Documentation: MODEL "DKWv_o"

Nikolay Gospodinov and Bin Wei

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1 Model

Suppose that there are a vector of four latent variables $x_t = (x_{1t}, x_{2t}, x_{3t})'$ and v_t that drive nominal and real yields as well as inflation. Their dynamics under the physical measure is

$$
\begin{aligned}
dx_t &= \mathcal{K}(\mu - x_t)\,dt + \Sigma dW_{x,t} && (1)\\
dv_t &= \kappa_v(\mu_v - v_t)\,dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t} + \rho dW_{\perp,t}\right) && (2)
\end{aligned}
$$

The nominal pricing kernel takes the form

$$dM_t^N/M_t^N = -r_t^N dt - \Lambda_{x,t}^{N\prime} dW_{x,t} - \Lambda_{v,t}^N dW_{v,t} - \Lambda_{\perp,t}^N dW_{\perp,t} \quad (3)$$

where the nominal short rate is

$$r_t^N = \rho_0^N + \rho_x^{N\prime} x_t + \rho_v^N v_t \quad (4)$$

and the vector of prices of risk is given by

$$
\begin{aligned}
\Lambda_{x,t}^N &= \lambda_{0,x}^N + \lambda_x^N x_t\\
\Lambda_{v,t}^N &= \gamma_v\sqrt{v_t}\\
\Lambda_{\perp,t}^N &= \gamma_v^{\perp}\sqrt{v_t}
\end{aligned}
$$

Let $q_t \equiv \log Q_t$ denote the log price level. The price level evolves as follows:

$$dq_t = \pi_t dt + \sigma_q' dW_{x,t} + \sqrt{v_t} dW_{\perp,t} \quad (5)$$

where the instantaneous expected inflation rate is given by

$$\pi_t = \rho_0^\pi + \rho_x^{\pi\prime} x_t + \rho_v^\pi v_t \quad (6)$$

Real Pricing Kernel. Based on the observation $M_t^R = M_t^N Q_t$, we have

$$
\begin{aligned}
dM_t^R/M_t^R &= dM_t^N/M_t^N + dQ_t/Q_t + \left(dM_t^N/M_t^N\right)\left(dQ_t/Q_t\right)\\
&= -r_t^N dt - \Lambda_{x,t}^{N\prime} dW_{x,t} - \Lambda_{v,t}^N dW_{v,t} - \Lambda_{\perp,t}^N dW_{\perp,t}\\
&\quad + \left[\pi_t + \frac{1}{2}\left(\sigma_q'\sigma_q + v_t\right) - \sigma_q'\Lambda_{x,t}^N - \Lambda_{\perp,t}^N\sqrt{v_t}\right] dt + \sigma_q' dW_{x,t} + \sqrt{v_t} dW_{\perp,t}\\
&= -r_t^R dt - \left(\Lambda_{x,t}^N - \sigma_q\right)' dW_{x,t} - \Lambda_{v,t}^N dW_{v,t} - \left(\Lambda_{\perp,t}^N - \sqrt{v_t}\right) dW_{\perp,t},
\end{aligned}
$$

where the real short rate is

$$
\begin{aligned}
r_t^R &= r_t^N - \pi_t + \left(\sigma_q' \Lambda_{x,t}^N + \sqrt{v_t}\Lambda_{\perp,t}^N\right) - \frac{1}{2}\left(\sigma_q'\sigma_q + v_t\right) \\
&\equiv \rho_0^R + \rho_x^{R\prime} x_t + \rho_v^R v_t
\end{aligned}
$$

with coefficients

$$
\begin{aligned}
\rho_0^R &= \rho_0^N - \rho_0^\pi - \frac{1}{2}\sigma_q'\sigma_q + \lambda_0'\sigma_q, \\
\rho_x^R &= \rho_x^N - \rho_x^\pi + \lambda_x^{N\prime}\sigma_q, \\
\rho_v^R &= \rho_v^N - \rho_v^\pi + \gamma_q - \frac{1}{2}.
\end{aligned}
$$

The real prices of risk can be derived as

$$
\begin{aligned}
\Lambda_{x,t}^R &= \Lambda_{x,t}^N - \sigma_q \equiv \lambda_0^R + \lambda_x^{R\prime} x_t, \\
\Lambda_{v,t}^R &= \Lambda_{v,t}^N = \gamma_v \sqrt{v_t}, \\
\Lambda_{\perp,t}^R &= \Lambda_{\perp,t}^N - \sqrt{v_t} = \left(\gamma_v^\perp - 1\right)\sqrt{v_t}
\end{aligned}
$$

where $\lambda_0^R = \lambda_0^N - \sigma_q$ and $\lambda_x^R = \lambda_x^N$.

Risk-Neutral Measure. The Radon-Nikodym derivative of the risk neutral measure $\mathbb{P}^*$ with respect to the physical measure $\mathbb{P}$ is given by

$$
\left(\frac{d\mathbb{P}^*}{d\mathbb{P}}\right)_{t,T} = \exp\left[-\frac{1}{2}\int_t^T \Lambda_s^{N\prime}\Lambda_s^N ds - \int_t^T \Lambda_s^{N\prime} dW_s\right] \tag{7}
$$

where

$$
\Lambda_t^N \equiv \begin{pmatrix} \Lambda_{x,t}^N \\ \Lambda_{v,t}^N \\ \Lambda_{\perp,t}^N \end{pmatrix}, W_t \equiv \begin{pmatrix} W_{x,t} \\ W_{v,t} \\ W_{\perp,t} \end{pmatrix}
$$

By the Girsanov theorem, $dW_t^* = dW_t + \Lambda_t^N dt$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. It implies that under the risk neutral measure, the dynamics is given by

$$
\begin{aligned}
dx_t &= \mathcal{K}\left(\mu - x_t\right)dt + \Sigma dW_{x,t} \\
&= \mathcal{K}\left(\mu - x_t\right)dt + \Sigma\left(dW_{x,t}^* - \Lambda_{x,t}^N dt\right) \\
&= \left[\left(\mathcal{K}\mu - \Sigma\lambda_0^N\right) - \left(\mathcal{K} + \Sigma\lambda_x^N\right)x_t\right]dt + \Sigma dW_{x,t}^* \\
&\equiv \mathcal{K}^*\left(\mu^* - x_t\right)dt + \Sigma dW_{x,t}^*,
\end{aligned}
$$

and

$$
\begin{aligned}
dv_t &= \kappa_v\left(\mu_v - v_t\right)dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}\left(dW_{v,t}^* - \Lambda_{v,t}^N dt\right) + \rho\left(dW_{\perp,t}^* - \Lambda_{\perp,t}^N dt\right)\right) \\
&\equiv \kappa_v^*\left(\mu_v^* - v_t\right)dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t}^* + \rho dW_{\perp,t}^*\right)
\end{aligned}
$$

and

$$
\begin{aligned}
dq_t &= \left(\rho_0^\pi + \rho_x^{\pi\prime}x_t + \rho_v^\pi v_t\right)dt + \sigma_q'\left(dW_{x,t}^* - \Lambda_{x,t}^N dt\right) + \sqrt{v_t}\left(dW_{\perp,t}^* - \Lambda_{\perp,t}^N dt\right) \\
&\equiv \left(\rho_0^{\pi*} + \rho_x^{\pi*\prime}x_t + \rho_v^{\pi*}v_t\right)dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t}dW_{\perp,t}^*,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{K}^* &= \mathcal{K}+\Sigma\lambda_x^N \\
\mathcal{K}^*\mu^* &= \mathcal{K}\mu-\Sigma\lambda_0^N \\
\kappa_v^* &= \kappa_v + \sqrt{1-\rho^2}\sigma_v\gamma_v + \rho\sigma_v\gamma_\perp \\
\kappa_v^*\mu_v^* &= \kappa_v\mu_v \\
\pi_t^* &= \rho_0^{\pi*} + \rho_x^{\pi*\prime}x_t + \rho_v^{\pi*}v_t \\
\rho_0^{\pi*} &= \rho_0^\pi - \lambda_0^{N\prime}\sigma_q \\
\rho_x^{\pi*} &= \rho_x^\pi - \lambda_x^{N\prime}\sigma_q \\
\rho_v^{\pi*} &= \rho_v^\pi - \gamma_\perp
\end{aligned}
$$

Forward Measure. The Radon-Nikodym derivative of the forward measure $\tilde{\mathbb{P}}$ with respect to the risk neutral measure $\mathbb{P}^*$ is given by

$$
\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right)_{t,T} = \frac{\exp\left(-\int_t^T r_s^N ds\right)}{P_{t,\tau}^N} \tag{8}
$$

and

$$
\Psi_t \equiv E_t^{\mathbb{P}^*}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right)_{0,T}\right] = E_t^{\mathbb{P}^*}\left[\frac{\exp\left(-\int_0^T r_s^N ds\right)}{P_{0,T}^N}\right] = \frac{P_{t,\tau}^N}{P_{0,T}^N}\exp\left(-\int_0^t r_s^N ds\right)
$$

We have

$$
\begin{aligned}
d\Psi_t &= \frac{\exp\left(-\int_0^t r_s^N ds\right)}{P_{0,T}^N}\left[dP_{t,\tau}^N - r_t^N P_{t,\tau}^N dt\right] \\
&= \Psi_t\left[B_\tau^{N\prime}\Sigma dW_{x,t} + C_\tau^N\sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t} + \rho dW_{\perp,t}\right)\right]
\end{aligned}
$$

By Girsanov's Theorem, we have

$$
d\tilde{W}_t = dW_t^* - \frac{d\Psi_t}{\Psi_t}\cdot dW_t^*
$$

or

$$
\begin{aligned}
d\tilde{W}_{x,t} &= dW_t^* - \Sigma' B_\tau^N dt \\
d\tilde{W}_{v,t} &= dW_{v,t}^* - \sqrt{1-\rho^2}\sigma_v C_\tau^N\sqrt{v_t}dt \\
d\tilde{W}_{\perp,t} &= dW_{\perp,t}^* - \rho\sigma_v C_\tau^N\sqrt{v_t}dt.
\end{aligned}
$$

The dynamics of the state variables under the risk neutral measure is given by

$$
\begin{aligned}
dx_t &= \mathcal{K}^*\left(\mu^* - x_t\right) + \Sigma dW_{x,t}^* \\
dv_t &= \kappa_v^*\left(\mu_v^* - v_t\right)dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t}^* + \rho dW_{\perp,t}^*\right) \\
dq_t &= \left(\rho_0^{\pi*} + \rho_x^{\pi*\prime}x_t + \rho_v^{\pi*}v_t\right)dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t}dW_{\perp,t}^*.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
dx_t &= \mathcal{K}^*\left(\mu^* - x_t\right)dt + \Sigma\left(d\tilde{W}_{x,t} + \Sigma' B_\tau^N dt\right) \\
&= \left(\mathcal{K}^*\mu^* + \Sigma\Sigma' B_\tau^N - \mathcal{K}^* x_t\right)dt + \Sigma d\tilde{W}_{x,t} \\
&\equiv \tilde{\mathcal{K}}\left(\tilde{\mu} - x_t\right)dt + \Sigma d\tilde{W}_{x,t},
\end{aligned}
$$

$$
\begin{aligned}
dv_t &= \kappa_v^*\left(\mu_v^* - v_t\right)dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t}^* + \rho dW_{\perp,t}^*\right) \\
&= \left[\kappa_v^*\mu_v^* - \left(\kappa_v^* - \sigma_v^2 C_\tau^N\right)v_t\right]dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}d\tilde{W}_{v,t} + \rho d\tilde{W}_{\perp,t}\right) \\
&\equiv \tilde{\kappa}_v\left(\tilde{\mu}_v - v_t\right)dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}d\tilde{W}_{v,t} + \rho d\tilde{W}_{\perp,t}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
dq_t &= \left(\rho_0^{\pi *} + \rho_x^{\pi *\prime}x_t + \rho_v^{\pi *}v_t\right)dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t}dW_{\perp,t}^* \\
&= \left(\left[\rho_0^{\pi *} + \sigma_q'\Sigma' B_\tau^N\right] + \rho_x^{\pi *\prime}x_t + \left[\rho_v^{\pi *} + \rho\sigma_v C_\tau^N\right]v_t\right)dt + \sigma_q' d\tilde{W}_{x,t} + \sqrt{v_t}d\tilde{W}_{\perp,t} \\
&\equiv \left(\rho_0^{\pi} + \rho_x^{\pi\prime}x_t + \rho_v^{\pi}v_t\right)dt + \sigma_q' d\tilde{W}_{x,t} + \sqrt{v_t}d\tilde{W}_{\perp,t}
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{K}} &= \mathcal{K}^* = \mathcal{K} + \Sigma\lambda_x^N \\
\tilde{\mathcal{K}}\tilde{\mu} &= \mathcal{K}^*\mu^* + \Sigma\Sigma' B_\tau^N = \mathcal{K}\mu - \Sigma\lambda_0^N + \Sigma\Sigma' B_\tau^N \\
\tilde{\kappa}_v &= \kappa_v^* - \sigma_v^2 C_\tau^N = \kappa_v + \sqrt{1-\rho^2}\sigma_v\gamma_v + \rho\sigma_v\gamma_\perp - \sigma_v^2 C_\tau^N \\
\tilde{\kappa}_v\tilde{\mu}_v &= \kappa_v^*\mu_v^* = \kappa_v\mu_v \\
\tilde{\pi}_t &= \widetilde{\rho}_0^{\pi} + \widetilde{\rho}_x^{\pi\prime}x_t + \widetilde{\rho}_v^{\pi}v_t \\
\widetilde{\rho}_0^{\pi} &= \rho_0^{\pi *} + \sigma_q'\Sigma' B_\tau^N = \rho_0^{\pi} - \lambda_0^{N\prime}\sigma_q + \sigma_q'\Sigma' B_\tau^N \\
\widetilde{\rho}_x^{\pi} &= \rho_x^{\pi *} = \rho_x^{\pi} - \lambda_x^{N\prime}\sigma_q \\
\widetilde{\rho}_v^{\pi} &= \rho_v^{\pi *} + \rho\sigma_v C_\tau^N = \rho_v^{\pi} - \gamma_\perp + \rho\sigma_v C_\tau^N
\end{aligned}
$$

Inflation Options. Under the put-call parity, we can show that

$$
E_t^{\tilde{\mathbb{P}}}\left[\frac{Q_{t+\tau}}{Q_t}\right] = \frac{P_{t,\tau,K}^{CAP} - P_{t,\tau,K}^{FLO}}{P_{t,\tau}^N} + (1+K)^\tau
$$

Note that

$$
\begin{aligned}
\frac{1}{\tau}\log E_t^{\tilde{\mathbb{P}}}\left[\frac{Q_{t+\tau}}{Q_t}\right] &= \frac{1}{\tau}\left(E_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau} - q_t\right] + \frac{1}{2}Var_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau} - q_t\right]\right) \\
&\equiv \mathcal{IE}_{t,\tau} + \frac{1}{2}\mathcal{IU}_{t,\tau}
\end{aligned}
$$

We first derive $\mathcal{IE}_{t,\tau} \equiv \frac{1}{\tau}E_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau} - q_t\right]$. Note

$$
\begin{aligned}
q_{t+\tau} - q_t &= \int_t^{t+\tau}\left[\left(\widetilde{\rho}_0^{\pi} + \widetilde{\rho}_x^{\pi\prime}x_s + \widetilde{\rho}_v^{\pi}v_s\right)ds + \sigma_q' d\widetilde{W}_{x,s} + \sqrt{v_s}d\widetilde{W}_{\perp,s}\right] \\
x_s &= \widetilde{\mu} + \exp\left(-\widetilde{\mathcal{K}}(s-t)\right)(x_t - \widetilde{\mu}) + \int_t^s \exp\left(-\widetilde{\mathcal{K}}(s-u)\right)\Sigma d\widetilde{W}_{x,u} \\
v_s &= \widetilde{\mu}_v + \exp\left(-\widetilde{\kappa}_v(s-t)\right)(v_t - \widetilde{\mu}_v) + \int_t^s \exp\left(-\widetilde{\kappa}_v(s-u)\right)\sigma_v\sqrt{v_u}\left(\sqrt{1-\rho^2}d\widetilde{W}_{v,u} + \rho d\widetilde{W}_{\perp,u}\right)
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{1}{\tau}\int_t^{t+\tau} E_t^{\tilde{\mathbb{P}}}\left[x_s\right]ds &= \frac{1}{\tau}\int_t^{t+\tau}\left[\widetilde{\mu} + \exp\left(-\widetilde{\mathcal{K}}(s-t)\right)(x_t - \widetilde{\mu})\right]ds \\
&= \widetilde{\mu} + \left(\tilde{\mathcal{K}}\tau\right)^{-1}\left(I - \exp\left(-\tilde{\mathcal{K}}\tau\right)\right)(x_t - \widetilde{\mu}) \\
&\equiv \tilde{a}_\tau^x + \tilde{b}_\tau^x x_t
\end{aligned}
$$

and similarly,

$$\frac{1}{\tau}\int_t^{t+\tau} E_t^{\tilde{\mathbb{P}}}\left[v_s\right] ds \equiv \tilde{a}_\tau^v + \tilde{b}_\tau^v v_t$$

where $\tilde{a}_\tau^x = \left(I - \tilde{b}_\tau^x\right)\tilde{\mu}$ and $\tilde{b}_\tau^x = \left(\tilde{\mathcal{K}}\tau\right)^{-1}\left(I - \exp\left(-\tilde{\mathcal{K}}\tau\right)\right)$, and $\tilde{a}_\tau^v$, $\tilde{b}_\tau^v$ are similarly defined. Therefore,

$$\mathcal{IE}_{t,\tau} \equiv \frac{1}{\tau}E_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau} - q_t\right] = \widetilde{\rho}_0^\pi + \widetilde{\rho}_x^{\pi\prime}\left(\tilde{a}_\tau^x + \tilde{b}_\tau^x x_t\right) + \widetilde{\rho}_v^\pi\left(\tilde{a}_\tau^v + \tilde{b}_\tau^v v_t\right)$$

Next we derive $\mathcal{IU}_{t,\tau} \equiv \frac{1}{\tau}Var_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau} - q_t\right]$. Note that:

$$\begin{aligned}
& \left[q_{t+\tau} - q_t\right] - E_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau} - q_t\right] \\
= \; & \int_t^{t+\tau}\left[\widetilde{\rho}_x^{\pi\prime}\left(x_s - E_t^{\tilde{\mathbb{P}}}\left[x_s\right]\right)ds + \widetilde{\rho}_v^\pi\left(v_s - E_t^{\tilde{\mathbb{P}}}\left[v_s\right]\right) + \sigma_q' d\widetilde{W}_{x,s} + \sqrt{v_s}d\widetilde{W}_{\perp,s}\right] \\
= \; & \int_t^{t+\tau}\widetilde{\rho}_x^{\pi\prime}\int_t^s \exp\left(-\widetilde{\mathcal{K}}\left(s-u\right)\right)\Sigma d\widetilde{W}_{x,u}ds \\
& + \int_t^{t+\tau}\widetilde{\rho}_v^\pi\int_t^s \exp\left(-\widetilde{\kappa}_v\left(s-u\right)\right)\sigma_v\sqrt{v_u}\left(\sqrt{1-\rho^2}d\widetilde{W}_{v,u} + \rho d\widetilde{W}_{\perp,u}\right)ds \\
& + \int_t^{t+\tau}\left[\sigma_q' d\widetilde{W}_{x,s} + \sqrt{v_s}d\widetilde{W}_{\perp,s}\right] \\
= \; & \widetilde{\rho}_x^{\pi\prime}\int_t^{t+\tau}\int_u^{t+\tau}\exp\left(-\widetilde{\mathcal{K}}\left(s-u\right)\right)ds\Sigma d\widetilde{W}_{x,u} \\
& + \widetilde{\rho}_v^\pi\sigma_v\int_t^{t+\tau}\int_u^{t+\tau}\exp\left(-\widetilde{\kappa}_v\left(s-u\right)\right)ds\sqrt{v_u}\left(\sqrt{1-\rho^2}d\widetilde{W}_{v,u} + \rho d\widetilde{W}_{\perp,u}\right) \\
& + \int_t^{t+\tau}\left[\sigma_q' d\widetilde{W}_{x,s} + \sqrt{v_s}d\widetilde{W}_{\perp,s}\right] \\
= \; & \widetilde{\rho}_x^{\pi\prime}\int_t^{t+\tau}\left(\widetilde{\mathcal{K}}\right)^{-1}\left(I - \exp\left(-\widetilde{\mathcal{K}}\left(t+\tau-s\right)\right)\right)\Sigma d\widetilde{W}_{x,s} \\
& + \widetilde{\rho}_v^\pi\sigma_v\int_t^{t+\tau}\left(\kappa_v\right)^{-1}\left(1 - \exp\left(-\widetilde{\kappa}_v\left(t+\tau-s\right)\right)\right)\sqrt{v_s}\left(\sqrt{1-\rho^2}d\widetilde{W}_{v,s} + \rho d\widetilde{W}_{\perp,s}\right) \\
& + \int_t^{t+\tau}\left[\sigma_q' d\widetilde{W}_{x,s} + \sqrt{v_s}d\widetilde{W}_{\perp,s}\right] \\
= \; & \int_t^{t+\tau}\left[\widetilde{\rho}_x^{\pi\prime}\left(\widetilde{\mathcal{K}}\right)^{-1}\left(I - \exp\left(-\widetilde{\mathcal{K}}\left(t+\tau-s\right)\right)\right) + \sigma_q'\right]\Sigma d\widetilde{W}_{x,s} \\
& + \int_t^{t+\tau}\left[\rho\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\left(1 - \exp\left(-\widetilde{\kappa}_v\left(t+\tau-s\right)\right)\right) + 1\right]\sqrt{v_s}d\widetilde{W}_{\perp,s} \\
& + \sqrt{1-\rho^2}\widetilde{\rho}_v^\pi\sigma_v\int_t^{t+\tau}\left(\widetilde{\kappa}_v\right)^{-1}\left(1 - \exp\left(-\widetilde{\kappa}_v\left(t+\tau-s\right)\right)\right)\sqrt{v_s}d\widetilde{W}_{v,s} \\
= \; & \int_t^{t+\tau}\left(\widehat{\sigma}_q - \Sigma'\exp\left(-\widetilde{\mathcal{K}}'\left(t+\tau-s\right)\right)\left(\widetilde{\mathcal{K}}'\right)^{-1}\widetilde{\rho}_x^\pi\right)' d\widetilde{W}_{x,s} \\
& + \int_t^{t+\tau}\left[\left(1 + \rho\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right) - \rho\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}e^{-\widetilde{\kappa}_v\left(t+\tau-s\right)}\right]\sqrt{v_s}d\widetilde{W}_{\perp,s} \\
& + \sqrt{1-\rho^2}\widetilde{\rho}_v^\pi\sigma_v\int_t^{t+\tau}\left(\widetilde{\kappa}_v\right)^{-1}\left(1 - \exp\left(-\widetilde{\kappa}_v\left(t+\tau-s\right)\right)\right)\sqrt{v_s}d\widetilde{W}_{v,s}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathcal{IU}_{t,\tau} &\equiv \frac{1}{\tau}Var_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau}-q_t\right]=\frac{1}{\tau}E_t^{\tilde{\mathbb{P}}}\left[\left(\left[q_{t+\tau}-q_t\right]-E_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau}-q_t\right]\right)^2\right] \\
&= \frac{1}{\tau}E_t^{\tilde{\mathbb{P}}}\left[\left(\begin{array}{c}\int_t^{t+\tau}\left(\widehat{\sigma}_q-\Sigma'\exp\left(-\widetilde{\mathcal{K}}'\left(t+\tau-s\right)\right)\left(\widetilde{\mathcal{K}}'\right)^{-1}\widetilde{\rho}_x^{\pi}\right)'d\widetilde{W}_{x,s} \\
+\int_t^{t+\tau}\left[\left(1+\rho\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)-\rho\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}e^{-\widetilde{\kappa}_v(t+\tau-s)}\right]\sqrt{v_s}d\widetilde{W}_{\perp,s} \\
+\sqrt{1-\rho^2}\widetilde{\rho}_v^{\pi}\sigma_v\int_t^{t+\tau}\left(\widetilde{\kappa}_v\right)^{-1}\left(1-\exp\left(-\widetilde{\kappa}_v\left(t+\tau-s\right)\right)\right)\sqrt{v_s}d\widetilde{W}_{v,s}\end{array}\right)^2\right] \\
&= \frac{1}{\tau}E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau}\left(\widehat{\sigma}_q-\Sigma'\exp\left(-\widetilde{\mathcal{K}}'\left(t+\tau-s\right)\right)\left(\widetilde{\mathcal{K}}'\right)^{-1}\widetilde{\rho}_x^{\pi}\right)'\left(\widehat{\sigma}_q-\Sigma'\exp\left(-\widetilde{\mathcal{K}}'\left(t+\tau-s\right)\right)\left(\widetilde{\mathcal{K}}'\right)^{-1}\widetilde{\rho}_x^{\pi}\right)ds\right. \\
&\quad +\int_t^{t+\tau}\left[\left(1+\rho\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)-\rho\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}e^{-\widetilde{\kappa}_v(t+\tau-s)}\right]^2v_sds \\
&\quad \left.+\left(\sqrt{1-\rho^2}\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\int_t^{t+\tau}\left(1-\exp\left(-\widetilde{\kappa}_v\left(t+\tau-s\right)\right)\right)^2v_sds\right] \\
&\equiv H_0+\frac{1}{\tau}\int_t^{t+\tau}\left\{\begin{array}{c}\left[\left(1+\rho\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2+\left(\sqrt{1-\rho^2}\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right] \\
-2\left[\left(1+\rho\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)\rho\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}+\left(\sqrt{1-\rho^2}\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]e^{-\widetilde{\kappa}_v(t+\tau-s)} \\
+\left[\left(\rho\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2+\left(\sqrt{1-\rho^2}\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]e^{-2\widetilde{\kappa}_v(t+\tau-s)}\end{array}\right\}E_t^{\tilde{\mathbb{P}}}\left[v_s\right]ds \\
&= H_0+\frac{1}{\tau}\int_t^{t+\tau}\left\{\begin{array}{c}\left[\left(1+\rho\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2+\left(\sqrt{1-\rho^2}\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right] \\
-2\left[\rho+\left(\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)\right]\left(\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)e^{-\widetilde{\kappa}_v(t+\tau-s)} \\
+\left[\left(\widetilde{\rho}_v^{\pi}\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]e^{-2\widetilde{\kappa}_v(t+\tau-s)}\end{array}\right\}\left\{\widetilde{\mu}_v+e^{-\widetilde{\kappa}_v(s-t)}\left(v_t-\widetilde{\mu}_v\right)\right\}ds \\
&\equiv H_0+\widetilde{\mu}_vH_1+H_2\left(v_t-\widetilde{\mu}_v\right)
\end{aligned}
$$

where

$$\widehat{\sigma}_q'=\sigma_q'+\widetilde{\rho}_x^{\pi\prime}\left(\mathcal{K}\right)^{-1}\Sigma \text{ or } \widehat{\sigma}_q=\sigma_q+\Sigma'\left(\mathcal{K}'\right)^{-1}\widetilde{\rho}_x^{\pi}$$

$$
\begin{aligned}
H_0 &\equiv \frac{1}{\tau}\int_t^{t+\tau}\left(\widehat{\sigma}_q-\Sigma'\exp\left(-\widetilde{\mathcal{K}}'\left(t+\tau-s\right)\right)\left(\widetilde{\mathcal{K}}'\right)^{-1}\widetilde{\rho}_x^{\pi}\right)'\left(\widehat{\sigma}_q-\Sigma'\exp\left(-\widetilde{\mathcal{K}}'\left(t+\tau-s\right)\right)\left(\widetilde{\mathcal{K}}'\right)^{-1}\widetilde{\rho}_x^{\pi}\right)ds \\
&= \widehat{\sigma}_q'\widehat{\sigma}_q-2\widehat{\sigma}_q'\Sigma'\left[\frac{1}{\tau}\int_t^{t+\tau}\exp\left(-\widetilde{\mathcal{K}}'\left(t+\tau-s\right)\right)ds\right]\left(\widetilde{\mathcal{K}}'\right)^{-1}\widetilde{\rho}_x^{\pi} \\
&\quad +\rho_x^{\pi\prime}\left(\widetilde{\mathcal{K}}\right)^{-1}\left[\frac{1}{\tau}\int_t^{t+\tau}\exp\left(-\widetilde{\mathcal{K}}\left(t+\tau-s\right)\right)\Sigma\Sigma'\exp\left(-\widetilde{\mathcal{K}}'\left(t+\tau-s\right)\right)ds\right]\left(\widetilde{\mathcal{K}}'\right)^{-1}\widetilde{\rho}_x^{\pi} \\
&= \widehat{\sigma}_q'\widehat{\sigma}_q-2\widehat{\sigma}_q'\Sigma'\left(\widetilde{\mathcal{K}}'\tau\right)^{-1}\left(I-\exp\left(-\widetilde{\mathcal{K}}'\tau\right)\right)\left(\widetilde{\mathcal{K}}'\right)^{-1}\widetilde{\rho}_x^{\pi}+\rho_x^{\pi\prime}\left(\widetilde{\mathcal{K}}\right)^{-1}\left(\frac{1}{\tau}\widetilde{\Omega}_{\tau}^x\right)\left(\widetilde{\mathcal{K}}'\right)^{-1}\widetilde{\rho}_x^{\pi}
\end{aligned}
$$

and

$$
\begin{aligned}
H_1 &\equiv \frac{1}{\tau}\int_t^{t+\tau}\left\{\begin{array}{c}\left[1+2\rho\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)+\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]\\ -2\left[\rho+\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)\right]\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)e^{-\widetilde{\kappa}_v(t+\tau-s)}\\ +\left[\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]e^{-2\widetilde{\kappa}_v(t+\tau-s)}\end{array}\right\}ds\\
&= \left[1+2\rho\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)+\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]\\
&\quad -2\left[\rho+\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)\right]\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)\left(\widetilde{\kappa}_v\tau\right)^{-1}\left(1-\exp\left(-\widetilde{\kappa}_v\tau\right)\right)\\
&\quad +\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\left(2\widetilde{\kappa}_v\tau\right)^{-1}\left(1-\exp\left(-2\widetilde{\kappa}_v\tau\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
H_2 &\equiv \frac{1}{\tau}\int_t^{t+\tau}\left\{\begin{array}{c}\left[1+2\rho\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)+\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]\\ -2\left[\rho+\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)\right]\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)e^{-\widetilde{\kappa}_v(t+\tau-s)}\\ +\left[\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]e^{-2\widetilde{\kappa}_v(t+\tau-s)}\end{array}\right\}e^{-\widetilde{\kappa}_v(s-t)}ds\\
&= e^{-\widetilde{\kappa}_v\tau}\frac{1}{\tau}\int_t^{t+\tau}\left\{\begin{array}{c}\left[1+2\rho\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)+\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]\\ -2\left[\rho+\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)\right]\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)e^{-\widetilde{\kappa}_v(t+\tau-s)}\\ +\left[\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]e^{-2\widetilde{\kappa}_v(t+\tau-s)}\end{array}\right\}e^{\widetilde{\kappa}_v(t+\tau-s)}ds\\
&= e^{-\widetilde{\kappa}_v\tau}\left\{\begin{array}{c}\left[1+2\rho\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)+\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]\left(\widetilde{\kappa}_v\tau\right)^{-1}\left(\exp\left(\widetilde{\kappa}_v\tau\right)-1\right)\\ -2\left[\rho+\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)\right]\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)\\ +\left[\left(\widetilde{\rho}_v^\pi\sigma_v\left(\widetilde{\kappa}_v\right)^{-1}\right)^2\right]\left(\widetilde{\kappa}_v\tau\right)^{-1}\left(1-\exp\left(-\widetilde{\kappa}_v\tau\right)\right)\end{array}\right\}
\end{aligned}
$$

where we denote $\widetilde{\Omega}_\tau^x \equiv \int_t^{t+\tau}\exp\left(-\widetilde{\mathcal{K}}\left(t+\tau-s\right)\right)\Sigma\Sigma'\exp\left(-\widetilde{\mathcal{K}}'\left(t+\tau-s\right)\right)ds = \int_0^\tau \exp\left(-\widetilde{\mathcal{K}}s\right)\Sigma\Sigma'\exp\left(-\widetilde{\mathcal{K}}'s\right)ds$, satisfying

$$
vec\left(\widetilde{\Omega}_\tau^x\right) = -\left[\left(\widetilde{\mathcal{K}}\otimes I\right)+\left(I\otimes\widetilde{\mathcal{K}}\right)\right]^{-1} vec\left(\exp\left(-\widetilde{\mathcal{K}}\tau\right)\Sigma\Sigma'\exp\left(-\widetilde{\mathcal{K}}'\tau\right)-\Sigma\Sigma'\right)
$$

Therefore,

$$
\begin{aligned}
\frac{1}{\tau}\log E_t^{\tilde{\mathbb{P}}}\left[\frac{Q_{t+\tau}}{Q_t}\right] &= \frac{1}{\tau}\left(E_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau}-q_t\right]+\frac{1}{2}Var_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau}-q_t\right]\right)\\
&= \mathcal{IE}_{t,\tau}+\frac{1}{2}\mathcal{IU}_{t,\tau}\\
&= \widetilde{\rho}_0^\pi+\widetilde{\rho}_x^{\pi\prime}\left(\tilde{a}_\tau^x+\tilde{b}_\tau^x x_t\right)+\widetilde{\rho}_v^\pi\left(\tilde{a}_\tau^v+\tilde{b}_\tau^v v_t\right)+\frac{1}{2}\left(H_0+\widetilde{\mu}_v H_1+H_2\left(v_t-\widetilde{\mu}_v\right)\right)\\
&\equiv \tilde{a}_\tau^\pi+\tilde{b}_\tau^{\pi\prime}x_t+\widetilde{c}_\tau^\pi v_t
\end{aligned}
$$

where

$$\begin{aligned}
\tilde{a}^{\pi}_{\tau} &= \widetilde{\rho}^{\pi}_{0} + \widetilde{\rho}^{\pi\prime}_{x} \tilde{a}^{x}_{\tau} + \widetilde{\rho}^{\pi}_{v} \tilde{a}^{v}_{\tau} + \frac{1}{2} \left(H_0 + \widetilde{\mu}_v \left(H_1 - H_2 \right) \right) \\
\tilde{b}^{\pi}_{\tau} &= \tilde{b}^{x\prime}_{\tau} \widetilde{\rho}^{\pi}_{x} \\
\widetilde{c}^{\pi}_{\tau} &= \tilde{b}^{v}_{\tau} \widetilde{\rho}^{\pi}_{v} + \frac{1}{2} H_2
\end{aligned}$$

We now turn to the pricing of inflation options. Under the approximation (or exact?)

$$q_{t+\tau} - q_t|_{\mathcal{F}_t} \sim N\left(\tau \cdot \mathcal{IE}_{t,\tau}, \tau \cdot \mathcal{IU}_{t,\tau} \right),$$

we can derive the pricing formula for inflation caps and floors:

$$\begin{aligned}
P^{CAP}_{t,\tau,K} &= \exp\left(-\tau y^{N}_{t,\tau} \right) E^{\tilde{\mathbb{P}}}_{t} \left[\left(\frac{Q_{t+\tau}}{Q_t} - (1+K)^{\tau} \right)^{+} \right] \\
&= P^{N}_{t,\tau} \left[\begin{array}{c} e^{\tau\left(\mathcal{IE}_{t,\tau} + \frac{1}{2}\mathcal{IU}_{t,\tau} \right)} \Phi\left(\frac{-\log(1+K) + (\mathcal{IE}_{t,\tau} + \mathcal{IU}_{t,\tau})}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}} \right) \\ -\left(1+K\right)^{\tau} \Phi\left(\frac{-\log(1+K) + \mathcal{IE}_{t,\tau}}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}} \right) \end{array} \right]
\end{aligned}$$

and

$$\begin{aligned}
P^{FLO}_{t,\tau,K} &= \exp\left(-\tau y^{N}_{t,\tau} \right) E^{\tilde{\mathbb{P}}}_{t} \left[\left((1+K)^{\tau} - \frac{Q_{t+\tau}}{Q_t} \right)^{+} \right] \\
&= P^{N}_{t,\tau} \left[\begin{array}{c} -e^{\tau\left(\mathcal{IE}_{t,\tau} + \frac{1}{2}\mathcal{IU}_{t,\tau} \right)} \Phi\left(-\frac{-\log(1+K) + (\mathcal{IE}_{t,\tau} + \mathcal{IU}_{t,\tau})}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}} \right) \\ +\left(1+K\right)^{\tau} \Phi\left(-\frac{-\log(1+K) + \mathcal{IE}_{t,\tau}}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}} \right) \end{array} \right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
p^{CAP}_{t,\tau,K} &\equiv \ln\left(P^{CAP}_{t,\tau,K} \right) \\
&= -\tau \left[a^{N}_{\tau} + b^{N\prime}_{\tau} x_t \right] + \ln\left(\left[\exp\left(h_0\left(\tau, x_t, v_t\right) \right) \Phi\left(h_1\left(\tau, x_t, v_t\right) \right) - (1+K)^{\tau} \Phi\left(h_2\left(\tau, x_t, v_t\right) \right) \right] \right) \qquad (9)
\end{aligned}$$

where

$$\begin{aligned}
h_0 &= \tau \left(\mathcal{IE}_{t,\tau} + \frac{1}{2} \mathcal{IU}_{t,\tau} \right) \\
h_1 &= \frac{-\log\left(1+K\right) + \left(\mathcal{IE}_{t,\tau} + \mathcal{IU}_{t,\tau} \right)}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}} \\
h_2 &= \frac{-\log\left(1+K\right) + \mathcal{IE}_{t,\tau}}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}}
\end{aligned}$$

implying

$$\begin{aligned}
h_{0,x} &= \tau \tilde{b}^{\pi}_{\tau} \\
h_{1,x} &= \frac{1}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}} \tau \tilde{b}^{\pi}_{\tau} \\
h_{2,x} &= \frac{1}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}} \tau \tilde{b}^{\pi}_{\tau}
\end{aligned}$$