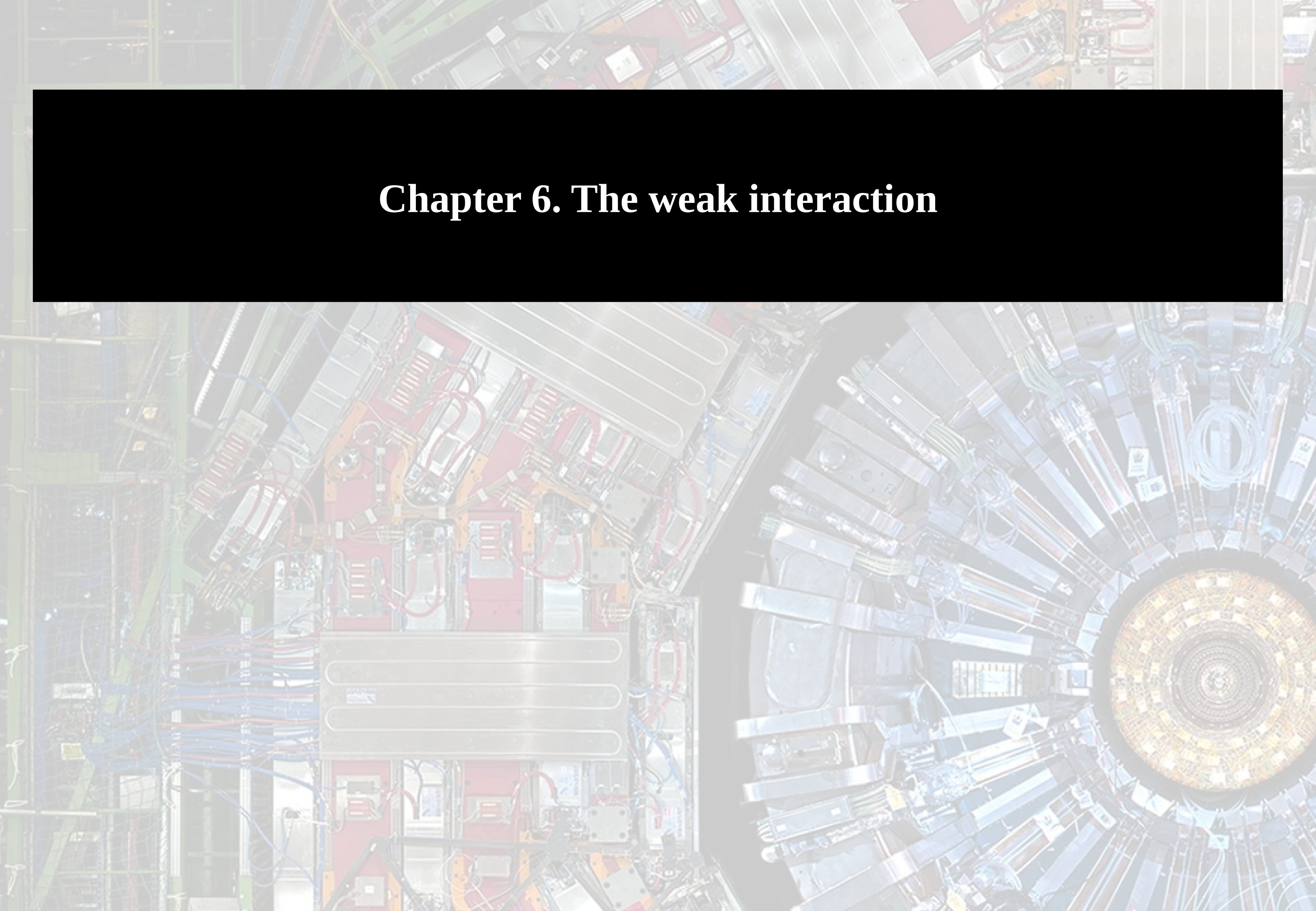
Chapter 6. The weak interaction



Parity

★ The parity operator performs spatial inversion through the origin:

$$\psi'(\vec{x},t) = \hat{P}\psi(\vec{x},t) = \psi(-\vec{x},t)$$

- applying $\hat{P}$ twice: $\hat{P}\hat{P}\psi(\vec{x},t) = \hat{P}\psi(-\vec{x},t) = \psi(\vec{x},t)$

  so $\hat{P}\hat{P} = I$ ➡ $\hat{P}^{-1} = \hat{P}$

- To preserve the normalisation of the wave-function

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\psi|\hat{P}^\dagger\hat{P}|\psi\rangle$$

$\hat{P}^\dagger\hat{P} = I$ ➡ $\hat{P}$ Unitary

- But since $\hat{P}\hat{P} = I$ $\hat{P} = \hat{P}^\dagger$ ➡ $\hat{P}$ Hermitian

  which implies Parity is an observable quantity. If the interaction Hamiltonian commutes with $\hat{P}$, parity is an observable conserved quantity

- If $\psi(\vec{x},t)$ is an eigenfunction of the parity operator with eigenvalue P

$$\hat{P}\psi(\vec{x},t) = P\psi(\vec{x},t) \quad ➡ \quad \hat{P}\hat{P}\psi(\vec{x},t) = P\hat{P}\psi(\vec{x},t) = P^2\psi(\vec{x},t)$$

since $\hat{P}\hat{P} = I$ $P^2 = 1$

➡ Parity has eigenvalues $P = \pm 1$

★ QED and QCD are invariant under parity

★ Experimentally observe that Weak Interactions do not conserve parity

Intrinsic Parities of fundamental particles:

Spin-1 Bosons

- From Gauge Field Theory can show that the gauge bosons have $P = -1$

$$P_\gamma = P_g = P_{W^+} = P_{W^-} = P_Z = -1$$

Spin-½ Fermions

- From the Dirac equation showed (handout 2):

  Spin ½ particles have opposite parity to spin ½ anti-particles

- Conventional choice: spin ½ particles have $P = +1$

$$P_{e^-} = P_{\mu^-} = P_{\tau^-} = P_\nu = P_q = +1$$

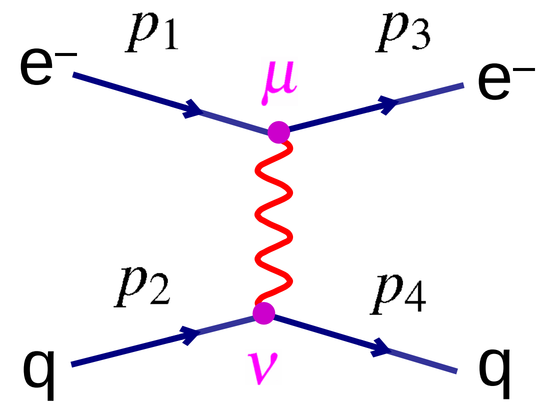
and anti-particles have opposite parity, i.e.

$$P_{e^+} = P_{\mu^+} = P_{\tau^+} = P_{\overline{\nu}} = P_{\overline{q}} = -1$$

★ For Dirac spinors it was shown (handout 2) that the parity operator is:

$$\boxed{\hat{P} = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}$$

Parity Conservation in QED and QCD



- **Consider the QED process** $e^-q \to e^-q$
- **The Feynman rules for QED give:**

$$-iM = [\overline{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}_q(p_4)ie\gamma^\nu u_q(p_2)]$$

- **Which can be expressed in terms of the electron and quark 4-vector currents:**

$$M = -\frac{e^2}{q^2}g_{\mu\nu}j_e^\mu j_q^\nu = -\frac{e^2}{q^2}j_e.j_q$$

with $j_e = \overline{u}_e(p_3)\gamma^\mu u_e(p_1)$ **and** $j_q = \overline{u}_q(p_4)\gamma^\mu u_q(p_2)$

★ **Consider the what happen to the matrix element under the parity transformation**

- **Spinors transform as** $\boxed{u \xrightarrow{\hat{P}} \hat{P}u = \gamma^0 u}$
- **Adjoint spinors transform as**

$$\overline{u} = u^\dagger\gamma^0 \xrightarrow{\hat{P}} (\hat{P}u)^\dagger\gamma^0 = u^\dagger\gamma^{0\dagger}\gamma^0 = u^\dagger\gamma^0\gamma^0 = \overline{u}\gamma^0$$

$$\boxed{\overline{u} \xrightarrow{\hat{P}} \overline{u}\gamma^0}$$

- **Hence** $j_e = \overline{u}_e(p_3)\gamma^\mu u_e(p_1) \xrightarrow{\hat{P}} \overline{u}_e(p_3)\gamma^0\gamma^\mu\gamma^0 u_e(p_1)$

★ Consider the components of the four-vector current

$\boxed{0:}$ $$j_e^0 \xrightarrow{\hat{P}} \bar{u}\gamma^0\gamma^0\gamma^0 u = \bar{u}\gamma^0 u = j_e^0$$ since $\gamma^0\gamma^0 = 1$

$\boxed{k=1,2,3:}$ $$j_e^k \xrightarrow{\hat{P}} \bar{u}\gamma^0\gamma^k\gamma^0 u = -\bar{u}\gamma^k\gamma^0\gamma^0 u = -\bar{u}\gamma^k u = -j_e^k$$ since $\gamma^0\gamma^k = -\gamma^k\gamma^0$

- The time-like component remains unchanged and the space-like components change sign
- Similarly $j_q^0 \xrightarrow{\hat{P}} j_q^0$ $\quad j_q^k \xrightarrow{\hat{P}} -j_q^k \quad k = 1,2,3$

★ Consequently the four-vector scalar product

$$j_e.j_q = j_e^0 j_q^0 - j_e^k j_q^k \xrightarrow{\hat{P}} j_e^0 j_q^0 - (-j_e^k)(-j_q^k) = j_e.j_q \quad k = 1,3$$

or $j^\mu \xrightarrow{\hat{P}} j_\mu$

$$j^\mu.j^\nu \xrightarrow{\hat{P}} j_\mu.j_\nu \xrightarrow{\hat{P}} j^\mu.j^\nu$$

QED Matrix Elements are Parity Invariant

➡ **Parity Conserved in QED**

★ The QCD vertex has the same form and, thus,

Parity Conserved in QCD

Parity Violation in β-Decay

★ The parity operator $\hat{P}$ corresponds to a discrete transformation $x \rightarrow -x,\ etc.$

★ Under the parity transformation:

Vectors change sign

$$\vec{r} \xrightarrow{\hat{P}} -\vec{r}$$

$$\vec{p} \xrightarrow{\hat{P}} -\vec{p} \qquad \left(p_x = \frac{\partial}{\partial x},\ etc.\right)$$

Axial-Vectors unchanged

$$\vec{L} \xrightarrow{\hat{P}} \vec{L} \qquad (\vec{L} = \vec{r} \wedge \vec{p})$$

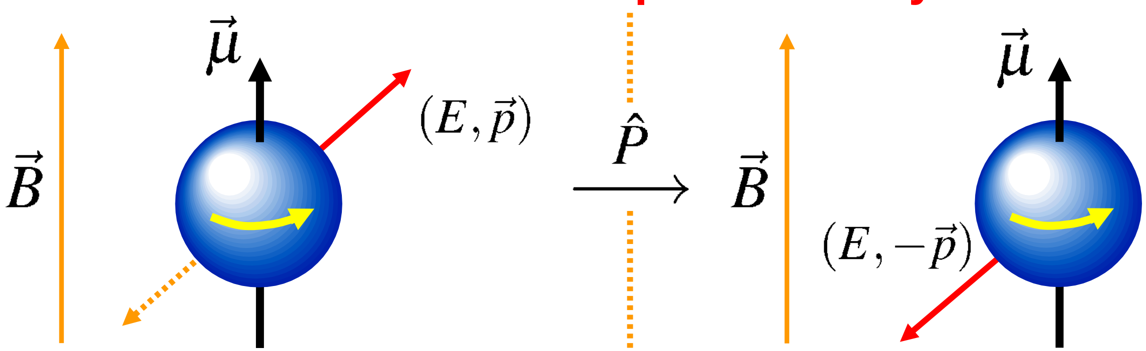
$$\vec{\mu} \xrightarrow{\hat{P}} \vec{\mu} \qquad (\vec{\mu} \propto \vec{L})$$

Note B is an axial vector
$\mathrm{d}\vec{B} \propto \vec{J} \wedge \vec{r}\,\mathrm{d}^3\vec{r}$

★ **1957:** C.S.Wu et al. studied beta decay of polarized cobalt-60 nuclei:

$$^{60}\mathrm{Co} \rightarrow {}^{60}Ni^{*} + e^{-} + \overline{\nu}_e$$

★ Observed electrons emitted preferentially in direction opposite to applied field



more e- in   c.f.

If parity were conserved: expect equal rate for producing e⁻ in directions along and opposite to the nuclear spin.

★ Conclude parity is violated in WEAK INTERACTION
→ that the WEAK interaction vertex is NOT of the form $\overline{u}_e \gamma^{\mu} u_{\nu}$

Bilinear Covariants

★ The requirement of Lorentz invariance of the matrix element severely restricts the form of the interaction vertex. QED and QCD are "VECTOR" interactions:

$$j^\mu = \overline{\psi}\gamma^\mu\phi$$

★ This combination transforms as a 4-vector (Handout 2 appendix V)

★ In general, there are only 5 possible combinations of two spinors and the gamma matrices that form Lorentz invariant currents, called "bilinear covariants":

Type	Form	Components	"Boson Spin"
♦ SCALAR	$\overline{\psi}\phi$	1	0
♦ PSEUDOSCALAR	$\overline{\psi}\gamma^5\phi$	1	0
♦ VECTOR	$\overline{\psi}\gamma^\mu\phi$	4	1
♦ AXIAL VECTOR	$\overline{\psi}\gamma^\mu\gamma^5\phi$	4	1
♦ TENSOR	$\overline{\psi}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\phi$	6	2

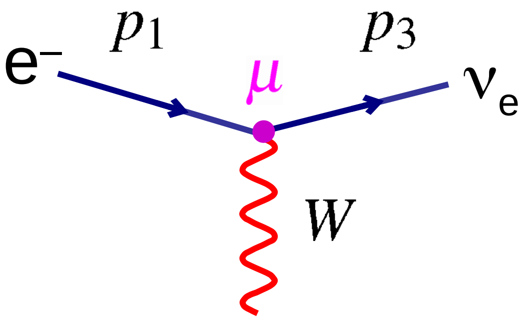
★ Note that in total the sixteen components correspond to the 16 elements of a general 4x4 matrix: "decomposition into Lorentz invariant combinations"

★ In QED the factor $g_{\mu\nu}$ arose from the sum over polarization states of the virtual photon (2 transverse + 1 longitudinal, 1 scalar) = (2J+1) + 1

★ Associate SCALAR and PSEUDOSCALAR interactions with the exchange of a SPIN-0 boson, etc. – no spin degrees of freedom

V-A Structure of the Weak Interaction

- ★ The most general form for the interaction between a fermion and a boson is a linear combination of bilinear covariants
- ★ For an interaction corresponding to the **exchange of a spin-1** particle the most general form is a linear combination of **VECTOR** and **AXIAL-VECTOR**
- ★ The form for WEAK interaction is <u>determined from experiment</u> to be **VECTOR – AXIAL-VECTOR** **(V – A)**



$$\boxed{j^\mu \propto \bar{u}_{\nu_e}(\gamma^\mu - \gamma^\mu\gamma^5)u_e}$$

V – A

- ★ Can this account for parity violation?
- ★ First consider parity transformation of a **pure AXIAL-VECTOR** current

$$j_A = \overline{\psi}\gamma^\mu\gamma^5\phi \qquad \textbf{with} \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3; \quad \gamma^5\gamma^0 = -\gamma^0\gamma^5$$

$$j_A = \overline{\psi}\gamma^\mu\gamma^5\phi \xrightarrow{\hat{P}} \overline{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\phi = -\overline{\psi}\gamma^0\gamma^\mu\gamma^0\gamma^5\phi$$

$$j_A^0 = \xrightarrow{\hat{P}} -\overline{\psi}\gamma^0\gamma^0\gamma^0\gamma^5\phi = -\overline{\psi}\gamma^0\gamma^5\phi = -j_A^0$$

$$j_A^k = \xrightarrow{\hat{P}} -\overline{\psi}\gamma^0\gamma^k\gamma^0\gamma^5\phi = +\overline{\psi}\gamma^k\gamma^5\phi = +j_A^k \qquad k = 1,2,3$$

$$\boxed{\text{or} \quad j_A^\mu \xrightarrow{\hat{P}} -j_{A\mu}}$$

- **The space-like components remain unchanged and the time-like components change sign (the opposite to the parity properties of a vector-current)**

$$\boxed{j_A^0 \xrightarrow{\hat{P}} -j_A^0; \quad j_A^k \xrightarrow{\hat{P}} +j_A^k;} \boxed{j_V^0 \xrightarrow{\hat{P}} +j_V^0; \quad j_V^k \xrightarrow{\hat{P}} -j_V^k}$$

- **Now consider the matrix elements**

$$M \propto g_{\mu\nu} j_1^\mu j_2^\nu = j_1^0 j_2^0 - \sum_{k=1,3} j_1^k j_2^k$$

- **For the combination of a two axial-vector currents**

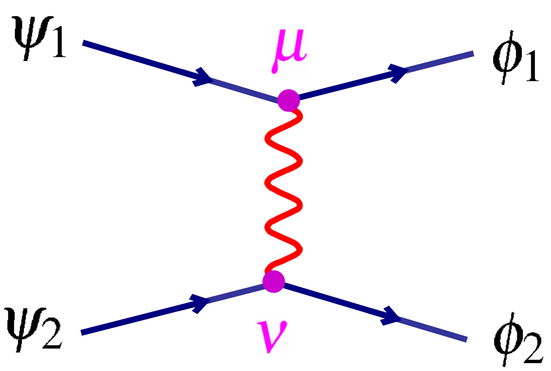
$$j_{A1} \cdot j_{A2} \xrightarrow{\hat{P}} (-j_1^0)(-j_2^0) - \sum_{k=1,3} (j_1^k)(j_2^k) = j_{A1} \cdot j_{A2}$$

- **Consequently parity is conserved for both a pure vector and pure axial-vector interactions**
- **However the combination of a vector current and an axial vector current**

$$j_{V1} \cdot j_{A2} \xrightarrow{\hat{P}} (j_1^0)(-j_2^0) - \sum_{k=1,3} (-j_1^k)(j_2^k) = -j_{V1} \cdot j_{A2}$$

changes sign under parity – can give parity violation !

★ **Now consider a general linear combination of VECTOR and AXIAL-VECTOR (note this is relevant for the Z-boson vertex)**



$$j_1 = \overline{\phi}_1 (g_V \gamma^\mu + g_A \gamma^\mu \gamma^5) \psi_1 = g_V j_1^V + g_A j_1^A$$

$$\frac{g_{\mu\nu}}{q^2 - m^2}$$

$$j_2 = \overline{\phi}_2 (g_V \gamma^\mu + g_A \gamma^\mu \gamma^5) \psi_2 = g_V j_2^V + g_A j_2^A$$

$$M_{fi} \propto j_1 . j_2 = g_V^2 j_1^V . j_2^V + g_A^2 j_1^A . j_2^A + g_V g_A (j_1^V . j_2^A + j_1^A . j_2^V)$$

- **Consider the parity transformation of this scalar product**

$$j_1 . j_2 \xrightarrow{\hat{P}} g_V^2 j_1^V . j_2^V + g_A^2 j_1^A . j_2^A - g_V g_A (j_1^V . j_2^A + j_1^A . j_2^V)$$

- **If either g_A or g_V is zero, Parity is conserved, i.e. parity conserved in a pure VECTOR or pure AXIAL-VECTOR interaction**
- **Relative strength of parity violating part** $\propto \dfrac{g_V g_A}{g_V^2 + g_A^2}$

$\boxed{\textbf{Maximal Parity Violation for V-A (or V+A)}}$

Chiral Structure of QED (Reminder)

★ **Recall (Handout 4) introduced CHIRAL projections operators**

$$P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)$$

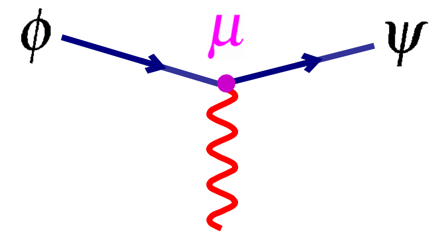
project out chiral right- and left- handed states

★ **In the ultra-relativistic limit, chiral states correspond to helicity states**

★ **Any spinor can be expressed as:**

$$\psi = \tfrac{1}{2}(1+\gamma^5)\psi + \tfrac{1}{2}(1-\gamma^5)\psi = P_R\psi + P_L\psi = \psi_R + \psi_L$$

- **The QED vertex** $\overline{\psi}\gamma^\mu\phi$ **in terms of chiral states:**

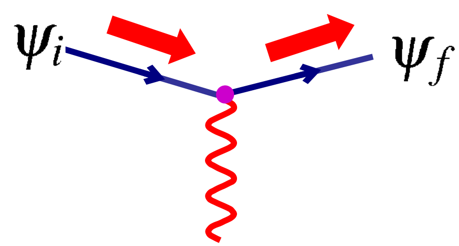


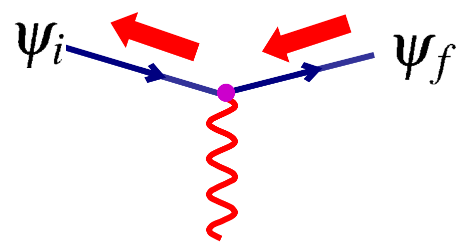
$$\overline{\psi}\gamma^\mu\phi = \overline{\psi}_R\gamma^\mu\phi_R + \overline{\psi}_R\gamma^\mu\phi_L + \overline{\psi}_L\gamma^\mu\phi_R + \overline{\psi}_L\gamma^\mu\phi_L$$

conserves chirality, e.g.

$$\begin{aligned}
\overline{\psi}_R\gamma^\mu\phi_L &= \tfrac{1}{2}\psi^\dagger(1+\gamma^5)\gamma^0\gamma^\mu\tfrac{1}{2}(1-\gamma^5)\phi \\
&= \tfrac{1}{4}\psi^\dagger\gamma^0(1-\gamma^5)\gamma^\mu(1-\gamma^5)\phi \\
&= \tfrac{1}{4}\overline{\psi}\gamma^\mu(1+\gamma^5)(1-\gamma^5)\phi = 0
\end{aligned}$$

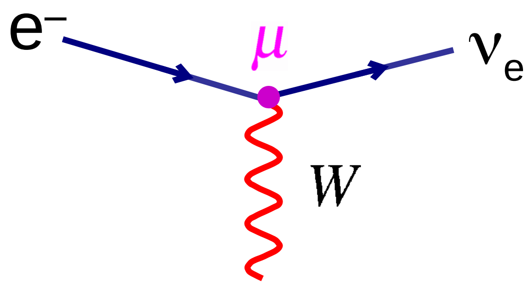
★ **In the ultra-relativistic limit only two helicity combinations are non-zero**





Helicity Structure of the WEAK Interaction

★ The charged current ($W^{\pm}$) weak vertex is:



$$\boxed{\frac{-ig_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5)}$$

★ Since $\frac{1}{2}(1-\gamma^5)$ projects out left-handed chiral particle states:

$$\overline{\psi}\tfrac{1}{2}\gamma^\mu(1-\gamma^5)\phi = \overline{\psi}\gamma^\mu\phi_L$$

(question 16)

★ Writing $\overline{\psi} = \overline{\psi}_R + \overline{\psi}_L$ and from discussion of QED, $\overline{\psi}_R\gamma^\mu\phi_L = 0$ gives

$$\overline{\psi}\tfrac{1}{2}\gamma^\mu(1-\gamma^5)\phi = \overline{\psi}_L\gamma^\mu\phi_L$$

Only the left-handed chiral components of particle spinors and right-handed chiral components of anti-particle spinors participate in charged current weak interactions

★ At very high energy $(E \gg m)$, the left-handed chiral components are helicity eigenstates :

$\frac{1}{2}(1-\gamma^5)u \Rightarrow$

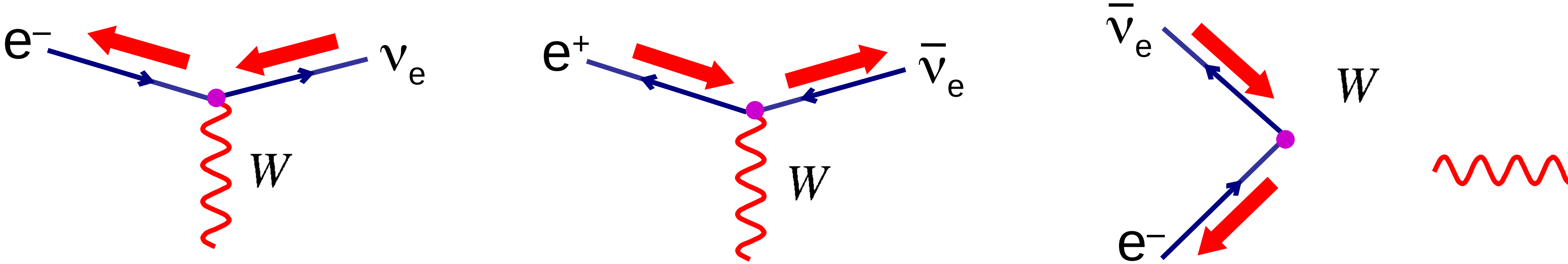
LEFT-HANDED PARTICLES
Helicity = -1

$\frac{1}{2}(1-\gamma^5)v \Rightarrow$

RIGHT-HANDED ANTI-PARTICLES
Helicity = +1

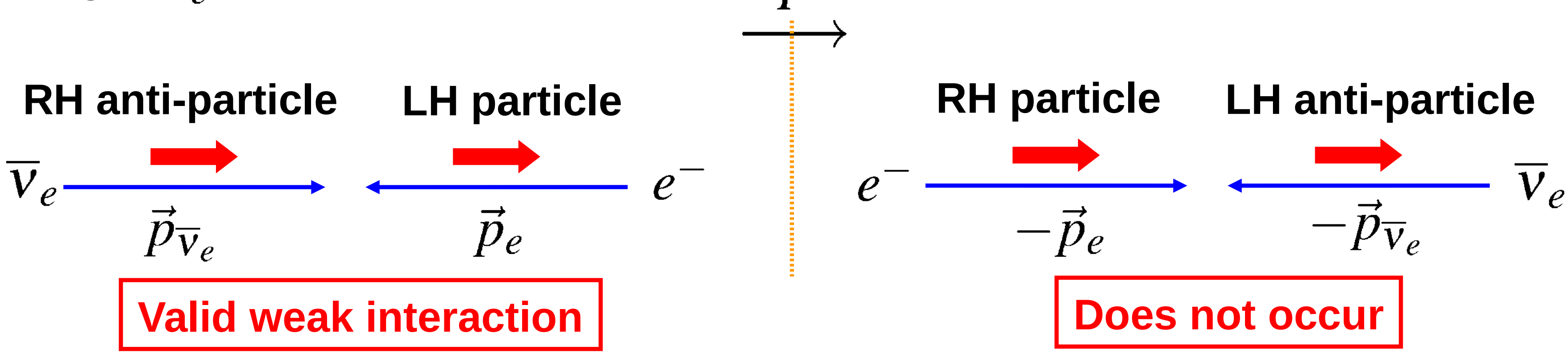
In the ultra-relativistic limit only left-handed particles and right-handed antiparticles participate in charged current weak interactions

e.g. In the relativistic limit, the only possible electron – neutrino interactions are:



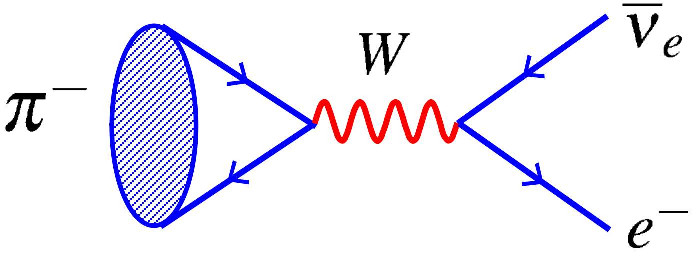
★ **The helicity dependence of the weak interaction ⟷ parity violation**

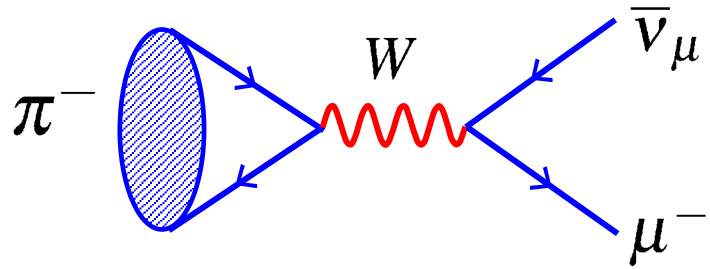
e.g. $\overline{\nu}_e + e^- \rightarrow W^-$



Helicity in Pion Decay

★ The decays of charged pions provide a good demonstration of the role of helicity in the weak interaction





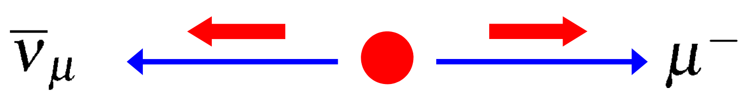
<u>**EXPERIMENTALLY:**</u>

$$\frac{\Gamma(\pi^- \to e^- \overline{\nu}_e)}{\Gamma(\pi^- \to \mu^- \overline{\nu}_\mu)} = 1.23 \times 10^{-4}$$

- Might expect the decay to electrons to dominate – due to increased phase space…. The opposite happens, the electron decay is helicity suppressed

★ Consider decay in pion rest frame.

- Pion is spin zero: so the spins of the $\overline{\nu}$ and μ are opposite
- Weak interaction only couples to RH chiral anti-particle states. Since neutrinos are (almost) massless, must be in RH Helicity state
- Therefore, to conserve angular mom. muon is emitted in a RH HELICITY state



- But only left-handed CHIRAL particle states participate in weak interaction

★ **The general right-handed helicity solution to the Dirac equation is**

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad \textbf{with} \quad c=\cos\frac{\theta}{2} \quad \textbf{and} \quad s=\sin\frac{\theta}{2}$$

- **project out the left-handed <u>chiral</u> part of the wave-function using**

$$P_L = \tfrac{1}{2}(1-\gamma^5) = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

giving

$$P_L u_\uparrow = \tfrac{1}{2}N\left(1-\frac{|\vec{p}|}{E+m}\right)\begin{pmatrix} c \\ e^{i\phi}s \\ -c \\ -e^{i\phi}s \end{pmatrix} = \tfrac{1}{2}N\left(1-\frac{|\vec{p}|}{E+m}\right)u_L$$

In the limit $m \ll E$ this tends to zero

- **similarly**

$$P_R u_\uparrow = \tfrac{1}{2}N\left(1+\frac{|\vec{p}|}{E+m}\right)\begin{pmatrix} c \\ e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix} = \tfrac{1}{2}N\left(1+\frac{|\vec{p}|}{E+m}\right)u_R$$

In the limit $m \ll E$, $P_R u_\uparrow \rightarrow u_R$

★ **Hence** $$\boxed{u_\uparrow} = P_R u_\uparrow + P_L u_\uparrow = \frac{1}{2}\left(1+\frac{|\vec{p}|}{E+m}\right)\boxed{u_R} + \frac{1}{2}\left(1-\frac{|\vec{p}|}{E+m}\right)\boxed{u_L}$$

RH Helicity — RH Chiral — LH Chiral

- In the limit $E \gg m$, as expected, the RH chiral and helicity states are identical
- Although only LH chiral particles participate in the weak interaction the contribution from RH **Helicity** states is not necessarily zero !

$$\overline{\nu}_\mu \longleftarrow \bullet \longrightarrow \mu^-$$

$m_\nu \approx 0$: RH Helicity $\equiv$ RH Chiral

$m_\mu \neq 0$: RH Helicity has LH Chiral Component

★ Expect matrix element to be proportional to **LH chiral component** of **RH Helicity** electron/muon **spinor**

$$M_{fi} \propto \frac{1}{2}\left(1-\frac{|\vec{p}|}{E+m}\right) = \boxed{\frac{m_\mu}{m_\pi + m_\mu}}$$

from the kinematics of pion decay at rest

★ Hence because the electron mass is much smaller than the pion mass the decay $\pi^- \to e^- \overline{\nu}_e$ **is heavily suppressed.**

Evidence for V-A

★ **The V-A nature of the charged current weak interaction vertex fits with experiment**

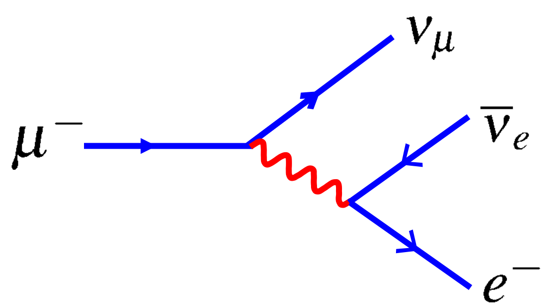
<u>EXAMPLE</u> charged pion decay **(question 17)**

- **Experimentally measure:** $$\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} = (1.230 \pm 0.004) \times 10^{-4}$$
- **Theoretical predictions (depend on Lorentz Structure of the interaction)**

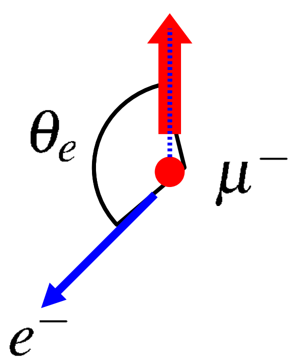
V-A $(\overline{\psi}\gamma^\mu(1-\gamma^5)\phi)$ **or** **V+A** $(\overline{\psi}\gamma^\mu(1+\gamma^5)\phi)$ ➡ $$\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} \approx 1.3 \times 10^{-4}$$

Scalar $(\overline{\psi}\phi)$ **or** **Pseudo-Scalar** $(\overline{\psi}\gamma^5\phi)$ ➡ $$\frac{\Gamma(\pi^- \to e^-\overline{\nu}_e)}{\Gamma(\pi^- \to \mu^-\overline{\nu}_\mu)} = 5.5$$

<u>EXAMPLE</u> muon decay



Measure electron energy and angular distributions relative to muon spin direction. Results expressed in terms of general S+P+V+A+T form in "Michel Parameters"



e.g. TWIST expt: 6×10^9 μ decays
Phys. Rev. Lett. 95 (2005) 101805

$$\rho = 0.75080 \pm 0.00105$$

V-A Prediction: $\rho = 0.75$

Weak Charged Current Propagator

★ The charged-current Weak interaction is different from QED and QCD in that it is mediated by massive W-bosons **(80.3 GeV)**

★ This results in a more complicated form for the propagator:

- in handout 4 showed that for the exchange of a massive particle:

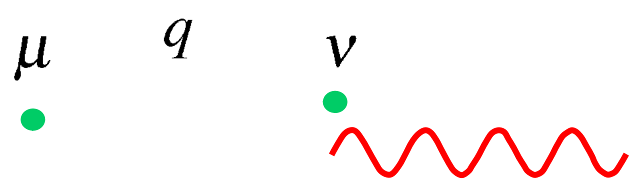
$$\underset{\text{massless}}{\frac{1}{q^2}} \longrightarrow \underset{\text{massive}}{\frac{1}{q^2-m^2}}$$

- In addition the sum over W boson polarization states modifies the numerator

W-boson propagator

spin 1 $W^{\pm}$

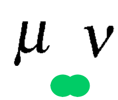
$$\frac{-i\left[g_{\mu\nu}-q_\mu q_\nu/m_W^2\right]}{q^2-m_W^2}$$



★ However in the limit where q^2 is small compared with $m_W = 80.3\,\mathrm{GeV}$ the interaction takes a simpler form.

W-boson propagator ($q^2 \ll m_W^2$)

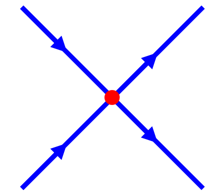
$$\frac{ig_{\mu\nu}}{m_W^2}$$



- The interaction appears point-like (i.e no q^2 dependence)

Connection to Fermi Theory

★ In 1934, before the discovery of parity violation, Fermi proposed, in analogy with QED, that the invariant matrix element for β-decay was of the form:



$$M_{fi} = G_F\, g_{\mu\nu}[\overline{\psi}\gamma^{\mu}\psi][\overline{\psi}\gamma^{\nu}\psi]$$

where $G_F = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$

- Note the absence of a propagator : i.e. this represents an interaction at a point

★ After the discovery of parity violation in 1957 this was modified to

$$M_{fi} = \frac{G_F}{\sqrt{2}} g_{\mu\nu}[\overline{\psi}\gamma^{\mu}(1-\gamma^5)\psi][\overline{\psi}\gamma^{\nu}(1-\gamma^5)\psi]$$

(the factor of √2 was included so the numerical value of G_F did not need to be changed)

★ Compare to the prediction for W-boson exchange

$$M_{fi} = [\tfrac{g_W}{\sqrt{2}}\overline{\psi}\tfrac{1}{2}\gamma^{\mu}(1-\gamma^5)\psi]\frac{g_{\mu\nu}-q_{\mu}q_{\nu}/m_W^2}{q^2-m_W^2}[\tfrac{g_W}{\sqrt{2}}\overline{\psi}\tfrac{1}{2}\gamma^{\nu}(1-\gamma^5)\psi]$$

which for $q^2 \ll m_W^2$ becomes:

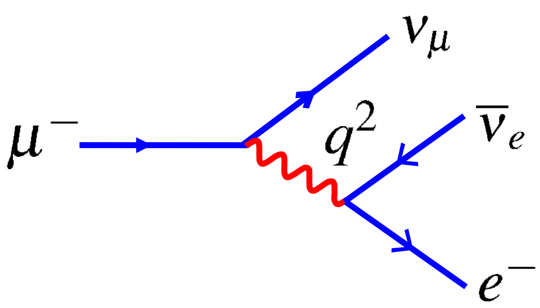
$$M_{fi} = \frac{g_W^2}{8m_W^2} g_{\mu\nu}[\overline{\psi}\gamma^{\mu}(1-\gamma^5)\psi][\overline{\psi}\gamma^{\nu}(1-\gamma^5)\psi]$$

$$\Rightarrow \boxed{\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}}$$

Still usually use G_F to express strength of weak interaction as the is the quantity that is precisely determined in muon decay

Strength of Weak Interaction

★ **Strength of weak interaction most precisely measured in muon decay**



- **Here** $q^2 < m_\mu \; (0.106\,\mathrm{GeV})$
- **To a very good approximation the W-boson propagator can be written**

$$\frac{-i\left[g_{\mu\nu} - q_\mu q_\nu / m_W^2\right]}{q^2 - m_W^2} \approx \frac{i g_{\mu\nu}}{m_W^2}$$

- **In muon decay measure** g_W^2 / m_W^2
- **Muon decay** ➡ $G_\mathrm{F} = 1.16639(1) \times 10^{-5}\,\mathrm{GeV}^{-2}$

$$\boxed{\frac{G_\mathrm{F}}{\sqrt{2}} = \frac{g_W^2}{8 m_W^2}}$$

★ **To obtain the intrinsic strength of weak interaction need to know mass of W-boson:** $m_W = 80.403 \pm 0.029\,\mathrm{GeV}$ **(see handout 14)**

➡ $$\alpha_W = \frac{g_W^2}{4\pi} = \frac{8 m_W^2 G_\mathrm{F}}{4\sqrt{2}\pi} = \frac{1}{30}$$

★ **The intrinsic strength of the weak interaction is similar to, but greater than, the EM interaction ! It is the massive W-boson in the propagator which makes it appear weak. For** $q^2 \gg m_W^2$ **weak interactions are more likely than EM.**

Summary

★ Weak interaction is of form Vector – Axial-vector (V-A)

$$\boxed{\frac{-ig_w}{\sqrt{2}}\frac{1}{2}\gamma^{\mu}(1-\gamma^{5})}$$

★ Consequently only left-handed chiral particle states and right-handed chiral anti-particle states participate in the weak interaction

➡ **MAXIMAL PARITY VIOLATION**

★ Weak interaction also violates Charge Conjugation symmetry

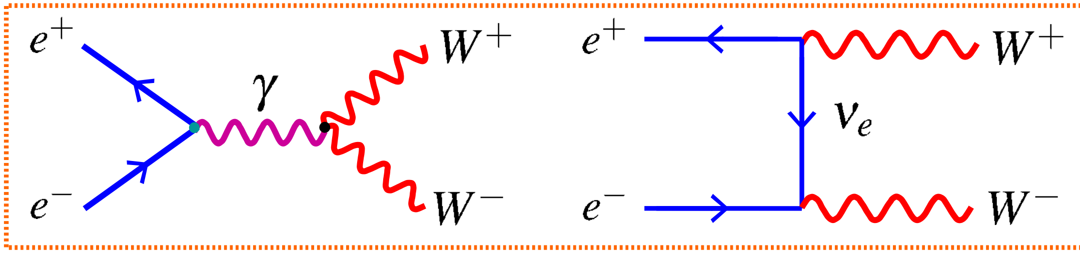
★ At low q^2 weak interaction is only weak because of the large W-boson mass

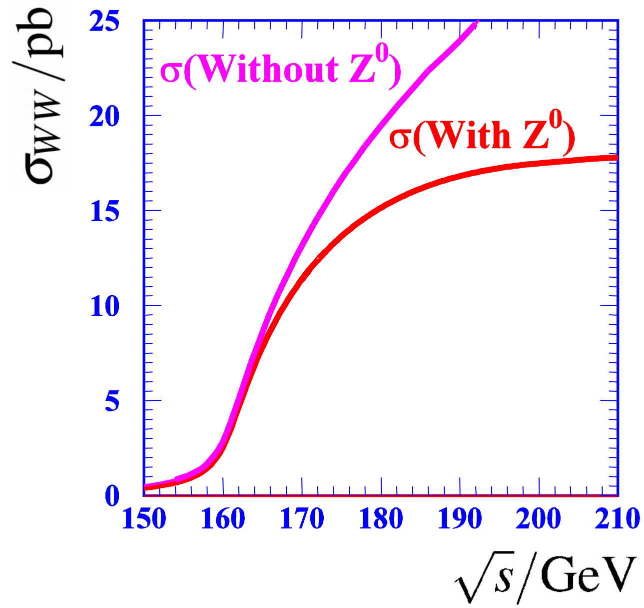
$$\boxed{\frac{G_{\mathrm{F}}}{\sqrt{2}}=\frac{g_W^2}{8m_W^2}}$$

★ Intrinsic strength of weak interaction is similar to that of QED

From W to Z

- ★ The $W^{\pm}$ bosons carry the EM charge - suggestive Weak are EM forces are related.
- ★ W bosons can be produced in e^+e^- annihilation

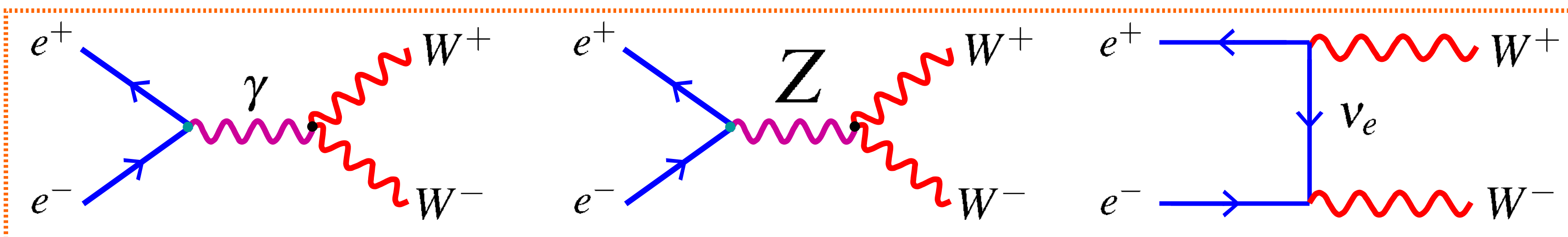




- ★ With just these two diagrams there is a problem: the cross section increases with C.o.M energy and at some point violates **QM unitarity**

UNITARITY VIOLATION: when QM calculation gives larger flux of W bosons than incoming flux of electrons/positrons

- ★ Problem can be "fixed" by introducing a new boson, the Z. The new diagram interferes negatively with the above two diagrams fixing the unitarity problem



$$|M_{\gamma WW} + M_{ZWW} + M_{\nu WW}|^2 < |M_{\gamma WW} + M_{\nu WW}|^2$$

- ★ Only works if **Z, γ, W** couplings are related: need **ELECTROWEAK UNIFICATION**

$SU(2)_L$: The Weak Interaction

- The Weak Interaction arises from SU(2) local phase transformations

$$\psi \to \psi' = \psi e^{i\vec{\alpha}(x).\frac{\vec{\sigma}}{2}}$$

where the $\vec{\sigma}$ are the generators of the SU(2) symmetry, i.e the three Pauli spin matrices

⇒ **3 Gauge Bosons** $W_1^\mu,\ W_2^\mu,\ W_3^\mu$

- The wave-functions have two components which, in analogy with isospin, are represented by "weak isospin"
- The fermions are placed in isospin doublets and the local phase transformation corresponds to

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix} \to \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}' = e^{i\vec{\alpha}(x).\frac{\vec{\sigma}}{2}} \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}$$

- Weak Interaction only couples to LH particles/RH anti-particles, hence only place LH particles/RH anti-particles in weak isospin doublets: $I_W = \frac{1}{2}$
RH particles/LH anti-particles placed in weak isospin singlets: $I_W = 0$

Weak Isospin

$\boxed{I_W = \frac{1}{2}}$ $\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L, \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L, \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L, \begin{pmatrix}u\\ d'\end{pmatrix}_L, \begin{pmatrix}c\\ s'\end{pmatrix}_L, \begin{pmatrix}t\\ b'\end{pmatrix}_L$ ← $\boxed{I_W^3 = +\frac{1}{2}}$ ← $\boxed{I_W^3 = -\frac{1}{2}}$

$\boxed{I_W = 0}$ $(\nu_e)_R,\ (e^-)_R, \ldots (u)_R,\ (d)_R, \ldots$

Note: RH/LH refer to chiral states

★ For simplicity only consider $\chi_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$

- The gauge symmetry specifies the form of the interaction: one term for each of the 3 generators of SU(2) – [note: here include interaction strength in current]

$$j^1_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_1 \chi_L \qquad j^2_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_2 \chi_L \qquad j^3_\mu = g_W \overline{\chi}_L \gamma^\mu \tfrac{1}{2}\sigma_3 \chi_L$$

★ The charged current W^+/W^- interaction enters as a linear combinations of W_1, W_2

$$W^{\pm\mu} = \frac{1}{\sqrt{2}}(W_1^\mu \pm W_2^\mu)$$

★ The $W^\pm$ interaction terms

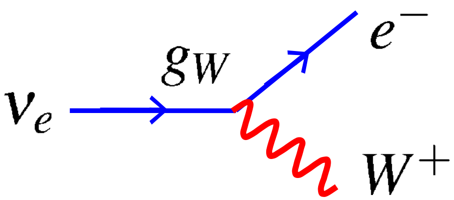
$$j^\mu_\pm = \frac{g_W}{\sqrt{2}}(j_1^\mu \pm i j_2^\mu) = \frac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \tfrac{1}{2}(\sigma_1 \pm i\sigma_2)\chi_L$$

★ Express in terms of the weak isospin ladder operators $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$

$$\boxed{j^\mu_\pm = \frac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \sigma_\pm \chi_L}$$

Origin of $\frac{1}{\sqrt{2}}$ in Weak CC

W⁺



corresponds to

$$\boxed{j^\mu_+ = \frac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \sigma_+ \chi_L}$$

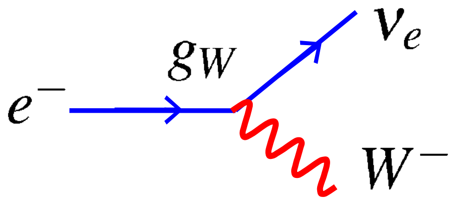
Bars indicates adjoint spinors

which can be understood in terms of the weak isospin doublet

$$j^\mu_+ = \frac{g_W}{\sqrt{2}} \overline{\chi}_L \gamma^\mu \sigma_+ \chi_L = \frac{g_W}{\sqrt{2}} (\overline{\nu}_L, \overline{e}_L)\gamma^\mu \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu \\ e \end{pmatrix}_L = \frac{g_W}{\sqrt{2}} \overline{\nu}_L \gamma^\mu e_L = \boxed{\frac{g_W}{\sqrt{2}} \overline{\nu} \gamma^\mu \tfrac{1}{2}(1-\gamma^5) e}$$

★ **Similarly**

W⁻



corresponds to $\boxed{j^{\mu}_{-} = \frac{g_W}{\sqrt{2}}\overline{\chi}_L\gamma^{\mu}\sigma_-\chi_L}$

$$j^{\mu}_{-} = \frac{g_W}{\sqrt{2}}\overline{\chi}_L\gamma^{\mu}\sigma_-\chi_L = \frac{g_W}{\sqrt{2}}(\overline{\nu}_L,\overline{e}_L)\gamma^{\mu}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}\nu\\ e\end{pmatrix}_L = \frac{g_W}{\sqrt{2}}\overline{e}_L\gamma^{\mu}\nu_L = \frac{g_W}{\sqrt{2}}\overline{e}\gamma^{\mu}\tfrac{1}{2}(1-\gamma^5)\nu$$

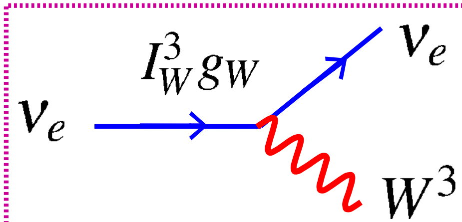
★ **However have an additional interaction due to W³**

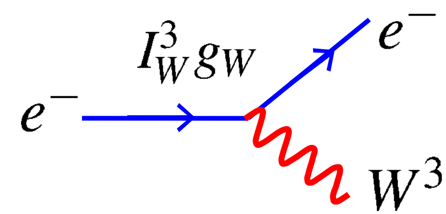
$$\boxed{j^{\mu}_{3} = g_W\overline{\chi}_L\gamma^{\mu}\tfrac{1}{2}\sigma_3\chi_L}$$

expanding this:

$$\boxed{I^3_W = \pm\tfrac{1}{2}}$$

$$j^{\mu}_{3} = g_W\frac{1}{2}(\overline{\nu}_L,\overline{e}_L)\gamma^{\mu}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\nu\\ e\end{pmatrix}_L = g_W\tfrac{1}{2}\overline{\nu}_L\gamma^{\mu}\nu_L - g_W\tfrac{1}{2}\overline{e}_L\gamma^{\mu}e_L$$





NEUTRAL CURRENT INTERACTIONS !

Electroweak Unification

- ★ Tempting to identify the W^3 as the Z
- ★ However this is not the case, have two physical neutral spin-1 gauge bosons, γ, Z and the W^3 is a mixture of the two,
- ★ Equivalently write the photon and Z in terms of the W^3 and a new neutral spin-1 boson the B
- ★ The <u>physical</u> bosons (the Z and photon field, A) are:

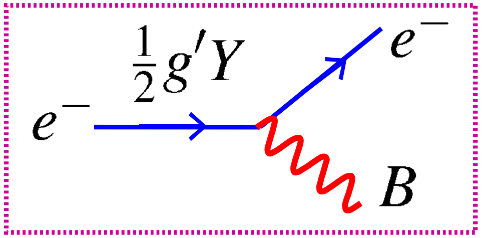
$$A_\mu = B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W$$
$$Z_\mu = -B_\mu \sin\theta_W + W_\mu^3 \cos\theta_W$$

θ_W is the weak mixing angle

- ★ The new boson is associated with a new gauge symmetry similar to that of electromagnetism : $U(1)_Y$
- ★ The charge of this symmetry is called WEAK HYPERCHARGE Y

$$\boxed{Y = 2Q - 2I_W^3}$$

Q is the EM charge of a particle
I_W^3 is the third comp. of weak isospin



- By convention the coupling to the B_μ is $\frac{1}{2}g'Y$

$e_L: \; Y = 2(-1) - 2(-\frac{1}{2}) = -1$ $\qquad \nu_L: \; Y = +1$

$e_R: \; Y = 2(-1) - 2(0) = -2$ $\qquad \nu_R: \; Y = 0$

(this identification of hypercharge in terms of Q and I_3 makes all of the following work out)

★ For this to work the coupling constants of the W^3, B, and photon must be related

e.g. consider contributions involving the neutral interactions of electrons:

$\boxed{\gamma}$ $$j_\mu^{em} = e\overline{\psi}Q_e\gamma_\mu\psi = e\bar{e}_L Q_e\gamma_\mu e_L + e\bar{e}_R Q_e\gamma_\mu e_R$$

$\boxed{W^3}$ $$j_\mu^{W^3} = -\frac{g_W}{2}\bar{e}_L\gamma_\mu e_L$$

$\boxed{B}$ $$j_\mu^{Y} = \frac{g'}{2}\overline{\psi}Y_e\gamma_\mu\psi = \frac{g'}{2}\bar{e}_L Y_{e_L}\gamma_\mu e_L + \frac{g'}{2}\bar{e}_R Y_{e_R}\gamma_\mu e_R$$

★ The relation $A_\mu = B_\mu\cos\theta_W + W_\mu^3\sin\theta_W$ is equivalent to requiring

$$\boxed{j_\mu^{em} = j_\mu^{Y}\cos\theta_W + j_\mu^{W^3}\sin\theta_W}$$

• Writing this in full:

$$e\bar{e}_L Q_e\gamma_\mu e_L + e\bar{e}_R Q_e\gamma_\mu e_R = \tfrac{1}{2}g'\cos\theta_W[\bar{e}_L Y_{e_L}\gamma_\mu e_L + \bar{e}_R Y_{e_R}\gamma_\mu e_R] - \tfrac{1}{2}g_W\sin\theta_W[\bar{e}_L\gamma_\mu e_L]$$

$$-e\bar{e}_L\gamma_\mu e_L - e\bar{e}_R\gamma_\mu e_R = \tfrac{1}{2}g'\cos\theta_W[-\bar{e}_L\gamma_\mu e_L - 2\bar{e}_R\gamma_\mu e_R] - \tfrac{1}{2}g_W\sin\theta_W[\bar{e}_L\gamma_\mu e_L]$$

which works if: $\boxed{e = g_W\sin\theta_W = g'\cos\theta_W}$ (i.e. equate coefficients of L and R terms)

★ Couplings of electromagnetism, the weak interaction and the interaction of the $U(1)_Y$ symmetry are therefore related.

The Z Boson

★ **In this model we can now derive the couplings of the Z Boson**

$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W \qquad \boxed{I^3_W}$$

for the electron $I^3_W = \frac{1}{2}$

$$j^Z_\mu = -\tfrac{1}{2}g'\sin\theta_W[\bar{e}_L Y_{e_L}\gamma_\mu e_L + \bar{e}_R Y_{e_R}\gamma_\mu e_R] - \tfrac{1}{2}g_W\cos\theta_W[e_L\gamma_\mu e_L]$$

- **Writing this in terms of weak isospin and charge:**

$$j^Z_\mu = -\tfrac{1}{2}g'\sin\theta_W[\bar{e}_L(2Q-2I^3_W)\gamma_\mu e_L + \bar{e}_R\underbrace{(2Q)}\gamma_\mu e_R] + I^3_W g_W\cos\theta_W[e_L\gamma_\mu e_L]$$

For RH chiral states $I_3=0$

- **Gathering up the terms for LH and RH chiral states:**

$$j^Z_\mu = \left[g' I^3_W\sin\theta_W - g'Q\sin\theta_W + g_W I^3_W\cos\theta_W\right]\bar{e}_L\gamma_\mu e_L - \left[g'Q\sin\theta_W\right]e_R\gamma_\mu e_R$$

- **Using:** $e = g_W\sin\theta_W = g'\cos\theta_W$ **gives**

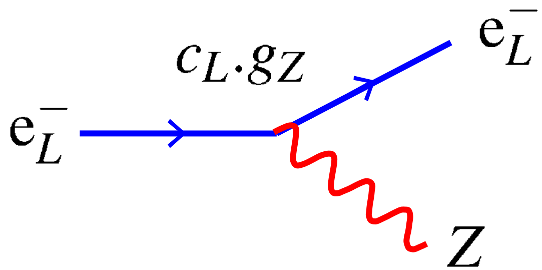
$$j^Z_\mu = \left[g'\frac{(I^3_W - Q\sin^2\theta_W)}{\sin\theta_W}\right]\bar{e}_L\gamma_\mu e_L - \left[g'\frac{Q\sin^2\theta_W}{\sin\theta_W}\right]e_R\gamma_\mu e_R$$

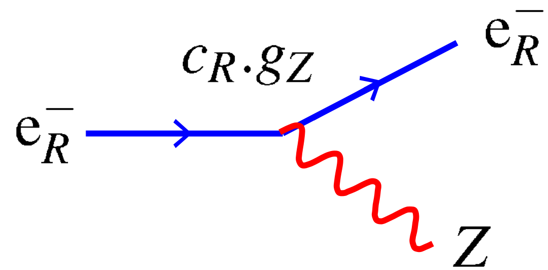
$$\boxed{j^Z_\mu = g_Z(I^3_W - Q\sin^2\theta_W)[\bar{e}_L\gamma_\mu e_L] - g_Z Q\sin^2\theta_W[e_R\gamma_\mu e_R]}$$

with $\boxed{e = g_Z\cos\theta_W\sin\theta_W}$ **i.e.** $\boxed{g_Z = \dfrac{g_W}{\cos\theta_W}}$

★ **Unlike for the Charged Current Weak interaction (W) the Z Boson couples to both LH and RH chiral components, but not equally…**

$$
\begin{aligned}
j^Z_\mu &= g_Z(I^3_W - Q\sin^2\theta_W)[\overline{\mathrm{e}}_L\gamma_\mu \mathrm{e}_L] - g_Z Q\sin^2\theta_W[\mathrm{e}_R\gamma_\mu e_R] \\
&= g_Z c_L[\overline{\mathrm{e}}_L\gamma_\mu \mathrm{e}_L] + g_Z c_R[\mathrm{e}_R\gamma_\mu e_R]
\end{aligned}
$$





$$\boxed{c_L = I^3_W - Q\sin^2\theta_W} \qquad \boxed{c_R = -Q\sin^2\theta_W}$$

W^3 part of Z couples only to LH components (like $W^\pm$)

B_μ part of Z couples equally to LH and RH components

★ **Use projection operators to obtain vector and axial vector couplings**

$$\overline{u}_L\gamma_\mu u_L = \overline{u}\gamma_\mu \tfrac{1}{2}(1-\gamma_5)u \qquad \overline{u}_R\gamma_\mu u_R = \overline{u}\gamma_\mu \tfrac{1}{2}(1+\gamma_5)u$$

$$j^Z_\mu = g_Z\overline{u}\gamma_\mu\left[c_L\tfrac{1}{2}(1-\gamma_5) + c_R\tfrac{1}{2}(1+\gamma_5))\right]u$$

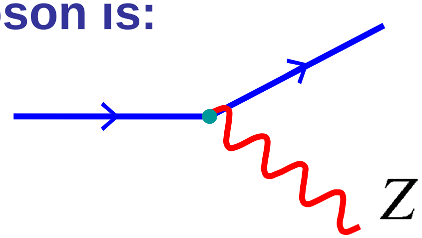
$$j^Z_\mu = \frac{g_Z}{2}\bar{u}\gamma_\mu\left[(c_L + c_R) + (c_R - c_L)\gamma_5)\right]u$$

★ **Which in terms of V and A components gives:** $$j^Z_\mu = \frac{g_Z}{2}\bar{u}\gamma_\mu\left[c_V - c_A\gamma_5\right]u$$

with $\boxed{c_V = c_L + c_R = I^3_W - 2Q\sin^2\theta_W}$ $\boxed{c_A = c_L - c_R = I^3_W}$

★ **Hence the vertex factor for the Z boson is:**

$$\boxed{-ig_Z\tfrac{1}{2}\gamma_\mu\left[c_V - c_A\gamma_5\right]}$$



★ **Using the experimentally determined value of the weak mixing angle:**

$\boxed{\sin^2\theta_W \approx 0.23}$

Fermion	Q	I^3_W	c_L	c_R	c_V	c_A
ν_e, ν_μ, ν_τ	0	$+\frac{1}{2}$	$+\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$
e^-, μ^-, τ^-	-1	$-\frac{1}{2}$	-0.27	0.23	-0.04	$-\frac{1}{2}$
u, c, t	$+\frac{2}{3}$	$+\frac{1}{2}$	0.35	-0.15	$+0.19$	$+\frac{1}{2}$
d, s, b	$-\frac{1}{3}$	$-\frac{1}{2}$	-0.42	0.08	-0.35	$-\frac{1}{2}$

Summary

★ The Standard Model interactions are mediated by spin-1 gauge bosons

★ The form of the interactions are completely specified by the assuming an underlying local phase transformation ➡ GAUGE INVARIANCE

$U(1)_{em}$	⇨	QED
$SU(2)_L$	⇨	Charged Current Weak Interaction + W^3
$SU(3)_{col}$	⇨	QCD

★ In order to "unify" the electromagnetic and weak interactions, introduced a new symmetry gauge symmetry : U(1) hypercharge

$U(1)_Y$ ⇨ B_μ

★ The physical Z boson and the photon are mixtures of the neutral W boson and B determined by the Weak Mixing angle

$$\boxed{\sin\theta_W \approx 0.23}$$

★ Have we really unified the EM and Weak interactions ? Well not really…

- Started with two independent theories with coupling constants g_W, e
- Ended up with coupling constants which are related but at the cost of introducing a new parameter in the Standard Model θ_W
- Interactions not unified from any higher theoretical principle… but it works!