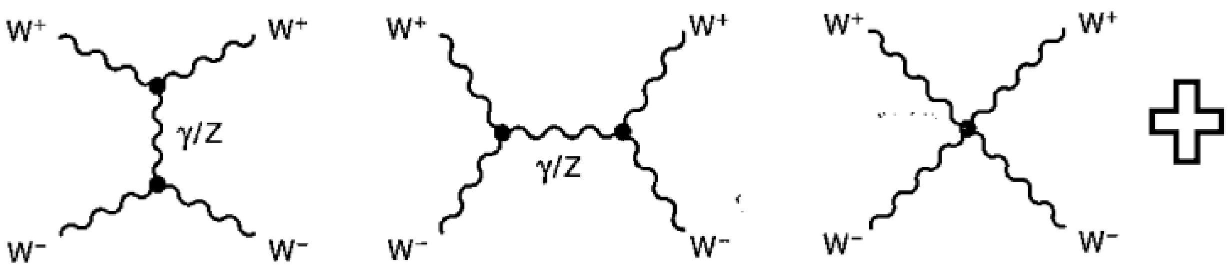
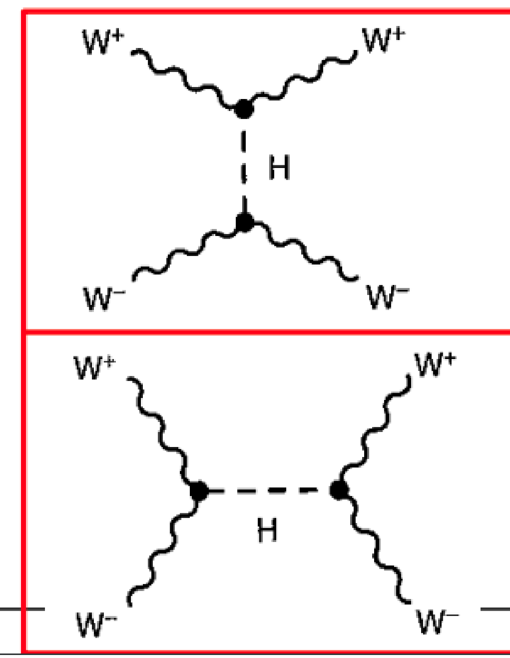
Chapter 8. The Higgs Boson

★ The Brout-Englert-Higgs mechanism (or Higgs mechanism) was introduced to allow gauge bosons to acquire mass without breaking the local gauge invariance. It also gives mass to the fundamental fermions.

't Hooft showed only LOCAL GAUGE INVARIANCE theories are renormalisable (the cancelation of all infinities is achieved with a finite number of interaction terms)

★ As the Z boson was introduced to explain the $e^+e^- \to W^+W^-$ cross section, a scalar boson is introduced to resolve the violation of unitarily in the $W^+W^- \to W^+W^-$ process (due to the extra contribution of the longitudinal degree of freedom).





Lagrangians in Quantum Field Theory

★ In QFT the entity is the "quantum field" : $\phi_i(t, x, y, z)$, a continuous function of the space-time coordinates (can be a scalar as the temperature, or vector as the electric field, …)
Particles are excitations of the field.

★ The dynamics of a quantum field is described by a Lagrangian density

$$\mathcal{L}\left(\phi_i, \partial_\mu \phi_i\right) \Rightarrow L = \int \mathcal{L}\, \mathrm{d}^3\mathbf{x}.$$

Spin 0 scalar field

$$\boxed{\mathcal{L}_S = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2.}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi \equiv \partial^0 \phi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial(\partial_k \phi)} = -\partial_k \phi \equiv \partial^k \phi,$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0. \Rightarrow \boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0,}$$

Euler-Lagrange eq.

Klein-Gordon Equation

Spin 1/2 field

★ Described by a four-component complex spinor: $\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \Psi_1 + i\Phi_1 \\ \Psi_2 + i\Phi_2 \\ \Psi_3 + i\Phi_3 \\ \Psi_4 + i\Phi_4 \end{pmatrix}.$

$$\boxed{\mathcal{L}_D = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi.}$$

The eight real fields can be expressed as a lineal combination of ψ and $\bar{\psi}$.

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_i)} = 0 \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\bar{\psi}_i} = i\gamma^\mu\partial_\mu\psi - m\psi,$$

$$\Rightarrow \quad \boxed{i\gamma^\mu(\partial_\mu\psi) - m\psi = 0.}$$

Dirac Equation

Spin 1 field

★ As the electromagnetic field, $A^{\mu} = (\phi, \mathbf{A})$

Using the $F^{\mu\nu}$ field-strength tensor $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$

★ The Lagrangian for a free photon field, and in the presence of a current

$$\boxed{\mathcal{L}_{EM} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}. \qquad \mathcal{L}_{EM} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^{\mu}A_{\mu}.} \Rightarrow \boxed{\partial_{\mu}F^{\mu\nu} = j^{\nu},}$$

Maxwell Equation

★ For a spin 1 field with mass,

$$\boxed{\mathcal{L}_{\mathrm{Proca}} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}m_{\gamma}^{2}A^{\mu}A_{\mu},}$$

Proca Equation

$$\Rightarrow \boxed{\partial_{\mu}\partial^{\mu}A^{\nu} - \partial^{\nu}(\partial_{\mu}A^{\mu}) + m^{2}A^{\nu} = j^{\nu}}$$

Particle masses

★ The required local gauge invariance of the SM theory is broken by the mass terms in the Lagragian.

$$\mathcal{L}_{\rm QED} \rightarrow \overline{\psi}(i\gamma^{\mu}\partial_{\mu} - m_{\rm e})\psi + e\overline{\psi}\gamma^{\mu}A_{\mu}\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m_{\gamma}^{2}A_{\mu}A^{\mu}.$$

In the gauge field:

- the transformation in the field: $A_{\mu} \rightarrow A'_{\mu} = A_{\mu} - \partial_{\mu}\chi$
- and in the mass term:

$$\tfrac{1}{2}m_{\gamma}^{2}A_{\mu}A^{\mu} \rightarrow \tfrac{1}{2}m_{\gamma}^{2}\left(A_{\mu} - \partial_{\mu}\chi\right)\left(A^{\mu} - \partial^{\mu}\chi\right) \neq \tfrac{1}{2}m_{\gamma}^{2}A_{\mu}A^{\mu},$$

In the fermion term:

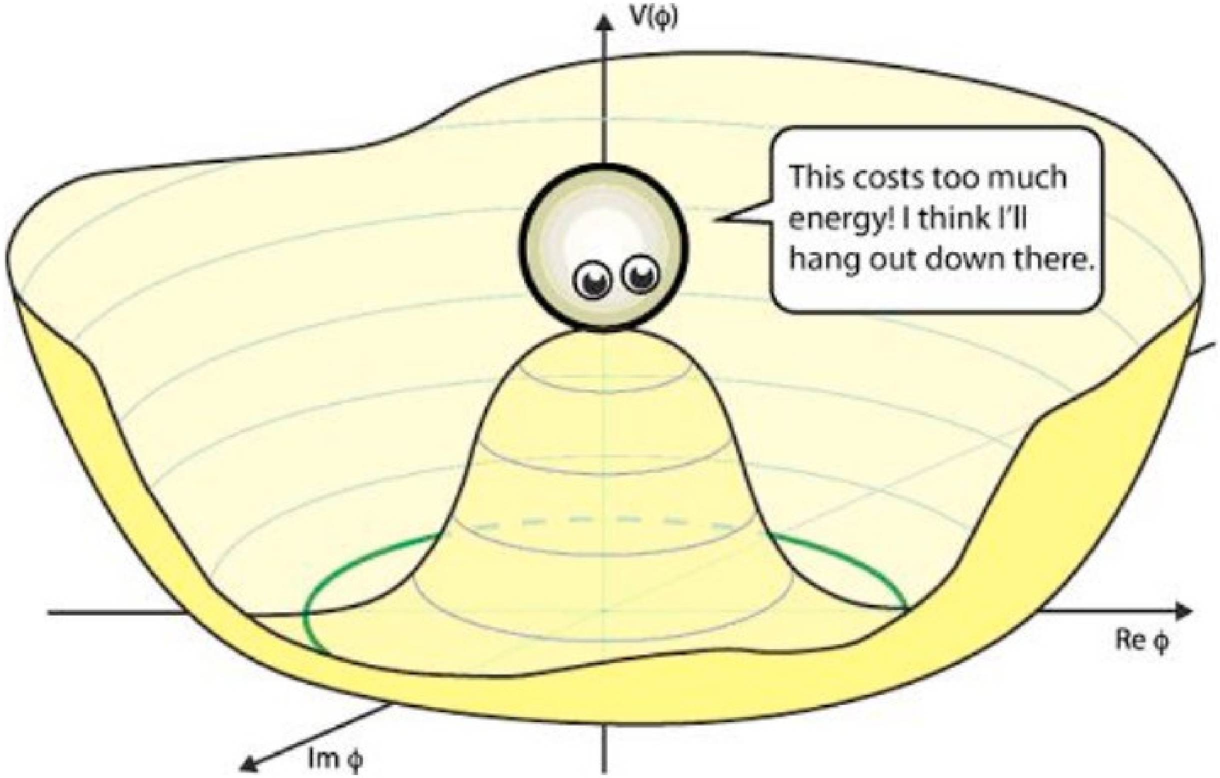
- writing the mass term in terms of the chiral states

$$\left.\begin{aligned} e_R &= P_R e_R \\ \overline{e}_R &= \overline{e}_R P_L \end{aligned}\right\} \quad \overline{e}_R e_R = 0\ldots$$

$$\begin{aligned} -m_{\rm e}\overline{\rm e}{\rm e} &= -m_{\rm e}\overline{\rm e}\left[\tfrac{1}{2}(1-\gamma^5) + \tfrac{1}{2}(1+\gamma^5)\right]{\rm e} \\ &= -m_{\rm e}\overline{\rm e}\left[\tfrac{1}{2}(1-\gamma^5){\rm e}_L + \tfrac{1}{2}(1+\gamma^5){\rm e}_R\right] \\ &= -m_{\rm e}(\overline{\rm e}_R{\rm e}_L + \overline{\rm e}_L{\rm e}_R). \end{aligned}$$

- .. but in $SU(2)_L$ the left-handed states transform as doublet while the right-handed states transform as singlet

The BEH (Higgs) mechanism



The BEH (Higgs) mechanism

① How mass terms for a scalar field arise from a broken symmetry

② How the mass of a gauge boson arise from a broken U(1) global gauge symmetry

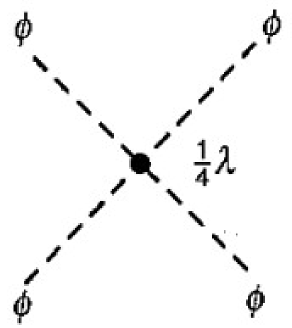
③ How the mass of a gauge boson arise from a broken U(1) local gauge symmetry

Extension of the breaking symmetry mechanism to the $SU(2)_L \times U(1)_Y$

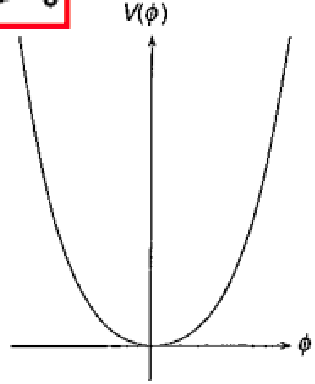
① Interacting Scalar Fields

The Lagrangian of a scalar field ϕ under a potential $V(\phi)$:

$$
\begin{aligned}
\mathcal{L} &= \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi) \\
&= \underbrace{\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)}_{\textit{Kinetic energy}} - \underbrace{\tfrac{1}{2}\mu^2\phi^2}_{\textit{Mass term}} - \underbrace{\tfrac{1}{4}\lambda\phi^4}_{\textit{Self-interaction}}.
\end{aligned}
$$



$\boxed{\mu^2 > 0}$



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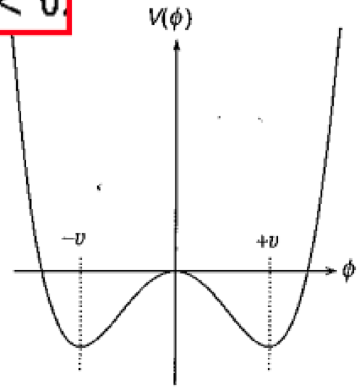


$$\boxed{V(\phi) = \mu^2\phi^2/2 + \lambda\phi^4/4}$$

- λ must be positive to have a finite minimum
- If $\mu^2 > 0$ the potential has a minimum at $\phi = 0$
- If $\mu^2 < 0$ the potential has minima at:

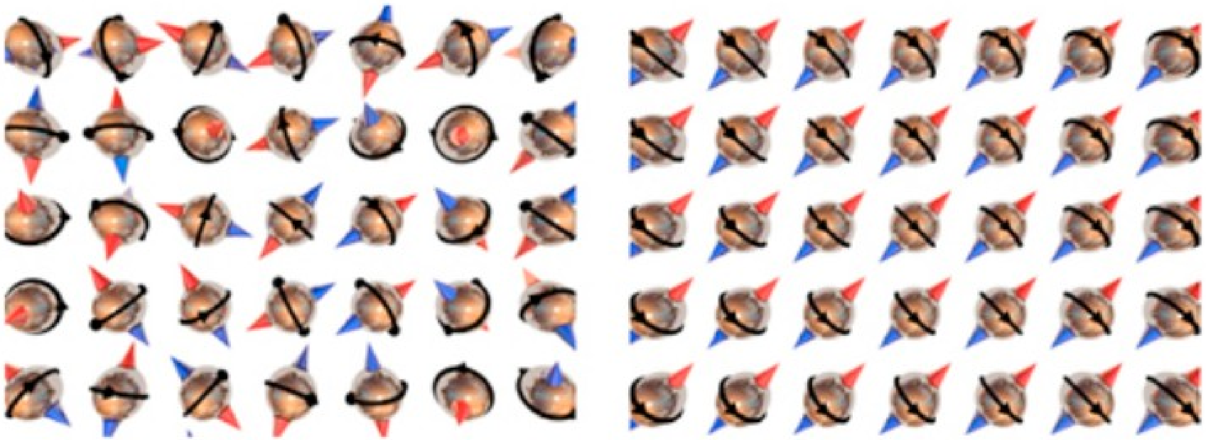
$$\boxed{\phi = \pm v = \pm\left|\sqrt{\frac{-\mu^2}{\lambda}}\right|}$$

The lowest energy state is non 0 and the ϕ field has a non-zero vacuum expectation value ***v***

★ The corresponding term in the Lagrangian can not be considered as mass term
★ The vacuum state of the field could be $\phi = +v$ or $\phi = -v$.
★ The choice of the vacuum breaks the symmetry of the Lagrangian

⇨ **Spontaneous symmetry breaking**

Classical example: **Imagine a collection of atoms arranged regularly in a plane, each atom having a small magnetic field that is constrained to point in one of two directions: it can either point upwards or downwards.**



Schematic depiction of the atoms in a ferromagnet.
Left: The symmetrical phase above the Curie temperature where all the magnetic fields of the atoms are randomly oriented.
Right: The asymmetrical phase below the Curie temperature where the magnetic fields of all the atoms are aligned in the same direction.

constant

★ **If the vacuum is chosen as $\phi = +v$, a small excitation around the minimum (*the particle state*) would be:**

$$\phi(x) = v + \eta(x)$$

The Lagrangian can be written as function of η

$$\mathcal{L}(\eta) = \tfrac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - V(\eta)$$
$$= \tfrac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \tfrac{1}{2}\mu^2(v+\eta)^2 - \tfrac{1}{4}\lambda(v+\eta)^4.$$

$\mu^2 = -\lambda v^2$

$$\mathcal{L}(\eta) = \tfrac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) \underbrace{- \lambda v^2 \eta^2}_{\frac{1}{2}m^2\phi^2 \text{ Mass term}} \underbrace{- \lambda v \eta^3 - \tfrac{1}{4}\lambda \eta^4}_{\text{Self-interactions}} \underbrace{+ \tfrac{1}{4}\lambda v^4}_{\text{Constant}}.$$

$$\boxed{m_\eta = \sqrt{2\lambda v^2} = \sqrt{-2\mu^2},}$$

★ **The new Lagrangian *L(η)* is the same as *L(ϕ)* but expressed as small excitations about the minimum**

$$\boxed{\mathcal{L}(\eta) = \tfrac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \tfrac{1}{2}m_\eta^2 \eta^2 - V(\eta), \quad \text{with} \quad V(\eta) = \lambda v \eta^3 + \tfrac{1}{4}\lambda \eta^4.}$$

② Breaking the U(1) global gauge symmetry

★ **Extending the real scalar field to a *complex scalar field*:** $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2),$

★ **The corresponding Lagrangian is:**

$$\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - V(\phi) \quad \text{with} \quad V(\phi) = \mu^2(\phi^*\phi) + \lambda(\phi^*\phi)^2.$$

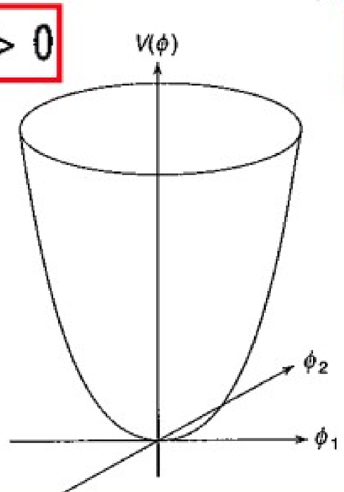
It is invariant under $\phi \rightarrow \phi' = e^{i\alpha}\phi$, **since** $\phi'^*\phi' = \phi^*\phi$

Global U(1) Symmetry

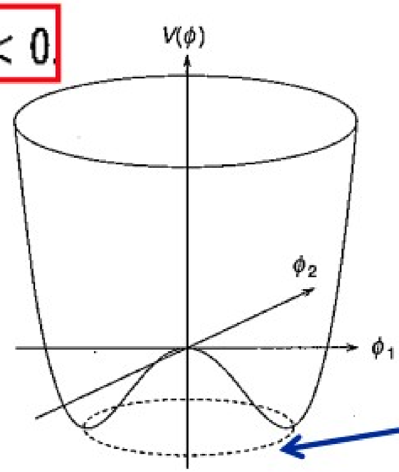
in terms of ϕ_1 and ϕ_2

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2.$$

$\mu^2 > 0$



$\mu^2 < 0$

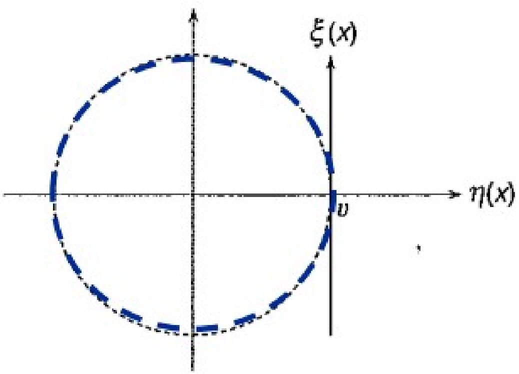


- **If $\mu^2 > 0$ the potential has a minimum at $\phi_1 = \phi_2 = 0$**
- **If $\mu^2 < 0$ the potential has an infinite set of minima**

$$\phi_1^2 + \phi_2^2 = \frac{-\mu^2}{\lambda} = v^2,$$

★ **Choosing a physical vacuum state that correspond to any given point of the cycle (the vacuum is degenerated), we break the global U(1) symmetry.**

★ **Let's choose** $(\phi_1, \phi_2) = (v, 0)$, **a point in the real direction, and consider small perturbation about the vacuum,**



$$\left.\begin{aligned} \phi_1(x) &= \eta(x) + v \\ \phi_2(x) &= \xi(x) \end{aligned}\right\} \quad \boxed{\phi = \frac{1}{\sqrt{2}}(\eta + v + i\xi).}$$

$$\phi^2 = \phi\phi^* = \frac{1}{2}\left[(v+\eta)^2 + \xi^2\right]$$

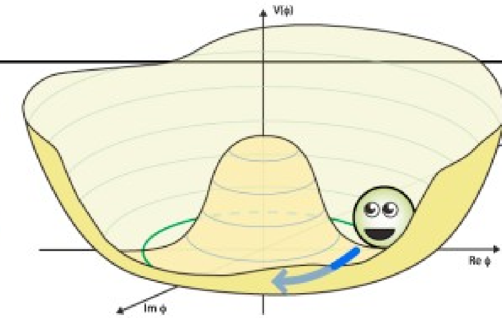
★ **The Lagrangian in terms of η and ξ** $\Longrightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi) - V(\eta, \xi),$

$$\begin{aligned} V(\eta,\xi) &= \mu^2\phi^2 + \lambda\phi^4 \\ &= -\frac{1}{2}\lambda v^2\left\{(v+\eta)^2 + \xi^2\right\} + \frac{1}{4}\lambda\left\{(v+\eta)^2 + \xi^2\right\}^2 \\ &= -\frac{1}{4}\lambda v^4 + \lambda v^2\eta^2 + \lambda v\eta^3 + \frac{1}{4}\lambda\eta^4 + \frac{1}{4}\lambda\xi^4 + \lambda v\eta\xi^2 + \frac{1}{2}\lambda\eta^2\xi^2. \end{aligned}$$

ϕ^2

$\mu^2 = -\lambda v^2,$

Mass term *Self-interactions*

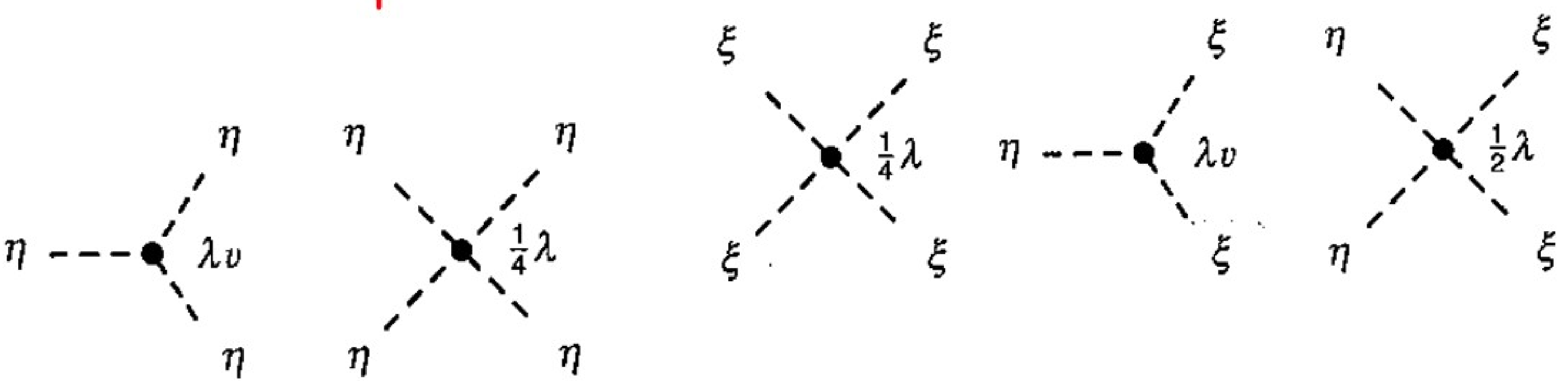


$$\boxed{\mathcal{L} = \tfrac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \tfrac{1}{2}m_\eta^2\eta^2 + \tfrac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi) - V_{int}(\eta,\xi),}$$

★ With a massless field ξ and a field η with $m_\eta = \sqrt{2\lambda v^2}$ corresponding to a excitation without change of potential (Goldstone boson) and a excitation that requires a change in the potential

★ V_{int} represents the scalar interactions obtained by breaking the symmetry for a complex field

$$V_{int}(\eta,\xi) = \underbrace{\lambda v\eta^3 + \tfrac{1}{4}\lambda\eta^4}_{} + \underbrace{\tfrac{1}{4}\lambda\xi^4 + \lambda v\eta\xi^2 + \tfrac{1}{2}\lambda\eta^2\xi^2}_{}.$$



③ Breaking a U(1) local gauge symmetry (The BEH mechanism)

★ Start with spontaneous breaking of a U(1) local gauge symmetry

★ Invariance under $\phi(x) \to \phi'(x) = e^{ig\chi(x)}\phi(x)$. requires $\partial_\mu \to D_\mu = \partial_\mu + igB_\mu$.

★ The resulting Lagrancian $\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - V(\phi^2)$, is invariant provided the field transforms as $B_\mu \to B'_\mu = B_\mu - \partial_\mu\chi(x)$.

★ The combined Lagrancian for ϕ and Bμ $\boxed{\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - \mu^2\phi^2 - \lambda\phi^4,}$

$$
\begin{aligned}
(D_\mu\phi)^*(D^\mu\phi) &= (\partial_\mu - igB_\mu)\phi^*(\partial^\mu + igB^\mu)\phi \\
&= (\partial_\mu\phi)^*(\partial^\mu\phi) - igB_\mu\phi^*(\partial^\mu\phi) + ig(\partial_\mu\phi^*)B^\mu\phi + g^2B_\mu B^\mu\phi^*\phi
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (\partial_\mu\phi)^*(\partial^\mu\phi) - \mu^2\phi^2 - \lambda\phi^4 \\
&- igB_\mu\phi^*(\partial^\mu\phi) + ig(\partial_\mu\phi^*)B^\mu\phi + g^2B_\mu B^\mu\phi^*\phi.
\end{aligned}
}
$$

★ As before, if $\mu^2 < 0$, we choose the physical vacuum state to be $\phi_1 + i\phi_2 = v$,

and we make small perturbations about the vacuum: $\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\xi(x))$.

$$\boxed{\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \lambda v^2\eta^2}_{\text{massive } \eta} + \underbrace{\tfrac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi)}_{\text{massless } \xi} - \underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}g^2v^2B_\mu B^\mu}_{\text{massive gauge field}} - V_{int} + gvB_\mu(\partial^\mu\xi),}$$

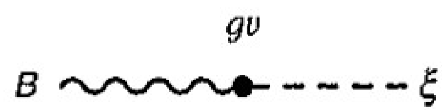
The underlying gauge symmetry is *hidden* but not removed !

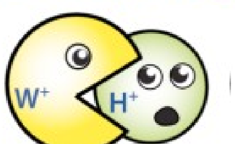
Interaction terms η, ξ and B

★ Counting degrees of freedom:

- The original *L* has 4 d.o.f. : ϕ_1, ϕ_2, and the two polarizations of the massless B field
- The new *L* has 4 + 1: the longitudinal polarization of the massive B field

★ Also, the new *L* has a new term: $gvB_\mu(\partial^\mu\xi)$,
The direct coupling of a Goldstone boson and the B field





Thus the B field acquires the new degree of freedom, the longitudinal polarization

★ The ξ field can be removed from the *L* by making an appropriate gauge transformation (since the *L* was local gauge invariant the physics does not change by choosing a given gauge !)

Note that one can re-write the *L* terms:

$$\tfrac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + gvB_\mu(\partial^\mu\xi) + \tfrac{1}{2}g^2v^2B_\mu B^\mu = \tfrac{1}{2}g^2v^2\left[B_\mu + \frac{1}{gv}(\partial_\mu\xi)\right]^2,$$

And make the gauge transformation:

$$B_\mu(x) \to B'_\mu(x) = B_\mu(x) + \frac{1}{gv}\partial_\mu\xi(x),$$

Unitary Gauge

The *L* becomes:

$$\boxed{\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta) - \lambda v^2\eta^2}_{\text{massive }\eta} + -\underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}g^2v^2B^{\mu\prime}B'_\mu}_{\text{massive gauge field}} - V_{int}.}$$

★ This choice of gauge is: $\chi(x) = -\xi(x)/gv$

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\xi(x)),$$

$$\phi(x) \to \phi'(x) = e^{-ig\frac{\xi(x)}{gv}}\phi(x) = e^{-i\xi(x)/v}\phi(x).$$

$$e^{ix} = 1 + ix - \frac{x^2}{2} - i\frac{x^3}{6} + \frac{x^4}{24} + \ldots$$

$$\phi(x) \approx \frac{1}{\sqrt{2}}\left[v + \eta(x)\right]e^{i\xi(x)/v}.$$

★ The choice of the *Unitary Gauge* implies to choose the complex scalar field ϕ entirely real

$$\phi(x) \to \phi'(x) = \frac{1}{\sqrt{2}} e^{-i\xi(x)/v} \left[v + \eta(x)\right] e^{i\xi(x)/v} = \frac{1}{\sqrt{2}}(v + \eta(x)).$$

Physical field

$$\boxed{\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x)) \equiv \frac{1}{\sqrt{2}}(v + h(x)).}$$

★ And the Lagrancian can be written as:

$$\begin{aligned}
\mathcal{L} &= (D_\mu\phi)^*(D^\mu\phi) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mu^2\phi^2 - \lambda\phi^4 \\
&= \tfrac{1}{2}(\partial_\mu - igB_\mu)(v+h)(\partial^\mu + igB^\mu)(v+h) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\mu^2(v+h)^2 - \tfrac{1}{4}\lambda(v+h)^4 \\
&= \tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) + \tfrac{1}{2}g^2 B_\mu B^\mu (v+h)^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \lambda v^2 h^2 - \lambda v h^3 - \tfrac{1}{4}\lambda h^4 + \tfrac{1}{4}\lambda v^4.
\end{aligned}$$

with all physical fields … and ignoring constants (v^n) …

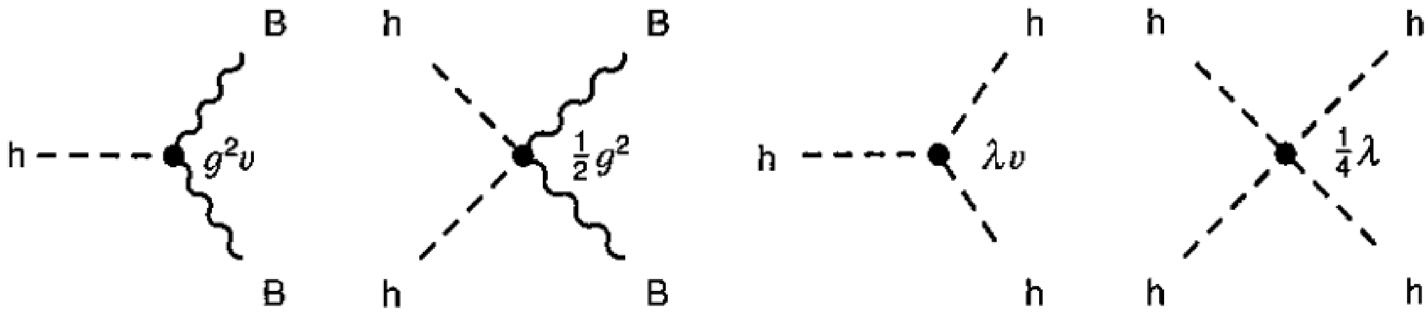
$$\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) - \lambda v^2 h^2}_{\text{massive } h \text{ scalar}} - \underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}g^2 v^2 B_\mu B^\mu}_{\text{massive gauge boson}}$$
$$+ \underbrace{g^2 v B_\mu B^\mu h + \tfrac{1}{2}g^2 B_\mu B^\mu h^2}_{h,B \text{ interactions}} - \underbrace{\lambda v h^3 - \tfrac{1}{4}\lambda h^4}_{h \text{ self-interactions}} .$$

a massive Higgs field & a massive B field

$m_B = g\,v,$ $\quad m_H = \sqrt{2\lambda}\,v.$

both masses related with v

★ **The interaction terms are:**



The Standard Model Higgs

- ★ In the Weinberg-Salam model the BEH mechanism is applied to the electro-weak $SU(2)_L \times U(1)_Y$ local gauge symmetry.

- ★ We need 3 Goldstone bosons, and there will be *at least* 1 massive boson as result of the excitation about the vacuum (4 degrees of freedom)

- ★ The simplest Higgs model consist of two complex scalar fields

$$\boxed{\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}}$$

- ★ The Lagrangian for this doublet of scalar fields, with $V(\phi)$ being the Higgs potential, is:

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - V(\phi), \qquad V(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2.$$

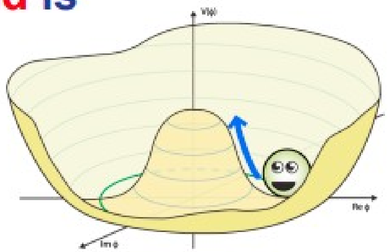
- ★ For $\mu^2 < 0$, and choosing the vacuum at the infinite set of minima satisfy

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

$$\phi^\dagger \phi = \tfrac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = \frac{v^2}{2} = -\frac{\mu^2}{2\lambda}.$$

★ As before, expanding the fields about the minimum $\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1(x) + i\phi_2(x) \\ v + \eta(x) + i\phi_4(x) \end{pmatrix}$.

★ The three massless Goldstone boson will provide the longitudinal polarization to the 3 Gauge field and, in the **unitary gauge the scalar Higgs field** is



$$\boxed{\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}.}$$

★ In $SU(2)_L \times U(1)_Y$, the covariant derivative acting on the Higgs doublet is:

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + ig_W \mathbf{T} \cdot \mathbf{W}_\mu + ig' \frac{Y}{2} B_\mu, \qquad D_\mu \phi = \frac{1}{2} \left[2\partial_\mu + \left(ig_W \boldsymbol{\sigma} \cdot \mathbf{W}_\mu + ig' B_\mu \right) \right] \phi,$$

$$\mathbf{T} = \tfrac{1}{2}\boldsymbol{\sigma} \qquad Y = 2\left(Q - I_W^{(3)}\right).$$

In the unitary gauge

$$D_\mu \phi = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2\partial_\mu + ig_W W_\mu^{(3)} + ig' B_\mu & ig_W [W_\mu^{(1)} - iW_\mu^{(2)}] \\ ig_W [W_\mu^{(1)} + iW_\mu^{(2)}] & 2\partial_\mu - ig_W W_\mu^{(3)} + ig' B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v + h \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} ig_W (W_\mu^{(1)} - iW_\mu^{(2)})(v + h) \\ (2\partial_\mu - ig_W W_\mu^{(3)} + ig' B_\mu)(v + h) \end{pmatrix}.$$

★ **The term in the L that generates the mass of the gauge fields is** $(D_\mu\phi)^\dagger(D^\mu\phi)$.

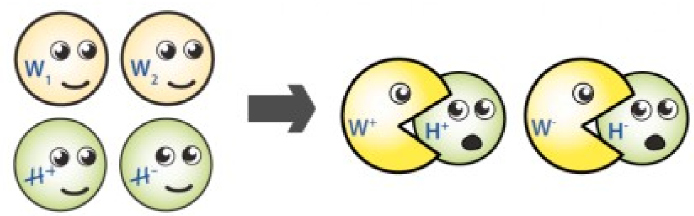
$$(D_\mu\phi)^\dagger(D^\mu\phi) = \tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) + \tfrac{1}{8}g_\mathrm{W}^2(W_\mu^{(1)} + iW_\mu^{(2)})(W^{(1)\mu} - iW^{(2)\mu})(v+h)^2$$
$$+ \tfrac{1}{8}(g_\mathrm{W}W_\mu^{(3)} - g'B_\mu)(g_\mathrm{W}W^{(3)\mu} - g'B^\mu)(v+h)^2.$$

★ **The mass terms are those quadratic in the gauge fields**

$$\tfrac{1}{8}v^2g_\mathrm{W}^2\left(W_\mu^{(1)}W^{(1)\mu} + W_\mu^{(2)}W^{(2)\mu}\right) + \tfrac{1}{8}v^2\left(g_\mathrm{W}W_\mu^{(3)} - g'B_\mu\right)\left(g_\mathrm{W}W^{(3)\mu} - g'B^\mu\right).$$

★ **For the spin 1 $W^{(1)}$ and $W^{(2)}$ fields:**

$$\tfrac{1}{2}m_\mathrm{W}^2W_\mu^{(1)}W^{(1)\mu} \quad \text{and} \quad \tfrac{1}{2}m_\mathrm{W}^2W_\mu^{(2)}W^{(2)\mu},$$



$$\boxed{m_\mathrm{W} = \tfrac{1}{2}g_\mathrm{W}v.}$$

The W mass is related to the coupling constant g_W and the vacuum expectation value of the Higgs

★ **And for the spin 1 W(3) and B fields:**

$$\frac{v^2}{8}\left(g_W W_\mu^{(3)} - g' B_\mu\right)\left(g_W W^{(3)\mu} - g' B^\mu\right) = \frac{v^2}{8}\begin{pmatrix} W_\mu^{(3)} & B_\mu \end{pmatrix}\begin{pmatrix} g_W^2 & -g_W g' \\ -g_W g' & g'^2 \end{pmatrix}\begin{pmatrix} W^{(3)\mu} \\ B^\mu \end{pmatrix}$$

$$\equiv \frac{v^2}{8}\begin{pmatrix} W_\mu^{(3)} & B_\mu \end{pmatrix}\mathbf{M}\begin{pmatrix} W^{(3)\mu} \\ B^\mu \end{pmatrix},$$

M **is a non-diagonal matrix that allows the mix of the fields**

The *physical gauge fields*, eigenstates of the free particle Hamiltonian, correspond to the basis where the matrix is diagonal

The masses of the physical field are obtained from the solutions of the characteristic equation: $\det(\mathbf{M} - \lambda I) = 0,$

$$(g_W^2 - \lambda)(g'^2 - \lambda) - g_W^2 g'^2 = 0, \quad \Longrightarrow \quad \begin{aligned} \lambda &= 0 \\ \lambda &= g_W^2 + g'^2. \end{aligned}$$

★ **Therefore, in the diagonal basis, the mass matrix *M* is**

$$\frac{1}{8}v^2\begin{pmatrix} A_\mu & Z_\mu \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & g_W^2 + g'^2 \end{pmatrix}\begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} A_\mu & Z_\mu \end{pmatrix}\begin{pmatrix} m_A^2 & 0 \\ 0 & m_Z^2 \end{pmatrix}\begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix},$$

$$\boxed{m_A = 0,} \qquad \boxed{m_Z = \tfrac{1}{2}v\sqrt{g_W^2 + g'^2}.}$$

A and Z are the physical neutral bosons
The massless neutral boson A is identified to the photon
The mass of the Z boson depends on the $SU(2)_L$ and $U(1)_Y$
Coupling constants and the Higgs vacuum expectation value

★ **The physical fields expressed in terms of the massless gauge bosons of $SU(2)_L \times U(1)_Y$ are:**

$$A_\mu = \frac{g' W_\mu^{(3)} + g_W B_\mu}{\sqrt{g_W^2 + g'^2}} \qquad Z_\mu = \frac{g_W W_\mu^{(3)} - g' B_\mu}{\sqrt{g_W^2 + g'^2}}$$

and using $\dfrac{g'}{g_W} = \tan\theta_W,$

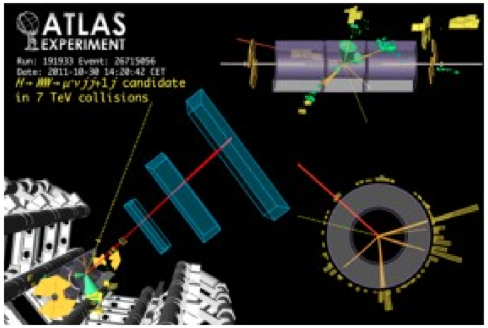
$$\boxed{\begin{aligned} A_\mu &= \cos\theta_W B_\mu + \sin\theta_W W_\mu^{(3)}, \\ Z_\mu &= -\sin\theta_W B_\mu + \cos\theta_W W_\mu^{(3)}, \end{aligned}}$$

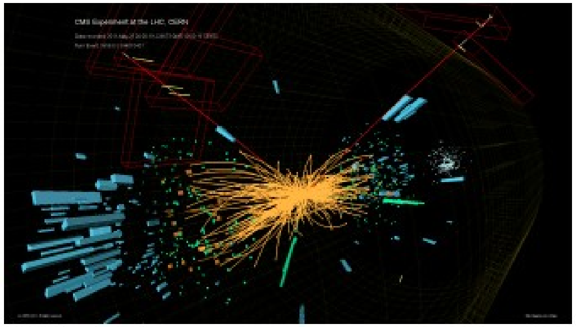
★ **In the GWS $SU(2)_L \times U(1)_Y$ model there are 4 free parameters: *g_W, g', λ and μ (*)***
Using the relation $m_W = \frac{1}{2} g_W v$ and the measured m_W and g_W the vacuum expectation value is

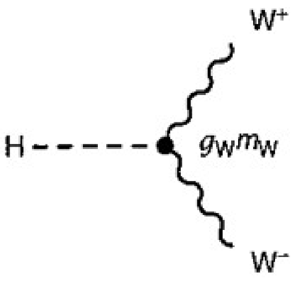
(*) $v^2 = \dfrac{-\mu^2}{\lambda} \quad m_H^2 = 2\lambda v^2.$

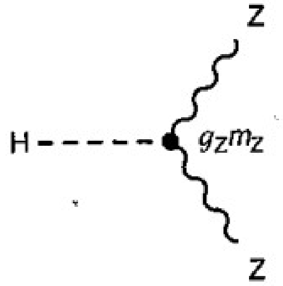
$\boxed{v = 246 \text{ GeV}}$

Coupling to the gauge bosons









$$(D_\mu\phi)^\dagger(D^\mu\phi) = \tfrac{1}{2}(\partial_\mu h)(\partial^\mu h) + \tfrac{1}{8}g_W^2(W_\mu^{(1)} + iW_\mu^{(2)})(W^{(1)\mu} - iW^{(2)\mu})(v+h)^2$$
$$+ \tfrac{1}{8}(g_W W_\mu^{(3)} - g' B_\mu)(g_W W^{(3)\mu} - g' B^\mu)(v+h)^2.$$

$$W^\pm = \tfrac{1}{\sqrt{2}}\left(W^{(1)} \mp iW^{(2)}\right).$$

$$\tfrac{1}{4}g_W^2 W_\mu^- W^{+\mu}(v+h)^2 = \underbrace{\tfrac{1}{4}g_W^2 v^2 W_\mu^- W^{+\mu}}_{\textit{Mass term}} + \underbrace{\tfrac{1}{2}g_W^2 v W_\mu^- W^{+\mu} h}_{\textit{Triplet coupling}} + \underbrace{\tfrac{1}{4}g_W^2 W_\mu^- W^{+\mu} hh}_{\textit{Quartic coupling}}.$$

$$g_{HWW} = \tfrac{1}{2}g_W^2 v \equiv g_W m_W.$$
$$g_{HZZ} = g_Z m_Z,$$

Fermion masses

- The BEH mechanism can be used to generate the mass of the fermions.
- Remember the mass term $-m\bar{\psi}\psi = -m\left(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R\right)$ does not respect the $SU(2)_L \times U(1)_Y$ symmetry

- Since the $\phi(x)$ is a doublet we can apply an infinitesimal local gauge transformation $\phi \to \phi' = (I + ig_W\epsilon(x)\cdot\mathbf{T})\phi.$, exactly the same as for the left-handed fermion wave function, ***L***

$$\bar{L} \equiv L^\dagger\gamma^0 \qquad \bar{L} \to \bar{L}' = \bar{L}(I - ig_W\epsilon(x)\cdot\mathbf{T}).$$

Therefore the product $\bar{L}\phi$ is invariant under $SU(2)_L$

- The product with the right-handed function (***R***), $\bar{L}\phi R$, and $(\bar{L}\phi R)^\dagger = \bar{R}\phi^\dagger L.$ are invariant under $SU(2)_L \times U(1)_Y$ symmetry.

Therefore terms like: $-g_f(\bar{L}\phi R + \bar{R}\phi^\dagger L)$ are local gauge invariant

For the electron doublet:

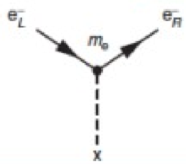
$$\boxed{\mathcal{L}_e = -g_e\left[\begin{pmatrix}\bar{\nu}_e & \bar{e}\end{pmatrix}_L\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}e_R + \bar{e}_R\begin{pmatrix}\phi^{+*} & \phi^{0*}\end{pmatrix}\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L\right],}$$

Yukawa coupling
Electron-Higgs

★ **After spontaneous symmetry breaking, and in the unitary gauge** $\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix},$

$$\mathcal{L}_e = \underbrace{-\frac{g_e}{\sqrt{2}} v\left(\bar{e}_L e_R + \bar{e}_R e_L\right)}_{\textit{Mass term}} \underbrace{- \frac{g_e}{\sqrt{2}} h\left(\bar{e}_L e_R + \bar{e}_R e_L\right)}_{\textit{e-H interaction}}.$$

$$g_e = \sqrt{2}\,\frac{m_e}{v}.$$



$$\boxed{\mathcal{L}_e = -m_e \bar{e}e - \frac{m_e}{v}\bar{e}eh.}$$

But this only works for down-type quarks !

Something special is required to give mass to up-type quarks (and neutrinos)

★ **It can be obtained by constructing the conjugate doublet $\phi_c(x)$, with v in the upper part of the doublet**

$$\phi_c = -i\sigma_2\phi^* = \begin{pmatrix} -\phi^{0*} \\ \phi^- \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\phi_3 + i\phi_4 \\ \phi_1 - i\phi_2 \end{pmatrix}.$$

★ A gauge invariant mass term for the up-type quarks is, for example $\overline{L}\phi_c R + \overline{R}\phi_c^{\dagger} L$,

$$\mathcal{L}_{\mathrm{u}} = g_{\mathrm{u}} \begin{pmatrix} \bar{\mathrm{u}} & \bar{\mathrm{d}} \end{pmatrix}_L \begin{pmatrix} -\phi^{0*} \\ \phi^{-} \end{pmatrix} \mathrm{u}_R + \text{ Hermitian conjugate,}$$

After symmetry breaking

$$\mathcal{L}_{\mathrm{u}} = -\frac{g_{\mathrm{u}}}{\sqrt{2}} v \left(\bar{\mathrm{u}}_L \mathrm{u}_R + \bar{\mathrm{u}}_R \mathrm{u}_L\right) - \frac{g_{\mathrm{u}}}{\sqrt{2}} h \left(\bar{\mathrm{u}}_L \mathrm{u}_R + \bar{\mathrm{u}}_R \mathrm{u}_L\right),$$

$g_{\mathrm{u}} = \sqrt{2} m_{\mathrm{u}}/v,$

$$\boxed{\mathcal{L}_{\mathrm{u}} = -m_{\mathrm{u}} \bar{\mathrm{u}}\mathrm{u} - \frac{m_{\mathrm{u}}}{v} \bar{\mathrm{u}}\mathrm{u}h.}$$

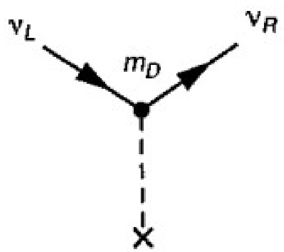
★ Thus, the gauge invariant mass terms for the Dirac fermions, with $g_{\mathrm{f}} = \sqrt{2}\frac{m_{\mathrm{f}}}{v}$,

$$\boxed{\mathcal{L} = -g_{\mathrm{f}} \left[\overline{L}\phi R + (\overline{L}\phi R)^{\dagger}\right] \quad \text{or} \quad \mathcal{L} = g_{\mathrm{f}} \left[\overline{L}\phi_c R + (\overline{L}\phi_c R)^{\dagger}\right],}$$

★ For the top-quark $g_f \sim 1$. One expect all g_f to be $O(1)$, but for neutrinos $g_f < 10^{-12}$

Neutrino masses

★ From the observation of the neutrino oscillations we know the neutrinos have mass. A new term in the Lagrangian can be introduced as done for the up-type quarks



$$\mathcal{L}_D = -m_D\left(\overline{\nu_R}\nu_L + \overline{\nu_L}\nu_R\right).$$

The right-handed neutrinos and left-handed antineutrinos exit !

★ Those states will transform as singlet and we can add these kind of terms add to the *L* w/o breaking the gauge invariance.
The left-handed antineutrino appears as the CP conjugate field $\psi^c = \hat{C}\hat{P}\psi = i\gamma^2\gamma^0\psi^*$,

$$\mathcal{L}_M = -\tfrac{1}{2}M\left(\overline{\nu_R^c}\nu_R + \overline{\nu_R}\nu_R^c\right),$$

Majorana mass term
(Majorana neutrinos)

★ The combined Dirac and Majorana (***DM***) mass term is:

The most general renormalisable expression of the neutrino mass *L*

$$\boxed{\mathcal{L}_{DM} = -\tfrac{1}{2}\begin{pmatrix}\overline{\nu_L} & \overline{\nu_R^c}\end{pmatrix}\begin{pmatrix}0 & m_D \\ m_D & M\end{pmatrix}\begin{pmatrix}\nu_L^c \\ \nu_R\end{pmatrix} + h.c.}$$

★ **The physical states correspond to the eigenstates of the diagonal M matrix**

$$\det(\mathbf{M} - \lambda I) = 0, \qquad \lambda^2 - M\lambda - m_D^2 = 0.$$

$$\boxed{m_\pm = \lambda_\pm = \frac{M \pm \sqrt{M^2 + 4m_D^2}}{2} = \frac{M \pm M\sqrt{1 + 4m_D^2/M^2}}{2}.}$$

if the Majorana mass M >> m_D

$$\boxed{m_\pm \approx \tfrac{1}{2}M \pm \tfrac{1}{2}\left(M + \frac{2m_D^2}{M}\right),}$$

$$|m_\nu| \approx \frac{m_D^2}{M}$$ ***Light neutrino O(1 GeV)***

$$m_N \approx M.$$ ***Heavy neutrino ~ 10^{11}***

The seesaw mechanism

The seesaw mechanism

★ If the Majorana mass exits, the mechanism predicts that for each type of SM neutrino there are a *VERY light neutrino* and a *VERY massive neutrino* state. The physical states are

$$\nu = \cos\theta(\nu_L + \nu_L^c) - \sin\theta(\nu_R + \nu_R^c) \quad \text{and} \quad N = \cos\theta(\nu_R + \nu_R^c) + \sin\theta(\nu_L + \nu_L^c),$$

if M >> m_D with : $\tan\theta \approx m_D/M.$

$$\boxed{\nu \approx (\nu_L + \nu_L^c) - \frac{m_D}{M}(\nu_R + \nu_R^c) \quad \text{and} \quad N \approx (\nu_R + \nu_R^c) + \frac{m_D}{M}(\nu_L + \nu_L^c),}$$

Giving similar phenomenology as in the SM !

★ The seesaw mechanism provides an explanation for the existence of very small masses of the neutrinos.

If the Majorana neutrinos exit we will see in double β decay experiments