \sqrt{v_t}\left(\sqrt{1-\rho^2}\, dW_{v,t}^* + \rho dW_{\perp,t}^*\right) && (10) \\
d\delta_t &= \kappa_\delta^*\left(\mu_\delta^* - \delta_t\right) dt + \sigma_\delta dW_{\delta,t}^* && (11) \\
ds_t &= \left(\phi_0^* + \phi_x^{*\prime} x_t + \phi_v^* v_t + \phi_\delta^* \delta_t + \phi_s^* s_t\right) dt + \sigma_s dW_{\delta,t}^* && (12)
\end{aligned}
$$

Oil futures. Let $s_t \equiv \log(S_t)$ denote the spot oil price.

$$
\begin{aligned}
F_{t,\tau}^{oil} &= E_t^*\left[\exp\left(s_{t+\tau}\right)\right] \\
&\equiv \exp\left(A_\tau^{oil} + B_\tau^{oil\prime} x_t + C_\tau^{oil} v_t + D_\tau^{oil}\delta_t + E_\tau^{oil} s_t\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\dot{A}_\tau^{oil} &= \left(\mathcal{K}^*\mu^*\right)' B_\tau^{oil} + \left(\kappa_v^*\mu_v^*\right) C_\tau^{oil} + \left(\kappa_\delta^*\mu_\delta^*\right) D_\tau^{oil} + \phi_0^* E_\tau^{oil} \\
&\quad + \frac{1}{2} B_\tau^{oil\prime}\Sigma\Sigma' B_\tau^{oil} + \frac{1}{2}\sigma_\delta^2\left(D_\tau^{oil}\right)^2 + \frac{1}{2}\sigma_s^2\left(E_\tau^{oil}\right)^2 \\
\dot{B}_\tau^{oil} &= -\left(\mathcal{K}^*\right)' B_\tau^{oil} + \phi_x^* E_\tau^{oil} \\
\dot{C}_\tau^{oil} &= -\kappa_v^* C_\tau^{oil} + \frac{1}{2}\sigma_v^2\left(C_\tau^{oil}\right)^2 + \phi_v^* E_\tau^{oil} \\
\dot{D}_\tau^{oil} &= -\kappa_\delta^* D_\tau^{oil} + \phi_\delta^* E_\tau^{oil} \\
\dot{E}_\tau^{oil} &= \phi_s^* E_\tau^{oil}
\end{aligned}
$$

and the initial conditions are: $A_0^{oil} = B_0^{oil} = C_0^{oil} = D_0^{oil} = 0$ and $E_0^{oil} = 1$.

For simplicity, we assume

$$
\rho_v^N = 0 \tag{13}
$$

which implies $\phi_v^* = \phi_v = 0$.

Consider the transformation

$$
\begin{aligned}
\mathcal{A} &= A_\tau^{oil} \\
\mathcal{B} &= \begin{pmatrix} B_\tau^{oil} \\ D_\tau^{oil} \\ E_\tau^{oil} - 1 \end{pmatrix} \\
\mathcal{C} &= C_\tau^{oil}
\end{aligned}
$$

Then $\mathcal{A}(0) = \mathcal{B}(0) = \mathcal{C}(0) = 0$ and

$$
\begin{aligned}
\dot{\mathcal{A}} &= \left(\phi_0^* + \frac{1}{2}\sigma_s^2\right) + \begin{pmatrix} \mathcal{K}^*\mu^* \\ \kappa_\delta^*\mu_\delta^* \\ \phi_0^* + \sigma_s^2 \end{pmatrix}' \mathcal{B} + \left(\kappa_v^*\mu_v^*\right)\mathcal{C} + \frac{1}{2}\mathcal{B}' \begin{pmatrix} \Sigma\Sigma' & & \\ & \sigma_\delta^2 & \\ & & \sigma_s^2 \end{pmatrix} \mathcal{B} \\
\dot{\mathcal{B}} &= \begin{pmatrix} \phi_x^* \\ \phi_\delta^* \\ \phi_s^* \end{pmatrix} + \begin{pmatrix} -\left(\mathcal{K}^*\right)' & 0 & \phi_x^* \\ 0 & -\kappa_\delta^* & \phi_\delta^* \\ 0 & 0 & \phi_s^* \end{pmatrix} \mathcal{B} \\
\dot{\mathcal{C}} &= -\kappa_v^*\mathcal{C} + \frac{1}{2}\sigma_v^2\mathcal{C}^2
\end{aligned}
$$

Based on the observation $M_t^R = M_t^N Q_t$, we have

$$
\begin{aligned}
dM_t^R/M_t^R &= dM_t^N/M_t^N + dQ_t/Q_t + \left(dM_t^N/M_t^N\right)\cdot\left(dQ_t/Q_t\right) \\
&= -r_t^N dt - \Lambda_{x,t}^{N\prime} dW_{x,t} - \Lambda_{v,t}^N dW_{v,t} - \Lambda_{\perp,t}^N dW_{\perp,t} - \Lambda_{\delta,t}^N dW_{\delta,t} \\
&\quad + \left[\pi_t + \frac{1}{2}\left(\sigma_q'\sigma_q + v_t\right) - \sigma_q'\Lambda_{x,t}^N - \sqrt{v_t}\Lambda_{\perp,t}^N\right] dt + \sigma_q' dW_{x,t} + \sqrt{v_t}\, dW_{\perp,t} \\
&\equiv -r_t^R dt - \Lambda_{x,t}^{R\prime} dW_{x,t} - \Lambda_{v,t}^R dW_{v,t} - \Lambda_{\perp,t}^R dW_{\perp,t} - \Lambda_{\delta,t}^R dW_{\delta,t}
\end{aligned}
$$

where

$$\begin{aligned}
r_t^R &= r_t^N - \left[\pi_t + \frac{1}{2}\left(\sigma_q'\sigma_q + v_t\right)\right] + \sigma_q'\Lambda_{x,t}^N + \sqrt{v_t}\Lambda_{\perp,t}^N \\
&= r_t^N - \left[\pi_t + \frac{1}{2}\left(\sigma_q'\sigma_q + v_t\right)\right] + \sigma_q'\left(\lambda_{0,x}^N + \lambda_x^{N\prime}x_t\right) + \gamma_v^{\perp}v_t \\
&\equiv \rho_0^R + \rho_x^{R\prime}x_t + \rho_v^R v_t + \rho_\delta^R\delta_t + \rho_s^R s_t \\
\Lambda_{x,t}^R &= \Lambda_{x,t}^N - \sigma_q \equiv \lambda_{0,x}^R + \lambda_x^{R\prime}x_t \\
\Lambda_{v,t}^R &= \Lambda_{v,t}^N = \gamma_v\sqrt{v_t}, \\
\Lambda_{\perp,t}^R &= \Lambda_{\perp,t}^N - \sqrt{v_t} = \left(\gamma_v^{\perp} - 1\right)\sqrt{v_t} \\
\Lambda_{\delta,t}^R &= \Lambda_{\delta,t}^N \equiv \lambda_{0,\delta}^R + \lambda_\delta^R\delta_t
\end{aligned}$$

and

$$\begin{aligned}
\rho_0^R &= \rho_0^N - \rho_0^\pi - \frac{1}{2}\sigma_q'\sigma_q + \lambda_{0,x}'\sigma_q \\
\rho_x^R &= \rho_x^N - \rho_x^\pi + \lambda_x^{N\prime}\sigma_q \\
\rho_v^R &= \rho_v^N - \rho_v^\pi + \gamma_v^{\perp} - \frac{1}{2} \\
\rho_\delta^R &= \rho_\delta^N - \rho_\delta^\pi \text{ and } \rho_s^R = \rho_s^N - \rho_s^\pi
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{0,x}^R &= \lambda_{0,x}^N - \sigma_q, \text{ and } \lambda_x^R = \lambda_x^N \\
\lambda_{0,\delta}^R &= \lambda_{0,\delta}^N, \text{ and } \lambda_\delta^R = \lambda_\delta^N
\end{aligned}$$

1.1 Dynamics under the Risk-Neutral Measure and Bond Pricing

Let

$$\Lambda_t^N \equiv \begin{pmatrix} \Lambda_{x,t}^N \\ \Lambda_{v,t}^N \\ \Lambda_{\perp,t}^N \\ \Lambda_{\delta,t}^N \end{pmatrix}, W_t \equiv \begin{pmatrix} W_{x,t} \\ W_{v,t} \\ W_{\perp,t} \\ W_{\delta,t} \end{pmatrix}.$$

Then, the Radon-Nikodym derivative of the risk neutral measure $\mathbb{P}^*$ with respect to the physical measure $\mathbb{P}$ is given by

$$\left(\frac{d\mathbb{P}^*}{d\mathbb{P}}\right)_{t,T} = \exp\left[-\frac{1}{2}\int_t^T \Lambda_s^{N\prime}\Lambda_s^N ds - \int_t^T \Lambda_s^{N\prime}dW_s\right] \tag{14}$$

By the Girsanov theorem, $dW_t^* = dW_t + \Lambda_t^N dt$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. It implies that under the risk neutral measure,

$$\begin{aligned}
dW_{x,t}^* &= dW_{x,t} + \Lambda_{x,t}^N dt \\
dW_{v,t}^* &= dW_{v,t} + \Lambda_{v,t}^N dt \\
dW_{\perp,t}^* &= dW_{\perp,t} + \Lambda_{\perp,t}^N dt \\
dW_{\delta,t}^* &= dW_{\delta,t} + \Lambda_{\delta,t}^N dt
\end{aligned}$$

Therefore,

$$
\begin{aligned}
dx_t &= \mathcal{K}(\mu - x_t)\,dt + \Sigma dW_{x,t} \\
&= \mathcal{K}(\mu - x_t)\,dt + \Sigma\left(dW^*_{x,t} - \Lambda^N_{x,t}dt\right) \\
&= \left[\left(\mathcal{K}\mu - \Sigma\lambda^N_0\right) - \left(\mathcal{K} + \Sigma\lambda^N_x\right)x_t\right]dt + \Sigma dW^*_{x,t} \\
&\equiv \mathcal{K}^*(\mu^* - x_t)\,dt + \Sigma dW^*_{x,t},
\end{aligned}
$$

$$
\begin{aligned}
dv_t &= \kappa_v(\mu_v - v_t)\,dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t} + \rho dW_{\perp,t}\right) \\
&= \kappa_v(\mu_v - v_t)\,dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}\left(dW^*_{v,t} - \Lambda^N_{v,t}dt\right) + \rho\left(dW^*_{\perp,t} - \Lambda^N_{\perp,t}dt\right)\right) \\
&= \left(\kappa_v\mu_v - \left[\kappa_v + \sqrt{1-\rho^2}\sigma_v\gamma_v + \rho\sigma_v\gamma_\perp\right]v_t\right)dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW^*_{v,t} + \rho dW^*_{\perp,t}\right) \\
&\equiv \kappa^*_v(\mu^*_v - v_t)\,dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW^*_{v,t} + \rho dW^*_{\perp,t}\right)
\end{aligned}
$$

$$
\begin{aligned}
d\delta_t &= \kappa^*_\delta(\mu^*_\delta - \delta_t)\,dt + \sigma_\delta dW^*_{\delta,t} \\
ds_t &= \left(\phi^*_0 + \phi^{*\prime}_x x_t + \phi^*_v v_t + \phi^*_\delta\delta_t + \phi^*_s s_t\right)dt + \sigma_s dW^*_{\delta,t}
\end{aligned}
$$

and

$$
\begin{aligned}
dq_t &= \pi_t dt + \sigma'_q dW_{x,t} + \sqrt{v_t}dW_{\perp,t} \\
&= \left(\rho^\pi_0 + \rho^{\pi\prime}_x x_t + \rho^\pi_v v_t + \rho^\pi_\delta\delta_t + \rho^\pi_s s_t\right)dt + \sigma'_q\left(dW^*_{x,t} - \Lambda^N_{x,t}dt\right) + \sqrt{v_t}\left(dW^*_{\perp,t} - \Lambda^N_{\perp,t}dt\right) \\
&= \left(\rho^\pi_0 + \rho^{\pi\prime}_x x_t + \rho^\pi_v v_t + \rho^\pi_\delta\delta_t + \rho^\pi_s s_t - \sigma'_q\left(\lambda^N_0 + \lambda^N_x x_t\right) - \gamma_\perp v_t\right)dt + \sigma'_q dW^*_{x,t} + \sqrt{v_t}dW^*_{\perp,t} \\
&= \left(\rho^\pi_0 - \lambda^{N\prime}_0\sigma_q + \left(\rho^\pi_x - \lambda^{N\prime}_x\sigma_q\right)' x_t + \left(\rho^\pi_v - \gamma_\perp\right)v_t + \rho^\pi_\delta\delta_t + \rho^\pi_s s_t\right)dt + \sigma'_q dW^*_{x,t} + \sqrt{v_t}dW^*_{\perp,t} \\
&\equiv \left(\rho^{\pi*}_0 + \rho^{\pi*\prime}_x x_t + \rho^{\pi*}_v v_t + \rho^{\pi*}_\delta\delta_t + \rho^{\pi*}_s s_t\right)dt + \sigma'_q dW^*_{x,t} + \sqrt{v_t}dW^*_{\perp,t},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{K}^* &= \mathcal{K}{+}\Sigma\lambda^N_x \\
\mathcal{K}^*\mu^* &= \mathcal{K}\mu{-}\Sigma\lambda^N_0 \\
\kappa^*_v &= \kappa_v + \sqrt{1-\rho^2}\sigma_v\gamma_v + \rho\sigma_v\gamma_\perp \\
\kappa^*_v\mu^*_v &= \kappa_v\mu_v \\
\kappa^*_\delta &= \kappa_\delta + \sigma_\delta\lambda^N_\delta \\
\kappa^*_\delta\mu^*_\delta &= \kappa_v\mu_v - \sigma_\delta\lambda^N_{0,\delta} \\
\pi^*_t &= \rho^{\pi*}_0 + \rho^{\pi*\prime}_x x_t + \rho^{\pi*}_v v_t + \rho^{\pi*}_\delta\delta_t + \rho^{\pi*}_s s_t \\
\rho^{\pi*}_0 &= \rho^\pi_0 - \lambda^{N\prime}_0\sigma_q \\
\rho^{\pi*}_x &= \rho^\pi_x - \lambda^{N\prime}_x\sigma_q \\
\rho^{\pi*}_v &= \rho^\pi_v - \gamma_\perp \\
\rho^{\pi*}_\delta &= \rho^\pi_\delta \text{ and } \rho^{\pi*}_s = \rho^\pi_s
\end{aligned}
$$

We first establish a well-known result of affine-form of nominal yields in Proposition 1 below.

Proposition 1 *Under this model, nominal and real bond prices take the exponential-affine form*

$$P^i_{t,\tau} = \exp\left(A^i_\tau + B^{i\prime}_\tau x_t + C^i_\tau v_t + D^i_\tau \delta_t + E^i_\tau s_t\right), \; i = N, R \tag{15}$$

and nominal and real yields take the affine form

$$y^i_{t,\tau} = a^i_\tau + b^{i\prime}_\tau x_t + c^i_\tau v_t + d^i_\tau \delta_t + e^i_\tau s_t, \; i = N, R \tag{16}$$

where $a^i_\tau \equiv -A^i_\tau/\tau$, $b^i_\tau \equiv -B^i_\tau/\tau$, $c^i_\tau \equiv -C^i_\tau/\tau$, $d^i_\tau \equiv -D^i_\tau/\tau$ *and* $e^i_\tau \equiv -E^i_\tau/\tau$, *and* A^i_τ, B^i_τ, C^i_τ, D^i_τ, E^i_τ $(i = N, R)$ *satisfy the following system of ODEs:*

$$\begin{aligned}
\frac{dA^i_\tau}{d\tau} &= -\rho^i_0 + \left(\mathcal{K}\mu - \Sigma\lambda^i_0\right)' B^i_\tau + \kappa^*_v \mu^*_v C^i_\tau + \kappa^*_\delta \mu^*_\delta D^i_\tau + \phi^*_0 E^i_\tau + \frac{1}{2} B^{i\prime}_\tau \Sigma\Sigma' B^i_\tau + \frac{1}{2}\sigma^2_\delta \left(D^i_\tau\right)^2 + \frac{1}{2}\sigma^2_s \left(E^i_\tau\right)^2, \textit{ with } A^i_0 = 0 \\
\frac{dB^i_\tau}{d\tau} &= -\rho^i_x - \left(\mathcal{K} + \Sigma\lambda^i_x\right)' B^i_\tau + \phi^*_x E^i_\tau, \textit{ with } B^i_0 = 0 \\
\frac{dC^i_\tau}{d\tau} &= -\rho^i_v - \kappa^*_v C^i_\tau + \frac{1}{2}\sigma^2_v \left(C^i_\tau\right)^2 + \phi^*_v E^i_\tau, \textit{ with } C^i_0 = 0 \\
\frac{dD^i_\tau}{d\tau} &= -\rho^i_\delta - \kappa^*_\delta D^i_\tau + \phi^*_\delta E^i_\tau, \textit{ with } D^i_0 = 0 \\
\frac{dE^i_\tau}{d\tau} &= -\rho^i_s + \phi^*_s E^i_\tau, \textit{ with } E^i_0 = 0
\end{aligned}$$

Consider the transformation

$$\begin{aligned}
\mathcal{A} &= A^i_\tau \\
\mathcal{B} &= \begin{pmatrix} B^i_\tau \\ D^i_\tau \\ E^i_\tau \end{pmatrix} \\
\mathcal{C} &= C^i_\tau
\end{aligned}$$

Then

$$\begin{aligned}
\dot{\mathcal{A}} &= -\rho^i_0 + \begin{pmatrix} \mathcal{K}\mu - \Sigma\lambda^i_0 \\ \kappa^*_\delta \mu^*_\delta \\ \phi^*_0 \end{pmatrix}' \mathcal{B} + \left(\kappa^*_v \mu^*_v\right)\mathcal{C} + \frac{1}{2}\mathcal{B}' \begin{pmatrix} \Sigma\Sigma' & & \\ & \sigma^2_\delta & \\ & & \sigma^2_s \end{pmatrix} \mathcal{B} \\
\dot{\mathcal{B}} &= \begin{pmatrix} -\rho^i_x \\ -\rho^i_\delta \\ -\rho^i_s \end{pmatrix} + \begin{pmatrix} -\left(\mathcal{K} + \Sigma\lambda^i_x\right)' & 0 & \phi^*_x \\ 0 & -\kappa^*_\delta & \phi^*_\delta \\ 0 & 0 & \phi^*_s \end{pmatrix} \mathcal{B} \\
\dot{\mathcal{C}} &= -\rho^i_v - \kappa^*_v \mathcal{C} + \frac{1}{2}\sigma^2_v \mathcal{C}^2
\end{aligned}$$

Survey Yield Forecast. To calculate survey yield forecasts, we need the following results:

$$\begin{aligned}
E_t\left[x_{t+\tau}\right] &= \mu + \exp\left(-\kappa\tau\right)\left(x_t - \mu\right) \\
E_t\left[v_{t+\tau}\right] &= \mu_v + \exp\left(-\kappa_v\tau\right)\left(v_t - \mu_v\right) \\
E_t\left[\delta_{t+\tau}\right] &= \mu_\delta + \exp\left(-\kappa_\delta\tau\right)\left(\delta_t - \mu_\delta\right)
\end{aligned}$$

Next we calculate, $E_t\left[s_{t+\tau}\right]$. Note that

$$\begin{aligned}
ds_t &= \left(\phi_0 + \phi'_x x_t + \phi_v v_t + \phi_\delta \delta_t + \phi_s s_t\right) dt + \sigma_s dW_{\delta,t} \\
d\left[\exp\left(-\phi_s t\right) s_t\right] &= \exp\left(-\phi_s t\right)\left[\left(\phi_0 + \phi'_x x_t + \phi_v v_t + \phi_\delta \delta_t\right) dt + \sigma_s dW_{\delta,t}\right]
\end{aligned}$$

and

$$\begin{aligned} & \exp\left(-\phi_s\left(t+\tau\right)\right)s_{t+\tau}-\exp\left(-\phi_s t\right)s_t \\ = & \int_t^{t+\tau}\exp\left(-\phi_s u\right)\left(\phi_0+\phi_x' x_u+\phi_v v_u+\phi_\delta\delta_u\right)du+\exp\left(-\phi_s u\right)\sigma_s dW_{\delta,u} \end{aligned}$$

implying

$$\begin{aligned} & \exp\left(-\phi_s\left(t+\tau\right)\right)E_t\left[s_{t+\tau}\right]-\exp\left(-\phi_s t\right)s_t \\ = & \int_t^{t+\tau}\exp\left(-\phi_s u\right)\left(\begin{array}{c} \phi_0+\phi_x'\left(\left[\mu+\exp\left(-\kappa\left(u-t\right)\right)\left(x_t-\mu\right)\right]\right) \\ +\phi_v\left(\left[\mu_v+\exp\left(-\kappa_v\left(u-t\right)\right)\left(v_t-\mu_v\right)\right]\right) \\ +\phi_\delta\left(\left[\mu_\delta+\exp\left(-\kappa_\delta\left(u-t\right)\right)\left(\delta_t-\mu_\delta\right)\right]\right) \end{array}\right)du \\ = & \frac{1}{\phi_s}\left(\exp\left(-\phi_s t\right)-\exp\left(-\phi_s\left(t+\tau\right)\right)\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right) \\ & +\exp\left(-\phi_s t\right)\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(I-\exp\left(-\left(\kappa+\phi_s I\right)\tau\right)\right)\left(x_t-\mu\right) \\ & +\exp\left(-\phi_s t\right)\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(1-\exp\left(-\left(\kappa_v+\phi_s\right)\tau\right)\right)\left(v_t-\mu_v\right) \\ & +\exp\left(-\phi_s t\right)\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(1-\exp\left(-\left(\kappa_\delta+\phi_s\right)\tau\right)\right)\left(\delta_t-\mu_\delta\right) \end{aligned}$$

or

$$\begin{aligned} & E_t\left[s_{t+\tau}\right] \\ = & \exp\left(\phi_s\tau\right)s_t+\frac{1}{\phi_s}\left(\exp\left(\phi_s\tau\right)-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right) \\ & +\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right)I-\exp\left(-\kappa\tau\right)\right)\left(x_t-\mu\right) \\ & +\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_v\tau\right)\right)\left(v_t-\mu_v\right) \\ & +\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_\delta\tau\right)\right)\left(\delta_t-\mu_\delta\right) \end{aligned}$$

and

$$Var_t\left[s_{t+\tau}\right]=E_t\left[\int_{-\tau}^0\exp\left(-2\phi_s u\right)\sigma_s^2 du\right]=\frac{1}{2\phi_s}\left(\exp\left(2\phi_s\tau\right)-1\right)\sigma_s^2$$

Therefore,

$$\begin{aligned} E_t^{svy}\left[y_{t+\tau,3m}^N\right] &= E_t^{mkt}\left[y_{t+\tau,3m}^N\right]+\epsilon_{t,\tau}^f \\ &= E_t\left[a_{3m}^N+b_{3m}^{N\prime}x_{t+\tau}+c_{3m}^N v_{t+\tau}+d_{3m}^N\delta_{t+\tau}+e_{3m}^N s_{t+\tau}\right]+\epsilon_{t,\tau}^f \\ &= a_{3m}^N+b_{3m}^{N\prime}\left[\mu+\exp\left(-\kappa\tau\right)\left(x_t-\mu\right)\right] \\ &\quad +c_{3m}^N\left[\mu_v+\exp\left(-\kappa_v\tau\right)\left(v_t-\mu_v\right)\right] \\ &\quad +d_{3m}^N\left[\mu_\delta+\exp\left(-\kappa_\delta\tau\right)\left(\delta_t-\mu_\delta\right)\right] \\ &\quad +e_{3m}^N\left[\begin{array}{c} \exp\left(\phi_s\tau\right)s_t+\frac{1}{\phi_s}\left(\exp\left(\phi_s\tau\right)-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right) \\ +\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right)I-\exp\left(-\kappa\tau\right)\right)\left(x_t-\mu\right) \\ +\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_v\tau\right)\right)\left(v_t-\mu_v\right) \\ +\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_\delta\tau\right)\right)\left(\delta_t-\mu_\delta\right) \end{array}\right]+\epsilon_{t,\tau}^f \\ &\equiv a_\tau^f+b_\tau^{f\prime}x_t+c_\tau^f v_t+d_\tau^f\delta_t+e_\tau^f s_t+\epsilon_{t,\tau}^f \end{aligned}$$

where

$$
\begin{aligned}
a_\tau^f &= a_{3m}^N + b_{3m}^{N\prime}\left[I-\exp\left(-\kappa\tau\right)\right]\mu + c_{3m}^N\left[1-\exp\left(-\kappa_v\tau\right)\right]\mu_v + d_{3m}^N\left[1-\exp\left(-\kappa_\delta\tau\right)\right]\mu_\delta \\
&\quad + e_{3m}^N \begin{bmatrix} \frac{1}{\phi_s}\left(\exp\left(\phi_s\tau\right)-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right) \\ -\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right)I-\exp\left(-\kappa\tau\right)\right)\mu \\ -\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_v\tau\right)\right)\mu_v \\ -\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_\delta\tau\right)\right)\mu_\delta \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
b_\tau^{f\prime} &= b_{3m}^{N\prime}\exp\left(-\kappa\tau\right) + e_{3m}^N\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right)I-\exp\left(-\kappa\tau\right)\right) \\
c_\tau^f &= c_{3m}^N\exp\left(-\kappa_v\tau\right) + e_{3m}^N\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_v\tau\right)\right) \\
d_\tau^f &= d_{3m}^N\exp\left(-\kappa_\delta\tau\right) + e_{3m}^N\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_\delta\tau\right)\right) \\
e_\tau^f &= e_{3m}^N\exp\left(\phi_s\tau\right)
\end{aligned}
$$

Lastly, we turn to the long-range forecast: $E_t^{mkt}\left[\overline{y}_{3m,T_1,T_2}^N\right]$ as follows. Note that

$$
\begin{aligned}
& E_t\left[\frac{1}{T_2-T_1}\int_{t+T_1}^{t+T_2} s_u du\right] \\
= {} & \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\begin{bmatrix} \exp\left(\phi_s u\right)s_t + \frac{1}{\phi_s}\left(\exp\left(\phi_s u\right)-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right) \\ +\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(\exp\left(\phi_s u\right)I-\exp\left(-\kappa u\right)\right)\left(x_t-\mu\right) \\ +\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(\exp\left(\phi_s u\right)-\exp\left(-\kappa_v u\right)\right)\left(v_t-\mu_v\right) \\ +\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(\exp\left(\phi_s u\right)-\exp\left(-\kappa_\delta u\right)\right)\left(\delta_t-\mu_\delta\right) \end{bmatrix} du \\
= {} & W_{T_1,T_2}^s s_t + \frac{1}{\phi_s}\left(W_{T_1,T_2}^s-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right) \\
& +\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(W_{T_1,T_2}^s I-W_{T_1,T_2}^x\right)\left(x_t-\mu\right) \\
& +\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(W_{T_1,T_2}^s-W_{T_1,T_2}^v\right)\left(v_t-\mu_v\right) \\
& +\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(W_{T_1,T_2}^s-W_{T_1,T_2}^\delta\right)\left(\delta_t-\mu_\delta\right)
\end{aligned}
$$

where

$$
\begin{aligned}
W_{T_1,T_2}^x &= \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\exp\left(-\kappa s\right)ds = \frac{1}{T_2-T_1}\kappa^{-1}\left(\exp\left(-\kappa T_1\right)-\exp\left(-\kappa T_2\right)\right) \\
W_{T_1,T_2}^v &= \frac{1}{T_2-T_1}\kappa_v^{-1}\left(\exp\left(-\kappa_v T_1\right)-\exp\left(-\kappa_v T_2\right)\right) \\
W_{T_1,T_2}^\delta &= \frac{1}{T_2-T_1}\kappa_\delta^{-1}\left(\exp\left(-\kappa_\delta T_1\right)-\exp\left(-\kappa_\delta T_2\right)\right) \\
W_{T_1,T_2}^s &= \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\exp\left(\phi_s u\right)du = \frac{1}{T_2-T_1}\phi_s^{-1}\left(\exp\left(\phi_s T_2\right)-\exp\left(\phi_s T_1\right)\right)
\end{aligned}
$$

Therefore,

$$\begin{aligned}
E_t^{svy}\left[\overline{y}_{3m,T_1,T_2}^N\right] &= E_t^{mkt}\left[\overline{y}_{3m,T_1,T_2}^N\right]+\epsilon_{t,LT}^f = E_t\left[\frac{1}{T_2-T_1}\int_{T_1}^{T_2} y_{s,3m}^N ds\right]+\epsilon_{t,LT}^f \\
&= E_t\left[\frac{1}{T_2-T_1}\int_{T_1}^{T_2}\left(a_{3m}^N+b_{3m}^{N\prime}x_s+c_{3m}^N v_s+d_{3m}^N\delta_s+e_{3m}^N s_s\right)ds\right]+\epsilon_{t,LT}^f \\
&= a_{3m}^N+b_{3m}^{N\prime}\left(I-W_{T_1,T_2}\right)\mu+b_{3m}^{N\prime}W_{T_1,T_2}x_t \\
&\quad +c_{3m}^N\left[\left(1-W_{T_1,T_2}^v\right)\mu_v+W_{T_1,T_2}^v v_t\right]+d_{3m}^N\left[\left(1-W_{T_1,T_2}^\delta\right)\mu_\delta+W_{T_1,T_2}^\delta\delta_t\right] \\
&\quad +e_{3m}^N\left[\begin{array}{c} W_{T_1,T_2}^s s_t+\frac{1}{\phi_s}\left(W_{T_1,T_2}^s-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right) \\ +\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(W_{T_1,T_2}^s I-W_{T_1,T_2}^x\right)\left(x_t-\mu\right) \\ +\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(W_{T_1,T_2}^s-W_{T_1,T_2}^v\right)\left(v_t-\mu_v\right) \\ +\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(W_{T_1,T_2}^s-W_{T_1,T_2}^\delta\right)\left(\delta_t-\mu_\delta\right)\end{array}\right]+\epsilon_{t,LT}^f \\
&\equiv a_{LT}^f+b_{LT}^{f\prime}x_t+c_{LT}^f v_t+d_{LT}^f\delta_t+\epsilon_{t,LT}^f
\end{aligned}$$

where

$$\begin{aligned}
a_{LT}^f &= a_{3m}^N+b_{3m}^{N\prime}\left(I-W_{T_1,T_2}^x\right)\mu+c_{3m}^N\left(1-W_{T_1,T_2}^v\right)\mu_v+d_{3m}^N\left(1-W_{T_1,T_2}^\delta\right)\mu_\delta \\
&\quad +e_{3m}^N\left[\begin{array}{c}\frac{1}{\phi_s}\left(W_{T_1,T_2}^s-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right)-\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(W_{T_1,T_2}^s I-W_{T_1,T_2}^x\right)\mu \\ -\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(W_{T_1,T_2}^s-W_{T_1,T_2}^v\right)\mu_v-\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(W_{T_1,T_2}^s-W_{T_1,T_2}^\delta\right)\mu_\delta\end{array}\right]
\end{aligned}$$

and

$$\begin{aligned}
b_{LT}^{f\prime} &= b_{3m}^{N\prime}W_{T_1,T_2}^x+e_{3m}^N\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(W_{T_1,T_2}^s I-W_{T_1,T_2}^x\right) \\
c_{LT}^f &= c_{3m}^N W_{T_1,T_2}^v+e_{3m}^N\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(W_{T_1,T_2}^s-W_{T_1,T_2}^v\right) \\
d_{LT}^f &= d_{3m}^N W_{T_1,T_2}^\delta+e_{3m}^N\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(W_{T_1,T_2}^s-W_{T_1,T_2}^\delta\right) \\
e_{LT}^f &= e_{3m}^N W_{T_1,T_2}^s
\end{aligned}$$

1.2 Dynamics under the Forward Measure and Inflation Option Pricing

Under the forward measure, we have

$$\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right)_{t,T}=\frac{\exp\left(-\int_t^T r_s^N ds\right)}{P_{t,\tau}^N}$$

and

$$\Psi_t\equiv E_t^{\mathbb{P}^*}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right)_{0,T}\right]=E_t^{\mathbb{P}^*}\left[\frac{\exp\left(-\int_0^T r_s^N ds\right)}{P_{0,T}^N}\right]=\frac{P_{t,\tau}^N}{P_{0,T}^N}\exp\left(-\int_0^t r_s^N ds\right)$$

We have

$$\begin{aligned}
d\Psi_t &= \frac{\exp\left(-\int_0^t r_s^N ds\right)}{P_{0,T}^N}\left[dP_{t,\tau}^N-r_t^N P_{t,\tau}^N dt\right] \\
&= \Psi_t\left[B_\tau^{N\prime}\Sigma dW_{x,t}+C_\tau^N\sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t}+\rho dW_{\perp,t}\right)+\left(D_\tau^N\sigma_\delta+E_\tau^N\sigma_s\right)dW_{\delta,t}\right]
\end{aligned}$$

By Girsanov's Theorem, we have

$$d\tilde{W}_t = dW_t^* - \frac{d\Psi_t}{\Psi_t} \cdot dW_t^*$$

or

$$\begin{aligned}
d\tilde{W}_{x,t} &= dW_t^* - \Sigma' B_\tau^N dt \\
d\tilde{W}_{v,t} &= dW_{v,t}^* - \sqrt{1-\rho^2}\sigma_v C_\tau^N \sqrt{v_t} dt \\
d\tilde{W}_{\perp,t} &= dW_{\perp,t}^* - \rho \sigma_v C_\tau^N \sqrt{v_t} dt. \\
d\tilde{W}_{\delta,t} &= dW_{\delta,t}^* - \left(\sigma_\delta D_\tau^N + \sigma_s E_\tau^N\right) dt
\end{aligned}$$

The dynamics of the state variables under the risk neutral measure is given by

$$\begin{aligned}
dx_t &= \left[\mathcal{K}^* \left(\mu^* - x_t\right)\right] dt + \Sigma dW_{x,t}^* \\
dv_t &= \kappa_v^* \left(\mu_v^* - v_t\right) dt + \sigma_v \sqrt{v_t} \left(\sqrt{1-\rho^2} dW_{v,t}^* + \rho dW_{\perp,t}^*\right) \\
dq_t &= \left(\rho_0^{\pi *} + \rho_x^{\pi * \prime} x_t + \rho_v^{\pi *} v_t + \rho_\delta^{\pi *} v_t\right) dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t} dW_{\perp,t}^* \\
d\delta_t &= \kappa_\delta^* \left(\mu_\delta^* - \delta_t\right) dt + \sigma_\delta dW_{\delta,t}^*
\end{aligned}$$

Therefore,

$$\begin{aligned}
d\delta_t &= \kappa_\delta^* \left(\mu_\delta^* - \delta_t\right) dt + \sigma_\delta dW_{\delta,t}^* \\
&= \kappa_\delta^* \left(\mu_\delta^* - \delta_t\right) dt + \sigma_\delta \left(d\tilde{W}_{\delta,t} + \left(\sigma_\delta D_\tau^N + \sigma_s E_\tau^N\right) dt\right) \\
&= \left(\kappa_\delta^* \mu_\delta^* + \sigma_\delta \left(\sigma_\delta D_\tau^N + \sigma_s E_\tau^N\right) - \kappa_\delta^* \delta_t\right) dt + \sigma_\delta d\tilde{W}_{\delta,t} \\
&\equiv \tilde{\kappa}_\delta \left(\tilde{\mu}_\delta - \delta_t\right) dt + \sigma_\delta d\tilde{W}_{\delta,t}
\end{aligned}$$

$$\begin{aligned}
dx_t &= \left[\mathcal{K}^* \left(\mu^* - x_t\right)\right] dt + \Sigma \left(d\tilde{W}_{x,t} + \Sigma' B_\tau^N dt\right) \\
&= \left(\mathcal{K}^* \mu^* + \Sigma \Sigma' B_\tau^N - \mathcal{K}^* x_t\right) dt + \Sigma d\tilde{W}_{x,t} \\
&\equiv \tilde{\mathcal{K}} \left(\tilde{\mu} - x_t\right) dt + \Sigma d\tilde{W}_{x,t},
\end{aligned}$$

$$\begin{aligned}
dv_t &= \kappa_v^* \left(\mu_v^* - v_t\right) dt + \sigma_v \sqrt{v_t} \left(\sqrt{1-\rho^2} dW_{v,t}^* + \rho dW_{\perp,t}^*\right) \\
&= \left[\kappa_v^* \mu_v^* - \left(\kappa_v^* - \sigma_v^2 C_\tau^N\right) v_t\right] dt + \sigma_v \sqrt{v_t} \left(\sqrt{1-\rho^2} d\tilde{W}_{v,t} + \rho d\tilde{W}_{\perp,t}\right) \\
&\equiv \tilde{\kappa}_v \left(\tilde{\mu}_v - v_t\right) dt + \sigma_v \sqrt{v_t} \left(\sqrt{1-\rho^2} d\tilde{W}_{v,t} + \rho d\tilde{W}_{\perp,t}\right)
\end{aligned}$$

$$\begin{aligned}
ds_t &= \left(\phi_0^* + \phi_x^{*\prime} x_t + \phi_v^* v_t + \phi_\delta^* \delta_t + \phi_s^* s_t\right) dt + \sigma_s dW_{\delta,t}^* \\
&= \left(\phi_0^* + \phi_x^{*\prime} x_t + \phi_v^* v_t + \phi_\delta^* \delta_t + \phi_s^* s_t\right) dt + \sigma_s \left(d\tilde{W}_{\delta,t} + \left(\sigma_\delta D_\tau^N + \sigma_s E_\tau^N\right) dt\right) \\
&= \left(\left[\phi_0^* + \sigma_s \left(\sigma_\delta D_\tau^N + \sigma_s E_\tau^N\right)\right] + \phi_x^{*\prime} x_t + \phi_v^* v_t + \phi_\delta^* \delta_t + \phi_s^* s_t\right) dt + \sigma_s d\tilde{W}_{\delta,t} \\
&\equiv \left(\widetilde{\phi}_0 + \widetilde{\phi}_x' x_t + \widetilde{\phi}_v v_t + \widetilde{\phi}_\delta \delta_t + \widetilde{\phi}_s s_t\right) dt + \sigma_s d\tilde{W}_{\delta,t}
\end{aligned}$$

and

$$\begin{aligned}
dq_t &= \left(\rho_0^{\pi*} + \rho_x^{\pi*\prime} x_t + \rho_v^{\pi*} v_t + \rho_\delta^{\pi*}\delta_t + \rho_s^{\pi*} s_t\right) dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t} dW_{\perp,t}^* \\
&= \left(\left[\rho_0^{\pi*} + \sigma_q'\Sigma' B_\tau^N\right] + \rho_x^{\pi*\prime} x_t + \left[\rho_v^{\pi*} + \rho\sigma_v C_\tau^N\right] v_t + \rho_\delta^{\pi*}\delta_t + \rho_s^{\pi*} s_t\right) dt + \sigma_q' d\tilde{W}_{x,t} + \sqrt{v_t} d\tilde{W}_{\perp,t} \\
&\equiv \left(\widetilde{\rho}_0^{\pi} + \widetilde{\rho}_x^{\pi\prime} x_t + \widetilde{\rho}_v^{\pi} v_t + \widetilde{\rho}_\delta^{\pi*}\delta_t + \widetilde{\rho}_s^{\pi*} s_t\right) dt + \sigma_q' d\tilde{W}_{x,t} + \sqrt{v_t} d\tilde{W}_{\perp,t}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathcal{K}} &= \mathcal{K}^* = \mathcal{K} + \Sigma\lambda_1^N, \\
\tilde{\mathcal{K}}\tilde{\mu} &= \mathcal{K}^*\mu^* + \Sigma\Sigma' B_\tau^N = \mathcal{K}\mu - \Sigma\lambda_0^N + \Sigma\Sigma' B_\tau^N, \\
\tilde{\kappa}_v &= \kappa_v^* - \sigma_v^2 C_\tau^N = \kappa_v + \sqrt{1-\rho^2}\sigma_v\gamma_v + \rho\sigma_v\gamma_\perp - \sigma_v^2 C_\tau^N, \\
\tilde{\kappa}_v\tilde{\mu}_v &= \kappa_v^*\mu_v^* = \kappa_v\mu_v, \\
\tilde{\kappa}_\delta &= \kappa_\delta^* = \kappa_\delta + \sigma_\delta\lambda_\delta^N \\
\tilde{\kappa}_\delta\tilde{\mu}_\delta &= \kappa_\delta^*\mu_\delta^* + \sigma_\delta\left(\sigma_\delta D_\tau^N + \sigma_s E_\tau^N\right) = \kappa_v\mu_v - \sigma_\delta\lambda_{0,\delta}^N + \sigma_\delta\left(\sigma_\delta D_\tau^N + \sigma_s E_\tau^N\right) \\
\tilde{\pi}_t &= \widetilde{\rho}_0^{\pi} + \widetilde{\rho}_x^{\pi\prime} x_t + \widetilde{\rho}_v^{\pi} v_t + \widetilde{\rho}_\delta^{\pi}\delta_t + \widetilde{\rho}_s^{\pi} s_t, \\
\widetilde{\rho}_0^{\pi} &= \rho_0^{\pi*} + \sigma_q'\Sigma' B_\tau^N = \rho_0^{\pi} - \lambda_0^{N\prime}\sigma_q + \sigma_q'\Sigma' B_\tau^N, \\
\widetilde{\rho}_x^{\pi} &= \rho_x^{\pi*} = \rho_x^{\pi} - \lambda_x^{N\prime}\sigma_q, \\
\widetilde{\rho}_v^{\pi} &= \rho_v^{\pi*} + \rho\sigma_v C_\tau^N = \rho_v^{\pi} - \gamma_\perp + \rho\sigma_v C_\tau^N. \\
\widetilde{\rho}_\delta^{\pi*} &= \rho_\delta^{\pi*} \text{ and } \widetilde{\rho}_s^{\pi*} = \rho_s^{\pi*}
\end{aligned}$$

Next, we compute the expected values of the state variables over the period t to $t+\tau$ under the forward measure. First, similar to the calculations under the risk neutral measure, we have

$$E_t^{\tilde{\mathbb{P}}}(x_s) = \tilde{\mu} + \exp\left(-\tilde{\mathcal{K}}(s-t)\right)(x_t - \tilde{\mu})$$

and

$$\begin{aligned}
\frac{1}{\tau}E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau} x_s ds\right] &= \frac{1}{\tau}\left[\int_t^{t+\tau}\left(\tilde{\mu} + \exp\left(-\tilde{\mathcal{K}}(s-t)\right)(x_t - \tilde{\mu})\right) ds\right] \\
= \quad \tilde{\mu} + \widetilde{W}_{0,\tau}^x (x_t - \tilde{\mu}) \equiv \tilde{a}_\tau^x + \tilde{b}_\tau^x x_t
\end{aligned}$$

where $\widetilde{W}_{T_1,T_2}^x$ is defined similarly as W_{T_1,T_2}^x except that dynamics under the forward measure is used instead, and

$$\tilde{a}_\tau^x = \left(I - \widetilde{W}_{0,\tau}^x\right)\tilde{\mu} \text{ and } \tilde{b}_\tau^x = \widetilde{W}_{0,\tau}^x \equiv \left(\tilde{\mathcal{K}}\tau\right)^{-1}\left(I - \exp\left(-\tilde{\mathcal{K}}\tau\right)\right)$$

Similarly (the results for δ_t are very similar and thus omitted), we have

$$\begin{aligned}
E_t^{\tilde{\mathbb{P}}}(v_s) &= \tilde{\mu}_v + \exp\left(-\tilde{\kappa}_v(s-t)\right)(v_t - \tilde{\mu}_v) \\
\frac{1}{\tau}E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau} v_s ds\right] &= \tilde{\mu}_v + \widetilde{W}_{0,\tau}^v (v_t - \tilde{\mu}_v) \equiv \tilde{a}_\tau^v + \tilde{b}_\tau^v v_t
\end{aligned}$$

where

$$\tilde{a}_\tau^v = \tilde{\mu}_v\left[1 - \widetilde{W}_{0,\tau}^v\right] \text{ and } \tilde{b}_\tau^v = \widetilde{W}_{0,\tau}^v \equiv (\tilde{\kappa}_v\tau)^{-1}\left(1 - \exp(-\tilde{\kappa}_v\tau)\right).$$

Furthermore,

$$
\begin{aligned}
& E_t^{\tilde{\mathbb{P}}}\left[s_s\right] \\
= & \exp\left(\widetilde{\phi}_s\left(s-t\right)\right)s_t+\frac{1}{\widetilde{\phi}_s}\left(\exp\left(\widetilde{\phi}_s\left(s-t\right)\right)-1\right)\left(\widetilde{\phi}_0+\widetilde{\phi}_x'\mu+\widetilde{\phi}_v\mu_v+\widetilde{\phi}_\delta\mu_\delta\right) \\
& +\widetilde{\phi}_x'\left(\kappa+\widetilde{\phi}_sI\right)^{-1}\left(\exp\left(\widetilde{\phi}_s\left(s-t\right)\right)I-\exp\left(-\kappa\left(s-t\right)\right)\right)\left(x_t-\mu\right) \\
& +\widetilde{\phi}_v\left(\kappa_v+\widetilde{\phi}_s\right)^{-1}\left(\exp\left(\widetilde{\phi}_s\left(s-t\right)\right)-\exp\left(-\kappa_v\left(s-t\right)\right)\right)\left(v_t-\mu_v\right) \\
& +\widetilde{\phi}_\delta\left(\kappa_\delta+\widetilde{\phi}_s\right)^{-1}\left(\exp\left(\widetilde{\phi}_s\left(s-t\right)\right)-\exp\left(-\kappa_\delta\left(s-t\right)\right)\right)\left(\delta_t-\mu_\delta\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& E_t^{\tilde{\mathbb{P}}}\left[\frac{1}{T_2-T_1}\int_{t+T_1}^{t+T_2}s_u du\right] \\
= & \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\left[\begin{array}{c}\exp\left(\widetilde{\phi}_su\right)s_t+\frac{1}{\widetilde{\phi}_s}\left(\exp\left(\widetilde{\phi}_su\right)-1\right)\left(\widetilde{\phi}_0+\widetilde{\phi}_x'\mu+\widetilde{\phi}_v\mu_v+\widetilde{\phi}_\delta\mu_\delta\right) \\ +\widetilde{\phi}_x'\left(\kappa+\widetilde{\phi}_sI\right)^{-1}\left(\exp\left(\widetilde{\phi}_su\right)I-\exp\left(-\kappa u\right)\right)\left(x_t-\mu\right) \\ +\widetilde{\phi}_v\left(\kappa_v+\widetilde{\phi}_s\right)^{-1}\left(\exp\left(\widetilde{\phi}_su\right)-\exp\left(-\kappa_vu\right)\right)\left(v_t-\mu_v\right) \\ +\widetilde{\phi}_\delta\left(\kappa_\delta+\widetilde{\phi}_s\right)^{-1}\left(\exp\left(\widetilde{\phi}_su\right)-\exp\left(-\kappa_\delta u\right)\right)\left(\delta_t-\mu_\delta\right)\end{array}\right]du \\
= & \widetilde{W}_{T_1,T_2}^s s_t+\frac{1}{\widetilde{\phi}_s}\left(\widetilde{W}_{T_1,T_2}^s-1\right)\left(\widetilde{\phi}_0+\widetilde{\phi}_x'\mu+\widetilde{\phi}_v\mu_v+\widetilde{\phi}_\delta\mu_\delta\right) \\
& +\widetilde{\phi}_x'\left(\kappa+\widetilde{\phi}_sI\right)^{-1}\left(\widetilde{W}_{T_1,T_2}^sI-\widetilde{W}_{T_1,T_2}^x\right)\left(x_t-\mu\right) \\
& +\widetilde{\phi}_v\left(\kappa_v+\widetilde{\phi}_s\right)^{-1}\left(\widetilde{W}_{T_1,T_2}^s-\widetilde{W}_{T_1,T_2}^v\right)\left(v_t-\mu_v\right) \\
& +\widetilde{\phi}_\delta\left(\kappa_\delta+\widetilde{\phi}_s\right)^{-1}\left(\widetilde{W}_{T_1,T_2}^s-\widetilde{W}_{T_1,T_2}^\delta\right)\left(\delta_t-\mu_\delta\right)
\end{aligned}
$$

implying

$$E_t^{\tilde{\mathbb{P}}}\left[\frac{1}{\tau}\int_t^{t+\tau}s_u du\right]\equiv\tilde{a}_\tau^s+\tilde{b}_\tau^{s,x}x_t+\widetilde{b}_\tau^{s,v}v_t+\widetilde{b}_\tau^{s,\delta}v_t+\widetilde{b}_\tau^{s,s}s_t$$

where

$$
\begin{aligned}
\widetilde{a}_\tau^s \equiv & \frac{1}{\widetilde{\phi}_s}\left(\widetilde{W}_{0,\tau}^s-1\right)\left(\widetilde{\phi}_0+\widetilde{\phi}_x'\mu+\widetilde{\phi}_v\mu_v+\widetilde{\phi}_\delta\mu_\delta\right) \\
& -\widetilde{\phi}_x'\left(\kappa+\widetilde{\phi}_sI\right)^{-1}\left(\widetilde{W}_{0,\tau}^sI-\widetilde{W}_{0,\tau}^x\right)\mu \\
& -\widetilde{\phi}_v\left(\kappa_v+\widetilde{\phi}_s\right)^{-1}\left(\widetilde{W}_{0,\tau}^s-\widetilde{W}_{0,\tau}^v\right)\mu_v \\
& -\widetilde{\phi}_\delta\left(\kappa_\delta+\widetilde{\phi}_s\right)^{-1}\left(\widetilde{W}_{0,\tau}^s-\widetilde{W}_{0,\tau}^\delta\right)\mu_\delta
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{b}_{\tau}^{s,x} &\equiv \widetilde{\phi}_x' \left(\kappa + \widetilde{\phi}_s I\right)^{-1} \left(\widetilde{W}_{0,\tau}^{s} I - \widetilde{W}_{0,\tau}^{x}\right) \\
\tilde{b}_{\tau}^{s,v} &\equiv \widetilde{\phi}_v \left(\kappa_v + \widetilde{\phi}_s\right)^{-1} \left(\widetilde{W}_{0,\tau}^{s} - \widetilde{W}_{0,\tau}^{v}\right) \\
\tilde{b}_{\tau}^{s,\delta} &\equiv \widetilde{\phi}_\delta \left(\kappa_\delta + \widetilde{\phi}_s\right)^{-1} \left(\widetilde{W}_{0,\tau}^{s} - \widetilde{W}_{0,\tau}^{\delta}\right) \\
\tilde{b}_{\tau}^{s,s} &\equiv \widetilde{W}_{0,\tau}^{s}
\end{aligned}
$$

We can now calculate inflation expectation and variance as follows:

$$
\begin{aligned}
& E_t^{\tilde{\mathbb{P}}}\left[\log\left(\frac{Q_{t+\tau}}{Q_t}\right)\right] = E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau} \tilde{\pi}_s ds\right] = E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau} \left(\widetilde{\rho}_0^{\pi} + \widetilde{\rho}_x^{\pi\prime} x_s + \widetilde{\rho}_v^{\pi} v_s + \widetilde{\rho}_\delta^{\pi} \delta_s + \widetilde{\rho}_s^{\pi} s_s\right) ds\right] \\
= \;& \tau \left(\begin{array}{c} \widetilde{\rho}_0^{\pi} + \widetilde{\rho}_x^{\pi\prime}\left(\tilde{a}_\tau^x + \tilde{b}_\tau^x x_t\right) + \widetilde{\rho}_v^{\pi}\left(\tilde{a}_\tau^v + \tilde{b}_\tau^v v_t\right) + \widetilde{\rho}_\delta^{\pi}\left(\tilde{a}_\tau^\delta + \tilde{b}_\tau^\delta \delta_t\right) \\ +\widetilde{\rho}_s^{\pi}\left(\tilde{a}_\tau^s + \tilde{b}_\tau^{s,x} x_t + \tilde{b}_\tau^{s,v} v_t + \tilde{b}_\tau^{s,\delta}\delta_t + \tilde{b}_\tau^{s,s} s_t\right) \end{array}\right) \\
\equiv \;& \tau\left(\tilde{a}_\tau^{\pi} + \tilde{b}_\tau^{\pi\prime} x_t + \tilde{c}_\tau^{\pi} v_t + \widetilde{c2}_\tau^{\pi}\delta_t + \widetilde{c3}_\tau^{\pi} s_t\right)
\end{aligned}
$$

and

$$
Var_t^{\tilde{\mathbb{P}}}\left[\log\left(\frac{Q_{t+\tau}}{Q_t}\right)\right] = E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau}\left(\sigma_q'\sigma_q + v_s\right) ds\right] = \tau\left(\sigma_q'\sigma_q + \tilde{a}_\tau^v + \tilde{b}_\tau^v v_t\right) \equiv \tau\left(\tilde{d}_\tau^{\pi} + \tilde{e}_\tau^{\pi} v_t\right),
$$

where

$$
\begin{aligned}
\tilde{a}_\tau^{\pi} &= \widetilde{\rho}_0^{\pi} + \widetilde{\rho}_x^{\pi\prime}\tilde{a}_\tau^x + \widetilde{\rho}_v^{\pi}\tilde{a}_\tau^v + \widetilde{\rho}_\delta^{\pi}\tilde{a}_\tau^\delta + \widetilde{\rho}_s^{\pi}\tilde{a}_\tau^s, \\
\tilde{b}_\tau^{\pi} &= \tilde{b}_\tau^{x\prime}\widetilde{\rho}_x^{\pi} + \tilde{b}_\tau^{s,x\prime}\widetilde{\rho}_s^{\pi}, \\
\tilde{c}_\tau^{\pi} &= \widetilde{\rho}_v^{\pi}\tilde{b}_\tau^v + \widetilde{\rho}_s^{\pi}\tilde{b}_\tau^{s,v}, \\
\widetilde{c2}_\tau^{\pi} &= \widetilde{\rho}_\delta^{\pi}\tilde{b}_\tau^\delta + \widetilde{\rho}_s^{\pi}\tilde{b}_\tau^{s,\delta} \\
\widetilde{c3}_\tau^{\pi} &= \widetilde{\rho}_s^{\pi}\tilde{b}_\tau^{s,s} \\
\tilde{d}_\tau^{\pi} &= \sigma_q'\sigma_q + \tilde{a}_\tau^v \\
\tilde{e}_\tau^{\pi} &= \tilde{b}_\tau^v.
\end{aligned}
$$

Finally, we turn to inflation option pricing. The price of a τ-maturity inflation cap with strike K is given by

$$
\begin{aligned}
P_{t,\tau,K}^{CAP} &= \exp\left(-\tau y_{t,\tau}^N\right) E_t^{\tilde{\mathbb{P}}}\left[\left(\frac{Q_{t+\tau}}{Q_t} - (1+K)^\tau\right)^+\right] \\
&= \exp\left(-\tau y_{t,\tau}^N\right) E_t^{\tilde{\mathbb{P}}}\left[\left(\exp\left(\log\left(\frac{Q_{t+\tau}}{Q_t}\right)\right) - (1+K)^\tau\right)^+\right] \\
&= \exp\left(-\tau y_{t,\tau}^N\right)\left[\exp\left(\tau\left(\left(\tilde{a}_\tau^{\pi} + \frac{\tilde{d}_\tau^{\pi}}{2}\right) + \tilde{b}_\tau^{\pi\prime} x_t + \left(\tilde{c}_\tau^{\pi} + \frac{\tilde{e}_\tau^{\pi}}{2}\right) v_t + \widetilde{c2}_\tau^{\pi}\delta_t + \widetilde{c3}_\tau^{\pi} s_t\right)\right)\right. \\
&\quad \times\Phi\left(\frac{\tau}{\sigma}\left[-\ln(1+K) + \left(\tilde{a}_\tau^{\pi} + \tilde{d}_\tau^{\pi}\right) + \tilde{b}_\tau^{\pi\prime} x_t + \left(\tilde{c}_\tau^{\pi} + \tilde{e}_\tau^{\pi}\right) v_t + \widetilde{c2}_\tau^{\pi}\delta_t + \widetilde{c3}_\tau^{\pi} s_t\right]\right) \\
&\quad \left. -(1+K)^\tau\,\Phi\left(\frac{\tau}{\sigma}\left[-\ln(1+K) + \tilde{a}_\tau^{\pi} + \tilde{b}_\tau^{\pi\prime} x_t + \tilde{c}_\tau^{\pi} v_t + \widetilde{c2}_\tau^{\pi}\delta_t + \widetilde{c3}_\tau^{\pi} s_t\right]\right)\right],
\end{aligned}
$$

and the price of a τ-maturity inflation cap with strike K is given by

$$\begin{aligned}
P_{t,\tau,K}^{FLO} &= \exp\left(-\tau y_{t,\tau}^{N}\right) E_t^{\tilde{\mathbb{P}}}\left[\left((1+K)^{\tau} - \frac{Q_{t+\tau}}{Q_t}\right)^{+}\right] \\
&= \exp\left(-\tau y_{t,\tau}^{N}\right) E_t^{\tilde{\mathbb{P}}}\left[\left((1+K)^{\tau} - \exp\left(\log\left(\frac{Q_{t+\tau}}{Q_t}\right)\right)\right)^{+}\right] \\
&= \exp\left(-\tau y_{t,\tau}^{N}\right)\Bigg[-\exp\left(\tau\left(\left(\tilde{a}_\tau^{\pi} + \frac{\tilde{d}_\tau^{\pi}}{2}\right) + \tilde{b}_\tau^{\pi\prime} x_t + \left(\tilde{c}_\tau^{\pi} + \frac{\tilde{e}_\tau^{\pi}}{2}\right) v_t\right)\right) \\
&\quad \times \Phi\left(-\frac{\tau}{\sigma}\left[-\ln(1+K) + \left(\tilde{a}_\tau^{\pi} + \tilde{d}_\tau^{\pi}\right) + \tilde{b}_\tau^{\pi\prime} x_t + \left(\tilde{c}_\tau^{\pi} + \tilde{e}_\tau^{\pi}\right) v_t\right]\right) \\
&\quad + (1+K)^{\tau}\,\Phi\left(-\frac{\tau}{\sigma}\left[-\ln(1+K) + \tilde{a}_\tau^{\pi} + \tilde{b}_\tau^{\pi\prime} x_t + \tilde{c}_\tau^{\pi} v_t\right]\right)\Bigg]
\end{aligned}$$

where we used the fact that for a normal random variable $\tilde{z} \sim N\left(\mu, \sigma^2\right)$,

$$\begin{aligned}
E\left[\left(a e^{\tilde{z}} - b\right)^{+}\right] &= a\exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(\frac{\ln(a/b) + \left(\mu + \sigma^2\right)}{\sigma}\right) - b\Phi\left(\frac{\ln(a/b) + \mu}{\sigma}\right), \\
E\left[\left(b - a e^{\tilde{z}}\right)^{+}\right] &= -a\exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(-\frac{\ln(a/b) + \left(\mu + \sigma^2\right)}{\sigma}\right) + b\Phi\left(-\frac{\ln(a/b) + \mu}{\sigma}\right).
\end{aligned}$$

Remark. We can derive option-implied inflation expectations as follows:

$$\begin{aligned}
& P_{t,\tau,K}^{CAP} - P_{t,\tau,K}^{FLO} \\
&= \exp\left(-\tau y_{t,\tau}^{N}\right) E_t^{\tilde{\mathbb{P}}}\left[\left(\frac{Q_{t+\tau}}{Q_t} - (1+K)^{\tau}\right)^{+}\right] - \exp\left(-\tau y_{t,\tau}^{N}\right) E_t^{\tilde{\mathbb{P}}}\left[\left((1+K)^{\tau} - \frac{Q_{t+\tau}}{Q_t}\right)^{+}\right] \\
&= \exp\left(-\tau y_{t,\tau}^{N}\right) E_t^{\tilde{\mathbb{P}}}\left[\frac{Q_{t+\tau}}{Q_t} - (1+K)^{\tau}\right]
\end{aligned}$$

1.3 State Dynamics

We first consider the discrete-time dynamics of the state variable q_t between time $t-\Delta t$ and t. From $dq_t = \pi_t dt + \sigma_q' dW_{x,t} + \sqrt{v_t} dW_{\perp,t}$, we have

$$q_t = q_{t-\Delta t} + \left(\rho_0^{\pi}\Delta t + \left(\rho_x^{\pi\prime}\Delta t\right) x_t + \left(\rho_v^{\pi}\Delta t\right) v_t + \left(\rho_\delta^{\pi}\Delta t\right)\delta_t + \left(\rho_s^{\pi}\Delta t\right) s_t\right) + \eta_t^q$$

where $\eta_t^q \equiv \int_{t-\Delta t}^{t}\left(\sigma_q' dW_{x,s} + \sqrt{v_s} dW_{\perp,s}\right) \sim N\left(0, \Omega_{t-\Delta t}^{q}\right)$ and

$$\begin{aligned}
\Omega_{t-\Delta t}^{q} &\equiv Var_{t-\Delta t}\left(\eta_t^q\right) = \sigma_q'\sigma_q\Delta t + E_{t-\Delta t}\left[\int_{t-\Delta t}^{t} v_s ds\right] \\
&= \sigma_q'\sigma_q\Delta t + \left[\int_0^{\Delta t}\left(\mu_v\left(1 - e^{-\kappa_v s}\right) + e^{-\kappa_v s} v_{t-\Delta t}\right) ds\right] \\
&= \sigma_q'\sigma_q\Delta t + \mu_v\Delta t + \left(v_{t-\Delta t} - \mu_v\right)\frac{1 - \exp\left(-\kappa_v\Delta t\right)}{\kappa_v}.
\end{aligned}$$

Similarly, for the state variable x_t we have

$$x_t = \exp\left(-\mathcal{K}\Delta t\right) x_{t-\Delta t} + \left(I - \exp\left(-\mathcal{K}\Delta t\right)\right)\mu + \eta_t^x,$$

where $\eta_t^x = \int_0^{\Delta t} \exp\left(-\kappa s\right)\Sigma dW_{x,s} \sim N\left(0, \Omega_{t-\Delta t}^x\right)$ and

$$\Omega_{t-\Delta t}^x \equiv Var_{t-\Delta t}\left(\eta_t^x\right) = \int_0^{\Delta t} \exp\left(-\mathcal{K}s\right)\Sigma\Sigma' \exp\left(-\mathcal{K}'s\right) ds = N\Xi N',$$

with $\mathcal{K} = NDN^{-1}$, $D = diag\left([d_1, ..., d_N]\right)$, and $\Xi_{i,j} = \left[(N^{-1}\Sigma)(N^{-1}\Sigma)'\right]_{i,j} \frac{1-\exp[-(d_i+d_j)\cdot t]}{(d_i+d_j)}$. The covariance matrix between η_t^x and η_t^q is given by

$$\Omega_{t-\Delta t}^{xq} = Cov_{t-\Delta t}\left[\eta_t^x, \eta_t^q\right] = \int_0^{\Delta t} \exp\left(-\mathcal{K}s\right)\Sigma\sigma_q ds = \mathcal{K}^{-1}\left(I - \exp\left(-\mathcal{K}s\Delta t\right)\right)\Sigma\sigma_q.$$

where I signifies the identity matrix.

Next, we consider the state variable v_t. From $dv_t = \kappa_v\left(\mu_v - v_t\right)dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t} + \rho dW_{\perp,t}\right)$, we have

$$v_t = e^{-\kappa_v\Delta t}v_{t-\Delta t} + \mu_v\left(1 - e^{-\kappa_v\Delta t}\right) + \eta_t^v$$

where $\eta_t^v = \sigma_v\int_{t-\Delta t}^t e^{\kappa_v(s-t)}\sqrt{v_s}\left(\sqrt{1-\rho^2}dB_s^v + \rho dB_s^{\perp}\right) \sim N\left(0, \Omega_{t-1}^v\right)$ and

$$\begin{aligned}
\Omega_{t-\Delta t}^v &= E_{t-\Delta t}\left[\sigma_v^2\int_{t-\Delta t}^t e^{2\kappa_v(s-t)}v_s ds\right] \\
&= \mu_v\sigma_v^2\frac{\left(1-e^{-\kappa_v\Delta t}\right)^2}{2\kappa_v} + v_{t-1}\sigma_v^2\frac{e^{-\kappa_v\Delta t} - e^{-2\kappa_v\Delta t}}{\kappa_v}
\end{aligned}$$

The covariance between η_t^v and η_t^x is zero by construction while the covariance between the inflation and volatility innovation terms is

$$\begin{aligned}
\Omega_{t-\Delta t}^{vq} &= Cov_{t-\Delta t}\left[\eta_t^v, \eta_t^q\right] = \rho\sigma_v E_{t-\Delta t}\left[\int_{t-\Delta t}^t e^{\kappa_v(s-t)}v_s ds\right] \\
&= \rho\sigma_v\left[\mu_v\frac{\left(1-e^{-\frac{\kappa_v}{2}\Delta t}\right)^2}{\kappa_v} + v_{t-\Delta t}\frac{e^{-\frac{\kappa_v}{2}\Delta t} - e^{-\kappa_v\Delta t}}{\kappa_v/2}\right].
\end{aligned}$$

Next, we consider the state variable δ_t. From $d\delta_t = \kappa_\delta\left(\mu_\delta - \delta_t\right)dt + \sigma_\delta dW_{\delta,t}$, we have

$$\delta_t = e^{-\kappa_\delta\Delta t}\delta_{t-\Delta t} + \mu_\delta\left(1 - e^{-\kappa_\delta\Delta t}\right) + \eta_t^\delta$$

where $\eta_t^\delta = \sigma_\delta\int_{t-\Delta t}^t e^{\kappa_\delta(u-t)}dW_{\delta,u} \sim N\left(0, \Omega_{t-\Delta t}^\delta\right)$ and

$$\Omega_{t-\Delta t}^\delta = E_{t-\Delta t}\left[\sigma_\delta^2\int_{t-\Delta t}^t e^{2\kappa_\delta(s-t)}ds\right] = \sigma_\delta^2\frac{1-e^{-2\kappa_\delta\Delta t}}{2\kappa_\delta}$$

Last, we consider the state variable s_t.

$$s_t - s_{t-\Delta t} = \left(\phi_0\Delta t + \left(\phi_x\Delta t\right)' x_{t-\Delta t} + \phi_v\Delta t v_{t-\Delta t} + \phi_\delta\Delta t\delta_{t-\Delta t} + \phi_s\Delta t s_{t-\Delta t}\right) + \eta_t^s$$

where $\eta_t^s = \sigma_s\int_{t-\Delta t}^t \exp\left(-\phi_s\left(u-t\right)\right)dW_{\delta,u} \sim N\left(0, \Omega_{t-\Delta t}^s\right)$ and

$$\Omega_{t-\Delta t}^s = E_{t-\Delta t}\left[\sigma_s^2\int_{t-\Delta t}^t e^{2(-\phi_s)(u-t)}du\right] = \sigma_s^2\frac{e^{2\phi_s\Delta t}-1}{2\phi_s}$$

and

$$
\begin{aligned}
\Omega_{t-\Delta t}^{\delta s} &= \sigma_\delta \sigma_s E_{t-\Delta t}\left[\int_{t-\Delta t}^{t} e^{\kappa_\delta (u-t)} dW_{\delta,u} \int_{t-\Delta t}^{t} e^{-\phi_s (u-t)} dW_{\delta,u}\right] \\
&= \sigma_\delta \sigma_s E_{t-\Delta t}\left[\int_{t-\Delta t}^{t} e^{(\kappa_\delta - \phi_s)(u-t)} du\right] \\
&= \sigma_\delta \sigma_s \frac{1 - e^{-(\kappa_\delta - \phi_s)\Delta t}}{\kappa_\delta - \phi_s}
\end{aligned}
$$

In summary, the dynamics of the state vector $Z_t = (q_t, x_t', v_t, \delta_t, s_t)'$ follows the VAR process

$$Z_t = \mathcal{A} + \mathcal{B} Z_{t-\Delta t} + \eta_t$$

where $\mathcal{A} = \begin{bmatrix} \rho_0^\pi \Delta t \\ (I - \exp(-\mathcal{K}\Delta t))\mu \\ \left(1 - e^{-\kappa_v \Delta t}\right)\mu_v \\ \left(1 - e^{-\kappa_\delta \Delta t}\right)\mu_\delta \\ \phi_0 \Delta t \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 1 & \rho_x^{\pi\prime}\Delta t & \rho_v^\pi \Delta t & \rho_\delta^\pi \Delta t & \rho_s^\pi \Delta t \\ 0 & \exp(-\mathcal{K}\Delta t) & & & \\ 0 & 0_{1\times 3} & e^{-\kappa_v \Delta t} & & \\ 0 & 0_{1\times 3} & 0 & e^{-\kappa_\delta \Delta t} & \\ 0 & \phi_x' \Delta t & \phi_v \Delta t & \phi_\delta \Delta t & 1 + \phi_s \Delta t \end{bmatrix}$, $\eta_t = \begin{pmatrix} \eta_t^q \\ \eta_t^x \\ \eta_t^v \\ \eta_t^\delta \\ \eta_t^s \end{pmatrix} \sim N(0, \Omega_{t-\Delta t})$ and $\Omega_{t-\Delta t} = \begin{bmatrix} \Omega_{t-\Delta t}^q & \Omega_{t-\Delta t}^{xq\prime} & \Omega_{t-\Delta t}^{vq} & & \\ \Omega_{t-\Delta t}^{xq} & \Omega_{t-\Delta t}^x & & & \\ \Omega_{t-\Delta t}^{vq} & & \Omega_{t-\Delta t}^v & & \\ & & & \Omega_{t-\Delta t}^\delta & \Omega_{t-\Delta t}^{\delta s} \\ & & & \Omega_{t-\Delta t}^{\delta s} & \Omega_{t-\Delta t}^s \end{bmatrix}$.