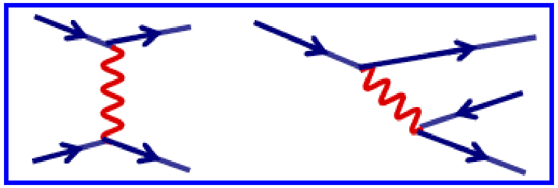
Chapter 4. Interactions as a exchange of particles

Cross Sections and Decay Rates



- In particle physics we are mainly concerned with particle <u>interactions</u> and <u>decays</u>, i.e. **transitions between states**
  - these are the experimental observables of particle physics
- Calculate <u>transition rates</u> from **Fermi's Golden Rule**

$$\boxed{\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)}$$

Γ_{fi} is number of transitions per unit time from initial state $|i\rangle$ to final state $\langle f|$ – **not Lorentz Invariant !**

T_{fi} is Transition **Matrix Element**

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j \neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \ldots$$

$\hat{H}$ is the perturbing Hamiltonian

$\rho(E_f)$ is density of final states

★ Rates depend on **MATRIX ELEMENT** and **DENSITY OF STATES**

the ME contains the fundamental particle physics

just kinematics

The Golden Rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- Rewrite the expression for density of states using a delta-function

$$\rho(E_f) = \left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_f} = \int \frac{\mathrm{d}n}{\mathrm{d}E}\delta(E-E_i)\mathrm{d}E$$ since $E_f = E_i$

Note : integrating over all final state energies but energy conservation now taken into account explicitly by delta function

- Hence the golden rule becomes: $\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)\mathrm{d}n$

the integral is over **all** "allowed" final states of any energy

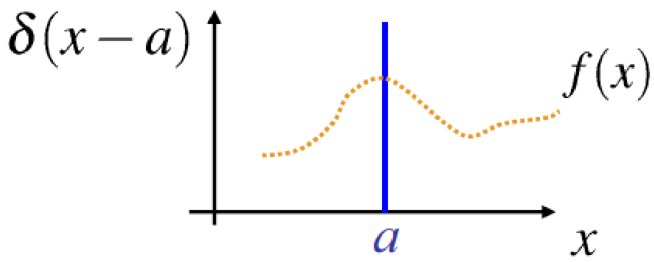
number of states

The density of states corresponds to the number of momentum states accessible to a particular decay
The number of states increases with the momentum of the final-state particles

➔ **The decay to lighter particles, which will be produced with larger momentum, are favored over decays to heavier particles**

Dirac δ Function

- In the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: **"infinitely narrow spike of unit area"**



$$\int_{-\infty}^{+\infty} \delta(x-a)\mathrm{d}x = 1$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

- Any function with the above properties can represent $\delta(x)$

e.g. $$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x^2}{2\sigma^2}\right)}$$ (an infinitesimally narrow Gaussian)

- In relativistic quantum mechanics delta functions prove extremely useful for integrals over phase space, e.g. in the decay $a \rightarrow 1 + 2$

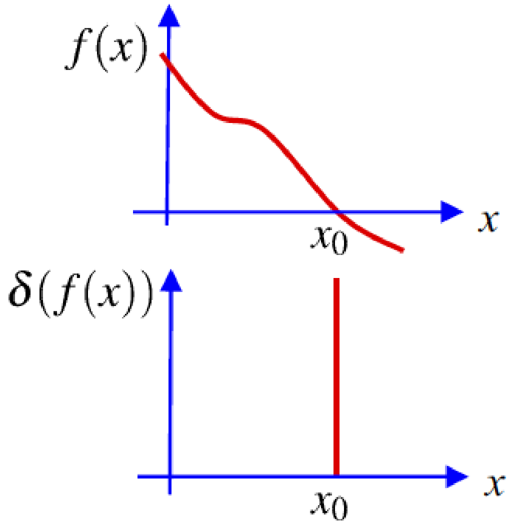
$$\int \ldots \delta(E_a - E_1 - E_2)\mathrm{d}E \quad \text{and} \quad \int \ldots \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\mathrm{d}^3\vec{p}$$

express energy and momentum conservation

★ We will soon need an expression for the **delta function of a function** $\delta(f(x))$

- Start from the definition of a delta function

$$\int_{y_1}^{y_2} \delta(y)\mathrm{d}y = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$



- Now express in terms of $y = f(x)$ where $f(x_0) = 0$ and then change variables

$$\int_{x_1}^{x_2} \delta(f(x))\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- From properties of the delta function (i.e. here only non-zero at x_0)

$$\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} \int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- Rearranging and expressing the **RHS** as a delta function

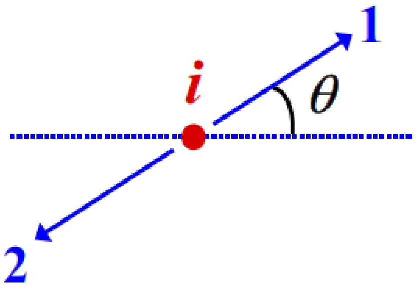
$$\int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \frac{1}{|\mathrm{d}f/\mathrm{d}x|_{x_0}} \int_{x_1}^{x_2} \delta(x - x_0)\mathrm{d}x$$

$$\Rightarrow \quad \boxed{\delta(f(x)) = \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0}^{-1} \delta(x - x_0)} \qquad (1)$$

Particle Decay Rates

- Consider the two-body decay

$$i \rightarrow 1 + 2$$



- Want to calculate the decay rate in first order perturbation theory using plane-wave descriptions of the particles (Born approximation):

$$\psi_1 = N e^{i(\vec{p}.\vec{r} - Et)} \qquad (\vec{k}.\vec{r} = \vec{p}.\vec{r} \quad \text{as} \quad \hbar = 1)$$
$$= N e^{-ip.x}$$

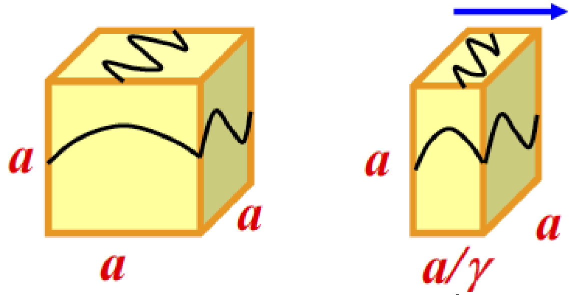
where N is the normalisation and $p.x = p^\mu x_\mu$

<u>**For decay rate calculation need to know:**</u>

- Wave-function normalisation
- Transition matrix element from perturbation theory
- Expression for the density of states

$\Bigg\}$ All in a Lorentz Invariant form

- In non-relativistic QM normalise to one particle/unit volume: $\int \psi^* \psi \mathrm{d}V = 1$
- When considering relativistic effects, volume contracts by $\gamma = E/m$



- Particle density therefore increases by $\gamma = E/m$
  - ★ Conclude that a relativistic invariant wave-function normalisation needs to be **proportional to E particles per unit volume**
- Usual convention: **Normalise to 2E particles/unit volume** $\int \psi'^* \psi' \mathrm{d}V = 2E$
- Previously used ψ normalised to 1 particle per unit volume $\int \psi^* \psi \mathrm{d}V = 1$
- Hence $\psi' = (2E)^{1/2}\psi$ is normalised to $2E$ per unit volume
- **Define** Lorentz Invariant Matrix Element, M_{fi}, in terms of the wave-functions normalised to $2E$ particles per unit volume

$$\boxed{M_{fi} = \langle \psi_1'.\psi_2'...|\hat{H}|...\psi_{n-1}'\psi_n' \rangle = (2E_1.2E_2.2E_3....2E_n)^{1/2} T_{fi}}$$

- **Non-relativistic: normalised to one particle in a cube of side a**

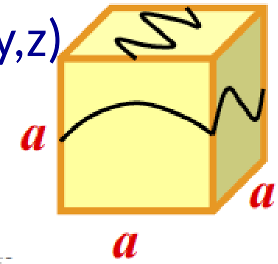
$$\int \psi\psi^* \mathrm{d}V = N^2 = 1/a^3 = 1 / V$$

- **Apply boundary conditions** $(\vec{p} = \hbar\vec{k})$:
- **Wave-function vanishing at box boundaries**
  ➡ **quantised particle momenta:**

$\psi(x+a,y,z) = \psi(x,y,z)$

➔ $e^{ipx} = e^{ip(x+a)}$

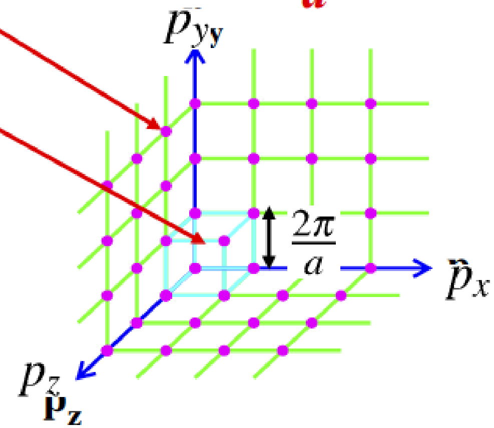
$$p_x = \frac{2\pi n_x}{a};\ p_y = \frac{2\pi n_y}{a};\ p_z = \frac{2\pi n_z}{a} \quad \equiv \quad pa = 2\pi$$



- **Volume of single state in momentum space:**

$d^3\mathbf{p} = dp_x dp_y dp_z$

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$



- **Normalising to one particle/unit volume gives number of states in element:** $\mathrm{d}^3\vec{p} = \mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z$

Average volume of a single state →

$$\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{\frac{(2\pi)^3}{V}} \times \frac{1}{V} = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$$

- **Therefore density of states in Golden rule:**

Number of states between E and E+dE: $\rho(E_f) = \left|\frac{\mathrm{d}n}{dE}\right|_{E_f} = \left|\frac{\mathrm{d}n}{\mathrm{d}|\vec{p}|}\frac{\mathrm{d}|\vec{p}|}{\mathrm{d}E}\right|_{E_f}$

with $p = \beta E$

- **Integrating over an elemental shell in momentum-space gives** $(\mathrm{d}^3\vec{p} = 4\pi p^2 \mathrm{d}p)$

$$\rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \beta$$

The Golden Rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- Rewrite the expression for density of states using a delta-function

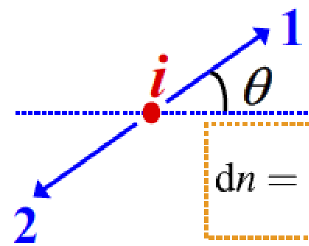
$$\rho(E_f) = \left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_f} = \int \frac{\mathrm{d}n}{\mathrm{d}E}\delta(E - E_i)\mathrm{d}E \qquad \text{since } E_f = E_i$$

Note : integrating over all final state energies but energy conservation now taken into account explicitly by delta function

- Hence the golden rule becomes: $\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)\mathrm{d}n$

the integral is over **all** "allowed" final states of any energy

- For dn in a two-body decay, only need to consider one particle : mom. conservation fixes the other



$$\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$$

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2)\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}$$

- However, can include momentum conservation explicitly by integrating over the momenta of both particles and using another δ-fn

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{\text{Energy cons.}}\underbrace{\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)}_{\text{Mom. cons.}}\underbrace{\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3}}_{\text{Density of states}}$$

- For the two body decay

$$i \rightarrow 1+2$$

$$\begin{aligned} M_{fi} &= \langle \psi_1' \psi_2' | \hat{H}' | \psi_i' \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} \langle \psi_1 \psi_2 | \hat{H}' | \psi_i \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} T_{fi} \end{aligned}$$

★ Now expressing T_{fi} in terms of M_{fi} gives

$$\boxed{\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}}$$

Note:

- M_{fi} uses relativistically normalised wave-functions. It is Lorentz Invariant
- $\dfrac{d^3\vec{p}}{(2\pi)^3 2E}$ is the Lorentz Invariant Phase Space for each final state particle; the factor of $2E$ arises from the wave-function normalisation
- This form of Γ_{fi} is simply a rearrangement of the original equation **but** the integral is now frame independent (i.e. L.I.)
- Γ_{fi} is inversely proportional to E_a, the energy of the decaying particle. This is exactly what one would expect from time dilation ($E_a = \gamma m$).
- Energy and momentum conservation in the delta functions

Decay Rate Calculations

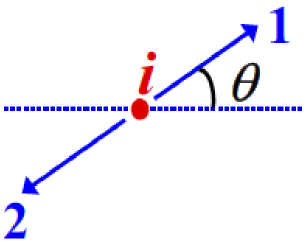
$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

★ **Because the integral is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient**

- **In the C.o.M. frame** $E_i = m_i$ **and** $\vec{p}_i = 0$ $\Rightarrow$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\delta^3(\vec{p}_1 + \vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$$

- **Integrating over** $\vec{p}_2$ **using the** δ**-function:**



$$\Rightarrow \Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\frac{d^3\vec{p}_1}{4E_1E_2}$$

now $E_2^2 = (m_2^2 + |\vec{p}_1|^2)$ **since the** δ**-function imposes** $\vec{p}_2 = -\vec{p}_1$

- **Writing** $d^3\vec{p}_1 = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$

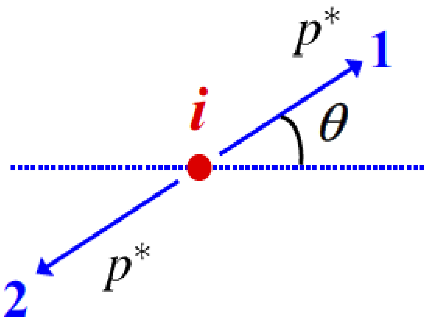
For convenience, here $|\vec{p}_1|$ **is written as** p_1

$$\Rightarrow \Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right)\frac{p_1^2 dp_1 d\Omega}{E_1E_2}$$

- Which can be written in the form $\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 g(p_1)\delta(f(p_1))\mathrm{d}p_1\mathrm{d}\Omega$ **(2)**

where $g(p_1) = p_1^2/(E_1E_2) = p_1^2(m_1^2+p_1^2)^{-1/2}(m_2^2+p_1^2)^{-1/2}$

and $f(p_1) = m_i - (m_1^2+p_1^2)^{1/2} - (m_2^2+p_1^2)^{1/2}$



Note:
- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1)=0$ determines the C.o.M momenta of the two decay products

i.e. $f(p_1)=0$ for $p_1 = p^*$

★ Eq. (2) can be integrated using the property of $\delta-$ function derived earlier (eq. (1))

$$\int g(p_1)\delta(f(p_1))\mathrm{d}p_1 = \frac{1}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}\int g(p_1)\delta(p_1-p^*)\mathrm{d}p_1 = \frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}$$

where p^* is the value for which $f(p^*)=0$

- All that remains is to evaluate $\mathrm{d}f/\mathrm{d}p_1$

$$\frac{\mathrm{d}f}{\mathrm{d}p_1} = -\frac{p_1}{(m_1^2+p_1^2)^{1/2}} - \frac{p_1}{(m_2^2+p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1+E_2}{E_1E_2}$$

giving:
$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left| \frac{E_1E_2}{p_1(E_1+E_2)}\frac{p_1^2}{E_1E_2}\right|_{p_1=p^*} d\Omega$$
$$= \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left| \frac{p_1}{E_1+E_2}\right|_{p_1=p^*} d\Omega$$

- But from $f(p_1)=0$, i.e. energy conservation: $E_1+E_2=m_i$

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i}\int |M_{fi}|^2 d\Omega$$

In the particle's rest frame $E_i = m_i$

$$\Rightarrow \quad \boxed{\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 d\Omega} \qquad (3)$$

VALID FOR ALL TWO-BODY DECAYS !

- p^* can be obtained from $f(p_1)=0$

$$(m_1^2+p^{*2})^{1/2} + (m_2^2+p^{*2})^{1/2} = m_i \qquad \text{(Question 3)}$$

$$\Rightarrow \quad p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]} \qquad \text{(now try Questions 4 \& 5)}$$

Cross section definition

$$\sigma = \frac{\text{no of interactions per unit time/per target}}{\text{incident flux}}$$

Flux = number of incident particles/ unit area/unit time

- The "cross section", σ, can be thought of as the <u>effective</u> cross-sectional area of the target particles for the interaction to occur.

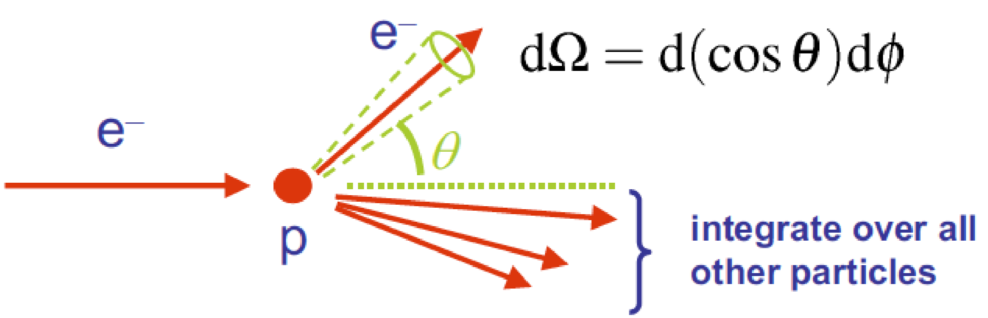
In general this has nothing to do with the physical size of the target. It is the underlying quantum probability that an interaction will occur.

Differential Cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$

or generally

$$\frac{d\sigma}{d\ldots}$$

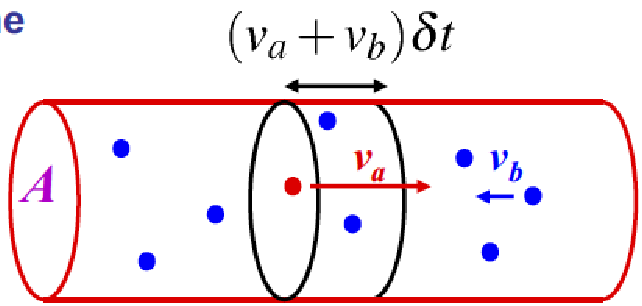


$$d\Omega = d(\cos\theta)d\phi$$

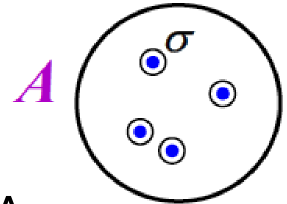
with $\boxed{\sigma = \int \frac{d\sigma}{d\Omega} d\Omega}$

example

- Consider a single particle of type a with velocity, v_a, traversing a region of area A containing n_b particles of type b per unit volume



In time δt a particle of type a traverses region containing $n_b(v_a + v_b)A\delta t =$ δN particles of type b



★ Interaction probability obtained from effective cross-sectional area occupied by the $n_b(v_a + v_b)A\delta t$ particles of type b

δP = (δN σ) / A

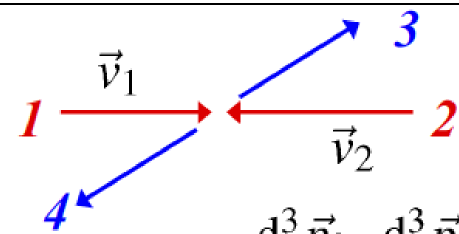
- Interaction Probability = $\dfrac{n_b(v_a + v_b)A\delta t\sigma}{A} = n_b v \delta t \sigma$ $\quad [v = v_a + v_b]$

Rate = δP / δt ➡ Rate per particle of type a = $n_b v \sigma$

- Consider volume V, total reaction rate = $(n_b v \sigma).(n_a V) = (n_b V)(n_a v)\sigma$ $= N_b \phi_a \sigma$

- As anticipated: **Rate = Flux x Number of targets x cross section**

Cross Section Calculations



- **Consider scattering process**

$$1 + 2 \to 3 + 4$$

- **Start from Fermi's Golden Rule:**

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}$$

where T_{fi} is the transition matrix for a normalisation of 1/unit volume

- **Now**

$$\begin{aligned} \text{Rate/Volume} &= (\text{flux of 1}) \times (\text{number density of 2}) \times \sigma \\ &= n_1(v_1 + v_2) \times n_2 \times \sigma \end{aligned}$$

$n_1 = n_2 = 1/V$
(V = 1)

- **For 1 target particle per unit volume** $\text{Rate} = (v_1 + v_2)\sigma$

$$\sigma = \frac{\Gamma_{fi}}{(v_1 + v_2)}$$

$$\sigma = \underbrace{\frac{(2\pi)^4}{v_1 + v_2}} \int \underbrace{|T_{fi}|^2} \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\underbrace{\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}}$$

the parts are not Lorentz Invariant

- To obtain a Lorentz Invariant form use wave-functions normalised to $2E$ particles per unit volume

$$\psi' = (2E)^{1/2}\psi$$

- Again define L.I. Matrix element $M_{fi} = (2E_1\, 2E_2\, 2E_3\, 2E_4)^{1/2} T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

- The integral is now written in a Lorentz invariant form
- The quantity $F = 2E_1 2E_2(v_1+v_1)$ can be written in terms of a four-vector scalar product and is therefore also Lorentz Invariant **(the Lorentz Inv. Flux)**

$$F = 4\left[(p_1^{\mu} p_{2\mu})^2 - m_1^2 m_2^2\right]^{1/2}$$

- Consequently cross section is a Lorentz Invariant quantity

<u>Two special cases of Lorentz Invariant Flux:</u>

- Centre-of-Mass Frame

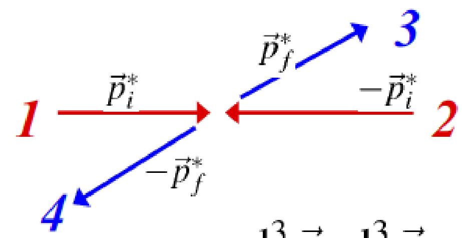
$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1E_2(|\vec{p}^*|/E_1+|\vec{p}^*|/E_2) \\ &= 4|\vec{p}^*|(E_1+E_2) \\ &= 4|\vec{p}^*|\sqrt{s} \end{aligned}$$

- Target (particle 2) at rest

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1m_2v_1 \\ &= 4E_1m_2(|\vec{p}_1|/E_1) \\ &= 4m_2|\vec{p}_1| \end{aligned}$$

2→2 Body Scattering in C.o.M. Frame

- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider 2→2 scattering in C.o.M. frame



- Start from

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

- Here $\vec{p}_1+\vec{p}_2=0$ and $E_1+E_2=\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s}-E_3-E_4)\delta^3(\vec{p}_3+\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

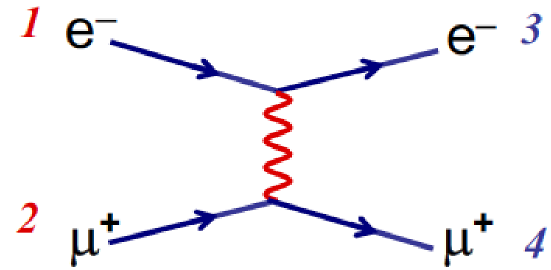
★ The integral is exactly the same integral that appeared in the particle decay calculation but with m_a replaced by $\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}}\frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 \mathrm{d}\Omega^*$$

$$\boxed{\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 \mathrm{d}\Omega^*}$$

- In the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$

$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s}\int |M_{fi}|^2 \mathrm{d}\Omega^*$$



- For calculating the total cross-section (which is Lorentz Invariant) the result on the previous page (eq. (4)) is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}|M_{fi}|^2\mathrm{d}\Omega^*$$

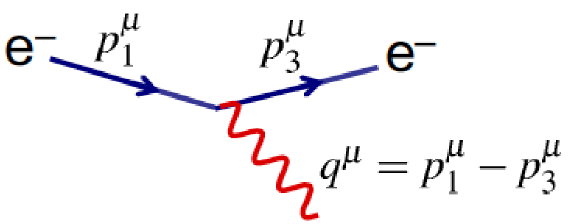
because the angles in $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^*$ refer to the C.o.M frame

- For the last calculation in this section, we need to find a L.I. expression for $\mathrm{d}\sigma$

★ Start by expressing $\mathrm{d}\Omega^*$ in terms of Mandelstam t i.e. the square of the four-momentum transfer

$$\boxed{t = q^2 = (p_1 - p_3)^2}$$

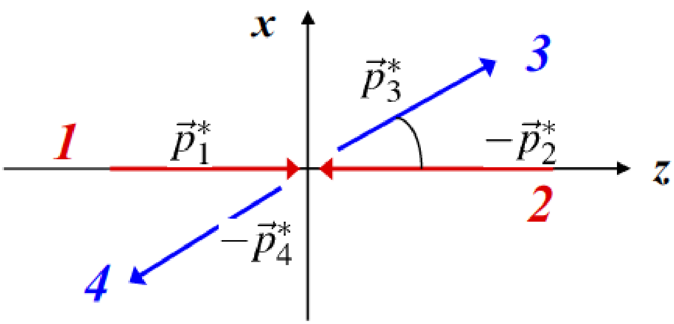
Product of four-vectors therefore L.I.



- Want to express $\mathrm{d}\Omega^*$ in terms of Lorentz Invariant $\mathrm{d}t$

  where $t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = m_1^2 + m_3^2 - 2p_1.p_3$

- **In C.o.M. frame:**



$$
\begin{aligned}
p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^*|) \\
p_3^{*\mu} &= (E_3^*, |\vec{p}_3^*|\sin\theta^*, 0, |\vec{p}_3^*|\cos\theta^*) \\
p_1^{\mu} p_{3\mu} &= E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \\
t &= m_1^2 + m_3^3 - E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*|\cos\theta^*
\end{aligned}
$$

giving $\quad \mathrm{d}t = 2|\vec{p}_1^*||\vec{p}_3^*|\mathrm{d}(\cos\theta^*)$

therefore $\quad \mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^* = \dfrac{\mathrm{d}t\,\mathrm{d}\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$

hence $\quad \mathrm{d}\sigma = \dfrac{1}{64\pi^2 s}\dfrac{|\vec{p}_3^*|}{|\vec{p}_1^*|}|M_{fi}|^2\mathrm{d}\Omega^* = \dfrac{1}{2\cdot 64\pi^2 s|\vec{p}_1^*|^2}|M_{fi}|^2\mathrm{d}\phi^*\mathrm{d}t$

- Finally, integrating over $\mathrm{d}\phi^*$ (assuming no ϕ^* dependence of $|M_{fi}|^2$) gives:

$$
\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}
$$

Lorentz Invariant differential cross section

- All quantities in the expression for $\mathrm{d}\sigma/\mathrm{d}t$ are Lorentz Invariant and therefore, it applies to **any rest frame**. It should be noted that $|\vec{p}_i^*|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

- As an example of how to use the invariant expression $\mathrm{d}\sigma/\mathrm{d}t$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$

1

e.g. electron or neutrino scattering

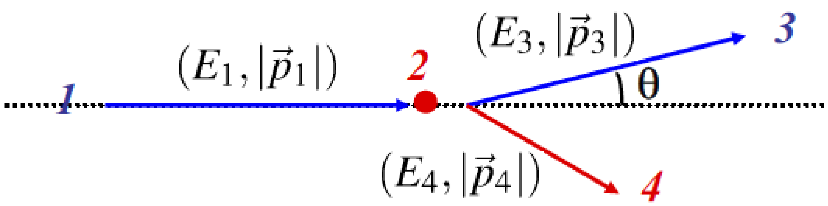
In this limit $$|\vec{p}_i^*|^2 = \frac{(s-m_2^2)^2}{4s}$$

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi(s-m_2^2)^2}|M_{fi}|^2} \qquad (m_1 = 0)$$

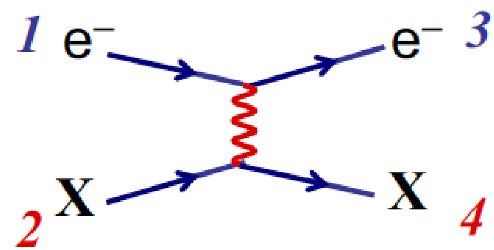
2→2 Body Scattering in Lab. Frame

2

- The other commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)
- First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: $m_1 = m_3 = 0, \quad m_2 = m_4 = M$



e.g.



- Wish to express the cross section in terms of scattering angle of the e⁻

$$d\Omega = 2\pi d(\cos\theta)$$

therefore $$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

Integrating over $d\phi$

- The rest is some rather tedious algebra…. start from four-momenta

$$p_1 = (E_1, 0, 0, E_1), \;\; p_2 = (M, 0, 0, 0), \;\; p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta), \;\; p_4 = (E_4, \vec{p}_4)$$

so here $$t = (p_1 - p_3)^2 = -2p_1.p_3 = -2E_1E_3(1-\cos\theta)$$

But from (E,p) conservation $p_1 + p_2 = p_3 + p_4$

and, therefore, can also express t in terms of particles 2 and 4

$$
\begin{aligned}
t &= (p_2 - p_4)^2 = 2M^2 - 2p_2.p_4 = 2M^2 - 2ME_4 \\
&= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)
\end{aligned}
$$

Note E_1 is a constant (the energy of the incoming particle) so

$$
\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2M\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}
$$

- Equating the two expressions for t gives $\boxed{E_3 = \dfrac{E_1 M}{M + E_1 - E_1\cos\theta}}$

so $$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{E_1^2 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3}{(E_1 M)}\right)^2 = \frac{E_3^2}{M}$$

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s - M^2)^2}|M_{fi}|^2
$$

using $s = (p_1 + p_2)^2 = M^2 + 2p_1.p_2 = M^2 + 2ME_1$

Particle 1 massless ➡ $(p_1^2 = 0)$

gives $(s - M^2) = 2ME_1$

➡ $$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2}$$

In limit $m_1 \to 0$

In this equation, E_3 is a function of θ:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}$$

giving
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2} \qquad (m_1 = 0)$$

General form for 2→2 Body Scattering in Lab. Frame

★ The calculation of the differential cross section for the case where m_1 can not be neglected is longer and contains no more "physics" (see appendix II). It gives:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{p_1 m_1}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2}$$

Again there is **only one independent variable**, θ, which can be seen from conservation of energy

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{\underbrace{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3} + m_4^2}$$

i.e. $|\vec{p}_3|$ is a function of θ

Summary

★ Used a Lorentz invariant formulation of Fermi's Golden Rule to derive decay rates and cross-sections in terms of the **Lorentz Invariant Matrix Element** (wave-functions normalised to 2E/Volume)

Main Results:

★ Particle decay:

$$\boxed{\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 \mathrm{d}\Omega}$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]}$$

★ Scattering cross section in C.o.M. frame:

$$\boxed{\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 \mathrm{d}\Omega^*}$$

★ Invariant differential cross section (valid in all frames):

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

Summary cont.

★ Differential cross section in the lab. frame (***m₁=0***)

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2} \Leftrightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2 |M_{fi}|^2}$$

★ Differential cross section in the lab. frame ($m_1 \neq 0$)

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{|\vec{p}_1|m_1}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2)-E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2}$$

with $E_1+m_2 = \sqrt{|\vec{p}_3|^2+m_3^2}+\sqrt{|\vec{p}_1|^2+|\vec{p}_3|^2-2|\vec{p}_1||\vec{p}_3|\cos\theta+m_4^2}$

Summary of the summary:

★ Have now dealt with **kinematics** of particle decays and cross sections
★ The **fundamental particle physics** is in the **matrix element**
★ The above equations are the basis for all calculations that follow

Interaction by Particle Exchange

- Calculate transition rates from **Fermi's Golden Rule**

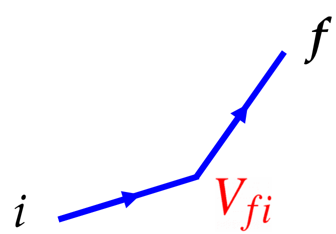
$$\boxed{\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)}$$

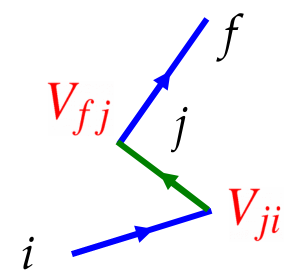
where T_{fi} is perturbation expansion for the Transition Matrix Element

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \ldots$$

- For particle scattering, the first two terms in the perturbation series can be viewed as:

"scattering in a potential"





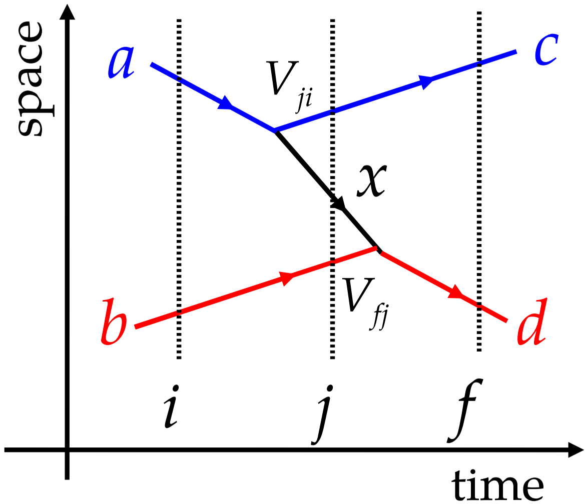
"scattering via an intermediate state"

- **"Classical picture" –** particles act as sources for fields which give rise a potential in which other particles scatter – **"action at a distance"**

- **"Quantum Field Theory picture" –** forces arise due to the exchange of virtual particles. No action at a distance + forces between particles now due to particles

- Consider the particle interaction $a+b \to c+d$ which occurs via an intermediate state corresponding to the exchange of particle x

- **One possible space-time picture of this process is:**



Initial state i : $a+b$
Final state f : $c+d$
Intermediate state j : $c+b+x$

- This time-ordered diagram corresponds to a "emitting" x and then b "absorbing" x

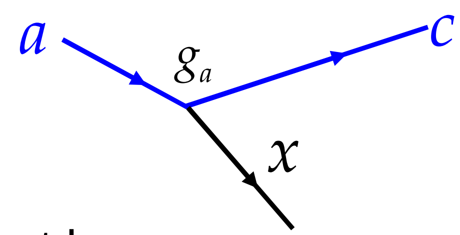
- **The corresponding term in the perturbation expansion is:**

$$T_{fi} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j}$$

$$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle\langle c+x|V|a\rangle}{(E_a+E_b)-(E_c+E_x+E_b)}$$

- T_{fi}^{ab} refers to the time-ordering where a emits x before b absorbs it

- Need an expression for $\langle c+x|V|a\rangle$ in non-invariant matrix element T_{fi}
  - Ultimately aiming to obtain Lorentz Invariant ME



- Recall T_{fi} is related to the invariant matrix element by

$$T_{fi} = \prod_k (2E_k)^{-1/2} M_{fi}$$

where k runs over all particles in the matrix element

- **Here we have**

V$_{\mathbf{ji}}$:

$$\langle c+x|V|a\rangle = \frac{M_{(a\to c+x)}}{(2E_a 2E_c 2E_x)^{1/2}}$$

$M_{(a\to c+x)}$ is the "Lorentz Invariant" matrix element for $a \to c + x$

★ **The simplest Lorentz Invariant quantity is a scalar, in this case**

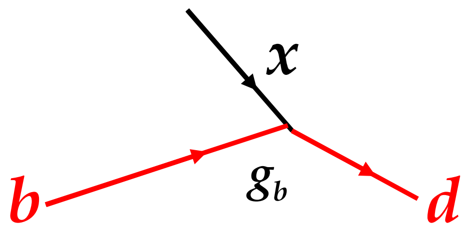
$$\langle c+x|V|a\rangle = \frac{g_a}{(2E_a 2E_c 2E_x)^{1/2}}$$

g_a is a measure of the strength of the interaction $a \to c + x$

Note : the matrix element is only LI in the sense that it is defined in terms of LI wave-function normalisations and that the form of the coupling is LI

Note : in this "illustrative" example g is not dimensionless.

Similarly $$\langle d|V|x+b\rangle = \frac{g_b}{(2E_b 2E_d 2E_x)^{1/2}}$$



Giving $$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle\langle c+x|V|a\rangle}{(E_a+E_b)-(E_c+E_x+E_b)}$$
$$= \frac{1}{2E_x}\cdot\frac{1}{(2E_a 2E_b 2E_c 2E_d)^{1/2}}\cdot\frac{g_a g_b}{(E_a-E_c-E_x)}$$

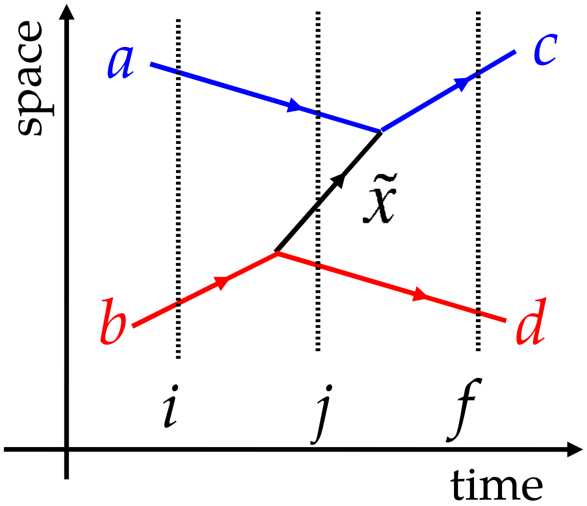
★ The "Lorentz Invariant" matrix element for the entire process is

$$M_{fi}^{ab} = (2E_a 2E_b 2E_c 2E_d)^{1/2} T_{fi}^{ab}$$
$$= \frac{1}{2E_x}\cdot\frac{g_a g_b}{(E_a-E_c-E_x)}$$

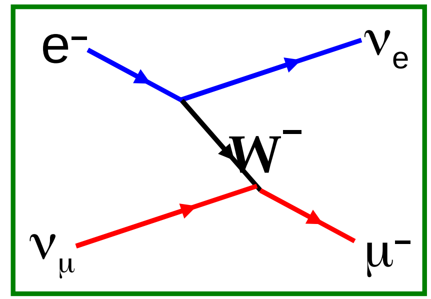
Note:

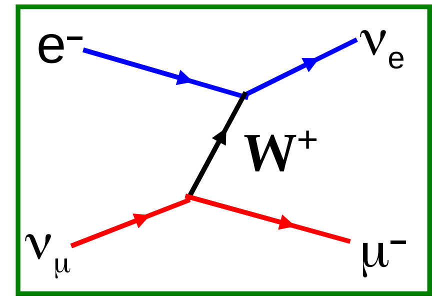
- M_{fi}^{ab} refers to the time-ordering where a emits x before b absorbs it
  It is not Lorentz invariant, order of events in time depends on frame
- Momentum is conserved at each interaction vertex but not energy
  $E_j \neq E_i$
- Particle x is "on-mass shell" i.e. $E_x^2 = \vec{p}_x^{\,2} + m^2$

★ But need to consider also the other time ordering for the process



- This **time-ordered** diagram corresponds to *b* "emitting" $\tilde{x}$ and then *a* absorbing $\tilde{x}$
  - $\tilde{x}$ is the anti-particle of x e.g.





- The **Lorentz invariant matrix element** for this time ordering is:

$$M_{fi}^{ba} = \frac{1}{2E_x} \cdot \frac{g_a g_b}{(E_b - E_d - E_x)}$$

★ In QM need to sum over matrix elements corresponding to same final state:

$$\begin{aligned} M_{fi} &= M_{fi}^{ab} + M_{fi}^{ba} \\ &= \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} + \frac{1}{E_b - E_d - E_x} \right) \\ &= \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} - \frac{1}{E_a - E_c + E_x} \right) \end{aligned}$$

Energy conservation:
$(E_a + E_b = E_c + E_d)$

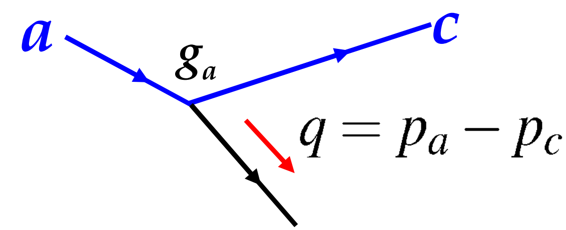
- **Which gives**

$$M_{fi} = \frac{g_a g_b}{2E_x} \cdot \frac{2E_x}{(E_a - E_c)^2 - E_x^2} = \frac{g_a g_b}{(E_a - E_c)^2 - E_x^2}$$

- **From 1st time ordering** $E_x^2 = \vec{p}_x^{\,2} + m_x^2 = (\vec{p}_a - \vec{p}_c)^2 + m_x^2$

giving

$$M_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (\vec{p}_a - \vec{p}_c)^2 - m_x^2} = \frac{g_a g_b}{(p_a - p_c)^2 - m_x^2}$$

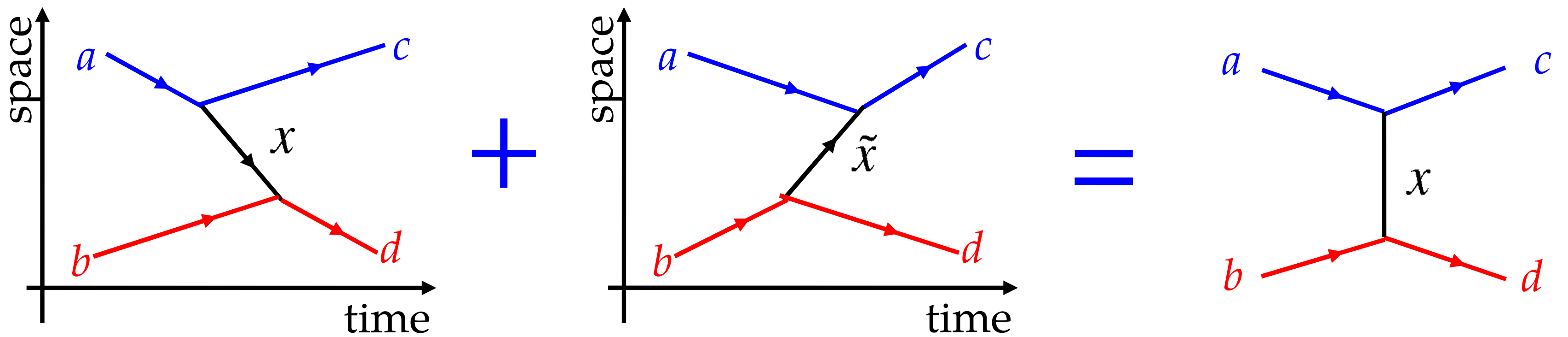


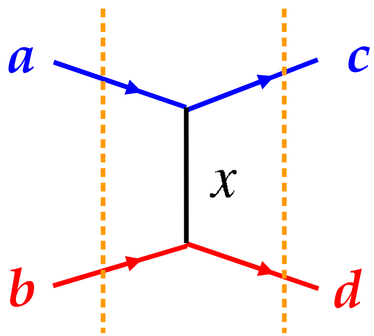
$$\boxed{M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}}$$

- After summing over all possible time orderings, M_{fi} is (as anticipated) Lorentz invariant. This is a remarkable result – the sum over all time orderings gives a frame independent matrix element.
- **Exactly the same result would have been obtained by considering the annihilation process**

Feynman Diagrams

- The sum over all possible time-orderings is represented by a **FEYNMAN diagram**





In a Feynman diagram:

- the LHS represents the initial state
- the RHS is the final state
- everything in between is "how the interaction happened"

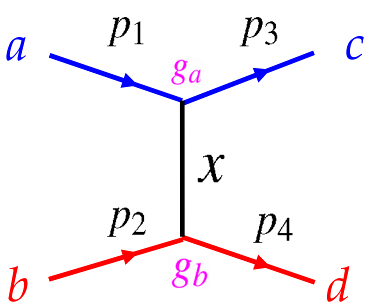
- It is important to remember that **energy and momentum** are conserved at each interaction vertex in the diagram.

- The facto $1/(q^2 - m_x^2)$ is the propagator; it arises naturally from the above discussion of interaction by particle exchange

★ **The matrix element:** $M_{fi} = \dfrac{g_a g_b}{q^2 - m_x^2}$ **depends on:**

- **The fundamental strength of the interaction at the two vertices** g_a, g_b

- **The four-momentum** q **, carried by the (virtual) particle which is determined from energy/momentum conservation at the vertices. Note** q^2 **can be either positive or negative.**



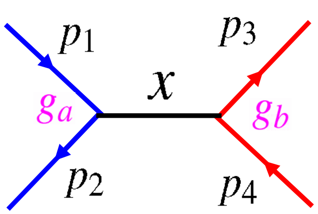
Here $q = p_1 - p_3 = p_4 - p_2 = t$ **"t-channel"**

For elastic scattering: $p_1 = (E, \vec{p}_1); \quad p_3 = (E, \vec{p}_3)$

$$q^2 = (E - E)^2 - (\vec{p}_1 - \vec{p}_3)^2$$

[also $q^2 = -4E_1E_2\sin^2(\theta/2)$]

$\boxed{q^2 < 0}$ **termed "space-like"** **(the sum of space components > time components)**



Here $q = p_1 + p_2 = p_3 + p_4 = s$ **"s-channel"**

In CoM: $p_1 = (E, \vec{p}); \quad p_2 = (E, -\vec{p})$

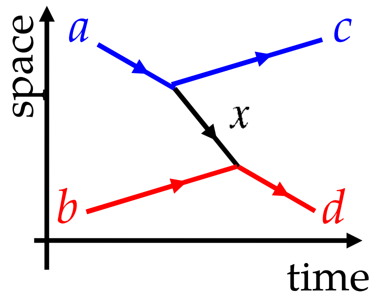
$$q^2 = (E + E)^2 - (\vec{p} - \vec{p})^2 = 4E^2$$

(the sum of time components > space component

$\boxed{q^2 > 0}$ **termed "time-like"**

Virtual Particles

"Time-ordered QM"



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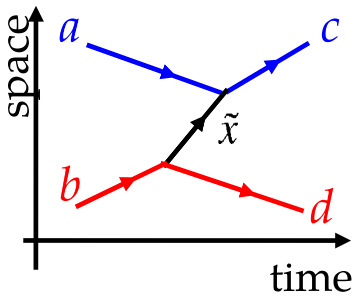

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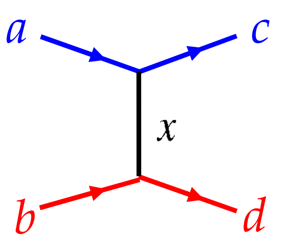
Feynman diagram



$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Momentum conserved at vertices
- Energy not conserved at vertices
- Exchanged particle "on mass shell"

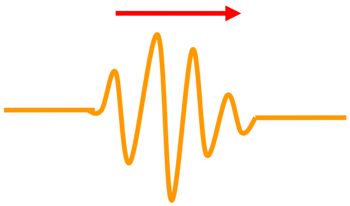
$$E_x^2 - |\vec{p}_x|^2 = m_x^2$$

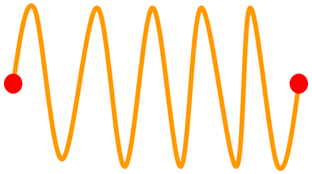
- Momentum AND energy conserved at interaction vertices
- Exchanged particle "off mass shell"

$$E_x^2 - |\vec{p}_x|^2 = q^2 \neq m_x^2$$

VIRTUAL PARTICLE

- Can think of observable "on mass shell" particles as propagating waves and unobservable virtual particles as normal modes between the source particles:





Quantum Electrodynamics (QED)

★ Now consider the interaction of an electron and tau lepton by the exchange of a photon. Although the general ideas we applied previously still hold, we now have to account for the spin of the electron/tau-lepton and also the spin (polarization) of the virtual photon.

(Non-examinable)

- The basic interaction between a photon and a charged particle can be introduced by making the minimal substitution (part II electrodynamics)

$$\vec{p} \rightarrow \vec{p} - q\vec{A}; \quad E \rightarrow E - q\phi$$

$\boxed{\text{(here } q = \text{charge)}}$

In QM:

$$\vec{p} = -i\vec{\nabla}; \qquad E = i\partial/\partial t$$

Therefore make substitution: $i\partial_\mu \rightarrow i\partial_\mu - qA_\mu$

where $A_\mu = (\phi, -\vec{A}); \qquad \partial_\mu = (\partial/\partial t, +\vec{\nabla})$

- The Dirac equation:

$$\gamma^\mu \partial_\mu \psi + im\psi = 0 \Rightarrow \gamma^\mu \partial_\mu \psi + iq\gamma^\mu A_\mu \psi + im\psi = 0$$

$$(\times i) \quad \Rightarrow \quad i\gamma^0 \frac{\partial \psi}{\partial t} + i\vec{\gamma}.\vec{\nabla}\psi - q\gamma^\mu A_\mu \psi - m\psi = 0$$

$$i\gamma^0\frac{\partial\psi}{\partial t} = \gamma^0\hat{H}\psi \;=\; m\psi - i\vec{\gamma}.\vec{\nabla}\psi + q\gamma^\mu A_\mu\psi$$

$$\times\gamma^0: \qquad \hat{H}\psi \;=\; \underbrace{(\gamma^0 m - i\gamma^0\vec{\gamma}.\vec{\nabla})}_{\text{Combined rest mass + K.E.}}\psi + \underbrace{q\gamma^0\gamma^\mu A_\mu}_{\text{Potential energy}}\psi$$

- We can identify the potential energy of a charged spin-half particle in an electromagnetic field as:

$$\boxed{\hat{V}_D = q\gamma^0\gamma^\mu A_\mu}$$

(note the A_0 term is just: $q\gamma^0\gamma^0 A_0 = q\phi$)

- The final complication is that we have to account for the photon polarization states.

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(\vec{p}.\vec{r}-Et)}$$

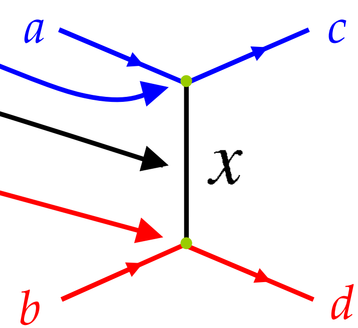
e.g. for a real photon propagating in the z direction we have two orthogonal transverse polarization states

$$\varepsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \varepsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

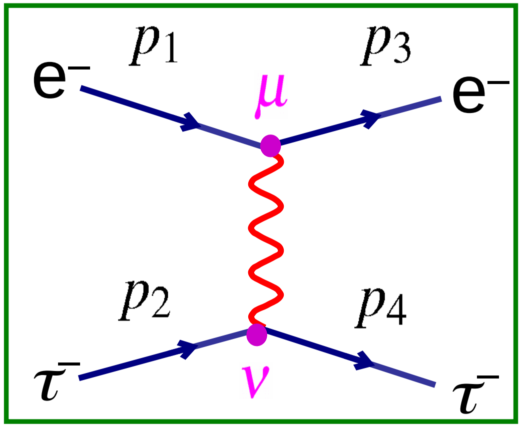
Could equally have chosen circularly polarized states

- Previously with the example of a simple spin-less interaction we had:

$$M = \underbrace{\langle \psi_c | V | \psi_a \rangle}_{= \, g_a} \frac{1}{q^2 - m_x^2} \underbrace{\langle \psi_d | V | \psi_b \rangle}_{= \, g_b}$$



★ In QED we could again go through the procedure of summing the time-orderings using Dirac spinors and the expression for $\hat{V}_D$ If we were to do this, remembering to sum over all photon polarizations, we would obtain:



$$M = \underbrace{\left[u_e^{\dagger}(p_3) q_e \gamma^0 \gamma^{\mu} u_e(p_1)\right]}_{\text{Interaction of } e^- \text{ with photon}} \underbrace{\sum_{\lambda} \frac{\varepsilon_{\mu}^{\lambda} (\varepsilon_{\nu}^{\lambda})^*}{q^2}}_{\text{Massless photon propagator summing over polarizations}} \underbrace{\left[u_{\tau}^{\dagger}(p_4) q_{\tau} \gamma^0 \gamma^{\nu} u_{\tau}(p_2)\right]}_{\text{Interaction of } \tau^- \text{ with photon}}$$

- All the physics of QED is in the above expression !

- The sum over the polarizations of the VIRTUAL photon has to include longitudinal and scalar contributions, i.e. 4 polarisation states

$$\varepsilon^{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \varepsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \varepsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \varepsilon^{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

and gives:

$$\sum_\lambda \varepsilon_\mu^\lambda (\varepsilon_\nu^\lambda)^* = -g_{\mu\nu}$$

This is not obvious – for the moment just take it on trust

and the invariant matrix element becomes:

(end of non-examinable section)

$$M = \left[u_e^\dagger(p_3) q_e \gamma^0 \gamma^\mu u_e(p_1)\right] \frac{-g_{\mu\nu}}{q^2} \left[u_\tau^\dagger(p_4) q_\tau \gamma^0 \gamma^\nu u_\tau(p_2)\right]$$

- Using the definition of the adjoint spinor $\overline{\psi} = \psi^\dagger \gamma^0$

$$\boxed{M = \left[\overline{u}_e(p_3) q_e \gamma^\mu u_e(p_1)\right] \frac{-g_{\mu\nu}}{q^2} \left[\overline{u}_\tau(p_4) q_\tau \gamma^\nu u_\tau(p_2)\right]}$$

★ This is a remarkably simple expression ! It is shown in Appendix V of Handout 2 that $\overline{u}_1 \gamma^\mu u_2$ transforms as a four vector. Writing

$$j_e^\mu = \overline{u}_e(p_3) \gamma^\mu u_e(p_1) \qquad j_\tau^\nu = \overline{u}_\tau(p_4) \gamma^\nu u_\tau(p_2)$$

$$M = -q_e q_\tau \frac{j_e \cdot j_\tau}{q^2}$$

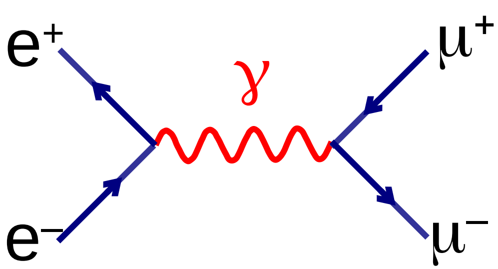
showing that M is Lorentz Invariant

Feynman Rules for QED

- It should be remembered that the expression

$$M = [\overline{u}_e(p_3)q_e\gamma^\mu u_e(p_1)]\frac{-g_{\mu\nu}}{q^2}[\overline{u}_\tau(p_4)q_\tau\gamma^\nu u_\tau(p_2)]$$

hides a lot of complexity. We have summed over all possible time-orderings and summed over all polarization states of the virtual photon. If we are then presented with a new Feynman diagram we don't want to go through the full calculation again. Fortunately this isn't necessary – can just write down matrix element using a set of simple rules



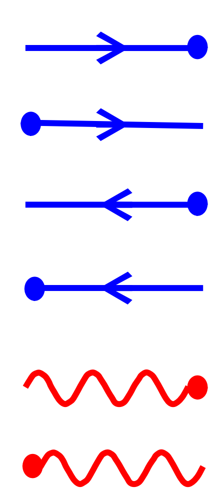
Basic Feynman Rules:

- Propagator factor for each internal line
  (i.e. each internal virtual particle)
- Dirac Spinor for each external line
  (i.e. each real incoming or outgoing particle)
- Vertex factor for each vertex

Basic Rules for QED

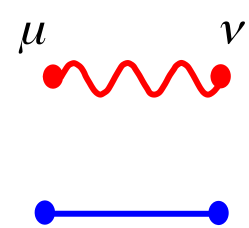
- External Lines

spin 1/2	incoming particle	$u(p)$
	outgoing particle	$\bar{u}(p)$
	incoming antiparticle	$\bar{v}(p)$
	outgoing antiparticle	$v(p)$
spin 1	incoming photon	$\varepsilon^{\mu}(p)$
	outgoing photon	$\varepsilon^{\mu}(p)^{*}$



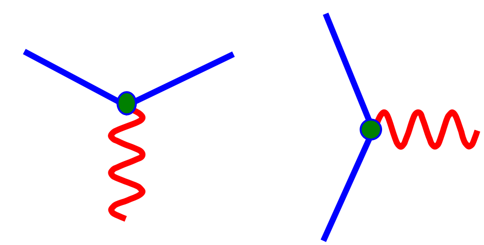
- Internal Lines (propagators)

spin 1	photon	$-\dfrac{ig_{\mu\nu}}{q^2}$
spin 1/2	fermion	$\dfrac{i(\gamma^{\mu}q_{\mu}+m)}{q^2-m^2}$

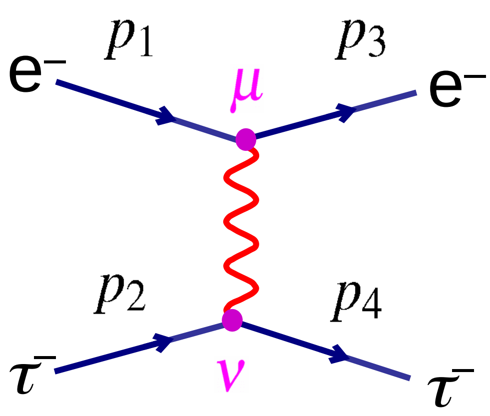


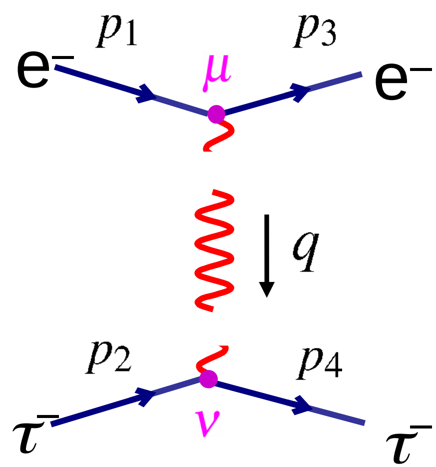
- Vertex Factors

spin 1/2	fermion (charge -|e|)	$ie\gamma^{\mu}$



- Matrix Element $-iM$ = product of all factors e.g.





$$\overline{u}_e(p_3)[ie\gamma^\mu]u_e(p_1)$$

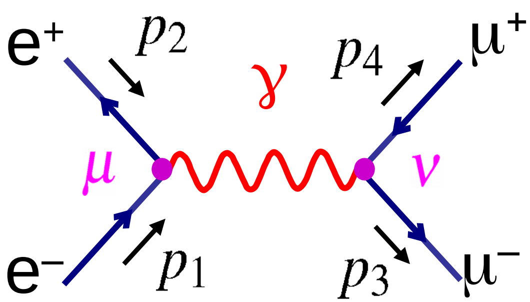
$$\frac{-ig_{\mu\nu}}{q^2}$$

$$\overline{u}_\tau(p_4)[ie\gamma^\nu]u_\tau(p_2)$$

$$-iM = [\overline{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}_\tau(p_4)ie\gamma^\nu u_\tau(p_2)]$$

- Which is the same expression as we obtained previously

e.g.



$$-iM = [\overline{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^\nu v(p_4)]$$

Note:

- At each vertex the adjoint spinor is written first
- Each vertex has a different index
- The $g_{\mu\nu}$ of the propagator connects the indices at the vertices

Summary

★ Interaction by particle exchange naturally gives rise to Lorentz Invariant Matrix Element of the form

$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

★ Derived the basic interaction in QED taking into account the spins of the fermions and polarization of the virtual photons:

$$-iM = [\overline{u}(p_3) ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\overline{u}(p_4) ie\gamma^\nu u(p_2)]$$

★ We now have all the elements to perform proper calculations in QED !

Appendix I : Lorentz Invariant Flux

NON-EXAMINABLE

- Collinear collision: $\quad a \longrightarrow \qquad \longleftarrow b$

$$v_a, \vec{p}_a \qquad\qquad v_b, \vec{p}_b$$

$$\begin{aligned} F = 2E_a 2E_b(v_a + v_b) &= 4E_aE_b\left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b}\right) \\ &= 4(|\vec{p}_a|E_b + |\vec{p}_b|E_a) \end{aligned}$$

To show this is Lorentz invariant, first consider

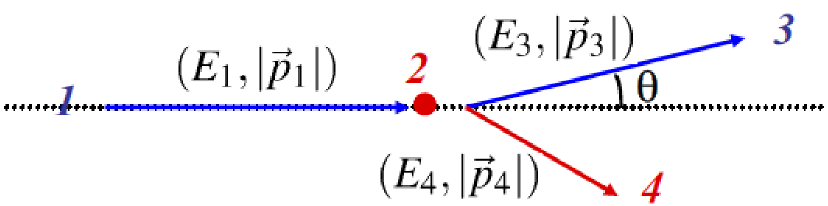
$$p_a.p_b = p_a^{\mu}p_{b\mu} = E_aE_b - \vec{p}_a.\vec{p}_b = E_aE_b + |\vec{p}_a||\vec{p}_b|$$

Giving

$$\begin{aligned} F^2/16 - (p_a^{\mu}p_{b\mu})^2 &= (|\vec{p}_a|E_b + |\vec{p}_b|E_a)^2 - (E_aE_b + |\vec{p}_a||\vec{p}_b|)^2 \\ &= |\vec{p}_a|^2(E_b^2 - |\vec{p}_b|^2) + E_a^2(|\vec{p}_b|^2 - E_b^2) \\ &= |\vec{p}_a|^2 m_b^2 - E_b^2 m_b^2 \\ &= -m_a^2 m_b^2 \\ F &= 4\left[(p_a^{\mu}p_{b\mu})^2 - m_a^2m_b^2\right]^{1/2} \end{aligned}$$

Appendix II : general 2→2 Body Scattering in lab frame

NON-EXAMINABLE



$$p_1 = (E_1, 0, 0, |\vec{p}_1|), \;\; p_2 = (M, 0, 0, 0), \;\; p_3 = (E_3, E_3 \sin\theta, 0, E_3 \cos\theta), \;\; p_4 = (E_4, \vec{p}_4)$$

again
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

But now the invariant quantity *t*:

$$\begin{aligned} t &= (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2p_2.p_4 = m_2^2 + m_4^2 - 2m_2 E_4 \\ &= m_2^2 + m_4^2 - 2m_2(E_1 + m_2 - E_3) \end{aligned}$$

$$\Rightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2m_2 \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}$$

Which gives $$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{d\sigma}{dt}$$

To determine $dE_3/d(\cos\theta)$, first differentiate $E_3^2 + |\vec{p}_3|^2 = m_3^2$

$$2E_3\frac{dE_3}{d(\cos\theta)} = 2|\vec{p}_3|\frac{d|\vec{p}_3|}{d(\cos\theta)} \qquad \text{(AII.1)}$$

Then equate $t = (p_1 - p_3)^2 = (p_4 - p_2)^2$ to give

$$m_1^2 + m_3^2 - 2(E_1E_3 - |\vec{p}_1||\vec{p}_3|\cos\theta) = m_4^2 + m_2^2 - 2m_2(E_1 + m_2 - E_3)$$

Differentiate wrt. $\cos\theta$

$$(E_1 + m_2)\frac{dE_3}{d\cos\theta} - |\vec{p}_1|\cos\theta\frac{d|\vec{p}_3|}{d\cos\theta} = |\vec{p}_1||\vec{p}_3|$$

Using (1) ➡ $$\frac{dE_3}{d(\cos\theta)} = \frac{|\vec{p}_1||\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \qquad \text{(AII.2)}$$

$$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{d\sigma}{dt} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

It is easy to show $|\vec{p}_i^*|\sqrt{s} = m_2|\vec{p}_1|$

$$\frac{d\sigma}{d\Omega} = \frac{dE_3}{d(\cos\theta)} \frac{m_2}{64\pi^2 m_2^2 |\vec{p}_1|^2} |M_{fi}|^2$$

and using (AII.2) obtain

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{p_1 m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2}$$