Documentation: MODEL_DKW_options

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1 Model

Suppose that there are a vector of four latent variables $x_t = (x_{1t}, x_{2t}, x_{3t})'$ and v_t that drive nominal and real yields as well as inflation. Their dynamics under the physical measure is

$$dx_t = \mathcal{K}\left(\mu - x_t\right) dt + \Sigma dW_{x,t} \tag{1}$$

The nominal pricing kernel takes the form

$$dM_t^N / M_t^N = -r_t^N dt - \Lambda_{x,t}^{N\prime} dW_{x,t} \tag{2}$$

where the nominal short rate is

$$r_t^N = \rho_0^N + \rho_x^{N\prime} x_t \tag{3}$$

and the vector of prices of risk is given by

$$\Lambda_{x,t}^N = \lambda_{0,x}^N + \lambda_x^N x_t$$

Let $q_t \equiv \log Q_t$ denote the log price level. The price level evolves as follows:

$$dq_t = \pi_t dt + \sigma_q' dW_{x,t} + \sigma_\perp dW_{\perp,t} \tag{4}$$

where the instantaneous expected inflation rate is given by

$$\pi_t = \rho_0^\pi + \rho_x^{\pi\prime} x_t \tag{5}$$

Note

$$\begin{aligned} q_{t+\tau} - q_t &= \int_t^{t+\tau} \left[\left(\rho_0^\pi + \rho_x^{\pi\prime} x_s\right) ds + \sigma_q' dW_{x,s} + \sigma_\perp dW_{\perp,s} \right] \\ x_s &= \mu + \exp\left(-\mathcal{K}\left(s-t\right)\right)\left(x_t - \mu\right) + \int_t^s \exp\left(-\mathcal{K}\left(s-u\right)\right) \Sigma dW_{x,u} \end{aligned}$$

implying

$$\begin{aligned} E_t\left[q_{t+\tau} - q_t\right] &= \int_t^{t+\tau} \left(\rho_0^\pi + \rho_x^{\pi\prime} E_t\left[x_s\right]\right) ds = \int_t^{t+\tau} \left(\rho_0^\pi + \rho_x^{\pi\prime}\left[\mu + e^{-\mathcal{K}(s-t)}\left(x_t - \mu\right)\right]\right) ds \\ &= \left(\rho_0^\pi + \rho_x^{\pi\prime}\mu\right)\tau + \rho_x^{\pi\prime}\left(\mathcal{K}\right)^{-1}\left(I - e^{-\mathcal{K}\tau}\right)\left(x_t - \mu\right) \end{aligned}$$

$$
\begin{aligned}
Var_t\left[q_{t+\tau}-q_t\right] &= E_t\left[\left(\left[q_{t+\tau}-q_t\right]-E_t\left[q_{t+\tau}-q_t\right]\right)^2\right] \\
&= E_t\left[\left(\int_t^{t+\tau}\left[\rho_x^{\pi\prime}\left(x_s-E_t\left[x_s\right]\right)ds+\sigma_q' dW_{x,s}+\sigma_\perp dW_{\perp,s}\right]\right)^2\right] \\
&= E_t\left[\left(\int_t^{t+\tau}\left[\rho_x^{\pi\prime}\int_t^s \exp\left(-\mathcal{K}\left(s-u\right)\right)\Sigma dW_{x,u}ds+\int_t^{t+\tau}\left[\sigma_q' dW_{x,s}+\sigma_\perp dW_{\perp,s}\right]\right]\right)^2\right] \\
&= E_t\left[\left(\rho_x^{\pi\prime}\int_t^{t+\tau}\int_t^s \exp\left(-\mathcal{K}\left(s-u\right)\right)\Sigma dW_{x,u}ds+\int_t^{t+\tau}\left[\sigma_q' dW_{x,s}+\sigma_\perp dW_{\perp,s}\right]\right)^2\right] \\
&= E_t\left[\left(\rho_x^{\pi\prime}\int_t^{t+\tau}\int_u^{t+\tau} \exp\left(-\mathcal{K}\left(s-u\right)\right)ds\Sigma dW_{x,u}+\int_t^{t+\tau}\left[\sigma_q' dW_{x,s}+\sigma_\perp dW_{\perp,s}\right]\right)^2\right] \\
&= E_t\left[\left(\rho_x^{\pi\prime}\int_t^{t+\tau}\left(\mathcal{K}\right)^{-1}\left(I-\exp\left(-\mathcal{K}\left(t+\tau-s\right)\right)\right)\Sigma dW_{x,s}+\int_t^{t+\tau}\left[\sigma_q' dW_{x,s}+\sigma_\perp dW_{\perp,s}\right]\right)^2\right] \\
&\equiv E_t\left[\left(\int_t^{t+\tau}\left[\left(\widehat{\sigma}_q-\Sigma' e^{-\mathcal{K}'(t+\tau-s)}\left(\mathcal{K}'\right)^{-1}\rho_x^\pi\right)' dW_{x,s}+\sigma_\perp dW_{\perp,s}\right]\right)^2\right] \\
&= \int_t^{t+\tau}\left(\widehat{\sigma}_q-\Sigma' e^{-\mathcal{K}'(t+\tau-s)}\left(\mathcal{K}'\right)^{-1}\rho_x^\pi\right)'\left(\widehat{\sigma}_q-\Sigma' e^{-\mathcal{K}'(t+\tau-s)}\left(\mathcal{K}'\right)^{-1}\rho_x^\pi\right)d_s+\sigma_\perp^2\tau \\
&= \left(\widehat{\sigma}_q'\widehat{\sigma}_q+\sigma_\perp^2\right)\tau-2\widehat{\sigma}_q'\Sigma'\left[\int_t^{t+\tau}e^{-\mathcal{K}'(t+\tau-s)}ds\right]\left(\mathcal{K}'\right)^{-1}\rho_x^\pi \\
&\quad +\rho_x^{\pi\prime}\left(\mathcal{K}\right)^{-1}\left[\int_t^{t+\tau}e^{-\mathcal{K}(t+\tau-s)}\Sigma\Sigma' e^{-\mathcal{K}'(t+\tau-s)}ds\right]\left(\mathcal{K}'\right)^{-1}\rho_x^\pi
\end{aligned}
$$

where

$$
\widehat{\sigma}_q'=\sigma_q'+\rho_x^{\pi\prime}\left(\mathcal{K}\right)^{-1}\Sigma \text{ or } \widehat{\sigma}_q=\sigma_q+\Sigma'\left(\mathcal{K}'\right)^{-1}\rho_x^\pi
$$

and suppose $\mathcal{K}$ can be diagonalized as $\mathcal{K}=NDN^{-1}$, where $D=diag\left([d_1,...,d_N]\right)$, then, denote $\Sigma^*=N^{-1}\Sigma$

$$
\begin{aligned}
\Omega_\tau^x &\equiv \int_t^{t+\tau}e^{-\mathcal{K}(t+\tau-s)}\Sigma\Sigma' e^{-\mathcal{K}'(t+\tau-s)}ds \\
&= \int_0^\tau e^{-\mathcal{K}s}\Sigma\Sigma' e^{-\mathcal{K}'s}ds \\
&= \int_0^\tau N\exp\left(-Ds\right)N^{-1}\Sigma\Sigma'\left(N^{-1}\right)'\exp\left(-Ds\right)N' ds \\
&= N\int_0^\tau \exp\left(-Ds\right)\Sigma^*\Sigma^{*\prime}\exp\left(-Ds\right)dsN' \\
&\equiv N\Theta N'
\end{aligned}
$$

The $(i,j)_{th}$ element of Θ is given by

$$
\begin{aligned}
\Theta_{i,j} &= \int_0^{\cdot t}e^{-d_i s}\left[\Sigma^*\Sigma^{*\prime}\right]_{i,j}e^{-d_j s}ds \\
&= \left[\Sigma^*\Sigma^{*\prime}\right]_{i,j}\int_0^{\cdot t}e^{-(d_i+d_j)s}ds \\
&= \left[\Sigma^*\Sigma^{*\prime}\right]_{i,j}\frac{1-\exp\left[-\left(d_i+d_j\right)\cdot t\right]}{\left(d_i+d_j\right)}
\end{aligned}
$$

Alternatively,

$$vec\left(\Omega_\tau^x\right) = -\left[\left(\mathcal{K}\otimes I\right)+\left(I\otimes\mathcal{K}\right)\right]^{-1} vec\left(e^{-\mathcal{K}\tau}\Sigma\Sigma' e^{-\mathcal{K}'\tau}-\Sigma\Sigma'\right)$$

Therefore,

$$\begin{aligned}
& Var_t\left[q_{t+\tau}-q_t\right] \\
= \;& \left(\widehat{\sigma}_q'\widehat{\sigma}_q+\sigma_\perp^2\right)\tau - 2\widehat{\sigma}_q'\Sigma'\left[\int_t^{t+\tau} e^{-\mathcal{K}'(t+\tau-s)}ds\right]\left(\mathcal{K}'\right)^{-1}\rho_x^\pi \\
& +\rho_x^{\pi\prime}\left(\mathcal{K}\right)^{-1}\left[\int_t^{t+\tau} e^{-\mathcal{K}(t+\tau-s)}\Sigma\Sigma' e^{-\mathcal{K}'(t+\tau-s)}ds\right]\left(\mathcal{K}'\right)^{-1}\rho_x^\pi \\
= \;& \left(\widehat{\sigma}_q'\widehat{\sigma}_q+\sigma_\perp^2\right)\tau - 2\widehat{\sigma}_q'\Sigma'\left(\mathcal{K}'\right)^{-1}\left(I-e^{-\mathcal{K}'\tau}\right)\left(\mathcal{K}'\right)^{-1}\rho_x^\pi \\
& +\rho_x^{\pi\prime}\left(\mathcal{K}\right)^{-1}\Omega_\tau^x\left(\mathcal{K}'\right)^{-1}\rho_x^\pi
\end{aligned}$$

Inflation options:

$$E_t^{\tilde{\mathbb{P}}}\left[\frac{Q_{t+\tau}}{Q_t}\right] = E_t^{\tilde{\mathbb{P}}}\left[\exp\left(q_{t+\tau}-q_t\right)\right] = \exp\left(E_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau}-q_t\right]+\frac{1}{2}Var_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau}-q_t\right]\right)$$

Thus

$$\begin{aligned}
& \frac{1}{\tau}\log\left(E_t^{\tilde{\mathbb{P}}}\left[\frac{Q_{t+\tau}}{Q_t}\right]\right) \\
= \;& \frac{1}{\tau}E_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau}-q_t\right]+\frac{1}{2\tau}Var_t^{\tilde{\mathbb{P}}}\left[q_{t+\tau}-q_t\right] \\
= \;& \frac{1}{\tau}\left[\left(\rho_0^\pi+\rho_x^{\pi\prime}\mu\right)\tau+\rho_x^{\pi\prime}\left(\mathcal{K}\right)^{-1}\left(I-e^{-\mathcal{K}\tau}\right)\left(x_t-\mu\right)\right] \\
& +\frac{1}{2\tau}\left[\begin{array}{c}\left(\widehat{\sigma}_q'\widehat{\sigma}_q+\sigma_\perp^2\right)\tau-2\widehat{\sigma}_q'\Sigma'\left(\mathcal{K}'\right)^{-1}\left(I-e^{-\mathcal{K}'\tau}\right)\left(\mathcal{K}'\right)^{-1}\rho_x^\pi \\ +\rho_x^{\pi\prime}\left(\mathcal{K}\right)^{-1}\Omega_\tau^x\left(\mathcal{K}'\right)^{-1}\rho_x^\pi\end{array}\right] \\
\equiv \;& \widetilde{a}_I^{option}+\widetilde{b}_I^{option\prime}x_t
\end{aligned}$$

Based on the observation $M_t^R = M_t^N Q_t$, we have

$$\begin{aligned}
dM_t^R/M_t^R &= dM_t^N/M_t^N + dQ_t/Q_t + \left(dM_t^N/M_t^N\right)\cdot\left(dQ_t/Q_t\right) \\
&= -r_t^N dt - \Lambda_{x,t}^{N\prime}dW_{x,t} + \left[\pi_t+\frac{1}{2}\left(\sigma_q'\sigma_q+\left(\sigma_q^\perp\right)^2\right)-\sigma_q'\Lambda_{x,t}^N\right]dt+\sigma_q'dW_{x,t} \\
&\equiv -r_t^R dt - \Lambda_{x,t}^{R\prime}dW_{x,t}
\end{aligned}$$

where

$$\begin{aligned}
r_t^R &= r_t^N - \left[\pi_t+\frac{1}{2}\left(\sigma_q'\sigma_q+\left(\sigma_q^\perp\right)^2\right)\right]+\sigma_q'\Lambda_{x,t}^N \\
&= r_t^N - \left[\pi_t+\frac{1}{2}\left(\sigma_q'\sigma_q+\left(\sigma_q^\perp\right)^2\right)\right]+\sigma_q'\left(\lambda_{0,x}^N+\lambda_x^{N\prime}x_t\right) \\
&\equiv \rho_0^R+\rho_x^{R\prime}x_t \\
\Lambda_{x,t}^R &= \Lambda_{x,t}^N-\sigma_q \equiv \lambda_{0,x}^R+\lambda_x^{R\prime}x_t
\end{aligned}$$

and

$$\begin{aligned}
\rho_0^R &= \rho_0^N - \rho_0^\pi - \frac{1}{2}\left(\sigma_q'\sigma_q + \left(\sigma_q^\perp\right)^2\right) + \lambda_{0,x}'\sigma_q \\
\rho_x^R &= \rho_x^N - \rho_x^\pi + \lambda_x^{N\prime}\sigma_q
\end{aligned}$$

and

$$\lambda_{0,x}^R = \lambda_{0,x}^N - \sigma_q, \text{ and } \lambda_x^R = \lambda_x^N$$

1.1 Dynamics under the Risk-Neutral Measure and Bond Pricing

Let

$$\Lambda_t^N \equiv \begin{pmatrix} \Lambda_{x,t}^N \\ \Lambda_{v,t}^N \\ \Lambda_{\perp,t}^N \\ \Lambda_{\delta,t}^N \end{pmatrix}, W_t \equiv \begin{pmatrix} W_{x,t} \\ W_{v,t} \\ W_{\perp,t} \\ W_{\delta,t} \end{pmatrix}.$$

Then, the Radon-Nikodym derivative of the risk neutral measure $\mathbb{P}^*$ with respect to the physical measure $\mathbb{P}$ is given by

$$\left(\frac{d\mathbb{P}^*}{d\mathbb{P}}\right)_{t,T} = \exp\left[-\frac{1}{2}\int_t^T \Lambda_s^{N\prime}\Lambda_s^N ds - \int_t^T \Lambda_s^{N\prime} dW_s\right] \tag{6}$$

By the Girsanov theorem, $dW_t^* = dW_t + \Lambda_t^N dt$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. It implies that under the risk neutral measure,

$$\begin{aligned}
dW_{x,t}^* &= dW_{x,t} + \Lambda_{x,t}^N dt \\
dW_{v,t}^* &= dW_{v,t} + \Lambda_{v,t}^N dt \\
dW_{\perp,t}^* &= dW_{\perp,t} + \Lambda_{\perp,t}^N dt \\
dW_{\delta,t}^* &= dW_{\delta,t} + \Lambda_{\delta,t}^N dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
dx_t &= \mathcal{K}\left(\mu - x_t\right)dt + \Sigma dW_{x,t} \\
&= \mathcal{K}\left(\mu - x_t\right)dt + \Sigma\left(dW_{x,t}^* - \Lambda_{x,t}^N dt\right) \\
&= \left[\left(\mathcal{K}\mu - \Sigma\lambda_0^N\right) - \left(\mathcal{K} + \Sigma\lambda_x^N\right)x_t\right]dt + \Sigma dW_{x,t}^* \\
&\equiv \mathcal{K}^*\left(\mu^* - x_t\right)dt + \Sigma dW_{x,t}^*,
\end{aligned}$$

and

$$\begin{aligned}
dq_t &= \pi_t dt + \sigma_q' dW_{x,t} + \sqrt{v_t}dW_{\perp,t} \\
&= \left(\rho_0^\pi + \rho_x^{\pi\prime}x_t\right)dt + \sigma_q'\left(dW_{x,t}^* - \Lambda_{x,t}^N dt\right) + \sigma_\perp\left(dW_{\perp,t}^* - \Lambda_{\perp,t}^N dt\right) \\
&= \left(\rho_0^\pi + \rho_x^{\pi\prime}x_t + \rho_v^\pi v_t + \rho_\delta^\pi\delta_t + \rho_s^\pi s_t - \sigma_q'\left(\lambda_0^N + \lambda_x^N x_t\right) - \gamma_\perp v_t\right)dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t}dW_{\perp,t}^* \\
&= \left(\rho_0^\pi - \lambda_0^{N\prime}\sigma_q + \left(\rho_x^\pi - \lambda_x^{N\prime}\sigma_q\right)' x_t + \left(\rho_v^\pi - \gamma_\perp\right)v_t + \rho_\delta^\pi\delta_t + \rho_s^\pi s_t\right)dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t}dW_{\perp,t}^* \\
&\equiv \left(\rho_0^{\pi*} + \rho_x^{\pi*\prime}x_t + \rho_v^{\pi*}v_t + \rho_\delta^{\pi*}\delta_t + \rho_s^{\pi*}s_t\right)dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t}dW_{\perp,t}^*,
\end{aligned}$$

where

$$
\begin{aligned}
\mathcal{K}^* &= \mathcal{K}+\Sigma\lambda_x^N \\
\mathcal{K}^*\mu^* &= \mathcal{K}\mu-\Sigma\lambda_0^N \\
\kappa_v^* &= \kappa_v + \sqrt{1-\rho^2}\sigma_v\gamma_v + \rho\sigma_v\gamma_\perp \\
\kappa_v^*\mu_v^* &= \kappa_v\mu_v \\
\kappa_\delta^* &= \kappa_\delta + \sigma_\delta\lambda_\delta^N \\
\kappa_\delta^*\mu_\delta^* &= \kappa_v\mu_v - \sigma_\delta\lambda_{0,\delta}^N \\
\pi_t^* &= \rho_0^{\pi*} + \rho_x^{\pi*\prime}x_t + \rho_v^{\pi*}v_t + \rho_\delta^{\pi*}\delta_t + \rho_s^{\pi*}s_t \\
\rho_0^{\pi*} &= \rho_0^\pi - \lambda_0^{N\prime}\sigma_q \\
\rho_x^{\pi*} &= \rho_x^\pi - \lambda_x^{N\prime}\sigma_q \\
\rho_v^{\pi*} &= \rho_v^\pi - \gamma_\perp \\
\rho_\delta^{\pi*} &= \rho_\delta^\pi \text{ and } \rho_s^{\pi*} = \rho_s^\pi
\end{aligned}
$$

We first establish a well-known result of affine-form of nominal yields in Proposition 1 below.

Proposition 1 *Under this model, nominal and real bond prices take the exponential-affine form*

$$
P_{t,\tau}^i = \exp\left(A_\tau^i + B_\tau^{i\prime}x_t + C_\tau^i v_t + D_\tau^i\delta_t + E_\tau^i s_t\right),\ \ i = N, R \tag{7}
$$

and nominal and real yields take the affine form

$$
y_{t,\tau}^i = a_\tau^i + b_\tau^{i\prime}x_t + c_\tau^i v_t + d_\tau^i\delta_t + e_\tau^i s_t,\ \ i = N, R \tag{8}
$$

where $a_\tau^i \equiv -A_\tau^i/\tau$, $b_\tau^i \equiv -B_\tau^i/\tau$, $c_\tau^i \equiv -C_\tau^i/\tau$, $d_\tau^i \equiv -D_\tau^i/\tau$ and $e_\tau^i \equiv -E_\tau^i/\tau$, and A_τ^i, B_τ^i, C_τ^i, D_τ^i, E_τ^i $(i = N, R)$ satisfy the following system of ODEs:

$$
\begin{aligned}
\frac{dA_\tau^i}{d\tau} &= -\rho_0^i + \left(\mathcal{K}\mu - \Sigma\lambda_0^i\right)' B_\tau^i + \kappa_v^*\mu_v^* C_\tau^i + \kappa_\delta^*\mu_\delta^* D_\tau^i + \phi_0^* E_\tau^i + \frac{1}{2}B_\tau^{i\prime}\Sigma\Sigma' B_\tau^i + \frac{1}{2}\sigma_\delta^2\left(D_\tau^i\right)^2 + \frac{1}{2}\sigma_s^2\left(E_\tau^i\right)^2,\ \textit{with } A_0^i = 0 \\
\frac{dB_\tau^i}{d\tau} &= -\rho_x^i - \left(\mathcal{K} + \Sigma\lambda_x^i\right)' B_\tau^i + \phi_x^* E_\tau^i,\ \textit{with } B_0^i = 0 \\
\frac{dC_\tau^i}{d\tau} &= -\rho_v^i - \kappa_v^* C_\tau^i + \frac{1}{2}\sigma_v^2\left(C_\tau^i\right)^2 + \phi_v^* E_\tau^i,\ \textit{with } C_0^i = 0 \\
\frac{dD_\tau^i}{d\tau} &= -\rho_\delta^i - \kappa_\delta^* D_\tau^i + \phi_\delta^* E_\tau^i,\ \textit{with } D_0^i = 0 \\
\frac{dE_\tau^i}{d\tau} &= -\rho_s^i + \phi_s^* E_\tau^i,\ \textit{with } E_0^i = 0
\end{aligned}
$$

Consider the transformation

$$
\begin{aligned}
\mathcal{A} &= A_\tau^i \\
\mathcal{B} &= \begin{pmatrix} B_\tau^i \\ D_\tau^i \\ E_\tau^i \end{pmatrix} \\
\mathcal{C} &= C_\tau^i
\end{aligned}
$$

Then

$$
\begin{aligned}
\dot{\mathcal{A}} &= -\rho_0^i + \begin{pmatrix} \mathcal{K}\mu - \Sigma\lambda_0^i \\ \kappa_\delta^* \mu_\delta^* \\ \phi_0^* \end{pmatrix}' \mathcal{B} + \left(\kappa_v^* \mu_v^*\right) \mathcal{C} + \frac{1}{2}\mathcal{B}' \begin{pmatrix} \Sigma\Sigma' & & \\ & \sigma_\delta^2 & \\ & & \sigma_s^2 \end{pmatrix} \mathcal{B} \\
\dot{\mathcal{B}} &= \begin{pmatrix} -\rho_x^i \\ -\rho_\delta^i \\ -\rho_s^i \end{pmatrix} + \begin{pmatrix} -\left(\mathcal{K} + \Sigma\lambda_x^i\right)' & 0 & \phi_x^* \\ 0 & -\kappa_\delta^* & \phi_\delta^* \\ 0 & 0 & \phi_s^* \end{pmatrix} \mathcal{B} \\
\dot{\mathcal{C}} &= -\rho_v^i - \kappa_v^* \mathcal{C} + \frac{1}{2}\sigma_v^2 \mathcal{C}^2
\end{aligned}
$$

Survey Yield Forecast. To calculate survey yield forecasts, we need the following results:

$$
\begin{aligned}
E_t\left[x_{t+\tau}\right] &= \mu + \exp\left(-\kappa\tau\right)\left(x_t - \mu\right) \\
E_t\left[v_{t+\tau}\right] &= \mu_v + \exp\left(-\kappa_v\tau\right)\left(v_t - \mu_v\right) \\
E_t\left[\delta_{t+\tau}\right] &= \mu_\delta + \exp\left(-\kappa_\delta\tau\right)\left(\delta_t - \mu_\delta\right)
\end{aligned}
$$

Next we calculate, $E_t\left[s_{t+\tau}\right]$. Note that

$$
\begin{aligned}
ds_t &= \left(\phi_0 + \phi_x' x_t + \phi_v v_t + \phi_\delta \delta_t + \phi_s s_t\right) dt + \sigma_s dW_{\delta,t} \\
d\left[\exp\left(-\phi_s t\right) s_t\right] &= \exp\left(-\phi_s t\right) \left[\left(\phi_0 + \phi_x' x_t + \phi_v v_t + \phi_\delta \delta_t\right) dt + \sigma_s dW_{\delta,t}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
& \exp\left(-\phi_s\left(t+\tau\right)\right) s_{t+\tau} - \exp\left(-\phi_s t\right) s_t \\
= & \int_t^{t+\tau} \exp\left(-\phi_s u\right) \left(\phi_0 + \phi_x' x_u + \phi_v v_u + \phi_\delta \delta_u\right) du + \exp\left(-\phi_s u\right) \sigma_s dW_{\delta,u}
\end{aligned}
$$

implying

$$
\begin{aligned}
& \exp\left(-\phi_s\left(t+\tau\right)\right) E_t\left[s_{t+\tau}\right] - \exp\left(-\phi_s t\right) s_t \\
= & \int_t^{t+\tau} \exp\left(-\phi_s u\right) \begin{pmatrix} \phi_0 + \phi_x'\left(\left[\mu + \exp\left(-\kappa\left(u-t\right)\right)\left(x_t - \mu\right)\right]\right) \\ +\phi_v\left(\left[\mu_v + \exp\left(-\kappa_v\left(u-t\right)\right)\left(v_t - \mu_v\right)\right]\right) \\ +\phi_\delta\left(\left[\mu_\delta + \exp\left(-\kappa_\delta\left(u-t\right)\right)\left(\delta_t - \mu_\delta\right)\right]\right) \end{pmatrix} du \\
= & \frac{1}{\phi_s}\left(\exp\left(-\phi_s t\right) - \exp\left(-\phi_s\left(t+\tau\right)\right)\right)\left(\phi_0 + \phi_x'\mu + \phi_v\mu_v + \phi_\delta\mu_\delta\right) \\
& + \exp\left(-\phi_s t\right)\phi_x'\left(\kappa + \phi_s I\right)^{-1}\left(I - \exp\left(-\left(\kappa + \phi_s I\right)\tau\right)\right)\left(x_t - \mu\right) \\
& + \exp\left(-\phi_s t\right)\phi_v\left(\kappa_v + \phi_s\right)^{-1}\left(1 - \exp\left(-\left(\kappa_v + \phi_s\right)\tau\right)\right)\left(v_t - \mu_v\right) \\
& + \exp\left(-\phi_s t\right)\phi_\delta\left(\kappa_\delta + \phi_s\right)^{-1}\left(1 - \exp\left(-\left(\kappa_\delta + \phi_s\right)\tau\right)\right)\left(\delta_t - \mu_\delta\right)
\end{aligned}
$$

or

$$
\begin{aligned}
& E_t\left[s_{t+\tau}\right] \\
= & \exp\left(\phi_s\tau\right) s_t + \frac{1}{\phi_s}\left(\exp\left(\phi_s\tau\right) - 1\right)\left(\phi_0 + \phi_x'\mu + \phi_v\mu_v + \phi_\delta\mu_\delta\right) \\
& + \phi_x'\left(\kappa + \phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right) I - \exp\left(-\kappa\tau\right)\right)\left(x_t - \mu\right) \\
& + \phi_v\left(\kappa_v + \phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right) - \exp\left(-\kappa_v\tau\right)\right)\left(v_t - \mu_v\right) \\
& + \phi_\delta\left(\kappa_\delta + \phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right) - \exp\left(-\kappa_\delta\tau\right)\right)\left(\delta_t - \mu_\delta\right)
\end{aligned}
$$

and

$$Var_t\left[s_{t+\tau}\right] = E_t\left[\int_{-\tau}^{0} \exp\left(-2\phi_s u\right)\sigma_s^2 du\right] = \frac{1}{2\phi_s}\left(\exp\left(2\phi_s\tau\right)-1\right)\sigma_s^2$$

Therefore,

$$\begin{aligned}
E_t^{svy}\left[y_{t+\tau,3m}^{N}\right] &= E_t^{mkt}\left[y_{t+\tau,3m}^{N}\right]+\epsilon_{t,\tau}^{f} \\
&= E_t\left[a_{3m}^{N}+b_{3m}^{N\prime}x_{t+\tau}+c_{3m}^{N}v_{t+\tau}+d_{3m}^{N}\delta_{t+\tau}+e_{3m}^{N}s_{t+\tau}\right]+\epsilon_{t,\tau}^{f} \\
&= a_{3m}^{N}+b_{3m}^{N\prime}\left[\mu+\exp\left(-\kappa\tau\right)\left(x_t-\mu\right)\right] \\
&\quad +c_{3m}^{N}\left[\mu_v+\exp\left(-\kappa_v\tau\right)\left(v_t-\mu_v\right)\right] \\
&\quad +d_{3m}^{N}\left[\mu_\delta+\exp\left(-\kappa_\delta\tau\right)\left(\delta_t-\mu_\delta\right)\right] \\
&\quad +e_{3m}^{N}\left[\begin{array}{c}
\exp\left(\phi_s\tau\right)s_t+\frac{1}{\phi_s}\left(\exp\left(\phi_s\tau\right)-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right) \\
+\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right)I-\exp\left(-\kappa\tau\right)\right)\left(x_t-\mu\right) \\
+\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_v\tau\right)\right)\left(v_t-\mu_v\right) \\
+\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_\delta\tau\right)\right)\left(\delta_t-\mu_\delta\right)
\end{array}\right]+\epsilon_{t,\tau}^{f} \\
&\equiv a_\tau^f+b_\tau^{f\prime}x_t+c_\tau^f v_t+d_\tau^f\delta_t+e_\tau^f s_t+\epsilon_{t,\tau}^{f}
\end{aligned}$$

where

$$\begin{aligned}
a_\tau^f &= a_{3m}^{N}+b_{3m}^{N\prime}\left[I-\exp\left(-\kappa\tau\right)\right]\mu+c_{3m}^{N}\left[1-\exp\left(-\kappa_v\tau\right)\right]\mu_v+d_{3m}^{N}\left[1-\exp\left(-\kappa_\delta\tau\right)\right]\mu_\delta \\
&\quad +e_{3m}^{N}\left[\begin{array}{l}
\frac{1}{\phi_s}\left(\exp\left(\phi_s\tau\right)-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right) \\
-\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right)I-\exp\left(-\kappa\tau\right)\right)\mu \\
-\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_v\tau\right)\right)\mu_v \\
-\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_\delta\tau\right)\right)\mu_\delta
\end{array}\right]
\end{aligned}$$

and

$$\begin{aligned}
b_\tau^{f\prime} &= b_{3m}^{N\prime}\exp\left(-\kappa\tau\right)+e_{3m}^{N}\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right)I-\exp\left(-\kappa\tau\right)\right) \\
c_\tau^f &= c_{3m}^{N}\exp\left(-\kappa_v\tau\right)+e_{3m}^{N}\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_v\tau\right)\right) \\
d_\tau^f &= d_{3m}^{N}\exp\left(-\kappa_\delta\tau\right)+e_{3m}^{N}\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_\delta\tau\right)\right) \\
e_\tau^f &= e_{3m}^{N}\exp\left(\phi_s\tau\right)
\end{aligned}$$

Lastly, we turn to the long-range forecast: $E_t^{mkt}\left[\overline{y}_{3m,T_1,T_2}^{N}\right]$ as follows. Note that

$$\begin{aligned}
&E_t\left[\frac{1}{T_2-T_1}\int_{t+T_1}^{t+T_2} s_u du\right] \\
&= \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\left[\begin{array}{c}
\exp\left(\phi_s u\right)s_t+\frac{1}{\phi_s}\left(\exp\left(\phi_s u\right)-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right) \\
+\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(\exp\left(\phi_s u\right)I-\exp\left(-\kappa u\right)\right)\left(x_t-\mu\right) \\
+\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(\exp\left(\phi_s u\right)-\exp\left(-\kappa_v u\right)\right)\left(v_t-\mu_v\right) \\
+\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(\exp\left(\phi_s u\right)-\exp\left(-\kappa_\delta u\right)\right)\left(\delta_t-\mu_\delta\right)
\end{array}\right]du \\
&= W_{T_1,T_2}^{s}s_t+\frac{1}{\phi_s}\left(W_{T_1,T_2}^{s}-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right) \\
&\quad +\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(W_{T_1,T_2}^{s}I-W_{T_1,T_2}^{x}\right)\left(x_t-\mu\right) \\
&\quad +\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(W_{T_1,T_2}^{s}-W_{T_1,T_2}^{v}\right)\left(v_t-\mu_v\right) \\
&\quad +\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(W_{T_1,T_2}^{s}-W_{T_1,T_2}^{\delta}\right)\left(\delta_t-\mu_\delta\right)
\end{aligned}$$

where

$$\begin{aligned}
W^x_{T_1,T_2} &= \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\exp(-\kappa s)\,ds = \frac{1}{T_2-T_1}\kappa^{-1}\left(\exp(-\kappa T_1)-\exp(-\kappa T_2)\right)\\
W^v_{T_1,T_2} &= \frac{1}{T_2-T_1}\kappa_v^{-1}\left(\exp(-\kappa_v T_1)-\exp(-\kappa_v T_2)\right)\\
W^\delta_{T_1,T_2} &= \frac{1}{T_2-T_1}\kappa_\delta^{-1}\left(\exp(-\kappa_\delta T_1)-\exp(-\kappa_\delta T_2)\right)\\
W^s_{T_1,T_2} &= \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\exp(\phi_s u)\,du = \frac{1}{T_2-T_1}\phi_s^{-1}\left(\exp(\phi_s T_2)-\exp(\phi_s T_1)\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
E_t^{svy}\left[\overline{y}^N_{3m,T_1,T_2}\right] &= E_t^{mkt}\left[\overline{y}^N_{3m,T_1,T_2}\right]+\epsilon^f_{t,LT} = E_t\left[\frac{1}{T_2-T_1}\int_{T_1}^{T_2} y^N_{s,3m}ds\right]+\epsilon^f_{t,LT}\\
&= E_t\left[\frac{1}{T_2-T_1}\int_{T_1}^{T_2}\left(a^N_{3m}+b^{N\prime}_{3m}x_s+c^N_{3m}v_s+d^N_{3m}\delta_s+e^N_{3m}s_s\right)ds\right]+\epsilon^f_{t,LT}\\
&= a^N_{3m}+b^{N\prime}_{3m}\left(I-W_{T_1,T_2}\right)\mu+b^{N\prime}_{3m}W_{T_1,T_2}x_t\\
&\quad +c^N_{3m}\left[\left(1-W^v_{T_1,T_2}\right)\mu_v+W^v_{T_1,T_2}v_t\right]+d^N_{3m}\left[\left(1-W^\delta_{T_1,T_2}\right)\mu_\delta+W^\delta_{T_1,T_2}\delta_t\right]\\
&\quad +e^N_{3m}\left[\begin{array}{c}
W^s_{T_1,T_2}s_t+\frac{1}{\phi_s}\left(W^s_{T_1,T_2}-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right)\\
+\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(W^s_{T_1,T_2}I-W^x_{T_1,T_2}\right)\left(x_t-\mu\right)\\
+\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(W^s_{T_1,T_2}-W^v_{T_1,T_2}\right)\left(v_t-\mu_v\right)\\
+\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(W^s_{T_1,T_2}-W^\delta_{T_1,T_2}\right)\left(\delta_t-\mu_\delta\right)
\end{array}\right]+\epsilon^f_{t,LT}\\
&\equiv a^f_{LT}+b^{f\prime}_{LT}x_t+c^f_{LT}v_t+d^f_{LT}\delta_t+\epsilon^f_{t,LT}
\end{aligned}$$

where

$$\begin{aligned}
a^f_{LT} &= a^N_{3m}+b^{N\prime}_{3m}\left(I-W^x_{T_1,T_2}\right)\mu+c^N_{3m}\left(1-W^v_{T_1,T_2}\right)\mu_v+d^N_{3m}\left(1-W^\delta_{T_1,T_2}\right)\mu_\delta\\
&\quad +e^N_{3m}\left[\begin{array}{c}
\frac{1}{\phi_s}\left(W^s_{T_1,T_2}-1\right)\left(\phi_0+\phi_x'\mu+\phi_v\mu_v+\phi_\delta\mu_\delta\right)-\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(W^s_{T_1,T_2}I-W^x_{T_1,T_2}\right)\mu\\
-\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(W^s_{T_1,T_2}-W^v_{T_1,T_2}\right)\mu_v-\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(W^s_{T_1,T_2}-W^\delta_{T_1,T_2}\right)\mu_\delta
\end{array}\right]
\end{aligned}$$

and

$$\begin{aligned}
b^{f\prime}_{LT} &= b^{N\prime}_{3m}W^x_{T_1,T_2}+e^N_{3m}\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(W^s_{T_1,T_2}I-W^x_{T_1,T_2}\right)\\
c^f_{LT} &= c^N_{3m}W^v_{T_1,T_2}+e^N_{3m}\phi_v\left(\kappa_v+\phi_s\right)^{-1}\left(W^s_{T_1,T_2}-W^v_{T_1,T_2}\right)\\
d^f_{LT} &= d^N_{3m}W^\delta_{T_1,T_2}+e^N_{3m}\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(W^s_{T_1,T_2}-W^\delta_{T_1,T_2}\right)\\
e^f_{LT} &= e^N_{3m}W^s_{T_1,T_2}
\end{aligned}$$

1.2 Dynamics under the Forward Measure and Inflation Option Pricing

Under the forward measure, we have

$$\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right)_{t,T} = \frac{\exp\left(-\int_t^T r^N_s ds\right)}{P^N_{t,\tau}}$$

and

$$\Psi_t \equiv E_t^{\mathbb{P}^*}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right)_{0,T}\right] = E_t^{\mathbb{P}^*}\left[\frac{\exp\left(-\int_0^T r_s^N ds\right)}{P_{0,T}^N}\right] = \frac{P_{t,\tau}^N}{P_{0,T}^N}\exp\left(-\int_0^t r_s^N ds\right)$$

We have

$$\begin{aligned} d\Psi_t &= \frac{\exp\left(-\int_0^t r_s^N ds\right)}{P_{0,T}^N}\left[dP_{t,\tau}^N - r_t^N P_{t,\tau}^N dt\right] \\ &= \Psi_t\left[B_\tau^{N\prime}\Sigma dW_{x,t}\right] \end{aligned}$$

By Girsanov's Theorem, we have

$$d\tilde{W}_t = dW_t^* - \frac{d\Psi_t}{\Psi_t}\cdot dW_t^*$$

or

$$d\tilde{W}_{x,t} = dW_t^* - \Sigma' B_\tau^N dt$$

The dynamics of the state variables under the risk neutral measure is given by

$$dx_t = \left[\mathcal{K}^*\left(\mu^* - x_t\right)\right]dt + \Sigma dW_{x,t}^*$$

Therefore,

$$\begin{aligned} dx_t &= \left[\mathcal{K}^*\left(\mu^* - x_t\right)\right]dt + \Sigma\left(d\tilde{W}_{x,t} + \Sigma' B_\tau^N dt\right) \\ &= \left(\mathcal{K}^*\mu^* + \Sigma\Sigma' B_\tau^N - \mathcal{K}^* x_t\right)dt + \Sigma d\tilde{W}_{x,t} \\ &\equiv \tilde{\mathcal{K}}\left(\tilde{\mu} - x_t\right)dt + \Sigma d\tilde{W}_{x,t}, \end{aligned}$$

$$\begin{aligned} dv_t &= \kappa_v^*\left(\mu_v^* - v_t\right)dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}dW_{v,t}^* + \rho dW_{\perp,t}^*\right) \\ &= \left[\kappa_v^*\mu_v^* - \left(\kappa_v^* - \sigma_v^2 C_\tau^N\right)v_t\right]dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}d\tilde{W}_{v,t} + \rho d\tilde{W}_{\perp,t}\right) \\ &\equiv \tilde{\kappa}_v\left(\tilde{\mu}_v - v_t\right)dt + \sigma_v\sqrt{v_t}\left(\sqrt{1-\rho^2}d\tilde{W}_{v,t} + \rho d\tilde{W}_{\perp,t}\right) \end{aligned}$$

$$\begin{aligned} ds_t &= \left(\phi_0^* + \phi_x^{*\prime}x_t + \phi_v^* v_t + \phi_\delta^*\delta_t + \phi_s^* s_t\right)dt + \sigma_s dW_{\delta,t}^* \\ &= \left(\phi_0^* + \phi_x^{*\prime}x_t + \phi_v^* v_t + \phi_\delta^*\delta_t + \phi_s^* s_t\right)dt + \sigma_s\left(d\tilde{W}_{\delta,t} + \left(\sigma_\delta D_\tau^N + \sigma_s E_\tau^N\right)dt\right) \\ &= \left(\left[\phi_0^* + \sigma_s\left(\sigma_\delta D_\tau^N + \sigma_s E_\tau^N\right)\right] + \phi_x^{*\prime}x_t + \phi_v^* v_t + \phi_\delta^*\delta_t + \phi_s^* s_t\right)dt + \sigma_s d\tilde{W}_{\delta,t} \\ &\equiv \left(\widetilde{\phi}_0 + \widetilde{\phi}_x' x_t + \widetilde{\phi}_v v_t + \widetilde{\phi}_\delta\delta_t + \widetilde{\phi}_s s_t\right)dt + \sigma_s d\tilde{W}_{\delta,t} \end{aligned}$$

and

$$\begin{aligned} dq_t &= \left(\rho_0^{\pi*} + \rho_x^{\pi*\prime}x_t + \rho_v^{\pi*}v_t + \rho_\delta^{\pi*}\delta_t + \rho_s^{\pi*}s_t\right)dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t}dW_{\perp,t}^* \\ &= \left(\left[\rho_0^{\pi*} + \sigma_q'\Sigma' B_\tau^N\right] + \rho_x^{\pi*\prime}x_t + \left[\rho_v^{\pi*} + \rho\sigma_v C_\tau^N\right]v_t + \rho_\delta^{\pi*}\delta_t + \rho_s^{\pi*}s_t\right)dt + \sigma_q' d\tilde{W}_{x,t} + \sqrt{v_t}d\tilde{W}_{\perp,t} \\ &\equiv \left(\widetilde{\rho}_0^{\pi} + \widetilde{\rho}_x^{\pi\prime}x_t + \widetilde{\rho}_v^{\pi}v_t + \widetilde{\rho}_\delta^{\pi*}\delta_t + \widetilde{\rho}_s^{\pi*}s_t\right)dt + \sigma_q' d\tilde{W}_{x,t} + \sqrt{v_t}d\tilde{W}_{\perp,t} \end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathcal{K}} &= \mathcal{K}^* = \mathcal{K}+\Sigma\lambda_1^N, \\
\tilde{\mathcal{K}}\tilde{\mu} &= \mathcal{K}^*\mu^* + \Sigma\Sigma' B_\tau^N = \mathcal{K}\mu - \Sigma\lambda_0^N + \Sigma\Sigma' B_\tau^N, \\
\tilde{\kappa}_v &= \kappa_v^* - \sigma_v^2 C_\tau^N = \kappa_v + \sqrt{1-\rho^2}\sigma_v\gamma_v + \rho\sigma_v\gamma_\perp - \sigma_v^2 C_\tau^N, \\
\tilde{\kappa}_v\tilde{\mu}_v &= \kappa_v^*\mu_v^* = \kappa_v\mu_v, \\
\tilde{\kappa}_\delta &= \kappa_\delta^* = \kappa_\delta + \sigma_\delta\lambda_\delta^N \\
\tilde{\kappa}_\delta\tilde{\mu}_\delta &= \kappa_\delta^*\mu_\delta^* + \sigma_\delta\left(\sigma_\delta D_\tau^N + \sigma_s E_\tau^N\right) = \kappa_v\mu_v - \sigma_\delta\lambda_{0,\delta}^N + \sigma_\delta\left(\sigma_\delta D_\tau^N + \sigma_s E_\tau^N\right) \\
\tilde{\pi}_t &= \widetilde{\rho}_0^\pi + \widetilde{\rho}_x^{\pi\prime} x_t + \widetilde{\rho}_v^\pi v_t + \widetilde{\rho}_\delta^\pi \delta_t + \widetilde{\rho}_s^\pi s_t, \\
\widetilde{\rho}_0^\pi &= \rho_0^{\pi*} + \sigma_q'\Sigma' B_\tau^N = \rho_0^\pi - \lambda_0^{N\prime}\sigma_q + \sigma_q'\Sigma' B_\tau^N, \\
\widetilde{\rho}_x^\pi &= \rho_x^{\pi*} = \rho_x^\pi - \lambda_x^{N\prime}\sigma_q, \\
\widetilde{\rho}_v^\pi &= \rho_v^{\pi*} + \rho\sigma_v C_\tau^N = \rho_v^\pi - \gamma_\perp + \rho\sigma_v C_\tau^N. \\
\widetilde{\rho}_\delta^{\pi*} &= \rho_\delta^{\pi*} \text{ and } \widetilde{\rho}_s^{\pi*} = \rho_s^{\pi*}
\end{aligned}$$

Next, we compute the expected values of the state variables over the period t to $t+\tau$ under the forward measure. First, similar to the calculations under the risk neutral measure, we have

$$E_t^{\tilde{\mathbb{P}}}(x_s) = \tilde{\mu} + \exp\left(-\tilde{\mathcal{K}}(s-t)\right)(x_t - \tilde{\mu})$$

and

$$\begin{aligned}
\frac{1}{\tau}E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau} x_s ds\right] &= \frac{1}{\tau}\left[\int_t^{t+\tau}\left(\tilde{\mu} + \exp\left(-\tilde{\mathcal{K}}(s-t)\right)(x_t-\tilde{\mu})\right)ds\right] \\
&= \tilde{\mu} + \widetilde{W}_{0,\tau}^x(x_t - \tilde{\mu}) \equiv \tilde{a}_\tau^x + \tilde{b}_\tau^x x_t
\end{aligned}$$

where $\widetilde{W}_{T_1,T_2}^x$ is defined similarly as W_{T_1,T_2}^x except that dynamics under the forward measure is used instead, and

$$\tilde{a}_\tau^x = \left(I - \widetilde{W}_{0,\tau}^x\right)\tilde{\mu} \text{ and } \tilde{b}_\tau^x = \widetilde{W}_{0,\tau}^x \equiv \left(\tilde{\mathcal{K}}\tau\right)^{-1}\left(I - \exp\left(-\tilde{\mathcal{K}}\tau\right)\right)$$

Similarly (the results for δ_t are very similar and thus omitted), we have

$$\begin{aligned}
E_t^{\tilde{\mathbb{P}}}(v_s) &= \tilde{\mu}_v + \exp\left(-\tilde{\kappa}_v(s-t)\right)(v_t - \tilde{\mu}_v) \\
\frac{1}{\tau}E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau} v_s ds\right] &= \tilde{\mu}_v + \widetilde{W}_{0,\tau}^v(v_t - \tilde{\mu}_v) \equiv \tilde{a}_\tau^v + \tilde{b}_\tau^v v_t
\end{aligned}$$

where

$$\tilde{a}_\tau^v = \tilde{\mu}_v\left[1 - \widetilde{W}_{0,\tau}^v\right] \text{ and } \tilde{b}_\tau^v = \widetilde{W}_{0,\tau}^v \equiv (\tilde{\kappa}_v\tau)^{-1}\left(1 - \exp(-\tilde{\kappa}_v\tau)\right).$$

Furthermore,

$$\begin{aligned}
& E_t^{\tilde{\mathbb{P}}}[s_s] \\
= \; & \exp\left(\widetilde{\phi}_s(s-t)\right)s_t + \frac{1}{\widetilde{\phi}_s}\left(\exp\left(\widetilde{\phi}_s(s-t)\right) - 1\right)\left(\widetilde{\phi}_0 + \widetilde{\phi}_x'\mu + \widetilde{\phi}_v\mu_v + \widetilde{\phi}_\delta\mu_\delta\right) \\
& + \widetilde{\phi}_x'\left(\kappa + \widetilde{\phi}_s I\right)^{-1}\left(\exp\left(\widetilde{\phi}_s(s-t)\right)I - \exp\left(-\kappa(s-t)\right)\right)(x_t - \mu) \\
& + \widetilde{\phi}_v\left(\kappa_v + \widetilde{\phi}_s\right)^{-1}\left(\exp\left(\widetilde{\phi}_s(s-t)\right) - \exp\left(-\kappa_v(s-t)\right)\right)(v_t - \mu_v) \\
& + \widetilde{\phi}_\delta\left(\kappa_\delta + \widetilde{\phi}_s\right)^{-1}\left(\exp\left(\widetilde{\phi}_s(s-t)\right) - \exp\left(-\kappa_\delta(s-t)\right)\right)(\delta_t - \mu_\delta)
\end{aligned}$$

and

$$E_t^{\tilde{\mathbb{P}}}\left[\frac{1}{T_2-T_1}\int_{t+T_1}^{t+T_2} s_u du\right]$$

$$= \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\left[\begin{array}{c} \exp\left(\widetilde{\phi}_s u\right)s_t + \frac{1}{\widetilde{\phi}_s}\left(\exp\left(\widetilde{\phi}_s u\right)-1\right)\left(\widetilde{\phi}_0 + \widetilde{\phi}_x'\mu + \widetilde{\phi}_v\mu_v + \widetilde{\phi}_\delta\mu_\delta\right) \\ +\widetilde{\phi}_x'\left(\kappa+\widetilde{\phi}_s I\right)^{-1}\left(\exp\left(\widetilde{\phi}_s u\right)I - \exp\left(-\kappa u\right)\right)(x_t-\mu) \\ +\widetilde{\phi}_v\left(\kappa_v+\widetilde{\phi}_s\right)^{-1}\left(\exp\left(\widetilde{\phi}_s u\right) - \exp\left(-\kappa_v u\right)\right)(v_t-\mu_v) \\ +\widetilde{\phi}_\delta\left(\kappa_\delta+\widetilde{\phi}_s\right)^{-1}\left(\exp\left(\widetilde{\phi}_s u\right) - \exp\left(-\kappa_\delta u\right)\right)(\delta_t-\mu_\delta) \end{array}\right]du$$

$$\begin{aligned} =\; & \widetilde{W}^s_{T_1,T_2}s_t + \frac{1}{\widetilde{\phi}_s}\left(\widetilde{W}^s_{T_1,T_2}-1\right)\left(\widetilde{\phi}_0 + \widetilde{\phi}_x'\mu + \widetilde{\phi}_v\mu_v + \widetilde{\phi}_\delta\mu_\delta\right) \\ & +\widetilde{\phi}_x'\left(\kappa+\widetilde{\phi}_s I\right)^{-1}\left(\widetilde{W}^s_{T_1,T_2}I - \widetilde{W}^x_{T_1,T_2}\right)(x_t-\mu) \\ & +\widetilde{\phi}_v\left(\kappa_v+\widetilde{\phi}_s\right)^{-1}\left(\widetilde{W}^s_{T_1,T_2} - \widetilde{W}^v_{T_1,T_2}\right)(v_t-\mu_v) \\ & +\widetilde{\phi}_\delta\left(\kappa_\delta+\widetilde{\phi}_s\right)^{-1}\left(\widetilde{W}^s_{T_1,T_2} - \widetilde{W}^\delta_{T_1,T_2}\right)(\delta_t-\mu_\delta) \end{aligned}$$

implying

$$E_t^{\tilde{\mathbb{P}}}\left[\frac{1}{\tau}\int_t^{t+\tau} s_u du\right] \equiv \tilde{a}^s_\tau + \tilde{b}^{s,x}_\tau x_t + \widetilde{b}^{s,v}_\tau v_t + \widetilde{b}^{s,\delta}_\tau v_t + \widetilde{b}^{s,s}_\tau s_t$$

where

$$\begin{aligned} \widetilde{a}^s_\tau \;\equiv\; & \frac{1}{\widetilde{\phi}_s}\left(\widetilde{W}^s_{0,\tau}-1\right)\left(\widetilde{\phi}_0 + \widetilde{\phi}_x'\mu + \widetilde{\phi}_v\mu_v + \widetilde{\phi}_\delta\mu_\delta\right) \\ & -\widetilde{\phi}_x'\left(\kappa+\widetilde{\phi}_s I\right)^{-1}\left(\widetilde{W}^s_{0,\tau}I - \widetilde{W}^x_{0,\tau}\right)\mu \\ & -\widetilde{\phi}_v\left(\kappa_v+\widetilde{\phi}_s\right)^{-1}\left(\widetilde{W}^s_{0,\tau} - \widetilde{W}^v_{0,\tau}\right)\mu_v \\ & -\widetilde{\phi}_\delta\left(\kappa_\delta+\widetilde{\phi}_s\right)^{-1}\left(\widetilde{W}^s_{0,\tau} - \widetilde{W}^\delta_{0,\tau}\right)\mu_\delta \end{aligned}$$

and

$$\begin{aligned} \tilde{b}^{s,x}_\tau &\equiv \widetilde{\phi}_x'\left(\kappa+\widetilde{\phi}_s I\right)^{-1}\left(\widetilde{W}^s_{0,\tau}I - \widetilde{W}^x_{0,\tau}\right) \\ \tilde{b}^{s,v}_\tau &\equiv \widetilde{\phi}_v\left(\kappa_v+\widetilde{\phi}_s\right)^{-1}\left(\widetilde{W}^s_{0,\tau} - \widetilde{W}^v_{0,\tau}\right) \\ \tilde{b}^{s,\delta}_\tau &\equiv \widetilde{\phi}_\delta\left(\kappa_\delta+\widetilde{\phi}_s\right)^{-1}\left(\widetilde{W}^s_{0,\tau} - \widetilde{W}^\delta_{0,\tau}\right) \\ \tilde{b}^{s,s}_\tau &\equiv \widetilde{W}^s_{0,\tau} \end{aligned}$$

We can now calculate inflation expectation and variance as follows:

$$E_t^{\tilde{\mathbb{P}}}\left[\log\left(\frac{Q_{t+\tau}}{Q_t}\right)\right] = E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau}\tilde{\pi}_s ds\right] = E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau}\left(\widetilde{\rho}^\pi_0 + \widetilde{\rho}^{\pi\prime}_x x_s + \widetilde{\rho}^\pi_v v_s + \widetilde{\rho}^\pi_\delta \delta_s + \widetilde{\rho}^\pi_s s_s\right)ds\right]$$

$$\begin{aligned} =\; & \tau\left(\begin{array}{c} \widetilde{\rho}^\pi_0 + \widetilde{\rho}^{\pi\prime}_x\left(\tilde{a}^x_\tau + \tilde{b}^x_\tau x_t\right) + \widetilde{\rho}^\pi_v\left(\tilde{a}^v_\tau + \tilde{b}^v_\tau v_t\right) + \widetilde{\rho}^\pi_\delta\left(\tilde{a}^\delta_\tau + \tilde{b}^\delta_\tau \delta_t\right) \\ +\widetilde{\rho}^\pi_s\left(\tilde{a}^s_\tau + \tilde{b}^{s,x}_\tau x_t + \widetilde{b}^{s,v}_\tau v_t + \widetilde{b}^{s,\delta}_\tau \delta_t + \widetilde{b}^{s,s}_\tau s_t\right) \end{array}\right) \\ \equiv\; & \tau\left(\tilde{a}^\pi_\tau + \tilde{b}^{\pi\prime}_\tau x_t + \tilde{c}^\pi_\tau v_t + \widetilde{c2}^\pi_\tau \delta_t + \widetilde{c3}^\pi_\tau s_t\right) \end{aligned}$$

and

$$Var_t^{\tilde{\mathbb{P}}}\left[\log\left(\frac{Q_{t+\tau}}{Q_t}\right)\right] = E_t^{\tilde{\mathbb{P}}}\left[\int_t^{t+\tau}\left(\sigma_q'\sigma_q + v_s\right)ds\right] = \tau\left(\sigma_q'\sigma_q + \tilde{a}_\tau^v + \tilde{b}_\tau^v v_t\right) \equiv \tau\left(\tilde{d}_\tau^\pi + \tilde{e}_\tau^\pi v_t\right),$$

where

$$\begin{aligned}
\tilde{a}_\tau^\pi &= \widetilde{\rho}_0^\pi + \widetilde{\rho}_x^{\pi\prime}\tilde{a}_\tau^x + \widetilde{\rho}_v^\pi\tilde{a}_\tau^v + \widetilde{\rho}_\delta^\pi\tilde{a}_\tau^\delta + \widetilde{\rho}_s^\pi\tilde{a}_\tau^s,\\
\tilde{b}_\tau^\pi &= \tilde{b}_\tau^{x\prime}\widetilde{\rho}_x^\pi + \tilde{b}_\tau^{s,x\prime}\widetilde{\rho}_s^\pi,\\
\tilde{c}_\tau^\pi &= \widetilde{\rho}_v^\pi\tilde{b}_\tau^v + \widetilde{\rho}_s^\pi\widetilde{b}_\tau^{s,v},\\
\widetilde{c2}_\tau^\pi &= \widetilde{\rho}_\delta^\pi\tilde{b}_\tau^\delta + \widetilde{\rho}_s^\pi\widetilde{b}_\tau^{s,\delta}\\
\widetilde{c3}_\tau^\pi &= \widetilde{\rho}_s^\pi\widetilde{b}_\tau^{s,s}\\
\tilde{d}_\tau^\pi &= \sigma_q'\sigma_q + \tilde{a}_\tau^v\\
\tilde{e}_\tau^\pi &= \tilde{b}_\tau^v.
\end{aligned}$$

Finally, we turn to inflation option pricing. The price of a τ-maturity inflation cap with strike K is given by

$$\begin{aligned}
P_{t,\tau,K}^{CAP} &= \exp\left(-\tau y_{t,\tau}^N\right)E_t^{\tilde{\mathbb{P}}}\left[\left(\frac{Q_{t+\tau}}{Q_t} - (1+K)^\tau\right)^+\right]\\
&= \exp\left(-\tau y_{t,\tau}^N\right)E_t^{\tilde{\mathbb{P}}}\left[\left(\exp\left(\log\left(\frac{Q_{t+\tau}}{Q_t}\right)\right) - (1+K)^\tau\right)^+\right]\\
&= \exp\left(-\tau y_{t,\tau}^N\right)\Bigg[\exp\left(\tau\left(\left(\tilde{a}_\tau^\pi + \frac{\tilde{d}_\tau^\pi}{2}\right) + \tilde{b}_\tau^{\pi\prime}x_t + \left(\tilde{c}_\tau^\pi + \frac{\tilde{e}_\tau^\pi}{2}\right)v_t + \widetilde{c2}_\tau^\pi\delta_t + \widetilde{c3}_\tau^\pi s_t\right)\right)\\
&\quad\times\Phi\left(\frac{\tau}{\sigma}\left[-\ln(1+K) + \left(\tilde{a}_\tau^\pi + \tilde{d}_\tau^\pi\right) + \tilde{b}_\tau^{\pi\prime}x_t + \left(\tilde{c}_\tau^\pi + \tilde{e}_\tau^\pi\right)v_t + \widetilde{c2}_\tau^\pi\delta_t + \widetilde{c3}_\tau^\pi s_t\right]\right)\\
&\quad-(1+K)^\tau\,\Phi\left(\frac{\tau}{\sigma}\left[-\ln(1+K) + \tilde{a}_\tau^\pi + \tilde{b}_\tau^{\pi\prime}x_t + \tilde{c}_\tau^\pi v_t + \widetilde{c2}_\tau^\pi\delta_t + \widetilde{c3}_\tau^\pi s_t\right]\right)\Bigg],
\end{aligned}$$

and the price of a τ-maturity inflation cap with strike K is given by

$$\begin{aligned}
P_{t,\tau,K}^{FLO} &= \exp\left(-\tau y_{t,\tau}^N\right)E_t^{\tilde{\mathbb{P}}}\left[\left((1+K)^\tau - \frac{Q_{t+\tau}}{Q_t}\right)^+\right]\\
&= \exp\left(-\tau y_{t,\tau}^N\right)E_t^{\tilde{\mathbb{P}}}\left[\left((1+K)^\tau - \exp\left(\log\left(\frac{Q_{t+\tau}}{Q_t}\right)\right)\right)^+\right]\\
&= \exp\left(-\tau y_{t,\tau}^N\right)\Bigg[-\exp\left(\tau\left(\left(\tilde{a}_\tau^\pi + \frac{\tilde{d}_\tau^\pi}{2}\right) + \tilde{b}_\tau^{\pi\prime}x_t + \left(\tilde{c}_\tau^\pi + \frac{\tilde{e}_\tau^\pi}{2}\right)v_t\right)\right)\\
&\quad\times\Phi\left(-\frac{\tau}{\sigma}\left[-\ln(1+K) + \left(\tilde{a}_\tau^\pi + \tilde{d}_\tau^\pi\right) + \tilde{b}_\tau^{\pi\prime}x_t + \left(\tilde{c}_\tau^\pi + \tilde{e}_\tau^\pi\right)v_t\right]\right)\\
&\quad+(1+K)^\tau\,\Phi\left(-\frac{\tau}{\sigma}\left[-\ln(1+K) + \tilde{a}_\tau^\pi + \tilde{b}_\tau^{\pi\prime}x_t + \tilde{c}_\tau^\pi v_t\right]\right)\Bigg]
\end{aligned}$$

where we used the fact that for a normal random variable $\widetilde{z} \sim N\left(\mu, \sigma^2\right)$,

$$
\begin{aligned}
E\left[\left(ae^{\widetilde{z}} - b\right)^+\right] &= a\exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(\frac{\ln(a/b) + \left(\mu + \sigma^2\right)}{\sigma}\right) - b\Phi\left(\frac{\ln(a/b)+\mu}{\sigma}\right), \\
E\left[\left(b - ae^{\widetilde{z}}\right)^+\right] &= -a\exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(-\frac{\ln(a/b) + \left(\mu + \sigma^2\right)}{\sigma}\right) + b\Phi\left(-\frac{\ln(a/b)+\mu}{\sigma}\right).
\end{aligned}
$$

Remark. We can derive option-implied inflation expectations as follows:

$$
\begin{aligned}
& P^{CAP}_{t,\tau,K} - P^{FLO}_{t,\tau,K} \\
= \; & \exp\left(-\tau y^N_{t,\tau}\right) E^{\tilde{\mathbb{P}}}_t\left[\left(\frac{Q_{t+\tau}}{Q_t} - (1+K)^\tau\right)^+\right] - \exp\left(-\tau y^N_{t,\tau}\right) E^{\tilde{\mathbb{P}}}_t\left[\left((1+K)^\tau - \frac{Q_{t+\tau}}{Q_t}\right)^+\right] \\
= \; & \exp\left(-\tau y^N_{t,\tau}\right) E^{\tilde{\mathbb{P}}}_t\left[\frac{Q_{t+\tau}}{Q_t} - (1+K)^\tau\right]
\end{aligned}
$$

1.3 State Dynamics

We first consider the discrete-time dynamics of the state variable q_t between time $t-\Delta t$ and t. From $dq_t = \pi_t dt + \sigma_q' dW_{x,t} + \sqrt{v_t} dW_{\perp,t}$, we have

$$
q_t = q_{t-\Delta t} + \left(\rho_0^\pi \Delta t + \left(\rho_x^{\pi\prime}\Delta t\right)x_t + \left(\rho_v^\pi \Delta t\right)v_t + \left(\rho_\delta^\pi \Delta t\right)\delta_t + \left(\rho_s^\pi\Delta t\right)s_t\right) + \eta_t^q
$$

where $\eta_t^q \equiv \int_{t-\Delta t}^t \left(\sigma_q' dW_{x,s} + \sqrt{v_s} dW_{\perp,s}\right) \sim N\left(0, \Omega^q_{t-\Delta t}\right)$ and

$$
\begin{aligned}
\Omega^q_{t-\Delta t} &\equiv Var_{t-\Delta t}\left(\eta_t^q\right) = \sigma_q'\sigma_q \Delta t + E_{t-\Delta t}\left[\int_{t-\Delta t}^t v_s ds\right] \\
&= \sigma_q'\sigma_q\Delta t + \left[\int_0^{\Delta t}\left(\mu_v\left(1 - e^{-\kappa_v s}\right) + e^{-\kappa_v s}v_{t-\Delta t}\right)ds\right] \\
&= \sigma_q'\sigma_q\Delta t + \mu_v\Delta t + \left(v_{t-\Delta t} - \mu_v\right)\frac{1-\exp\left(-\kappa_v\Delta t\right)}{\kappa_v}.
\end{aligned}
$$

Similarly, for the state variable x_t we have

$$
x_t = \exp\left(-\mathcal{K}\Delta t\right)x_{t-\Delta t} + \left(I - \exp\left(-\mathcal{K}\Delta t\right)\right)\mu + \eta_t^x,
$$

where $\eta_t^x = \int_0^{\Delta t}\exp\left(-\kappa s\right)\Sigma dW_{x,s} \sim N\left(0, \Omega^x_{t-\Delta t}\right)$ and

$$
\Omega^x_{t-\Delta t} \equiv Var_{t-\Delta t}\left(\eta_t^x\right) = \int_0^{\Delta t}\exp\left(-\mathcal{K}s\right)\Sigma\Sigma'\exp\left(-\mathcal{K}'s\right)ds = N\Xi N',
$$

with $\mathcal{K} = NDN^{-1}$, $D = diag\left([d_1, ..., d_N]\right)$, and $\Xi_{i,j} = \left[(N^{-1}\Sigma)(N^{-1}\Sigma)'\right]_{i,j}\frac{1-\exp[-(d_i+d_j)\cdot t]}{(d_i+d_j)}$. The covariance matrix between η_t^x and η_t^q is given by

$$
\Omega^{xq}_{t-\Delta t} = Cov_{t-\Delta t}\left[\eta_t^x, \eta_t^q\right] = \int_0^{\Delta t}\exp\left(-\mathcal{K}s\right)\Sigma\sigma_q ds = \mathcal{K}^{-1}\left(I - \exp\left(-\mathcal{K}s\Delta t\right)\right)\Sigma\sigma_q.
$$

where I signifies the identity matrix.

Next, we consider the state variable v_t. From $dv_t = \kappa_v (\mu_v - v_t) dt + \sigma_v \sqrt{v_t} \left(\sqrt{1-\rho^2} dW_{v,t} + \rho dW_{\perp,t} \right)$, we have

$$v_t = e^{-\kappa_v \Delta t} v_{t-\Delta t} + \mu_v \left(1 - e^{-\kappa_v \Delta t}\right) + \eta_t^v$$

where $\eta_t^v = \sigma_v \int_{t-\Delta t}^t e^{\kappa_v (s-t)} \sqrt{v_s} \left(\sqrt{1-\rho^2} dB_s^v + \rho dB_s^\perp \right) \sim N\left(0, \Omega_{t-1}^v\right)$ and

$$\begin{aligned} \Omega_{t-\Delta t}^v &= E_{t-\Delta t} \left[\sigma_v^2 \int_{t-\Delta t}^t e^{2\kappa_v (s-t)} v_s ds \right] \\ &= \mu_v \sigma_v^2 \frac{\left(1 - e^{-\kappa_v \Delta t}\right)^2}{2\kappa_v} + v_{t-1} \sigma_v^2 \frac{e^{-\kappa_v \Delta t} - e^{-2\kappa_v \Delta t}}{\kappa_v} \end{aligned}$$

The covariance between η_t^v and η_t^x is zero by construction while the covariance between the inflation and volatility innovation terms is

$$\begin{aligned} \Omega_{t-\Delta t}^{vq} &= Cov_{t-\Delta t}\left[\eta_t^v, \eta_t^q\right] = \rho \sigma_v E_{t-\Delta t} \left[\int_{t-\Delta t}^t e^{\kappa_v (s-t)} v_s ds \right] \\ &= \rho \sigma_v \left[\mu_v \frac{\left(1 - e^{-\frac{\kappa_v}{2} \Delta t}\right)^2}{\kappa_v} + v_{t-\Delta t} \frac{e^{-\frac{\kappa_v}{2} \Delta t} - e^{-\kappa_v \Delta t}}{\kappa_v / 2} \right]. \end{aligned}$$

Next, we consider the state variable δ_t. From $d\delta_t = \kappa_\delta (\mu_\delta - \delta_t) dt + \sigma_\delta dW_{\delta,t}$, we have

$$\delta_t = e^{-\kappa_\delta \Delta t} \delta_{t-\Delta t} + \mu_\delta \left(1 - e^{-\kappa_\delta \Delta t}\right) + \eta_t^\delta$$

where $\eta_t^\delta = \sigma_\delta \int_{t-\Delta t}^t e^{\kappa_\delta (u-t)} dW_{\delta,u} \sim N\left(0, \Omega_{t-\Delta t}^\delta\right)$ and

$$\Omega_{t-\Delta t}^\delta = E_{t-\Delta t} \left[\sigma_\delta^2 \int_{t-\Delta t}^t e^{2\kappa_\delta (s-t)} ds \right] = \sigma_\delta^2 \frac{1 - e^{-2\kappa_\delta \Delta t}}{2\kappa_\delta}$$

Last, we consider the state variable s_t.

$$s_t - s_{t-\Delta t} = \left(\phi_0 \Delta t + (\phi_x \Delta t)' x_{t-\Delta t} + \phi_v \Delta t v_{t-\Delta t} + \phi_\delta \Delta t \delta_{t-\Delta t} + \phi_s \Delta t s_{t-\Delta t} \right) + \eta_t^s$$

where $\eta_t^s = \sigma_s \int_{t-\Delta t}^t \exp\left(-\phi_s (u-t)\right) dW_{\delta,u} \sim N\left(0, \Omega_{t-\Delta t}^s\right)$ and

$$\Omega_{t-\Delta t}^s = E_{t-\Delta t} \left[\sigma_s^2 \int_{t-\Delta t}^t e^{2(-\phi_s)(u-t)} du \right] = \sigma_s^2 \frac{e^{2\phi_s \Delta t} - 1}{2\phi_s}$$

and

$$\begin{aligned} \Omega_{t-\Delta t}^{\delta s} &= \sigma_\delta \sigma_s E_{t-\Delta t} \left[\int_{t-\Delta t}^t e^{\kappa_\delta (u-t)} dW_{\delta,u} \int_{t-\Delta t}^t e^{-\phi_s (u-t)} dW_{\delta,u} \right] \\ &= \sigma_\delta \sigma_s E_{t-\Delta t} \left[\int_{t-\Delta t}^t e^{(\kappa_\delta - \phi_s)(u-t)} du \right] \\ &= \sigma_\delta \sigma_s \frac{1 - e^{-(\kappa_\delta - \phi_s)\Delta t}}{\kappa_\delta - \phi_s} \end{aligned}$$

In summary, the dynamics of the state vector $Z_t = (q_t, x_t', v_t, \delta_t, s_t)'$ follows the VAR process

$$Z_t = \mathcal{A} + \mathcal{B} Z_{t-\Delta t} + \eta_t$$

where $\mathcal{A} = \begin{bmatrix} \rho_0^\pi \Delta t \\ \left(I - \exp\left(-\mathcal{K}\Delta t\right)\right)\mu \\ \left(1 - e^{-\kappa_v \Delta t}\right)\mu_v \\ \left(1 - e^{-\kappa_\delta \Delta t}\right)\mu_\delta \\ \phi_0 \Delta t \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 1 & \rho_x^{\pi\prime}\Delta t & \rho_v^\pi \Delta t & \rho_\delta^\pi \Delta t & \rho_s^\pi \Delta t \\ 0 & \exp\left(-\mathcal{K}\Delta t\right) & & & \\ 0 & 0_{1\times 3} & e^{-\kappa_v \Delta t} & & \\ 0 & 0_{1\times 3} & 0 & e^{-\kappa_\delta \Delta t} & \\ 0 & \phi_x' \Delta t & \phi_v \Delta t & \phi_\delta \Delta t & 1 + \phi_s \Delta t \end{bmatrix}$, $\eta_t = \begin{pmatrix} \eta_t^q \\ \eta_t^x \\ \eta_t^v \\ \eta_t^\delta \\ \eta_t^s \end{pmatrix} \sim N\left(0, \Omega_{t-\Delta t}\right)$ and $\Omega_{t-\Delta t} = \begin{bmatrix} \Omega_{t-\Delta t}^q & \Omega_{t-\Delta t}^{xq\prime} & \Omega_{t-\Delta t}^{vq} & & \\ \Omega_{t-\Delta t}^{xq} & \Omega_{t-\Delta t}^x & & & \\ \Omega_{t-\Delta t}^{vq} & & \Omega_{t-\Delta t}^v & & \\ & & & \Omega_{t-\Delta t}^\delta & \Omega_{t-\Delta t}^{\delta s} \\ & & & \Omega_{t-\Delta t}^{\delta s} & \Omega_{t-\Delta t}^s \end{bmatrix}$.