

The background of the slide is a composite image of particle detector components. On the left, there are racks of electronic modules with numerous blue and red cables connected to them. On the right, there is a large, circular detector structure with a complex, multi-layered design, featuring a central golden-colored circular area surrounded by various blue and metallic components.

Chapter 3. The Dirac equation

Non-Relativistic QM (Revision)

- For particle physics need a relativistic formulation of quantum mechanics. But first take a few moments to review the non-relativistic formulation QM
- Take as the starting point non-relativistic energy:

$$E = T + V = \frac{\vec{p}^2}{2m} + V$$

- In QM we identify the energy and momentum operators:

$$\vec{p} \rightarrow -i\vec{\nabla}, \quad E \rightarrow i\frac{\partial}{\partial t}$$

which gives the time dependent Schrödinger equation (take $V=0$ for simplicity)

$$-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t} \quad (\text{S1})$$

with plane wave solutions: $\psi = Ne^{i(\vec{p}\cdot\vec{r}-Et)}$ where $\begin{cases} -i\nabla\psi = \vec{p}\psi \\ i\frac{\partial\psi}{\partial t} = E\psi \end{cases}$

- The SE is first order in the time derivatives and second order in spatial derivatives – and is manifestly **not Lorentz invariant**.
- In what follows we will use probability density/current extensively. For the non-relativistic case these are derived as follows

$$(\text{S1})^* \rightarrow -\frac{1}{2m}\vec{\nabla}^2\psi^* = -i\frac{\partial\psi^*}{\partial t} \quad (\text{S2})$$

$$\begin{aligned}\psi^* \times (\mathbf{S1}) - \psi \times (\mathbf{S2}) : \quad & -\frac{1}{2m} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) = i \left(\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} \right) \\ & -\frac{1}{2m} \vec{\nabla} \cdot (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) = i \frac{\partial}{\partial t} (\psi^* \psi)\end{aligned}$$

- Which by comparison with the continuity equation

$$\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

leads to the following expressions for probability density and current:

$$\rho = \psi^* \psi = |\psi|^2 \qquad \vec{j} = \frac{1}{2mi} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$$

- For a plane wave $\psi = N e^{i(\vec{p} \cdot \vec{r} - Et)}$

$$\rho = |N|^2 \quad \text{and} \quad \vec{j} = |N|^2 \frac{\vec{p}}{m} = |N|^2 \vec{v}$$

- ★ The number of particles per unit volume is $|N|^2$

- ★ For $|N|^2$ particles per unit volume moving at velocity \vec{v} , have $|N|^2 |\vec{v}|$ passing through a unit area per unit time (particle flux). Therefore \vec{j} is a vector in the particle's direction with magnitude equal to the **flux**.

The Klein-Gordon Equation

- Applying $\vec{p} \rightarrow -i\vec{\nabla}$, $E \rightarrow i\partial/\partial t$ to the relativistic equation for energy:

$$E^2 = |\vec{p}|^2 + m^2 \quad (\text{KG1})$$

gives the Klein-Gordon equation:

$$\frac{\partial^2 \psi}{\partial t^2} = \vec{\nabla}^2 \psi - m^2 \psi \quad (\text{KG2})$$

- Using $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \rightarrow \partial^\mu \partial_\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

KG can be expressed compactly as

$$(\partial^\mu \partial_\mu + m^2) \psi = 0 \quad (\text{KG3})$$

- For plane wave solutions, $\psi = Ne^{i(\vec{p} \cdot \vec{r} - Et)}$, the KG equation gives:

$$-E^2 \psi = -|\vec{p}|^2 \psi - m^2 \psi$$

$$\rightarrow E = \pm \sqrt{|\vec{p}|^2 + m^2}$$

- ★ Not surprisingly, the KG equation has negative energy solutions – this is just what we started with in eq. KG1
- ♦ Historically the –ve energy solutions were viewed as problematic. But for the KG there is also a problem with the probability density...

- Proceeding as before to calculate the probability and current densities:

$$\text{(KG2)*} \quad \frac{\partial^2 \psi^*}{\partial t^2} = \vec{\nabla}^2 \psi^* - m^2 \psi^* \quad \text{(KG4)}$$

$$\psi^* \times \text{(KG2)} - \psi \times \text{(KG4)} :$$

$$\begin{aligned} \psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2} &= \psi^* (\nabla^2 \psi - m^2 \psi) - \psi (\nabla^2 \psi^* - m^2 \psi^*) \\ \frac{\partial}{\partial t} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) &= \vec{\nabla} \cdot (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) \end{aligned}$$

- Which, again, by comparison with the continuity equation allows us to identify

(i to ensure ρ is always real)

$$\rho = i \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \quad \text{and} \quad \vec{j} = i (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$$

- For a plane wave $\psi = N e^{i(\vec{p} \cdot \vec{r} - Et)}$

$$\rho = 2E|N|^2 \quad \text{and} \quad \vec{j} = |N|^2 \vec{p}$$

- ★ Particle densities are proportional to E . We might have anticipated this from the previous discussion of Lorentz invariant phase space (i.e. density of $1/V$ in the particles rest frame will appear as E/V in a frame where the particle has energy E due to length contraction).

The Dirac Equation

★ Historically, it was thought that there were **two** main problems with the Klein-Gordon equation:

- ♦ Negative energy solutions
- ♦ The negative **particle densities** associated with these solutions

$$\rho = 2E|N|^2$$

★ We now know that in Quantum Field Theory these problems are overcome and the KG equation **is used** to describe **spin-0** particles (inherently single particle description \equiv multi-particle quantum excitations of a scalar field).

Nevertheless:



- ★ These problems motivated Dirac (1928) to search for a different formulation of relativistic quantum mechanics in which all **particle densities are positive**.
- ★ The resulting wave equation had solutions which not only solved this problem but also fully describe the intrinsic spin and magnetic moment of the electron!

The Dirac Equation :

- **Schrödinger eqn:** $-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t}$ **1st order in** $\partial/\partial t$
2nd order in $\partial/\partial x, \partial/\partial y, \partial/\partial z$
- **Klein-Gordon eqn:** $(\partial^\mu\partial_\mu + m^2)\psi = 0$ **2nd order throughout**

- **Dirac looked for an alternative which was 1st order throughout:**

$$\hat{H}\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi = i\frac{\partial\psi}{\partial t} \quad (\text{D1})$$

where \hat{H} is the Hamiltonian operator and, as usual, $\vec{p} = -i\vec{\nabla}$

- **Writing (D1) in full:**

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi = \left(i\frac{\partial}{\partial t}\right)\psi$$

“squaring” this equation gives

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi = -\frac{\partial^2\psi}{\partial t^2}$$

- **Which can be expanded in gory details as...**

$$\begin{aligned}
-\frac{\partial^2 \psi}{\partial t^2} = & -\alpha_x^2 \frac{\partial^2 \psi}{\partial x^2} - \alpha_y^2 \frac{\partial^2 \psi}{\partial y^2} - \alpha_z^2 \frac{\partial^2 \psi}{\partial z^2} + \beta^2 m^2 \psi \\
& -(\alpha_x \alpha_y + \alpha_y \alpha_x) \frac{\partial^2 \psi}{\partial x \partial y} - (\alpha_y \alpha_z + \alpha_z \alpha_y) \frac{\partial^2 \psi}{\partial y \partial z} - (\alpha_z \alpha_x + \alpha_x \alpha_z) \frac{\partial^2 \psi}{\partial z \partial x} \\
& -(\alpha_x \beta + \beta \alpha_x) m \frac{\partial \psi}{\partial x} - (\alpha_y \beta + \beta \alpha_y) m \frac{\partial \psi}{\partial y} - (\alpha_z \beta + \beta \alpha_z) m \frac{\partial \psi}{\partial z}
\end{aligned}$$

- For this to be a reasonable formulation of relativistic QM, a free particle must also obey $E^2 = \vec{p}^2 + m^2$, i.e. it must satisfy the **Klein-Gordon** equation:

$$-\frac{\partial^2 \psi}{\partial t^2} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial z^2} + m^2 \psi$$

- Hence for the Dirac Equation to be consistent with the KG equation require:

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1 \quad (D2)$$

$$\alpha_j \beta + \beta \alpha_j = 0 \quad (D3)$$

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 0 \quad (j \neq k) \quad (D4)$$

- ★ Immediately we see that the α_j and β cannot be numbers. Require 4 mutually anti-commuting matrices
- ★ Must be (at least) 4x4 matrices (see Appendix I)

- Consequently the wave-function must be a **four-component Dirac Spinor**

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

A consequence of introducing an equation that is 1st order in time/space derivatives is that the wave-function has new degrees of freedom !

- For the Hamiltonian $\hat{H}\psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi = i\partial\psi/\partial t$ to be Hermitian requires

$$\alpha_x = \alpha_x^\dagger; \quad \alpha_y = \alpha_y^\dagger; \quad \alpha_z = \alpha_z^\dagger; \quad \beta = \beta^\dagger;$$

E is always real !

(D5)

i.e. they require four anti-commuting Hermitian 4x4 matrices.

- At this point it is convenient to introduce an explicit representation for $\vec{\alpha}, \beta$. It should be noted that physical results do not depend on the particular representation – everything is in the commutation relations.
- A convenient choice is based on the Pauli spin matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- The matrices are Hermitian and anti-commute with each other

Dirac Equation: Probability Density and Current

- Now consider probability density/current – this is where the perceived problems with the Klein-Gordon equation arose.
- Start with the Dirac equation

$$-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + m\beta \psi = i \frac{\partial \psi}{\partial t} \quad (\text{D6})$$

and its **Hermitian** conjugate

$$+i \frac{\partial \psi^\dagger}{\partial x} \alpha_x^\dagger + i \frac{\partial \psi^\dagger}{\partial y} \alpha_y^\dagger + i \frac{\partial \psi^\dagger}{\partial z} \alpha_z^\dagger + m\psi^\dagger \beta^\dagger = -i \frac{\partial \psi^\dagger}{\partial t} \quad (\text{D7})$$

- Consider $\psi^\dagger \times (\text{D6}) - (\text{D7}) \times \psi$ remembering α, β are Hermitian \rightarrow

$$\psi^\dagger \left(-i\alpha_x \frac{\partial \psi}{\partial x} - i\alpha_y \frac{\partial \psi}{\partial y} - i\alpha_z \frac{\partial \psi}{\partial z} + \beta m \psi \right) - \left(i \frac{\partial \psi^\dagger}{\partial x} \alpha_x + i \frac{\partial \psi^\dagger}{\partial y} \alpha_y + i \frac{\partial \psi^\dagger}{\partial z} \alpha_z + m \psi^\dagger \beta \right) \psi = i \psi^\dagger \frac{\partial \psi}{\partial t} + i \frac{\partial \psi^\dagger}{\partial t} \psi$$

$$\rightarrow \underbrace{\psi^\dagger \left(\alpha_x \frac{\partial \psi}{\partial x} + \alpha_y \frac{\partial \psi}{\partial y} + \alpha_z \frac{\partial \psi}{\partial z} \right)} + \underbrace{\left(\frac{\partial \psi^\dagger}{\partial x} \alpha_x + \frac{\partial \psi^\dagger}{\partial y} \alpha_y + \frac{\partial \psi^\dagger}{\partial z} \alpha_z \right) \psi} + \frac{\partial (\psi^\dagger \psi)}{\partial t} = 0$$

- Now using the identity:

$$\psi^\dagger \alpha_x \frac{\partial \psi}{\partial x} + \frac{\partial \psi^\dagger}{\partial x} \alpha_x \psi \equiv \frac{\partial (\psi^\dagger \alpha_x \psi)}{\partial x}$$

gives the continuity equation

$$\vec{\nabla} \cdot (\psi^\dagger \vec{\alpha} \psi) + \frac{\partial (\psi^\dagger \psi)}{\partial t} = 0$$

(D8)

where $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$

- The probability density and current can be identified as:

$$\rho = \psi^\dagger \psi \quad \text{and} \quad \vec{j} = \psi^\dagger \vec{\alpha} \psi$$

where $\rho = \psi^\dagger \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 > 0$

- Unlike the KG equation, the Dirac equation has probability densities which are **always positive**.
- In addition, the solutions to the Dirac equation are **the four component Dirac Spinors**. A great success of the Dirac equation is that these components naturally give rise to the property of intrinsic spin.
- It can be shown that Dirac spinors represent spin-half particles (appendix II - next) with an intrinsic magnetic moment of

$$\vec{\mu} = \frac{q}{m} \vec{S}$$

(appendix III)

Appendix II : Spin

- For a Dirac spinor is orbital angular momentum a good quantum number?
i.e. does $L = \vec{r} \wedge \vec{p}$ commute with the Hamiltonian?

$$\begin{aligned}[H, \vec{L}] &= [\vec{\alpha} \cdot \vec{p} + \beta m, \vec{r} \wedge \vec{p}] \\ &= [\vec{\alpha} \cdot \vec{p}, \vec{r} \wedge \vec{p}]\end{aligned}$$

Consider the x component of L :

$$\begin{aligned}[H, L_x] &= [\vec{\alpha} \cdot \vec{p}, (\vec{r} \wedge \vec{p})_x] \\ &= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, y p_z - z p_y]\end{aligned}$$

The only non-zero contributions come from: $[x, p_x] = [y, p_y] = [z, p_z] = i$

$$\begin{aligned}[H, L_x] &= \alpha_y p_z [p_y, y] - \alpha_z p_y [p_z, z] \\ &= -i(\alpha_y p_z - \alpha_z p_y) \\ &= -i(\vec{\alpha} \wedge \vec{p})_x\end{aligned}$$

Therefore

$$[H, \vec{L}] = -i\vec{\alpha} \wedge \vec{p}$$

(A.1)

- ★ Hence the angular momentum does not commute with the Hamiltonian and is not a constant of motion

Introduce a new 4x4 operator:

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

where $\vec{\sigma}$ are the Pauli spin matrices: i.e.

$$\Sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \Sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad \Sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now consider the commutator

$$[H, \vec{\Sigma}] = [\vec{\alpha} \cdot \vec{p} + \beta m, \vec{\Sigma}]$$

here
$$[\beta, \vec{\Sigma}] = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} - \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = 0$$

and hence
$$[H, \vec{\Sigma}] = [\vec{\alpha} \cdot \vec{p}, \vec{\Sigma}]$$

Consider the x comp:
$$\begin{aligned} [H, \Sigma_x] &= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \Sigma_x] \\ &= p_x [\alpha_x, \Sigma_x] + p_y [\alpha_y, \Sigma_x] + p_z [\alpha_z, \Sigma_x] \end{aligned}$$

Taking each of the commutators in turn:

$$[\alpha_x, \Sigma_x] = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = 0$$

$$[\alpha_y, \Sigma_x] = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_y \sigma_y - \sigma_y \sigma_x \\ \sigma_y \sigma_x - \sigma_x \sigma_y & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2i\sigma_z \\ -2i\sigma_z & 0 \end{pmatrix}$$

$$= -2i\alpha_z$$

$$[\alpha_z, \Sigma_x] = 2i\alpha_y$$

Hence $[H, \Sigma_x] = p_x[\alpha_x, \Sigma_x] + p_y[\alpha_y, \Sigma_x] + p_z[\alpha_z, \Sigma_x]$

$$= -2ip_y\alpha_x + 2ip_z\alpha_y$$

$$= 2i(\vec{\alpha} \wedge \vec{p})_x$$

$$[H, \vec{\Sigma}] = 2i\vec{\alpha} \wedge \vec{p}$$

- Hence the observable corresponding to the operator $\vec{\Sigma}$ is also **not** a constant of motion. However, referring back to (A.1)

$$[H, \vec{S}] = \frac{1}{2} [H, \vec{\Sigma}] = i\vec{\alpha} \wedge \vec{p} = -[H, \vec{L}]$$

Therefore:

$$[H, \vec{L} + \vec{S}] = 0$$

- Because

$$\vec{S} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

the commutation relationships for \vec{S} are the same as for the $\vec{\sigma}$, e.g.

$[S_x, S_y] = iS_z$. Furthermore both S^2 and S_z are diagonal

$$S^2 = \frac{1}{4} (\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2) = \frac{3}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad S_z = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Consequently $S^2 \psi = S(S+1) \psi = \frac{3}{4}$ and for a particle travelling along the z direction $S_z \psi = \pm \frac{1}{2} \psi$
- ★ S has all the properties of spin in quantum mechanics and therefore the Dirac equation provides a natural account of the intrinsic angular momentum of fermions

Covariant Notation: the Dirac γ Matrices

- The Dirac equation can be written more elegantly by introducing the four Dirac gamma matrices:

$$\gamma^0 \equiv \beta; \quad \gamma^1 \equiv \beta \alpha_x; \quad \gamma^2 \equiv \beta \alpha_y; \quad \gamma^3 \equiv \beta \alpha_z$$

Premultiply the Dirac equation (D6) by β

$$i\beta \alpha_x \frac{\partial \psi}{\partial x} + i\beta \alpha_y \frac{\partial \psi}{\partial y} + i\beta \alpha_z \frac{\partial \psi}{\partial z} - \beta^2 m \psi = -i\beta \frac{\partial \psi}{\partial t}$$

→
$$i\gamma^1 \frac{\partial \psi}{\partial x} + i\gamma^2 \frac{\partial \psi}{\partial y} + i\gamma^3 \frac{\partial \psi}{\partial z} - m \psi = -i\gamma^0 \frac{\partial \psi}{\partial t}$$

using $\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ this can be written compactly as:

$$\boxed{(i\gamma^\mu \partial_\mu - m) \psi = 0}$$

(D9)

- ★ **NOTE:** it is important to realise that the **Dirac gamma matrices** are not **four-vectors** - they are constant matrices which remain invariant under a Lorentz transformation. However it can be shown that the Dirac equation is itself Lorentz covariant (see Appendix IV)

Properties of the γ matrices

- From the properties of the α and β matrices (D2)-(D4) immediately obtain:

$$(\gamma^0)^2 = \beta^2 = 1 \quad \text{and} \quad (\gamma^1)^2 = \beta \alpha_x \beta \alpha_x = -\alpha_x \beta \beta \alpha_x = -\alpha_x^2 = -1$$

- The full set of relations is

$$\begin{aligned} (\gamma^0)^2 &= 1 \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 &= -1 \\ \gamma^0 \gamma^j + \gamma^j \gamma^0 &= 0 \\ \gamma^j \gamma^k + \gamma^k \gamma^j &= 0 \quad (j \neq k) \end{aligned}$$

which can be expressed as:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad (\text{defines the algebra})$$

- Are the gamma matrices Hermitian?

- β is Hermitian so γ^0 is Hermitian.

- The α matrices are also Hermitian, giving

$$\gamma^{1\dagger} = (\beta \alpha_x)^\dagger = \alpha_x^\dagger \beta^\dagger = \alpha_x \beta = -\beta \alpha_x = -\gamma^1$$

- Hence $\gamma^1, \gamma^2, \gamma^3$ are anti-Hermitian

$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{1\dagger} = -\gamma^1, \quad \gamma^{2\dagger} = -\gamma^2, \quad \gamma^{3\dagger} = -\gamma^3$$

Pauli-Dirac Representation

- From now on we will use the Pauli-Dirac representation of the gamma matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \quad \text{which when written in full are}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}; \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Using the gamma matrices $\rho = \psi^\dagger \psi$ and $\vec{j} = \psi^\dagger \vec{\alpha} \psi$ can be written as:

$$j^\mu = (\rho, \vec{j}) = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

where j^μ is the **four-vector current**.

(The proof that j^μ is indeed a four vector is given in **Appendix V**.)

- In terms of the four-vector current the continuity equation becomes

$$\partial_\mu j^\mu = 0$$

- Finally the expression for the four-vector current

$$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

can be simplified by introducing the **adjoint spinor**

The Adjoint Spinor

- The adjoint spinor is defined as

$$\bar{\psi} = \psi^\dagger \gamma^0$$

i.e. $\bar{\psi} = \psi^\dagger \gamma^0 = (\psi^*)^T \gamma^0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$$\bar{\psi} = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

- In terms the adjoint spinor the four vector current can be written:

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$

★ We will use this expression in deriving the Feynman rules for the Lorentz invariant matrix element for the fundamental interactions.

- ★ That's enough notation, start to investigate the free particle solutions of the Dirac equation...

Dirac Equation: Free Particle at Rest

- Look for **free particle** solutions to the Dirac equation of form:

$$\psi = u(E, \vec{p}) e^{i(\vec{p} \cdot \vec{r} - Et)}$$

where $u(\vec{p}, E)$, which is a constant four-component spinor which must satisfy the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

- Consider the derivatives of the free particle solution

$$\partial_0 \psi = \frac{\partial \psi}{\partial t} = -iE \psi; \quad \partial_1 \psi = \frac{\partial \psi}{\partial x} = ip_x \psi, \quad \dots$$

substituting these into the Dirac equation gives:

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u = 0$$

which can be written:

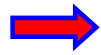
$$(\gamma^\mu p_\mu - m)u = 0 \tag{D10}$$

- This is the Dirac equation in “momentum” – note it contains no derivatives.
- For a **particle at rest** $\vec{p} = 0$

and $\psi = u(E, 0) e^{-iEt}$

eq. (D10) \longrightarrow

$$E\gamma^0 u - mu = 0$$



$$E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \quad (\text{D11})$$

- This equation has four orthogonal solutions:

$$u_1(m,0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad u_2(m,0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad u_3(m,0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad u_4(m,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(D11) \Rightarrow

$$E = m$$

(D11) \Rightarrow

$$E = -m$$

still have **NEGATIVE ENERGY SOLUTIONS**

(Question 6)

- Including the time dependence from $\psi = u(E,0)e^{-iEt}$ gives

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}; \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}; \quad \psi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt}; \quad \text{and} \quad \psi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt}$$

Two spin states with **E>0**

Two spin states with **E<0**

★ In QM mechanics can't just discard the **E<0** solutions as unphysical as we require a complete set of states - i.e. 4 SOLUTIONS

Dirac Equation: Plane Wave Solutions

- Now aim to find general plane wave solutions: $\psi = u(E, \vec{p})e^{i(\vec{p} \cdot \vec{r} - Et)}$
- Start from Dirac equation (D10): $(\gamma^\mu p_\mu - m)u = 0$

and use $\gamma^\mu p_\mu - m = E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 - m$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \cdot \vec{p} - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} (E - m)I & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -(E + m)I \end{pmatrix}$$

Note

$$\vec{\sigma} \cdot \vec{p} = p_x \sigma_x + p_y \sigma_y + p_z \sigma_z$$

Note in the above equation the 4x4 matrix is written in terms of four 2x2 sub-matrices

$$\begin{pmatrix} \begin{pmatrix} + & - \\ - & + \end{pmatrix} & \begin{pmatrix} + & - \\ - & + \end{pmatrix} \\ \begin{pmatrix} + & - \\ - & + \end{pmatrix} & \begin{pmatrix} + & - \\ - & + \end{pmatrix} \end{pmatrix}$$

- Writing the four component spinor as

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$(\gamma^\mu p_\mu - m)u = 0 \rightarrow \begin{pmatrix} (E - m)I & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -(E + m)I \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Giving two coupled simultaneous equations for u_A, u_B

$$\left. \begin{aligned} (\vec{\sigma} \cdot \vec{p})u_B &= (E - m)u_A \\ (\vec{\sigma} \cdot \vec{p})u_A &= (E + m)u_B \end{aligned} \right\}$$

(D12)

Expanding

$$\vec{\sigma} \cdot \vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z$$

$$\vec{\sigma} \cdot \vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

• Therefore (D12)

$$\left. \begin{aligned} (\vec{\sigma} \cdot \vec{p}) u_B &= (E - m) u_A \\ (\vec{\sigma} \cdot \vec{p}) u_A &= (E + m) u_B \end{aligned} \right\}$$

gives

$$u_B = \frac{\vec{\sigma} \cdot \vec{p}}{E + m} u_A = \frac{1}{E + m} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A$$

- Solutions can be obtained by making the arbitrary (but simplest) choices for u_A

i.e.

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

giving

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}; \quad \text{and} \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \quad \text{where N is the wave-function normalisation}$$

NOTE: For $\vec{p} = 0$ these correspond to the $E > 0$ particle at rest solutions

- ★ The choice of u_A is arbitrary, but this isn't an issue since we can express any other choice as a linear combination. It is analogous to choosing a basis for spin which could be eigenfunctions of S_x , S_y or S_z

Repeating for $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives the solutions u_3 and u_4

★ The four solutions are: $\psi_i = u_i(E, \vec{p}) e^{i(\vec{p} \cdot \vec{r} - Et)}$

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}; \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}; \quad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}; \quad u_4 = N_4 \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

- If any of these solutions is put back into the Dirac equation, as expected, we obtain

$$E^2 = \vec{p}^2 + m^2$$

which doesn't in itself identify the negative energy solutions.

- **One rather subtle point:** One could ask the question whether we can interpret **all four** solutions as positive energy solutions. The answer is no. If we take all solutions to have the same value of E , i.e. $E = +|E|$, only two of the solutions are found to be independent.
- There are only four independent solutions when the two **are taken to have $E < 0$** .

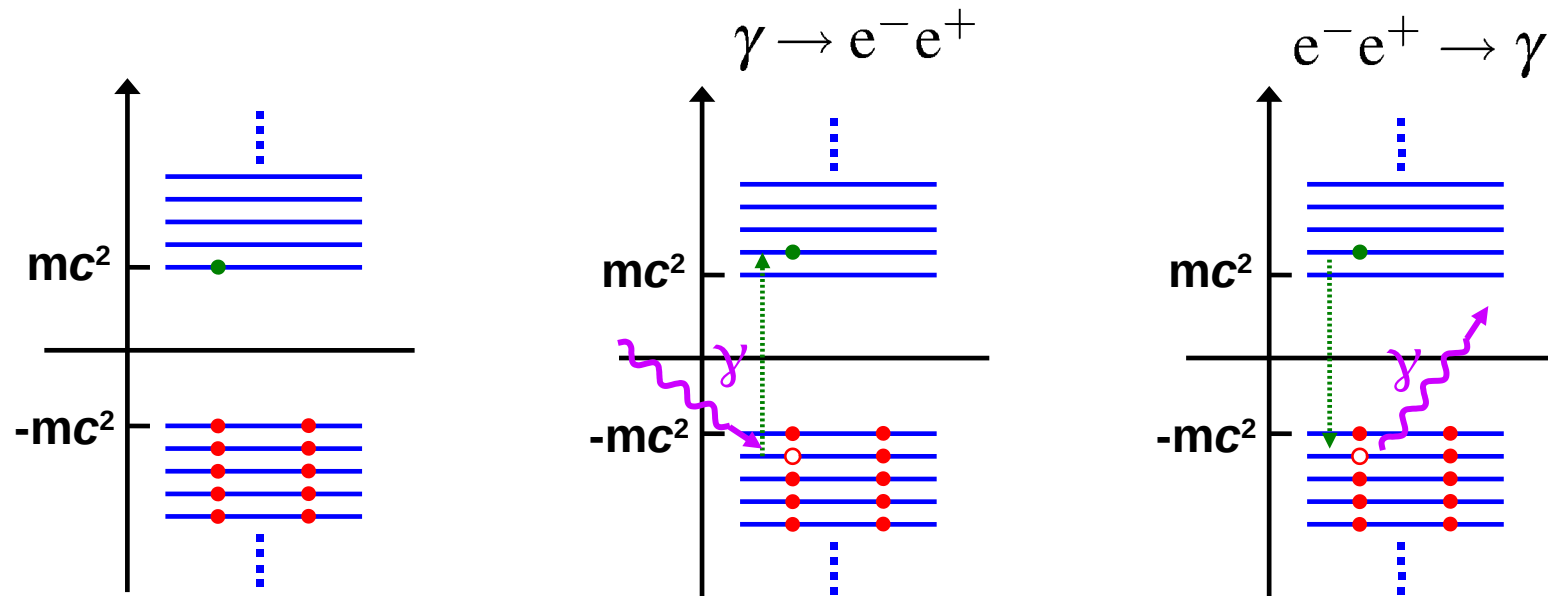
- ★ To identify which solutions have **$E < 0$** energy refer back to particle at rest (eq. D11).
- For $\vec{p} = 0$: u_1, u_2 correspond to the **$E > 0$** particle at rest solutions
 u_3, u_4 correspond to the **$E < 0$** particle at rest solutions

★ So u_1, u_2 are the +ve energy solutions and u_3, u_4 are the -ve energy solutions

Interpretation of -ve Energy Solutions

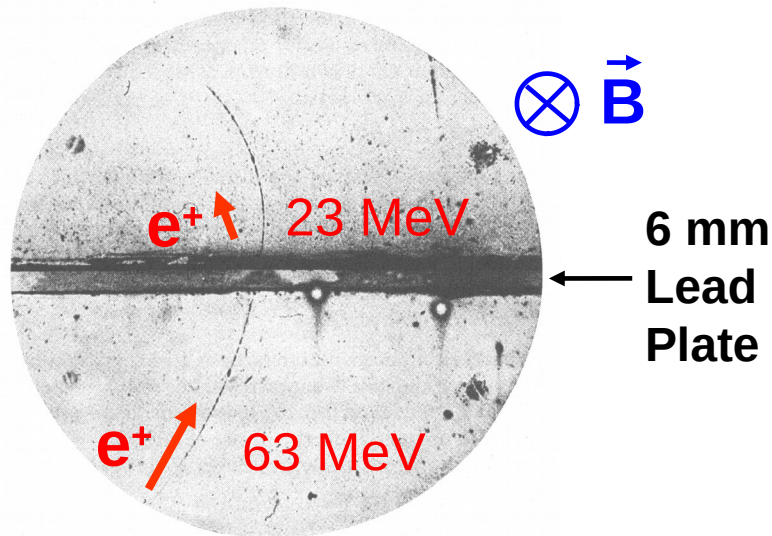
- ★ The Dirac equation has negative energy solutions. Unlike the KG equation these have positive probability densities. But how should -ve energy solutions be interpreted? Why don't all +ve energy electrons fall into to the lower energy -ve energy states?

Dirac Interpretation: the vacuum corresponds to all -ve energy states being full with the Pauli exclusion principle preventing electrons falling into -ve energy states. Holes in the -ve energy states correspond to +ve energy anti-particles with opposite charge. Provides a picture for pair-production and annihilation.



Discovery of the Positron

★ Cosmic ray track in cloud chamber:

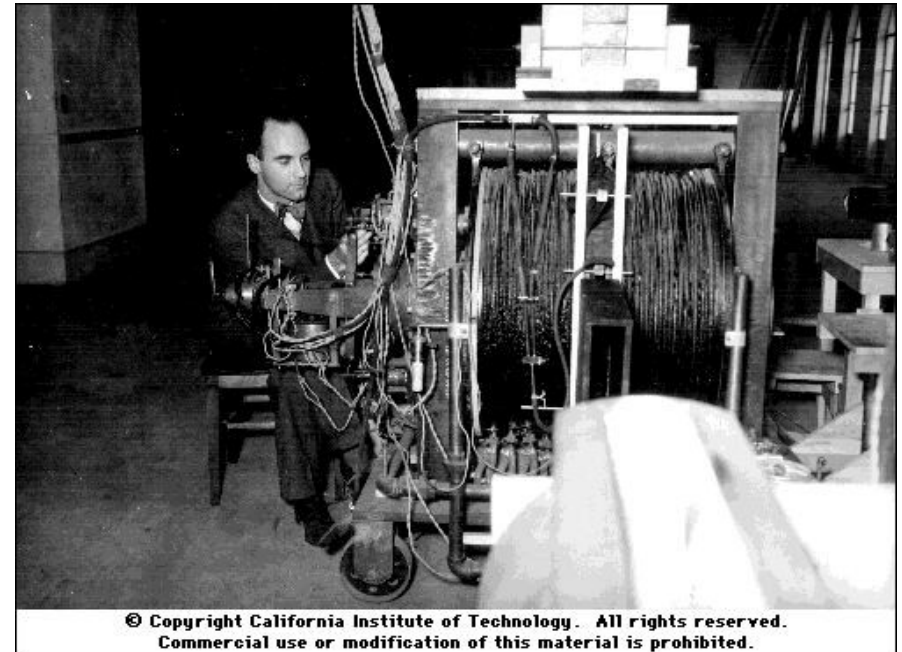


- e^+ enters at bottom, slows down in the lead plate – know direction
- Curvature in B-field shows that it is a positive particle
- Can't be a proton as would have stopped in the lead



Provided Verification of Predictions of Dirac Equation

C.D.Anderson, Phys Rev 43 (1933) 491



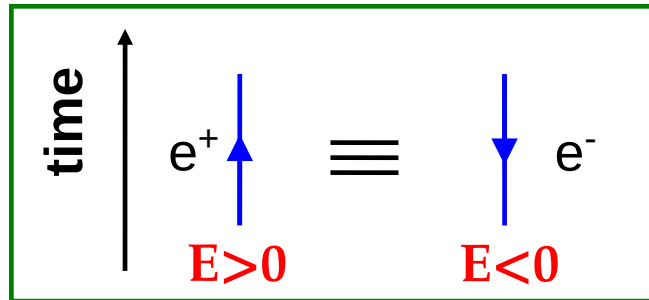
- ★ Anti-particle solutions exist ! But the picture of the vacuum corresponding to the state where all –ve energy states are occupied is rather unsatisfactory, what about bosons (no exclusion principle),....

Feynman-Stückelberg Interpretation

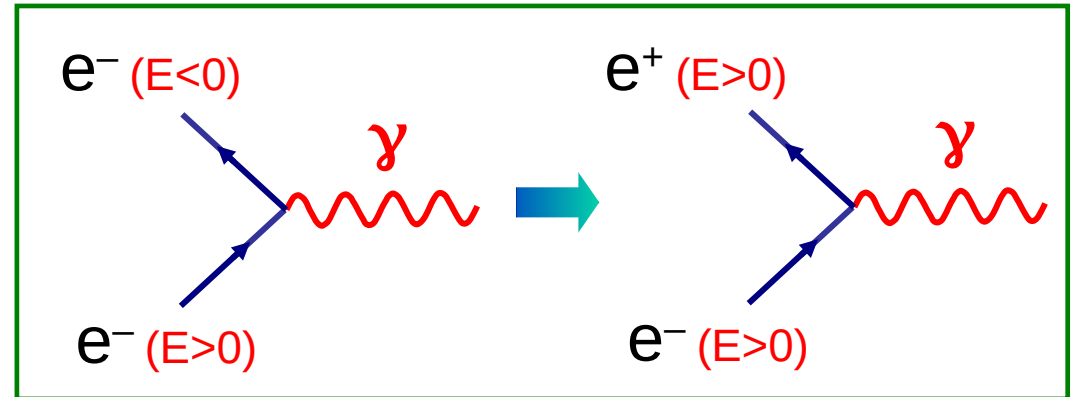
- ★ There are many problems with the Dirac interpretation of anti-particles and it is best viewed as of historical interest – don't take it too seriously.

Feynman-Stückelberg Interpretation:

- ★ Interpret a negative energy solution as **a negative energy particle** which propagates **backwards in time** or equivalently a positive energy **anti-particle** which propagates **forwards in time**



$$e^{-i(-E)(-t)} \rightarrow e^{-iEt}$$



NOTE: in the Feynman diagram the arrow on the anti-particle remains in the backwards in time direction to label it an anti-particle solution.

- ★ At this point it become more convenient to work with anti-particle wave-functions with $E = \sqrt{|\vec{p}|^2 + m^2}$ motivated by this interpretation

Anti-Particle Spinors

- Want to redefine our –ve energy solutions such that: $E = |\sqrt{|\vec{p}|^2 + m^2}|$
i.e. the energy of the **physical anti-particle**.

We can look at this in two ways:

—| Start from the negative energy solutions

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}; \quad u_4 = N_4 \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ -\frac{p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

Where E is understood to be negative

- Can simply “define” anti-particle wave-function by flipping the sign of E and \vec{p} following the Feynman-Stückelburg interpretation:

$$v_1(E, \vec{p})e^{-i(\vec{p} \cdot \vec{r} - Et)} = u_4(-E, -\vec{p})e^{i(\vec{p} \cdot \vec{r} - Et)}$$

$$v_2(E, \vec{p})e^{-i(\vec{p} \cdot \vec{r} - Et)} = u_3(-E, -\vec{p})e^{i(\vec{p} \cdot \vec{r} - Et)}$$

where E is now understood to be positive, $E = |\sqrt{|\vec{p}|^2 + m^2}|$

Anti-Particle Spinors

Find negative energy plane wave solutions to the Dirac equation of the form: $\psi = v(E, \vec{p}) e^{-i(\vec{p} \cdot \vec{r} - Et)}$ where $E = \sqrt{|\vec{p}|^2 + m^2}$

- Note that although $E > 0$ these are still negative energy solutions in the sense that

$$\hat{H}v_1 = i \frac{\partial}{\partial t} v_1 = -E v_1$$

- Solving the Dirac equation $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$$\Rightarrow (-\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z - m)v = 0$$

$$\boxed{(\gamma^\mu p_\mu + m)v = 0} \quad (\text{D13})$$

□ The Dirac equation in terms of momentum for ANTI-PARTICLES (c.f. D10)

- Proceeding as before:
$$\left. \begin{aligned} (\vec{\sigma} \cdot \vec{p})v_A &= (E - m)v_B \\ (\vec{\sigma} \cdot \vec{p})v_B &= (E + m)v_A \end{aligned} \right\} \text{etc., ...}$$

$$\Rightarrow v_1 = N'_1 \begin{pmatrix} \frac{p_x - ip_y}{E + m} \\ \frac{-p_z}{E + m} \\ 0 \\ 1 \end{pmatrix}; \quad v_2 = N'_2 \begin{pmatrix} \frac{p_z}{E + m} \\ \frac{p_x + ip_y}{E + m} \\ 1 \\ 0 \end{pmatrix}$$

- The same wave-functions that were written down on the previous page.

Particle and anti-particle Spinors

★ Four solutions of form: $\psi_i = u_i(E, \vec{p}) e^{i(\vec{p} \cdot \vec{r} - Et)}$

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}; \quad u_2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}; \quad u_3 = N \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}; \quad u_4 = N \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

$$E = + \left| \sqrt{|\vec{p}|^2 + m^2} \right| \qquad E = - \left| \sqrt{|\vec{p}|^2 + m^2} \right|$$

★ Four solutions of form $\psi_i = v_i(E, \vec{p}) e^{-i(\vec{p} \cdot \vec{r} - Et)}$

$$v_1 = N \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}; \quad v_2 = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}; \quad v_3 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \end{pmatrix}; \quad v_4 = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \end{pmatrix}$$

$$E = + \left| \sqrt{|\vec{p}|^2 + m^2} \right| \qquad E = - \left| \sqrt{|\vec{p}|^2 + m^2} \right|$$

★ Since we have a four component spinor, only four are linearly independent

- Could choose to work with $\{u_1, u_2, u_3, u_4\}$ or $\{v_1, v_2, v_3, v_4\}$ or ...
- Natural to use choose +ve energy solutions

$$\{u_1, u_2, v_1, v_2\}$$

Wave-Function Normalisation

- From **handout 1** want to normalise wave-functions to $2E$ particles per unit volume

- Consider $\psi = u_1 e^{+i(\vec{p} \cdot \vec{r} - Et)}$

Probability density $\rho = \psi^\dagger \psi = (\psi^*)^T \psi = u_1^\dagger u_1$

$$\begin{aligned} u_1^\dagger u_1 &= |N|^2 \left(1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2 + p_y^2}{(E+m)^2} \right) \\ &= |N|^2 \left(\frac{(E+m)^2 + |\vec{p}|^2}{(E+m)^2} \right) = |N|^2 \left(\frac{(E+m)^2 + E^2 - m^2}{(E+m)^2} \right) \\ &= |N|^2 \frac{2E^2 + 2Em}{(E+m)^2} = |N|^2 \frac{2E}{E+m} \end{aligned}$$

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}$$

which for the desired $2E$ particles per unit volume, requires that

$$N = \sqrt{E+m}$$

- Obtain same value of N for u_1, u_2, v_1, v_2

Charge Conjugation

- In the part II Relativity and Electrodynamics course it was shown that the motion of a charged particle in an electromagnetic field $A^\mu = (\phi, \vec{A})$ can be obtained by making the *minimal substitution*

$$\vec{p} \rightarrow \vec{p} - e\vec{A}; \quad E \rightarrow E - e\phi$$

with

$$\vec{p} = -i\vec{\nabla}; \quad E = i\partial/\partial t$$

this can be written

$$\partial_\mu \rightarrow \partial_\mu + ieA_\mu$$

and the Dirac equation becomes:

$$\gamma^\mu (\partial_\mu + ieA_\mu) \psi + im\psi = 0$$

- Taking the complex conjugate and pre-multiplying by $-i\gamma^2$

$$\Rightarrow -i\gamma^2 \gamma^{\mu*} (\partial_\mu - ieA_\mu) \psi^* - m\gamma^2 \psi^* = 0$$

But $\gamma^{0*} = \gamma^0; \gamma^{1*} = \gamma^1; \gamma^{2*} = -\gamma^2; \gamma^{3*} = \gamma^3$ and $\gamma^2 \gamma^{\mu*} = -\gamma^\mu \gamma^2$

$$\Rightarrow \gamma^\mu (\partial_\mu - ieA_\mu) \underbrace{i\gamma^2 \psi^*}_{\text{orange}} + im \underbrace{i\gamma^2 \psi^*}_{\text{orange}} = 0 \quad \text{(D14)}$$

- Define the charge conjugation operator:

$$\psi' = \hat{C}\psi = i\gamma^2 \psi^*$$

D14 becomes:

$$\gamma^\mu (\partial_\mu - ieA_\mu) \psi' + im\psi' = 0$$

- Comparing to the original equation

$$\gamma^\mu (\partial_\mu + ieA_\mu) \psi + im\psi = 0$$

we see that the spinor ψ' describes a particle of the same mass but with opposite charge, i.e. an **anti-particle** !

$$\hat{C} \longrightarrow \text{particle spinor} \rightleftharpoons \text{anti-particle spinor}$$

- Now consider the action of \hat{C} on the free particle wave-function:

$$\psi = u_1 e^{i(\vec{p} \cdot \vec{r} - Et)}$$

$$\psi' = \hat{C}\psi = i\gamma^2 \psi^* = i\gamma^2 u_1^* e^{-i(\vec{p} \cdot \vec{r} - Et)}$$

$$i\gamma^2 u_1^* = i \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}^* = \sqrt{E+m} \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v_1$$

hence $\psi = u_1 e^{i(\vec{p} \cdot \vec{r} - Et)} \xrightarrow{\hat{C}} \psi' = v_1 e^{-i(\vec{p} \cdot \vec{r} - Et)}$

similarly $\psi = u_2 e^{i(\vec{p} \cdot \vec{r} - Et)} \xrightarrow{\hat{C}} \psi' = v_2 e^{-i(\vec{p} \cdot \vec{r} - Et)}$

- ★ Under the charge conjugation operator the particle spinors u_1 and u_2 transform to the anti-particle spinors v_1 and v_2

Using the anti-particle solutions

- There is a **subtle** but **important** point about the anti-particle solutions written as

$$\psi = v(E, \vec{p}) e^{-i(\vec{p} \cdot \vec{r} - Et)}$$

Applying normal QM operators for momentum and energy $\hat{p} = -i\vec{\nabla}$, $\hat{H} = i\partial/\partial t$ gives $\hat{H}v_1 = i\partial v_1/\partial t = -Ev_1$ and $\hat{p}v_1 = -i\vec{\nabla}v_1 = -\vec{p}v_1$

- ★ But have **defined** solutions to have **E>0**

- ★ Hence the quantum mechanical operators giving the **physical** energy and momenta of the **anti-particle** solutions are:

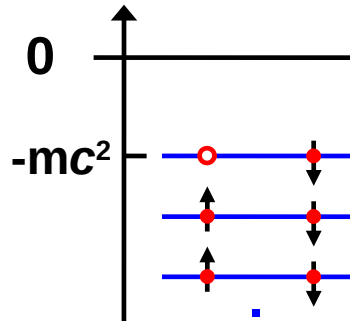
$$\hat{H}^{(v)} = -i\partial/\partial t \quad \text{and} \quad \hat{p}^{(v)} = i\vec{\nabla}$$

- Under the transformation $(E, \vec{p}) \rightarrow (-E, -\vec{p})$: $\vec{L} = \vec{r} \wedge \vec{p} \rightarrow -\vec{L}$

Conservation of **total** angular momentum $[H, \vec{L} + \vec{S}] = 0 \quad \Rightarrow \quad \boxed{\hat{S}^{(v)} \rightarrow -\hat{S}}$

★ The **physical spin** of the **anti-particle solutions** is given by $\hat{S}^{(v)} = -\hat{S}$

In the hole picture:



A spin-up hole leaves the negative energy sea in a spin down state

Summary of Solutions to the Dirac Equation

- The normalised free **PARTICLE** solutions to the Dirac equation:

$$\psi = u(E, \vec{p}) e^{+i(\vec{p} \cdot \vec{r} - Et)} \quad \text{satisfy} \quad (\gamma^\mu p_\mu - m)u = 0$$

with

$$u_1 = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}; \quad u_2 = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

- The **ANTI-PARTICLE** solutions in terms of the physical energy and momentum:

$$\psi = v(E, \vec{p}) e^{-i(\vec{p} \cdot \vec{r} - Et)} \quad \text{satisfy} \quad (\gamma^\mu p_\mu + m)v = 0$$

with

$$v_1 = \sqrt{E+m} \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}; \quad v_2 = \sqrt{E+m} \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

For these states the spin is given by $\hat{S}^{(v)} = -\hat{S}$

- For both particle and anti-particle solutions: $E = \sqrt{|\vec{p}|^2 + m^2}$

(Now try question 7 – mainly about 4 vector current)

Spin States

- In general the spinors u_1, u_2, v_1, v_2 are not Eigenstates of \hat{S}_z

$$\hat{S}_z = \frac{1}{2}\Sigma_z = \frac{1}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{Appendix II})$$

- However particles/anti-particles travelling in the z-direction: $p_z = \pm |\vec{p}|$

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{\pm|\vec{p}|}{E+m} \\ 0 \end{pmatrix}; \quad u_2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{\mp|\vec{p}|}{E+m} \\ 0 \end{pmatrix}; \quad v_1 = N \begin{pmatrix} 0 \\ 1 \\ \frac{\mp|\vec{p}|}{E+m} \\ 1 \end{pmatrix}; \quad v_2 = N \begin{pmatrix} \frac{\pm|\vec{p}|}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

are Eigenstates of \hat{S}_z

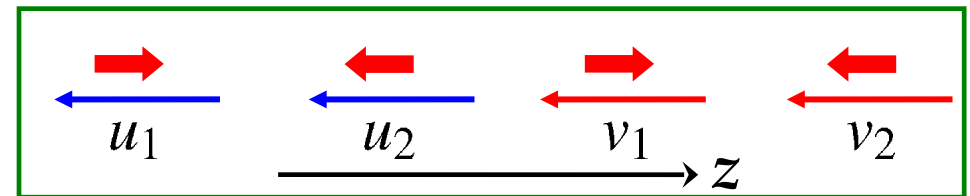
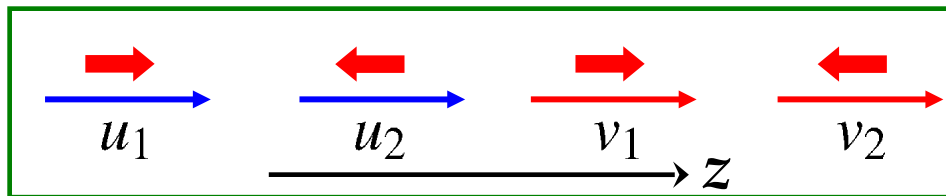
$$\hat{S}_z u_1 = +\frac{1}{2} u_1$$

$$\hat{S}_z u_2 = -\frac{1}{2} u_2$$

$$\hat{S}_z^{(v)} v_1 = -\hat{S}_z v_1 = +\frac{1}{2} v_1$$

$$\hat{S}_z^{(v)} v_2 = -\hat{S}_z v_2 = -\frac{1}{2} v_2$$

Note the change of sign of \hat{S} when dealing with antiparticle spinors



- ★ Spinors u_1, u_2, v_1, v_2 are only eigenstates of \hat{S}_z for $p_z = \pm |\vec{p}|$

Pause for Breath...

- Have found solutions to the Dirac equation which are also eigenstates \hat{S}_z but only for particles travelling along the z axis.
- Not a particularly useful basis
- More generally, want to label our states in terms of “good quantum numbers”, i.e. a set of commuting observables.
- Can't use z component of spin: $[\hat{H}, \hat{S}_z] \neq 0$ (Appendix II)
- Introduce a new concept “HELICITY”

Helicity plays an important role in much that follows

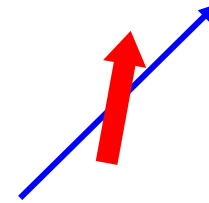
Helicity

- ★ The component of a particles spin along its direction of flight is a good quantum number:

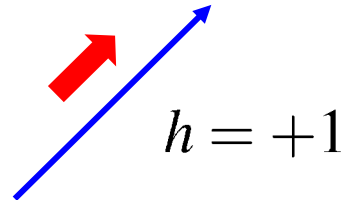
$$[\hat{H}, \hat{S} \cdot \hat{p}] = 0$$

- ★ Define the component of a particles spin along its direction of flight as **HELICITY**:

$$h \equiv \frac{\vec{S} \cdot \vec{p}}{|\vec{S}| |\vec{p}|} = \frac{2\vec{S} \cdot \vec{p}}{|\vec{p}|} = \frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|}$$



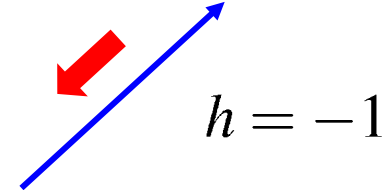
- If we make a measurement of the component of spin of a spin-half particle along any axis it can take two values $\pm 1/2$, consequently the eigenvalues of the helicity operator for a spin-half particle are: ± 1



$$h = +1$$

Often termed:

“right-handed”



$$h = -1$$

“left-handed”

- ★ **NOTE:** these are **“RIGHT-HANDED”** and **LEFT-HANDED** HELICITY eigenstates
- ★ In handout 4 we will discuss **RH** and **LH** CHIRAL eigenstates. Only in the limit $v \approx c$ are the **HELICITY** eigenstates the same as the **CHIRAL** eigenstates

Helicity Eigenstates

- ★ Wish to find solutions of Dirac equation which are also eigenstates of Helicity:

$$(\vec{\Sigma} \cdot \hat{p})u_{\uparrow} = +u_{\uparrow} \quad (\vec{\Sigma} \cdot \hat{p})u_{\downarrow} = -u_{\downarrow}$$

where u_{\uparrow} and u_{\downarrow} are **right** and **left handed** helicity states and **green** \hat{p} is the **unit vector** in the direction of the particle.

- The eigenvalue equation:

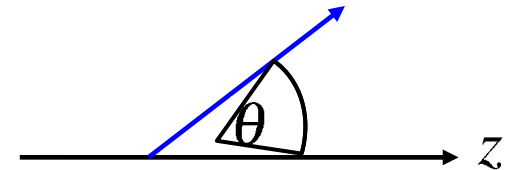
$$\begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \pm \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

gives the coupled equations:

$$\left. \begin{aligned} (\vec{\sigma} \cdot \hat{p})u_A &= \pm u_A \\ (\vec{\sigma} \cdot \hat{p})u_B &= \pm u_B \end{aligned} \right\} \quad \text{(D15)}$$

- Consider a particle propagating in (θ, ϕ) direction

$$\hat{p} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$



$$\vec{\sigma} \cdot \hat{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \cos \phi - i \sin \theta \sin \phi \\ \sin \theta \cos \phi + i \sin \theta \sin \phi & -\cos \theta \end{pmatrix}$$

$$\vec{\sigma} \cdot \hat{p} = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}$$

- Writing **either** $u_A = \begin{pmatrix} a \\ b \end{pmatrix}$ **or** $u_B = \begin{pmatrix} a \\ b \end{pmatrix}$ then **(D15)** gives the relation

$$\begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix} \quad (\text{For helicity } \pm 1)$$

So for the components of **BOTH** u_A and u_B

$$\frac{b}{a} = \frac{\pm 1 - \cos \theta}{\sin \theta} e^{i\phi}$$

- For the **right-handed helicity state, i.e. helicity +1:**

$$\frac{b}{a} = \frac{1 - \cos \theta}{\sin \theta} e^{i\phi} = \frac{2 \sin^2 \left(\frac{\theta}{2} \right)}{2 \sin \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta}{2} \right)} e^{i\phi} = e^{i\phi} \frac{\sin \left(\frac{\theta}{2} \right)}{\cos \left(\frac{\theta}{2} \right)}$$

→

$$u_{A\uparrow} \propto \begin{pmatrix} \cos \left(\frac{\theta}{2} \right) \\ e^{i\phi} \sin \left(\frac{\theta}{2} \right) \end{pmatrix} \quad u_{B\uparrow} \propto \begin{pmatrix} \cos \left(\frac{\theta}{2} \right) \\ e^{i\phi} \sin \left(\frac{\theta}{2} \right) \end{pmatrix}$$

- Putting in the constants of proportionality gives:

$$u_{\uparrow} = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} \kappa_1 \cos \left(\frac{\theta}{2} \right) \\ \kappa_1 e^{i\phi} \sin \left(\frac{\theta}{2} \right) \\ \kappa_2 \cos \left(\frac{\theta}{2} \right) \\ \kappa_2 e^{i\phi} \sin \left(\frac{\theta}{2} \right) \end{pmatrix}$$

- From the Dirac Equation (D12) we also have

$$\begin{aligned}
 (\vec{\sigma} \cdot \vec{p}) u_A &= (E + m) u_B \\
 u_B &= \frac{\vec{\sigma} \cdot \vec{p}}{E + m} u_A = \frac{|\vec{p}|}{E + m} (\underbrace{\vec{\sigma} \cdot \hat{p}}_{\text{Helicity}}) u_A = \pm \frac{|\vec{p}|}{E + m} u_A
 \end{aligned}
 \tag{D16}$$

- ★ (D15) determines the relative normalisation of u_A and u_B , i.e. here

$$\begin{aligned}
 u_B &= +1 \frac{|\vec{p}|}{E + m} u_A \\
 \Rightarrow u_{\uparrow} &= N \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi} \sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m} \cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m} e^{i\phi} \sin\left(\frac{\theta}{2}\right) \end{pmatrix}
 \end{aligned}$$

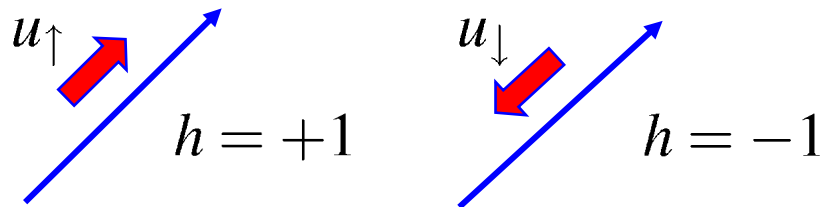
- The **negative helicity particle** state is obtained in the same way.
- The **anti-particle** states can also be obtained in the same manner although it must be remembered that $\hat{S}^{(v)} = -\hat{S}$

$$\text{i.e.} \quad \hat{h}^{(v)} = -(\vec{\Sigma} \cdot \hat{p}) \quad \rightarrow \quad (\vec{\Sigma} \cdot \hat{p}) v_{\uparrow} = -v_{\uparrow}$$

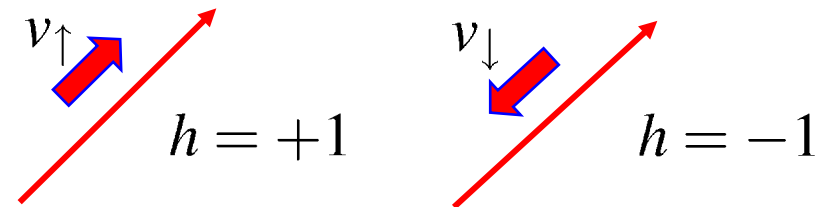
★ The particle and anti-particle helicity eigenstates are:

$u_{\uparrow} = N \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi} \sin\left(\frac{\theta}{2}\right) \\ \frac{ \vec{p} }{E+m} \cos\left(\frac{\theta}{2}\right) \\ \frac{ \vec{p} }{E+m} e^{i\phi} \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$	$u_{\downarrow} = N \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi} \cos\left(\frac{\theta}{2}\right) \\ \frac{ \vec{p} }{E+m} \sin\left(\frac{\theta}{2}\right) \\ -\frac{ \vec{p} }{E+m} e^{i\phi} \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$
$v_{\uparrow} = N \begin{pmatrix} \frac{ \vec{p} }{E+m} \sin\left(\frac{\theta}{2}\right) \\ -\frac{ \vec{p} }{E+m} e^{i\phi} \cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi} \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$	$v_{\downarrow} = N \begin{pmatrix} \frac{ \vec{p} }{E+m} \cos\left(\frac{\theta}{2}\right) \\ \frac{ \vec{p} }{E+m} e^{i\phi} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi} \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$

particles



anti-particles



★ For all four states, normalising to $2E$ particles/Volume again gives $N = \sqrt{E + m}$

★ The helicity eigenstates will be used extensively in the calculations that follow.

Intrinsic Parity of Dirac Particles

- ★ Before leaving the Dirac equation, consider parity
- ★ The parity operation is defined as spatial inversion through the origin:

$$x' \equiv -x; \quad y' \equiv -y; \quad z' \equiv -z; \quad t' \equiv t$$

- Consider a Dirac spinor, $\psi(x, y, z, t)$, which satisfies the Dirac equation

$$i\gamma^1 \frac{\partial \psi}{\partial x} + i\gamma^2 \frac{\partial \psi}{\partial y} + i\gamma^3 \frac{\partial \psi}{\partial z} - m\psi = -i\gamma^0 \frac{\partial \psi}{\partial t} \quad (\text{D17})$$

- Under the parity transformation: $\psi'(x', y', z', t') = \hat{P}\psi(x, y, z, t)$

Try $\hat{P} = \gamma^0 \quad \psi'(x', y', z', t') = \gamma^0 \psi(x, y, z, t)$

$$(\gamma^0)^2 = 1 \quad \text{so} \quad \psi(x, y, z, t) = \gamma^0 \psi'(x', y', z', t')$$

(D17) $\rightarrow i\gamma^1 \gamma^0 \frac{\partial \psi'}{\partial x} + i\gamma^2 \gamma^0 \frac{\partial \psi'}{\partial y} + i\gamma^3 \gamma^0 \frac{\partial \psi'}{\partial z} - m\gamma^0 \psi' = -i\gamma^0 \gamma^0 \frac{\partial \psi'}{\partial t}$

- Expressing derivatives in terms of the primed system:

$$-i\gamma^1 \gamma^0 \frac{\partial \psi'}{\partial x'} - i\gamma^2 \gamma^0 \frac{\partial \psi'}{\partial y'} - i\gamma^3 \gamma^0 \frac{\partial \psi'}{\partial z'} - m\gamma^0 \psi' = -i\gamma^0 \gamma^0 \frac{\partial \psi'}{\partial t'}$$

Since γ^0 anti-commutes with $\gamma^1, \gamma^2, \gamma^3$:

$$+i\gamma^0 \gamma^1 \frac{\partial \psi'}{\partial x'} + i\gamma^0 \gamma^2 \frac{\partial \psi'}{\partial y'} + i\gamma^0 \gamma^3 \frac{\partial \psi'}{\partial z'} - m\gamma^0 \psi' = -i \frac{\partial \psi'}{\partial t'}$$

Pre-multiplying by $\gamma^0 \Rightarrow i\gamma^1 \frac{\partial \psi'}{\partial x'} + i\gamma^2 \frac{\partial \psi'}{\partial y'} + i\gamma^3 \frac{\partial \psi'}{\partial z'} - m\psi' = -i\gamma^0 \frac{\partial \psi'}{\partial t'}$

- Which is the Dirac equation in the new coordinates.
- ★ There for under parity transformations the form of the Dirac equation is unchanged **provided** Dirac spinors transform as

$$\psi \rightarrow \hat{P}\psi = \pm \gamma^0 \psi$$

(note the above algebra doesn't depend on the choice of $\hat{P} = \pm \gamma^0$)

- For a particle/anti-particle at rest the solutions to the Dirac Equation are:

$$\psi = u_1 e^{-imt}; \quad \psi = u_2 e^{-imt}; \quad \psi = v_1 e^{+imt}; \quad \psi = v_2 e^{+imt}$$

with $u_1 = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad u_2 = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad v_1 = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \quad v_2 = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix};$

$$\hat{P}u_1 = \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \pm u_1 \quad \text{etc.} \rightarrow \begin{matrix} \hat{P}u_1 = \pm u_1 & \hat{P}v_1 = \mp v_1 \\ \hat{P}u_2 = \pm u_2 & \hat{P}v_2 = \mp v_2 \end{matrix}$$

- ★ Hence an **anti-particle** at rest has **opposite intrinsic parity** to a **particle** at rest.
- ★ Convention: particles are chosen to have +ve parity; corresponds to choosing

$$\hat{P} = +\gamma^0$$

Summary

- ★ The formulation of relativistic quantum mechanics starting from the linear Dirac equation

$$\hat{H}\psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi = i\frac{\partial \psi}{\partial t}$$

➡ New degrees of freedom : found to describe Spin $\frac{1}{2}$ particles

- ★ In terms of 4x4 gamma matrices the Dirac Equation can be written:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

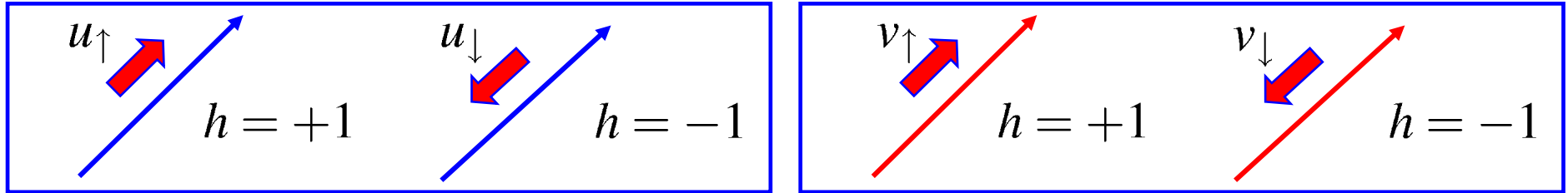
- ★ Introduces the 4-vector current and adjoint spinor:

$$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi = \bar{\psi} \gamma^\mu \psi$$

- ★ With the Dirac equation: **forced to have two positive energy and two negative energy solutions**
- ★ Feynman-Stückelberg interpretation: -ve energy particle solutions propagating backwards in time correspond to physical +ve energy anti-particles propagating forwards in time

$$u_1, u_2, v_1, v_2$$

★ Most useful basis: particle and anti-particle helicity eigenstates



★ In terms of 4-component spinors, the charge conjugation and parity operations are:

$$\psi \rightarrow \hat{C}\psi = i\gamma^2\psi^\dagger$$

$$\psi \rightarrow \hat{P}\psi = \gamma^0\psi$$

★ Now have all we need to know about a relativistic description of particles... next discuss particle interactions and QED.

Appendix I : Dimensions of the Dirac Matrices

non-examinable

Starting from $\hat{H}\psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi = i\frac{\partial \psi}{\partial t}$

For \hat{H} to be Hermitian for all \vec{p} requires $\alpha_i = \alpha_i^\dagger \quad \beta = \beta^\dagger$

To recover the KG equation: $\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$

$$\beta \alpha_j + \alpha_j \beta = 0$$

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 0 \quad (j \neq k)$$

Consider

$$Tr(B^\dagger AB) = B_{ij}^\dagger A_{jk} B_{ki}$$

with $B^\dagger B = 1$

$$= B_{ki} B_{ij}^\dagger A_{jk}$$

$$= \delta_{jk} A_{jk}$$

$$= Tr(A)$$

Therefore

$$Tr(\alpha) = Tr(\alpha_j^\dagger \alpha_i \alpha_j)$$

$$= -Tr(\alpha_j^\dagger \alpha_j \alpha_i) \quad \text{(using commutation relation)}$$

$$= -Tr(\alpha_i)$$

$$\Rightarrow Tr(\alpha_i) = 0$$

similarly

$$Tr(\beta) = 0$$

We can now show that the matrices are of even dimension by considering the eigenvalue equation, e.g.

$$\alpha \vec{x} = \lambda \vec{x}$$
$$\vec{x}^\dagger \vec{x} = \vec{x} \alpha^\dagger \alpha \vec{x} = \lambda^* \lambda \vec{x}^\dagger \vec{x}$$

Eigenvalues of a Hermitian matrix are real so $\lambda^2 = 1 \rightarrow \lambda = \pm 1$

but $Tr(\alpha) = \sum_i \lambda_i$

Since the α_i, β are trace zero Hermitian matrices with eigenvalues of ± 1 they must be of even dimension

For N=2 the 3 Pauli spin matrices satisfy

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 0 \quad (j \neq i)$$

But we require 4 anti-commuting matrices. Consequently the α_i, β of the Dirac equation must be of dimension 4, 6, 8,..... The simplest choice for is to assume that the α_i, β are of dimension 4.

Appendix II : Spin

non-examinable

- For a Dirac spinor is orbital angular momentum a good quantum number?
i.e. does $L = \vec{r} \wedge \vec{p}$ commute with the Hamiltonian?

$$\begin{aligned}[H, \vec{L}] &= [\vec{\alpha} \cdot \vec{p} + \beta m, \vec{r} \wedge \vec{p}] \\ &= [\vec{\alpha} \cdot \vec{p}, \vec{r} \wedge \vec{p}]\end{aligned}$$

Consider the x component of L :

$$\begin{aligned}[H, L_x] &= [\vec{\alpha} \cdot \vec{p}, (\vec{r} \wedge \vec{p})_x] \\ &= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, y p_z - z p_y]\end{aligned}$$

The only non-zero contributions come from: $[x, p_x] = [y, p_y] = [z, p_z] = i$

$$\begin{aligned}[H, L_x] &= \alpha_y p_z [p_y, y] - \alpha_z p_y [p_z, z] \\ &= -i(\alpha_y p_z - \alpha_z p_y) \\ &= -i(\vec{\alpha} \wedge \vec{p})_x\end{aligned}$$

Therefore

$$[H, \vec{L}] = -i\vec{\alpha} \wedge \vec{p}$$

(A.1)

- ★ Hence the angular momentum does not commute with the Hamiltonian and is not a constant of motion

Introduce a new 4x4 operator:

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

where $\vec{\sigma}$ are the Pauli spin matrices: i.e.

$$\Sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \Sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad \Sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now consider the commutator

$$[H, \vec{\Sigma}] = [\vec{\alpha} \cdot \vec{p} + \beta m, \vec{\Sigma}]$$

here
$$[\beta, \vec{\Sigma}] = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} - \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = 0$$

and hence
$$[H, \vec{\Sigma}] = [\vec{\alpha} \cdot \vec{p}, \vec{\Sigma}]$$

Consider the x comp:

$$\begin{aligned} [H, \Sigma_x] &= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \Sigma_x] \\ &= p_x [\alpha_x, \Sigma_x] + p_y [\alpha_y, \Sigma_x] + p_z [\alpha_z, \Sigma_x] \end{aligned}$$

Taking each of the commutators in turn:

$$[\alpha_x, \Sigma_x] = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = 0$$

$$[\alpha_y, \Sigma_x] = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_y \sigma_y - \sigma_y \sigma_x \\ \sigma_y \sigma_x - \sigma_x \sigma_y & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2i\sigma_z \\ -2i\sigma_z & 0 \end{pmatrix}$$

$$= -2i\alpha_z$$

$$[\alpha_z, \Sigma_x] = 2i\alpha_y$$

Hence $[H, \Sigma_x] = p_x[\alpha_x, \Sigma_x] + p_y[\alpha_y, \Sigma_x] + p_z[\alpha_z, \Sigma_x]$

$$= -2ip_y\alpha_x + 2ip_z\alpha_y$$

$$= 2i(\vec{\alpha} \wedge \vec{p})_x$$

$$[H, \vec{\Sigma}] = 2i\vec{\alpha} \wedge \vec{p}$$

- Hence the observable corresponding to the operator $\vec{\Sigma}$ is also **not** a constant of motion. However, referring back to (A.1)

$$[H, \vec{S}] = \frac{1}{2} [H, \vec{\Sigma}] = i\vec{\alpha} \wedge \vec{p} = -[H, \vec{L}]$$

Therefore:

$$[H, \vec{L} + \vec{S}] = 0$$

- Because

$$\vec{S} = \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

the commutation relationships for \vec{S} are the same as for the $\vec{\sigma}$, e.g. $[S_x, S_y] = iS_z$. Furthermore both S^2 and S_z are diagonal

$$S^2 = \frac{1}{4} (\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2) = \frac{3}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad S_z = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Consequently $S^2 \psi = S(S+1) \psi = \frac{3}{4}$ and for a particle travelling along the z direction $S_z \psi = \pm \frac{1}{2} \psi$
- ★ S has all the properties of spin in quantum mechanics and therefore the Dirac equation provides a natural account of the intrinsic angular momentum of fermions

Appendix III : Magnetic Moment

non-examinable

- In the part II Relativity and Electrodynamics course it was shown that the motion of a charged particle in an electromagnetic field $A^\mu = (\phi, \vec{A})$ can be obtained by making the *minimal substitution*

$$\vec{p} \rightarrow \vec{p} - q\vec{A}; \quad E \rightarrow E - q\phi$$

- Applying this to equations (D12)

$$(\vec{\sigma} \cdot \vec{p} - q\vec{\sigma} \cdot \vec{A})u_B = (E - m - q\phi)u_A \quad (\text{A.2})$$

$$(\vec{\sigma} \cdot \vec{p} - q\vec{\sigma} \cdot \vec{A})u_A = (E + m - q\phi)u_B$$

Multiplying (A.2) by $(E + m - q\phi)$

$$\begin{aligned} (\vec{\sigma} \cdot \vec{p} - q\vec{\sigma} \cdot \vec{A})(E + m - q\phi)u_B &= (E - m - q\phi)(E + m - q\phi)u_A \\ (\vec{\sigma} \cdot \vec{p} - q\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{p} - q\vec{\sigma} \cdot \vec{A})u_A &= (T - q\phi)(T + 2m - q\phi)u_A \end{aligned} \quad (\text{A.3})$$

where kinetic energy $T = E - m$

- In the non-relativistic limit $T \ll m$ (A.3) becomes

$$\begin{aligned} (\vec{\sigma} \cdot \vec{p} - q\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{p} - q\vec{\sigma} \cdot \vec{A})u_A &\approx 2m(T - q\phi)u_A \\ \left[(\vec{\sigma} \cdot \vec{p})^2 - q(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{p}) - q(\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{A}) + q^2(\vec{\sigma} \cdot \vec{A})^2 \right] u_A &\approx 2m(T - q\phi)u_A \end{aligned} \quad (\text{A.4})$$

- **Now** $\vec{\sigma} \cdot \vec{A} = \begin{pmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{pmatrix}$; $\vec{\sigma} \cdot \vec{B} = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$;

which leads to $(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot (\vec{A} \wedge \vec{B})$

and $(\vec{\sigma} \cdot \vec{A})^2 = |\vec{A}|^2$

- **The operator on the LHS of (A.4):**

$$= \vec{p}^2 - q \left[\vec{A} \cdot \vec{p} + i\vec{\sigma} \cdot \vec{A} \wedge \vec{p} + \vec{p} \cdot \vec{A} + i\vec{\sigma} \cdot \vec{p} \wedge \vec{A} \right] + q^2 \vec{A}^2$$

$$= (\vec{p} - q\vec{A})^2 - iq\vec{\sigma} \cdot [\vec{A} \wedge \vec{p} + \vec{p} \wedge \vec{A}]$$

$$= (\vec{p} - q\vec{A})^2 - q^2 \vec{\sigma} \cdot [\vec{A} \cdot \vec{\nabla} + \vec{\nabla} \cdot \vec{A}] \quad \vec{p} = -i\vec{\nabla}$$

$$= (\vec{p} - q\vec{A})^2 - q\vec{\sigma} \cdot (\vec{\nabla} \wedge \vec{A}) \quad (\vec{\nabla} \wedge \vec{A})\psi = \vec{\nabla} \wedge (\vec{A}\psi) + \vec{A} \wedge (\vec{\nabla}\psi)$$

$$= (\vec{p} - q\vec{A})^2 - q\vec{\sigma} \cdot \vec{B} \quad \vec{B} = \vec{\nabla} \wedge \vec{A}$$

- ★ **Substituting back into (A.4) gives the Schrödinger-Pauli equation for the motion of a non-relativistic spin 1/2 particle in an EM field**

$$\left[\frac{1}{2m} (\vec{p} - q\vec{A})^2 - \frac{q}{2m} \vec{\sigma} \cdot \vec{B} + q\phi \right] u_A = T u_A$$

$$\left[\frac{1}{2m} (\vec{p} - q\vec{A})^2 - \frac{q}{2m} \vec{\sigma} \cdot \vec{B} + q\phi \right] u_A = T u_A$$

- Since the energy of a magnetic moment in a field \vec{B} is $-\vec{\mu} \cdot \vec{B}$ we can identify the intrinsic magnetic moment of a spin $\frac{1}{2}$ particle to be:

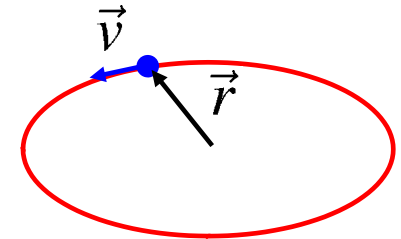
$$\vec{\mu} = \frac{q}{2m} \vec{\sigma}$$

In terms of the spin: $\vec{S} = \frac{1}{2} \vec{\sigma}$

$$\vec{\mu} = \frac{q}{m} \vec{S}$$

- Classically, for a charged particle current loop

$$\mu = \frac{q}{2m} \vec{L}$$



- The intrinsic magnetic moment of a spin half Dirac particle is twice that expected from classical physics. This is often expressed in terms of the **gyromagnetic** ratio is $g=2$.

$$\vec{\mu} = g \frac{q}{2m} \vec{S}$$

Appendix IV : Covariance of Dirac Equation

non-examinable

- For a Lorentz transformation we wish to demonstrate that the Dirac Equation is covariant i.e.

$$i\gamma^\mu \partial_\mu \psi = m\psi \quad (\text{A.5})$$

transforms to

$$i\gamma^\mu \partial'_\mu \psi' = m\psi' \quad (\text{A.6})$$

where

$$\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu} = \left(\frac{\partial}{\partial t'}, \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'} \right)$$

and

$$\psi'(x') = S\psi(x) \quad \text{is the transformed spinor.}$$

- The covariance of the Dirac equation will be established if the 4x4 matrix S exists.
- Consider a Lorentz transformation with the primed frame moving with velocity v along the x axis

where

$$\partial'_\mu = \Lambda_\mu^\nu \partial_\nu$$
$$\Lambda_\nu^\mu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With this transformation equation (A.6)

$$i\gamma^\nu \partial'_\nu \psi' = m\psi'$$

$$\Rightarrow i\gamma^\nu \Lambda_\nu^\mu \partial_\mu S\psi = mS\psi$$

which should be compared to the matrix S multiplying (A.5)

$$iS\gamma^\mu \partial_\mu \psi = mS\psi$$

★ Therefore the covariance of the Dirac equation will be demonstrated if we can find a matrix S such that

$$i\gamma^\nu \Lambda_\nu^\mu \partial_\mu S\psi = iS\gamma^\mu \partial_\mu \psi$$

$$\Rightarrow \gamma^\nu \Lambda_\nu^\mu S \partial_\mu \psi = S\gamma^\mu \partial_\mu \psi$$

$$\Rightarrow \boxed{S\gamma^\mu = \gamma^\nu S \Lambda_\nu^\mu} \quad (\text{A.7})$$

- Considering each value of $\mu = 0, 1, 2, 3$

$$\left. \begin{aligned} S\gamma^0 &= \gamma\gamma^0 S - \beta\gamma\gamma^1 S \\ S\gamma^1 &= -\beta\gamma\gamma^0 S + \gamma\gamma^1 S \\ S\gamma^2 &= \gamma^2 S \\ S\gamma^3 &= \gamma^3 S. \end{aligned} \right\} \quad \begin{array}{l} \text{where } \gamma = (1 - \beta^2)^{-1/2} \\ \text{and } \beta = v/c \end{array}$$

- It is easy (although tedious) to demonstrate that the matrix:

$$S = aI + b\gamma^0\gamma^1 \quad \text{with} \quad a = \sqrt{\frac{1}{2}(\gamma + 1)}, \quad b = \sqrt{\frac{1}{2}(\gamma - 1)}$$

satisfies the above simultaneous equations

NOTE: For a transformation along in the $-x$ direction $b = -\sqrt{\frac{1}{2}(\gamma - 1)}$

- ★ To summarise, under a Lorentz transformation a spinor $\psi(x)$ transforms to $\psi'(x') = S\psi(x)$. This transformation preserves the mathematical form of the Dirac equation

Appendix V : Transformation of Dirac Current

non-examinable

- ★ The Dirac current $j^\mu = \bar{\psi}\gamma^\mu\psi$ plays an important rôle in the description of particle interactions. Here we consider its transformation properties.
- Under a Lorentz transformation we have $\psi' = S\psi$
and for the adjoint spinor: $\bar{\psi}' = \psi'^\dagger\gamma^0 = S\psi^\dagger\gamma^0 = \psi^\dagger S^\dagger\gamma^0$
- First consider the transformation properties of $\bar{\psi}'\psi'$

$$\bar{\psi}'\psi' = \psi^\dagger S^\dagger\gamma^0 S\psi$$

where $S^\dagger = aI + b\gamma^{1\dagger}\gamma^{0\dagger} = aI - b\gamma^1\gamma^0$

giving
$$\begin{aligned} S^\dagger\gamma^0 S &= (aI - b\gamma^1\gamma^0)\gamma^0(aI + b\gamma^0\gamma^1) \\ &= a^2\gamma^0 - b^2\gamma^1\gamma^0\gamma^0\gamma^0\gamma^1 + ab\gamma^0\gamma^0\gamma^1 - b\gamma^1\gamma^0\gamma^0 \\ &= a^2\gamma^0 + b^2\gamma^0(\gamma^0)^2(\gamma^1)^2 + ab\gamma^1 - ab\gamma^1 \\ &= (a^2 - b^2)\gamma^0 \\ &= \gamma^0 \end{aligned}$$

hence
$$\bar{\psi}'\psi' = \psi^\dagger S^\dagger\gamma^0 S\psi = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi$$

- ★ The product $\bar{\psi}\psi$ is therefore a Lorentz invariant. More generally, the product $\bar{\psi}_1\psi_2$ is Lorentz covariant

★ Now consider

$$j'^{\mu} = \overline{\psi'} \gamma^{\mu} \psi'$$

$$= (\psi^{\dagger} S^{\dagger} \gamma^0) \gamma^{\mu} S \psi$$

- To evaluate this wish to express $\gamma^{\mu} S$ in terms of $S \gamma^{\mu}$

(A.7)

$$S \gamma^{\mu} = \gamma^{\nu} S \Lambda_{\nu}^{\mu}$$

→

$$S \gamma^{\mu} \Lambda_{\mu}^{\rho} = \gamma^{\nu} S \Lambda_{\nu}^{\mu} \Lambda_{\mu}^{\rho} = \gamma^{\nu} S \delta_{\nu}^{\rho} = \gamma^{\rho} S$$

where we used $\Lambda_{\nu}^{\mu} \Lambda_{\mu}^{\rho} = \delta_{\nu}^{\rho}$

- Rearranging the labels and reordering gives:

$$\gamma^{\mu} S = \Lambda_{\nu}^{\mu} S \gamma^{\nu}$$

$$j'^{\mu} = (\psi^{\dagger} S^{\dagger} \gamma^0) \gamma^{\mu} S \psi = \psi^{\dagger} S^{\dagger} \gamma^0 (\Lambda_{\nu}^{\mu} S \gamma^{\nu}) \psi$$

$$= \Lambda_{\nu}^{\mu} \psi^{\dagger} (S^{\dagger} \gamma^0 S) \gamma^{\nu} \psi = \Lambda_{\nu}^{\mu} \psi^{\dagger} \gamma^0 \gamma^{\nu} \psi$$

$$= \Lambda_{\nu}^{\mu} \overline{\psi} \gamma^{\nu} \psi = \Lambda_{\nu}^{\mu} j^{\nu}$$

→

$$\overline{\psi'} \gamma^{\mu} \psi = \Lambda_{\nu}^{\mu} \overline{\psi} \gamma^{\nu} \psi$$

- ★ Hence the Dirac current, $\overline{\psi} \gamma^{\mu} \psi$, transforms as a four-vector