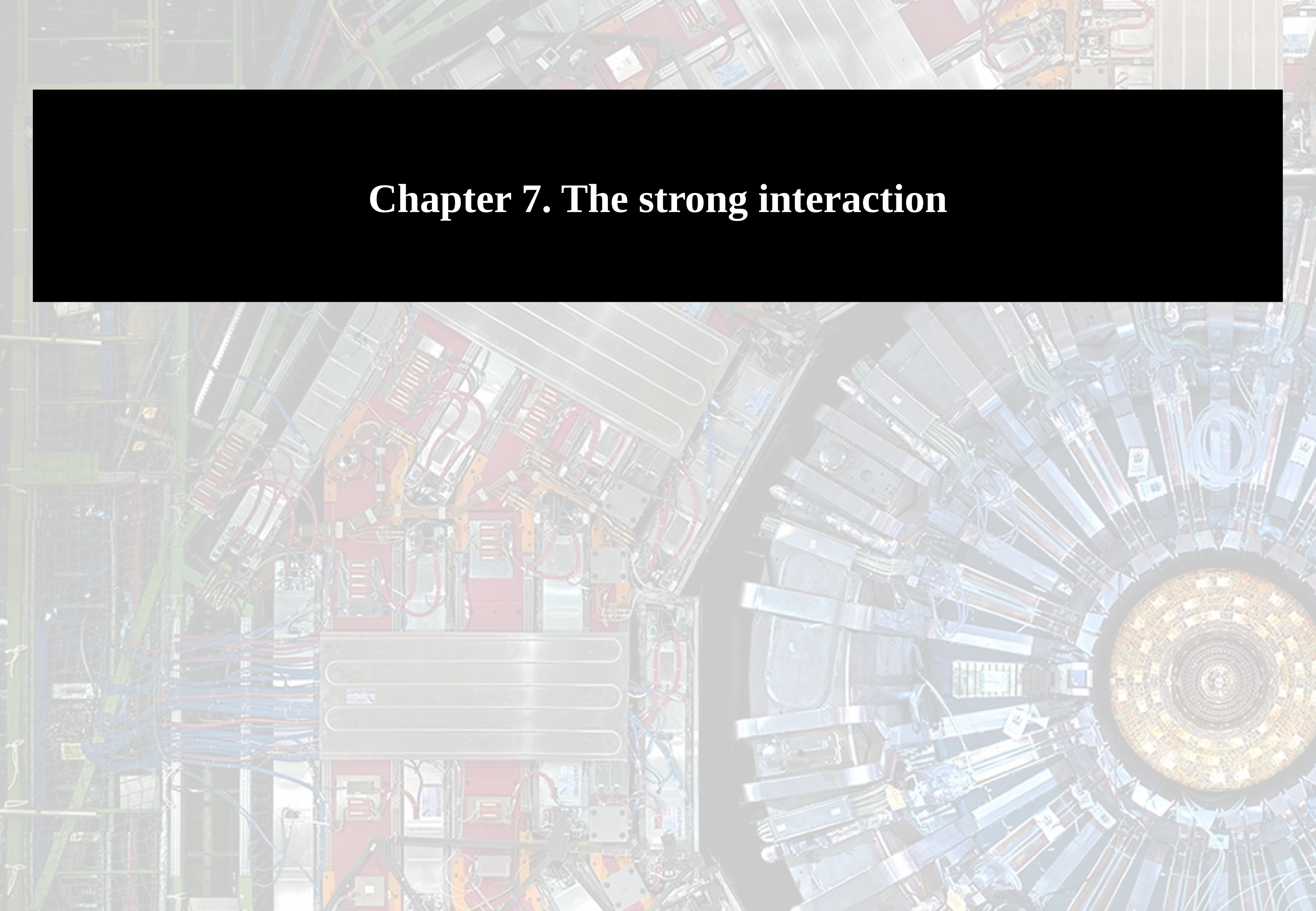


Chapter 7. The strong interaction

The Local Gauge Principle

(see the Appendices A, B and C for more details)

★ All the interactions between fermions and spin-1 bosons in the SM are specified by the principle of LOCAL GAUGE INVARIANCE

★ To arrive at QED, require physics to be invariant under the local phase transformation of particle wave-functions

$$\psi \rightarrow \psi' = \psi e^{iq\chi(x)}$$

★ Note that the change of phase depends on the space-time coordinate: $\chi(t,\vec{x})$

- Under this transformation the Dirac Equation transforms as

$$\boxed{i\gamma^\mu \partial_\mu \psi - m\psi = 0} \Rightarrow \boxed{i\gamma^\mu(\partial_\mu + iq\partial_\mu \chi)\psi - m\psi = 0}$$

- To make "physics", i.e. the Dirac equation, invariant under this local phase transformation FORCED to introduce a massless gauge boson, A_μ .

+ The Dirac equation has to be modified to include this new field:

$$\boxed{i\gamma^\mu(\partial_\mu - qA_\mu)\psi - m\psi = 0}$$

- The modified Dirac equation is invariant under local phase transformations if:

$$\boxed{A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \chi}$$

Gauge Invariance

★ For physics to remain unchanged – must have **GAUGE INVARIANCE** of the new field, i.e. physical predictions unchanged for $A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$

★ Hence the principle of invariance under local phase transformations completely specifies the interaction between a fermion and the gauge boson (i.e. photon):

$$i\gamma^\mu(\partial_\mu \psi - \boxed{qA_\mu})\psi - m\psi = 0$$

⇨ interaction vertex: $i\gamma^\mu q A_\mu$ (see p.111)

⇨ **QED !**

★ The local phase transformation of QED is a unitary **U(1)** transformation

$\psi \to \psi' = \hat{U}\psi$ i.e. $\psi \to \psi' = \psi e^{iq\chi(x)}$ with $U^\dagger U = 1$

Now extend this idea…

From QED to QCD

★ Suppose there is another fundamental symmetry of the universe, say

"invariance under SU(3) local phase transformations"

- i.e. require invariance under $\psi \to \psi' = \psi e^{ig\vec{\lambda}.\vec{\theta}(x)}$ where

$\vec{\lambda}$ are the eight 3x3 Gell-Mann matrices introduced in handout 7

$\vec{\theta}(x)$ are 8 functions taking different values at each point in space-time

⇨ 8 spin-1 gauge bosons

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

wave function is now a vector in COLOUR SPACE

⇨ $\boxed{\text{QCD !}}$

★ QCD is fully specified by require invariance under SU(3) **local** phase transformations

> Corresponds to rotating states in colour space about an axis whose direction is different at every space-time point

⇨ interaction vertex: $-\frac{1}{2} i g_s \lambda^a \gamma^\mu$

★ Predicts 8 massless gauge bosons – the gluons (one for each λ)

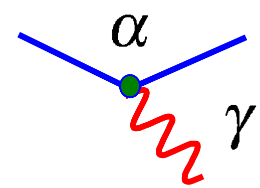
★ Also predicts exact form for interactions between gluons, i.e. the 3 and 4 gluon vertices – the details are beyond the level of this course

Colour in QCD

★ The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with 3 conserved “colour” charges

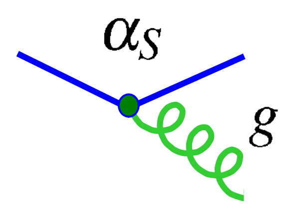
In QED:

- the electron carries one unit of charge $-e$
- the anti-electron carries one unit of anti-charge $+e$
- the force is mediated by a massless “gauge boson” – the photon



In QCD:

- quarks carry colour charge: r, g, b
- anti-quarks carry anti-charge: $\overline{r}, \overline{g}, \overline{b}$
- The force is mediated by massless gluons



★ In QCD, the strong interaction is invariant under rotations in colour space

$$r \leftrightarrow b;\ r \leftrightarrow g;\ b \leftrightarrow g$$

i.e. the same for all three colours

⇒ $\boxed{\text{SU(3) colour symmetry}}$

- This is an **exact** symmetry, unlike the approximate uds flavour symmetry discussed previously.

★ Represent r, g, b **SU(3)** colour states by:

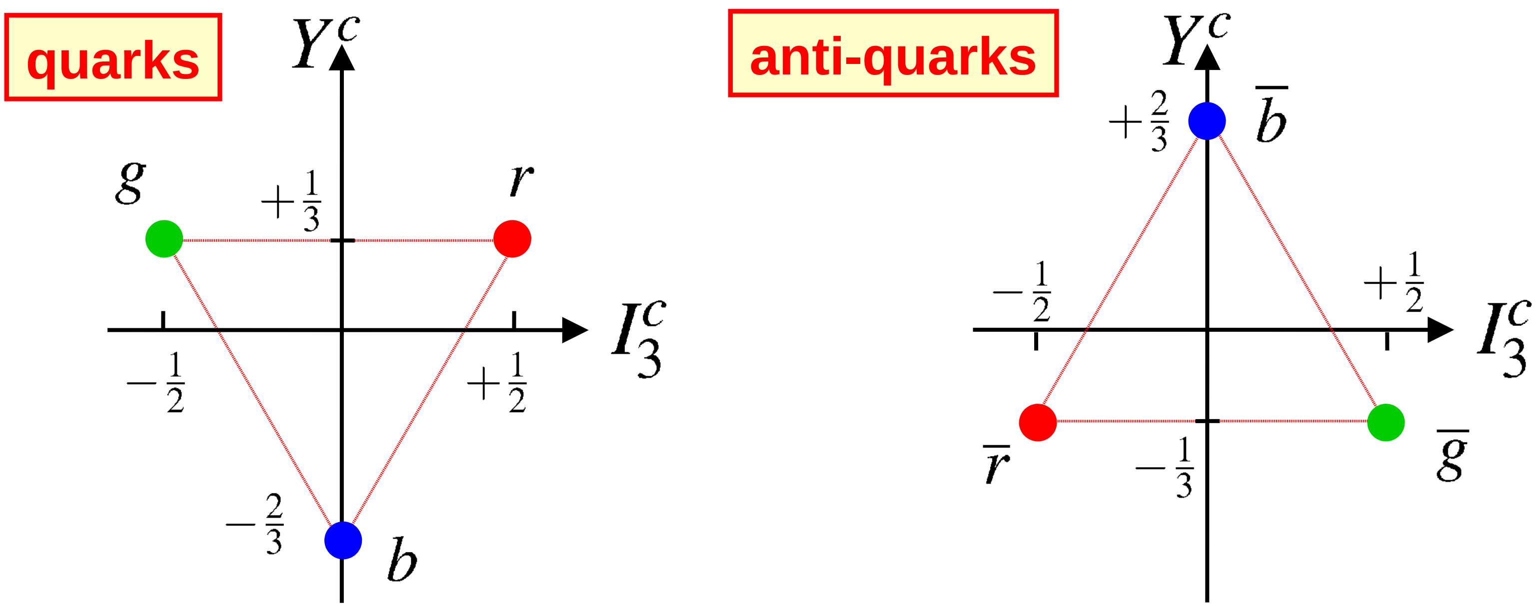
$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

★ Colour states can be labelled by two quantum numbers:

- I_3^c colour isospin
- Y^c colour hypercharge

Exactly analogous to labelling u,d,s flavour states by I_3 and Y

★ Each quark (anti-quark) can have the following colour quantum numbers:



Colour Confinement

★ It is believed (although not yet proven) that all observed free particles are "colourless"

- i.e. never observe a free quark (which would carry colour charge)
- consequently quarks are always found in bound states colourless hadrons

★ **Colour Confinement Hypothesis:**

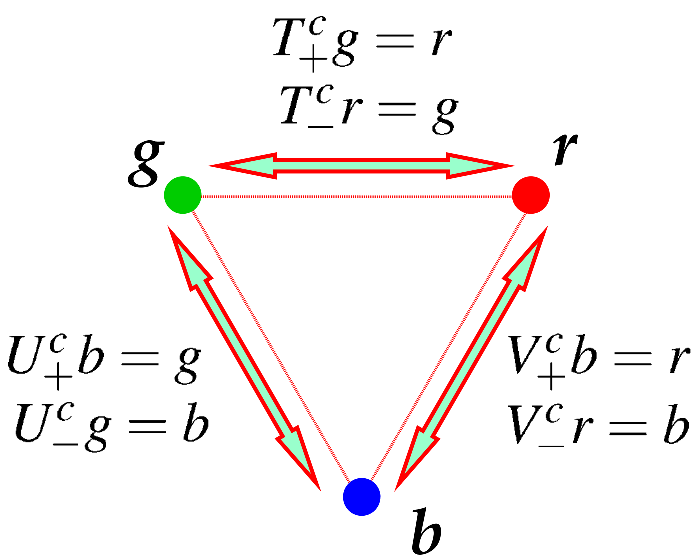
> only colour singlet states can exist as free particles

★ All hadrons must be "colourless" i.e. colour singlets

★ To construct colour wave-functions for hadrons can apply results for SU(3) flavour symmetry to SU(3) colour with replacement

$$\boxed{\begin{array}{l} u \longrightarrow r \\ d \longrightarrow g \\ s \longrightarrow b \end{array}}$$

★ just as for uds flavour symmetry can define colour ladder operators



Colour Singlets

★ It is important to understand what is meant by a **singlet** state

★ Consider spin states obtained from two spin 1/2 particles.

- Four spin combinations: $\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$
- Gives four eigenstates of $\hat{S}^2,\ \hat{S}_z$ $\qquad (2 \otimes 2 = 3 \oplus 1)$

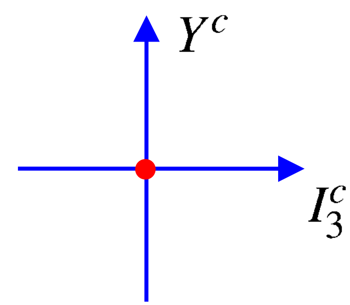
$$\begin{aligned} |1,+1\rangle &= \uparrow\uparrow \\ |1,0\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1,-1\rangle &= \downarrow\downarrow \end{aligned} \quad \boxed{\text{spin-1 triplet}} \quad \oplus \quad |0,0\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \quad \boxed{\text{spin-0 singlet}}$$

★ The singlet state is "spinless": it has zero angular momentum, is invariant under SU(2) spin transformations and spin ladder operators yield zero

$$S_{\pm}|0,0\rangle = 0$$

★ In the same way **COLOUR SINGLETS** are "colourless" combinations:

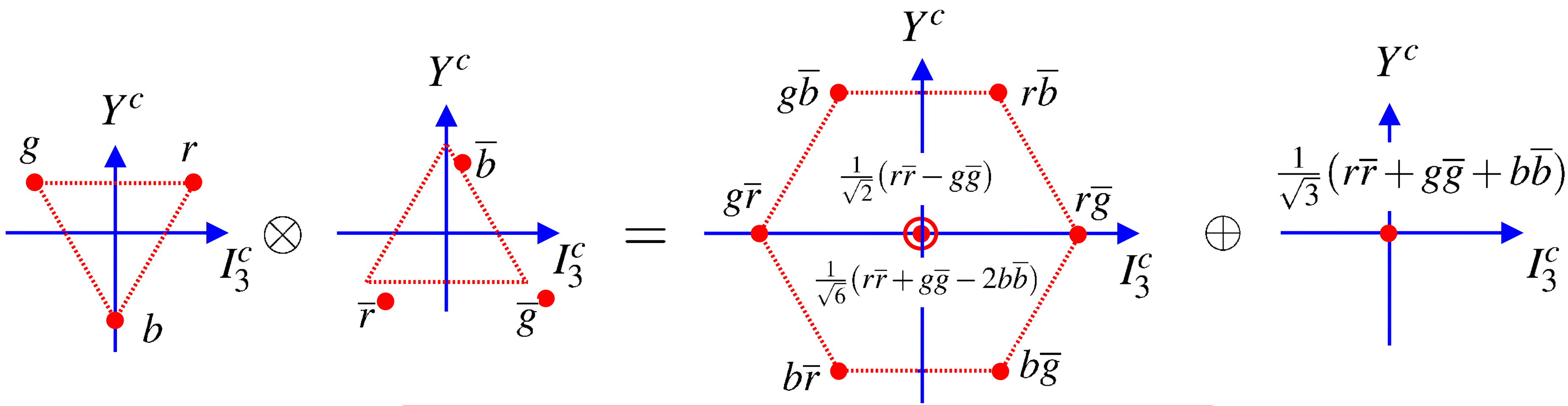
- they have zero colour quantum numbers $\quad I_3^c = 0,\ Y^c = 0$
- invariant under SU(3) colour transformations
- ladder operators $\quad T_{\pm},\ U_{\pm},\ V_{\pm}$ all yield zero



★ NOT sufficient to have $\ I_3^c = 0,\ Y^c = 0\ $: does not mean that state is a singlet

Meson Colour Wave-function

- ★ **Consider colour wave-functions for $q\overline{q}$**
- ★ **The combination of colour with anti-colour is mathematically identical to construction of meson wave-functions with uds flavour symmetry**



➡ **Coloured octet and a <u>colourless</u> singlet**

- **Colour confinement implies that hadrons only exist in colour singlet states so the colour wave-function for mesons is:**

$$\boxed{\psi_c^{q\overline{q}} = \frac{1}{\sqrt{3}}\left(r\overline{r}+g\overline{g}+b\overline{b}\right)}$$

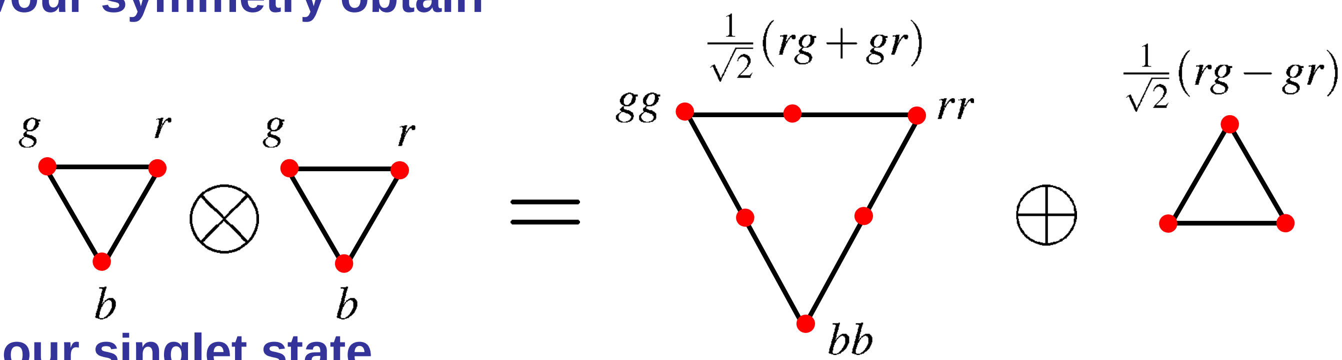
- ★ **Can we have a $qq\overline{q}$ state ? i.e. by adding a quark to the above octet can we form a state with $Y^c=0;\ I_3^c=0$. The answer is clear no.**

➡ $qq\overline{q}$ **bound states do not exist in nature.**

Baryon Colour Wave-function

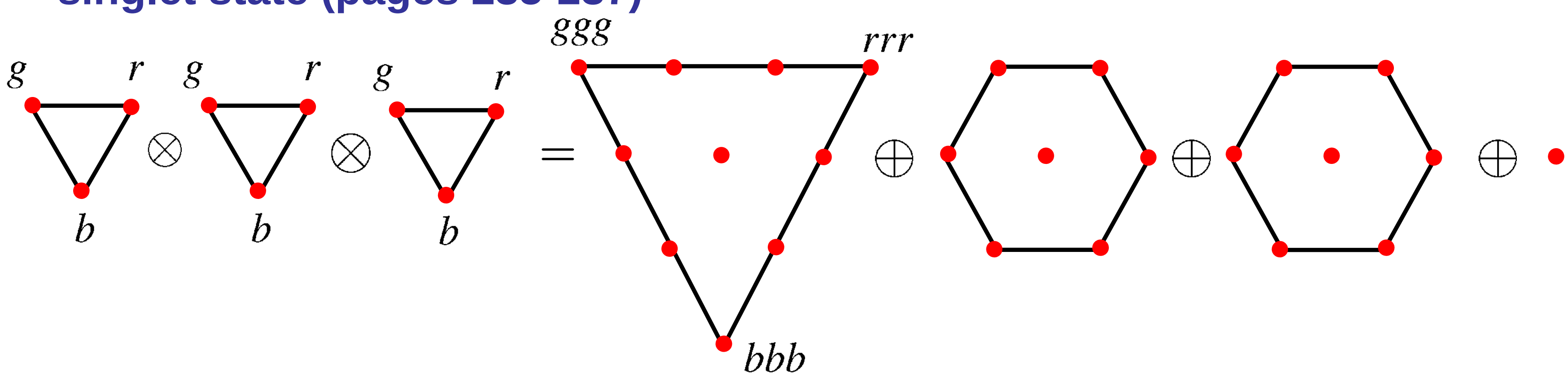
★ Do **qq** bound states exist ? This is equivalent to asking whether it possible to form a colour singlet from two colour triplets ?

- Following the discussion of construction of baryon wave-functions in SU(3) flavour symmetry obtain



- No **qq** colour singlet state
- Colour confinement ➡ bound states of **qq** do not exist

★ BUT combination of three quarks (three colour triplets) gives a colour singlet state (pages 235-237)



★ **The singlet colour wave-function is:**

$$\boxed{\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)}$$

Check this is a colour singlet…

- **It has** $I_3^c = 0,\ Y^c = 0$ **: a necessary but not sufficient condition**
- **Apply ladder operators, e.g.** T_+ **(recall** $T_+ g = r$ **)**

$$T_+ \psi_c^{qqq} = \frac{1}{\sqrt{6}}(rrb - rbr + rbr - rrb + brr - brr) = 0$$

- **Similarly** $T_- \psi_c^{qqq} = 0;\quad V_\pm \psi_c^{qqq} = 0;\quad U_\pm \psi_c^{qqq} = 0;$

★ **Colourless singlet - therefore qqq bound states exist !**

⟹ **Anti-symmetric colour wave-function**

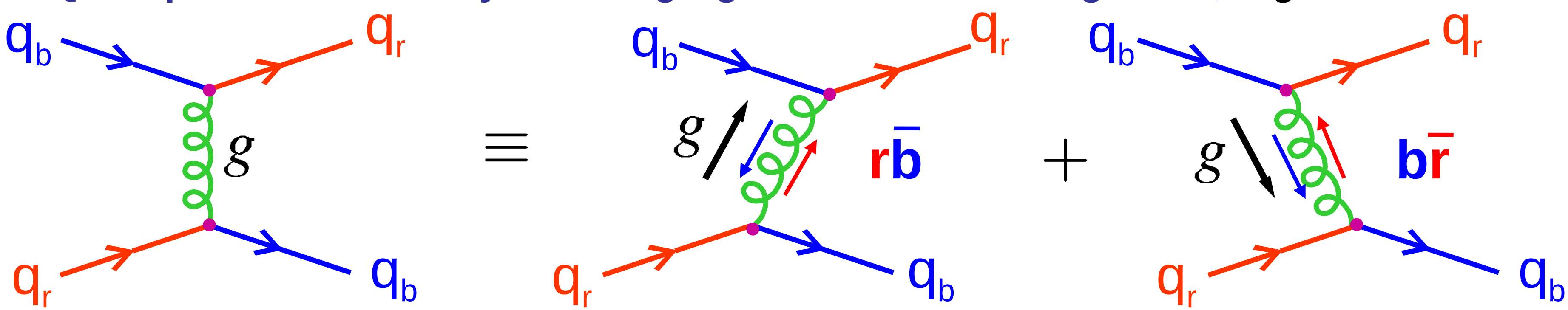
Allowed Hadrons i.e. the possible colour singlet states

- $q\bar{q},\ qqq$ **Mesons and Baryons**
- $q\bar{q}q\bar{q},\ qqqq\bar{q}$ **Exotic states, e.g. pentaquarks**

To date all confirmed hadrons are either mesons or baryons. However, some recent (but not entirely convincing) "evidence" for pentaquark states

Gluons

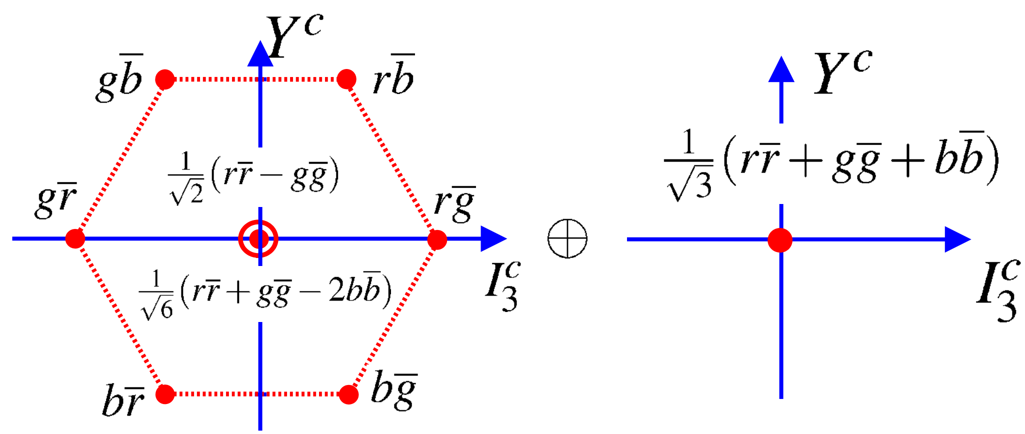
★ In QCD quarks interact by exchanging virtual massless gluons, e.g.



★ Gluons carry colour and anti-colour, e.g.

★ Gluon colour wave-functions (colour + anti-colour) are the same as those obtained for mesons (also colour + anti-colour)

⇨ OCTET + “COLOURLESS” SINGLET



★ So we might expect 9 physical gluons:

OCTET: $r\bar{g},\ r\bar{b},\ g\bar{r},\ g\bar{b},\ b\bar{r},\ b\bar{g},\ \frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g}),\ \frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$

SINGLET: $\frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})$

★ **BUT**, colour confinement hypothesis:

only colour singlet states can exist as free particles	$\Rightarrow$	**Colour singlet gluon** would be unconfined. It would behave like a strongly interacting photon ➡ infinite range Strong force.

★ **Empirically**, the strong force is short range and therefore know that the physical gluons are confined. The colour singlet state does not exist in nature !

NOTE: this is not entirely ad hoc. In the context of gauge field theory (see minor option) the strong interaction arises from a fundamental SU(3) symmetry. The gluons arise from the generators of the symmetry group (the Gell-Mann λ matrices). There are 8 such matrices ➡ 8 gluons. Had nature "chosen" a U(3) symmetry, would have 9 gluons, the additional gluon would be the colour singlet state and QCD would be an unconfined long-range force.

NOTE: the "gauge symmetry" determines the exact nature of the interaction

$\Rightarrow$ FEYNMAN RULES

Gluon-Gluon Interactions

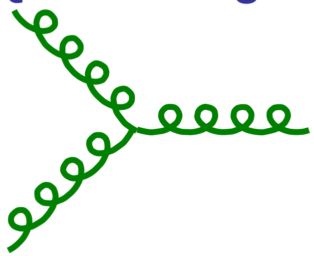
★ In QED the photon does not carry the charge of the EM interaction (photons are electrically neutral)

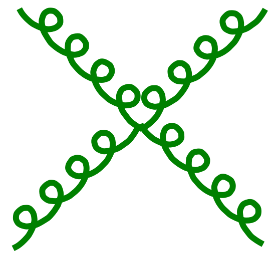
★ In contrast, in QCD the gluons do carry colour charge

⇒ **Gluon Self-Interactions**

★ Two new vertices (no QED analogues)

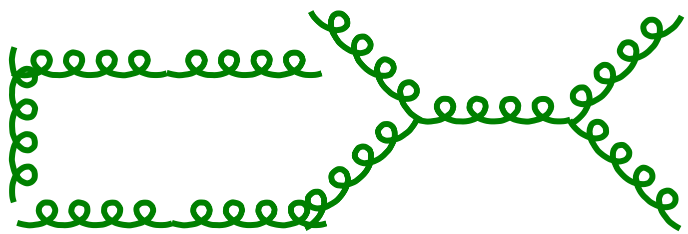
triple-gluon vertex



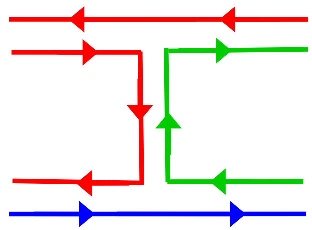


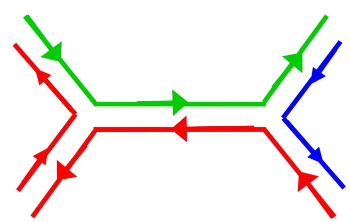
quartic-gluon vertex

★ In addition to quark-quark scattering, therefore can have gluon-gluon scattering



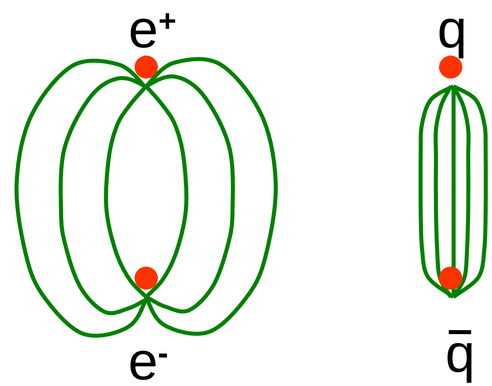
e.g. possible way of arranging the colour flow



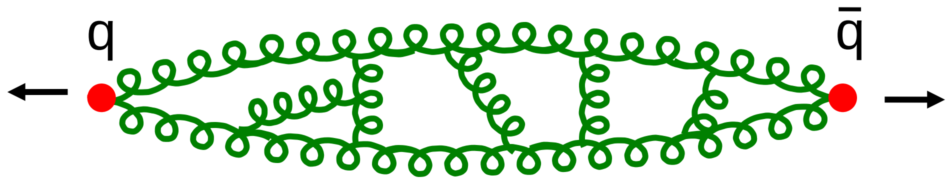


Gluon self-Interactions and Confinement

- ★ Gluon self-interactions are believed to give rise to colour confinement
- ★ Qualitative picture:
  - Compare QED with QCD
  - In QCD "gluon self-interactions squeeze lines of force into a flux tube"



- ★ What happens when try to separate two coloured objects e.g. $q\bar{q}$



- Form a flux tube of interacting gluons of approximately constant energy density $\sim 1\,\mathrm{GeV/fm}$

$$\Rightarrow \quad V(r) \sim \lambda r$$

- Require infinite energy to separate coloured objects to infinity
- Coloured quarks and gluons are always confined within colourless states
- In this way QCD provides a plausible explanation of confinement – but not yet proven (although there has been recent progress with Lattice QCD)

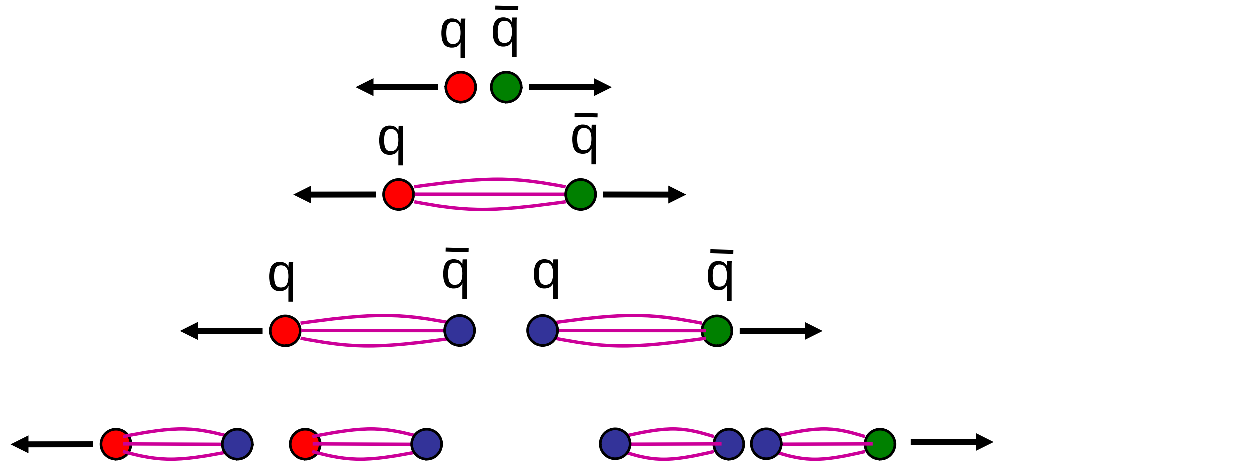
Hadronisation and Jets

★ **Consider a quark and anti-quark produced in electron positron annihilation**

i) Initially Quarks separate at high velocity

ii) Colour flux tube forms between quarks

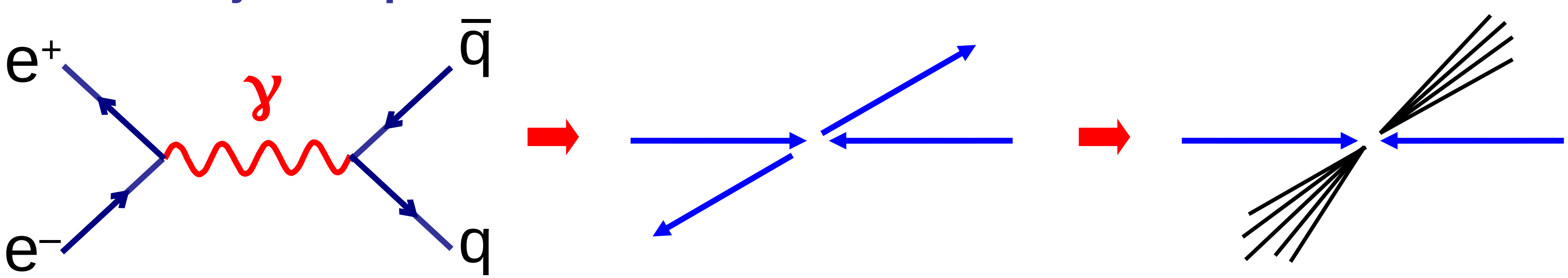
iii) Energy stored in the flux tube sufficient to produce $q\bar{q}$ pairs



iv) Process continues until quarks pair up into jets of colourless hadrons

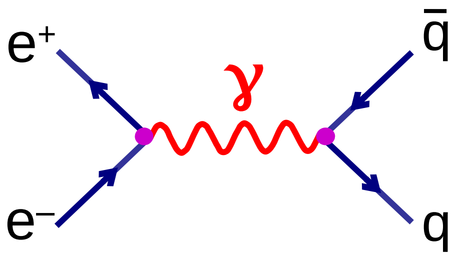
★ **This process is called hadronisation. It is not (yet) calculable.**

★ **The main consequence is that at collider experiments quarks and gluons observed as jets of particles**



QCD and Colour in e^+e^- Collisions

★ e^+e^- colliders are an excellent place to study QCD

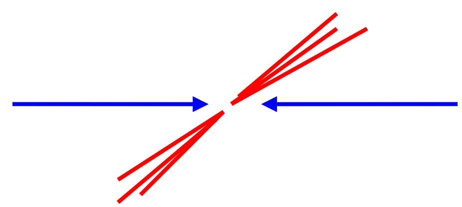


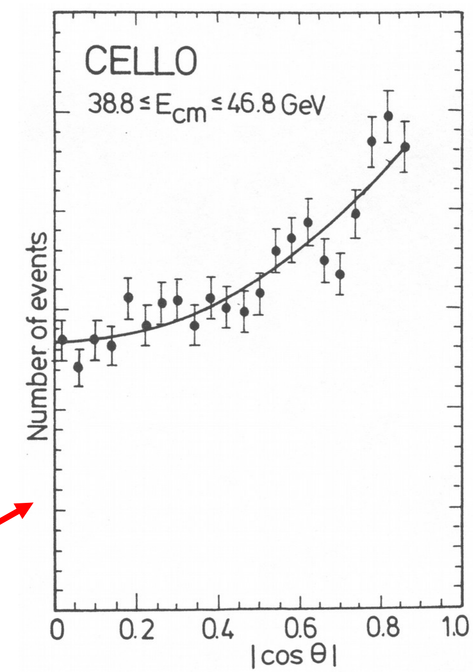
★ **Well defined production of quarks**
- QED process well-understood
- no need to know parton structure functions
- + experimentally very clean – no proton remnants

★ In handout 5 obtained expressions for the $e^+e^- \to \mu^+\mu^-$ cross-section

$$\sigma = \frac{4\pi\alpha^2}{3s} \qquad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

- In e^+e^- collisions produce all quark flavours for which $\sqrt{s} > 2m_q$
- In general, i.e. unless producing a $q\bar{q}$ bound state, produce jets of hadrons
- Usually can't tell which jet came from the quark and came from anti-quark





H.J.Behrend et al., Phys Lett 183B (1987) 400

★ Angular distribution of jets $\propto (1+\cos^2\theta)$

➡ **Quarks are spin ½**

★ Colour is conserved and quarks are produced as $r\bar{r},\ g\bar{g},\ b\bar{b}$

★ For a **single quark flavour** and **single colour**

$$\sigma(e^+e^- \to q_i\bar{q}_i) = \frac{4\pi\alpha^2}{3s}Q_q^2$$

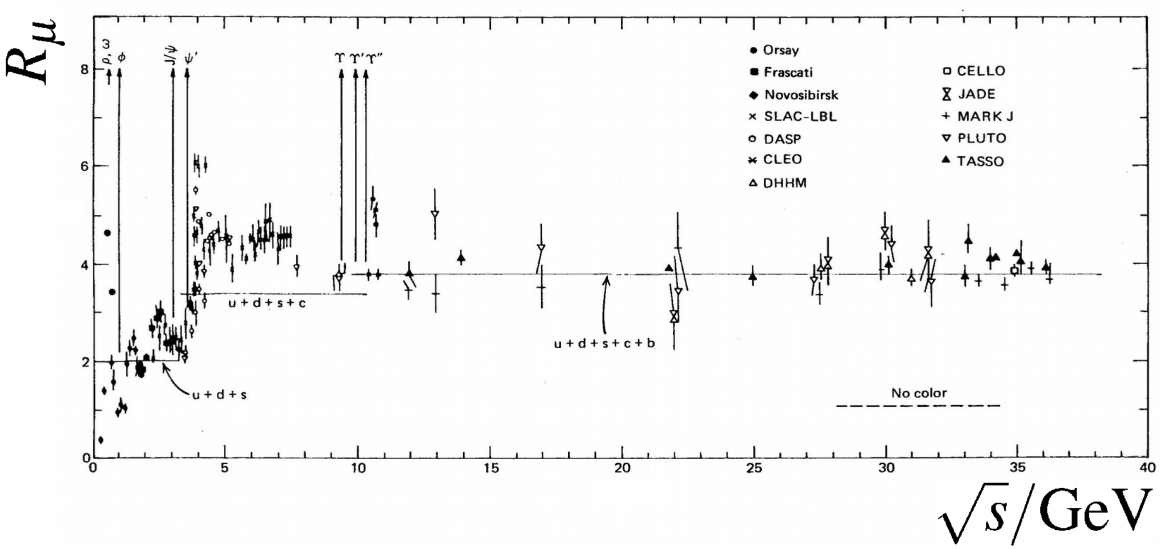
- Experimentally observe jets of hadrons:

$$\sigma(e^+e^- \to \text{hadrons}) = 3\sum_{u,d,s,..}\frac{4\pi\alpha^2}{3s}Q_q^2$$

Factor 3 comes from colours

- Usual to express as ratio compared to $\sigma(e^+e^- \to \mu^+\mu^-)$

$$R_\mu = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_{u,d,s,..}Q_q^2$$



u,d,s: $R_\mu = 3\times\left(\frac{1}{9}+\frac{4}{9}+\frac{1}{9}\right) = 2$

u,d,s,c: $R_\mu = \frac{10}{3}$

u,d,s,c,b: $R_\mu = \frac{11}{3}$

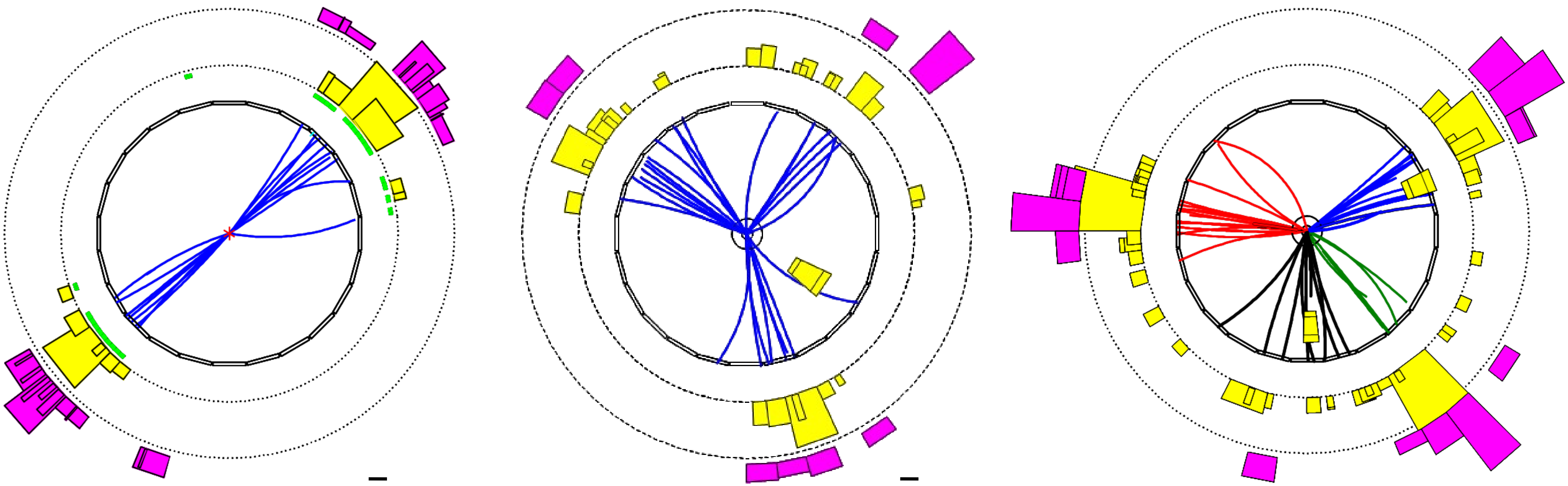
★ Data consistent with expectation **with factor 3 from colour**

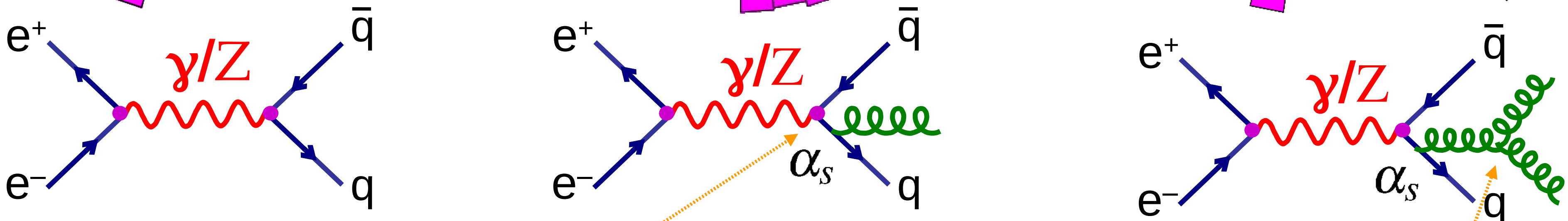
Jet production in e+e- Collisions

★ e^+e^- **colliders are also a good place to study gluons**

$e^+e^- \to q\bar{q} \to 2\text{jets}$ $\quad$ $e^+e^- \to q\bar{q}g \to 3\text{jets}$ $\quad$ $e^+e^- \to q\bar{q}gg \to 4\text{jets}$

OPAL at LEP (1989-2000)





Experimentally:

- **Three jet rate ➡ measurement of** α_s
- **Angular distributions ➡ gluons are spin-1**
- **Four-jet rate and distributions ➡ QCD has an underlying SU(3) symmetry**

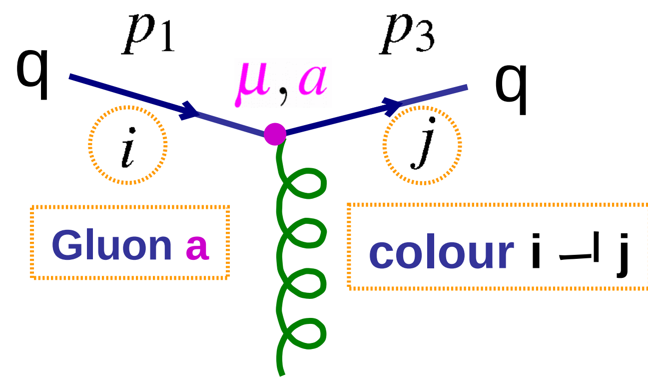
The Quark – Gluon Interaction

- **Representing the colour part of the fermion wave-functions by:**

$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

- **Particle wave-functions** $u(p) \longrightarrow c_i u(p)$
- **The QCD** qqg **vertex is written:**

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



- **Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices**

- **Isolating the colour part:**

$$c_j^\dagger\lambda^a c_i = c_j^\dagger \begin{pmatrix}\lambda^a_{1i}\\ \lambda^a_{2i}\\ \lambda^a_{3i}\end{pmatrix} = \lambda^a_{ji}$$

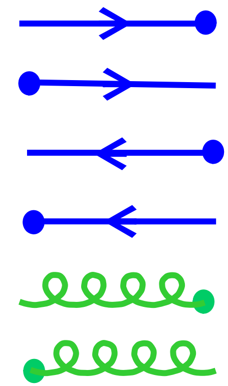
- **Hence the fundamental quark - gluon QCD interaction can be written**

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Feynman Rules for QCD

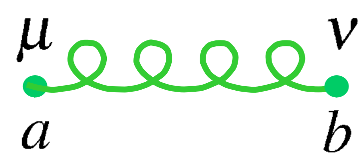
- **External Lines**

spin 1/2	incoming quark	$u(p)$	
	outgoing quark	$\bar{u}(p)$	
	incoming anti-quark	$\bar{v}(p)$	
	outgoing anti-quark	$v(p)$	
spin 1	incoming gluon	$\varepsilon^{\mu}(p)$	
	outgoing gluon	$\varepsilon^{\mu}(p)^{*}$	



- **Internal Lines (propagators)**

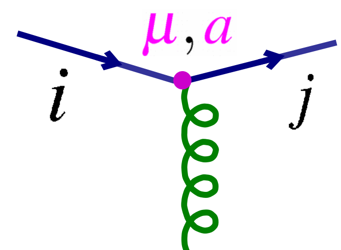
  spin 1 gluon $\boxed{\dfrac{-ig_{\mu\nu}}{q^2}\delta^{ab}}$



a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

  spin 1/2 quark $\boxed{-ig_s\frac{1}{2}\lambda^{a}_{ji}\gamma^{\mu}}$



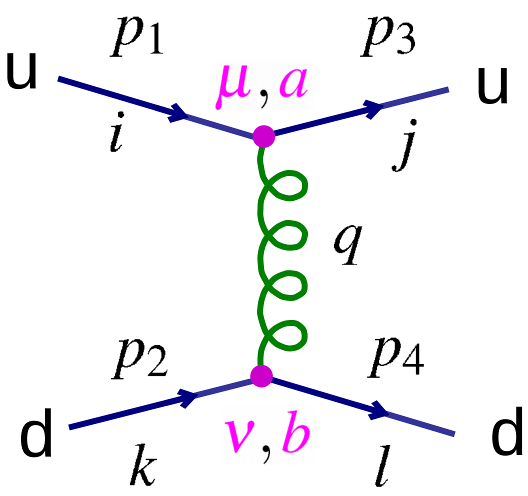
i, j = 1,2,3 are quark colours,

λ^{a} a = 1,2,..8 are the Gell-Mann SU(3) matrices

- **+ 3 gluon and 4 gluon interaction vertices**
- **Matrix Element -iM = product of all factors**

Matrix Element for quark-quark scattering

★ Consider QCD scattering of an up and a down quark



- The incoming and out-going quark colours are labelled by $i, j, k, l = \{1, 2, 3\}$ (or $\{r, g, b\}$)
- In terms of colour this scattering is $$ik \to jl$$
- The 8 different gluons are accounted for by the colour indices $a, b = 1, 2, \ldots, 8$
- NOTE: the δ-function in the propagator ensures $a = b$, i.e. the gluon "emitted" at a is the same as that "absorbed" at b

★ Applying the Feynman rules:

$$-iM = \left[\bar{u}_u(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u_u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\bar{u}_d(p_4)\{-\tfrac{1}{2}ig_s\lambda^b_{lk}\gamma^\nu\}u_d(p_2)\right]$$

where summation over a and b (and μ and ν) is implied.

★ Summing over a and b using the δ-function gives:

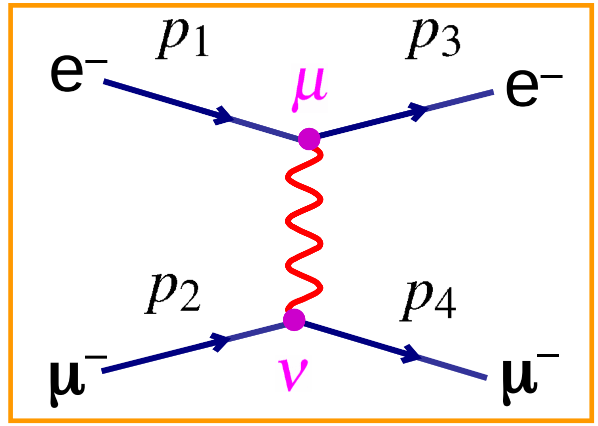
$$\boxed{M = -\frac{g_s^2}{4}\underbrace{\lambda^a_{ji}\lambda^a_{lk}}_{\text{Sum over all 8 gluons (repeated indices)}}\frac{1}{q^2}g_{\mu\nu}[\bar{u}_u(p_3)\gamma^\mu u_u(p_1)][\bar{u}_d(p_4)\gamma^\nu u_d(p_2)]}$$

QCD vs QED

QED

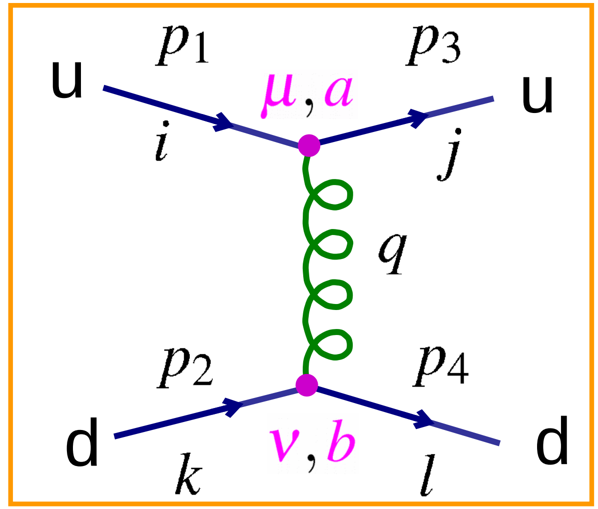
$$-iM = [\overline{u}(p_3)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)ie\gamma^\nu u(p_2)]$$

$$\boxed{M = -e^2\frac{1}{q^2}g_{\mu\nu}[\overline{u}(p_3)\gamma^\mu u(p_1)][\overline{u}(p_4)\gamma^\nu u(p_2)]}$$



QCD

$$\boxed{M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]}$$



★ **QCD Matrix Element = QED Matrix Element with:**

- $\boxed{e^2 \rightarrow g_s^2}$ **or equivalently** $\boxed{\alpha = \frac{e^2}{4\pi} \rightarrow \alpha_s = \frac{g_s^2}{4\pi}}$

+ QCD Matrix Element includes an additional “colour factor”

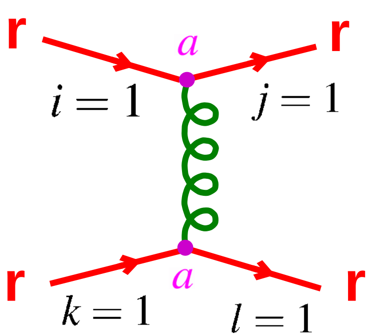
$$\boxed{C(ik \rightarrow jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}}$$

Evaluation of QCD Colour Factors

- **QCD colour factors reflect the gluon states that are involved**

$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons: $r\bar{g}\,,\, g\bar{r}$	$r\bar{b}\,,\, b\bar{r}$	$g\bar{b}\,,\, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\quad \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

Configurations involving a single colour

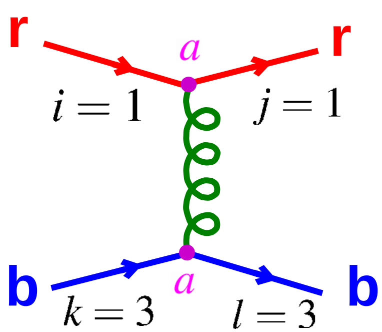


- **Only matrices with non-zero entries in 11 position are involved**

$$
\begin{aligned}
C(rr \to rr) &= \frac{1}{4} \sum_{a=1}^{8} \lambda^a_{11} \lambda^a_{11} = \frac{1}{4} (\lambda^3_{11} \lambda^3_{11} + \lambda^8_{11} \lambda^8_{11}) \\
&= \frac{1}{4} \left(1 + \frac{1}{3} \right) = \frac{1}{3}
\end{aligned}
$$

Similarly find $\boxed{C(rr \to rr) = C(gg \to gg) = C(bb \to bb) = \frac{1}{3}}$

Other configurations where quarks don't change colour e.g. $rb \to rb$



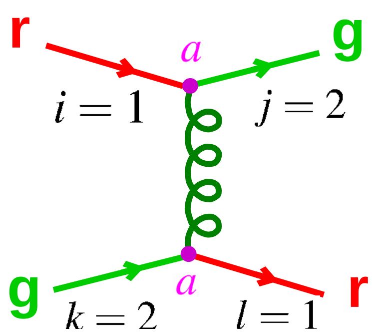
- Only matrices with non-zero entries in **11** and **33** position are involved

$$C(rb \to rb) = \frac{1}{4}\sum_{a=1}^{8} \lambda_{11}^{a}\lambda_{33}^{a} = \frac{1}{4}(\lambda_{11}^{8}\lambda_{33}^{8})$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}}.\frac{-2}{\sqrt{3}}\right) = -\frac{1}{6}$$

Similarly $C(rb \to rb) = C(rg \to rg) = C(gr \to gr) = C(gb \to gb) = C(br \to br) = C(bg \to bg) = -\frac{1}{6}$

Configurations where quarks swap colours e.g. $rg \to gr$



- Only matrices with non-zero entries in **12** and **21** position are involved

Gluons $r\bar{g}$, $g\bar{r}$

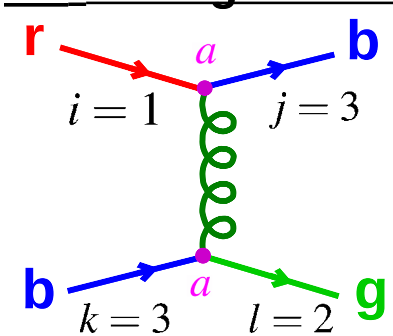
$$C(rg \to gr) = \frac{1}{4}\sum_{a=1}^{8} \lambda_{21}^{a}\lambda_{12}^{a} = \frac{1}{4}(\lambda_{21}^{1}\lambda_{12}^{1} + \lambda_{21}^{2}\lambda_{12}^{2})$$

$$= \frac{1}{4}(i(-i)+1) = \frac{1}{2}$$

$\hat{T}_{+}^{(ij)}\hat{T}_{-}^{(kl)}$

$$C(rb \to br) = C(rg \to gr) = C(gr \to rg) = C(gb \to bg) = C(br \to rb) = C(bg \to gb) = \frac{1}{2}$$

Configurations involving 3 colours e.g. $rb \to bg$



- Only matrices with non-zero entries in the **13** and **32** position
- But none of the λ matrices have non-zero entries in the **13** and **32** positions. Hence the colour factor is zero

★ colour is conserved

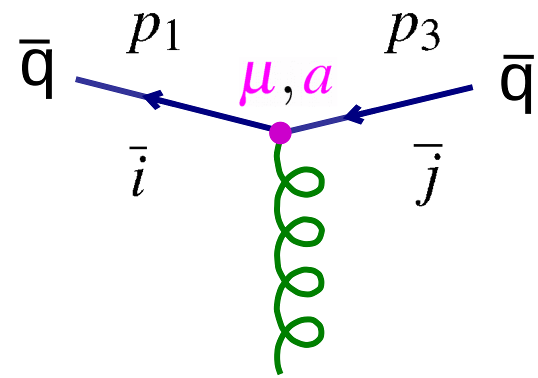
Colour Factors : Quarks vs Anti-Quarks

- **Recall the colour part of wave-function:**

$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

- **The QCD** qqg **vertex was written:**

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



★ **Now consider the anti-quark vertex**

- **The QCD** $\bar{q}\bar{q}$g **vertex is:**

$$\overline{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3)$$

Note that the incoming anti-particle now enters on the LHS of the expression

- **For which the colour part is**

$$c_i^\dagger\lambda^a c_j = c_i^\dagger\begin{pmatrix}\lambda^a_{1j}\\ \lambda^a_{2j}\\ \lambda^a_{3j}\end{pmatrix} = \lambda^a_{ij}$$

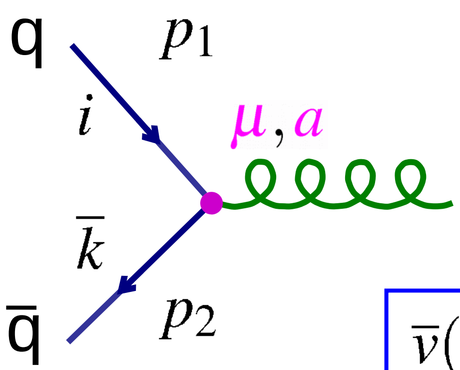
i.e indices *ij* are swapped with respect to the quark case

- **Hence** $\boxed{\overline{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3) \equiv \overline{v}(p_1)\{-\tfrac{1}{2}ig_s\lambda^a_{ij}\gamma^\mu\}v(p_3)}$

- **c.f. the quark - gluon QCD interaction**

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

★ **Finally we can consider the quark – anti-quark annihilation**



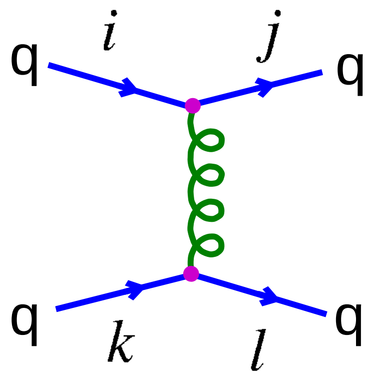
QCD vertex: $\overline{v}(p_2)c_k^\dagger\{-\frac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$

with $c_k^\dagger\lambda^a c_i = \lambda_{ki}^a$

$$\boxed{\overline{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda_{ki}^a\gamma^\mu\}u(p_1)}$$

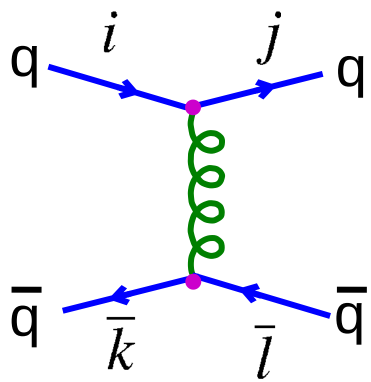
- Consequently the colour factors for the different diagrams are:

e.g.



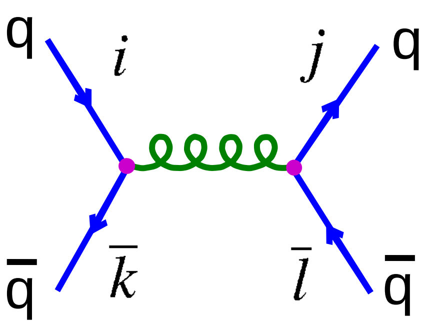
$$\boxed{C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}}$$

$$C(rr \to rr) = \tfrac{1}{3}$$
$$C(rg \to rg) = -\tfrac{1}{6}$$
$$C(rg \to gr) = \tfrac{1}{2}$$



$$\boxed{C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{kl}}$$

$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = -\tfrac{1}{6}$$
$$C(r\bar{r} \to g\bar{g}) = \tfrac{1}{2}$$



$$\boxed{C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ki}\lambda^a_{jl}}$$

$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = \tfrac{1}{2}$$
$$C(r\bar{r} \to g\bar{g}) = -\tfrac{1}{6}$$

Colour index of adjoint spinor comes first

Summary

★ Superficially QCD very similar to QED

★ But gluon self-interactions are believed to result in colour confinement

★ All hadrons are colour singlets which explains why only observe

Mesons **Baryons**

★ A low energies $\alpha_S \sim 1$

➡ Can't use perturbation theory !

Non-Perturbative regime

★ Where calculations can be performed, QCD provides a good description of relevant experimental data