Documentation: MODEL_FRBA_oil

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1 Model

Suppose that there are a vector of four latent variables $x_t = (x_{1t}, x_{2t}, x_{3t})'$ that drive nominal and real yields as well as inflation. Their dynamics under the physical measure is

$$dx_t = \mathcal{K}(\mu - x_t)\,dt + \Sigma dW_{x,t} \tag{1}$$

and

$$\begin{aligned} d\delta_t &= \kappa_\delta(\mu_\delta - \delta_t)\,dt + \sigma_\delta dW_{\delta,t} &(2)\\ ds_t &= \left(r_t^N - \delta_t - \frac{1}{2}\sigma_s^2 + \sigma_s\Lambda_{\delta,t}^N\right)dt + \sigma_s dW_{\delta,t} &(3)\end{aligned}$$

The nominal pricing kernel takes the form

$$dM_t^N/M_t^N = -r_t^N dt - \Lambda_{x,t}' dW_{x,t} - \Lambda_{\delta,t} dW_{\delta,t} \tag{4}$$

where the nominal short rate is

$$r_t^N = \rho_0^N + \rho_x^{N\prime} x_t + \rho_\delta^N \delta_t + \rho_s^N s_t \tag{5}$$

and the vector of prices of risk is given by

$$\begin{aligned}\Lambda_{x,t}^N &= \lambda_{0,x}^N + \lambda_x^N x_t &(6)\\ \Lambda_{\delta,t}^N &= \lambda_{0,\delta}^N + \lambda_\delta^N \delta_t &(7)\end{aligned}$$

Let $q_t \equiv \log Q_t$ denote the log price level. The price level evolves as follows:

$$dq_t = \pi_t dt + \sigma_q' dW_{x,t} + \sigma_q^\perp dW_{\perp,t} \tag{8}$$

where the instantaneous expected inflation rate is given by

$$\pi_t = \rho_0^\pi + \rho_x^{\pi\prime} x_t + \rho_\delta^\pi \delta_t + \rho_s^\pi s_t \tag{9}$$

Under the risk-neutral measure,

$$\begin{aligned} d\delta_t &= \kappa_\delta(\mu_\delta - \delta_t)\,dt + \sigma_\delta dW_{\delta,t}\\ &= \kappa_\delta(\mu_\delta - \delta_t)\,dt + \sigma_\delta\left(dW_{\delta,t}^* - \Lambda_{\delta,t}^N dt\right)\\ &\equiv \kappa_\delta^*(\mu_\delta^* - \delta_t)\,dt + \sigma_\delta dW_{\delta,t}^*\end{aligned}$$

and

$$
\begin{aligned}
ds_t &= \left(r_t^N - \delta_t - \frac{1}{2}\sigma_s^2 + \sigma_s \Lambda_{\delta,t}^N\right) dt + \sigma_s dW_{\delta,t} \\
&\equiv \left(\phi_0 + \phi_x' x_t + \phi_\delta \delta_t + \phi_s s_t\right) dt + \sigma_s dW_{\delta,t} \\
&\equiv \left(\phi_0^* + \phi_x^{*\prime} x_t + \phi_\delta^* \delta_t + \phi_s^* s_t\right) dt + \sigma_s dW_{\delta,t}^*
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_\delta^* &= \kappa_\delta + \sigma_\delta \lambda_\delta^N \\
\kappa_\delta^* \mu_\delta^* &= \kappa_\delta \mu_\delta - \sigma_\delta \lambda_{0,\delta}^N
\end{aligned}
$$

$$
\begin{aligned}
\phi_0 &= \rho_0^N - \frac{1}{2}\sigma_s^2 + \sigma_s \lambda_{0,\delta}^N \\
\phi_x &= \rho_x^N \\
\phi_\delta &= \rho_\delta^N - 1 + \sigma_s \lambda_\delta^N \\
\phi_s &= \rho_s^N
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_0^* &= \rho_0^N - \frac{1}{2}\sigma_s^2 = \phi_0 - \sigma_s \lambda_{0,\delta}^N \\
\phi_x^* &= \rho_x^N \\
\phi_\delta^* &= \rho_\delta^N - 1 = \phi_\delta - \sigma_s \lambda_\delta^N \\
\phi_s^* &= \rho_s^N
\end{aligned}
$$

Remark. We initialize λ_δ^N so that $\phi_\delta = 0 = \rho_\delta^N - 1 + \sigma_s \lambda_\delta^N$, or $\lambda_\delta^N = \left(1 - \rho_\delta^N\right)/\sigma_s$. Furthermore, as we show later, the long run mean of s_t is

$$
s_\infty \equiv -\frac{1}{\phi_s}\left(\phi_0 + \phi_x' \mu + \phi_v \mu_v + \phi_\delta \mu_\delta\right)
$$

Given the other parameters including ϕ_s, we can initialize $\lambda_{0,\delta}^N$ by matching sample mean of s_t, denoted by $\widehat{s}$, to its long run mean s_∞. Given that $\phi_\delta = 0$ via initialization and $\mu = 0$ by normalization and $\phi_v = \rho_v^N = 0$ by restriction, we have

$$
\begin{aligned}
\phi_0 &= -\left(\phi_v \mu_v + \phi_s \widehat{s}\right) = \rho_0^N - \frac{1}{2}\sigma_s^2 + \sigma_s \lambda_{0,\delta}^N \\
&\Rightarrow \lambda_{0,\delta}^N = -\frac{\phi_v \mu_v + \phi_s \widehat{s} + \rho_0^N - \frac{1}{2}\sigma_s^2}{\sigma_s} \\
&\Rightarrow \lambda_{0,\delta}^N = -\frac{\phi_s \widehat{s} + \rho_0^N - \frac{1}{2}\sigma_s^2}{\sigma_s}
\end{aligned}
$$

NOTE:

$$
\begin{aligned}
dx_t &= \mathcal{K}^* \left(\mu^* - x_t\right) dt + \Sigma dW_{x,t}^* &\quad (10) \\
d\delta_t &= \kappa_\delta^* \left(\mu_\delta^* - \delta_t\right) dt + \sigma_\delta dW_{\delta,t}^* &\quad (11) \\
ds_t &= \left(\phi_0^* + \phi_x^{*\prime} x_t + \phi_\delta^* \delta_t + \phi_s^* s_t\right) dt + \sigma_s dW_{\delta,t}^* &\quad (12)
\end{aligned}
$$

Oil futures. Let $s_t \equiv \log(S_t)$ denote the spot oil price.

$$
\begin{aligned}
F_{t,\tau}^{oil} &= E_t^*\left[\exp\left(s_{t+\tau}\right)\right] \\
&\equiv \exp\left(A_\tau^{oil} + B_\tau^{oil\prime} x_t + D_\tau^{oil}\delta_t + E_\tau^{oil} s_t\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\dot{A}_\tau^{oil} &= \left(\mathcal{K}^*\mu^*\right)' B_\tau^{oil} + \left(\kappa_\delta^*\mu_\delta^*\right) D_\tau^{oil} + \phi_0^* E_\tau^{oil} \\
&\quad + \frac{1}{2} B_\tau^{oil\prime}\Sigma\Sigma' B_\tau^{oil} + \frac{1}{2}\sigma_\delta^2\left(D_\tau^{oil}\right)^2 + \frac{1}{2}\sigma_s^2\left(E_\tau^{oil}\right)^2 \\
\dot{B}_\tau^{oil} &= -\left(\mathcal{K}^*\right)' B_\tau^{oil} + \phi_x^* E_\tau^{oil} \\
\dot{D}_\tau^{oil} &= -\kappa_\delta^* D_\tau^{oil} + \phi_\delta^* E_\tau^{oil} \\
\dot{E}_\tau^{oil} &= \phi_s^* E_\tau^{oil}
\end{aligned}
$$

and the initial conditions are: $A_0^{oil} = B_0^{oil} = D_0^{oil} = 0$ and $E_0^{oil} = 1$.

Consider the transformation

$$
\begin{aligned}
\mathcal{A} &= A_\tau^{oil} \\
\mathcal{B} &= \begin{pmatrix} B_\tau^{oil} \\ D_\tau^{oil} \\ E_\tau^{oil} - 1 \end{pmatrix}
\end{aligned}
$$

Then $\mathcal{A}(0) = \mathcal{B}(0) = \mathcal{C}(0) = 0$ and

$$
\begin{aligned}
\dot{\mathcal{A}} &= \left(\phi_0^* + \frac{1}{2}\sigma_s^2\right) + \begin{pmatrix} \mathcal{K}^*\mu^* \\ \kappa_\delta^*\mu_\delta^* \\ \phi_0^* + \sigma_s^2 \end{pmatrix}' \mathcal{B} + \frac{1}{2}\mathcal{B}' \begin{pmatrix} \Sigma\Sigma' & & \\ & \sigma_\delta^2 & \\ & & \sigma_s^2 \end{pmatrix} \mathcal{B} \\
\dot{\mathcal{B}} &= \begin{pmatrix} \phi_x^* \\ \phi_\delta^* \\ \phi_s^* \end{pmatrix} + \begin{pmatrix} -\left(\mathcal{K}^*\right)' & 0 & \phi_x^* \\ 0 & -\kappa_\delta^* & \phi_\delta^* \\ 0 & 0 & \phi_s^* \end{pmatrix} \mathcal{B}
\end{aligned}
$$

Based on the observation $M_t^R = M_t^N Q_t$, we have

$$
\begin{aligned}
dM_t^R/M_t^R &= dM_t^N/M_t^N + dQ_t/Q_t + \left(dM_t^N/M_t^N\right)\cdot\left(dQ_t/Q_t\right) \\
&= -r_t^N dt - \Lambda_{x,t}^{N\prime} dW_{x,t} - \Lambda_{\delta,t}^N dW_{\delta,t} \\
&\quad + \left[\pi_t + \frac{1}{2}\left(\sigma_q'\sigma_q + \left(\sigma_q^\perp\right)^2\right) - \sigma_q'\Lambda_{x,t}^N\right] dt + \sigma_q' dW_{x,t} + \sigma_q^\perp dW_{\perp,t} \\
&\equiv -r_t^R dt - \Lambda_{x,t}^{R\prime} dW_{x,t} - \Lambda_{\delta,t}^R dW_{\delta,t}
\end{aligned}
$$

where

$$
\begin{aligned}
r_t^R &= r_t^N - \left[\pi_t + \frac{1}{2}\left(\sigma_q'\sigma_q + \left(\sigma_q^\perp\right)^2\right)\right] + \sigma_q'\Lambda_{x,t}^N \equiv \rho_0^R + \rho_x^{R\prime} x_t + \rho_\delta^R \delta_t + \rho_s^R s_t \\
\Lambda_{x,t}^R &= \Lambda_{x,t}^N - \sigma_q \equiv \lambda_{0,x}^R + \lambda_x^{R\prime} x_t \\
\Lambda_{\delta,t}^R &= \Lambda_{\delta,t}^N \equiv \lambda_{0,\delta}^R + \lambda_\delta^R \delta_t
\end{aligned}
$$

and

$$\begin{aligned}
\rho_0^R &= \rho_0^N - \rho_0^\pi - \frac{1}{2}\left(\sigma_q'\sigma_q + \left(\sigma_q^\perp\right)^2\right) + \lambda_{0,x}'\sigma_q \\
\rho_x^R &= \rho_x^N - \rho_x^\pi + \lambda_x^{N\prime}\sigma_q \\
\rho_\delta^R &= \rho_\delta^N - \rho_\delta^\pi \\
\rho_s^R &= \rho_s^N - \rho_s^\pi
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{0,x}^R &= \lambda_{0,x}^N - \sigma_q, \text{ and } \lambda_x^R = \lambda_x^N \\
\lambda_{0,\delta}^R &= \lambda_{0,\delta}^N, \text{ and } \lambda_\delta^R = \lambda_\delta^N
\end{aligned}$$

1.1 Dynamics under the Risk-Neutral Measure and Bond Pricing

Let

$$\Lambda_t^N \equiv \begin{pmatrix} \Lambda_{x,t}^N \\ \Lambda_{\delta,t}^N \end{pmatrix}, W_t \equiv \begin{pmatrix} W_{x,t} \\ W_{\delta,t} \end{pmatrix}.$$

Then, the Radon-Nikodym derivative of the risk neutral measure $\mathbb{P}^*$ with respect to the physical measure $\mathbb{P}$ is given by

$$\left(\frac{d\mathbb{P}^*}{d\mathbb{P}}\right)_{t,T} = \exp\left[-\frac{1}{2}\int_t^T \Lambda_s^{N\prime}\Lambda_s^N ds - \int_t^T \Lambda_s^{N\prime} dW_s\right] \tag{13}$$

By the Girsanov theorem, $dW_t^* = dW_t + \Lambda_t^N dt$ is a standard Brownian motion under the risk-neutral probability measure $\mathbb{P}^*$. It implies that under the risk neutral measure,

$$\begin{aligned}
dW_{x,t}^* &= dW_{x,t} + \Lambda_{x,t}^N dt \\
dW_{\delta,t}^* &= dW_{\delta,t} + \Lambda_{\delta,t}^N dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
dx_t &= \mathcal{K}(\mu - x_t)\,dt + \Sigma dW_{x,t} \\
&= \mathcal{K}(\mu - x_t)\,dt + \Sigma\left(dW_{x,t}^* - \Lambda_{x,t}^N dt\right) \\
&= \left[\left(\mathcal{K}\mu - \Sigma\lambda_0^N\right) - \left(\mathcal{K} + \Sigma\lambda_x^N\right)x_t\right]dt + \Sigma dW_{x,t}^* \\
&\equiv \mathcal{K}^*(\mu^* - x_t)\,dt + \Sigma dW_{x,t}^*,
\end{aligned}$$

$$\begin{aligned}
d\delta_t &= \kappa_\delta^*(\mu_\delta^* - \delta_t)\,dt + \sigma_\delta dW_{\delta,t}^* \\
ds_t &= \left(\phi_0^* + \phi_x^{*\prime}x_t + \phi_\delta^*\delta_t + \phi_s^* s_t\right)dt + \sigma_s dW_{\delta,t}^*
\end{aligned}$$

and

$$\begin{aligned}
dq_t &= \pi_t dt + \sigma_q' dW_{x,t} + \sigma_q^\perp dW_{\perp,t} \\
&= \left(\rho_0^\pi + \rho_x^{\pi\prime}x_t + \rho_\delta^\pi\delta_t + \rho_s^\pi s_t\right)dt + \sigma_q'\left(dW_{x,t}^* - \Lambda_{x,t}^N dt\right) + \sigma_q^\perp dW_{\perp,t}^* \text{ (suppose } dW_{\perp,t}^* = dW_{\perp,t}) \\
&= \left(\rho_0^\pi + \rho_x^{\pi\prime}x_t + \rho_\delta^\pi\delta_t + \rho_s^\pi s_t - \sigma_q'\left(\lambda_{0,x}^N + \lambda_x^N x_t\right)\right)dt + \sigma_q' dW_{x,t}^* + \sigma_q^\perp dW_{\perp,t}^* \\
&= \left(\rho_0^\pi - \lambda_{0,x}^{N\prime}\sigma_q + \left(\rho_x^\pi - \lambda_x^{N\prime}\sigma_q\right)' x_t + \rho_\delta^\pi\delta_t + \rho_s^\pi s_t\right)dt + \sigma_q' dW_{x,t}^* + \sigma_q^\perp dW_{\perp,t}^* \\
&\equiv \left(\rho_0^{\pi*} + \rho_x^{\pi*\prime}x_t + \rho_\delta^{\pi*}\delta_t + \rho_s^{\pi*}s_t\right)dt + \sigma_q' dW_{x,t}^* + \sqrt{v_t}\,dW_{\perp,t}^*,
\end{aligned}$$

where

$$
\begin{aligned}
\mathcal{K}^* &= \mathcal{K}+\Sigma\lambda_x^N \\
\mathcal{K}^*\mu^* &= \mathcal{K}\mu-\Sigma\lambda_{0,x}^N \\
\kappa_\delta^* &= \kappa_\delta + \sigma_\delta\lambda_\delta^N \\
\kappa_\delta^*\mu_\delta^* &= \kappa_v\mu_v - \sigma_\delta\lambda_{0,\delta}^N \\
\pi_t^* &= \rho_0^{\pi*} + \rho_x^{\pi*\prime}x_t + \rho_\delta^{\pi*}\delta_t + \rho_s^{\pi*}s_t \\
\rho_0^{\pi*} &= \rho_0^\pi - \lambda_{0,x}^{N\prime}\sigma_q \\
\rho_x^{\pi*} &= \rho_x^\pi - \lambda_x^{N\prime}\sigma_q \\
\rho_\delta^{\pi*} &= \rho_\delta^\pi \text{ and } \rho_s^{\pi*} = \rho_s^\pi
\end{aligned}
$$

We first establish a well-known result of affine-form of nominal yields in Proposition 1 below.

Proposition 1 *Under this model, nominal and real bond prices take the exponential-affine form*

$$
P_{t,\tau}^i = \exp\left(A_\tau^i + B_\tau^{i\prime}x_t + D_\tau^i\delta_t + E_\tau^i s_t\right), \ \ i = N, R \tag{14}
$$

and nominal and real yields take the affine form

$$
y_{t,\tau}^i = a_\tau^i + b_\tau^{i\prime}x_t + d_\tau^i\delta_t + e_\tau^i s_t, \ \ i = N, R \tag{15}
$$

where $a_\tau^i \equiv -A_\tau^i/\tau$, $b_\tau^i \equiv -B_\tau^i/\tau$, $d_\tau^i \equiv -D_\tau^i/\tau$, *and* $e_\tau^i \equiv -E_\tau^i/\tau$, *and* A_τ^i, B_τ^i, D_τ^i, E_τ^i $(i = N, R)$ *satisfy the following system of ODEs:*

$$
\begin{aligned}
\frac{dA_\tau^i}{d\tau} &= -\rho_0^i + \left(\mathcal{K}\mu - \Sigma\lambda_0^i\right)' B_\tau^i + \kappa_\delta^*\mu_\delta^* D_\tau^i + \phi_0^* E_\tau^i + \frac{1}{2}B_\tau^{i\prime}\Sigma\Sigma' B_\tau^i + \frac{1}{2}\sigma_\delta^2\left(D_\tau^i\right)^2 + \frac{1}{2}\sigma_s^2\left(E_\tau^i\right)^2, \textit{ with } A_0^i = 0 \\
\frac{dB_\tau^i}{d\tau} &= -\rho_x^i - \left(\mathcal{K} + \Sigma\lambda_x^i\right)' B_\tau^i + \phi_x^* E_\tau^i, \textit{ with } B_0^i = 0 \\
\frac{dD_\tau^i}{d\tau} &= -\rho_\delta^i - \kappa_\delta^* D_\tau^i + \phi_\delta^* E_\tau^i, \textit{ with } D_0^i = 0 \\
\frac{dE_\tau^i}{d\tau} &= -\rho_s^i + \phi_s^* E_\tau^i, \textit{ with } E_0^i = 0
\end{aligned}
$$

Consider the transformation

$$
\begin{aligned}
\mathcal{A} &= A_\tau^i \\
\mathcal{B} &= \begin{pmatrix} B_\tau^i \\ D_\tau^i \\ E_\tau^i \end{pmatrix}
\end{aligned}
$$

Then

$$
\begin{aligned}
\dot{\mathcal{A}} &= -\rho_0^i + \begin{pmatrix} \mathcal{K}\mu - \Sigma\lambda_0^i \\ \kappa_\delta^*\mu_\delta^* \\ \phi_0^* \end{pmatrix}' \mathcal{B} + \frac{1}{2}\mathcal{B}' \begin{pmatrix} \Sigma\Sigma' & & \\ & \sigma_\delta^2 & \\ & & \sigma_s^2 \end{pmatrix} \mathcal{B} \\
\dot{\mathcal{B}} &= \begin{pmatrix} -\rho_x^i \\ -\rho_\delta^i \\ -\rho_s^i \end{pmatrix} + \begin{pmatrix} -\left(\mathcal{K} + \Sigma\lambda_x^i\right)' & 0 & \phi_x^* \\ 0 & -\kappa_\delta^* & \phi_\delta^* \\ 0 & 0 & \phi_s^* \end{pmatrix} \mathcal{B}
\end{aligned}
$$

Survey Yield Forecast. To calculate survey yield forecasts, we need the following results:

$$\begin{aligned} E_t\left[x_{t+\tau}\right] &= \mu+\exp\left(-\kappa\tau\right)\left(x_t-\mu\right) \\ E_t\left[\delta_{t+\tau}\right] &= \mu_\delta+\exp\left(-\kappa_\delta\tau\right)\left(\delta_t-\mu_\delta\right) \end{aligned}$$

Next we calculate, $E_t\left[s_{t+\tau}\right]$. Note that

$$\begin{aligned} ds_t &= \left(\phi_0+\phi_x' x_t+\phi_\delta\delta_t+\phi_s s_t\right)dt+\sigma_s dW_{\delta,t} \\ d\left[\exp\left(-\phi_s t\right)s_t\right] &= \exp\left(-\phi_s t\right)\left[\left(\phi_0+\phi_x' x_t+\phi_\delta\delta_t\right)dt+\sigma_s dW_{\delta,t}\right] \end{aligned}$$

and

$$\begin{aligned} & \exp\left(-\phi_s\left(t+\tau\right)\right)s_{t+\tau}-\exp\left(-\phi_s t\right)s_t \\ = & \int_t^{t+\tau}\left[\exp\left(-\phi_s u\right)\left(\phi_0+\phi_x' x_u+\phi_\delta\delta_u\right)du+\exp\left(-\phi_s u\right)\sigma_s dW_{\delta,u}\right] \end{aligned}$$

implying

$$\begin{aligned} & \exp\left(-\phi_s\left(t+\tau\right)\right)E_t\left[s_{t+\tau}\right]-\exp\left(-\phi_s t\right)s_t \\ = & \int_t^{t+\tau}\exp\left(-\phi_s u\right)\begin{pmatrix} \phi_0+\phi_x'\left(\left[\mu+\exp\left(-\kappa\left(u-t\right)\right)\left(x_t-\mu\right)\right]\right) \\ +\phi_\delta\left(\left[\mu_\delta+\exp\left(-\kappa_\delta\left(u-t\right)\right)\left(\delta_t-\mu_\delta\right)\right]\right)\end{pmatrix}du \\ = & \frac{1}{\phi_s}\left(\exp\left(-\phi_s t\right)-\exp\left(-\phi_s\left(t+\tau\right)\right)\right)\left(\phi_0+\phi_x'\mu+\phi_\delta\mu_\delta\right) \\ & +\exp\left(-\phi_s t\right)\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(I-\exp\left(-\left(\kappa+\phi_s I\right)\tau\right)\right)\left(x_t-\mu\right) \\ & +\exp\left(-\phi_s t\right)\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(1-\exp\left(-\left(\kappa_\delta+\phi_s\right)\tau\right)\right)\left(\delta_t-\mu_\delta\right) \end{aligned}$$

or

$$\begin{aligned} & E_t\left[s_{t+\tau}\right] \\ = & \exp\left(\phi_s\tau\right)s_t+\frac{1}{\phi_s}\left(\exp\left(\phi_s\tau\right)-1\right)\left(\phi_0+\phi_x'\mu+\phi_\delta\mu_\delta\right) \\ & +\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right)I-\exp\left(-\kappa\tau\right)\right)\left(x_t-\mu\right) \\ & +\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_\delta\tau\right)\right)\left(\delta_t-\mu_\delta\right) \end{aligned}$$

and

$$Var_t\left[s_{t+\tau}\right]=E_t\left[\int_{-\tau}^0\exp\left(-2\phi_s u\right)\sigma_s^2 du\right]=\frac{1}{2\phi_s}\left(\exp\left(2\phi_s\tau\right)-1\right)\sigma_s^2$$

Therefore,

$$\begin{aligned} E_t^{svy}\left[y_{t+\tau,3m}^N\right] &= E_t^{mkt}\left[y_{t+\tau,3m}^N\right]+\epsilon_{t,\tau}^f \\ &= E_t\left[a_{3m}^N+b_{3m}^{N\prime}x_{t+\tau}+d_{3m}^N\delta_{t+\tau}+e_{3m}^N s_{t+\tau}\right]+\epsilon_{t,\tau}^f \\ &= a_{3m}^N+b_{3m}^{N\prime}\left[\mu+\exp\left(-\kappa\tau\right)\left(x_t-\mu\right)\right] \\ & \quad +d_{3m}^N\left[\mu_\delta+\exp\left(-\kappa_\delta\tau\right)\left(\delta_t-\mu_\delta\right)\right] \\ & \quad +e_{3m}^N\left[\begin{array}{l} \exp\left(\phi_s\tau\right)s_t+\frac{1}{\phi_s}\left(\exp\left(\phi_s\tau\right)-1\right)\left(\phi_0+\phi_x'\mu+\phi_\delta\mu_\delta\right) \\ +\phi_x'\left(\kappa+\phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right)I-\exp\left(-\kappa\tau\right)\right)\left(x_t-\mu\right) \\ +\phi_\delta\left(\kappa_\delta+\phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right)-\exp\left(-\kappa_\delta\tau\right)\right)\left(\delta_t-\mu_\delta\right)\end{array}\right]+\epsilon_{t,\tau}^f \\ &\equiv a_\tau^f+b_\tau^{f\prime}x_t+d_\tau^f\delta_t+e_\tau^f s_t+\epsilon_{t,\tau}^f \end{aligned}$$

where

$$
\begin{aligned}
a_\tau^f &= a_{3m}^N + b_{3m}^{N\prime}\left[I - \exp\left(-\kappa\tau\right)\right]\mu + d_{3m}^N\left[1 - \exp\left(-\kappa_\delta\tau\right)\right]\mu_\delta \\
&\quad + e_{3m}^N \begin{bmatrix} \frac{1}{\phi_s}\left(\exp\left(\phi_s\tau\right) - 1\right)\left(\phi_0 + \phi_x^\prime\mu + \phi_\delta\mu_\delta\right) \\ -\phi_x^\prime\left(\kappa + \phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right)I - \exp\left(-\kappa\tau\right)\right)\mu \\ -\phi_\delta\left(\kappa_\delta + \phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right) - \exp\left(-\kappa_\delta\tau\right)\right)\mu_\delta \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
b_\tau^{f\prime} &= b_{3m}^{N\prime}\exp\left(-\kappa\tau\right) + e_{3m}^N\phi_x^\prime\left(\kappa + \phi_s I\right)^{-1}\left(\exp\left(\phi_s\tau\right)I - \exp\left(-\kappa\tau\right)\right) \\
d_\tau^f &= d_{3m}^N\exp\left(-\kappa_\delta\tau\right) + e_{3m}^N\phi_\delta\left(\kappa_\delta + \phi_s\right)^{-1}\left(\exp\left(\phi_s\tau\right) - \exp\left(-\kappa_\delta\tau\right)\right) \\
e_\tau^f &= e_{3m}^N\exp\left(\phi_s\tau\right)
\end{aligned}
$$

Lastly, we turn to the long-range forecast: $E_t^{mkt}\left[\overline{y}_{3m,T_1,T_2}^N\right]$ as follows. Note that

$$
\begin{aligned}
& E_t\left[\frac{1}{T_2 - T_1}\int_{t+T_1}^{t+T_2} s_u du\right] \\
= \;& \frac{1}{T_2 - T_1}\int_{T_1}^{T_2}\begin{bmatrix} \exp\left(\phi_s u\right)s_t + \frac{1}{\phi_s}\left(\exp\left(\phi_s u\right) - 1\right)\left(\phi_0 + \phi_x^\prime\mu + \phi_\delta\mu_\delta\right) \\ +\phi_x^\prime\left(\kappa + \phi_s I\right)^{-1}\left(\exp\left(\phi_s u\right)I - \exp\left(-\kappa u\right)\right)\left(x_t - \mu\right) \\ +\phi_\delta\left(\kappa_\delta + \phi_s\right)^{-1}\left(\exp\left(\phi_s u\right) - \exp\left(-\kappa_\delta u\right)\right)\left(\delta_t - \mu_\delta\right) \end{bmatrix} du \\
= \;& W_{T_1,T_2}^s s_t + \frac{1}{\phi_s}\left(W_{T_1,T_2}^s - 1\right)\left(\phi_0 + \phi_x^\prime\mu + \phi_\delta\mu_\delta\right) \\
& +\phi_x^\prime\left(\kappa + \phi_s I\right)^{-1}\left(W_{T_1,T_2}^s I - W_{T_1,T_2}^x\right)\left(x_t - \mu\right) \\
& +\phi_\delta\left(\kappa_\delta + \phi_s\right)^{-1}\left(W_{T_1,T_2}^s - W_{T_1,T_2}^\delta\right)\left(\delta_t - \mu_\delta\right)
\end{aligned}
$$

where

$$
\begin{aligned}
W_{T_1,T_2}^x &= \frac{1}{T_2 - T_1}\int_{T_1}^{T_2}\exp\left(-\kappa s\right)ds = \frac{1}{T_2 - T_1}\kappa^{-1}\left(\exp\left(-\kappa T_1\right) - \exp\left(-\kappa T_2\right)\right) \\
W_{T_1,T_2}^\delta &= \frac{1}{T_2 - T_1}\kappa_\delta^{-1}\left(\exp\left(-\kappa_\delta T_1\right) - \exp\left(-\kappa_\delta T_2\right)\right) \\
W_{T_1,T_2}^s &= \frac{1}{T_2 - T_1}\int_{T_1}^{T_2}\exp\left(\phi_s u\right)du = \frac{1}{T_2 - T_1}\phi_s^{-1}\left(\exp\left(\phi_s T_2\right) - \exp\left(\phi_s T_1\right)\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E_t^{svy}\left[\overline{y}^N_{3m,T_1,T_2}\right] &= E_t^{mkt}\left[\overline{y}^N_{3m,T_1,T_2}\right] + \epsilon^f_{t,LT} \\
&= E_t\left[\frac{1}{T_2-T_1}\int_{T_1}^{T_2} y^N_{s,3m}ds\right] + \epsilon^f_{t,LT} \\
&= E_t\left[\frac{1}{T_2-T_1}\int_{T_1}^{T_2}\left(a^N_{3m} + b^{N\prime}_{3m}x_s + d^N_{3m}\delta_s + e^N_{3m}s_s\right)ds\right] + \epsilon^f_{t,LT} \\
&= a^N_{3m} + b^{N\prime}_{3m}\left[\left(I - W^x_{T_1,T_2}\right)\mu + W^x_{T_1,T_2}x_t\right] \\
&\quad + d^N_{3m}\left[\left(1 - W^\delta_{T_1,T_2}\right)\mu_\delta + W^\delta_{T_1,T_2}\delta_t\right] \\
&\quad + e^N_{3m}\left[\begin{array}{c} W^s_{T_1,T_2}s_t + \frac{1}{\phi_s}\left(W^s_{T_1,T_2} - 1\right)\left(\phi_0 + \phi_x'\mu + \phi_\delta\mu_\delta\right) \\ +\phi_x'\left(\kappa + \phi_s I\right)^{-1}\left(W^s_{T_1,T_2}I - W^x_{T_1,T_2}\right)\left(x_t - \mu\right) \\ +\phi_\delta\left(\kappa_\delta + \phi_s\right)^{-1}\left(W^s_{T_1,T_2} - W^\delta_{T_1,T_2}\right)\left(\delta_t - \mu_\delta\right) \end{array}\right] + \epsilon^f_{t,LT} \\
&\equiv a^f_{LT} + b^{f\prime}_{LT}x_t + d^f_{LT}\delta_t + \epsilon^f_{t,LT}
\end{aligned}
$$

where

$$
\begin{aligned}
a^f_{LT} &= a^N_{3m} + b^{N\prime}_{3m}\left(I - W^x_{T_1,T_2}\right)\mu + d^N_{3m}\left(1 - W^\delta_{T_1,T_2}\right)\mu_\delta \\
&\quad + e^N_{3m}\left[\begin{array}{c} \frac{1}{\phi_s}\left(W^s_{T_1,T_2} - 1\right)\left(\phi_0 + \phi_x'\mu + \phi_\delta\mu_\delta\right) \\ -\phi_x'\left(\kappa + \phi_s I\right)^{-1}\left(W^s_{T_1,T_2}I - W^x_{T_1,T_2}\right)\mu \\ -\phi_\delta\left(\kappa_\delta + \phi_s\right)^{-1}\left(W^s_{T_1,T_2} - W^\delta_{T_1,T_2}\right)\mu_\delta \end{array}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
b^{f\prime}_{LT} &= b^{N\prime}_{3m}W^x_{T_1,T_2} + e^N_{3m}\phi_x'\left(\kappa + \phi_s I\right)^{-1}\left(W^s_{T_1,T_2}I - W^x_{T_1,T_2}\right) \\
d^f_{LT} &= d^N_{3m}W^\delta_{T_1,T_2} + e^N_{3m}\phi_\delta\left(\kappa_\delta + \phi_s\right)^{-1}\left(W^s_{T_1,T_2} - W^\delta_{T_1,T_2}\right) \\
e^f_{LT} &= e^N_{3m}W^s_{T_1,T_2}
\end{aligned}
$$

1.2 State Dynamics

We first consider the discrete-time dynamics of the state variable q_t between time $t-\Delta t$ and t. From $dq_t = \pi_t dt + \sigma_q' dW_{x,t} + \sqrt{v_t}dW_{\perp,t}$, we have

$$
q_t = q_{t-\Delta t} + \left(\rho_0^\pi\Delta t + \left(\rho_x^{\pi\prime}\Delta t\right)x_t + \left(\rho_\delta^\pi\Delta t\right)\delta_t + \left(\rho_s^\pi\Delta t\right)s_t\right) + \eta_t^q
$$

where $\eta_t^q \equiv \int_{t-\Delta t}^t\left(\sigma_q' dW_{x,s} + \sigma_q^\perp dW_{\perp,s}\right) \sim N\left(0,\Omega^q\right)$ and

$$
\Omega^q \equiv Var_{t-\Delta t}\left(\eta_t^q\right) = \left(\sigma_q'\sigma_q + \left(\sigma_q^\perp\right)^2\right)\Delta t
$$

Similarly, for the state variable x_t we have

$$
x_t = \exp\left(-\mathcal{K}\Delta t\right)x_{t-\Delta t} + \left(I - \exp\left(-\mathcal{K}\Delta t\right)\right)\mu + \eta_t^x,
$$

where $\eta_t^x = \int_0^{\Delta t}\exp\left(-\kappa s\right)\Sigma dW_{x,s} \sim N\left(0,\Omega^x\right)$ and

$$
\Omega^x \equiv Var_{t-\Delta t}\left(\eta_t^x\right) = \int_0^{\Delta t}\exp\left(-\mathcal{K}s\right)\Sigma\Sigma'\exp\left(-\mathcal{K}'s\right)ds = N\Xi N',
$$

with $\mathcal{K} = NDN^{-1}$, $D = diag\left([d_1, ..., d_N]\right)$, and $\Xi_{i,j} = \left[(N^{-1}\Sigma)(N^{-1}\Sigma)'\right]_{i,j} \frac{1-\exp[-(d_i+d_j)\cdot t]}{(d_i+d_j)}$. The covariance matrix between η_t^x and η_t^q is given by

$$\Omega^{xq} = Cov_{t-\Delta t}\left[\eta_t^x, \eta_t^q\right] = \int_0^{\Delta t} \exp\left(-\mathcal{K}s\right)\Sigma\sigma_q ds = \mathcal{K}^{-1}\left(I - \exp\left(-\mathcal{K}\Delta t\right)\right)\Sigma\sigma_q.$$

where I signifies the identity matrix.

Next, we consider the state variable δ_t. From $d\delta_t = \kappa_\delta\left(\mu_\delta - \delta_t\right)dt + \sigma_\delta dW_{\delta,t}$, we have

$$\delta_t = e^{-\kappa_\delta \Delta t}\delta_{t-\Delta t} + \mu_\delta\left(1 - e^{-\kappa_\delta\Delta t}\right) + \eta_t^\delta$$

where $\eta_t^\delta = \sigma_\delta \int_{t-\Delta t}^t e^{\kappa_\delta(u-t)} dW_{\delta,u} \sim N\left(0, \Omega^\delta\right)$ and

$$\Omega^\delta = E_{t-\Delta t}\left[\sigma_\delta^2 \int_{t-\Delta t}^t e^{2\kappa_\delta(s-t)}ds\right] = \sigma_\delta^2 \frac{1 - e^{-2\kappa_\delta\Delta t}}{2\kappa_\delta}$$

Last, we consider the state variable s_t. Note

$$d\left[\exp\left(-\phi_s t\right)s_t\right] = \exp\left(-\phi_s t\right)\left[\left(\phi_0 + \phi_x' x_t + \phi_\delta\delta_t\right)dt + \sigma_s dW_{\delta,t}\right]$$

we have

$$\begin{aligned} s_t &= e^{\phi_s\Delta t}s_{t-\Delta t} + \left(\phi_0 + \phi_x' x_{t-\Delta t} + \phi_\delta\delta_{t-\Delta t}\right)\frac{\exp\left(\phi_s\Delta t\right) - 1}{\phi_s} + \eta_t^s \\ &\approx \left(1 + \phi_s\Delta t\right)s_{t-\Delta t} + \left(\phi_0\Delta t + \left(\phi_x'\Delta t\right)x_{t-\Delta t} + \left(\phi_\delta\Delta t\right)\delta_{t-\Delta t}\right) + \eta_t^s \end{aligned}$$

where $\eta_t^s = \sigma_s\int_{t-\Delta t}^t \exp\left(-\phi_s\left(u-t\right)\right)dW_{\delta,u} \sim N\left(0, \Omega^s\right)$ and

$$\Omega^s = E_{t-\Delta t}\left[\sigma_s^2 \int_{t-\Delta t}^t e^{2(-\phi_s)(u-t)}du\right] = \sigma_s^2\frac{e^{2\phi_s\Delta t} - 1}{2\phi_s}$$

and

$$\begin{aligned} \Omega^{\delta s} &= \sigma_\delta\sigma_s E_{t-\Delta t}\left[\int_{t-\Delta t}^t e^{\kappa_\delta(u-t)}dW_{\delta,u}\int_{t-\Delta t}^t e^{-\phi_s(u-t)}dW_{\delta,u}\right] \\ &= \sigma_\delta\sigma_s E_{t-\Delta t}\left[\int_{t-\Delta t}^t e^{(\kappa_\delta - \phi_s)(u-t)}du\right] \\ &= \sigma_\delta\sigma_s\frac{1 - e^{-(\kappa_\delta-\phi_s)\Delta t}}{\kappa_\delta - \phi_s} \end{aligned}$$

In summary, the dynamics of the state vector $Z_t = \left(q_t, x_t', \delta_t, s_t\right)'$ follows the VAR process

$$Z_t = \mathcal{A} + \mathcal{B}Z_{t-\Delta t} + \eta_t$$

where $\mathcal{A} = \begin{bmatrix} \rho_0^\pi\Delta t \\ \left(I - \exp\left(-\mathcal{K}\Delta t\right)\right)\mu \\ \left(1 - e^{-\kappa_\delta\Delta t}\right)\mu_\delta \\ \phi_0\Delta t \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 1 & \rho_x^{\pi\prime}\Delta t & \rho_\delta^\pi\Delta t & \rho_s^\pi\Delta t \\ 0 & \exp\left(-\mathcal{K}\Delta t\right) & 0 & 0_{3\times 1} \\ 0 & 0_{1\times 3} & e^{-\kappa_\delta\Delta t} & 0 \\ 0 & \phi_x'\Delta t & \phi_\delta\Delta t & 1 + \phi_s\Delta t \end{bmatrix}$, $\eta_t = \begin{pmatrix} \eta_t^q \\ \eta_t^x \\ \eta_t^\delta \\ \eta_t^s \end{pmatrix} \sim$ $N\left(0, \Omega\right)$ and $\Omega = \begin{bmatrix} \Omega^q & \Omega^{xq\prime} & & \\ \Omega^{xq} & \Omega^x & & \\ & & \Omega^\delta & \Omega^{\delta s} \\ & & \Omega^{\delta s} & \Omega^s \end{bmatrix}$.