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Math for Machine Learning

Linear algebra - Week 4

W4 Lesson 1

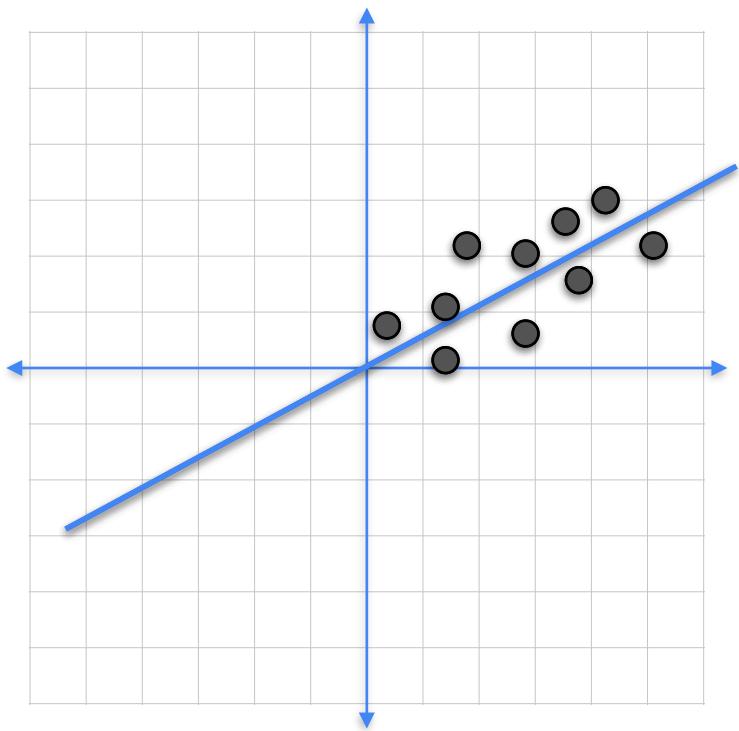


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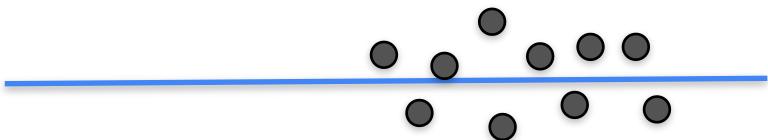
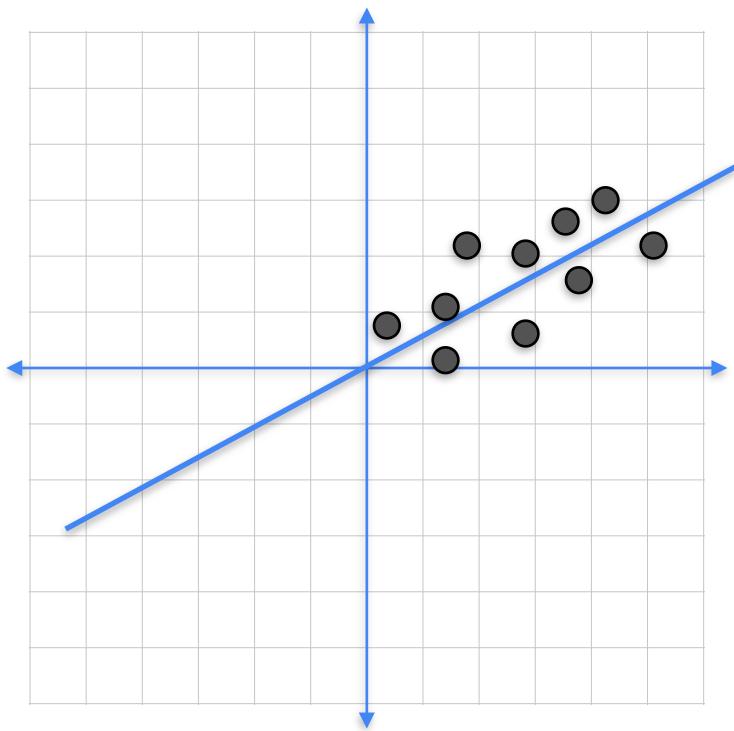
Determinants and Eigenvectors

Machine learning motivation

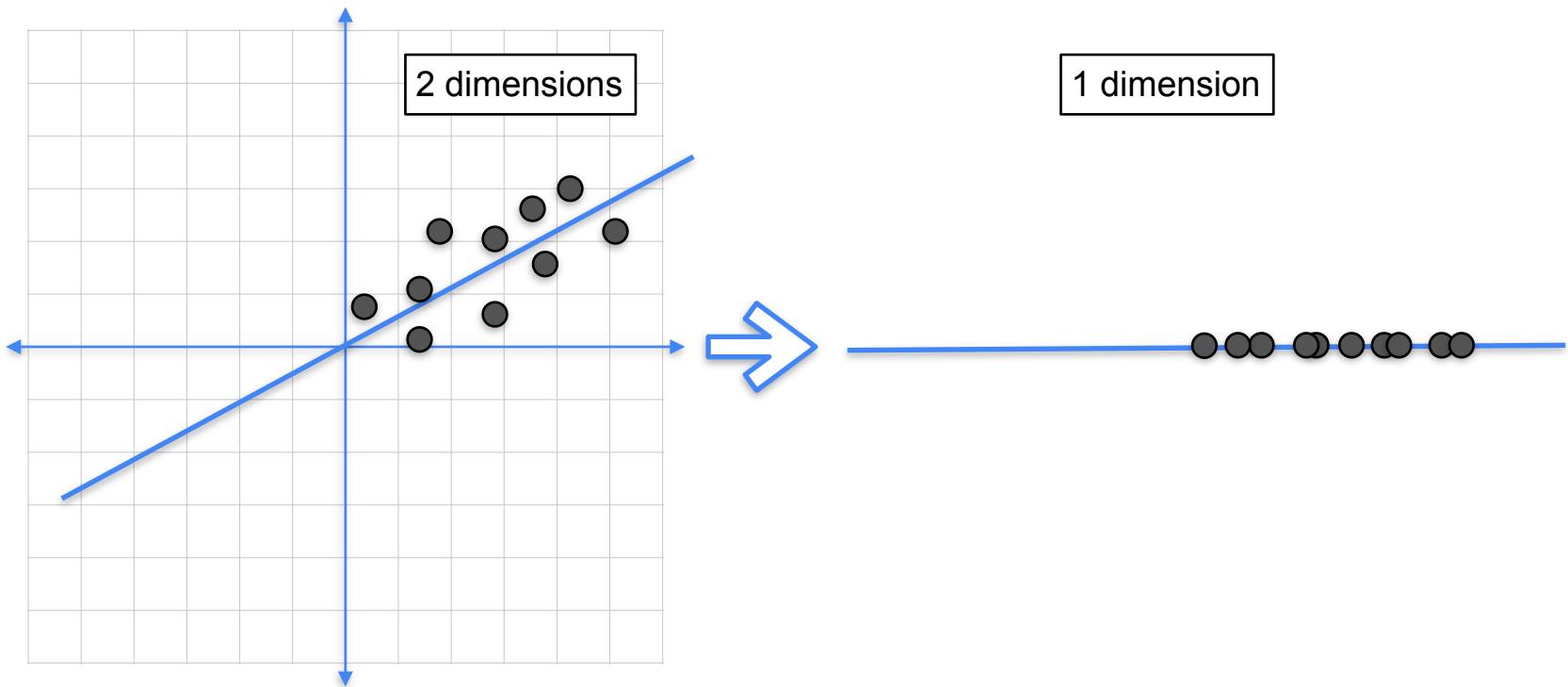
Principal Component Analysis



Principal Component Analysis

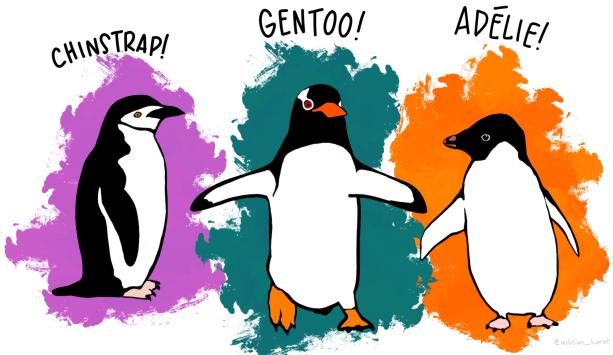


Principal Component Analysis



Principal Component Analysis

- Reduce dimensions (columns) of dataset
- Preserve as much information as possible



species	culmen_length_mm	culmen_depth_mm	flipper_length_mm	body_mass_g	PC1	PC2	species
Adelie	40.6	17.2	187.0	3475.0	1.353843	-0.422253	Adelie
Adelie	38.9	17.8	181.0	3625.0	1.760446	-0.350965	Adelie
Adelie	35.7	16.9	185.0	3150.0	2.005766	-1.113797	Adelie
Gentoo	50.0	15.3	220.0	5550.0	-2.585758	0.061768	Gentoo
Adelie	34.5	18.1	187.0	2900.0	2.438111	-0.786227	Adelie

What to expect?



Linear transformation



Characterize your transformation

What to expect?



Linear transformation



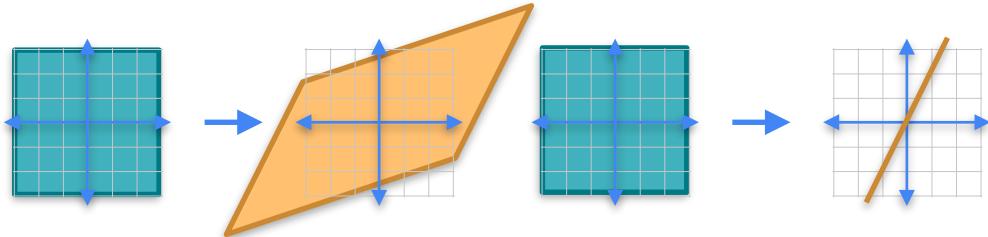
Singular / Non-singular



Characterize your transformation

Non-singular

3	1
1	2



Singular

1	1
2	2

What to expect?



Linear transformation

1

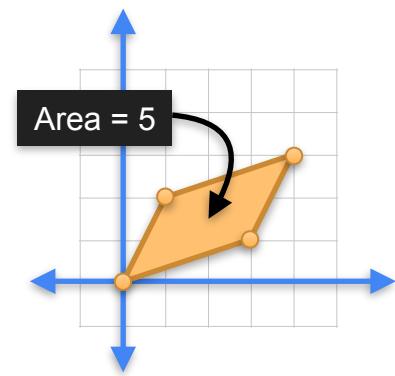
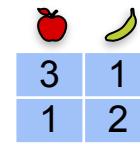
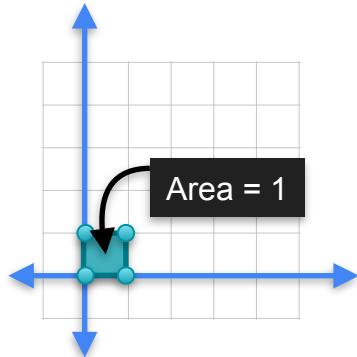
2

3

Determinant as area



Characterize your transformation



What to expect?



Linear transformation

1

2

3

Properties of determinant



Characterize your transformation

$$\begin{matrix} 3 & 1 \\ 1 & 2 \end{matrix} \cdot \begin{matrix} 1 & 1 \\ -2 & 1 \end{matrix} = \begin{matrix} 1 & 4 \\ -3 & 3 \end{matrix}$$

Det = 5 Det = 3 Det = 15
 = 5 \cdot 3

$$\begin{matrix} 3 & 1 \\ 1 & 2 \end{matrix}^{-1} = \begin{matrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{matrix}$$

$$\text{Det} = 5 \qquad \text{Det} = 0.2 = \frac{1}{5}$$

What to expect?



Linear transformation



Singular / Non-singular



Determinant as area

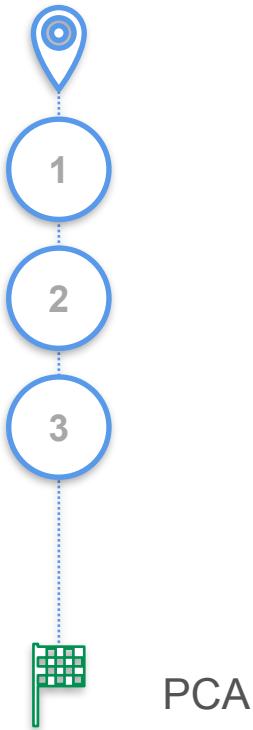
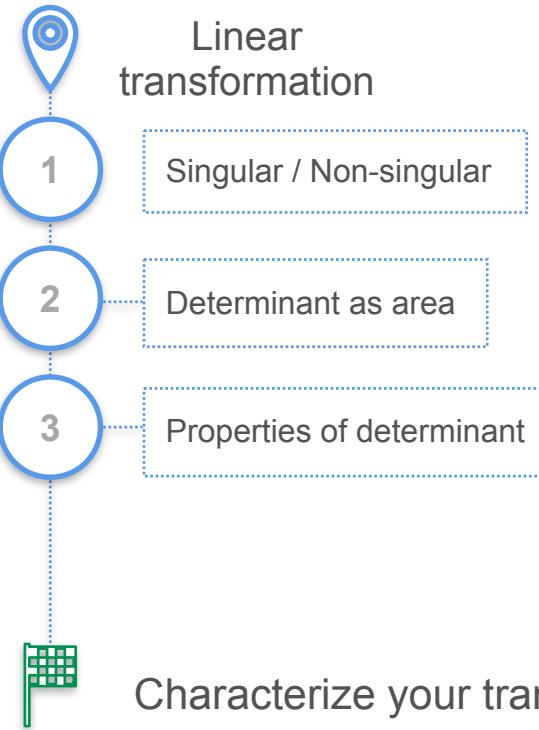


Properties of determinant

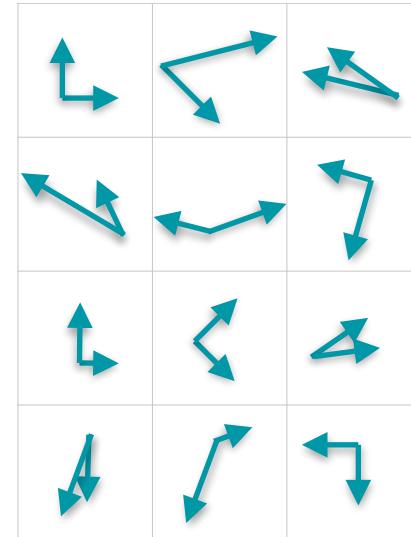
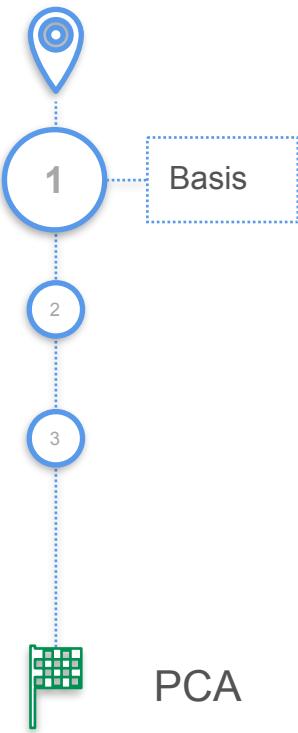
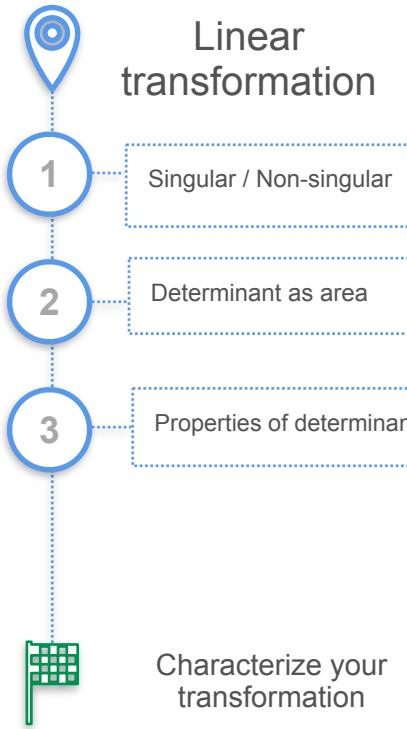


Characterize your transformation

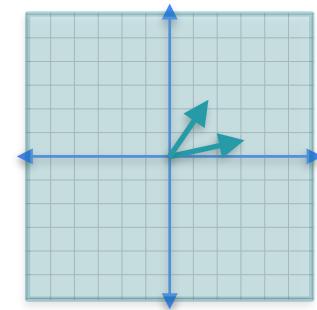
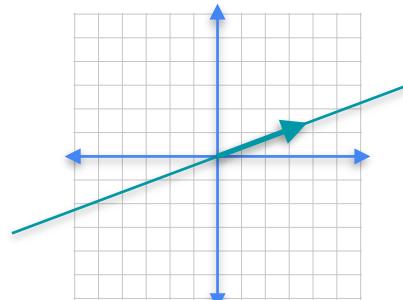
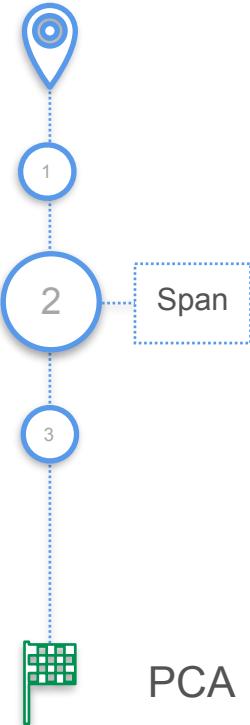
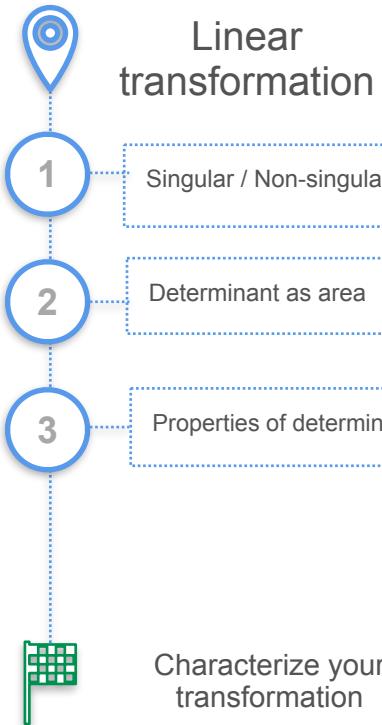
What to expect?



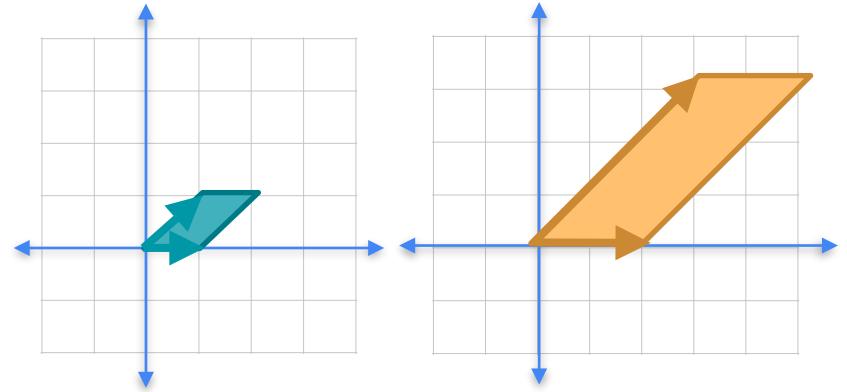
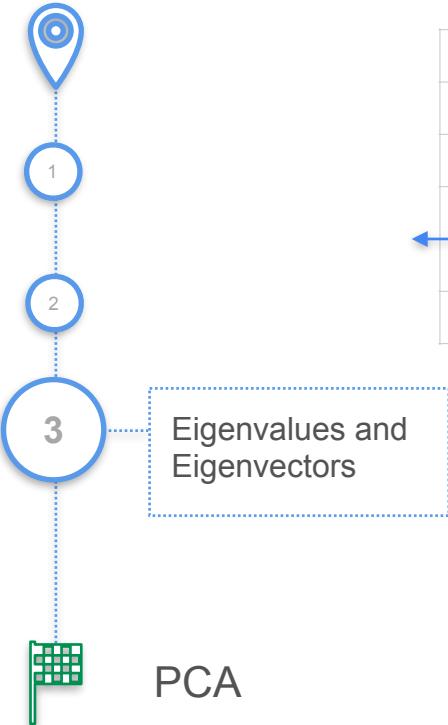
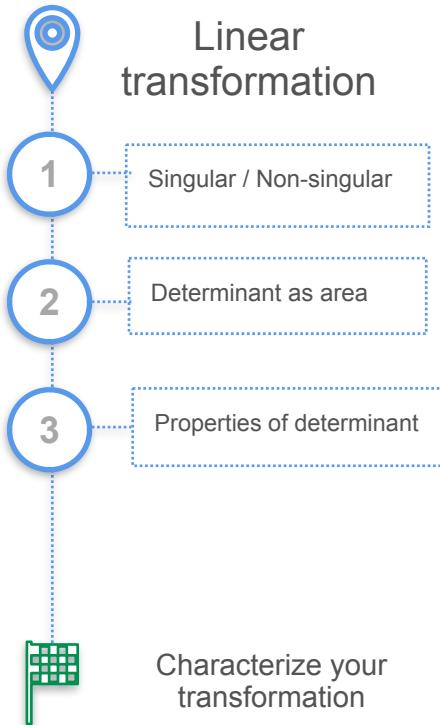
What to expect?



What to expect?



What to expect?



$$\begin{matrix} 2 & 1 & 1 \\ 0 & 3 & 0 \end{matrix} = \begin{matrix} 2 \\ 0 \end{matrix}$$

$(1,0) \rightarrow (2,0)$

$$A v_1 = \lambda_1 v_1$$

What to expect?



Linear transformation

1

Singular / Non-singular

2

Determinant as area

3

Properties of determinant



Characterize your transformation



1

Basis

2

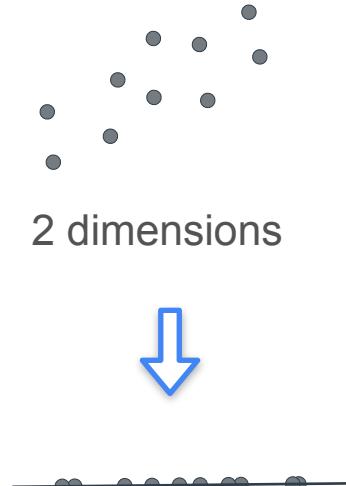
Span

3

Eigenvalues and
Eigenvectors



PCA



2 dimensions



1 dimension

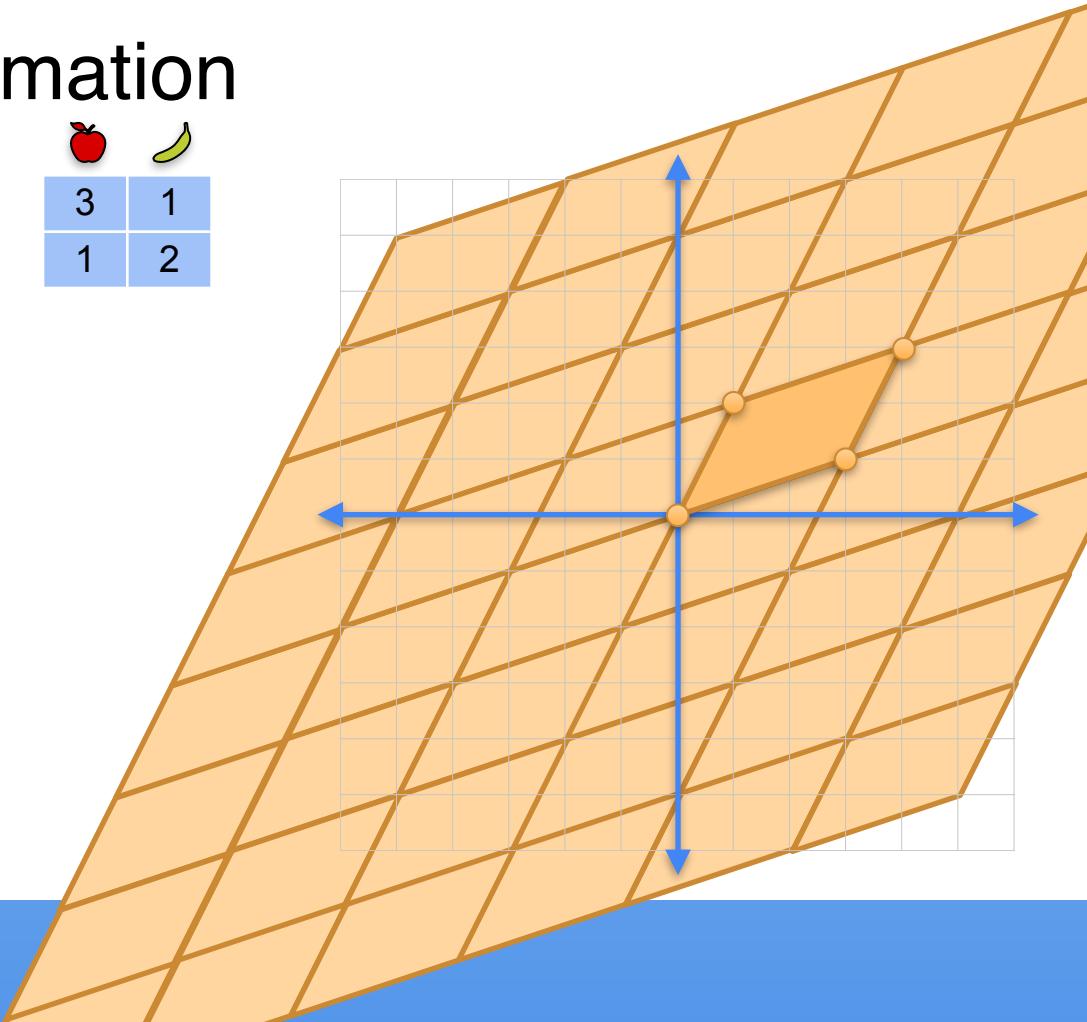
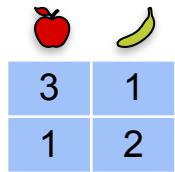
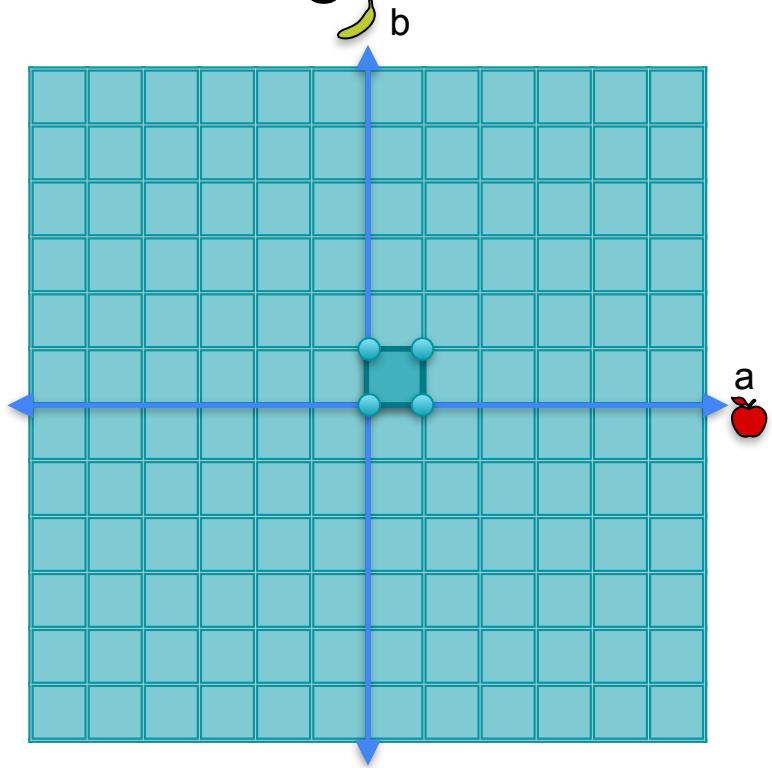


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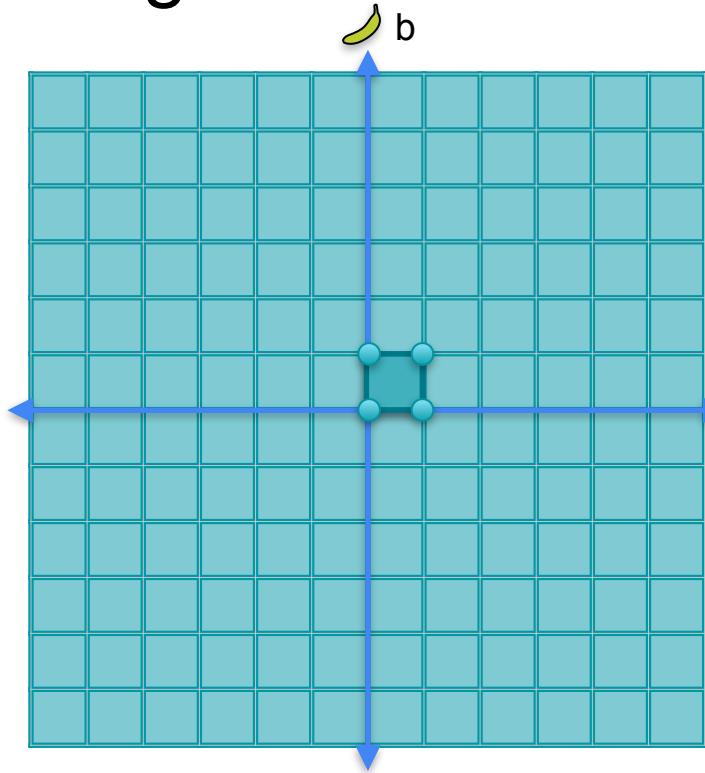
Determinants and Eigenvectors

Singularity and rank of linear transformations

Non-singular transformation



Singular transformation

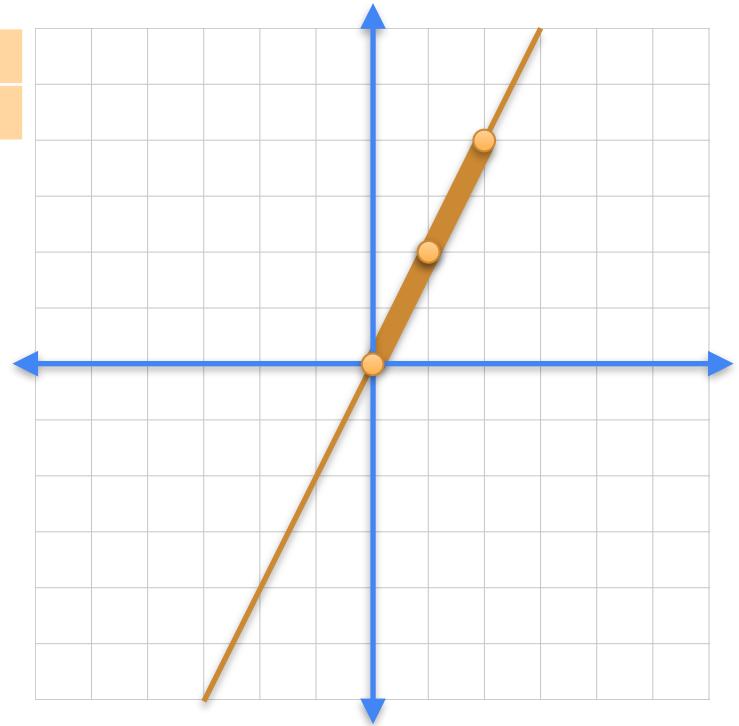


Two icons: an apple and a banana. Below them is a matrix multiplication example:

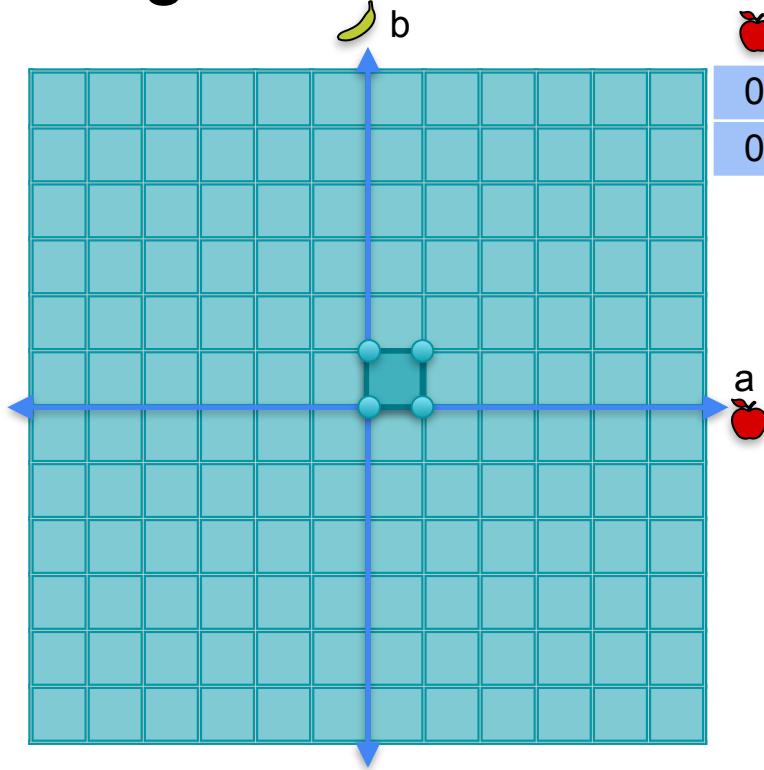
$$\begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix} \times \begin{matrix} 1 \\ 1 \end{matrix} = \begin{matrix} 2 \\ 4 \end{matrix}$$

Mapping table:

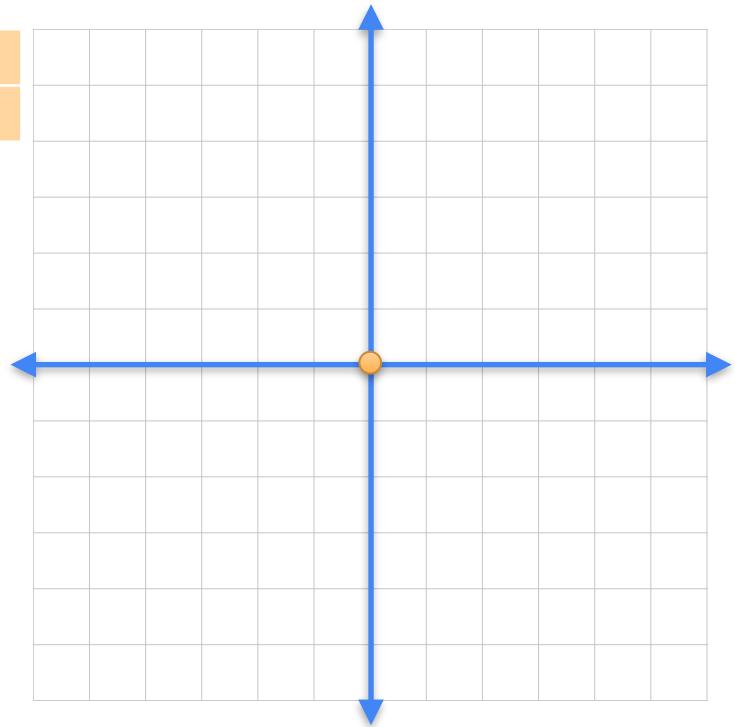
(0,0)	\rightarrow	(0,0)
(1,0)	\rightarrow	(1,2)
(0,1)	\rightarrow	(1,2)
(1,1)	\rightarrow	(2,4)



Singular transformation



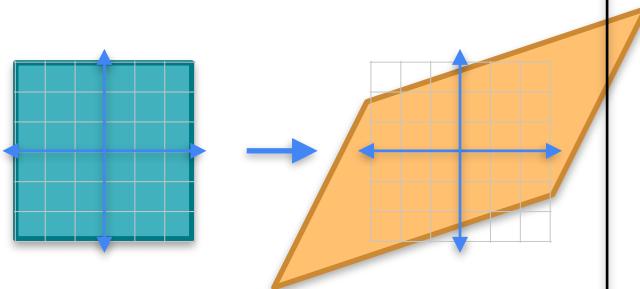
$$\begin{array}{cc} \text{apple} & \text{banana} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} a \\ b \end{matrix} = \begin{matrix} 0 \\ 0 \end{matrix} \end{array}$$



Singular and non-singular transformations

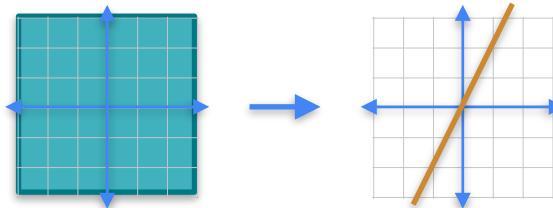
Non-singular

3	1
1	2



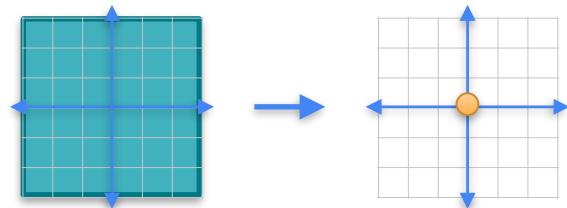
Singular

1	1
2	2



Singular

0	0
0	0



Rank of linear transformations

Rank 2

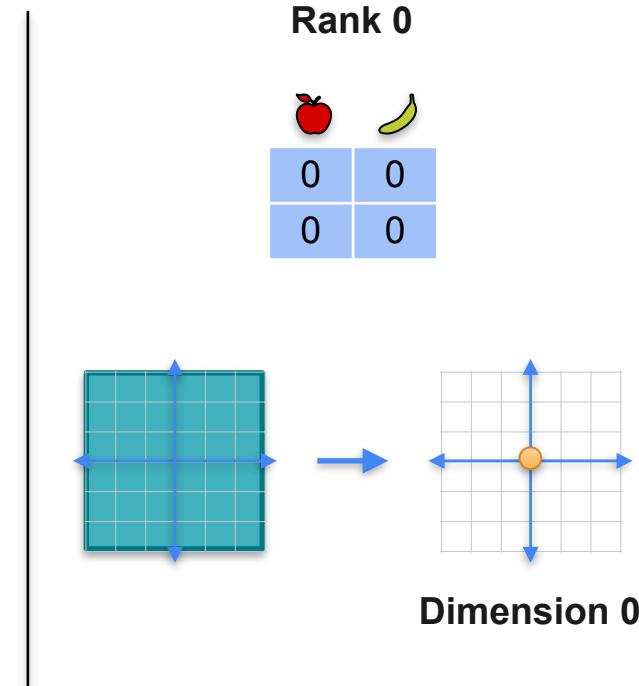
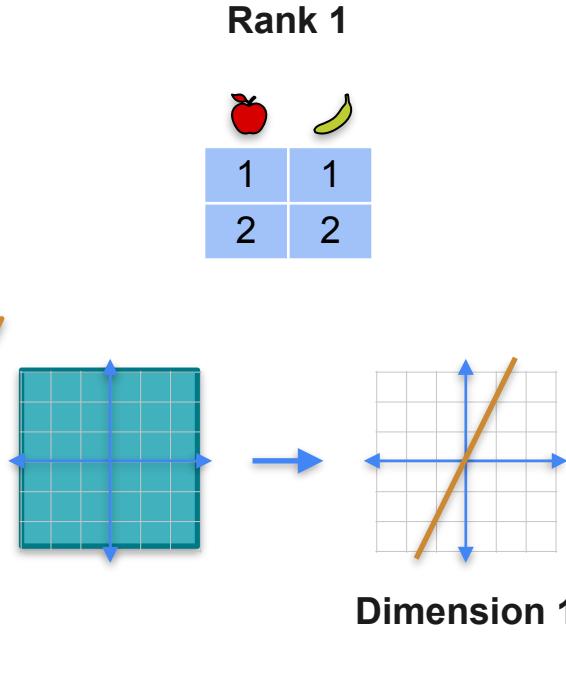
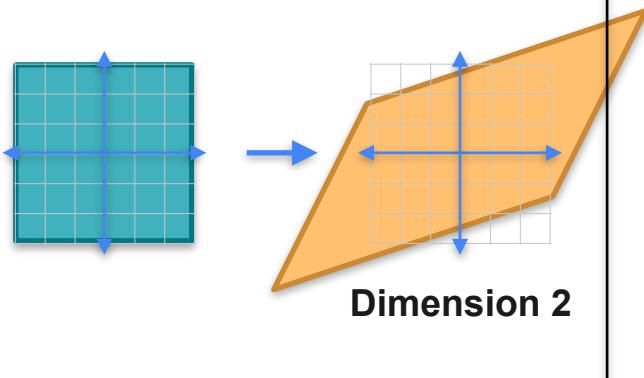
3	1
1	2

Rank 1

1	1
2	2

Rank 0

0	0
0	0



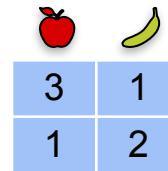
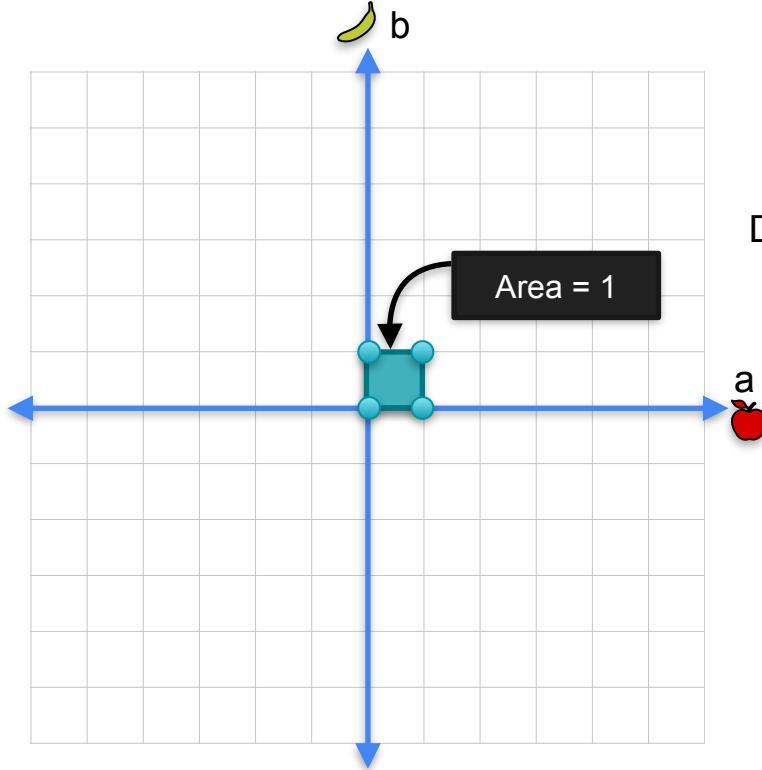


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Determinants and Eigenvectors

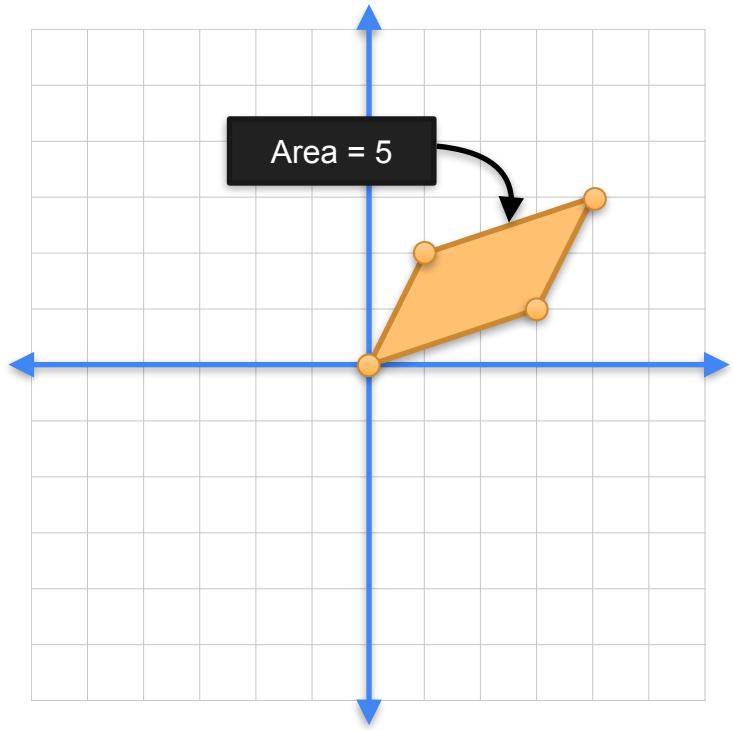
Determinant as an area

Determinant as an area

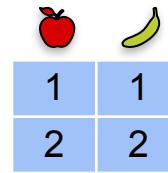
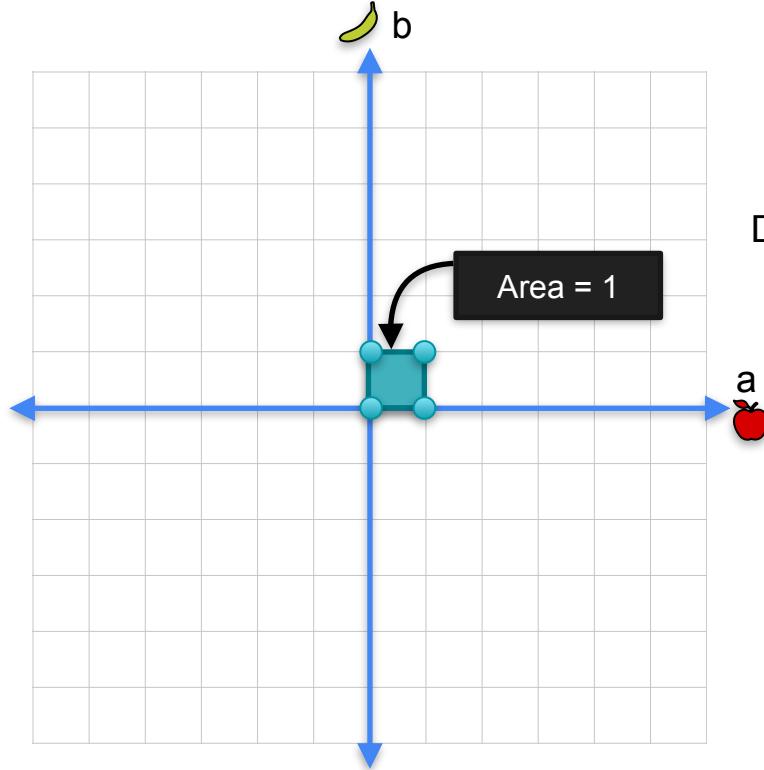


$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

$$\text{Det} = 5$$

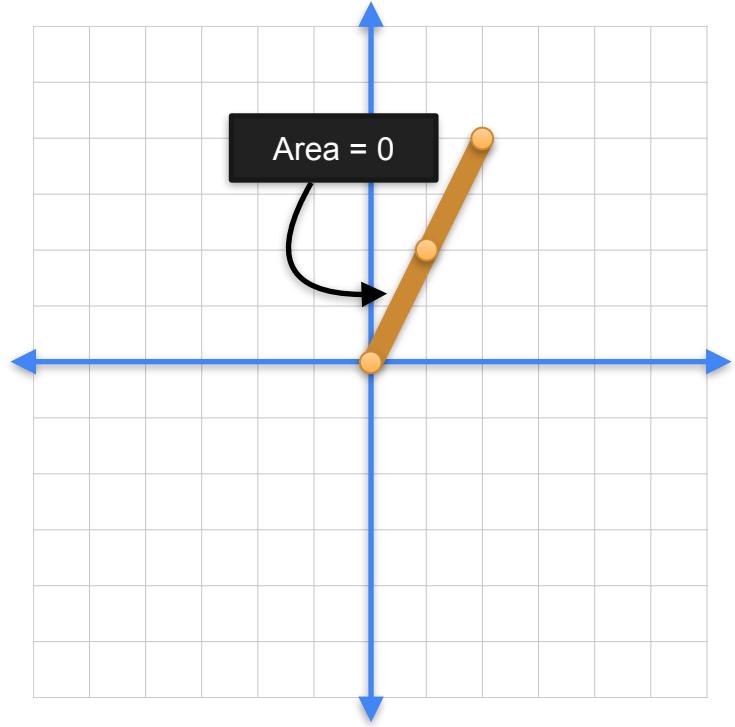


Determinant as an area

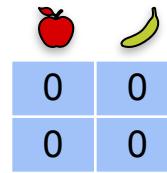
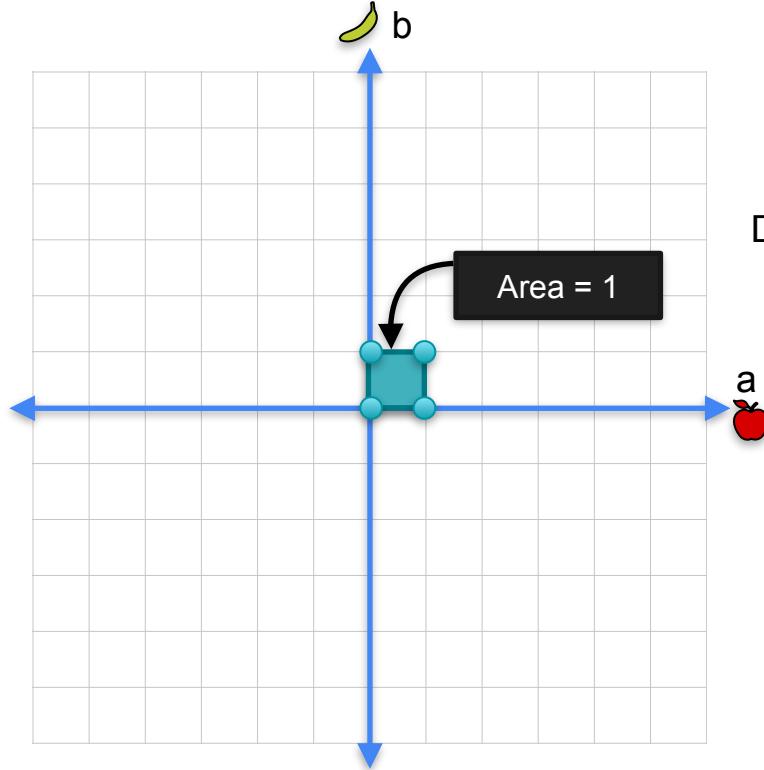


$$\text{Det} = 1 \cdot 2 - 1 \cdot 2$$

$$\text{Det} = 0$$

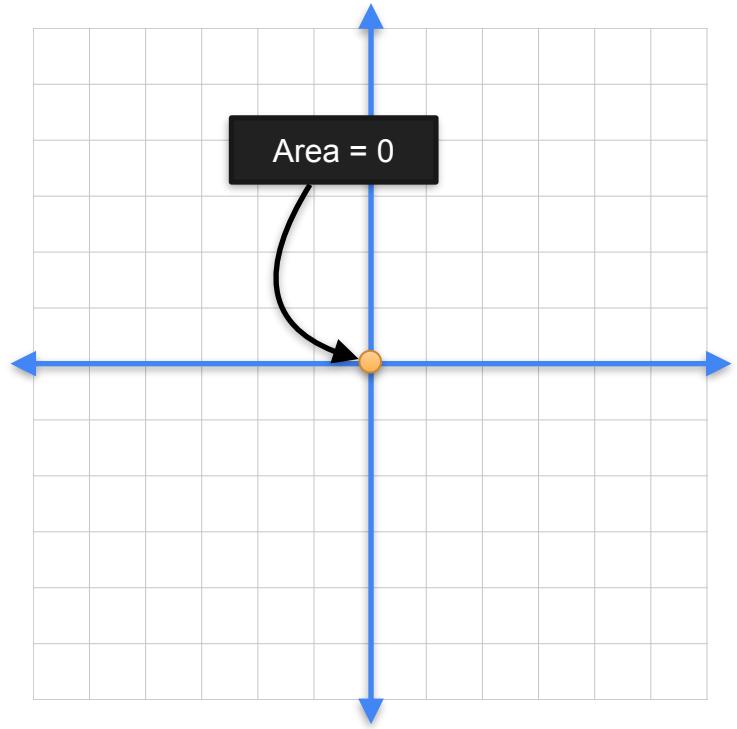


Determinant as an area



$$\text{Det} = 0 \cdot 0 - 0 \cdot 0$$

$$\text{Det} = 0$$

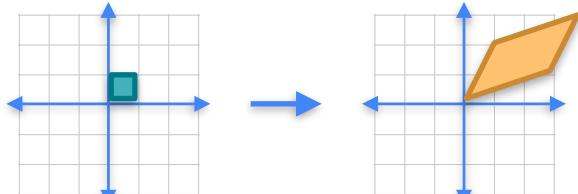


Determinant as an area

Non-singular

3	1
1	2

$$\text{Determinant} = 5$$

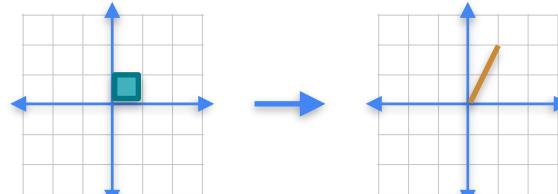


$$\text{Area} = 5$$

Singular

1	1
2	2

$$\text{Determinant} = 0$$

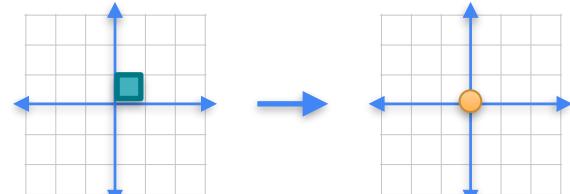


$$\text{Area} = 0$$

Singular

0	0
0	0

$$\text{Determinant} = 0$$



$$\text{Area} = 0$$

Negative determinants?

	
3	1
1	2

$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

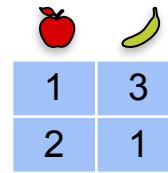
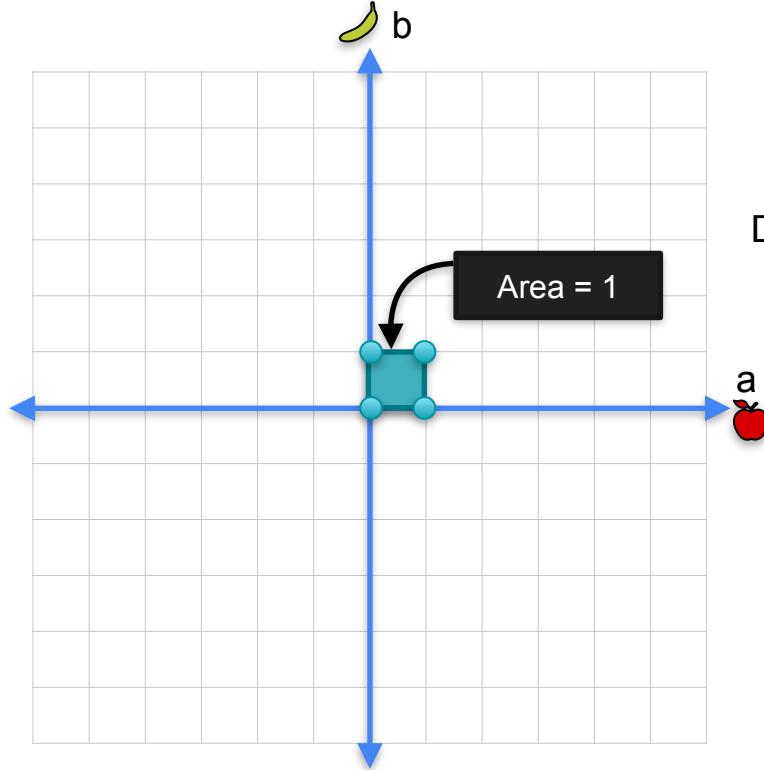
$$\text{Det} = 5$$

	
1	3
2	1

$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

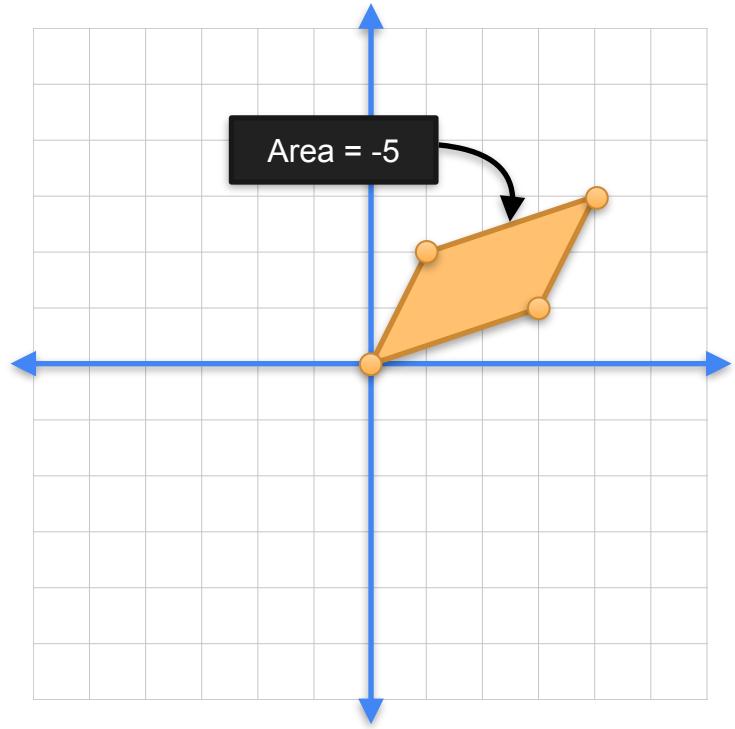
$$\text{Det} = -5$$

Determinant as an area

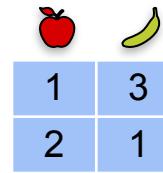
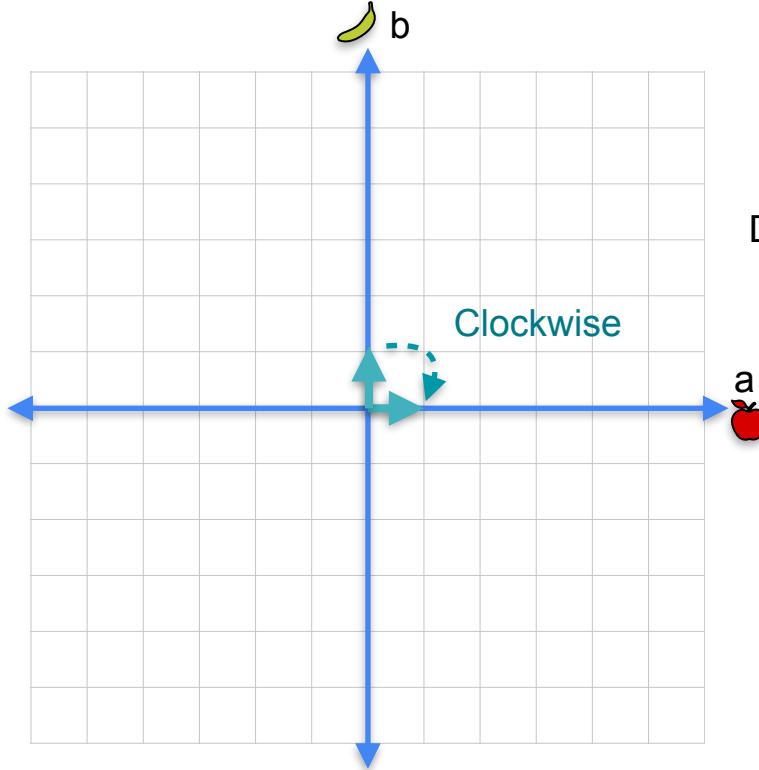


$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$



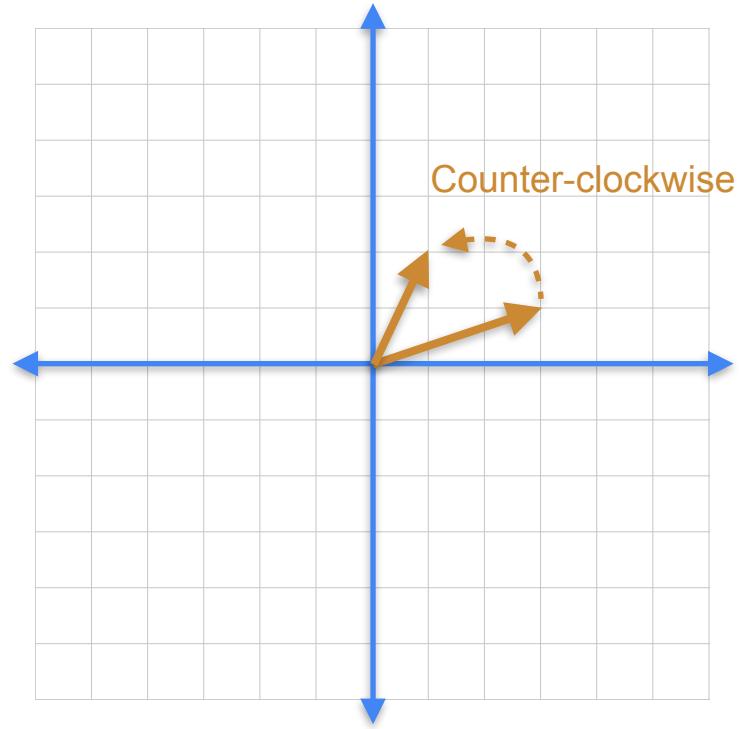
Determinant as an area



$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$

Negative





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Determinants and Eigenvectors

Determinant of a product

Determinant of a product

3	1
1	2

5	2
1	2

=

16	8
7	6

$$\det = 5$$

$$3 \cdot 2 - 1 \cdot 1$$

$$\det = 8$$

$$5 \cdot 2 - 2 \cdot 1$$

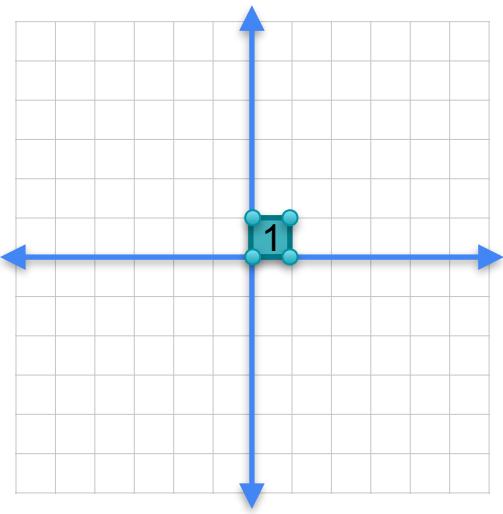
$$\det = 40$$

$$16 \cdot 6 - 8 \cdot 7$$

Determinant of a product

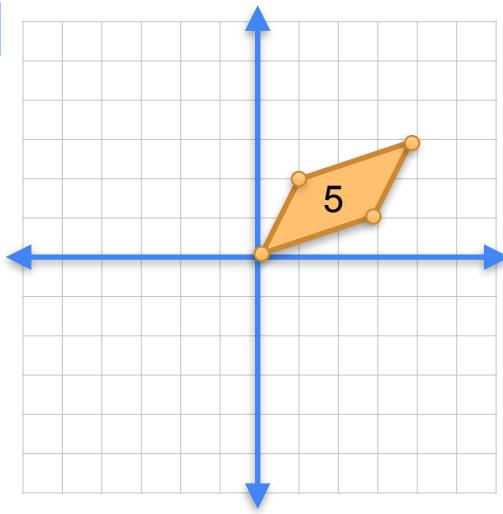
$$\det(AB) = \det(A) \det(B)$$

Determinant of a product



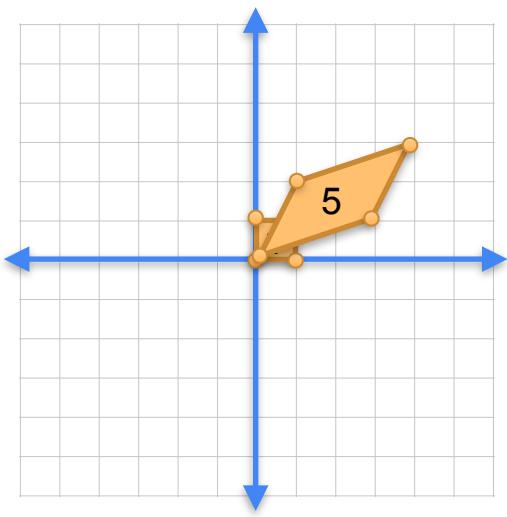
$$\begin{array}{|c|c|} \hline 3 & 1 \\ \hline 1 & 2 \\ \hline \end{array}$$

Det = 5



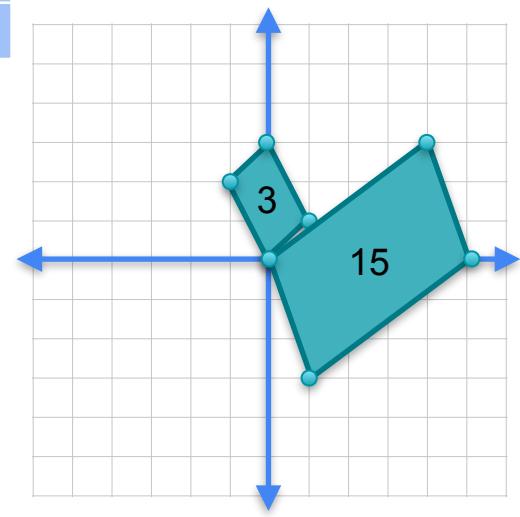
Area blows up by 5

Determinant of a product



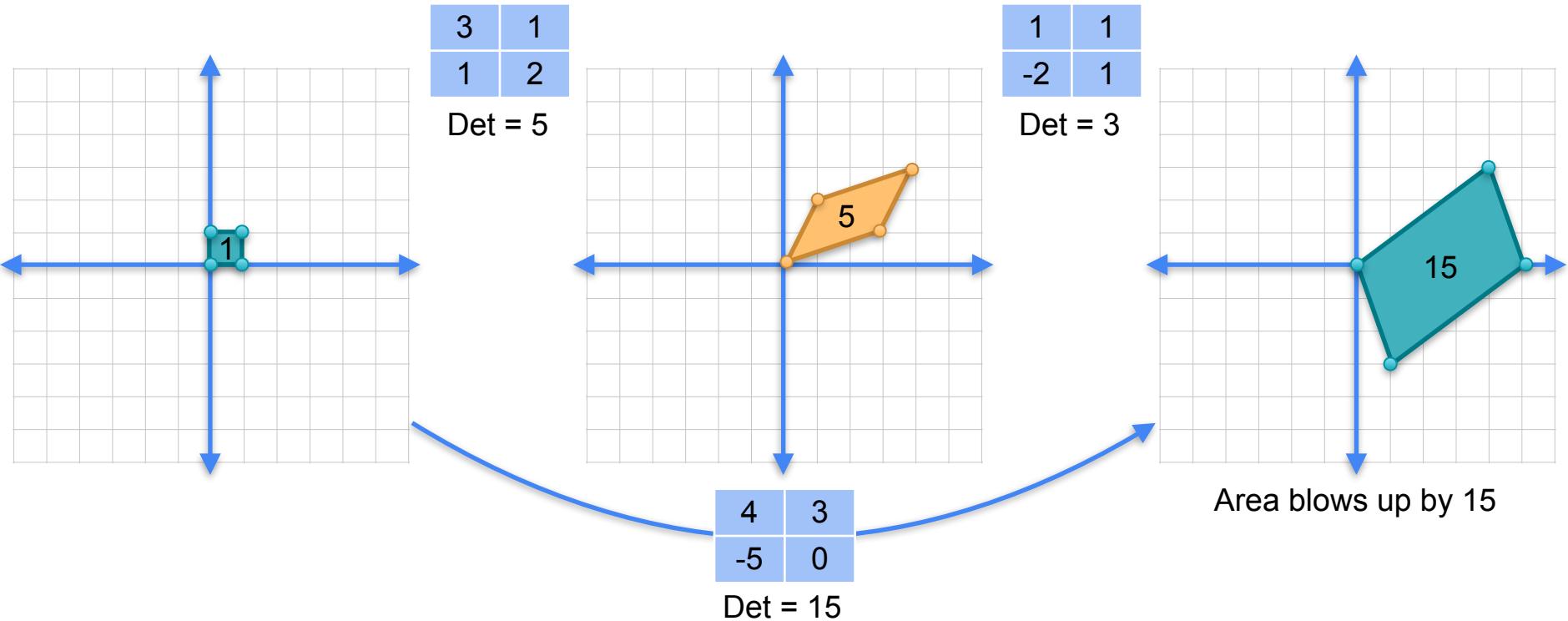
$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline -2 & 1 \\ \hline \end{array}$$

Det = 3



Area blows up by 3

Determinant of a product



Quiz

- The product of a singular and a non-singular matrix (in any order) is:
 - Singular
 - Non-singular
 - Could be either one

Solution

- If A is non-singular and B is singular, then $\det(AB) = \det(A) \times \det(B) = 0$, since $\det(B) = 0$. Therefore $\det(AB) = 0$, so AB is **singular**.

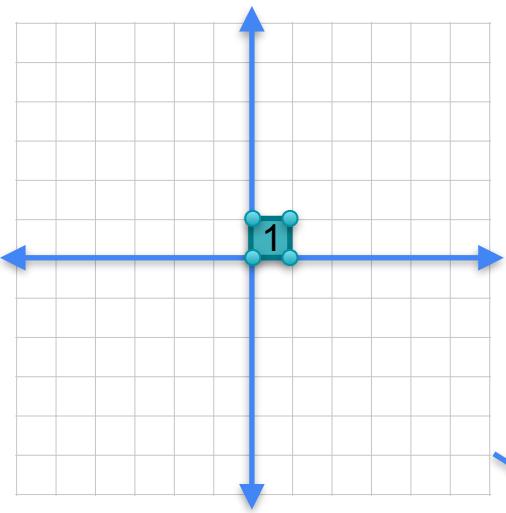
When one factor is zero

$$5 \cdot 0 = 0$$

When one factor is singular...

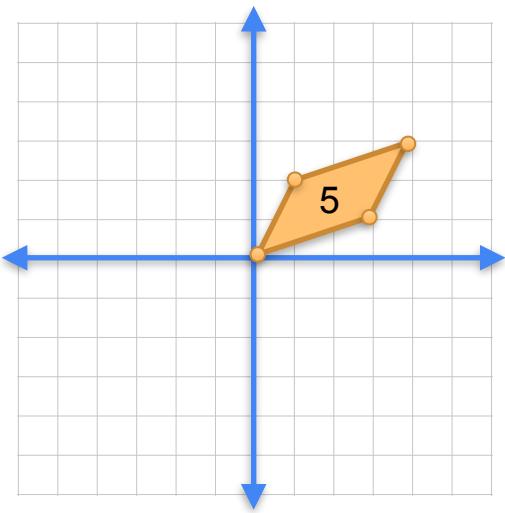
Non-singular	Singular	Singular
$\begin{matrix} 3 & 1 \\ 1 & 2 \end{matrix}$	$\begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix}$	$\begin{matrix} 4 & 8 \\ 3 & 6 \end{matrix}$
$\text{Det} = 5$	$\text{Det} = 0$	$\text{Det} = 0$

If one factor is singular...



3	1
1	2

Det = 5

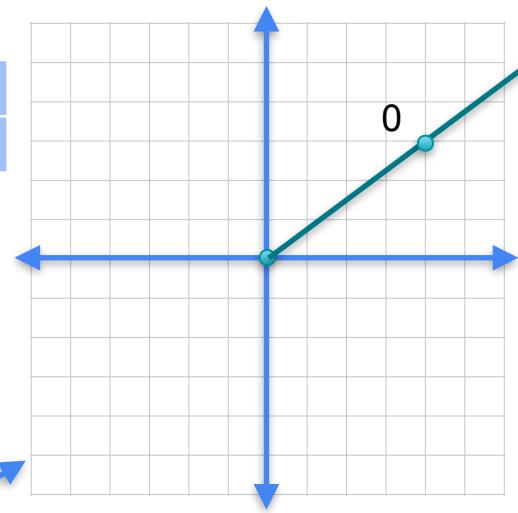


1	2
1	2

Det = 0

4	8
3	6

Det = 0



Area blows up by 0



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Determinants and Eigenvectors

Determinant of inverse

Quiz

- Find the determinants of the following matrices

0.4	-0.2
-0.2	0.6

0.25	-0.25
-0.125	0.625

Solution

$$\text{Det} \begin{array}{|c|c|} \hline 0.4 & -0.2 \\ \hline -0.2 & 0.6 \\ \hline \end{array} = (0.4)(0.6) - (-0.2)(-0.2) = 0.2$$

$$\text{Det} \begin{array}{|c|c|} \hline 0.25 & -0.25 \\ \hline -0.125 & 0.625 \\ \hline \end{array} = (0.25)(0.625) - (-0.125)(-0.25) = 0.125$$

Determinant of an inverse

$$\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}^{-1} = \begin{vmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{vmatrix}$$

$$\det = 5$$

$$5^{-1} = 0.2$$

$$\begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix}^{-1} = \begin{vmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{vmatrix}$$

$$\det = 8$$

$$8^{-1} = 0.125$$

$$\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} ? & ? \\ ? & ? \end{vmatrix}$$

$$\det = 0$$

$$0^{-1} = ???$$

Determinant of an inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Why?

$$\det(AB) = \det(A) \det(B)$$

Why is this?

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\begin{aligned} \det(AA^{-1}) &= \det(A) \det(A^{-1}) \\ \det(I) &= \det(A) \det(A^{-1}) \\ 1 &= \det(A) \det(A^{-1}) \\ 1 &= \frac{1}{\det(A)} \end{aligned}$$

Determinant of the identity matrix

$$\det \begin{array}{|cc|} \hline 1 & 0 \\ 0 & 1 \\ \hline \end{array} = 1 \cdot 1 - 0 \cdot 0 = 1$$

$$\det(I) = 1$$

W4 Lesson 2

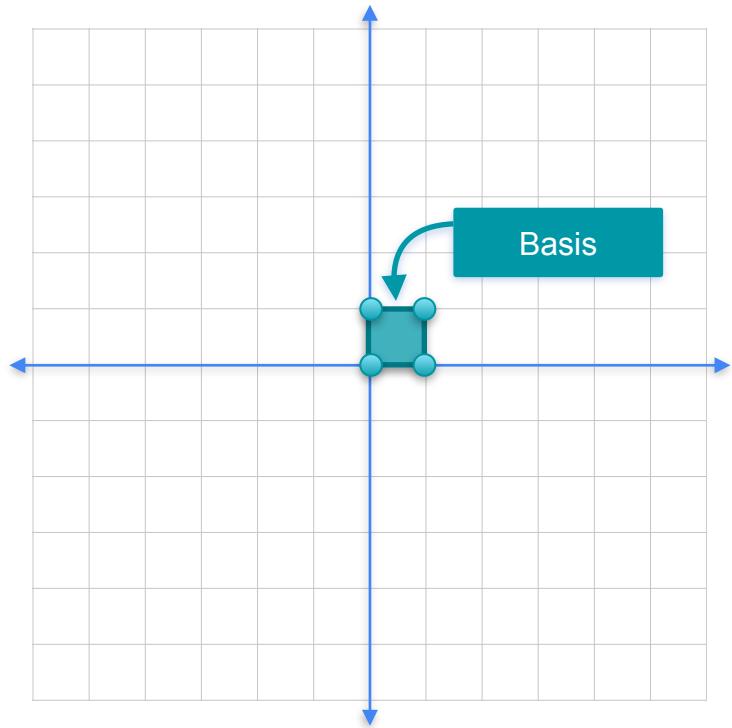


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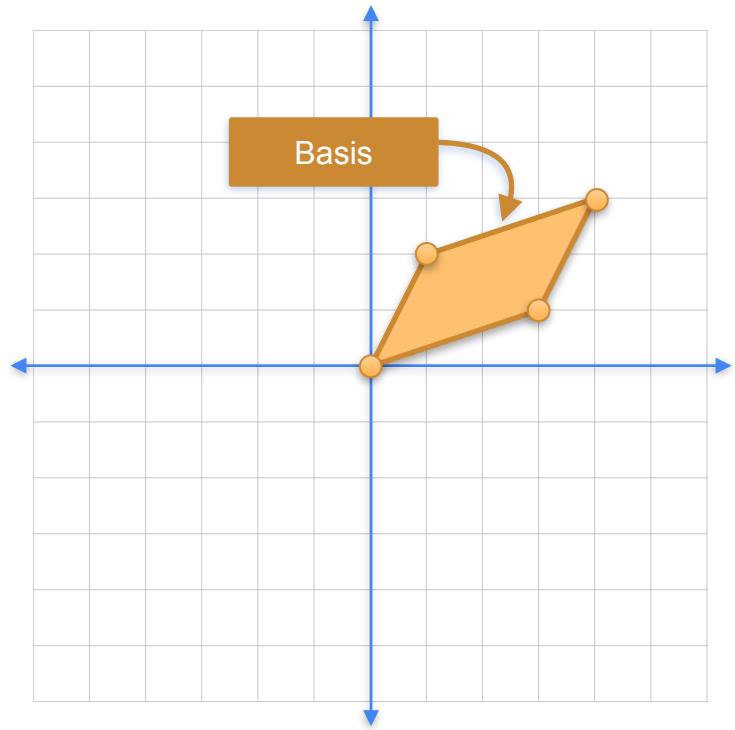
Determinants and Eigenvectors

Bases

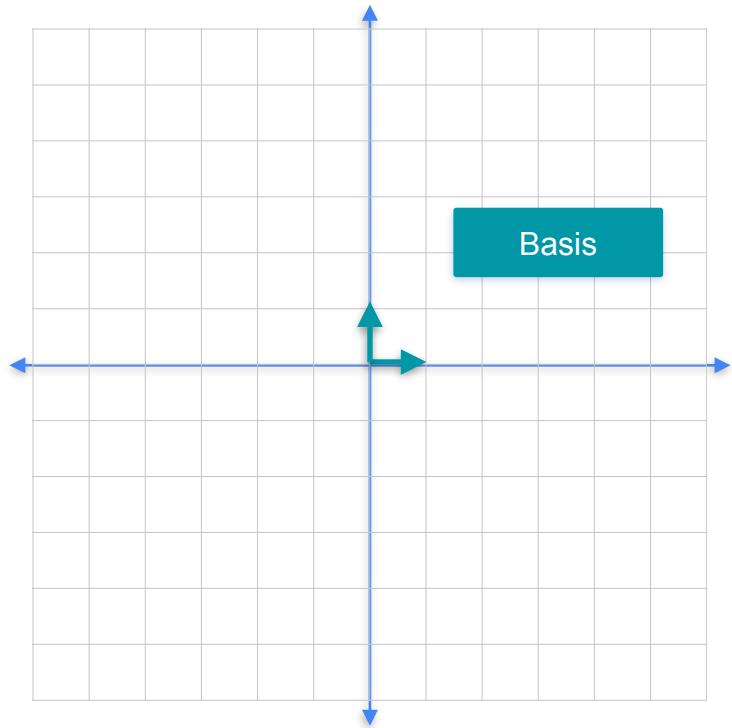
Bases



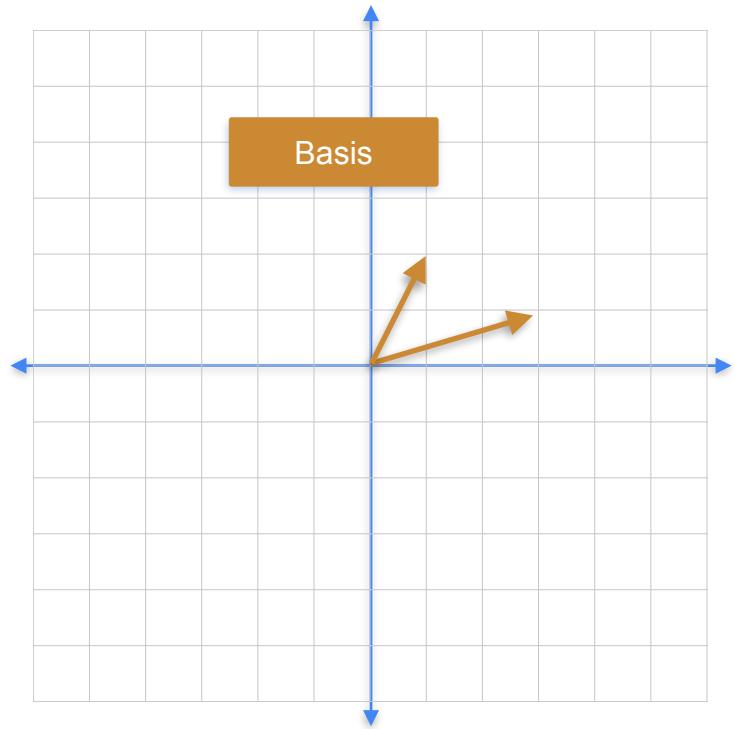
3	1
1	2



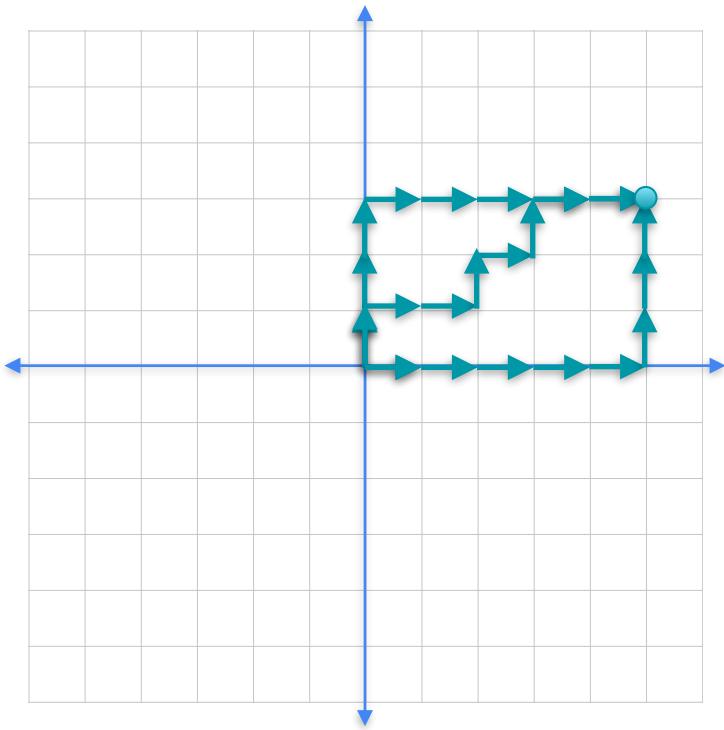
Bases



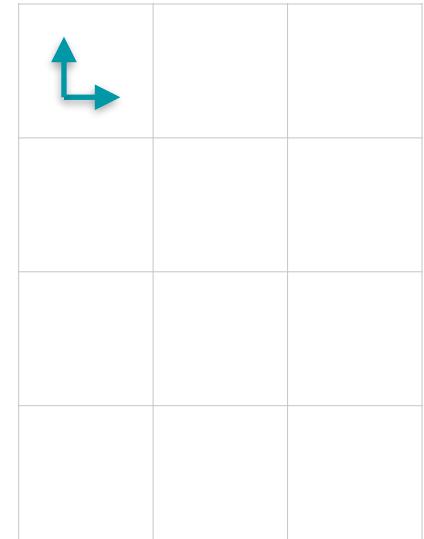
3	1
1	2



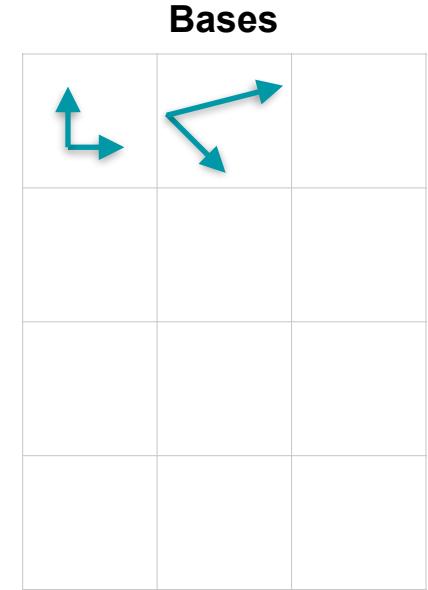
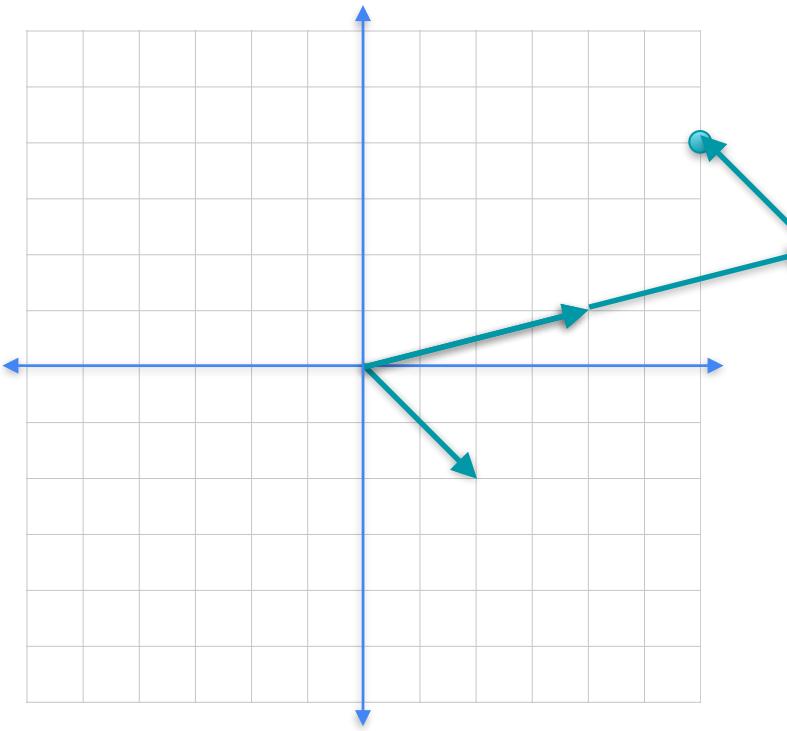
Bases



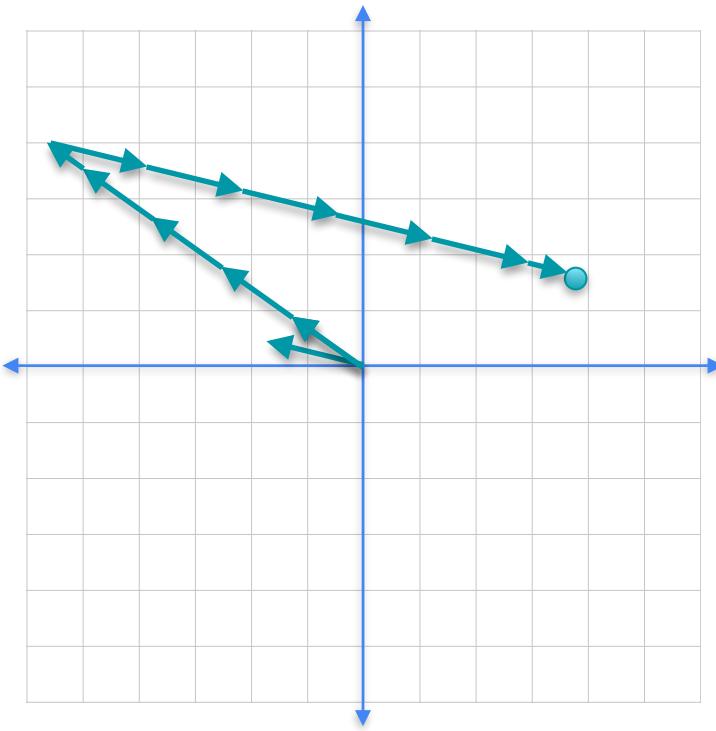
Bases



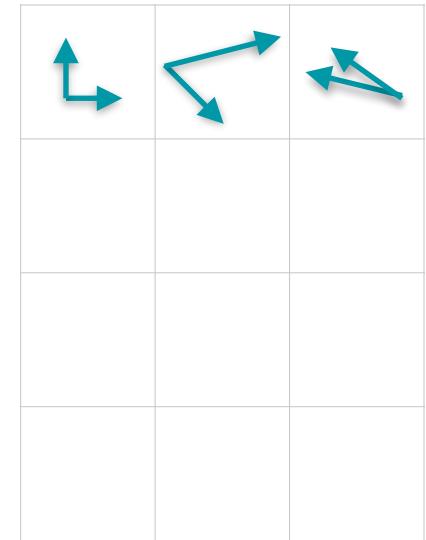
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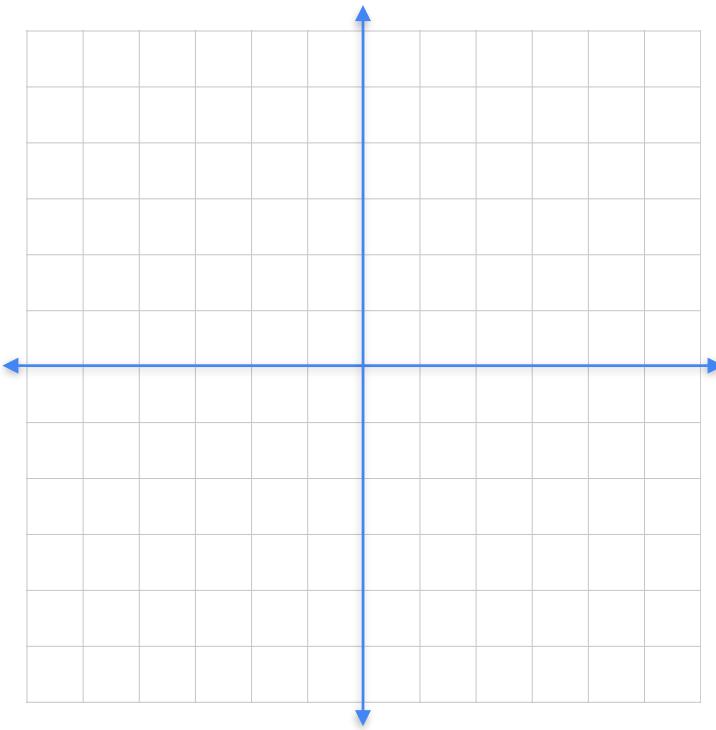
Bases



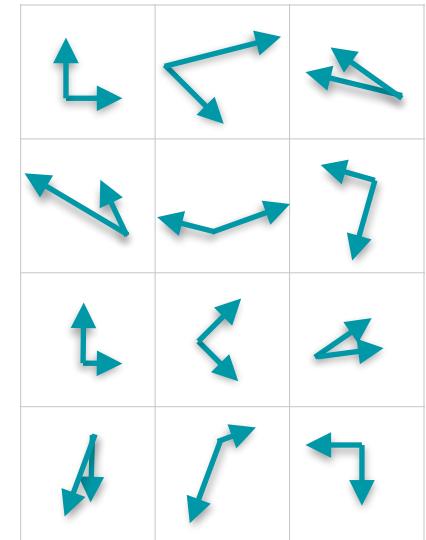
Bases



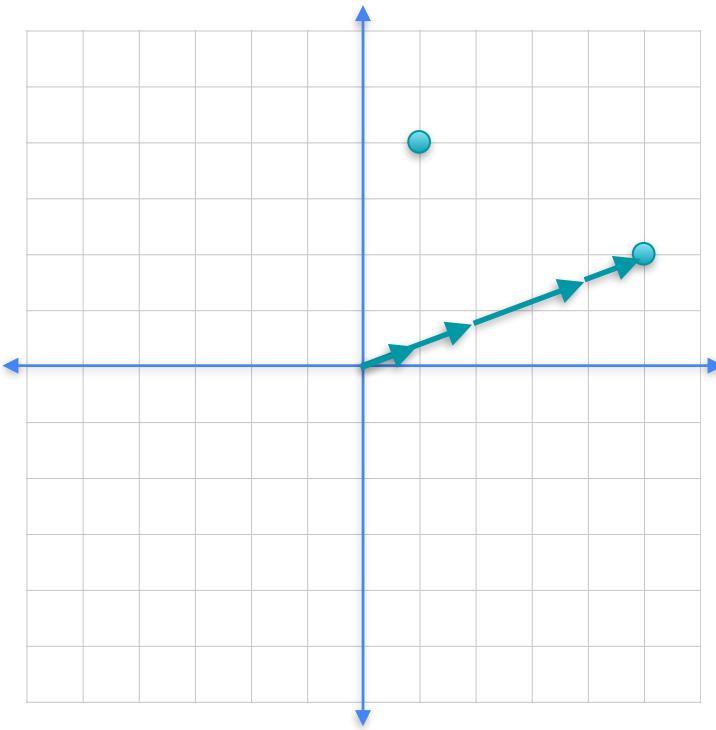
Bases



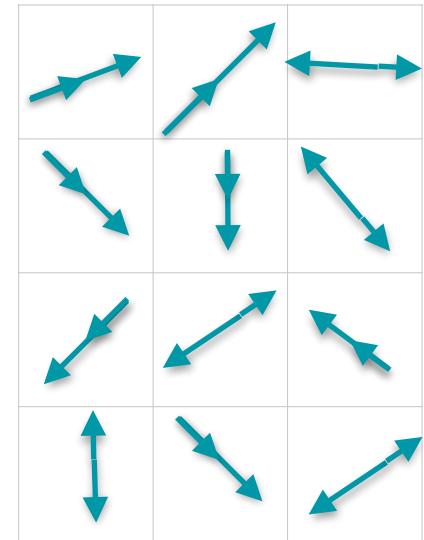
Bases



What is not a basis?



Not bases



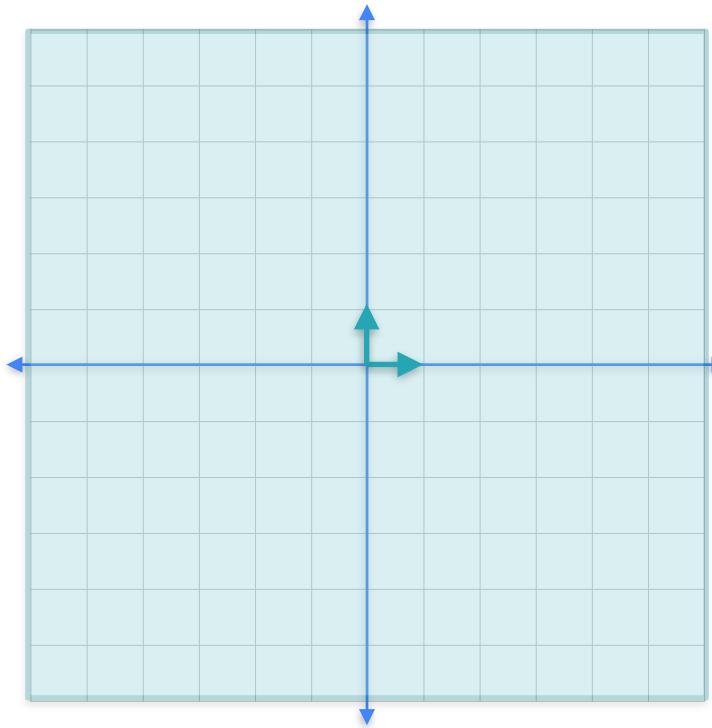


DeepLearning.AI

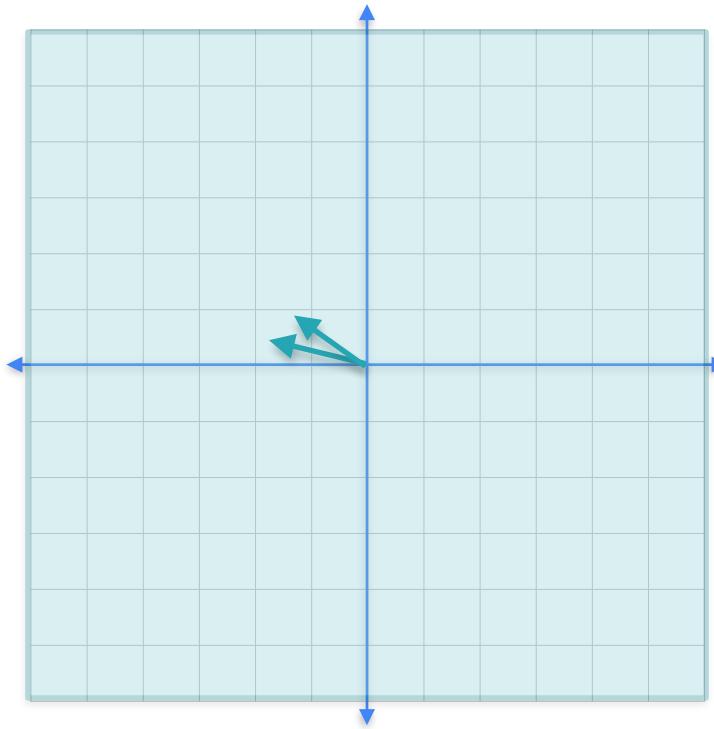
Determinants and Eigenvectors

Span

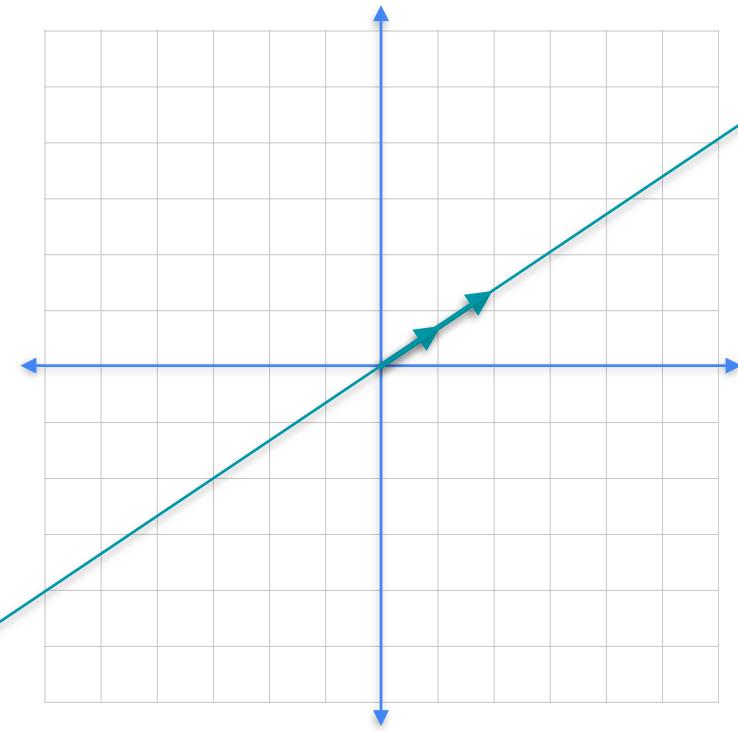
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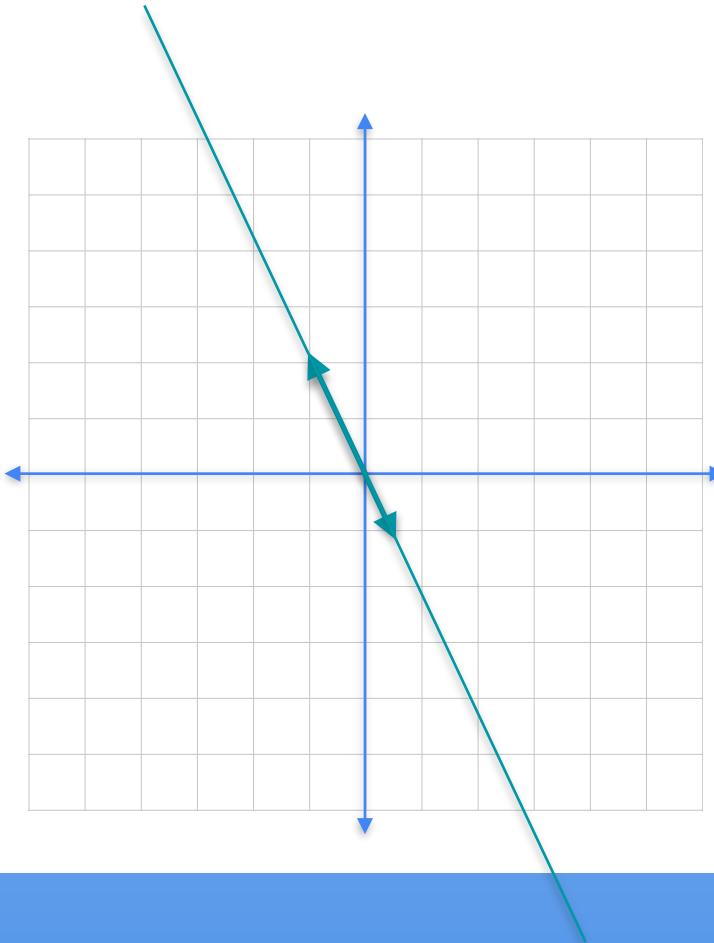
Span



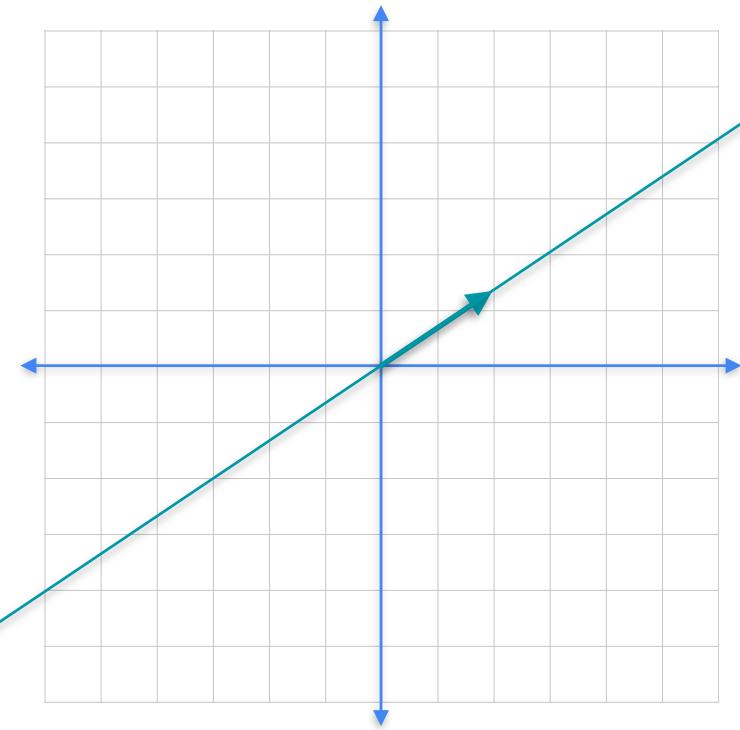
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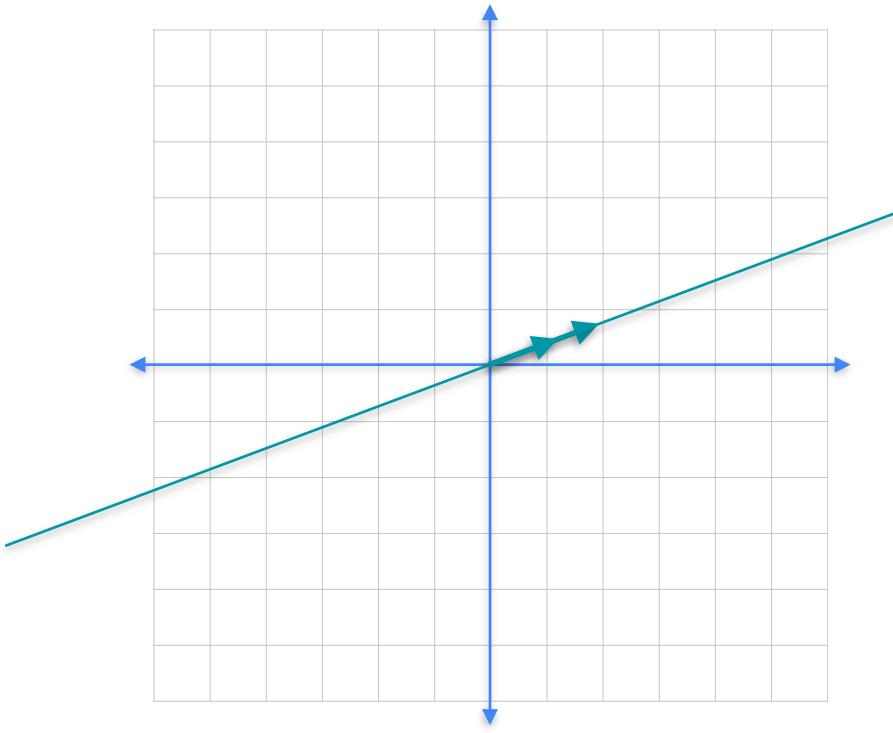
Span



Span

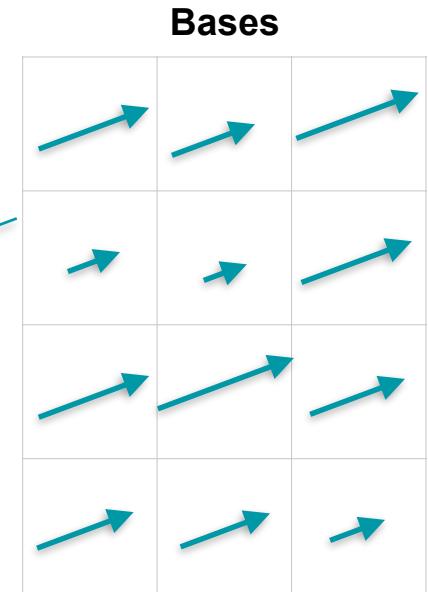
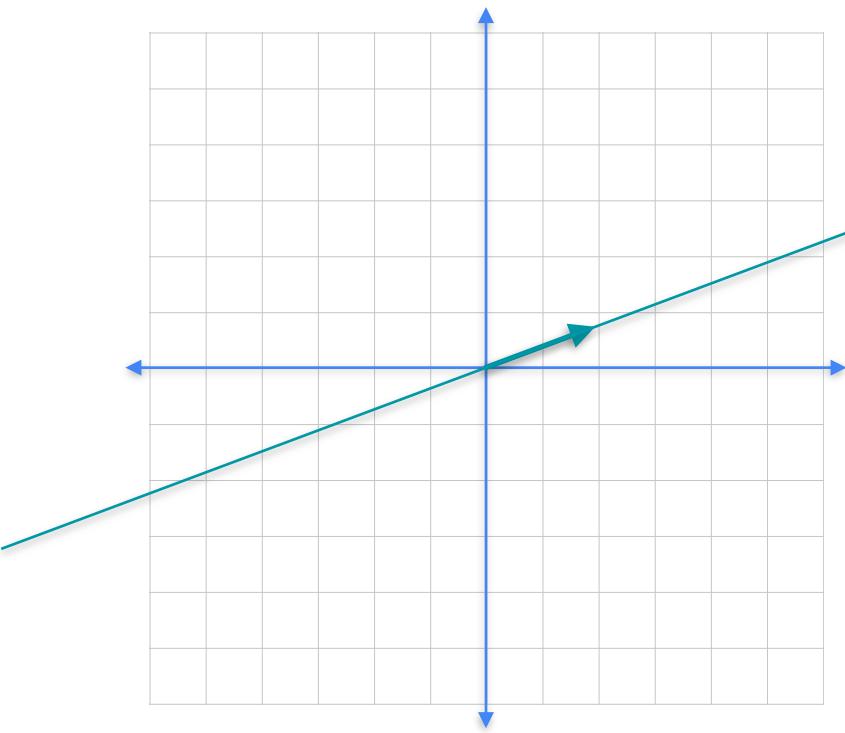


Is this a basis?

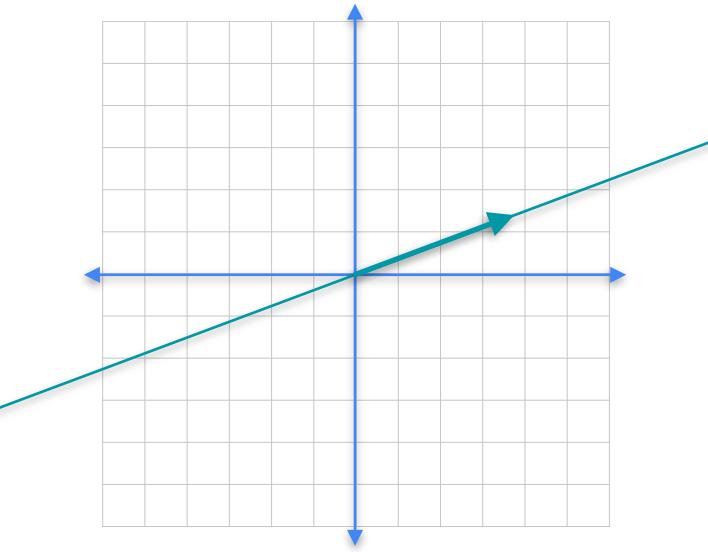


No

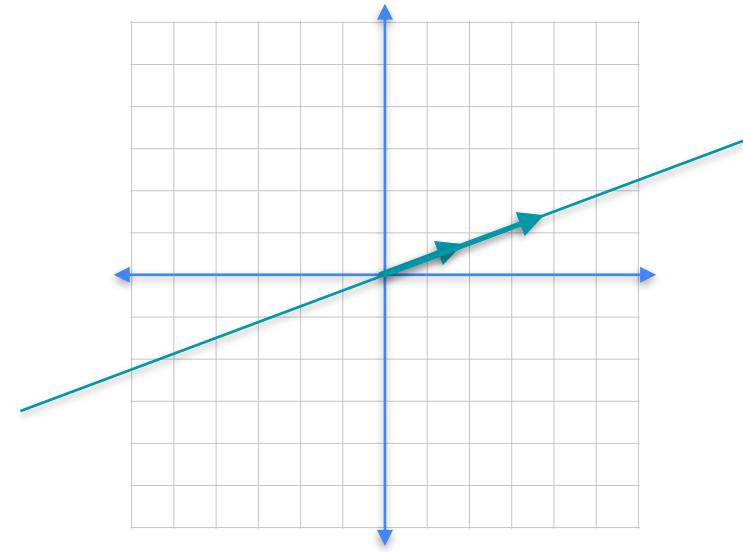
Is this a basis for something?



A basis is a minimal spanning set

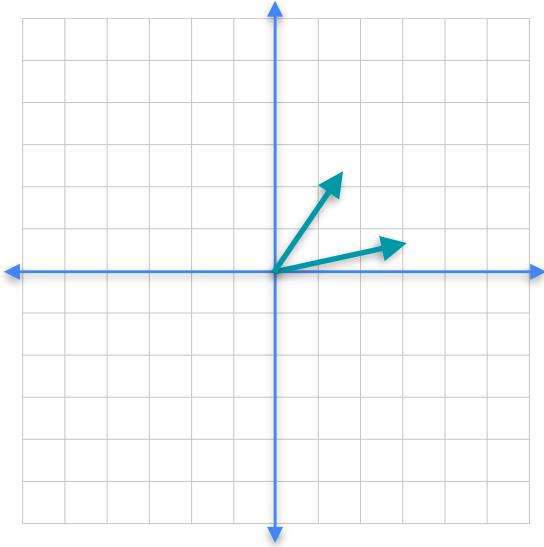


Basis

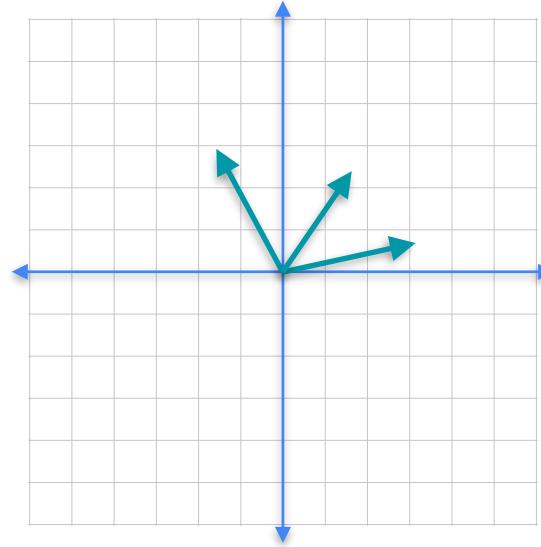


Not a basis

A basis is a minimal spanning set

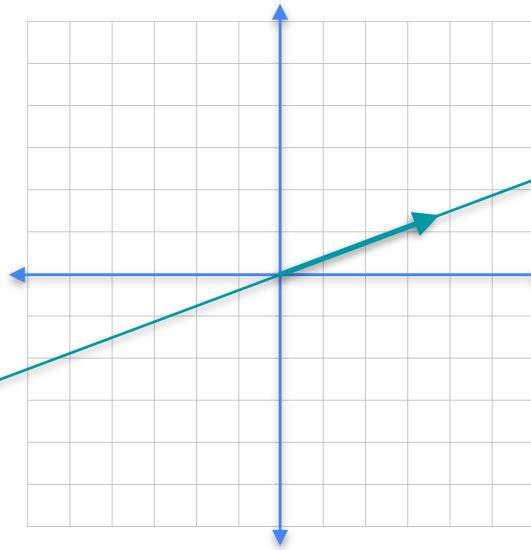


Basis

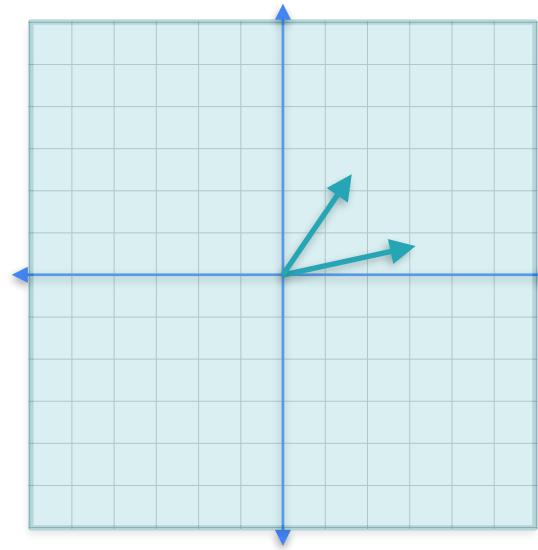


Not a basis

Number of elements in the basis is the dimension

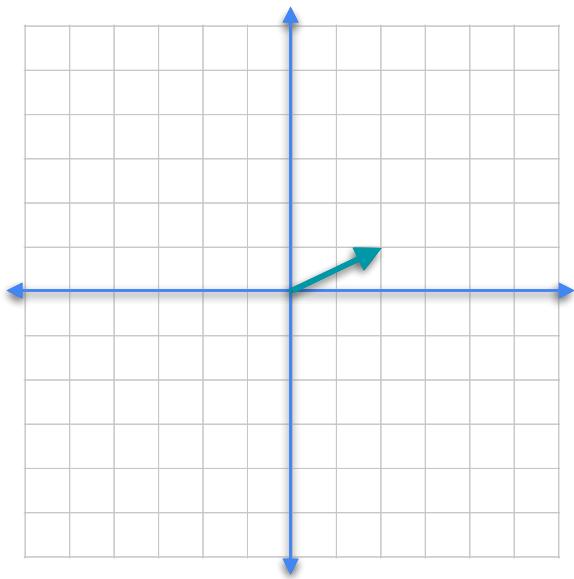


Dimensions: 1
1 element in the basis



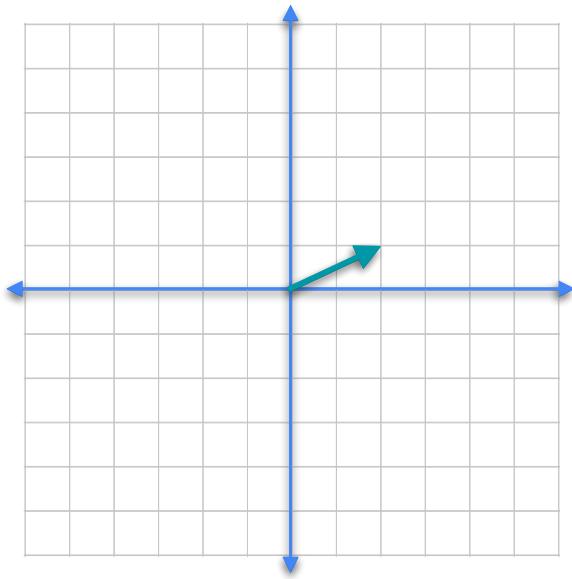
Dimensions: 2
2 elements in the basis

Linearly independent and linearly dependent vectors

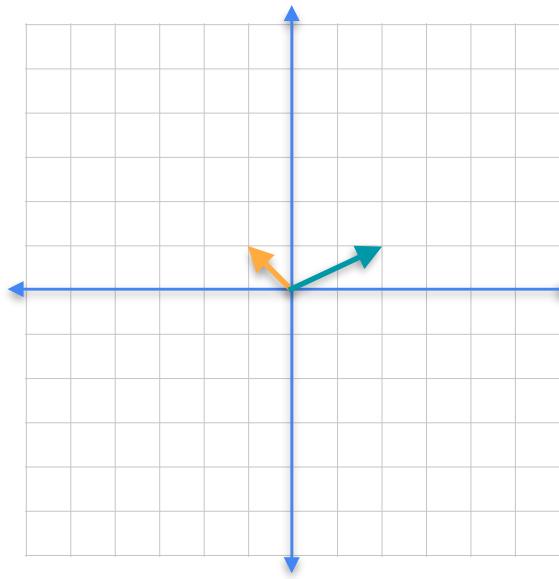


Linearly independent

Linearly independent and linearly dependent vectors

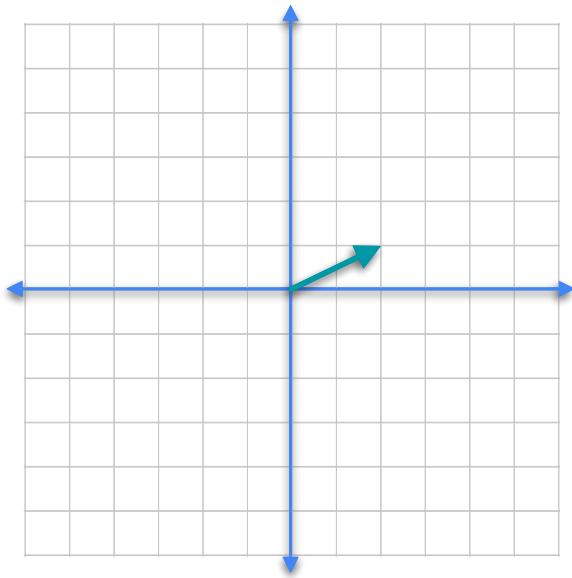


Linearly independent

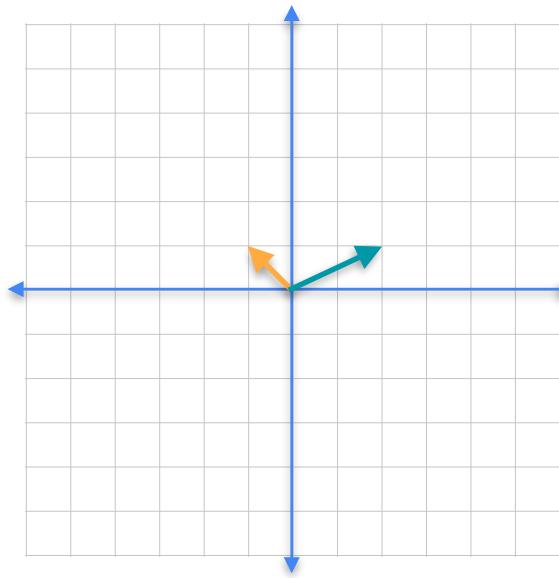


Linearly independent

Linearly independent and linearly dependent vectors

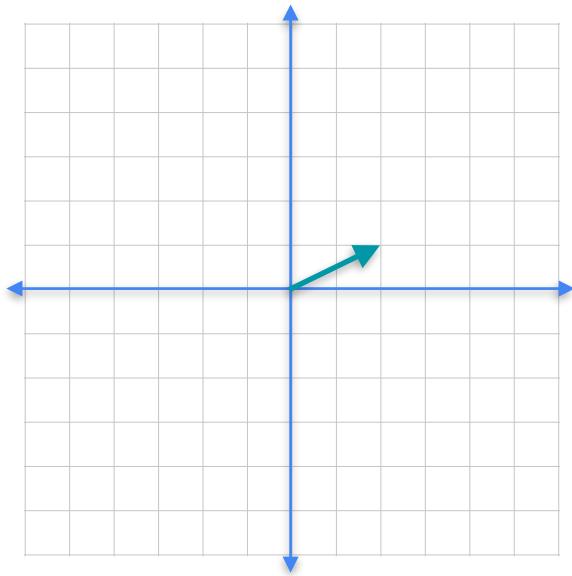


Linearly independent

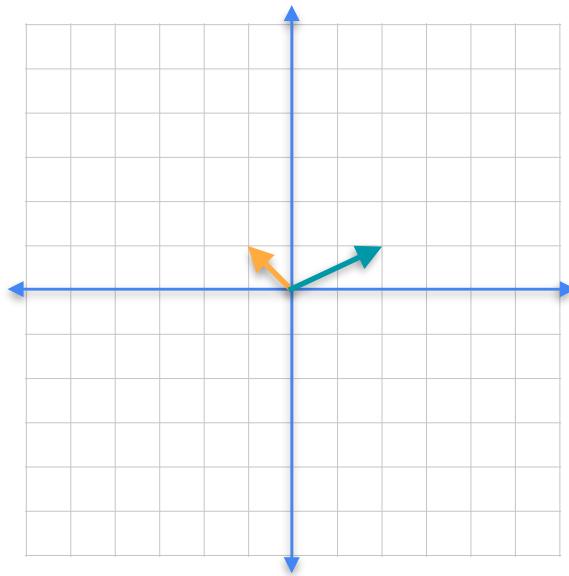


Linearly independent

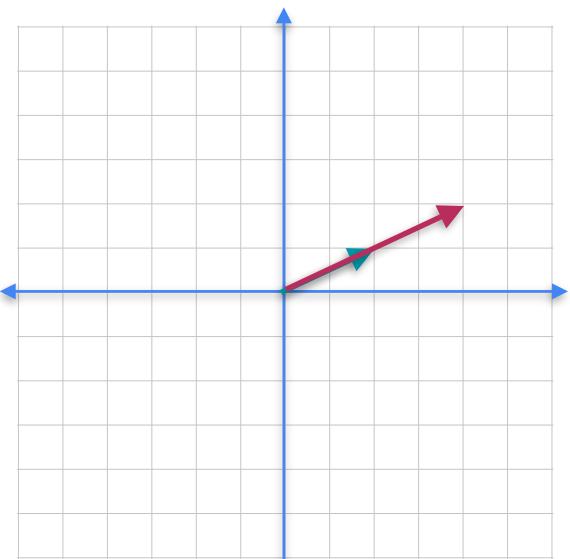
Linearly independent and linearly dependent vectors



Linearly independent

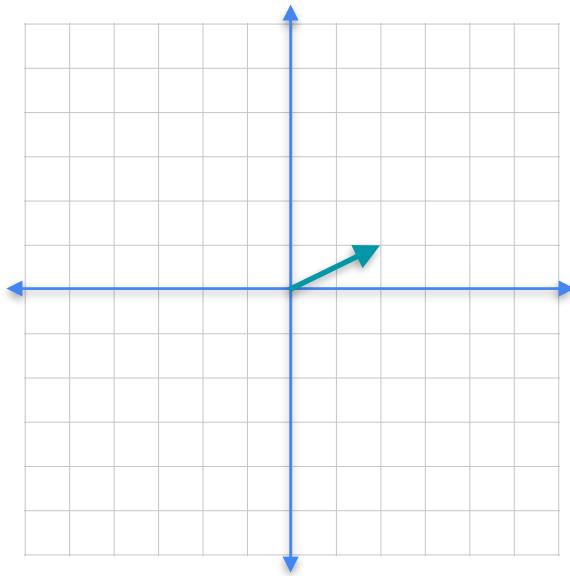


Linearly independent

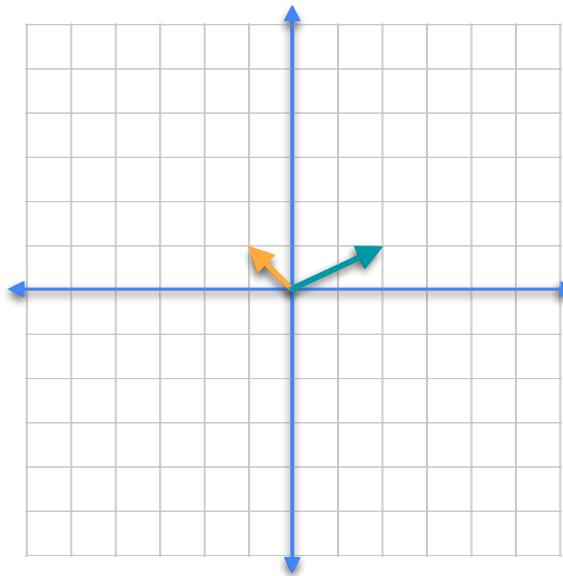


Linearly dependent

Linearly independent and linearly dependent vectors

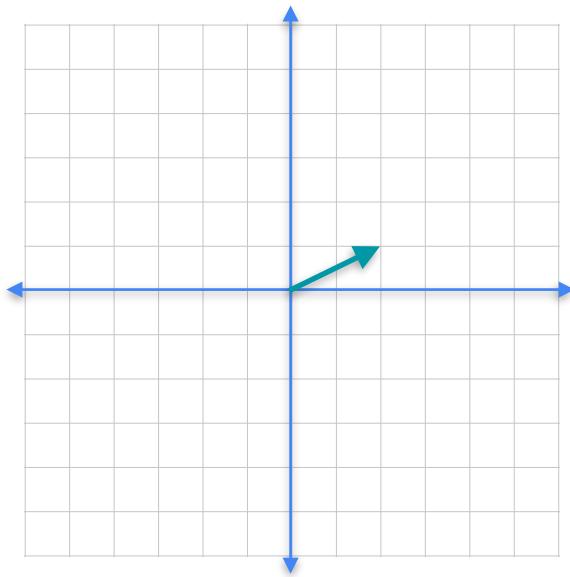


Linearly independent

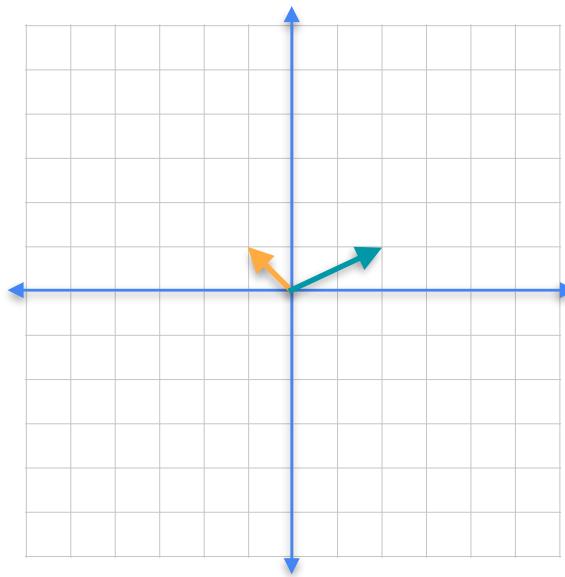


Linearly independent

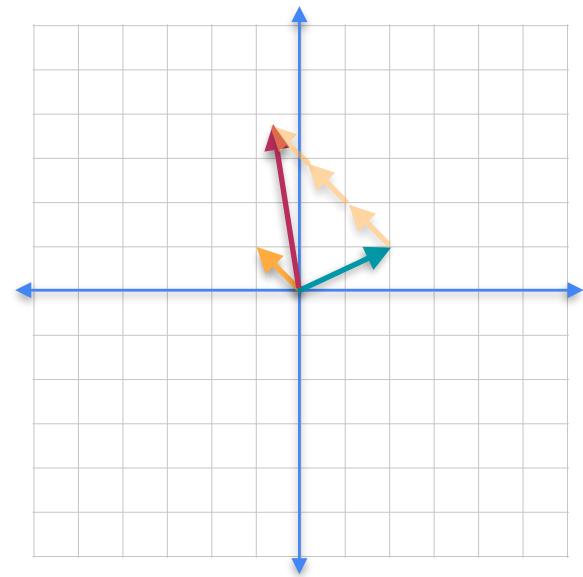
Linearly independent and linearly dependent vectors



Linearly independent

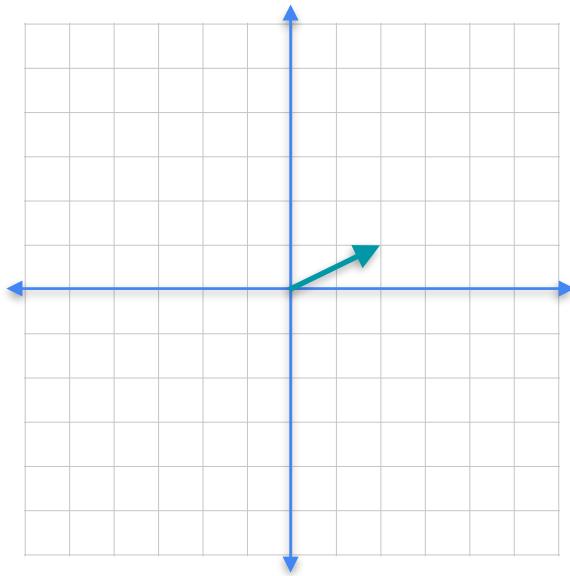


Linearly independent

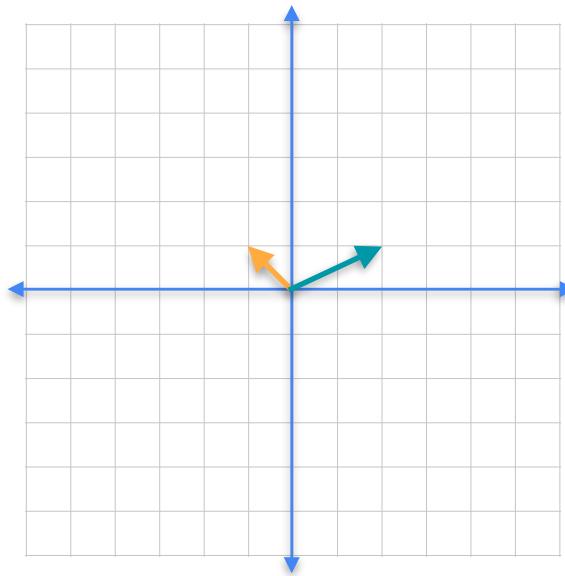


Linearly dependent

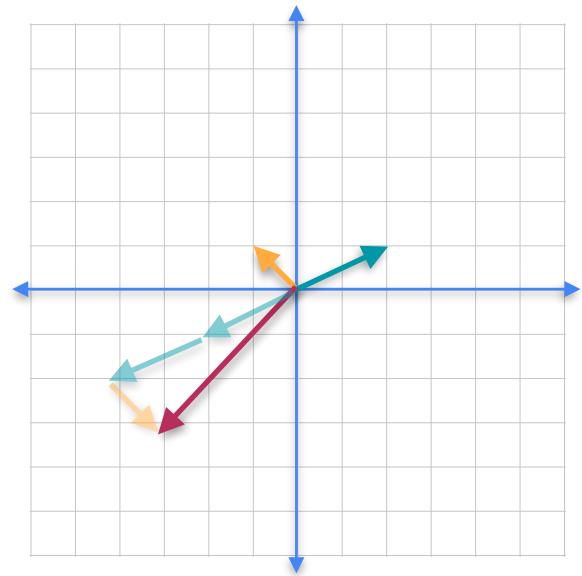
Linearly independent and linearly dependent vectors



Linearly independent

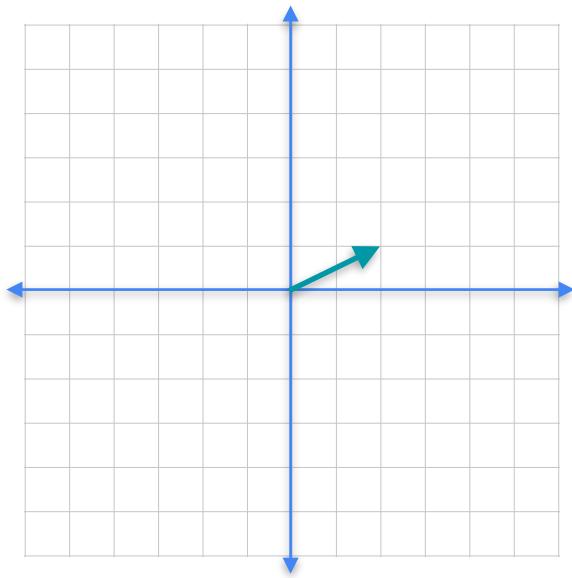


Linearly independent

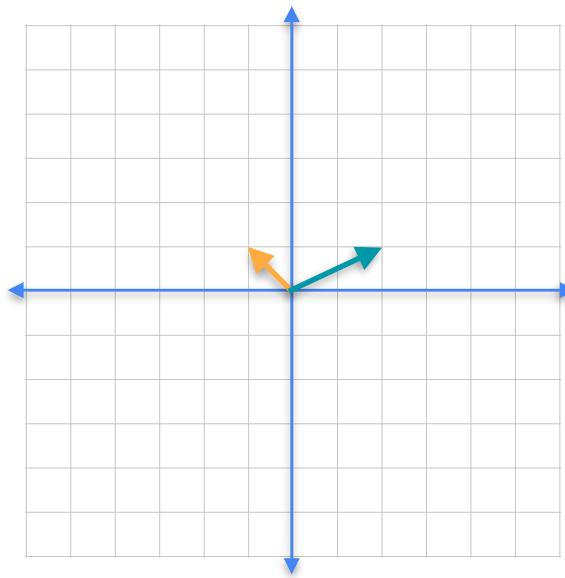


Linearly dependent

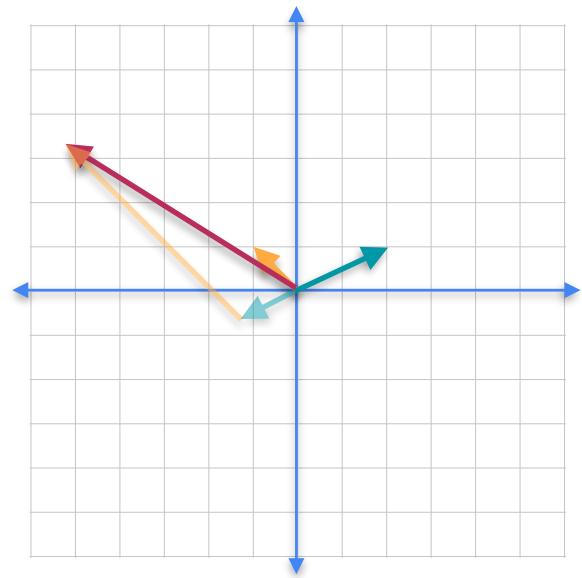
Linearly independent and linearly dependent vectors



Linearly independent

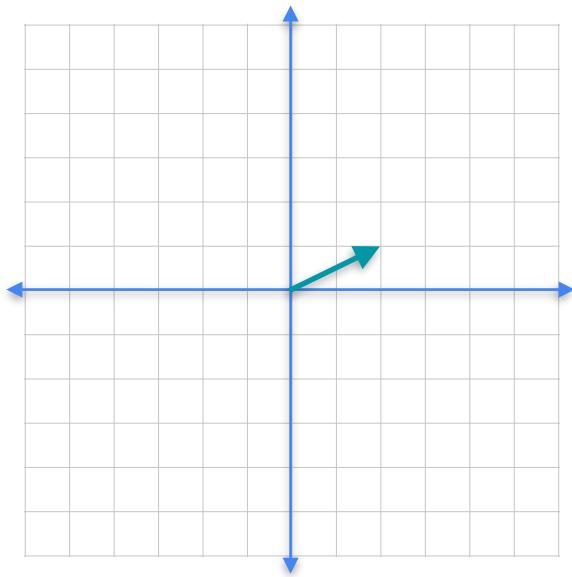


Linearly independent

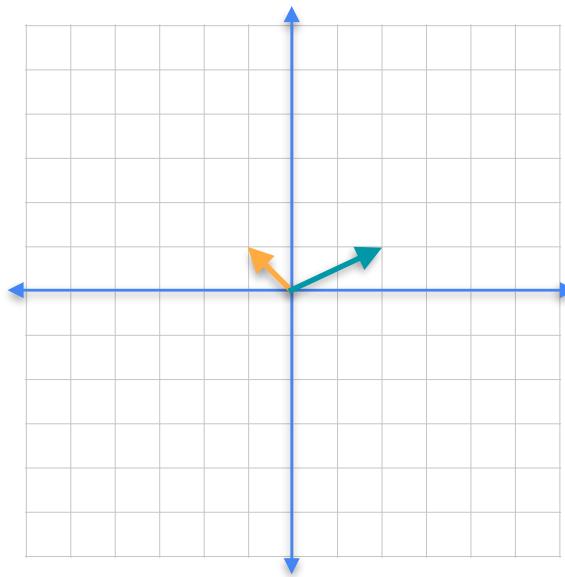


Linearly dependent

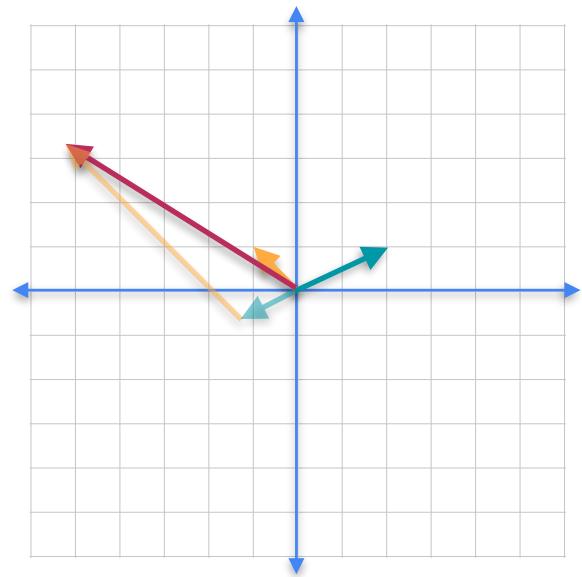
Linearly independent and linearly dependent vectors



Linearly independent

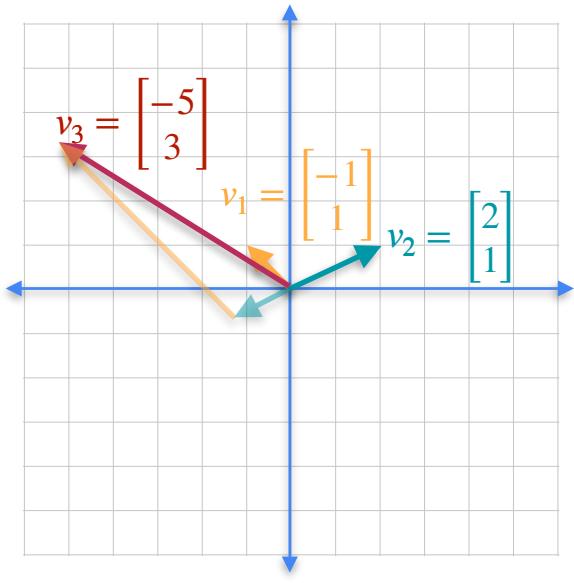


Linearly independent



Linearly dependent

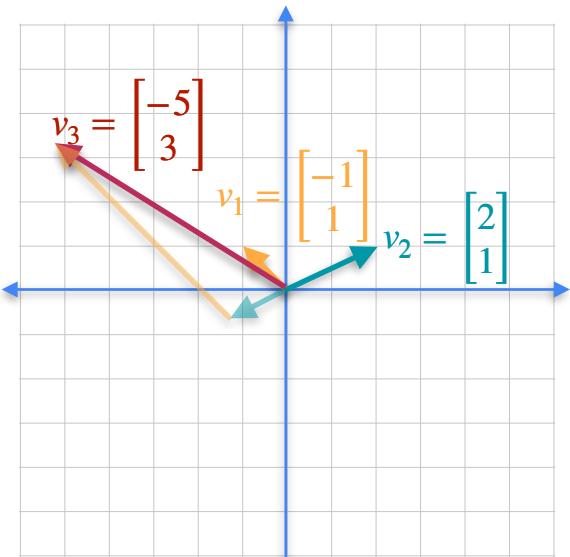
Let's see how to check for linear dependence



$$\alpha + \beta =$$

Linearly dependent

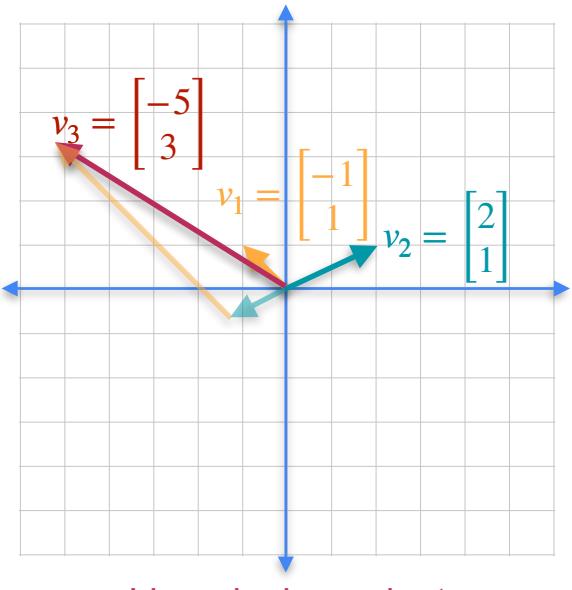
Let's see how to check for linear dependence



Linearly dependent

$$\alpha v_1 + \beta v_2 = v_3$$

Let's see how to check for linear dependence

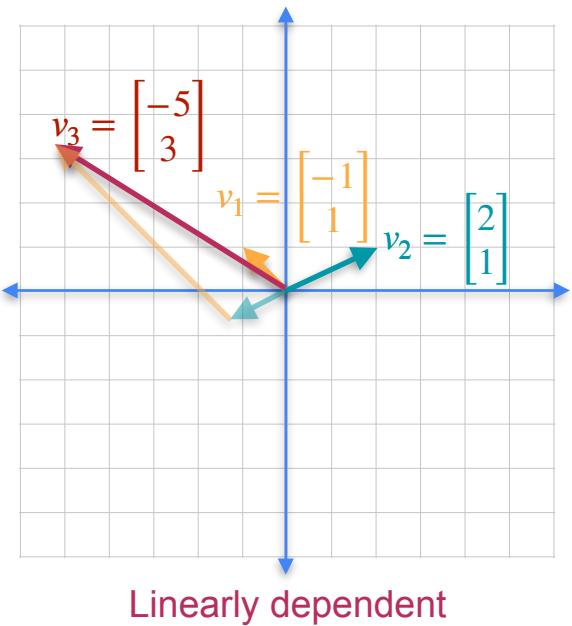


$$\alpha v_1 + \beta v_2 = v_3$$
$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

- 1
- 2

$$-\alpha + 2\beta = -5$$
$$\alpha + \beta = 3$$

Let's see how to check for linear dependence



$$\alpha v_1 + \beta v_2 = v_3$$
$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

v_3 is a linear combination
of v_1 and v_2

$$\begin{array}{l} 1 \\ 2 \end{array} \quad \begin{aligned} -\alpha + 2\beta &= -5 \\ \alpha + \beta &= 3 \end{aligned}$$

$$\begin{array}{l} 1 \\ 2 \end{array} \quad \begin{array}{l} + \\ 3\beta = -2 \end{array} \rightarrow \beta = -\frac{2}{3}$$
$$\alpha - \frac{2}{3} = 3 \rightarrow \alpha = \frac{11}{3}$$

Quiz

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

1 -1 = Linearly dependent

Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly dependent

Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Not a basis!

Linearly independent

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

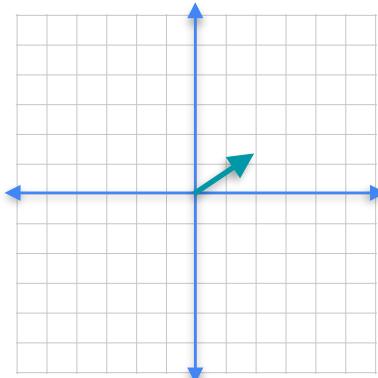
Basis: a formal definition

A basis is a set of vectors that:

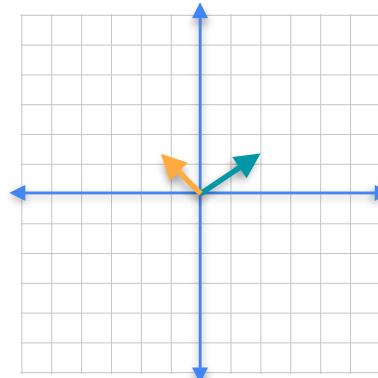
- Spans a vector space
- Is linearly independent



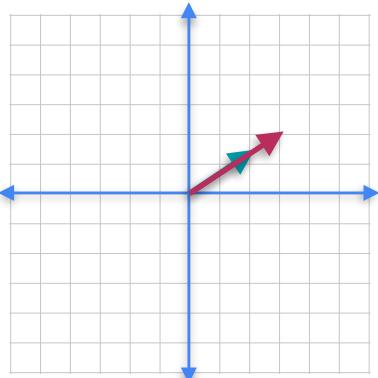
Not all sets of N vectors are a basis
for N-dimensional space



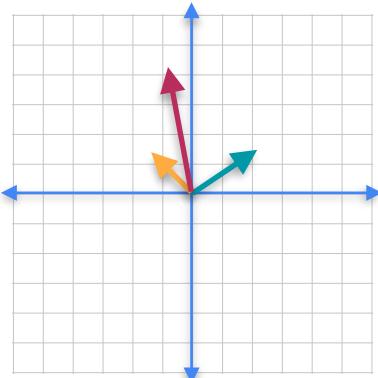
Spans a line
Linearly independent
Is a basis



Spans the plane
Linearly independent
Is a basis



Spans a line
Linearly dependent
Not a basis



Spans the plane
Linearly dependent
Not a basis

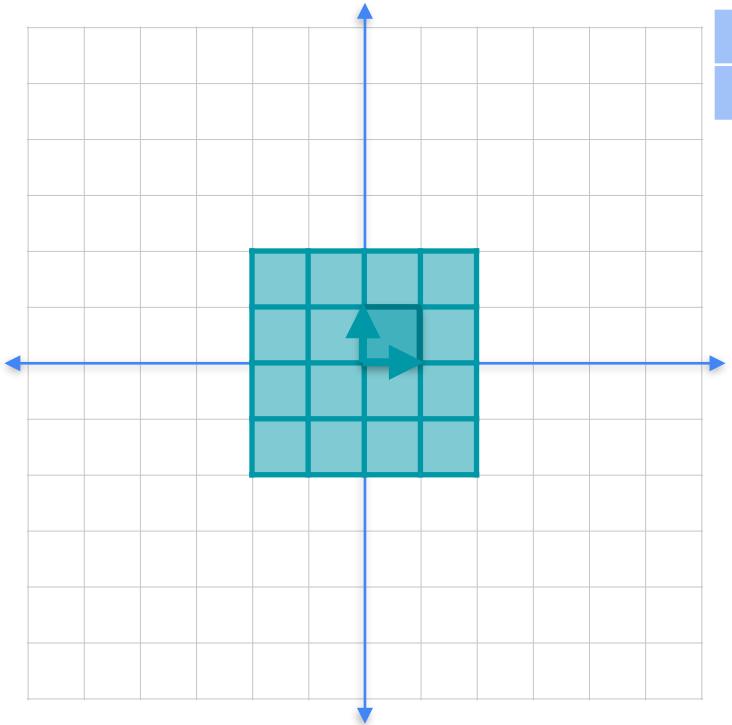


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Determinants and Eigenvectors

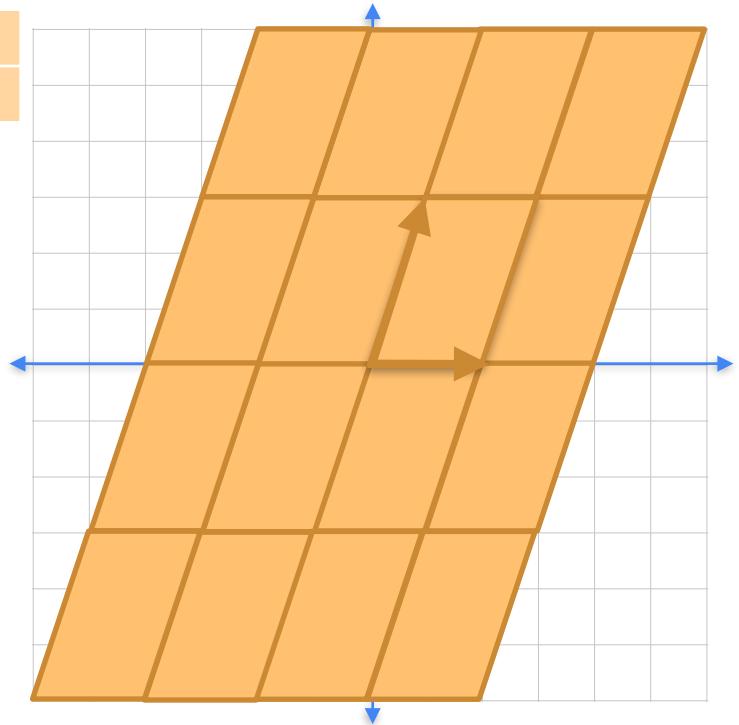
Eigenbasis

Basis

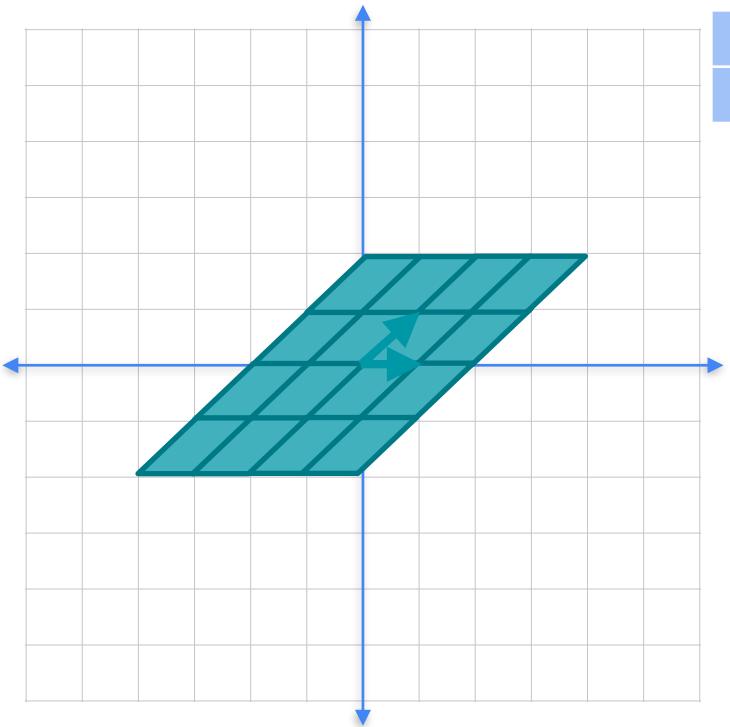


$$\begin{matrix} 2 & 1 & 0 \\ 0 & 3 & 1 \end{matrix} = \begin{matrix} 1 \\ 3 \end{matrix}$$

$$\begin{aligned} (1,0) &\rightarrow (2,0) \\ (0,1) &\rightarrow (1,3) \end{aligned}$$

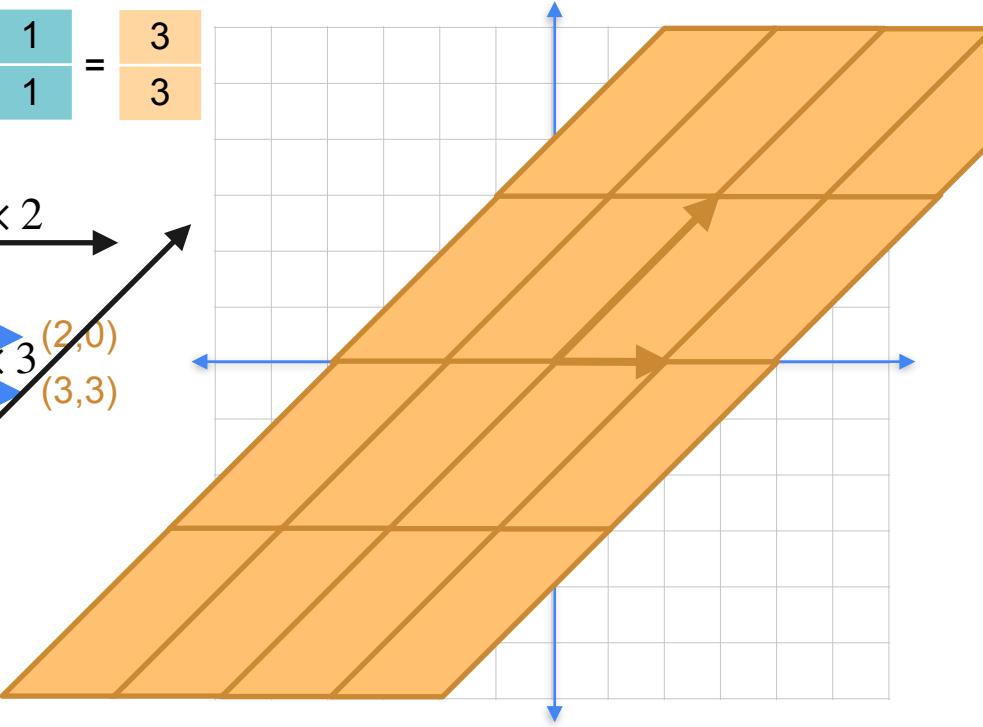


Eigenbasis

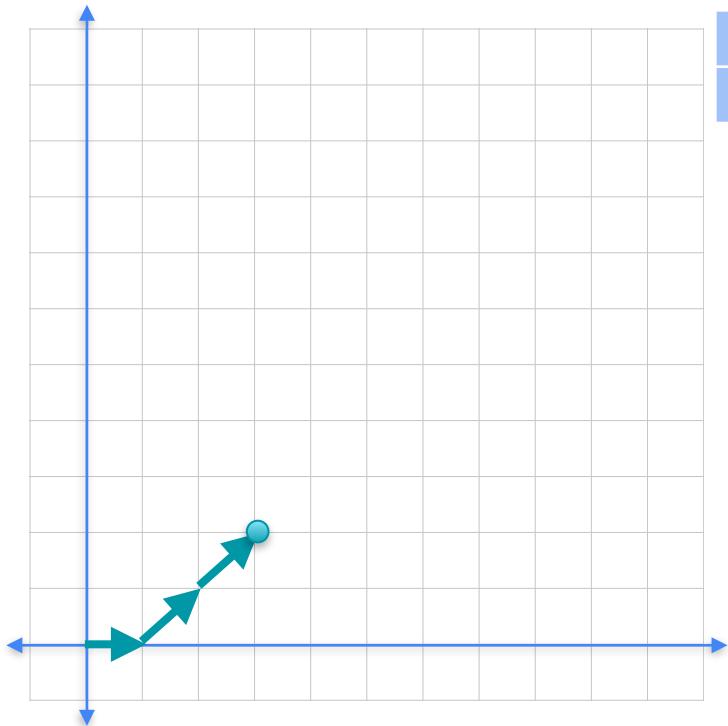


$$\begin{matrix} 2 & 1 & 1 \\ 0 & 3 & 1 \end{matrix} = \begin{matrix} 3 \\ 3 \end{matrix}$$

$$\begin{array}{c} \times 2 \\ \hline (1,0) \rightarrow (2,0) \\ (1,1) \rightarrow (3,3) \end{array}$$

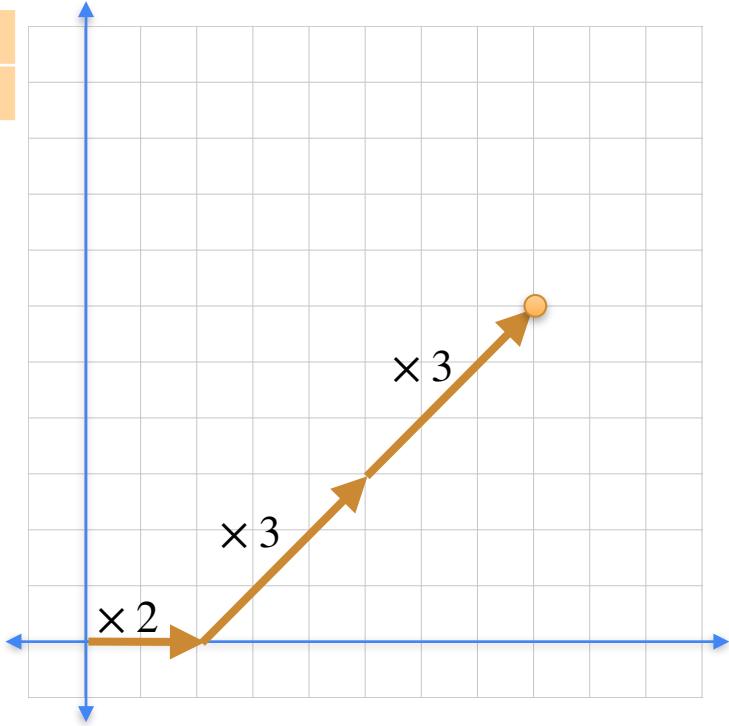


Eigenbasis



$$\begin{matrix} 2 & 1 & 3 \\ 0 & 3 & 2 \end{matrix} = \begin{matrix} 8 \\ 6 \end{matrix}$$

$(3,2) \rightarrow (8,6)$



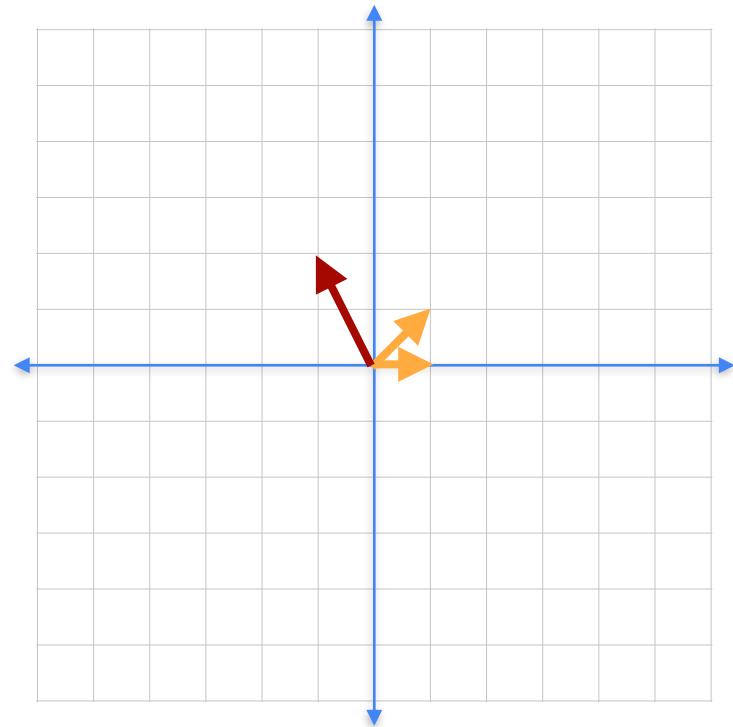


DeepLearning.AI

Determinants and Eigenvectors

Eigenvalues and Eigenvectors

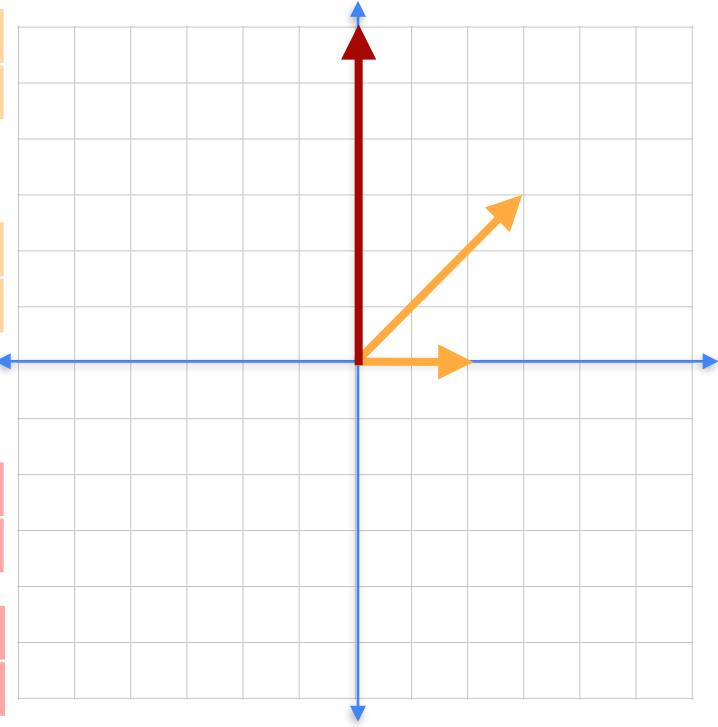
Eigenvalues and eigenvectors



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



Eigenvalues and eigenvectors

$$\begin{matrix} 2 & 1 & 1 \\ 0 & 3 & 0 \end{matrix} = \begin{matrix} 2 & 1 \\ 0 & 0 \end{matrix}$$

8 multiplications

2 multiplications

Matrix Multiplication
More work

$$A v_1$$

Scalar Multiplication
Less work

$$\lambda_1 v_1$$

First pair

$$\begin{matrix} 2 & 1 & 1 \\ 0 & 3 & 1 \end{matrix} = \begin{matrix} 3 & 1 \\ 1 & 1 \end{matrix}$$

$$A v_2$$

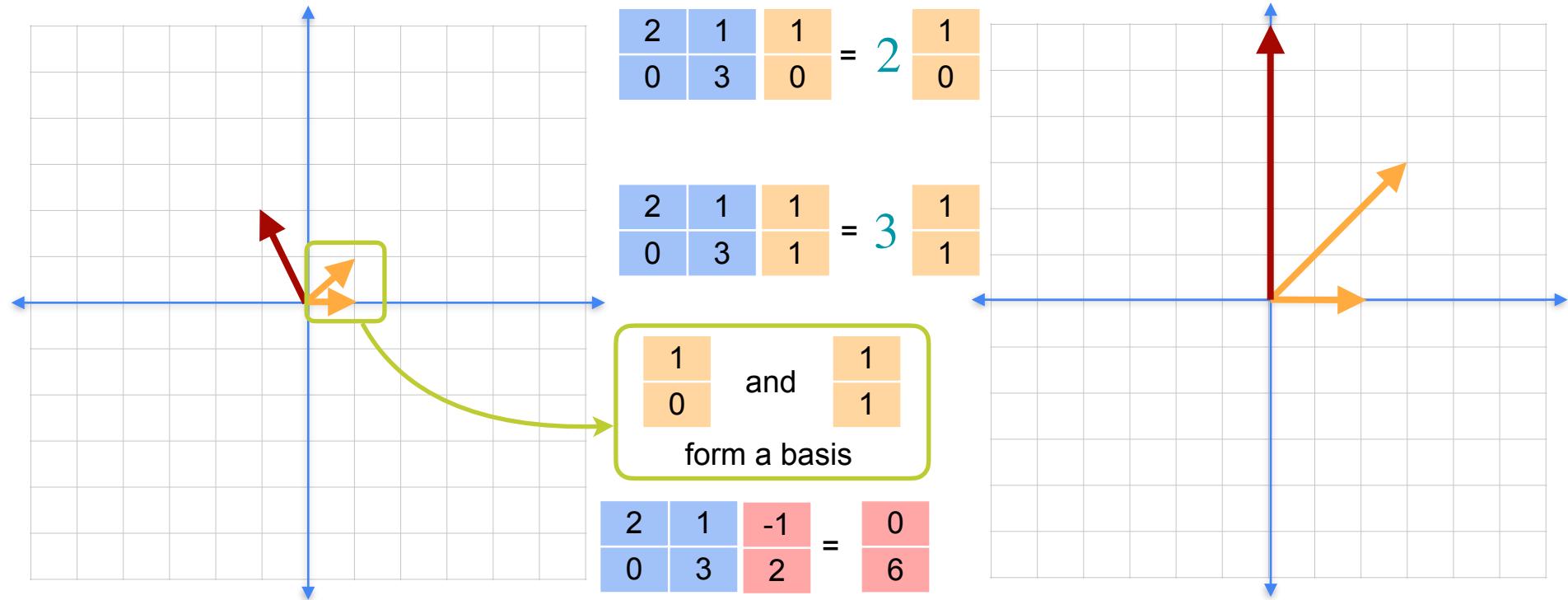
$$\lambda_2 v_2$$

Second pair

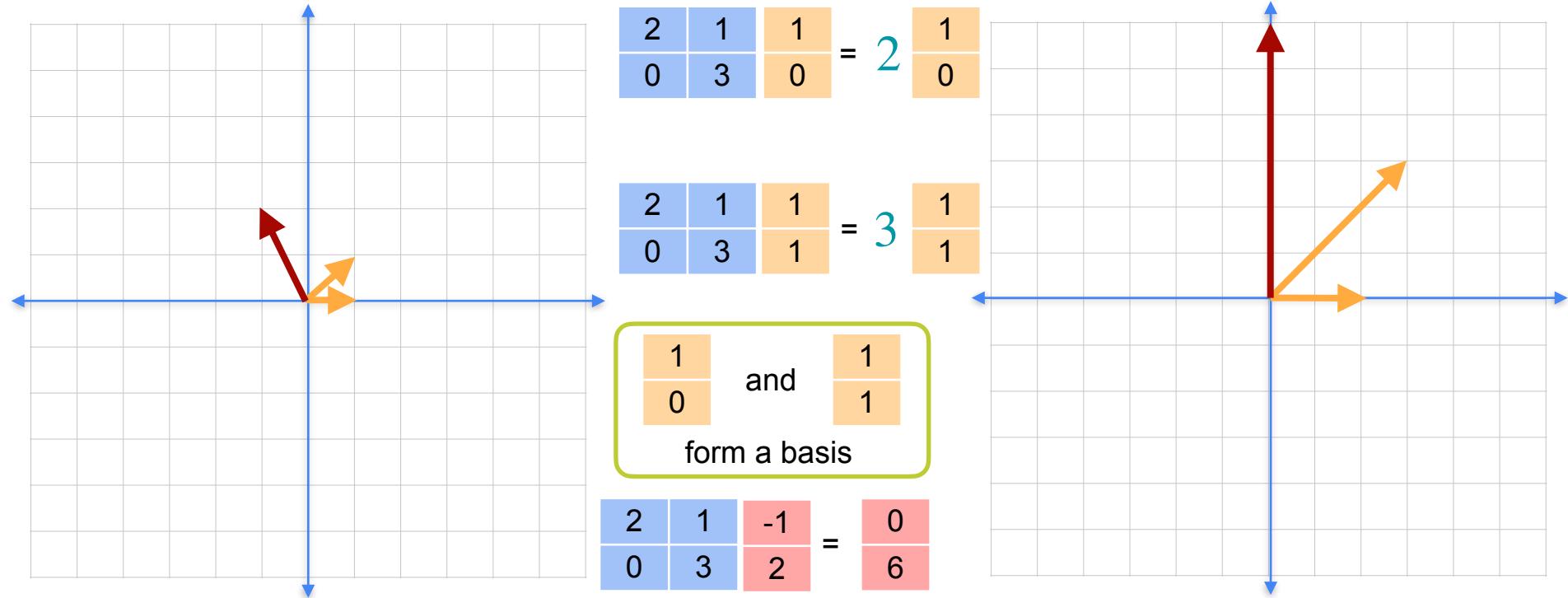
Eigenvalues

Eigenvectors

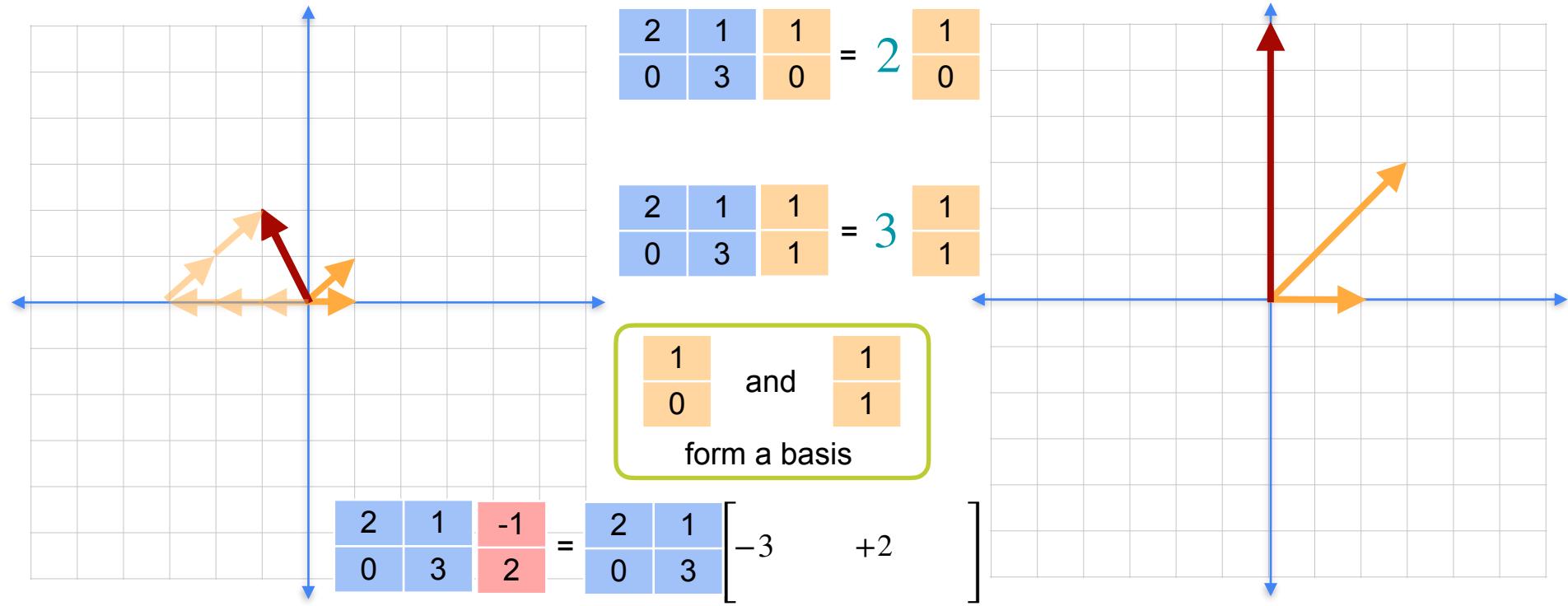
Eigenvalues and eigenvectors



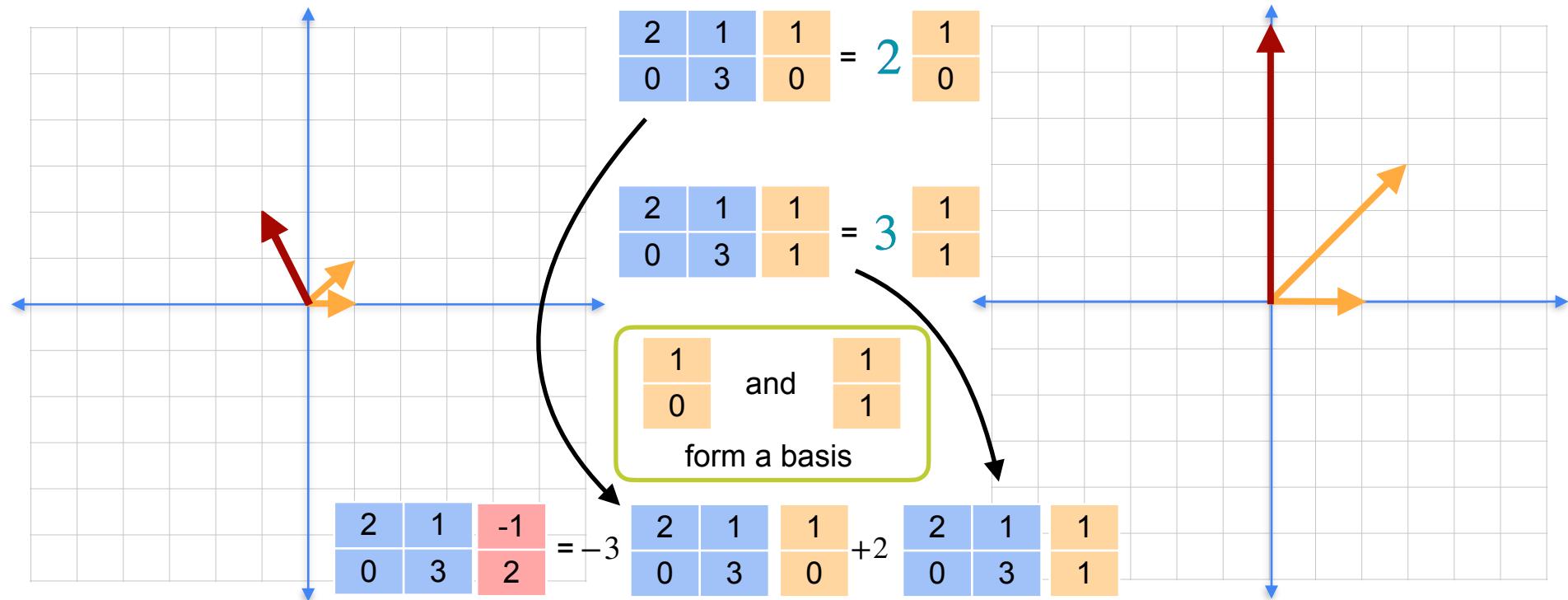
Eigenvalues and eigenvectors



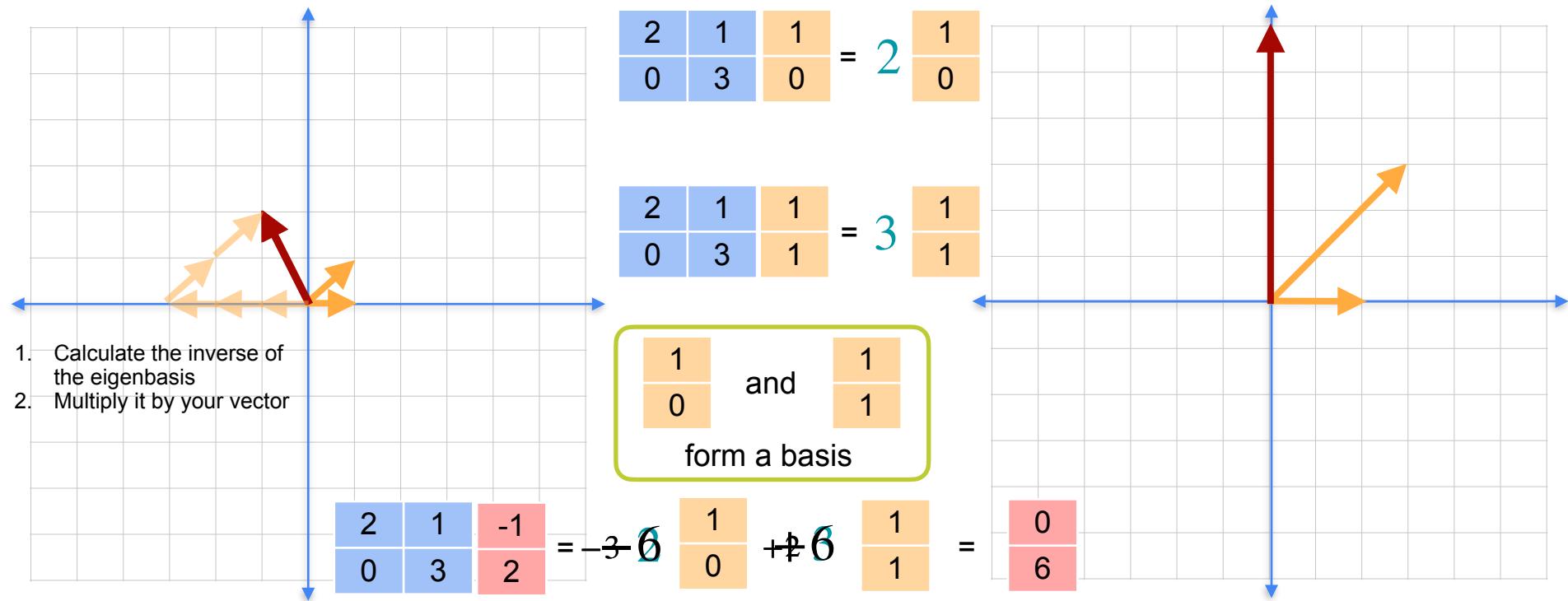
Eigenvalues and eigenvectors



Eigenvalues and eigenvectors



Eigenvalues and eigenvectors



Eigenvalues and eigenvectors

- $Av = \lambda v$ for each eigenvector / eigenvalue
- Eigenvectors: direction of stretch
- Eigenvalues: how much stretch
- Eigenbasis: the set of a matrix's eigenvectors, can be arranged as a matrix with one eigenvector in each column
- Save work and characterize a transformation

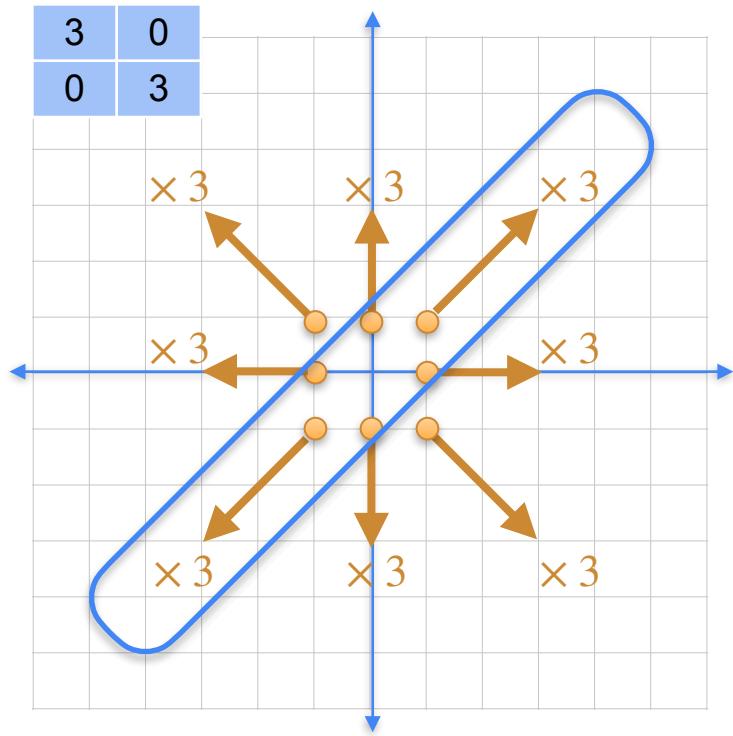
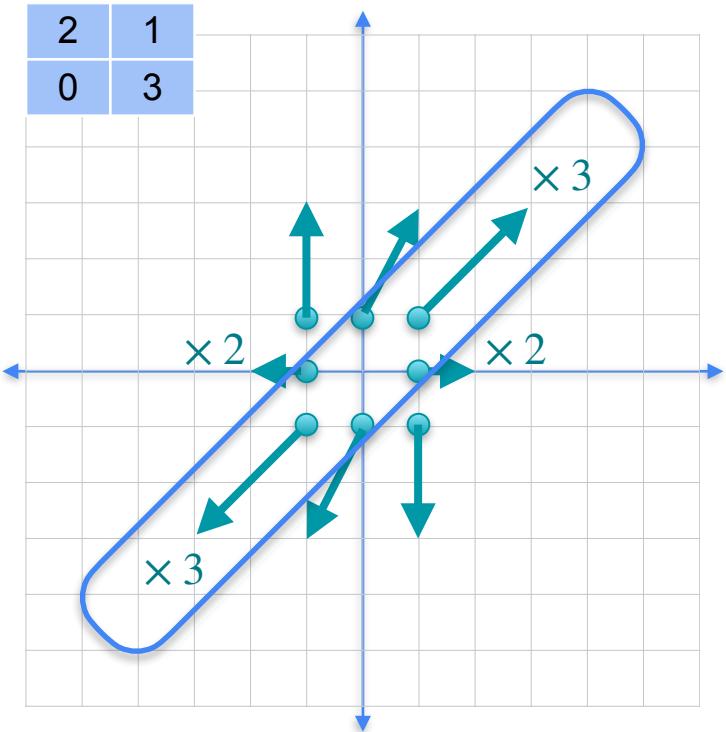


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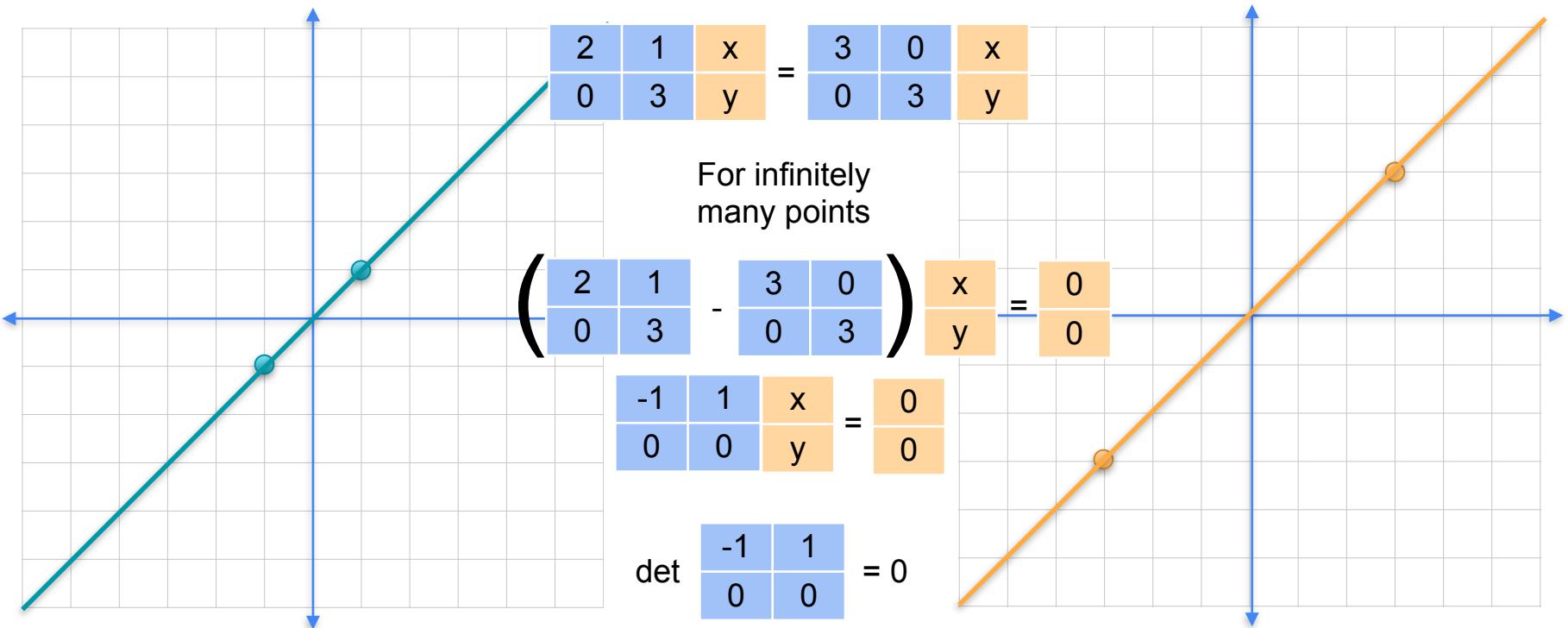
Determinants and Eigenvectors

Calculating eigenvalues and eigenvectors

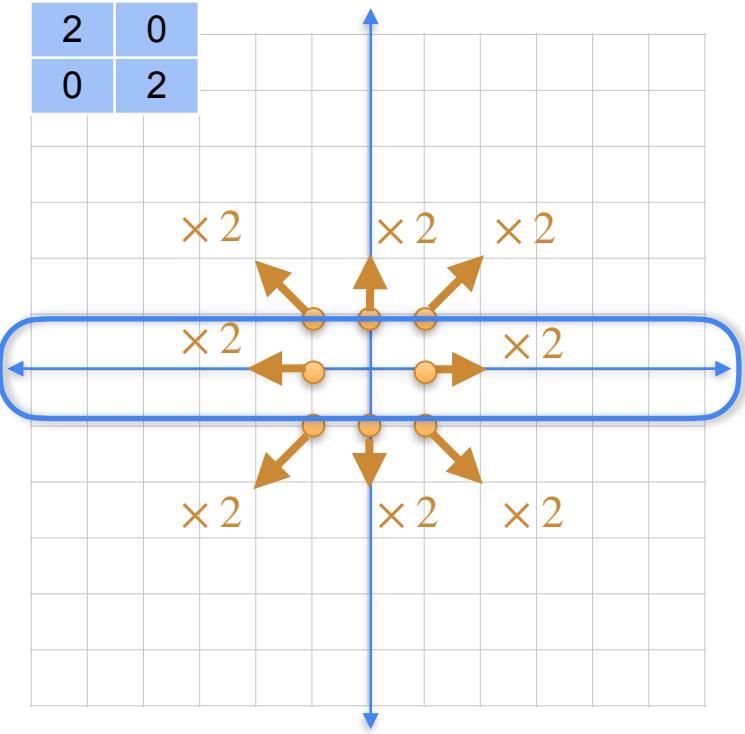
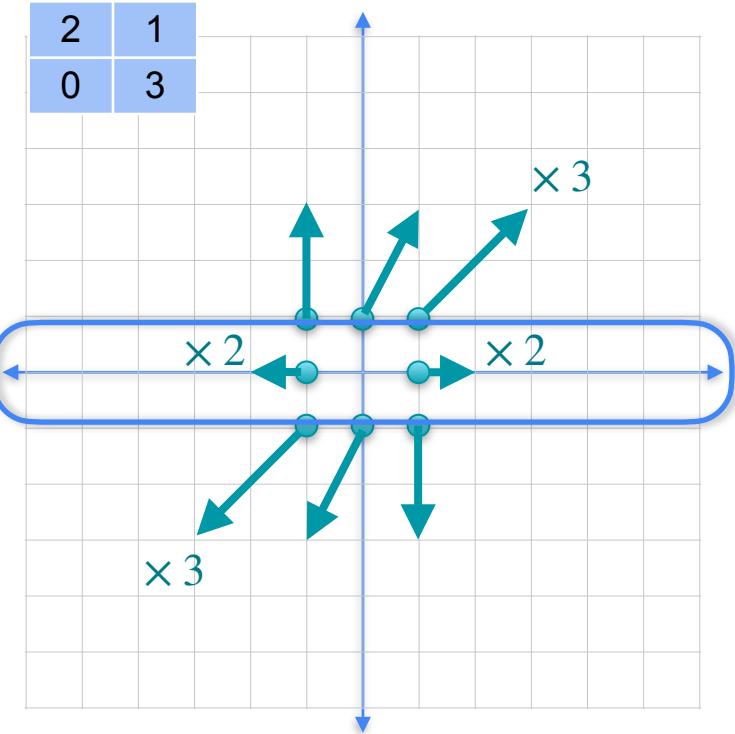
Finding eigenvalues



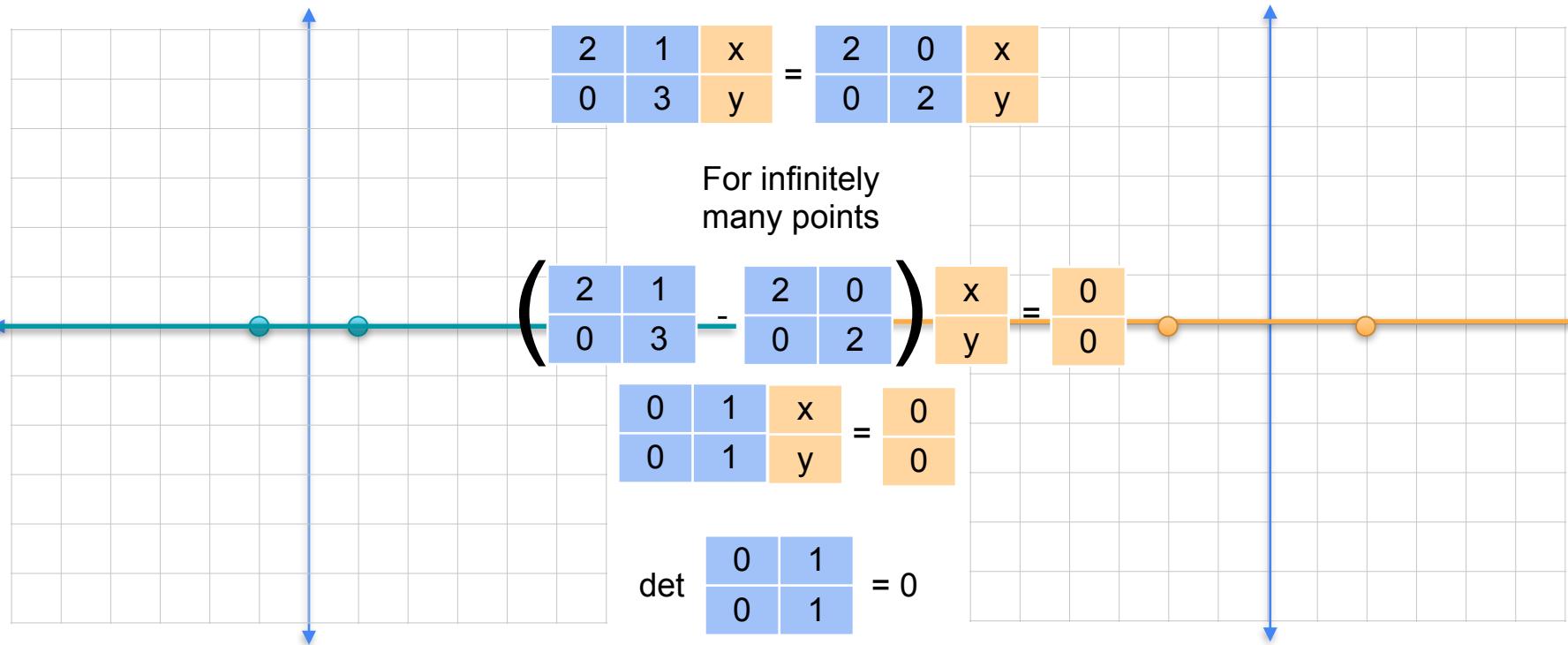
Finding eigenvalues



Finding eigenvalues



Finding eigenvalues



Finding eigenvalues

If λ is an eigenvalue:

$$\begin{array}{|c|c|c|} \hline 2 & 1 & x \\ \hline 0 & 3 & y \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \lambda & 0 & x \\ \hline 0 & \lambda & y \\ \hline \end{array}$$

For infinitely many (x,y)

$$\begin{array}{|c|c|c|} \hline 2-\lambda & 1 & x \\ \hline 0 & 3-\lambda & y \\ \hline \end{array} = \begin{array}{|c|c|} \hline 0 \\ \hline 0 \\ \hline \end{array}$$

Has infinitely many solutions

$$\det \begin{array}{|c|c|} \hline 2-\lambda & 1 \\ \hline 0 & 3-\lambda \\ \hline \end{array} = 0$$

Characteristic polynomial

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

$$\lambda = 2$$

$$\lambda = 3$$

Finding eigenvectors

Eigenvalues: $\lambda = 2$
 $\lambda = 3$

Solve the equations

$$\begin{array}{|c|c|c|} \hline 2 & 1 & x \\ \hline 0 & 3 & y \\ \hline \end{array} = 2 \begin{array}{|c|c|} \hline x \\ \hline y \\ \hline \end{array}$$
$$2x + y = 2x$$
$$0x + 3y = 2y$$
$$x = 1$$
$$y = 0$$
$$\begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 2 & 1 & x \\ \hline 0 & 3 & y \\ \hline \end{array} = 3 \begin{array}{|c|c|} \hline x \\ \hline y \\ \hline \end{array}$$
$$2x + y = 3x$$
$$0x + 3y = 3y$$
$$x = 1$$
$$y = 1$$
$$\begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array}$$

Quiz

- Find the eigenvalues and eigenvectors of this matrix:

9	4
4	3

Solution

- Eigenvalues: 11, 1
- Eigenvectors: (2,1), (-1,2)

9	4
4	3

- The characteristic polynomial is

$$\det \begin{array}{|cc|} \hline 9-\lambda & 4 \\ 4 & 3-\lambda \\ \hline \end{array} = (9 - \lambda)(3 - \lambda) - 4 \cdot 4 = 0$$

- Which factors as $\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1)$

- The solutions are $\lambda = 11$
 $\lambda = 1$

Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Characteristic polynomial: $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 2 - \lambda & 1 & -1 \\ 1 & -\lambda & -3 \\ -1 & -3 & -\lambda \end{bmatrix} = 0$$

$$(2 - \lambda)\lambda^2 + 3 + 3 - 9(2 - \lambda) + \lambda + \lambda = -\lambda^3 + 2\lambda^2 + 11\lambda - 12 = 0$$

$$-(\lambda + 3)(\lambda - 1)(\lambda - 4) = 0$$

Eigenvalues: $-3, 1, 4$

Finding eigenvalues

$$A = \begin{matrix} & \begin{matrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{matrix} \end{matrix} \quad \text{Eigenvalues: } -3, 1, 4$$

$$Av = \lambda v$$

$$\underbrace{\begin{matrix} 2 & 1 & -1 & x_1 \\ 1 & 0 & -3 & x_2 \\ -1 & -3 & 0 & x_3 \end{matrix}}_{= 4} = 4 \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$$
$$\begin{matrix} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{matrix} = \begin{matrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{matrix}$$

Finding eigenvalues

$$A = \begin{array}{|ccc|} \hline & 2 & 1 & -1 \\ & 1 & 0 & -3 \\ & -1 & -3 & 0 \\ \hline \end{array}$$

$$Av = \lambda v$$

$$\underbrace{\begin{array}{|ccc|c|} \hline & 2 & 1 & -1 & x_1 \\ & 1 & 0 & -3 & x_2 \\ & -1 & -3 & 0 & x_3 \\ \hline \end{array}}_{\begin{array}{l} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{array}} = 4 \begin{array}{|c|} \hline x_1 \\ x_2 \\ x_3 \\ \hline \end{array}$$

Eigenvalues: $-3, 1, 4$

$$\begin{array}{rcl} 2x_1 + x_2 - x_3 & = & 4x_1 \\ x_1 - 3x_3 & = & 4x_2 \\ -x_1 - 3x_2 & = & 4x_3 \end{array}$$

$$\begin{array}{rcl} R_1 & -2x_1 + x_2 - x_3 & = 0 \\ R_2 & x_1 - 4x_2 - 3x_3 & = 0 \\ R_3 & -x_1 - 3x_2 - 4x_3 & = 0 \end{array}$$

$$\begin{array}{rcl} R_2 + R_3 & & 3R_1 + R_3 \\ -7x_2 - 7x_3 = 0 & & -7x_1 - 7x_3 = 0 \\ x_2 = -x_3 & & x_1 = -x_3 \end{array}$$

$$\begin{array}{l} x_1 = k \\ x_2 = k \\ x_3 = -k \end{array}$$

infinite solutions
of this form

$$\begin{array}{ll} x_1 = 1 & x_1 = 2 \\ x_2 = 1 & x_2 = 2 \\ x_3 = -1 & x_3 = -2 \end{array}$$

this works! so does this!

Eigenvector:

$$\begin{array}{|c|} \hline 1 \\ 1 \\ -1 \\ \hline \end{array}$$

Finding eigenvalues

$$A = \begin{matrix} \begin{array}{|c|c|c|} \hline 2 & 1 & -1 \\ \hline 1 & 0 & -3 \\ \hline -1 & -3 & 0 \\ \hline \end{array} \end{matrix}$$

Eigenvalues $\lambda_1 = 4$ $\lambda_2 = 1$ $\lambda_3 = -3$

Eigenvectors

$\begin{matrix} 1 \\ 1 \\ -1 \end{matrix}$	$\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}$	$\begin{matrix} 2 \\ -1 \\ 1 \end{matrix}$
--	---	--

Note on dimensions

Eigenvalues → Determinant → Square Matrix

9	4
4	3



9	4	5
4	3	-2





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Determinants and Eigenvectors

**On the number of
eigenvectors**

Number of eigenvectors

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

3 by 3 matrix

?

3 distinct eigenvalues

?

3 distinct eigenvectors

Eigenvalues

$$\lambda_1 = 4 \quad \lambda_2 = 1 \quad \lambda_3 = -3$$



Eigenvectors

$$\begin{array}{ccc} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \end{array}$$

Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix}$$

Characteristic polynomial = $\det(A - \lambda I) = \det$

$$\begin{pmatrix} 2 - \lambda & 0 & 0 \\ 1 & 4 - \lambda & 0.5 \\ 0 & 0 & 2 - \lambda \end{pmatrix}$$

$$(2 - \lambda)^2(4 - \lambda) + 0 + 0 - 0 - 0 - 0 = 0$$

Eigenvalues: 4, 2, 2 Repeated eigenvalue

Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 0 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{pmatrix}} = 4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{pmatrix}$$

Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 0 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{matrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{matrix}} = 4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{pmatrix}$$

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 2x_3 &= 4x_3 \end{aligned}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ -2x_3 &= 0 \\ \rightarrow x_1 &= 0 \\ x_2 &= \text{any number} \\ \rightarrow x_3 &= 0 \end{aligned}$$

Eigenvector

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 0 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{pmatrix}} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 0 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{matrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{matrix}} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$$x_1 = 2x_2 - 0.5x_3$$

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \begin{aligned} x_1 &= 2 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}$$

Point in different directions
Different eigenvectors

Repeated eigenvalues - Example 1

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues $\lambda_1 = 4$ $\lambda_2 = 2$ $\lambda_3 = 2$

Eigenvectors

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$
---	---	---

Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{matrix} \begin{array}{|ccc|} \hline & 2 & 0 \\ & -1 & 4 \\ & 4 & 0 \\ \hline & 0 & -0.5 \\ & 2 & \\ \hline \end{array} \end{matrix}$$

Characteristic polynomial = $\det(A - \lambda I)$ = \det

$$\begin{matrix} \begin{array}{|ccc|} \hline & 2-\lambda & 0 & 0 \\ & 1 & 4-\lambda & 0.5 \\ & -4 & 0 & 2-\lambda \\ \hline \end{array} \end{matrix}$$

$$(2 - \lambda)^2(4 - \lambda) + 0 + 0 - 0 - 0 - 0$$

Eigenvalues: 4, 2, 2 Repeated eigenvalue

Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 4 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{pmatrix}} = 4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{pmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 4 & 0 & 2 & x_3 \end{pmatrix}}_{2x_1} = 4 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{pmatrix}$$

$$\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{pmatrix}$$

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 4x_1 + 2x_3 &= 4x_3 \end{aligned}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ 4x_1 - 2x_3 &= 0 \end{aligned}$$

$$x_1 = 0 \quad x_3 = 0 \quad x_2 = \text{any number}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

Same as before!

Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 4 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{pmatrix}} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 4 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{pmatrix}} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 & 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 & k \\ 4x_1 &= 0 & 4k \end{aligned}$$

$$x_1 = 0 \qquad x_3 = 4x_2$$

$$\begin{array}{c|c} \begin{matrix} 0 \\ 1 \\ 4 \end{matrix} & \begin{matrix} x_1 = 0 \\ x_2 = 1 \\ x_3 = 4 \end{matrix} \\ \hline \begin{matrix} 0 \\ 0.5 \\ 2 \end{matrix} & \begin{matrix} x_1 = 0 \\ x_2 = 0.5 \\ x_3 = 2 \end{matrix} \end{array}$$

$$\begin{pmatrix} 0 \\ k \\ 4k \end{pmatrix}$$

On the same line
Same eigenvector

Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\underbrace{\begin{pmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 4 & 0 & 2 & x_3 \end{pmatrix}}_{\begin{pmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{pmatrix}} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 4x_1 &= 0 \end{aligned}$$

$$x_1 = 0 \qquad x_3 = 4x_2$$

$$\begin{pmatrix} 0 \\ k \\ 4k \end{pmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{pmatrix}$$

Eigenvalues

$$\lambda_1 = 4$$

$$\lambda_2 = 2$$

$$\lambda_3 = 2$$

Eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

Can't create an eigenbasis
from this matrix

Summary

a	b
c	d

Eigenvalues

$$\lambda_1, \lambda_2$$

a	b	c
d	e	f
g	h	i

$$\lambda_1, \lambda_2, \lambda_3$$

If $\lambda_1 \neq \lambda_2$  2 eigenvectors
(2 different directions)

If $\lambda_1 = \lambda_2$  1 eigenvector
(1 direction)
2 eigenvectors
(2 different directions)

If $\lambda_1 \neq \lambda_2 \neq \lambda_3$  3 eigenvectors
(3 different directions)

If $\lambda_1 = \lambda_2 \neq \lambda_3$  2 eigenvectors
(2 different directions)
 3 eigenvectors
(3 different directions)

If $\lambda_1 = \lambda_2 = \lambda_3$  1 eigenvector
(1 direction)
 2 eigenvectors
(2 different directions)
 3 eigenvectors
(3 different directions)



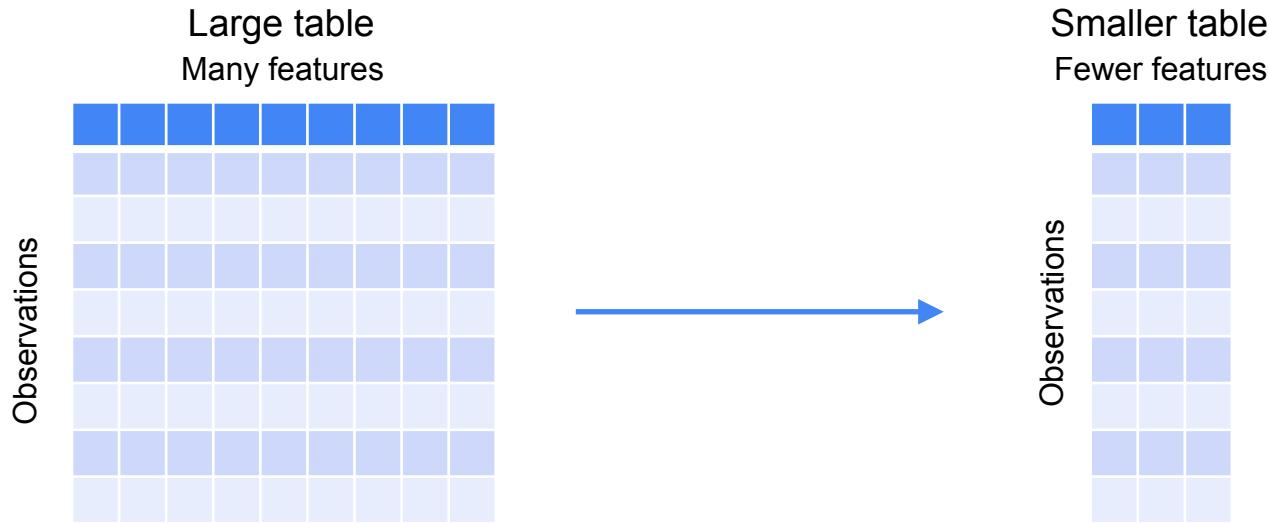
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Determinants and Eigenvectors

**Dimensionality reduction
and projection**

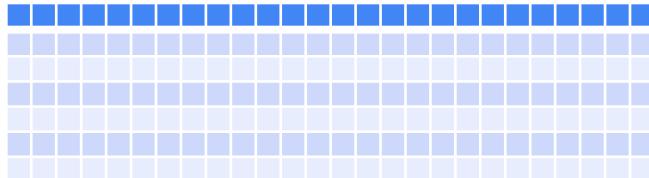
Dimensionality Reduction

- Reduce dimensions (# of columns) of dataset
- Preserve as much information as possible



Dimensionality Reduction

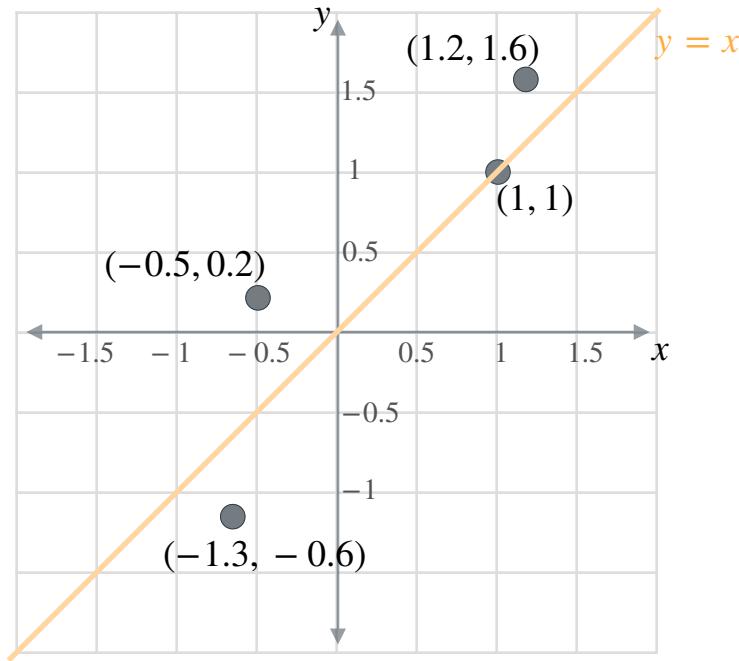
- Leads to smaller datasets
- Easier to visualize



Customer Age	Account Age	Days Since Login	Total Purchases	Total \$ Spent
23	1 month	10 days	1	\$100
71	45 months	2 days	Easy approach - just delete columns Loses valuable information	\$100
54	30 months	15 days	2	\$70
36	22 months	12 days	4	\$210

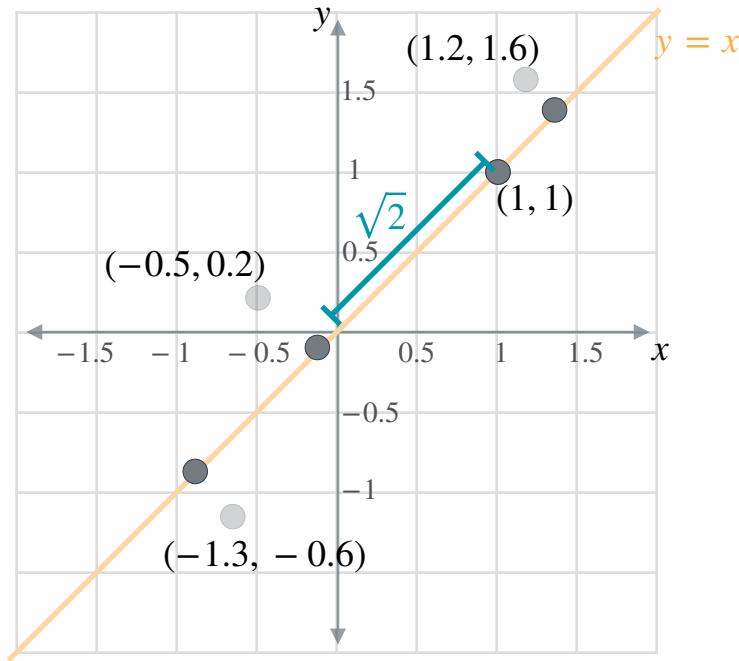
Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6



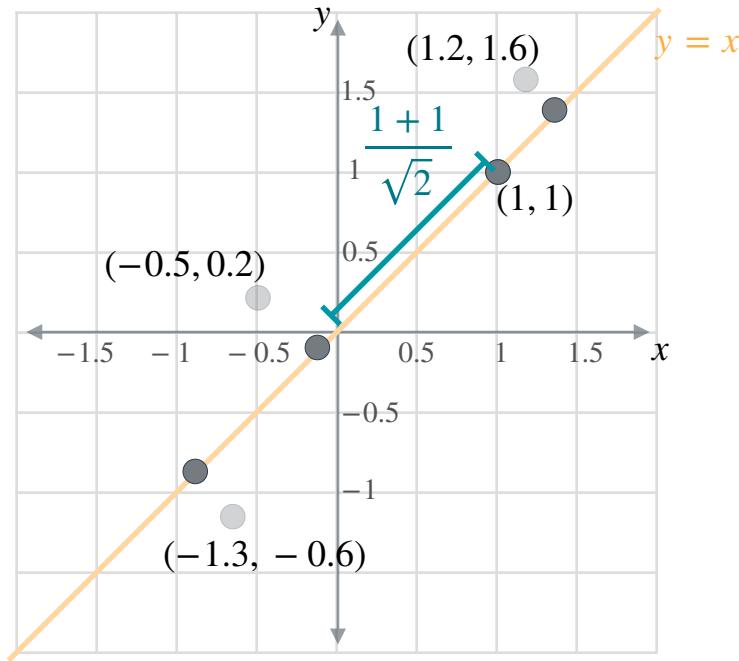
Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6



Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6



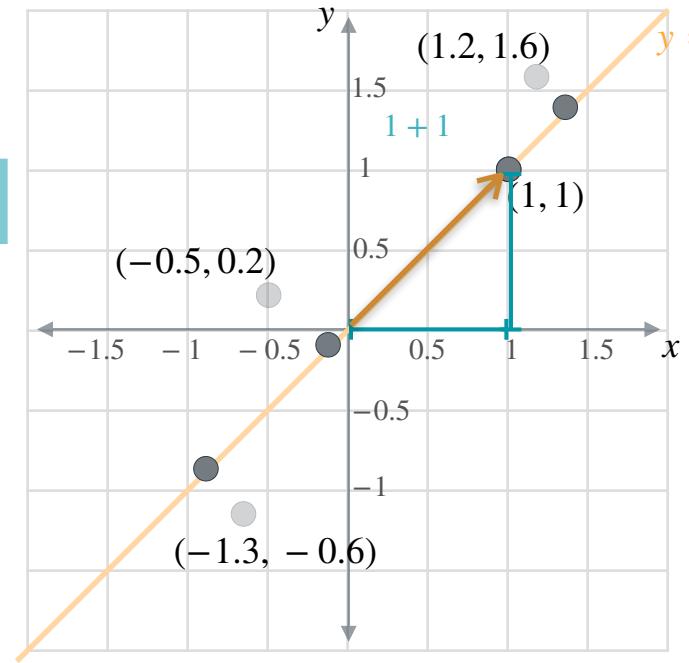
Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

1
1

=

$$(1 + 1)$$



Projections

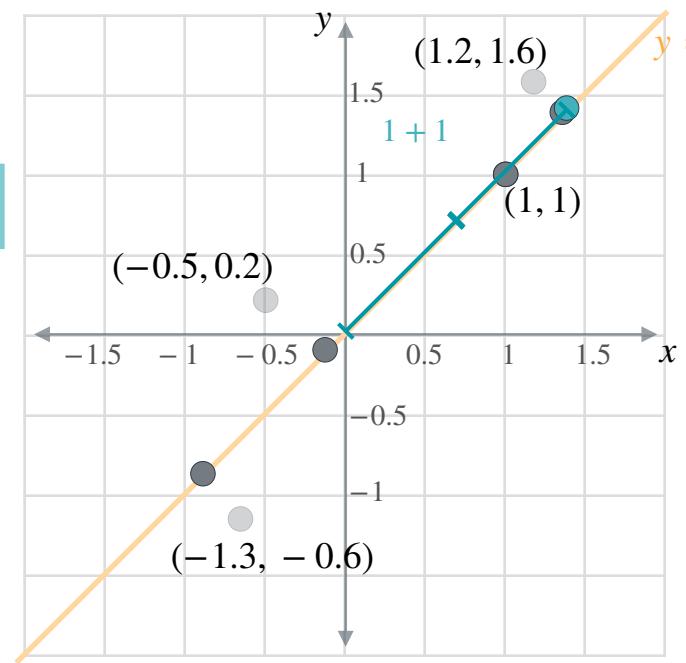
x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

→

1
1

=

$$(1 + 1)$$

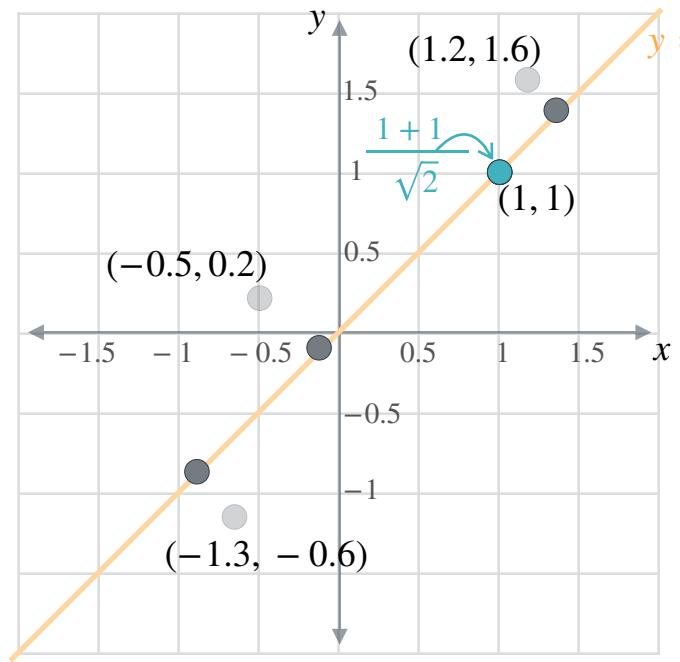


Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$(1+1)/\sqrt{2}$$



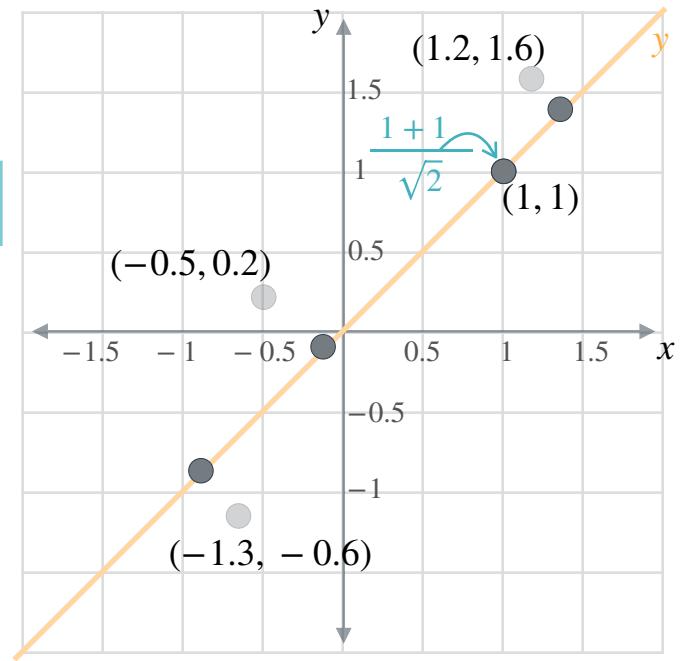
Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

Norm of 1

$$\frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}} = \frac{1}{\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_2}$$

$$(1+1)/\sqrt{2}$$

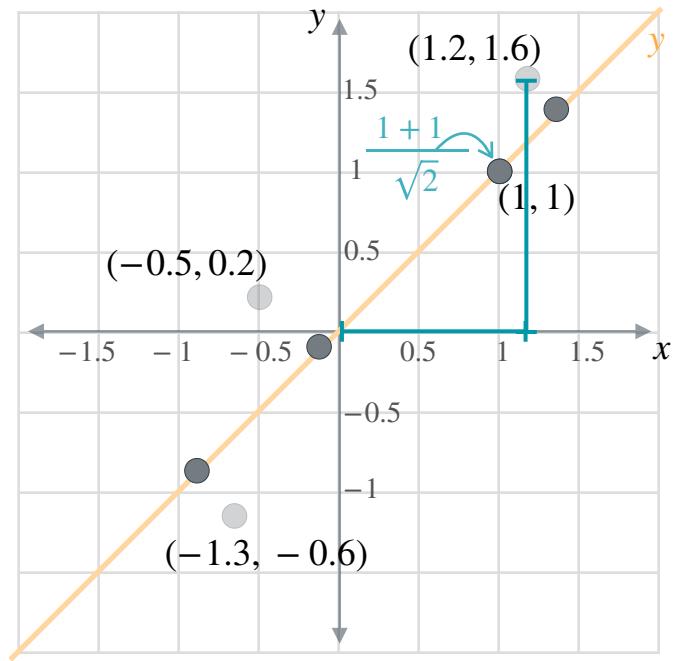


Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$\begin{matrix} (1+1)/\sqrt{2} \\ 1 \end{matrix}$$

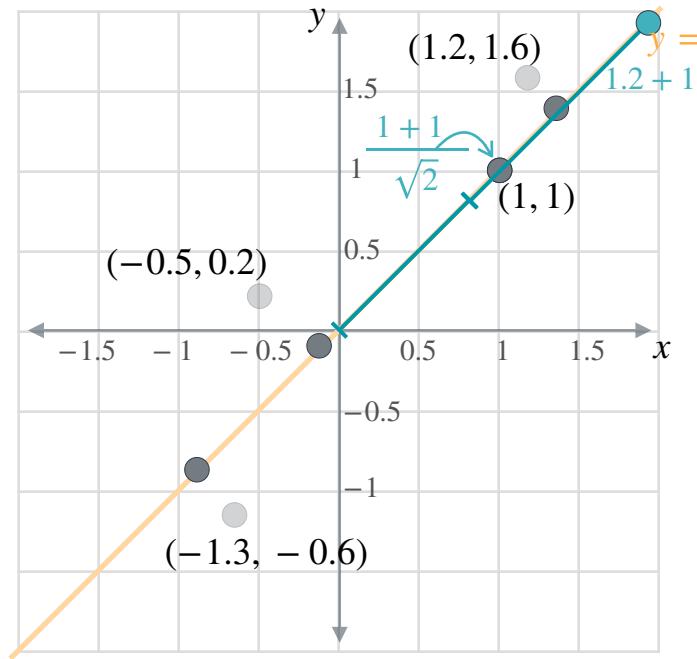


Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$\begin{matrix} (1+1)/\sqrt{2} \\ (1.2+1.6) \end{matrix}$$

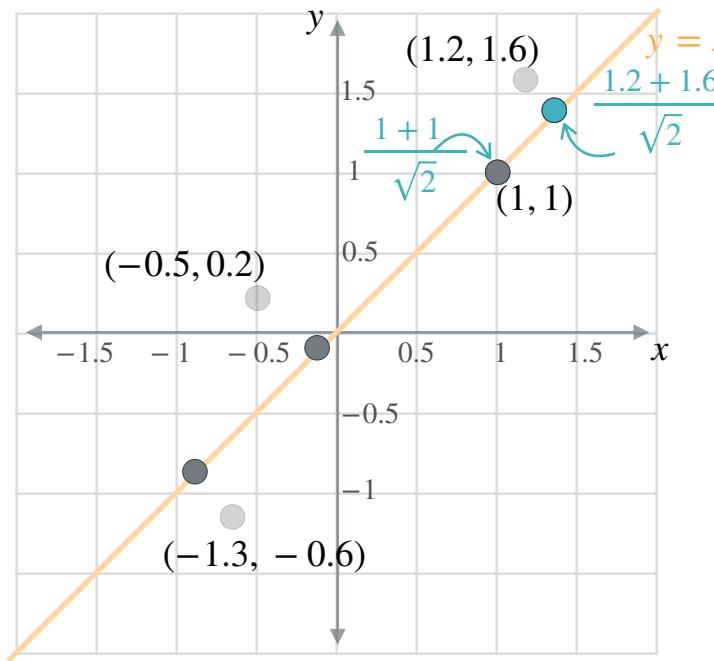


Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$\begin{matrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \end{matrix}$$

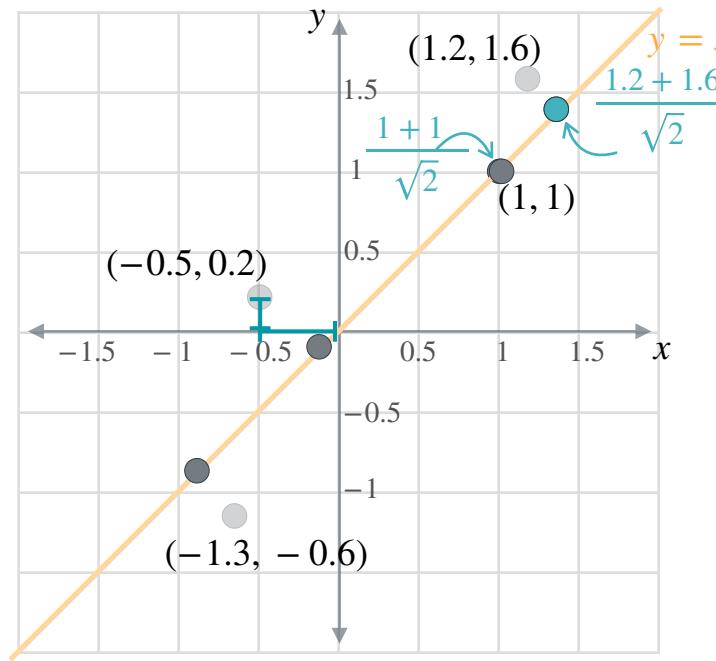


Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$\begin{matrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \end{matrix}$$

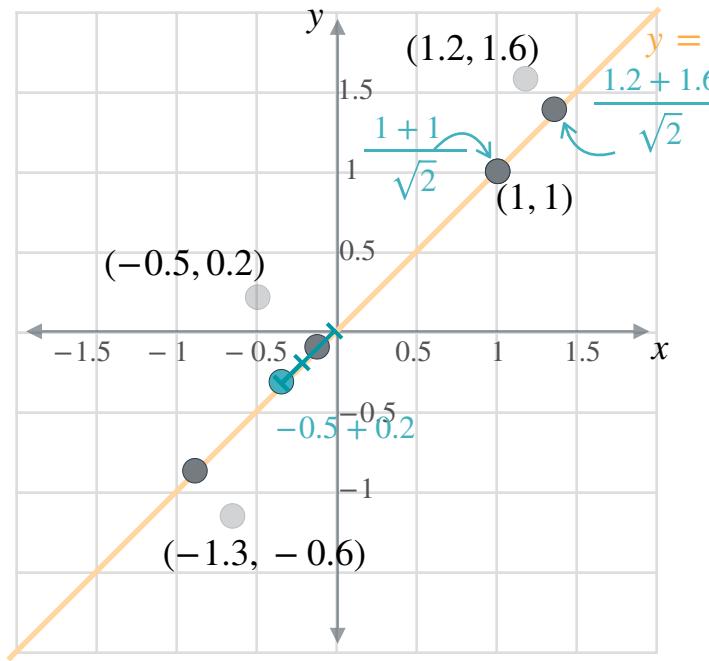


Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$\begin{matrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2) \end{matrix}$$

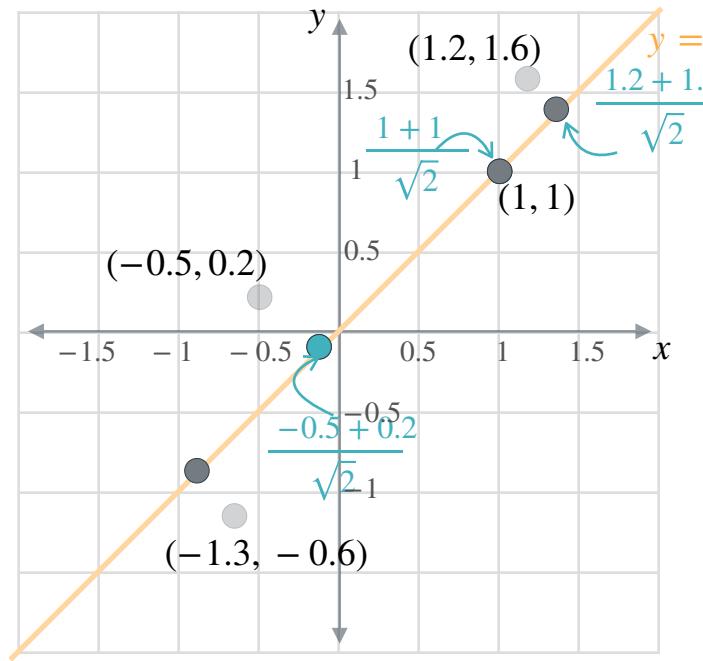


Projections

x	y
1.0	1.0
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$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

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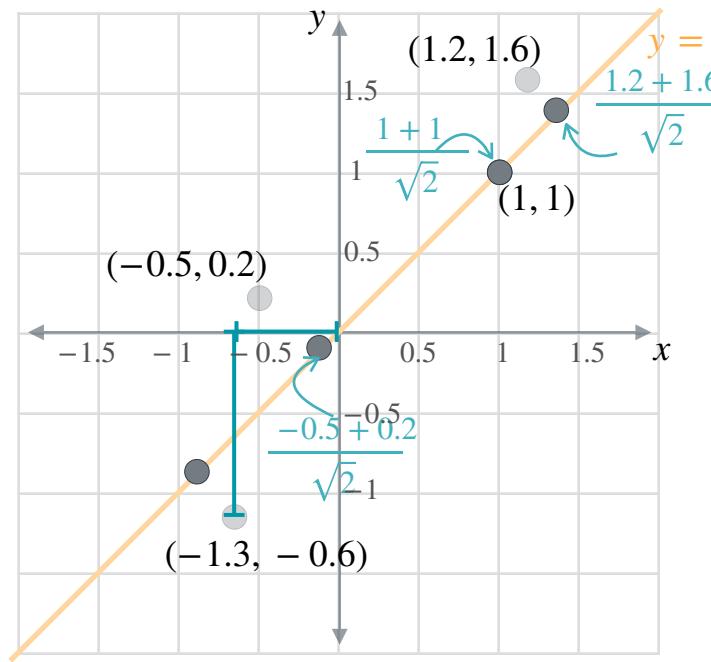


Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
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$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$\begin{matrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2)/\sqrt{2} \end{matrix}$$

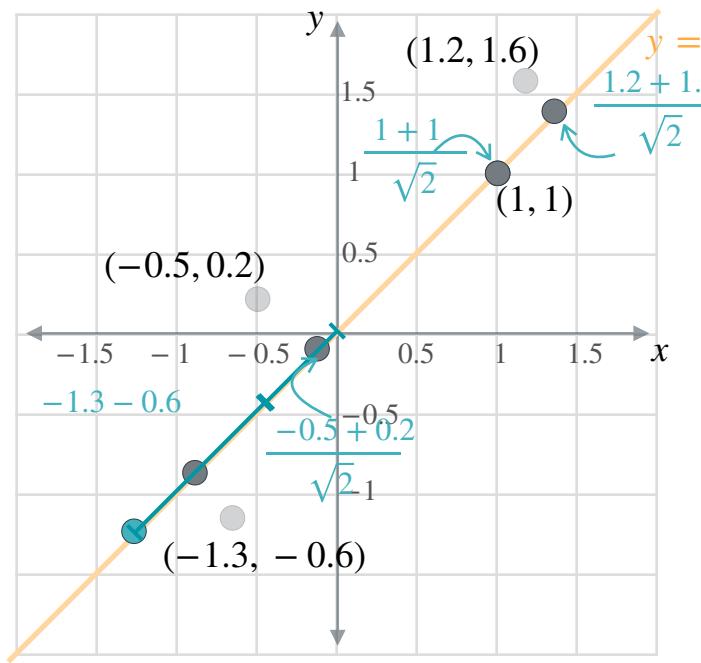


Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

$$\begin{matrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2)/\sqrt{2} \\ (-1.3-0.6) \end{matrix}$$



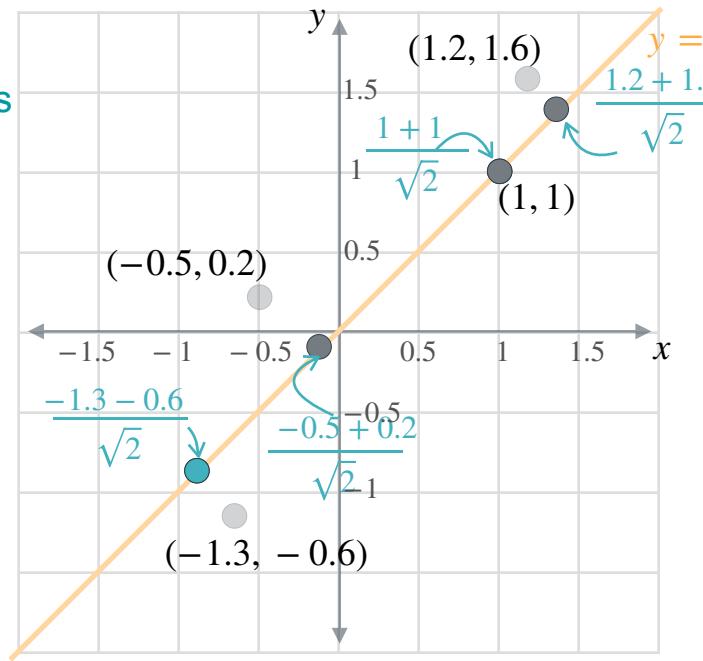
Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{matrix} 1 \\ 1 \end{matrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

$$\begin{aligned} & (1+1)/\sqrt{2} \\ & (1.2+1.6)/\sqrt{2} \\ & (-0.5+0.2)/\sqrt{2} \\ & (-1.3-0.6)/\sqrt{2} \end{aligned}$$



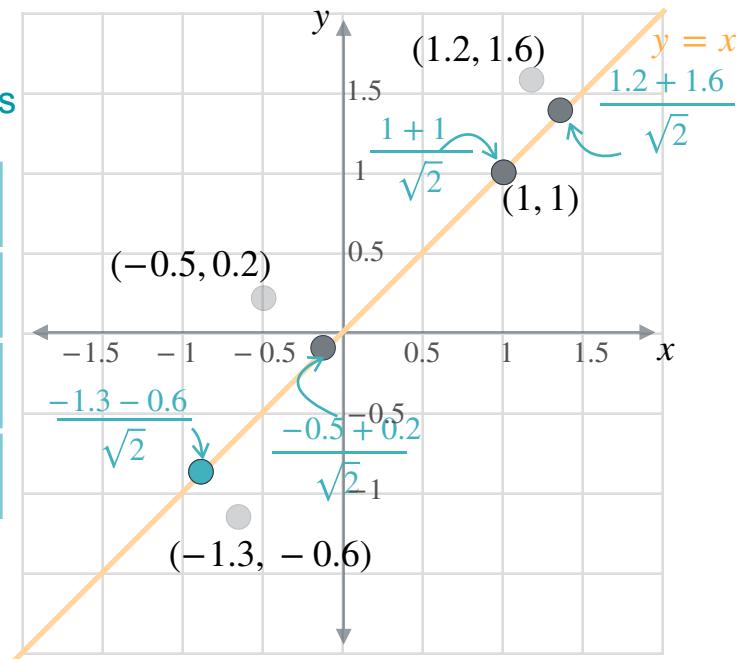
Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

1.4142
1.9799
-0.2121
-1.344

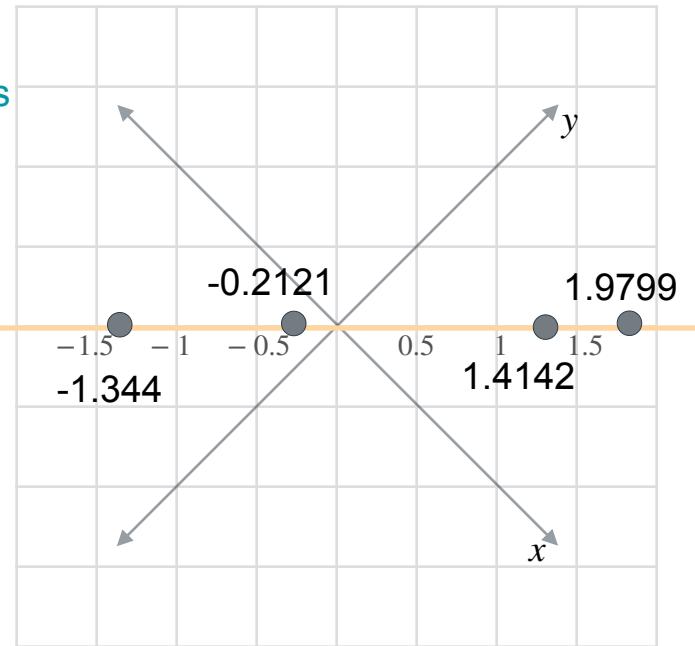


Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates
1.4142
1.9799
-0.2121
-1.344



Projections

To project a matrix A onto a vector v

$$A_P = A \frac{v}{\|v\|_2}$$

$r \times 1 \quad r \times c \quad c \times 1$

Projections

To project a matrix A onto vectors v_1 and v_2

$$A_P = A \underbrace{\begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}}_{V}$$

$r \times 2$ $r \times c$ $c \times 2$

Projections

To project a matrix A onto vectors v_1 and v_2

$$A_P = A \underbrace{\begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}}_{V}$$

$r \times 2$ $\boxed{r} \times c$ $c \times \boxed{2}$

Projections

To project a matrix A onto vectors v_1 and v_2

$$A_P = \textcolor{brown}{A} \textcolor{green}{V}$$

$$r \times 2 \quad r \times c \quad c \times 2$$

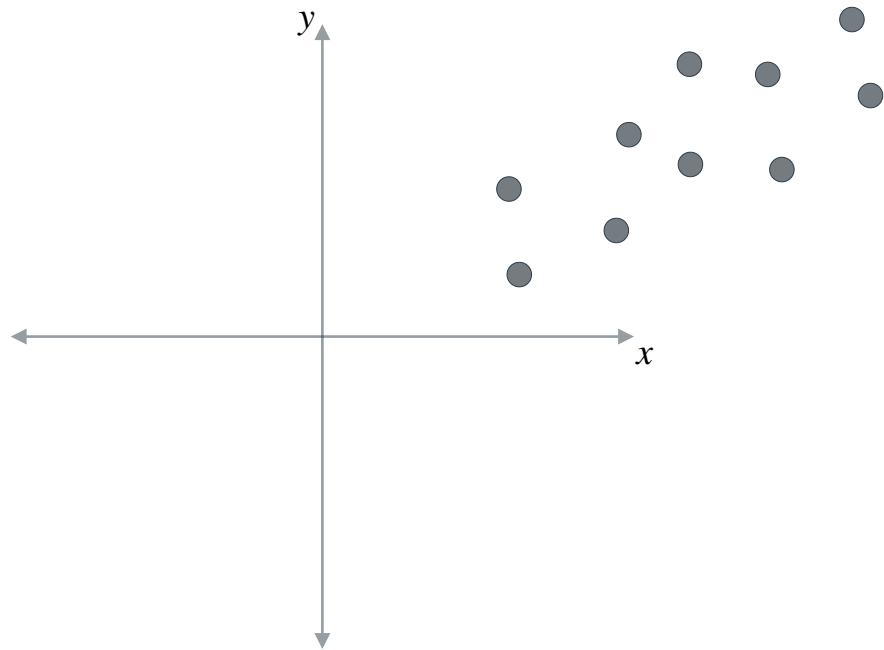


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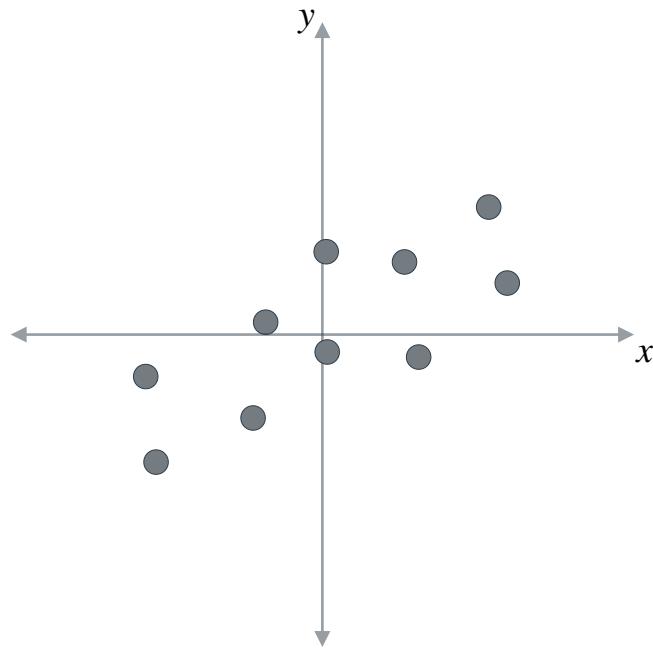
Determinants and Eigenvectors

Motivating PCA

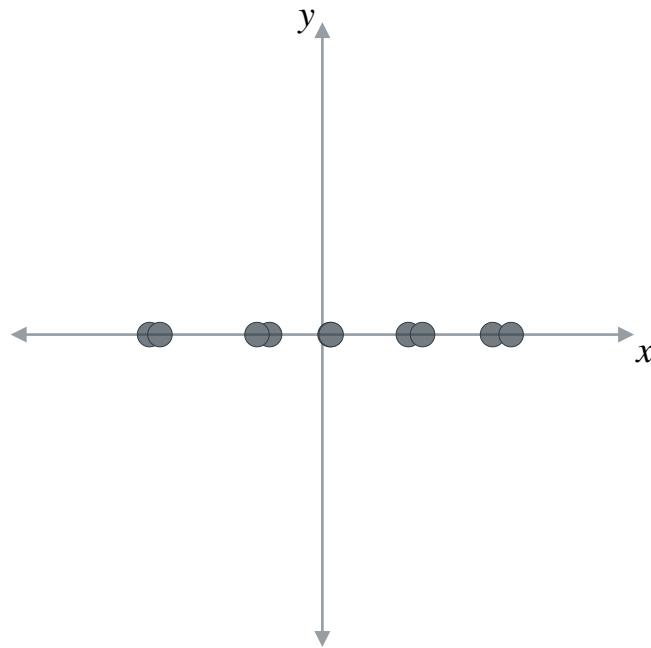
Dimensionality Reduction



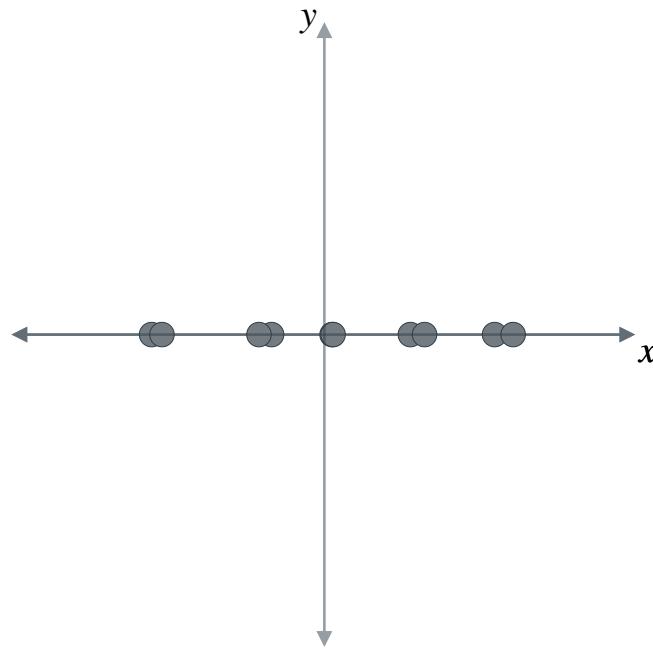
Principal Component Analysis (PCA)



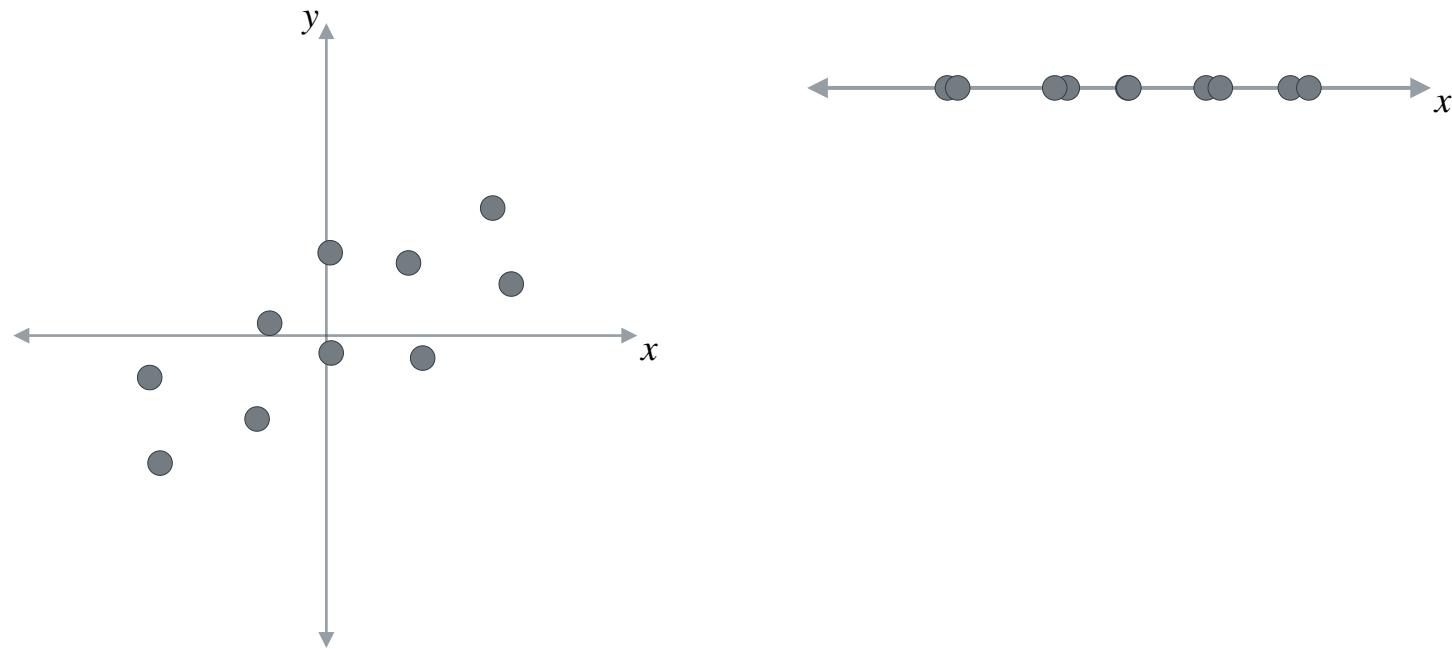
Principal Component Analysis (PCA)



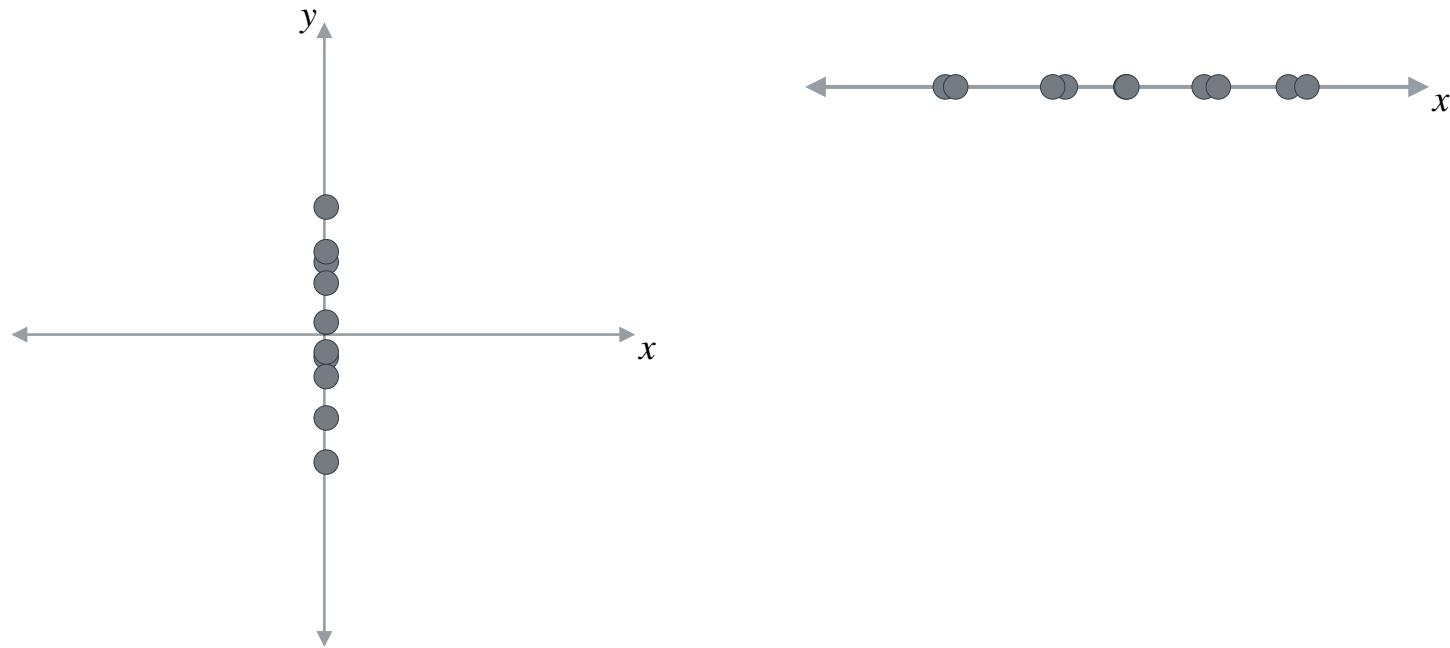
Principal Component Analysis (PCA)



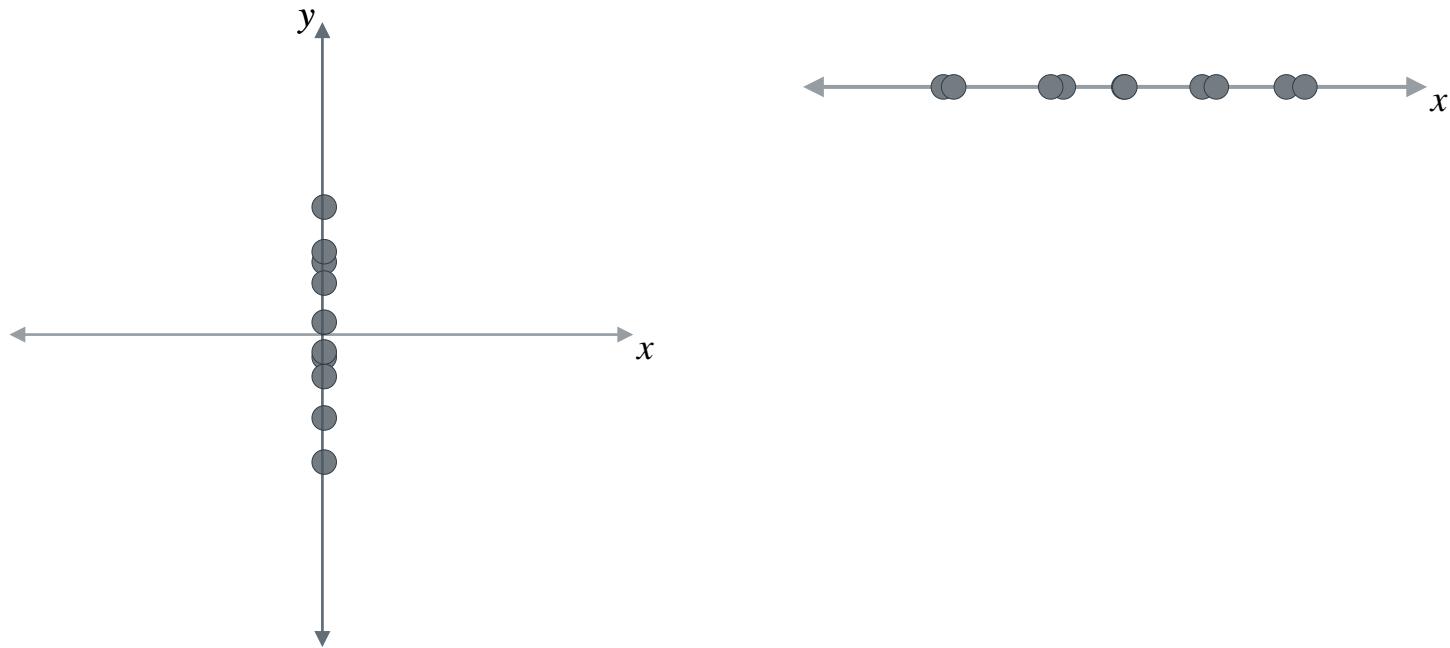
Principal Component Analysis (PCA)



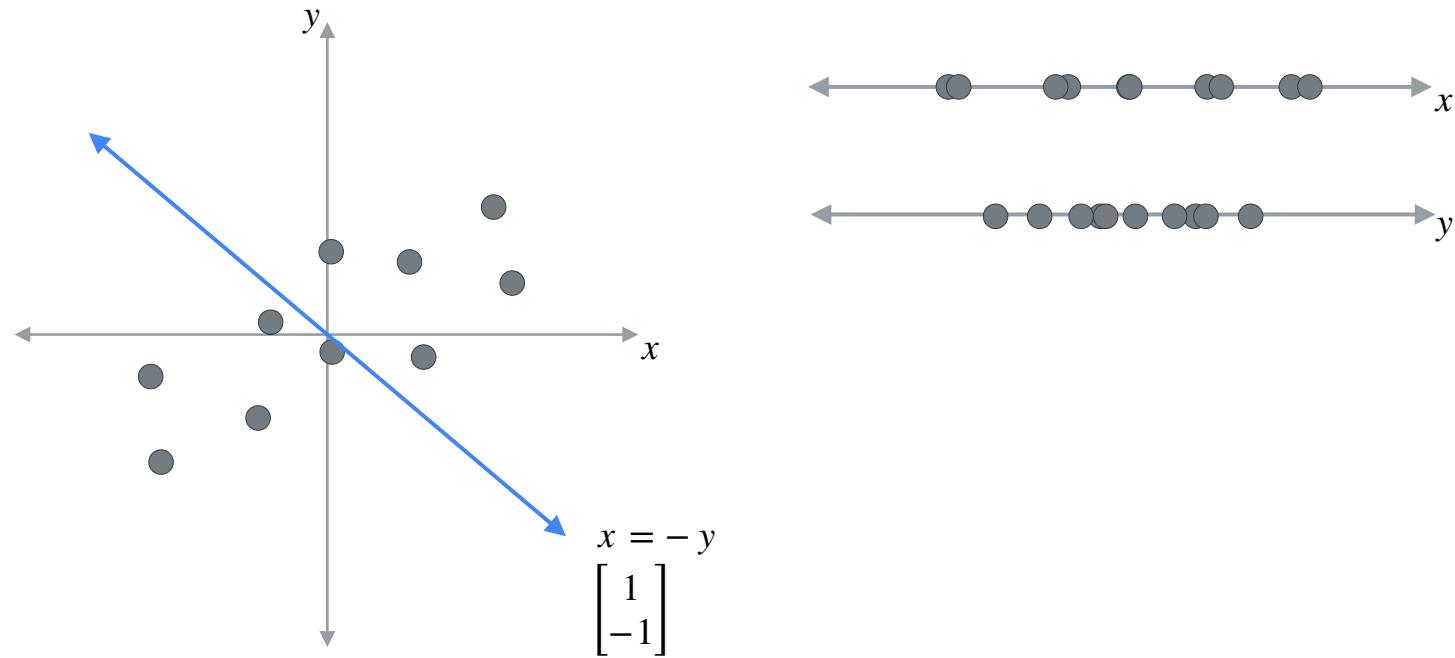
Principal Component Analysis (PCA)



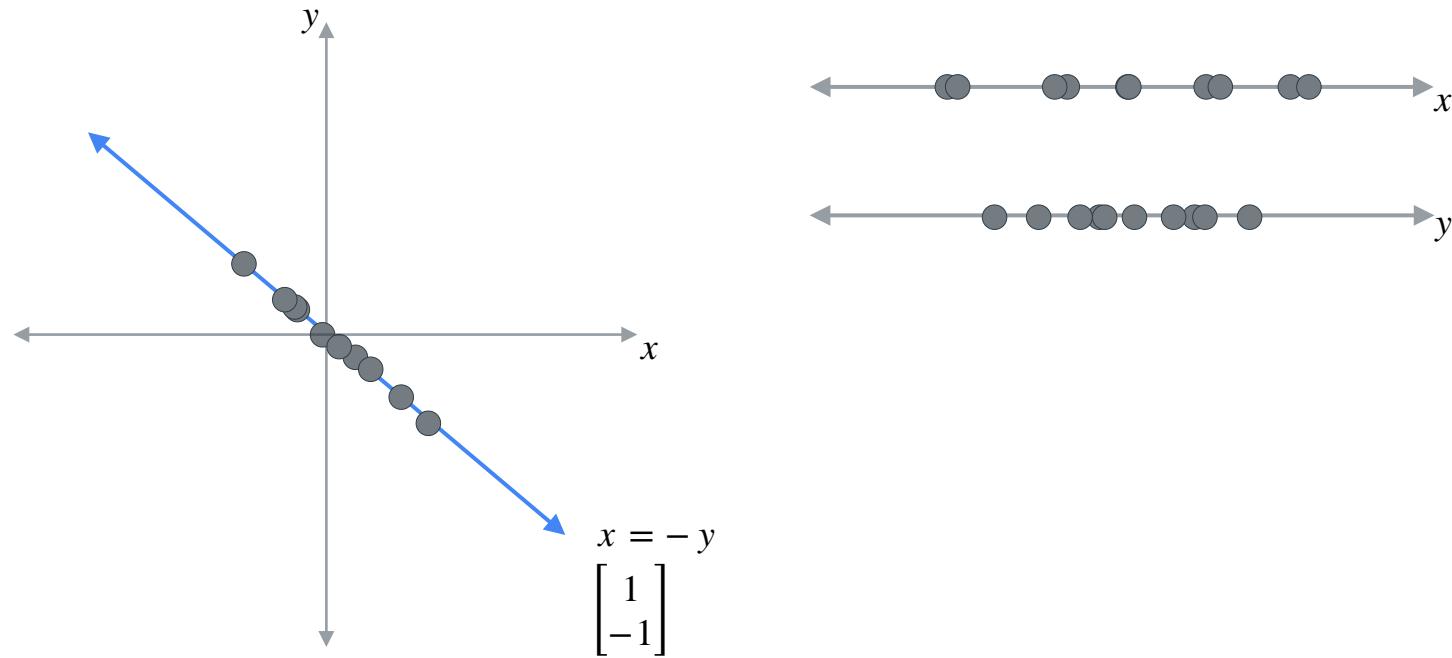
Principal Component Analysis (PCA)



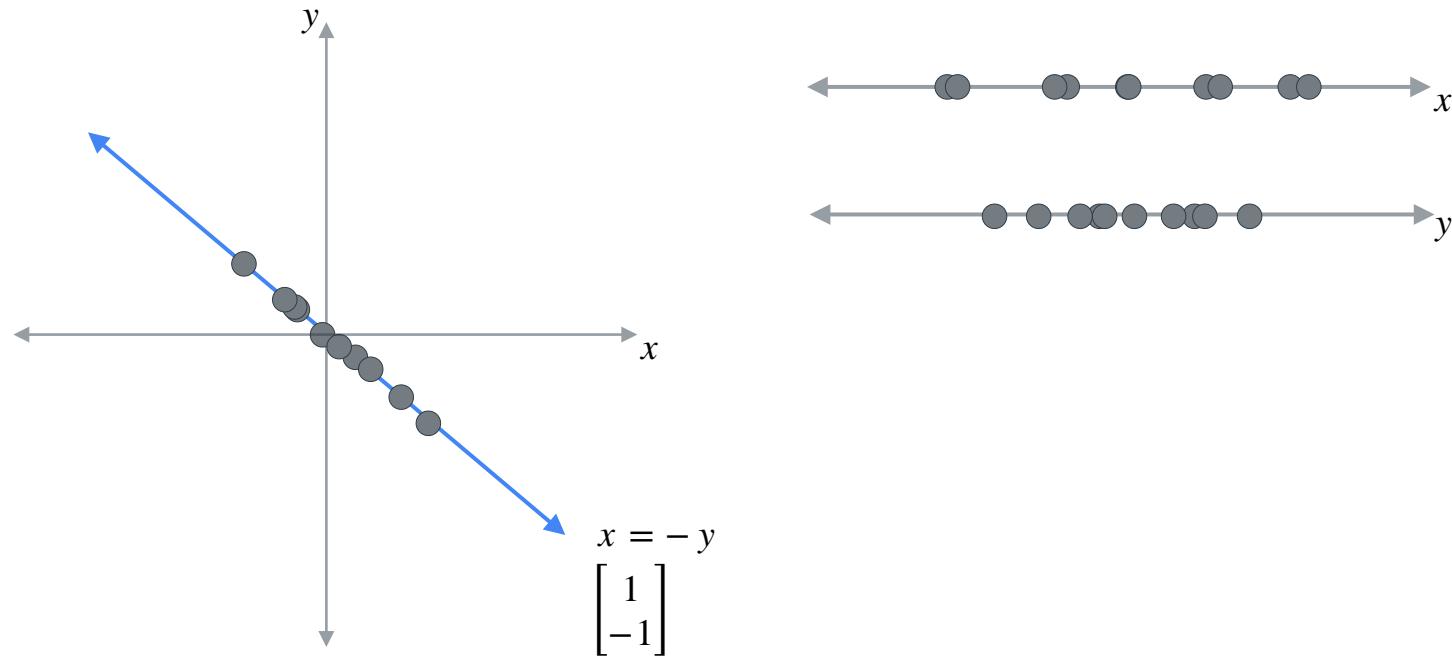
Principal Component Analysis (PCA)



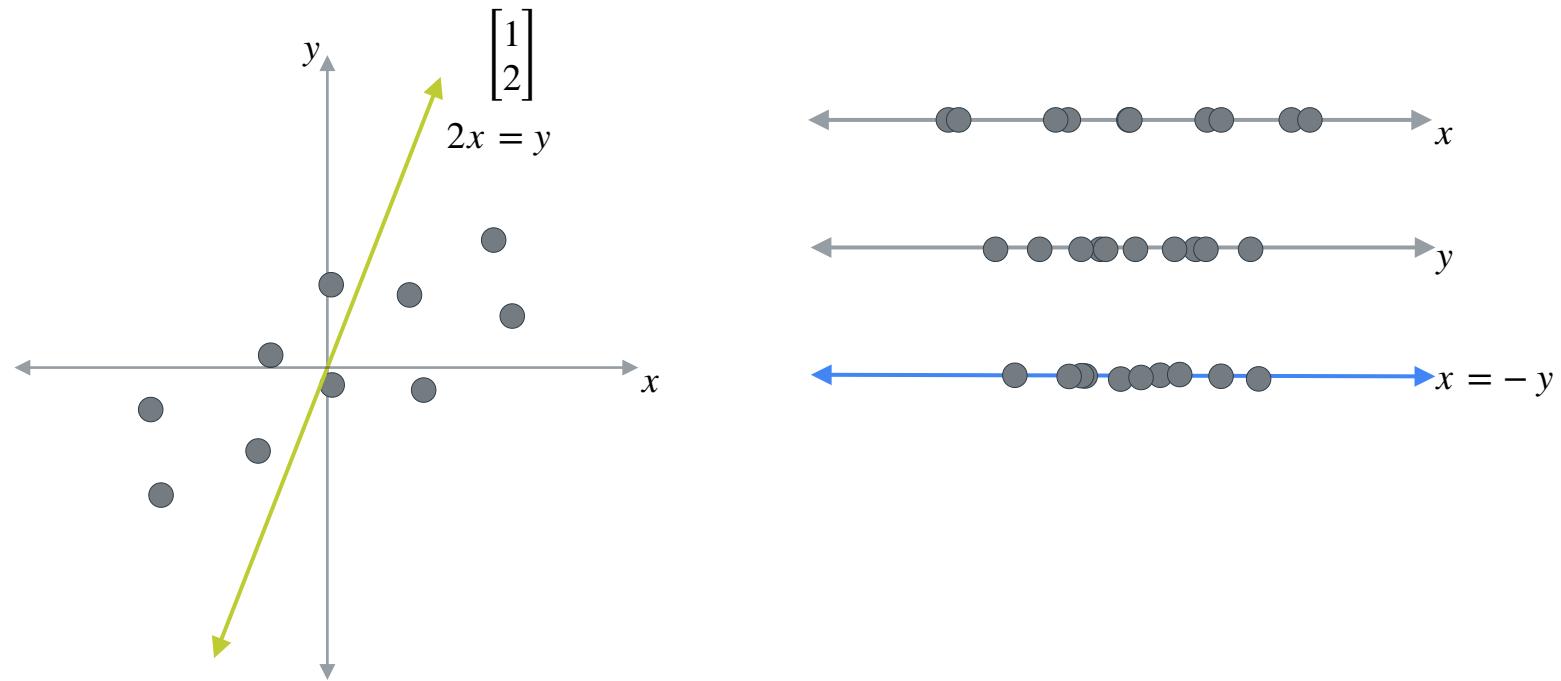
Principal Component Analysis (PCA)



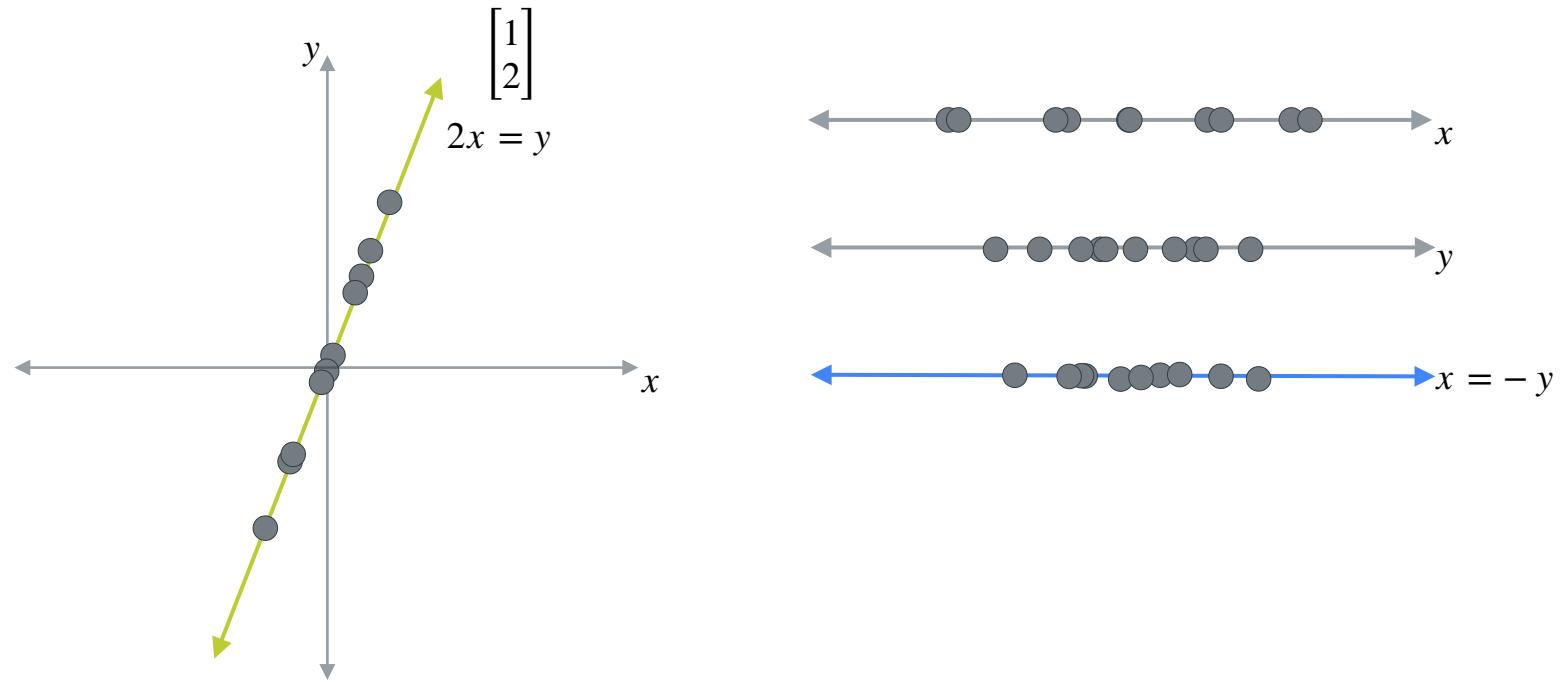
Principal Component Analysis (PCA)



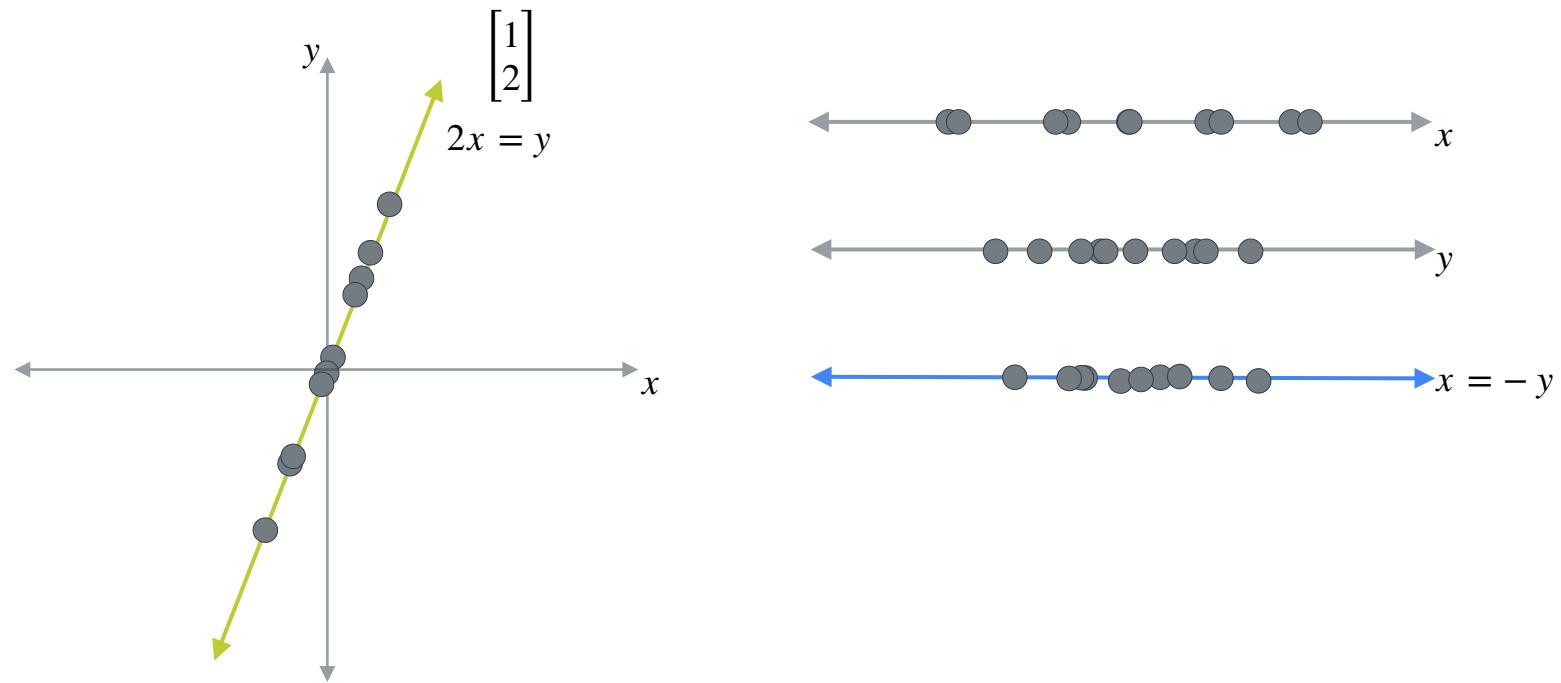
Principal Component Analysis (PCA)



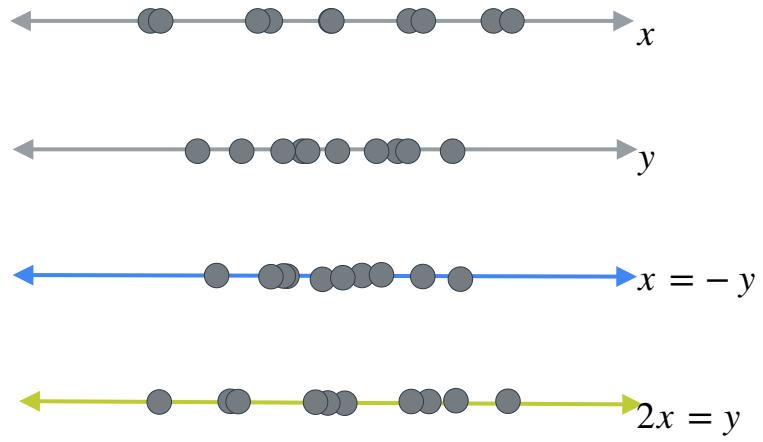
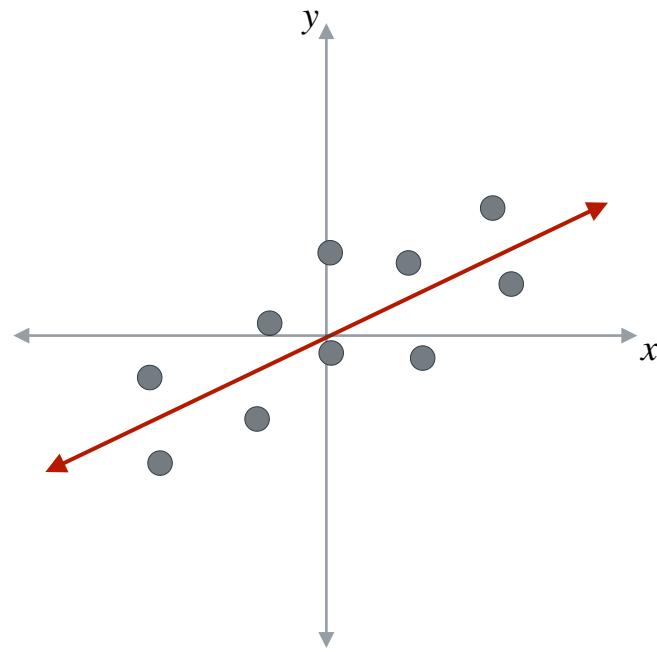
Principal Component Analysis (PCA)



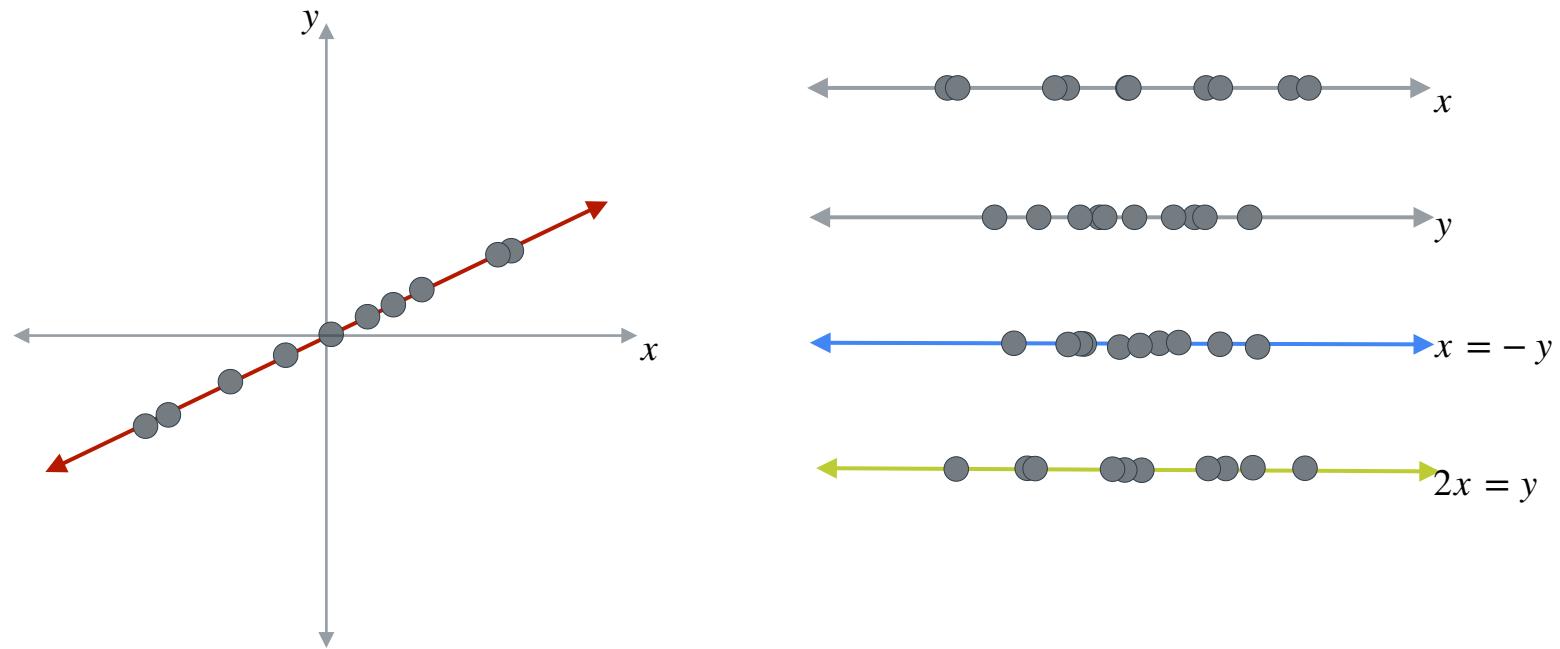
Principal Component Analysis (PCA)



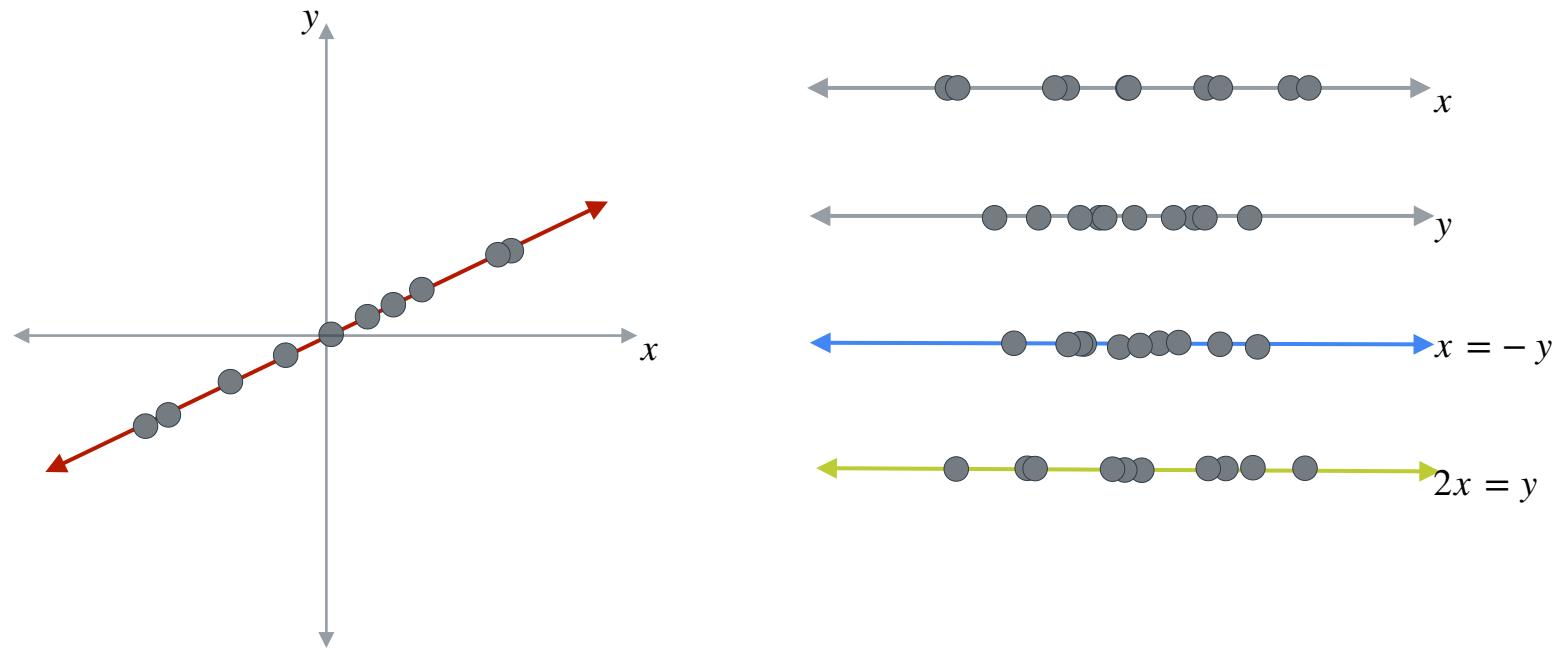
Principal Component Analysis (PCA)



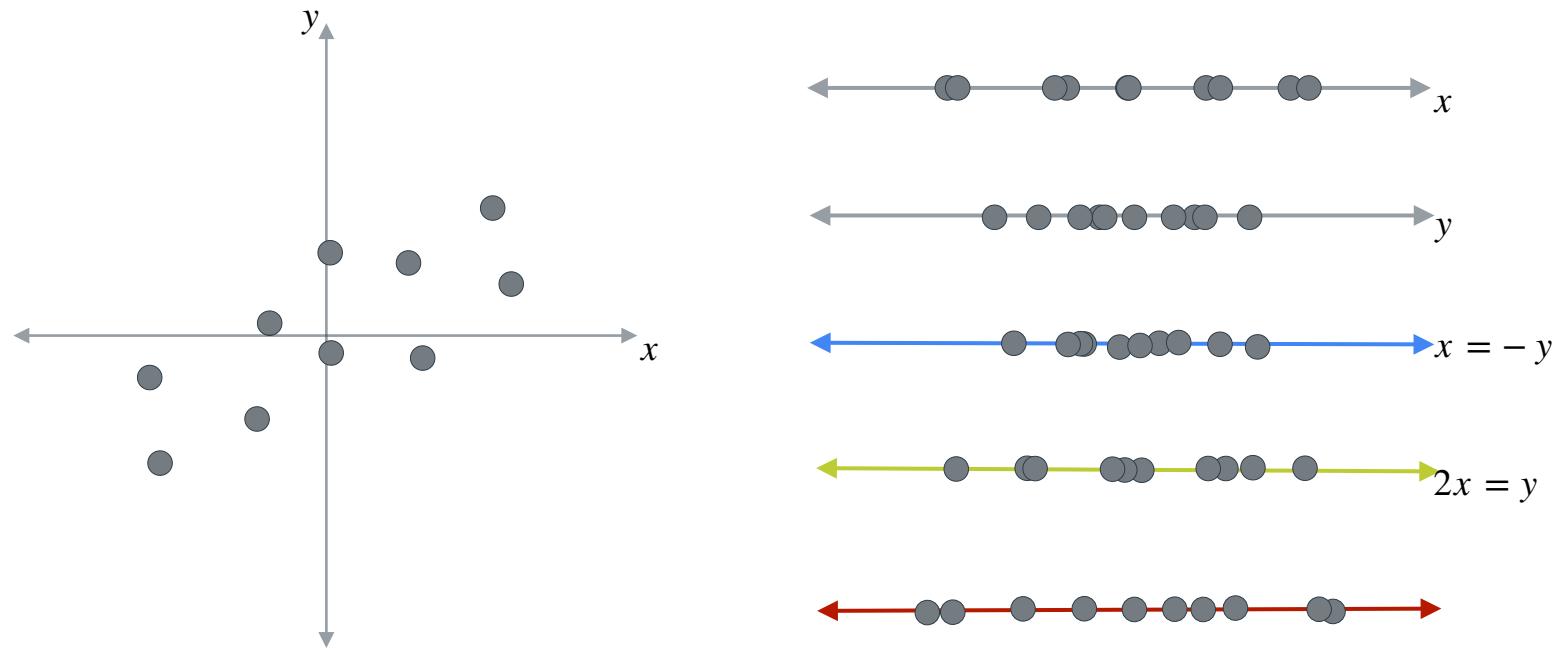
Principal Component Analysis (PCA)



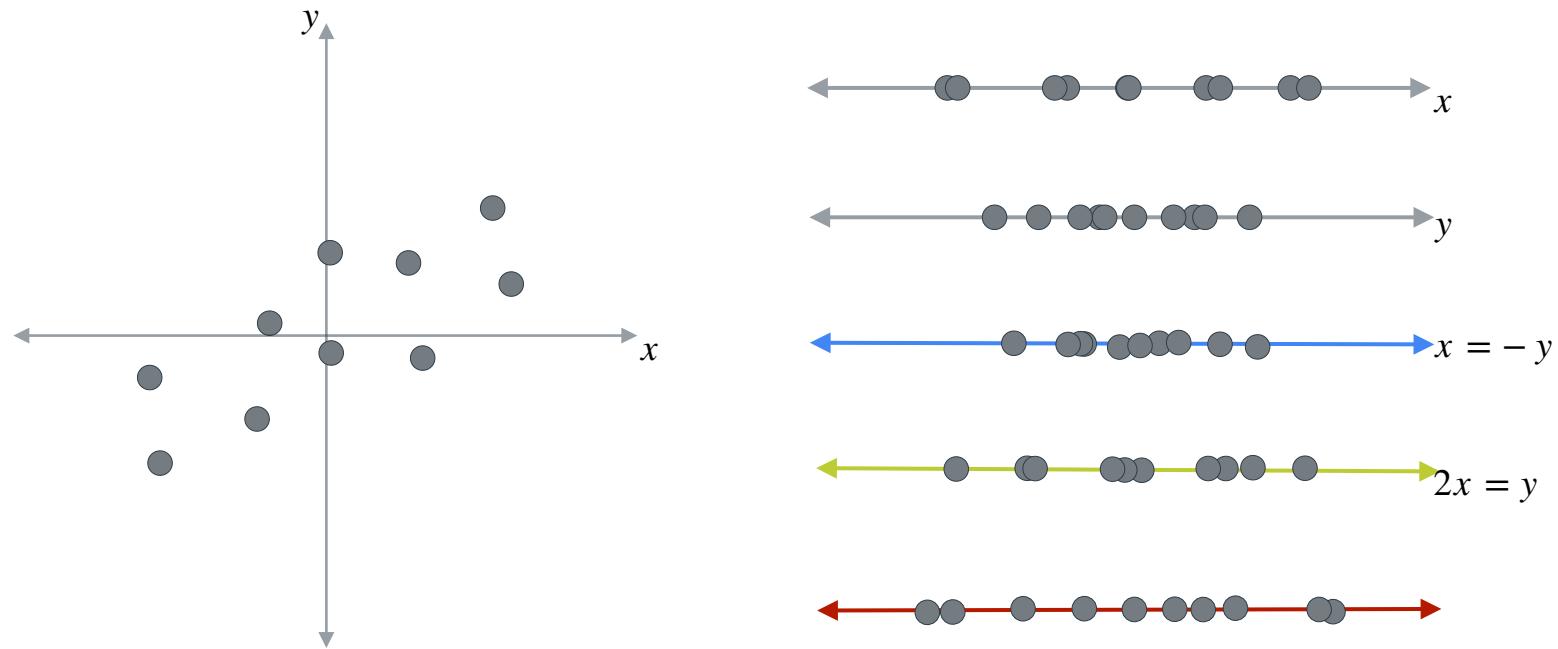
Principal Component Analysis (PCA)



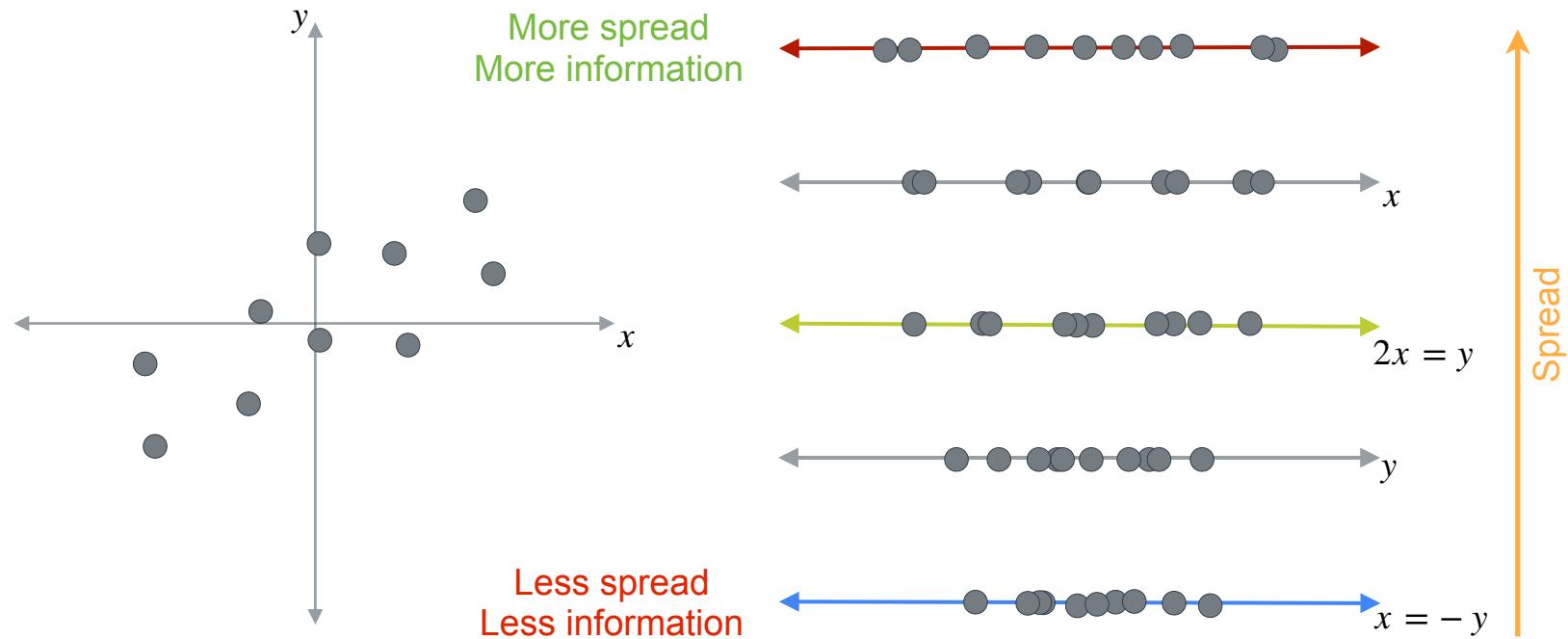
Principal Component Analysis (PCA)



Principal Component Analysis (PCA)

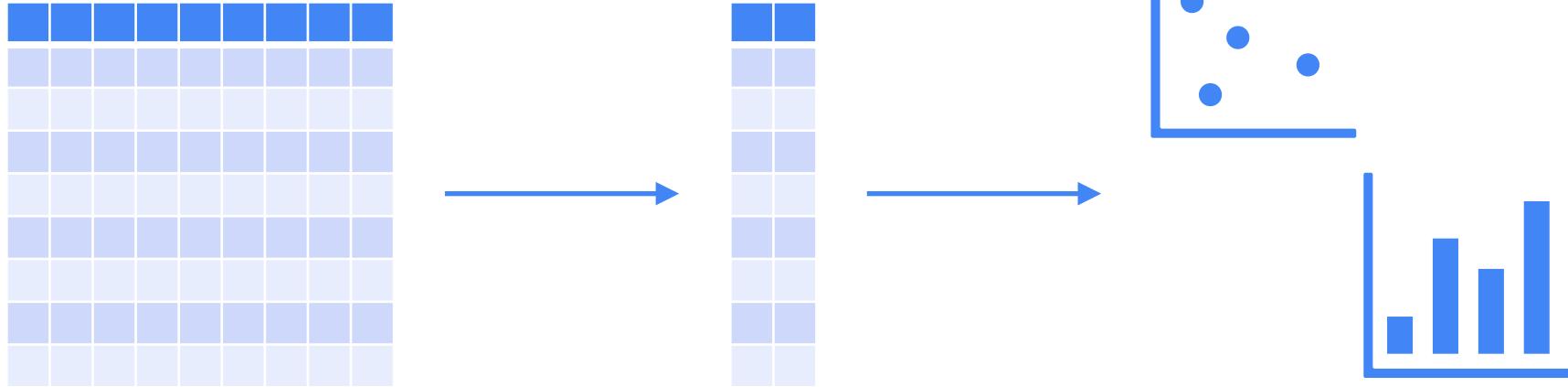


Principal Component Analysis (PCA)



Benefits of Dimensionality Reduction

- Easier dataset to manage
- PCA reduces dimensions while minimizing information loss
- Simpler visualization





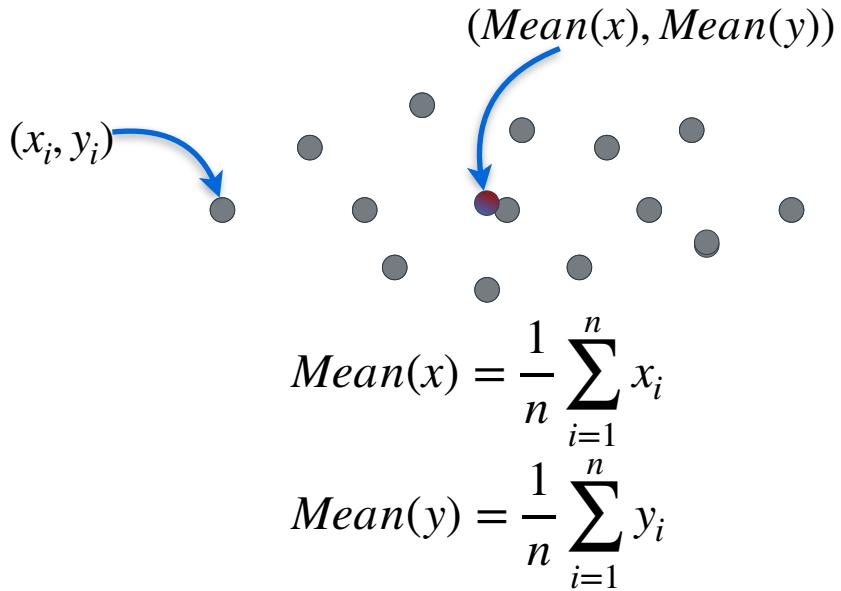
DeepLearning.AI

Determinants and Eigenvectors

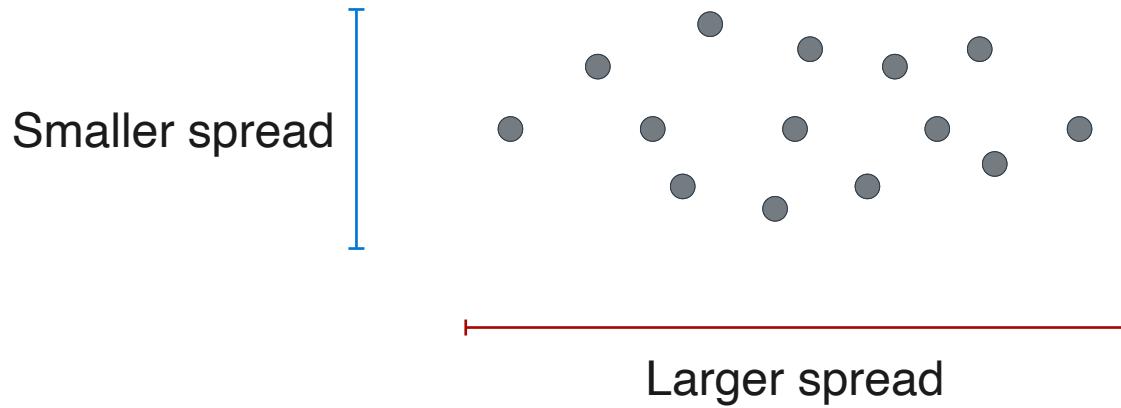
Variance and covariance

Mean

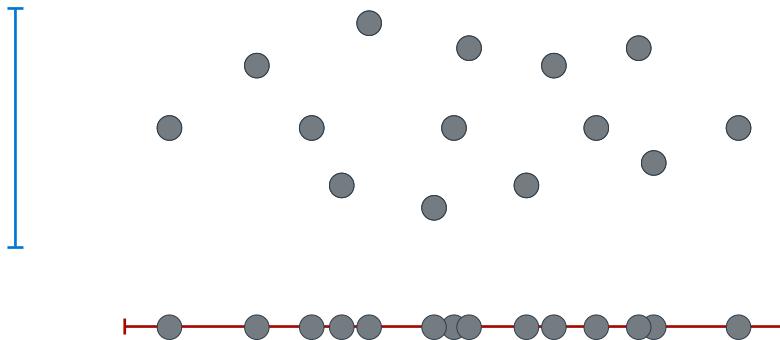
“The average of the data”



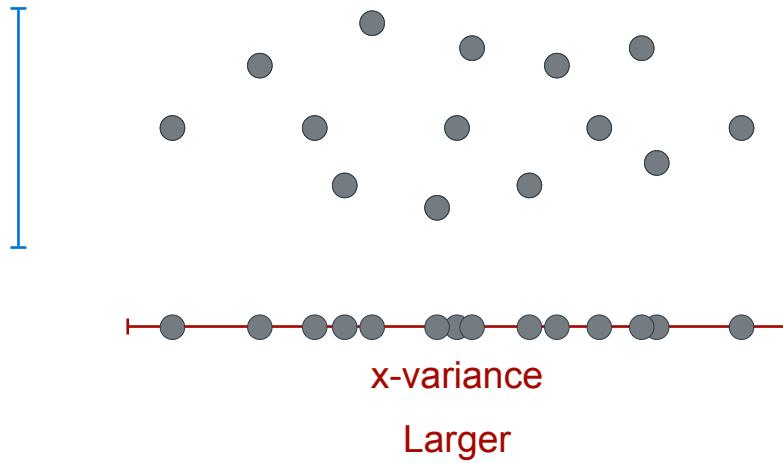
Variance



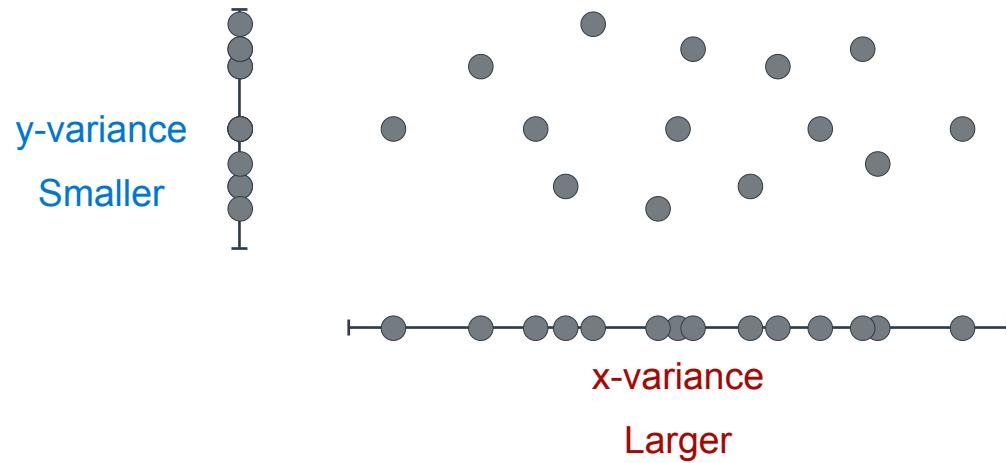
Variance



Variance



Variance



Variance

$$Variance(x) = \frac{1}{n - 1} \sum_{i=1}^n (x_i - Mean(x))^2 = 16$$

	x_i	$x_i - Mean(x)$	$(x_i - Mean(x))^2$
1	10	1	1
2	4	-5	25
3	11	2	4
4	14	5	25
5	6	-3	9

→ 64

$$Mean(x) = 9$$

Variance

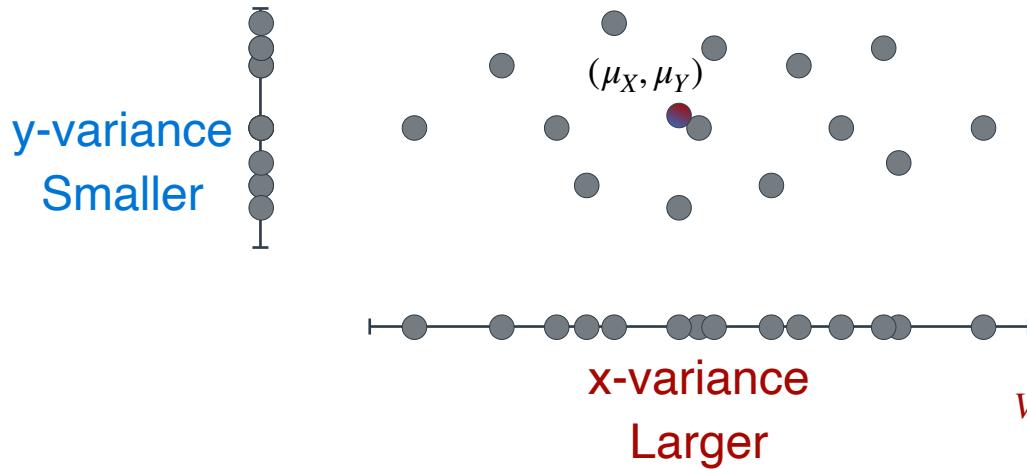
$$Variance(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \text{Mean}(x))^2$$

$$Var(x) \quad \mu_x$$

“The average squared distance from the mean”

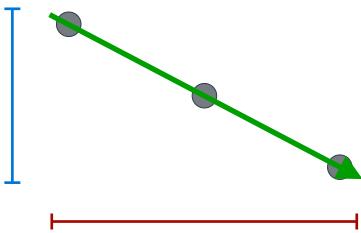
Variance

$$Var(y) = \frac{1}{n-1} \sum_{i=1}^n (y_i - \mu_Y)^2$$



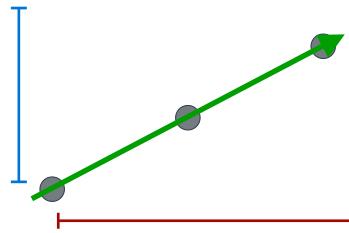
$$Var(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)^2$$

Problem



Negative covariance

Solution: Covariance



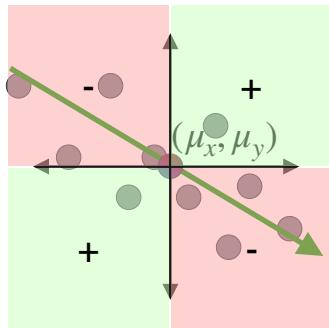
Positive covariance

Covariance

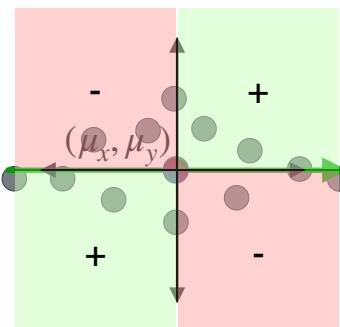
“Take the average”

$$Cov(x, y) = \frac{1}{n - 1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y)$$

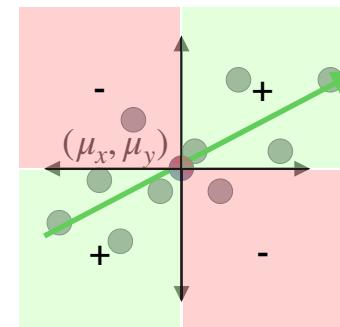
$$Var(x) = \frac{1}{n - 1} \sum_{i=1}^n (x_i - \mu_x)^2$$



negative covariance



covariance zero
(or very small)

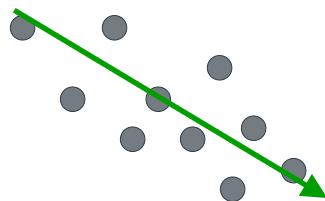


positive covariance

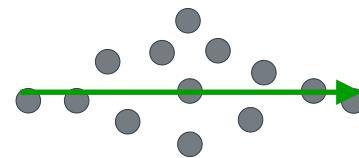
Covariance

$$Cov(x, y) = \frac{1}{n - 1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y)$$

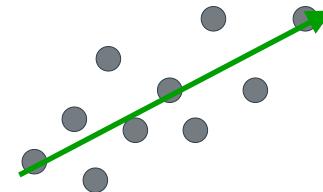
“The direction of the relationship between two variables”



negative covariance



covariance zero
(or very small)



positive covariance

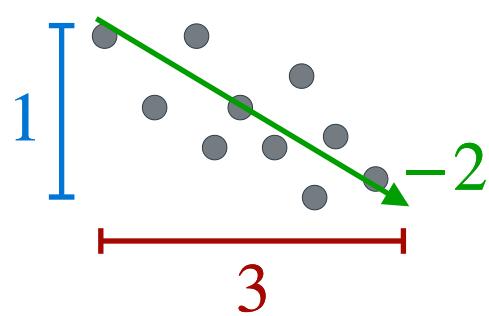


DeepLearning.AI

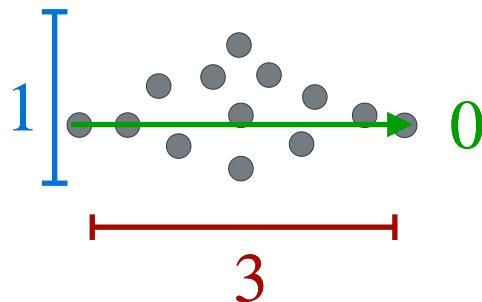
Determinants and Eigenvectors

The covariance matrix

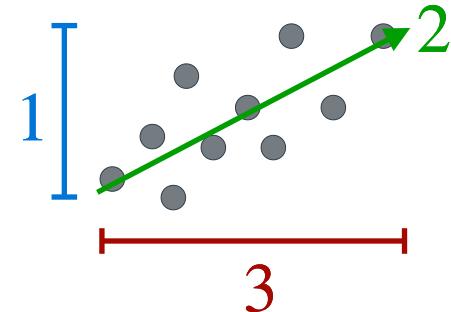
Covariance matrix



$$\begin{bmatrix} & \\ & \end{bmatrix}$$

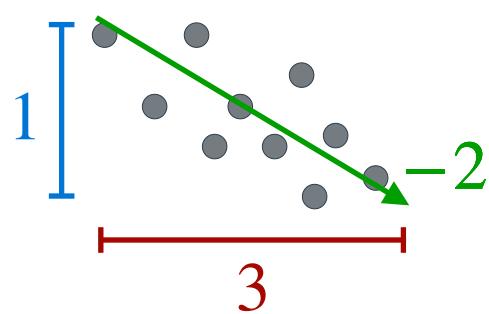


$$\begin{bmatrix} & \\ & \end{bmatrix}$$

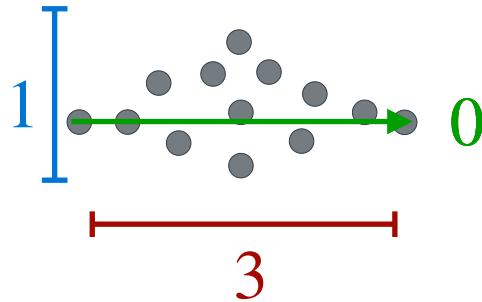


$$\begin{bmatrix} & \\ & \end{bmatrix}$$

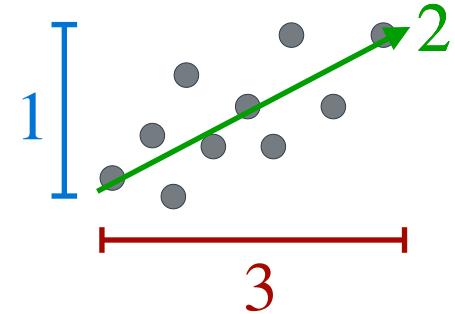
Covariance matrix



$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

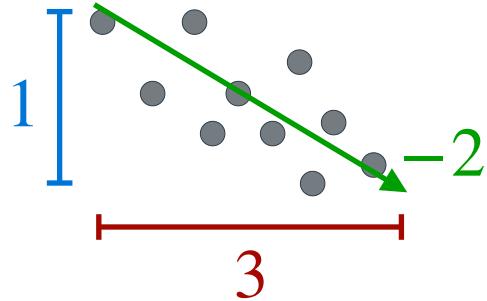


$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

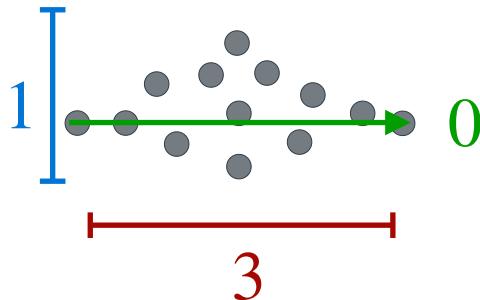


$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

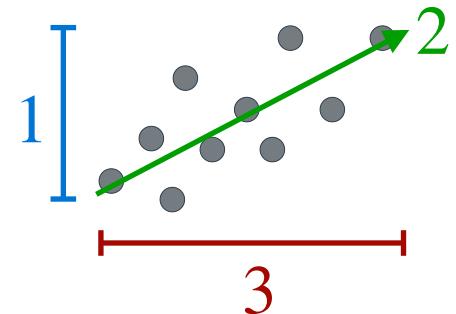
Covariance matrix



$$\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

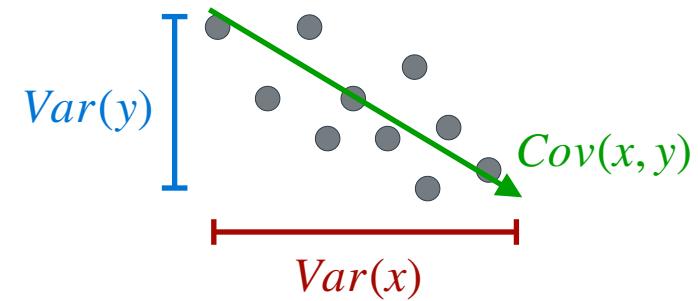


$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

Covariance matrix



$$C = \begin{bmatrix} x & y \\ Cov(x, x) & Cov(x, y) \\ Cov(y, x) & Cov(y, y) \end{bmatrix}$$

$$Cov(x, x) = Var(x)$$

Covariance matrix

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n - 1} (\mathbf{x} - \boldsymbol{\mu})^T (\mathbf{x} - \boldsymbol{\mu})$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n - 1}(A - \mu)^T(A - \mu)$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$\begin{aligned} C &= \frac{1}{n-1}(A - \mu)^T(A - \mu) = \frac{1}{n-1} \left(\begin{array}{c|c} & - \\ \hline - & \end{array} \right)^T \left(\begin{array}{c|c} & - \\ \hline - & \end{array} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \end{aligned}$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \end{aligned}$$

Covariance matrix

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &\quad \boxed{2} \times n \qquad \qquad \qquad n \times \boxed{2} \end{aligned}$$

Covariance matrix

$$\begin{aligned} C &= \frac{1}{n-1}(A - \mu)^T(A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &\quad (x_1 - \mu_x)(x_1 - \mu_x) + (x_2 - \mu_x)(x_2 - \mu_x) + \dots + (x_n - \mu_x)(x_n - \mu_x) \end{aligned}$$

Covariance matrix

$$\begin{aligned} C &= \frac{1}{n-1}(A - \mu)^T(A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &\quad \sum_{i=1}^n (x_i - \mu_x)^2 = \text{Var}(x) \end{aligned}$$

Covariance matrix

$$\begin{aligned} C &= \frac{1}{n-1}(A - \mu)^T(A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & \\ & \ddots \\ & & Var(y) \end{bmatrix} \\ &\quad \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)^2 = Var(x) \end{aligned}$$

Covariance matrix

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \textcolor{red}{Var(x)} \\ \vdots \end{bmatrix} \end{aligned}$$

$(x_1 - \mu_x)(y_1 - \mu_y) + (x_2 - \mu_x)(y_2 - \mu_y) + \dots + (x_n - \mu_x)(y_n - \mu_y)$

Covariance matrix

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) \\ \text{Var}(y) \end{bmatrix} \\ &\quad \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y) = \text{Cov}(x, y) \end{aligned}$$

Covariance matrix

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & Cov(x, y) \\ Cov(y, x) & Var(y) \end{bmatrix} \\ &= \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y) = Cov(x, y) \end{aligned}$$

Covariance matrix

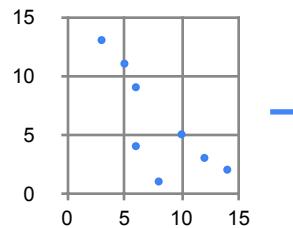
$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \textcolor{red}{Var(x)} & \textcolor{green}{Cov(x,y)} \\ \textcolor{green}{Cov(y,x)} & \textcolor{blue}{Var(y)} \end{bmatrix} \end{aligned}$$

Covariance matrix

$$\begin{aligned} C = \frac{1}{n-1}(A - \mu)^T(A - \mu) &= \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_n \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \textcolor{red}{Var(x)} & \textcolor{green}{Cov(x,y)} \\ \textcolor{green}{Cov(y,x)} & \textcolor{blue}{Var(y)} \end{bmatrix} \end{aligned}$$

Matrix formula

$$A - \mu = \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \quad C = \frac{1}{n-1}(A - \mu)^T(A - \mu)$$



A

x	y
10	5
12	3
6	9
6	4
5	11
14	2
8	1
3	13

$A - \mu$

$x - \mu_x$	$y - \mu_y$
2	-1
4	-3
-2	3
-2	-2
-3	5
6	-4
0	-5
-5	8

$$\frac{1}{7}$$

$(A - \mu)^T$

$x - \mu_x$	2	4	-2	-2	-3	6	0	-5
$y - \mu_y$	-1	-3	3	-2	5	-4	-5	8

$A - \mu$

$x - \mu_x$	$y - \mu_y$
2	-1
4	-3
-2	3
-2	-2
-3	5
6	-4
0	-5
-5	8

$$\mu_x = 8 \quad \mu_y = 6$$

$$C = \begin{bmatrix} 14 & -11.86 \\ -11.86 & 19.71 \end{bmatrix}$$

Matrix formula

$$A = \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix} \quad C = \frac{1}{n-1} (A - \mu)^T (A - \mu)$$

1. Arrange data with a different feature in each column
2. Calculate column averages
3. Subtract each average from their respective column to generate $A - \mu$
4. $\frac{1}{n-1} (A - \mu)^T (A - \mu)$ gives the covariance matrix C

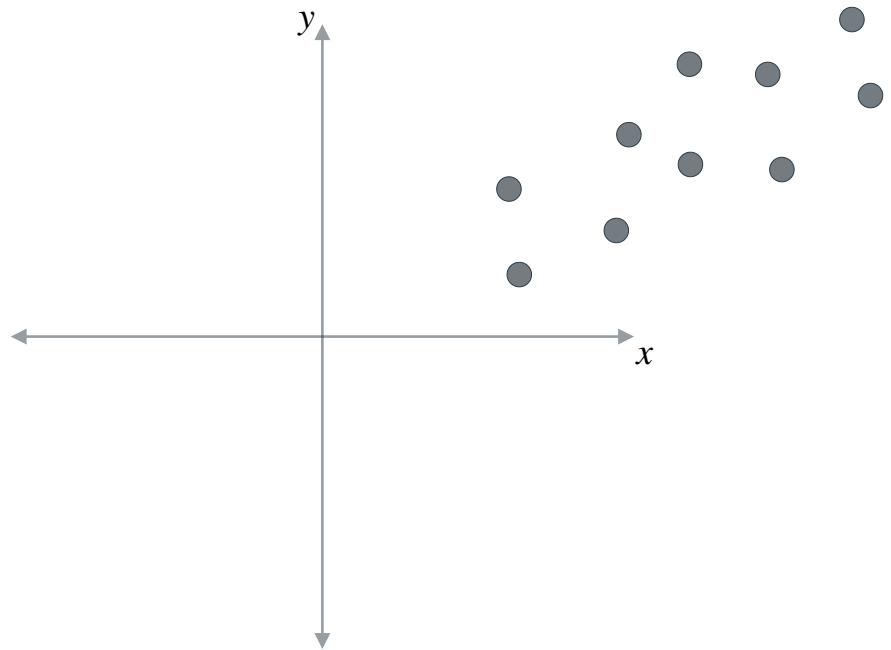


DeepLearning.AI

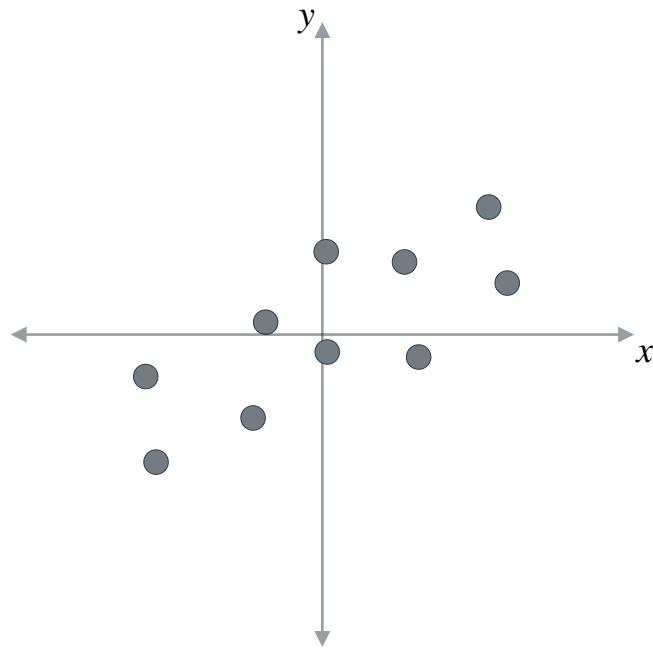
Determinants and Eigenvectors

PCA - Overview

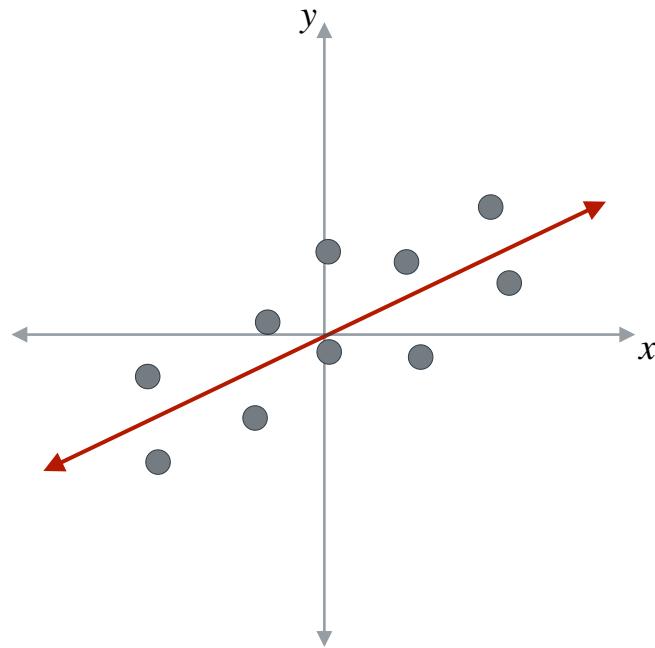
Principal Component Analysis (PCA)



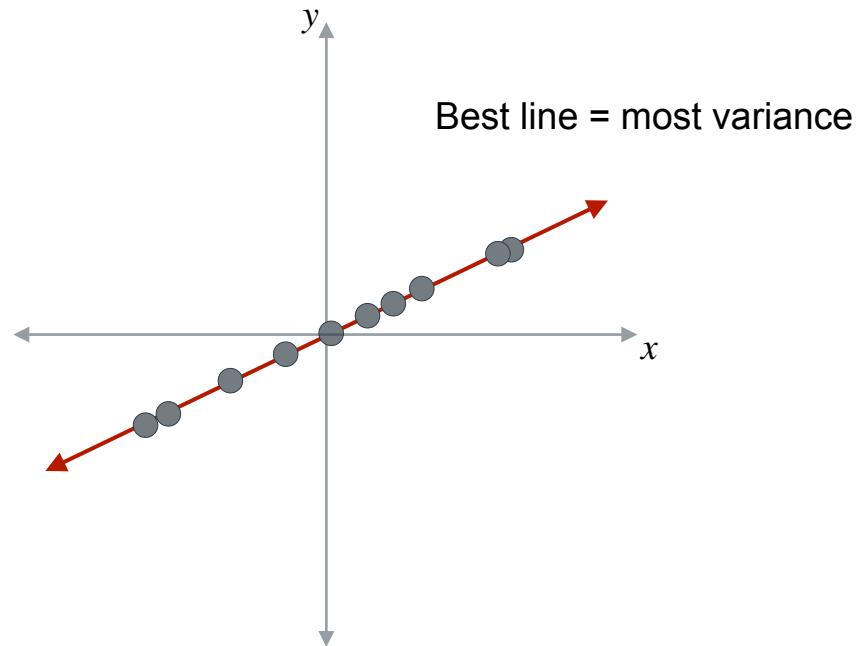
Principal Component Analysis (PCA)



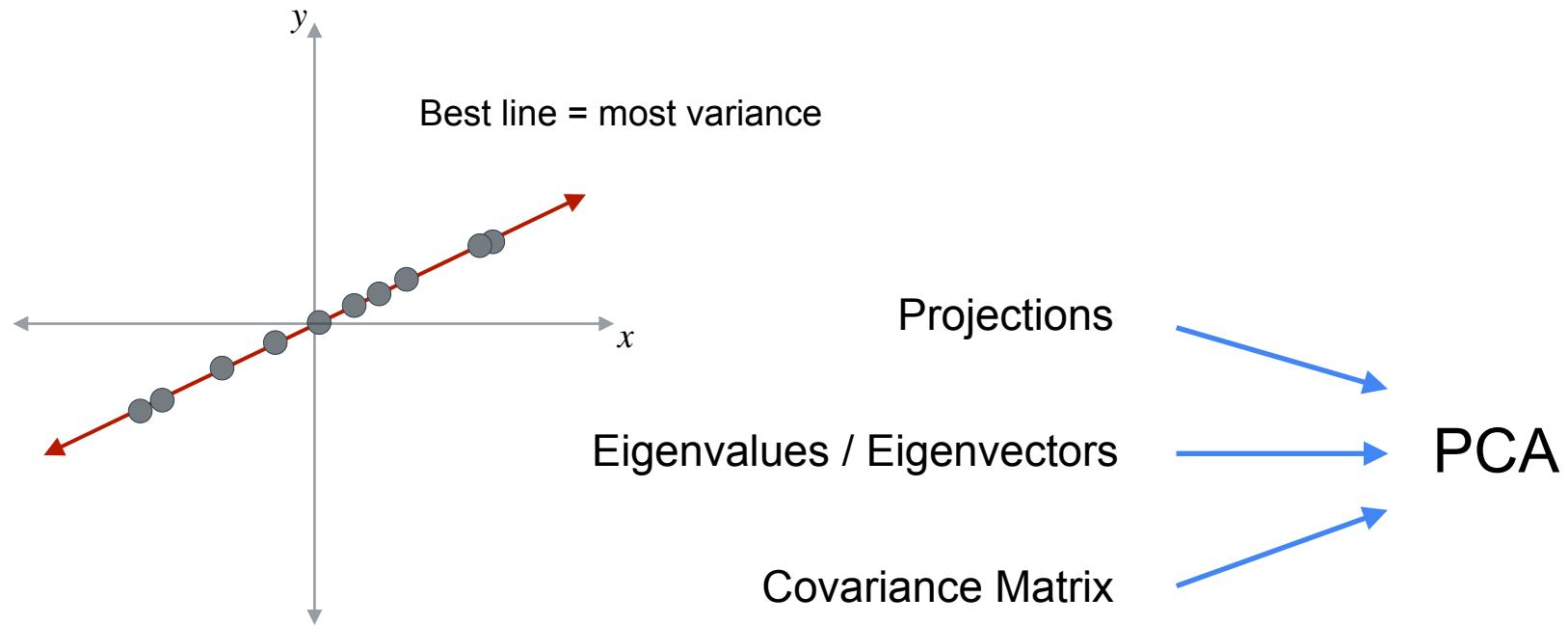
Principal Component Analysis (PCA)



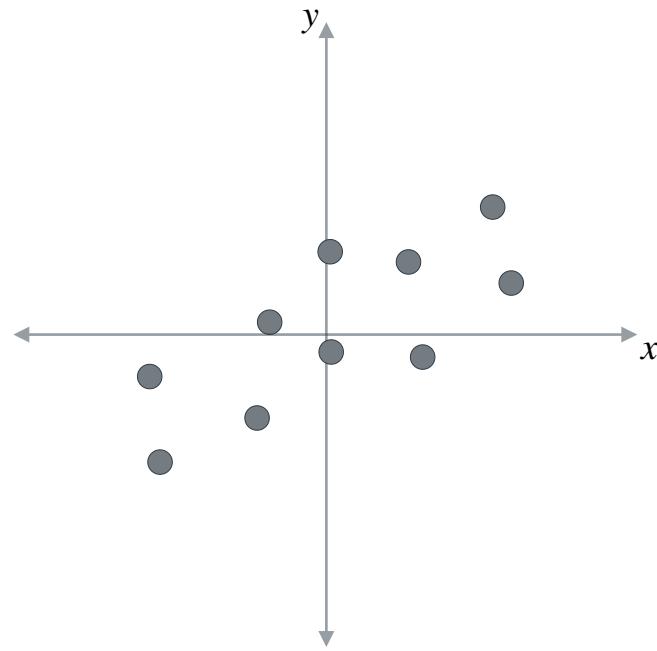
Principal Component Analysis (PCA)



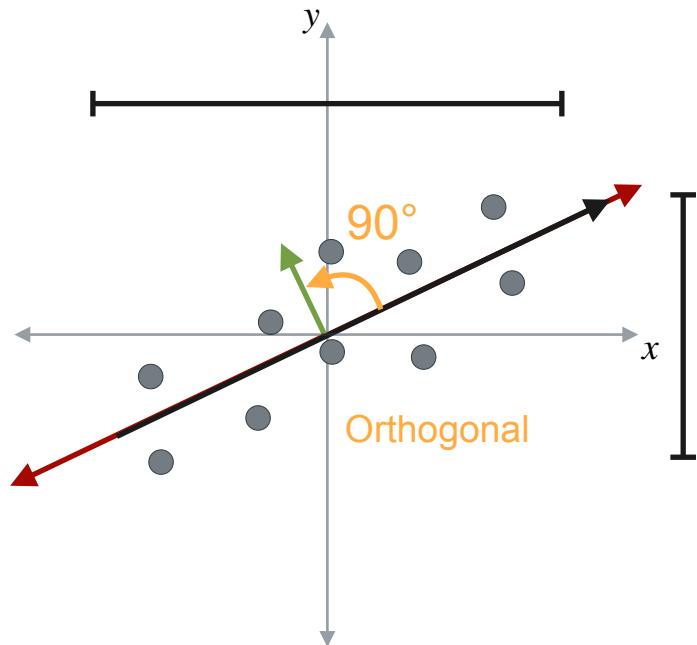
Principal Component Analysis (PCA)



Principal Component Analysis (PCA)



Principal Component Analysis (PCA)



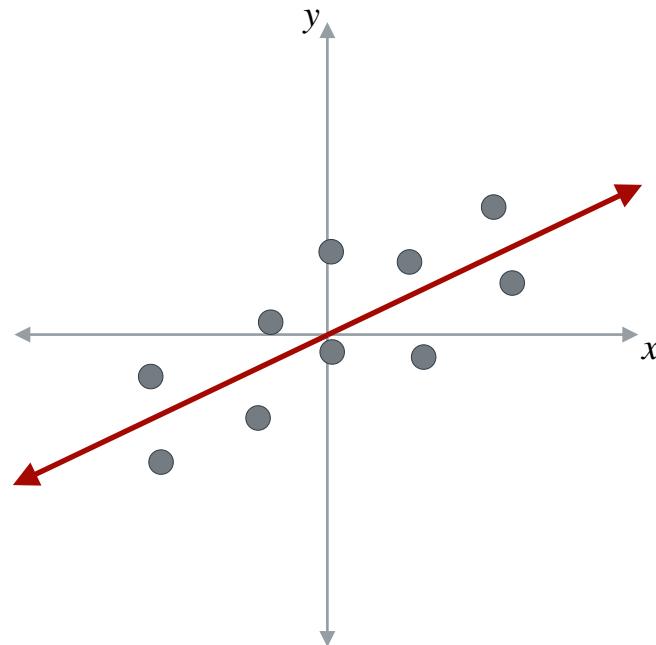
$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

Eigenvalues (magnitude)

Eigenvectors (direction)

The matrix C is shown as a 2x2 matrix with elements 9, 4, 4, and 3. The numbers 9 and 4 are circled in cyan. To the right of the matrix are two pinkish-red arrows representing eigenvectors. The first arrow is horizontal and the second is diagonal. Below the matrix is a yellow trophy icon with the number 11 next to it. The text "Eigenvalues (magnitude)" is positioned below the trophy, and "Eigenvectors (direction)" is positioned to the right of the arrows.

Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

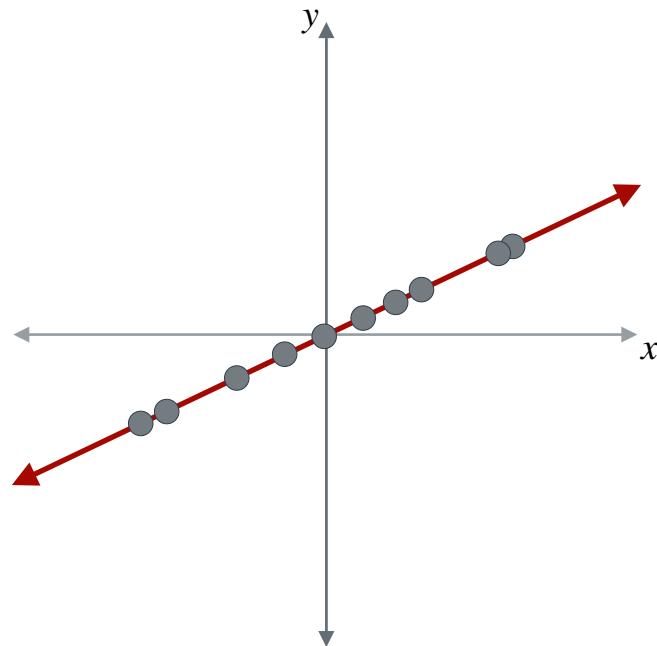
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors
(direction)

$$11$$

Eigenvalues
(magnitude)

Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

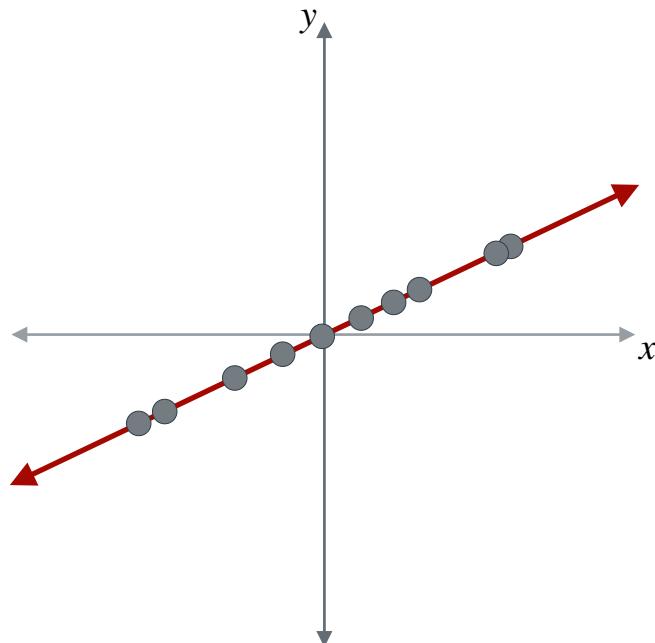
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors
(direction)

$$11$$

Eigenvalues
(magnitude)

Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

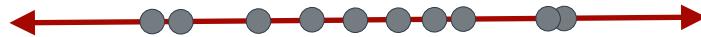
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors
(direction)

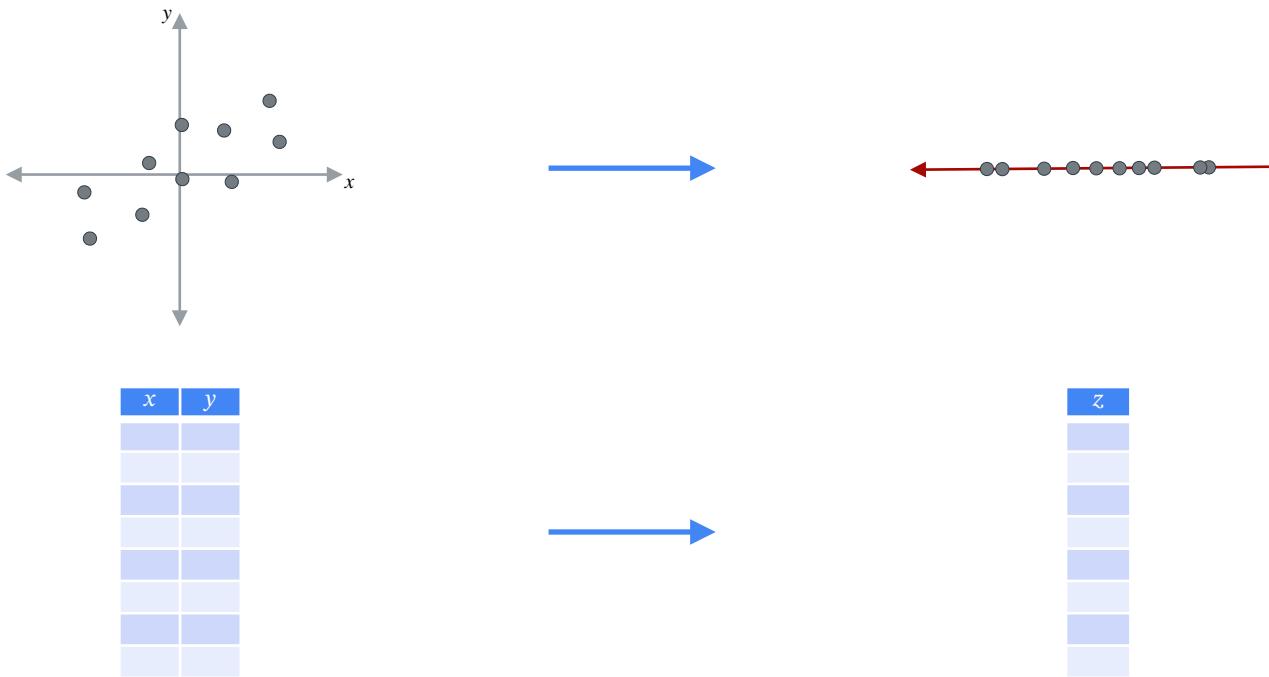
$$11$$

Eigenvalues
(magnitude)

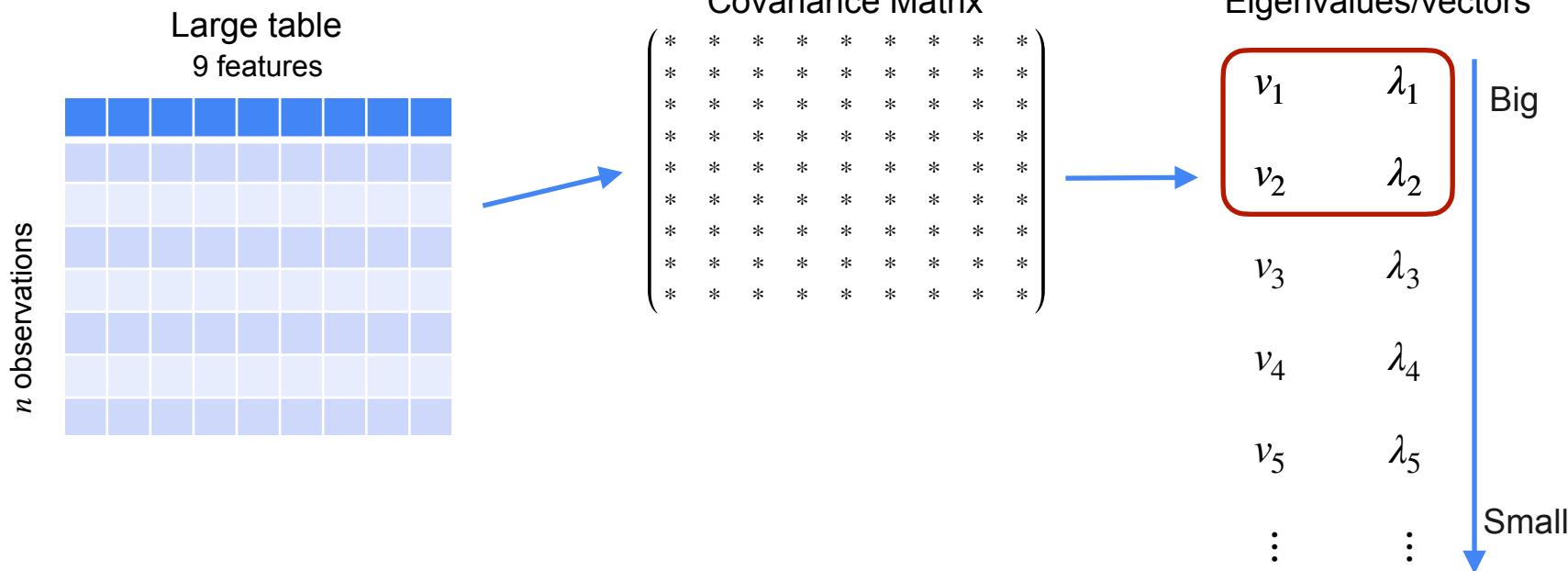
Principal Component Analysis (PCA)



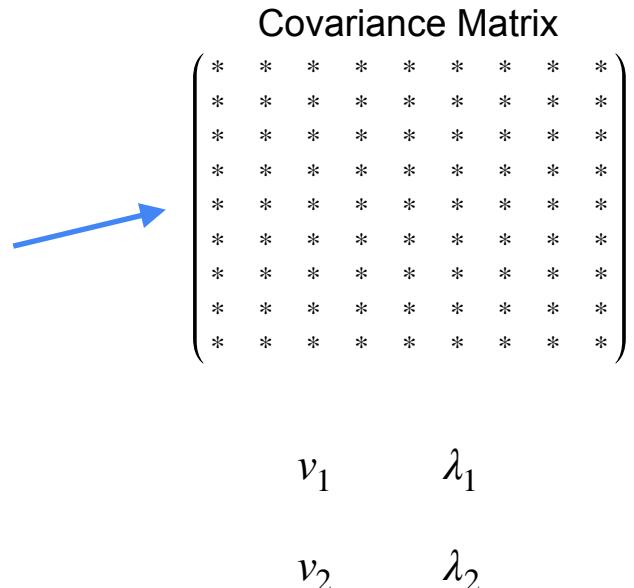
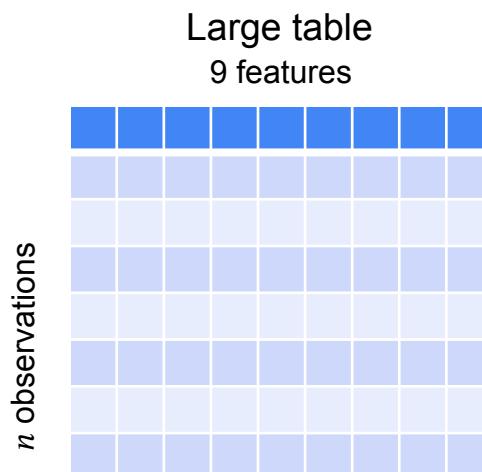
Principal Component Analysis (PCA)



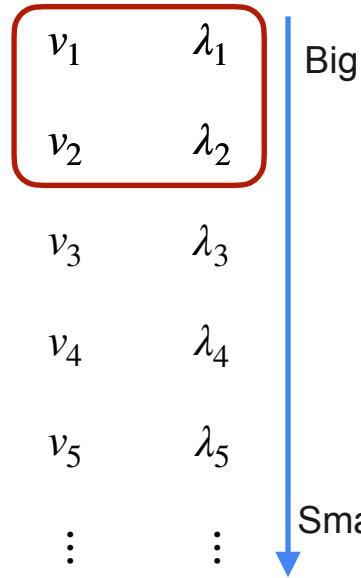
PCA: Principal Component Analysis



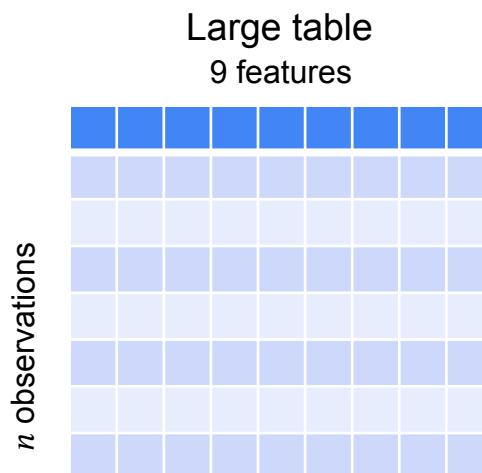
PCA: Principal Component Analysis



Eigenvalues/vectors



PCA: Principal Component Analysis



Covariance Matrix

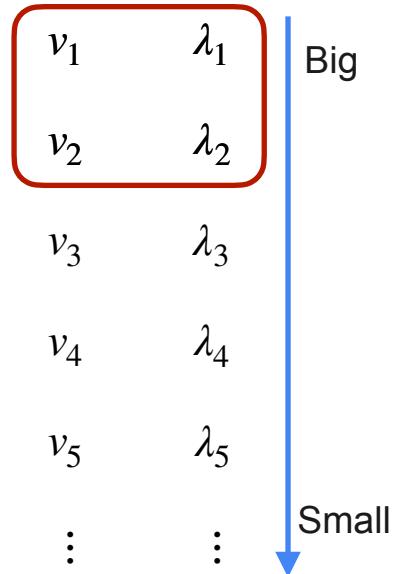
$$\begin{pmatrix} * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{pmatrix}$$

$$\begin{array}{ll} v_1 & \lambda_1 \\ v_2 & \lambda_2 \end{array}$$

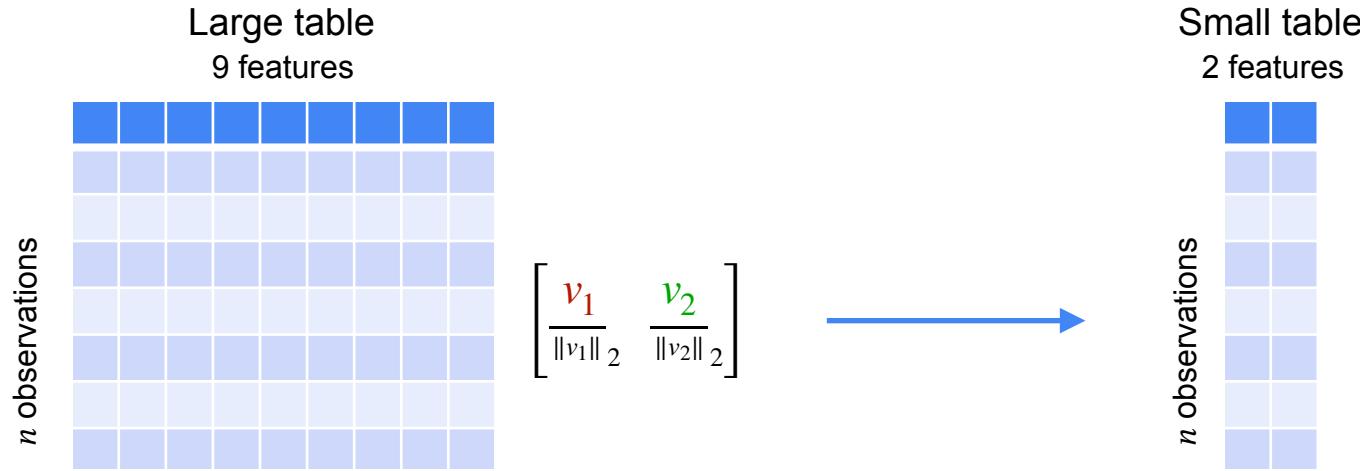
$$\begin{array}{ll} v_3 & \lambda_3 \\ v_4 & \lambda_4 \end{array}$$

$$\begin{array}{ll} v_5 & \lambda_5 \\ \vdots & \vdots \end{array}$$

Eigenvalues/vectors



PCA: Principal Component Analysis



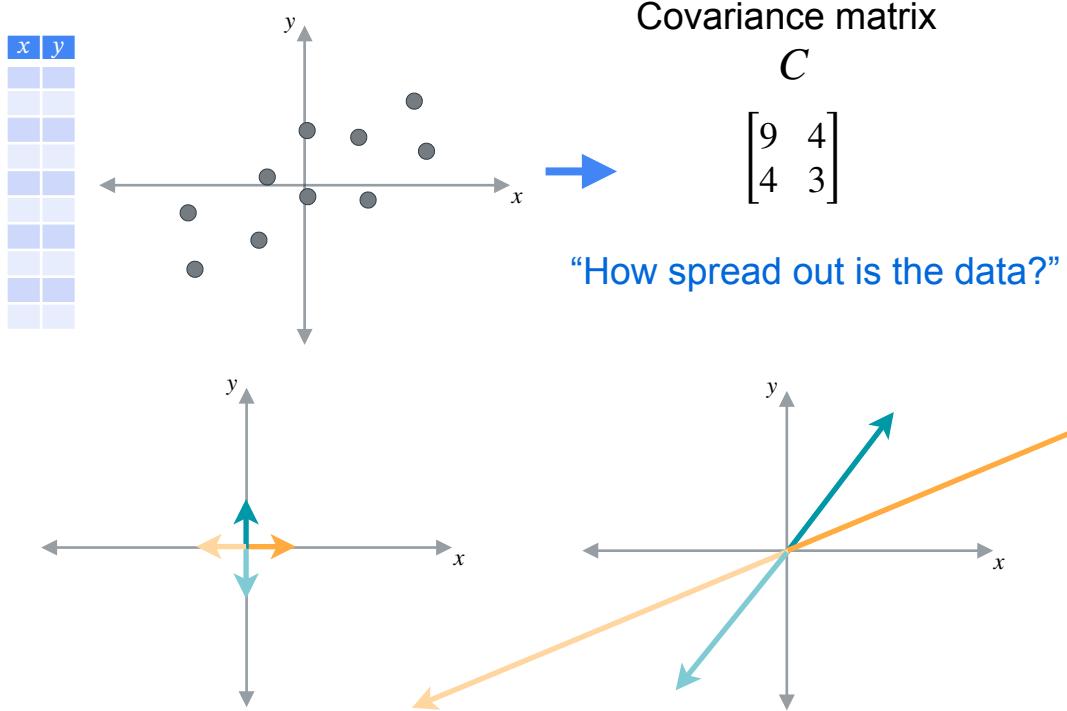


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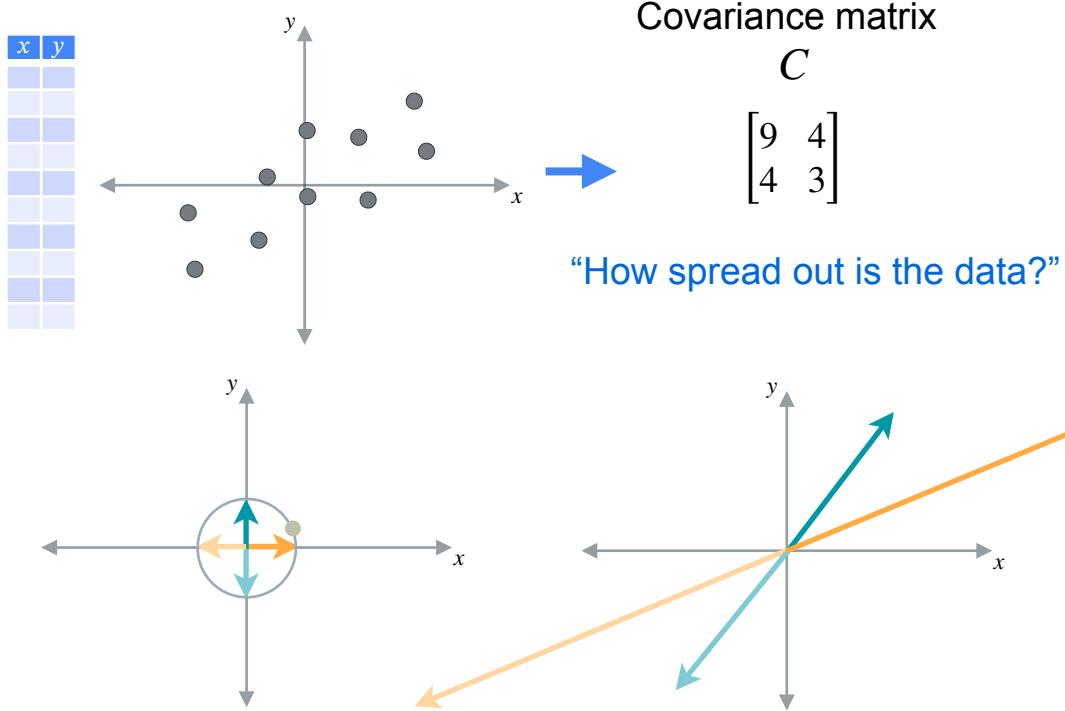
Determinants and Eigenvectors

PCA - Why it works

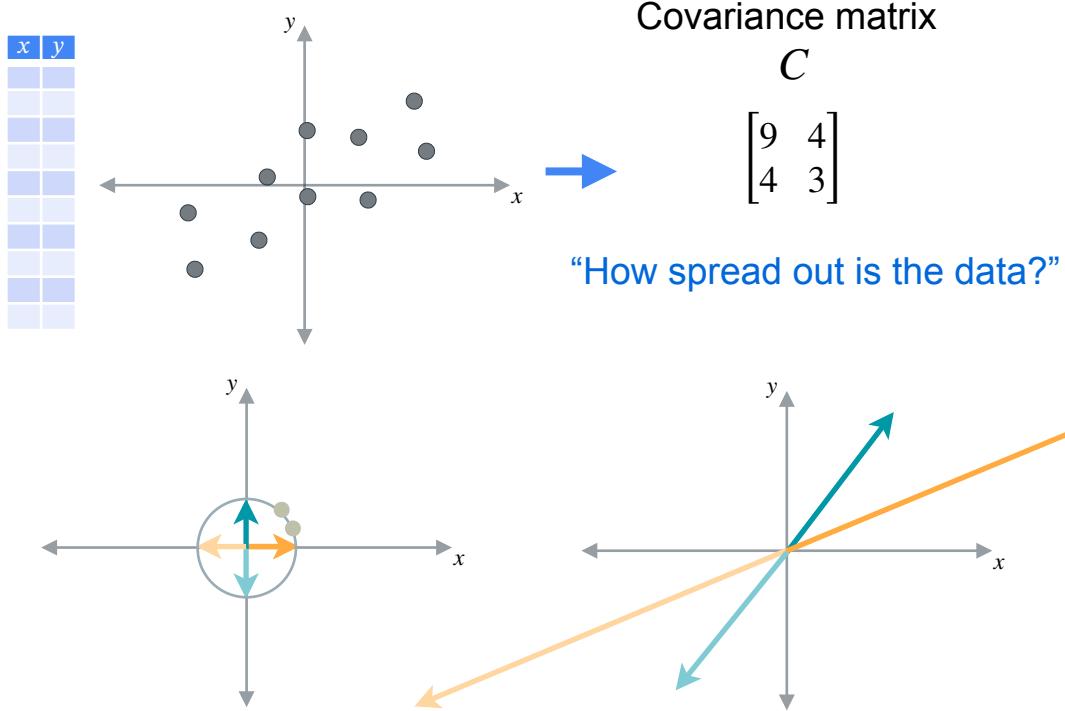
PCA: Why It Works



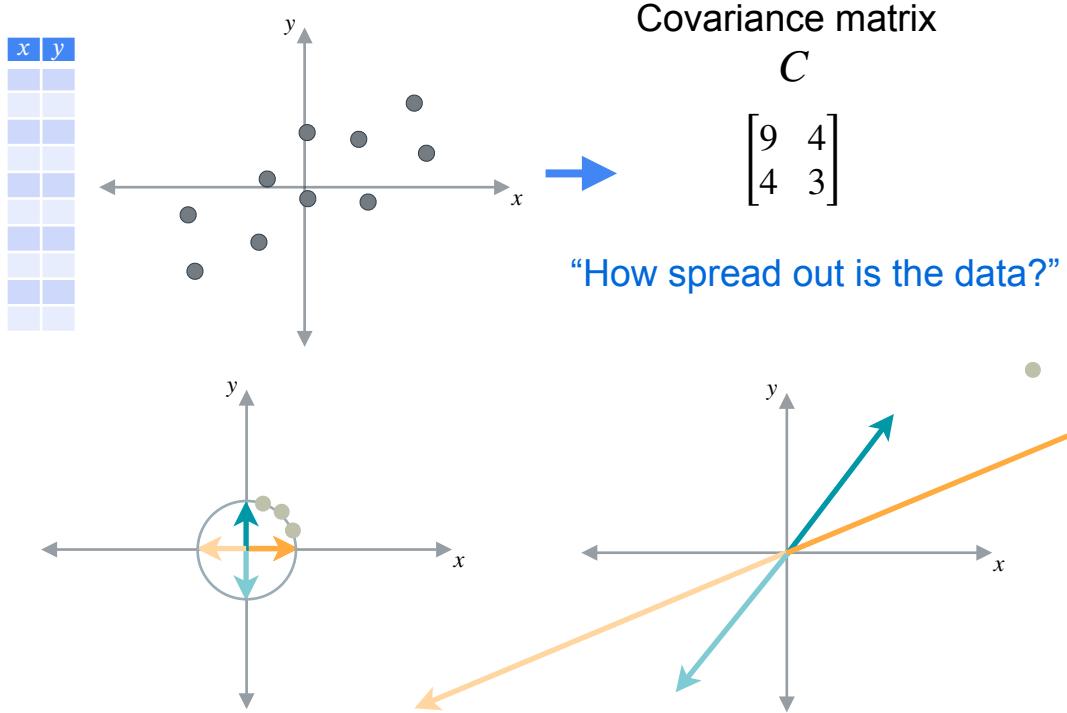
PCA: Why It Works



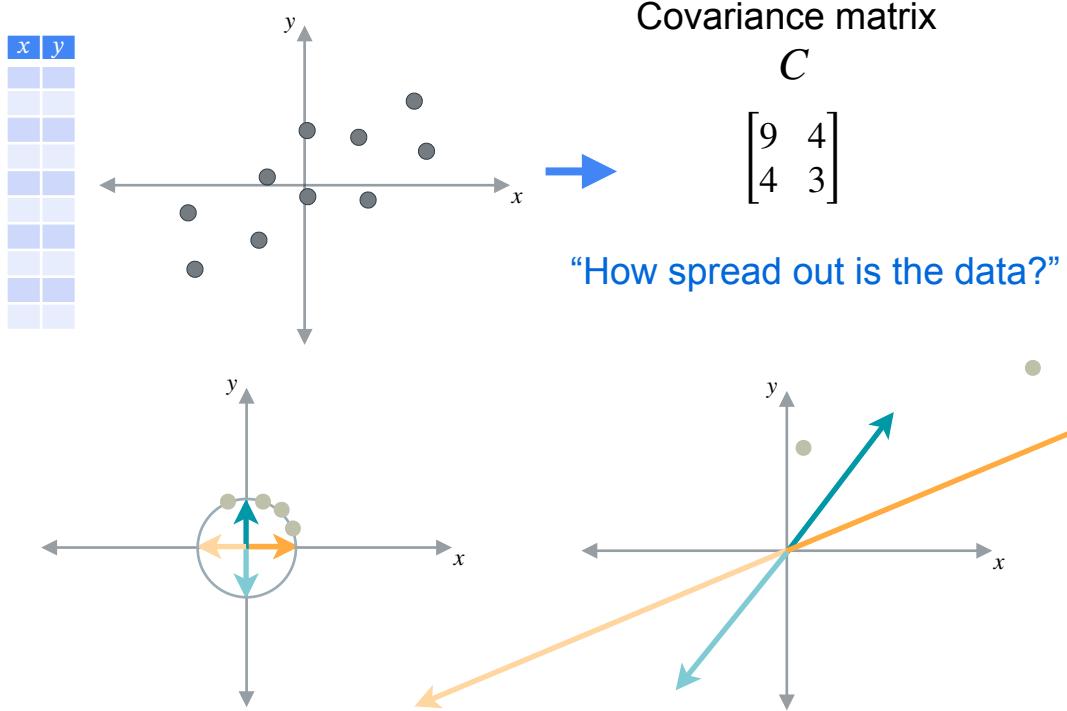
PCA: Why It Works



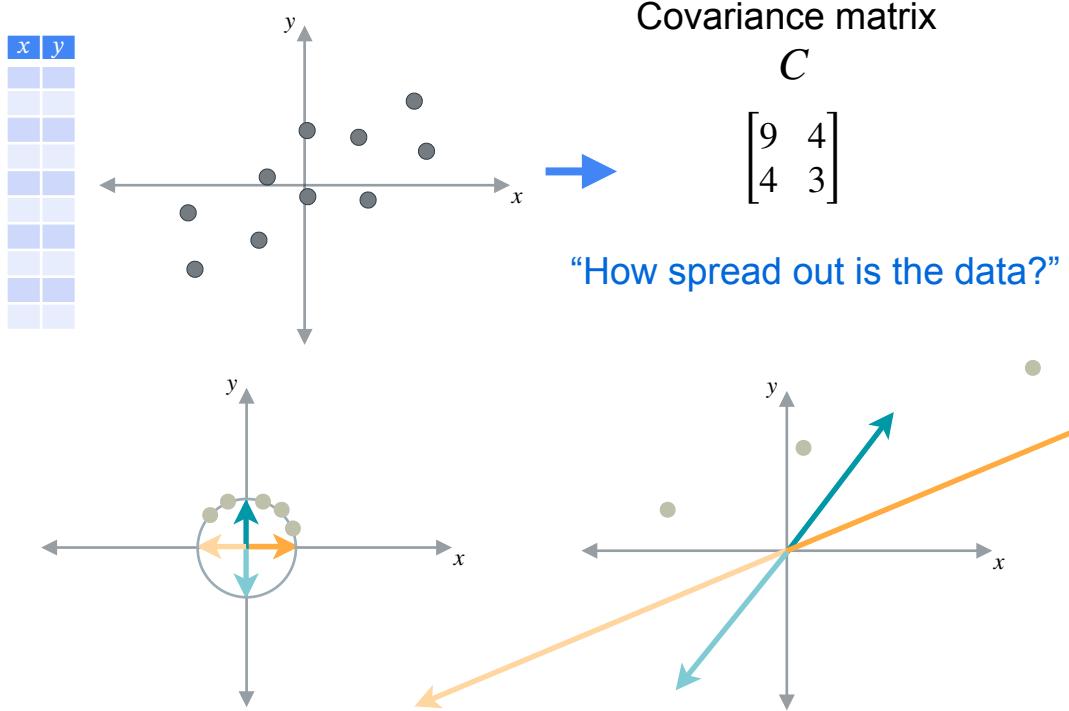
PCA: Why It Works



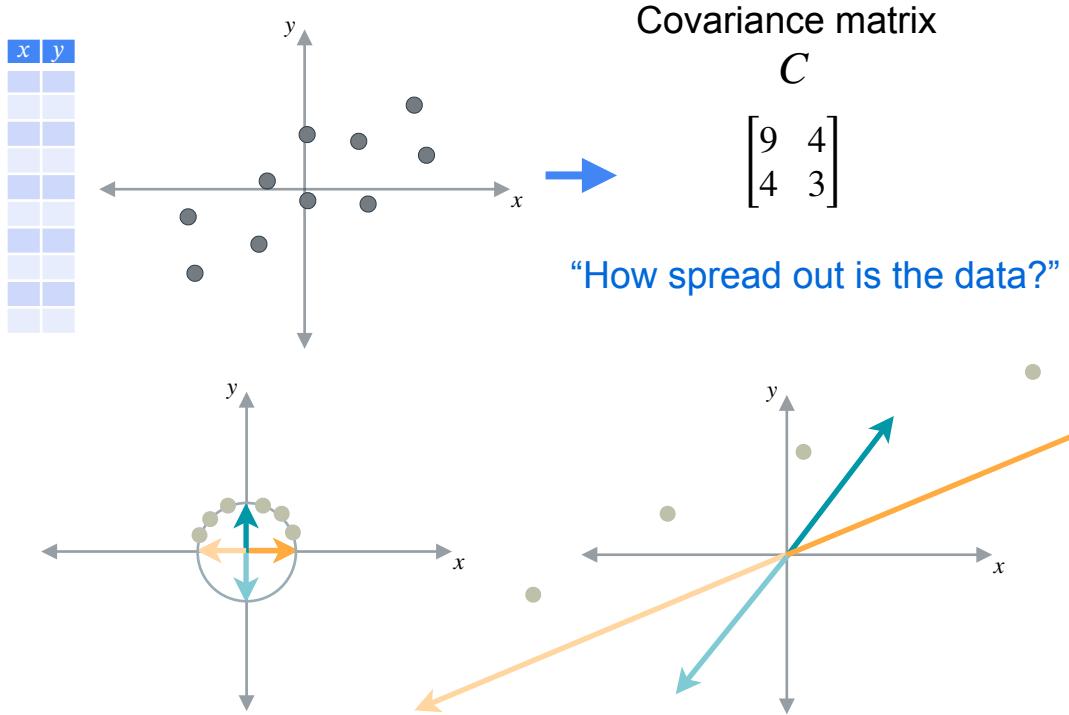
PCA: Why It Works



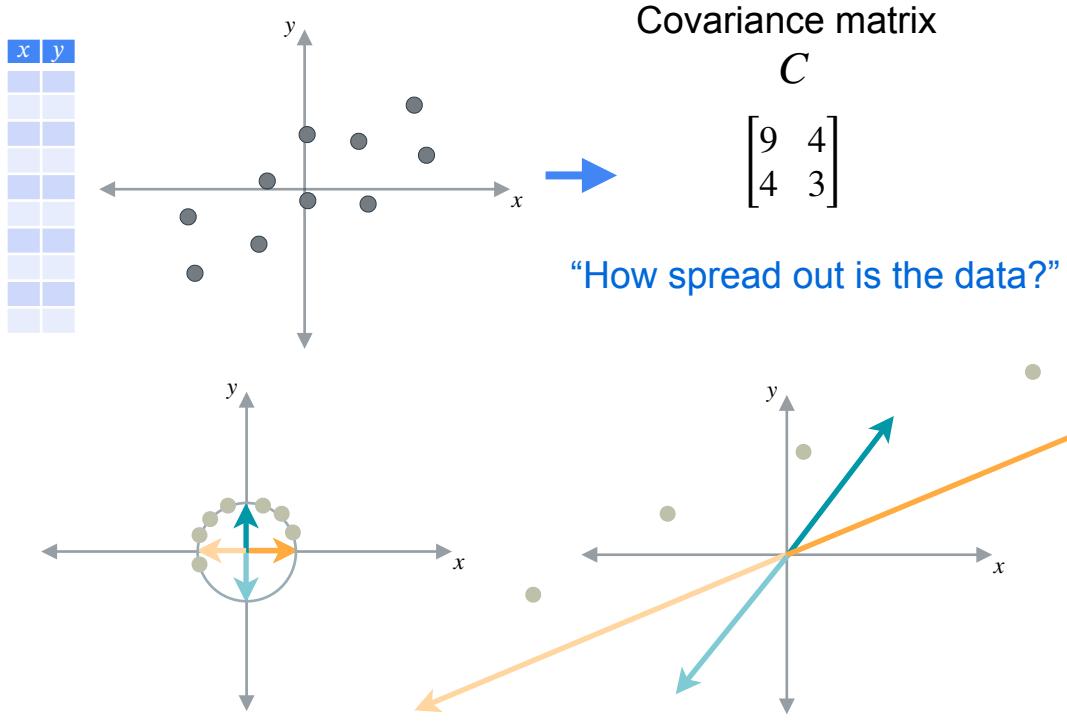
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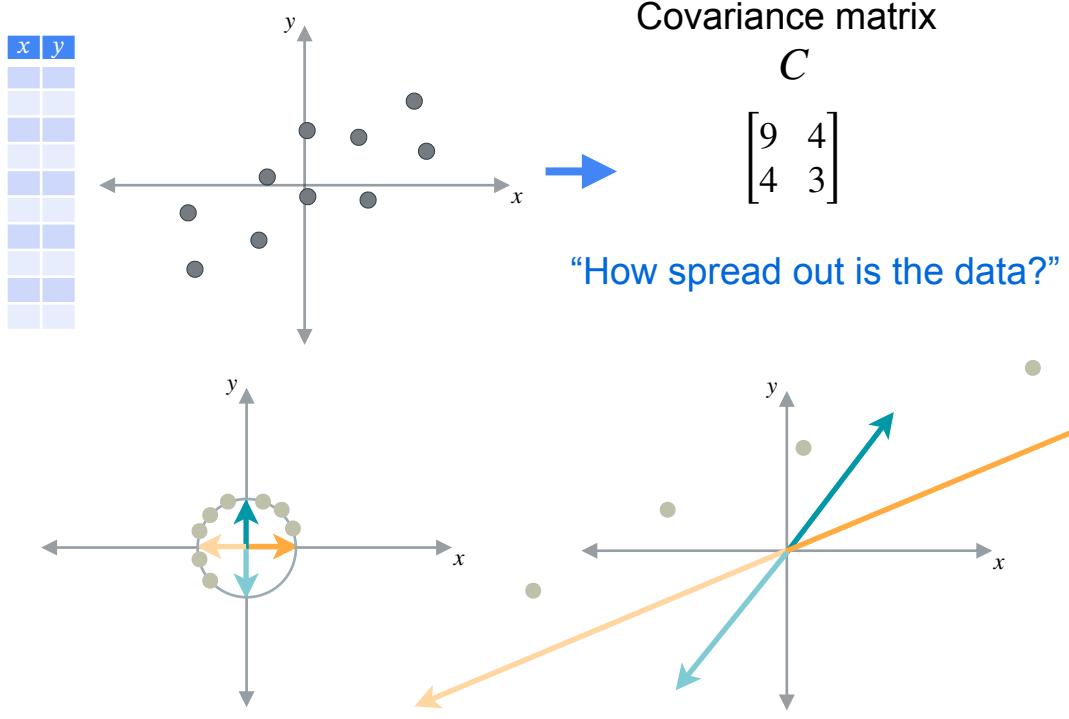
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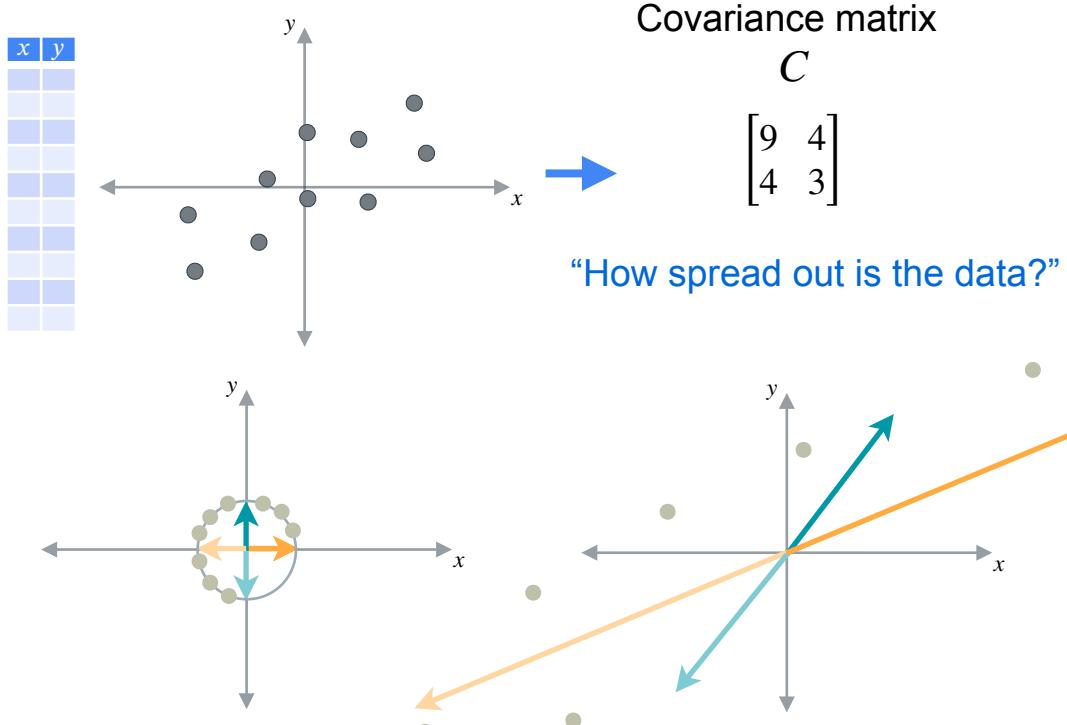
PCA: Why It Works



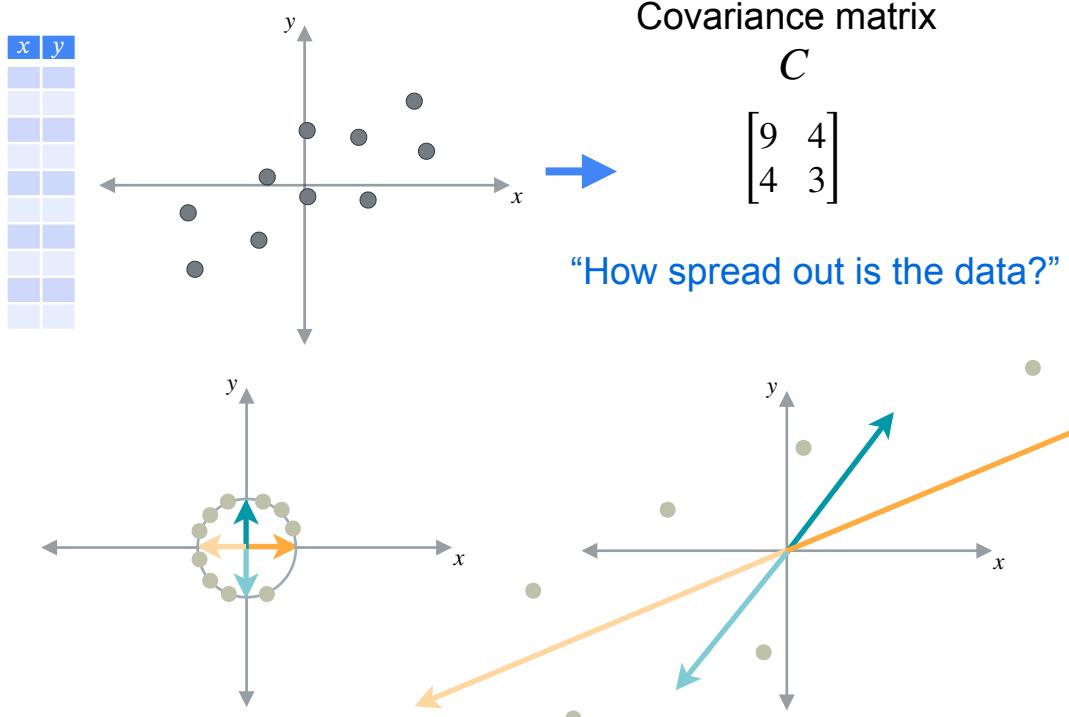
PCA: Why It Works



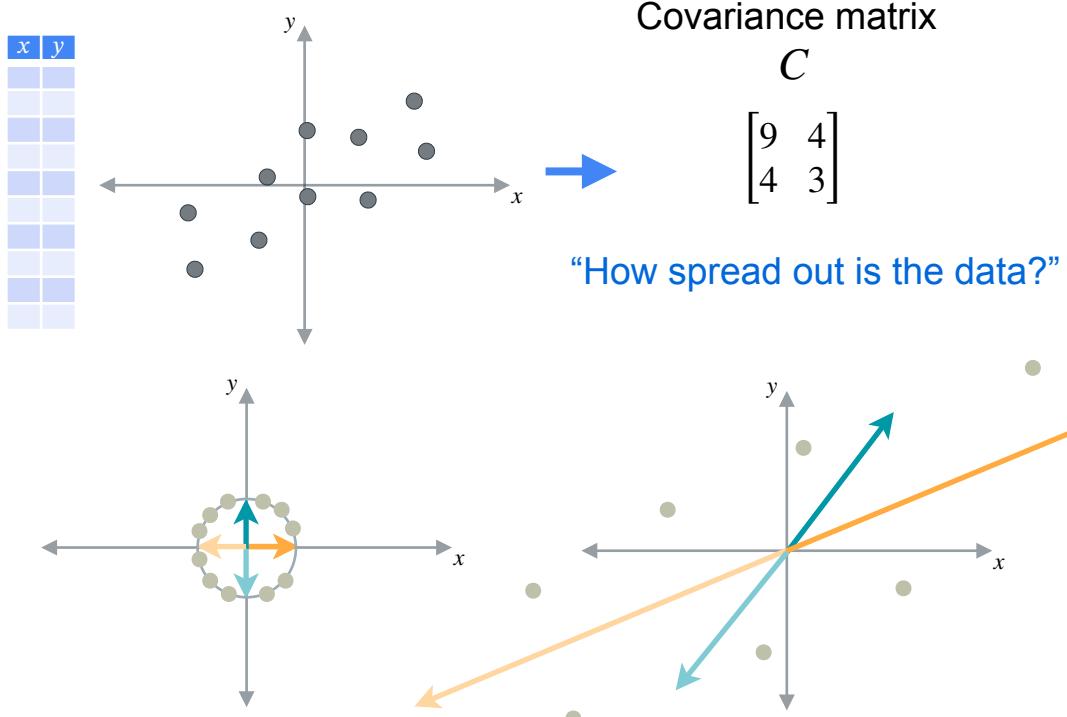
PCA: Why It Works



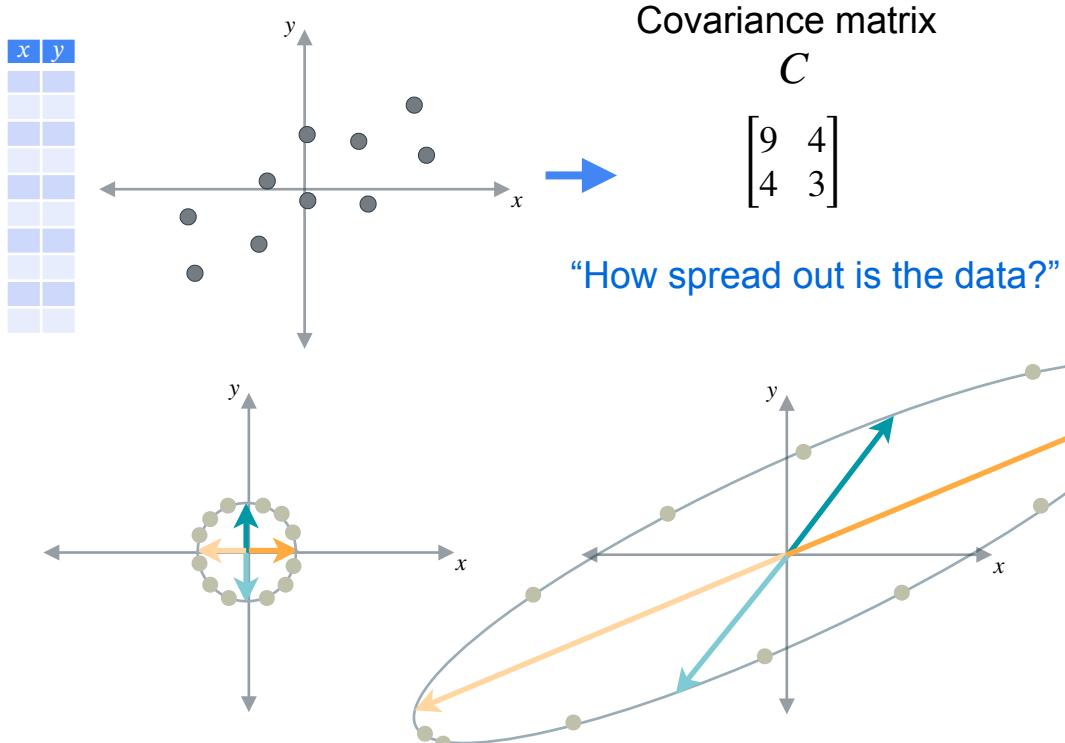
PCA: Why It Works



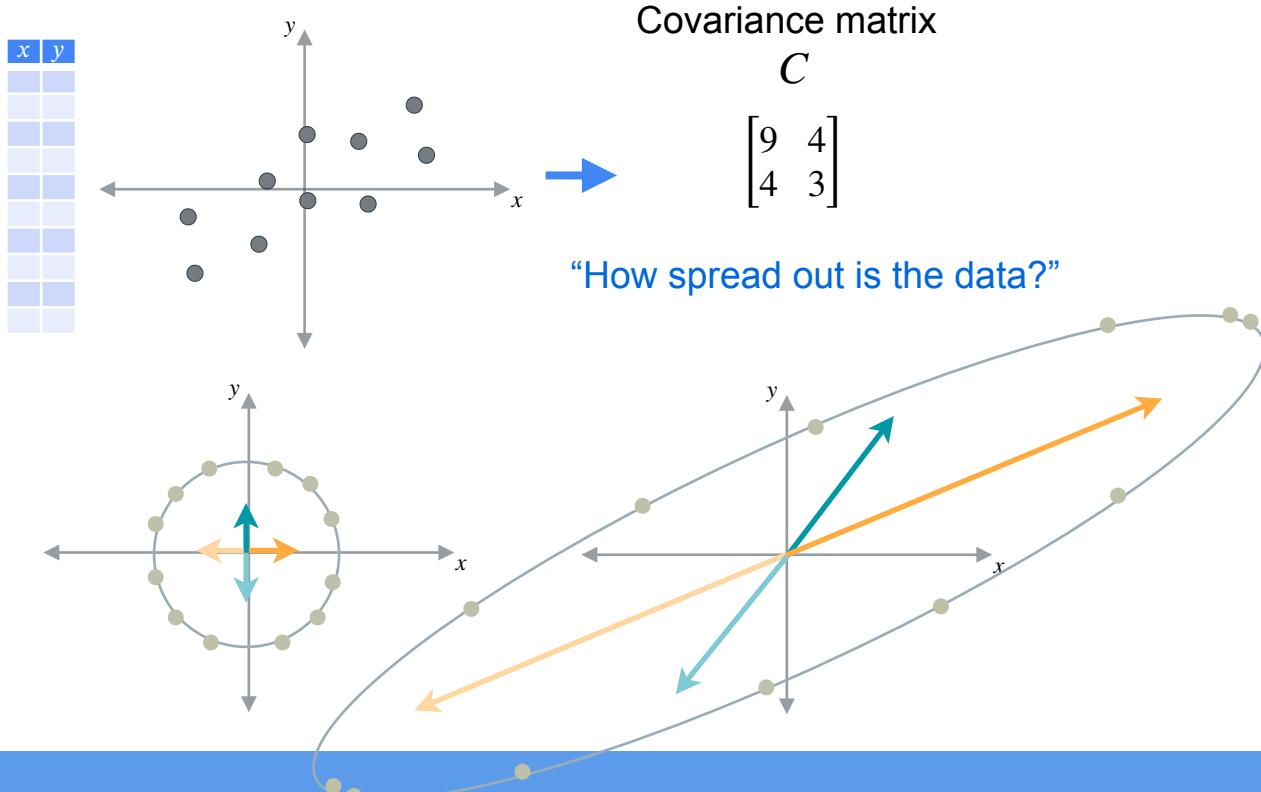
PCA: Why It Works



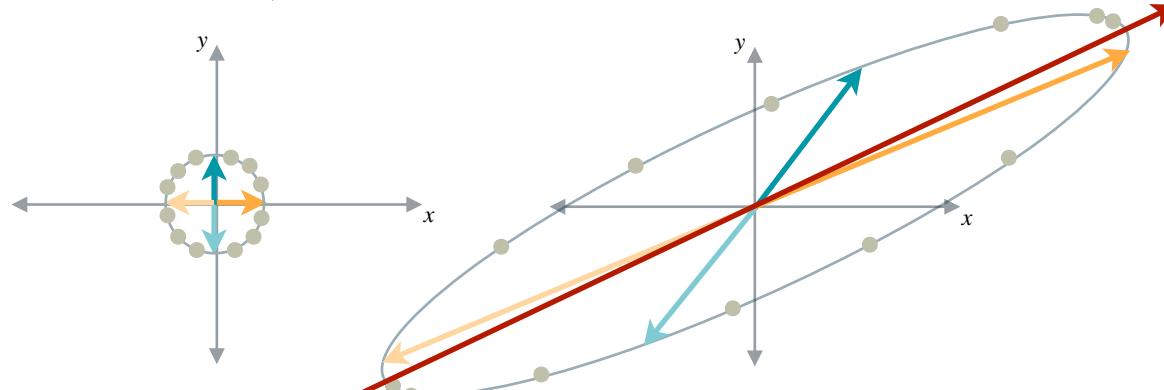
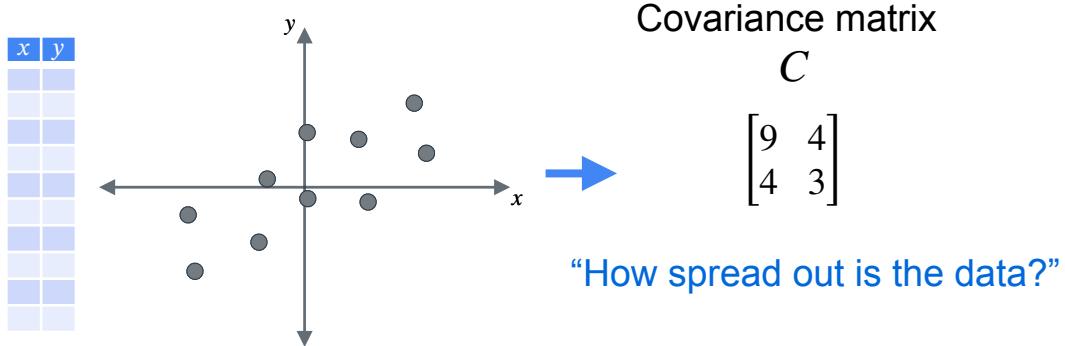
PCA: Why It Works



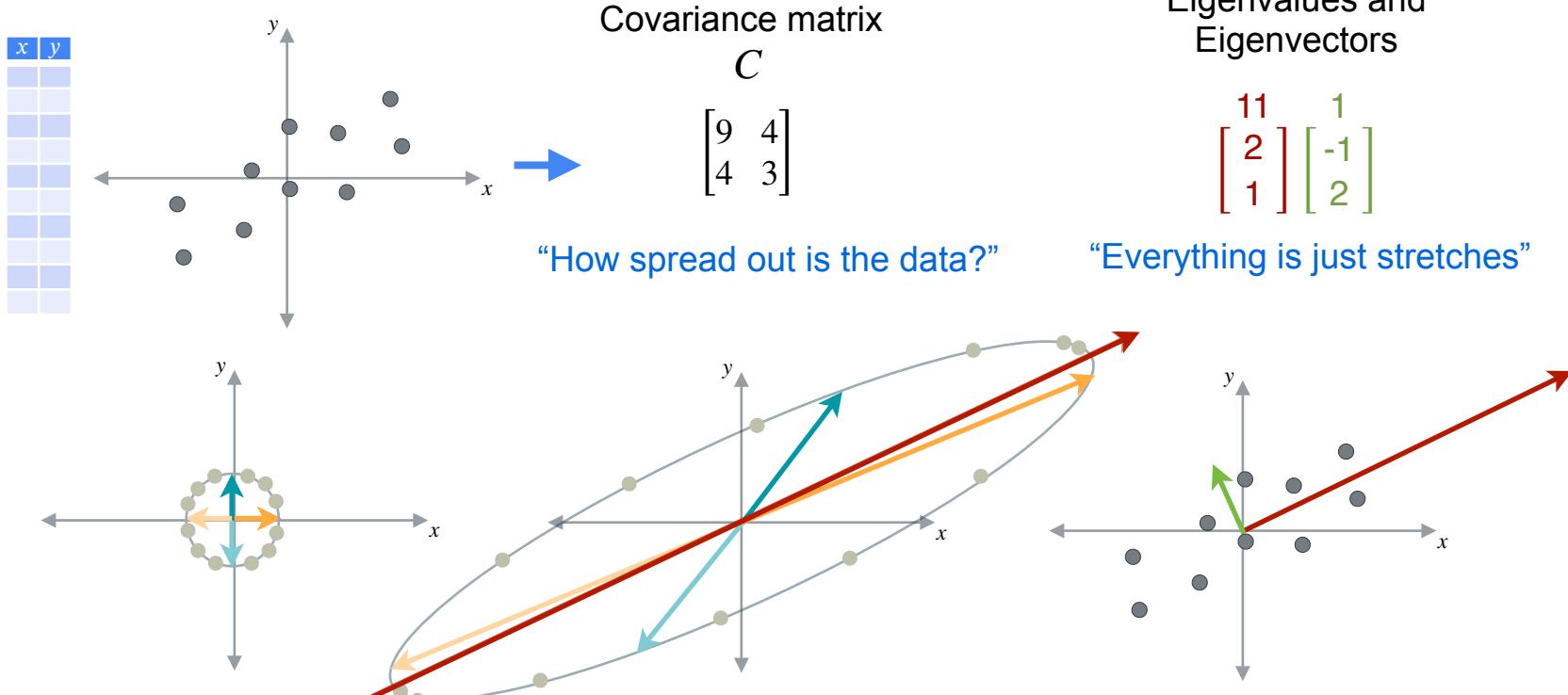
PCA: Why It Works



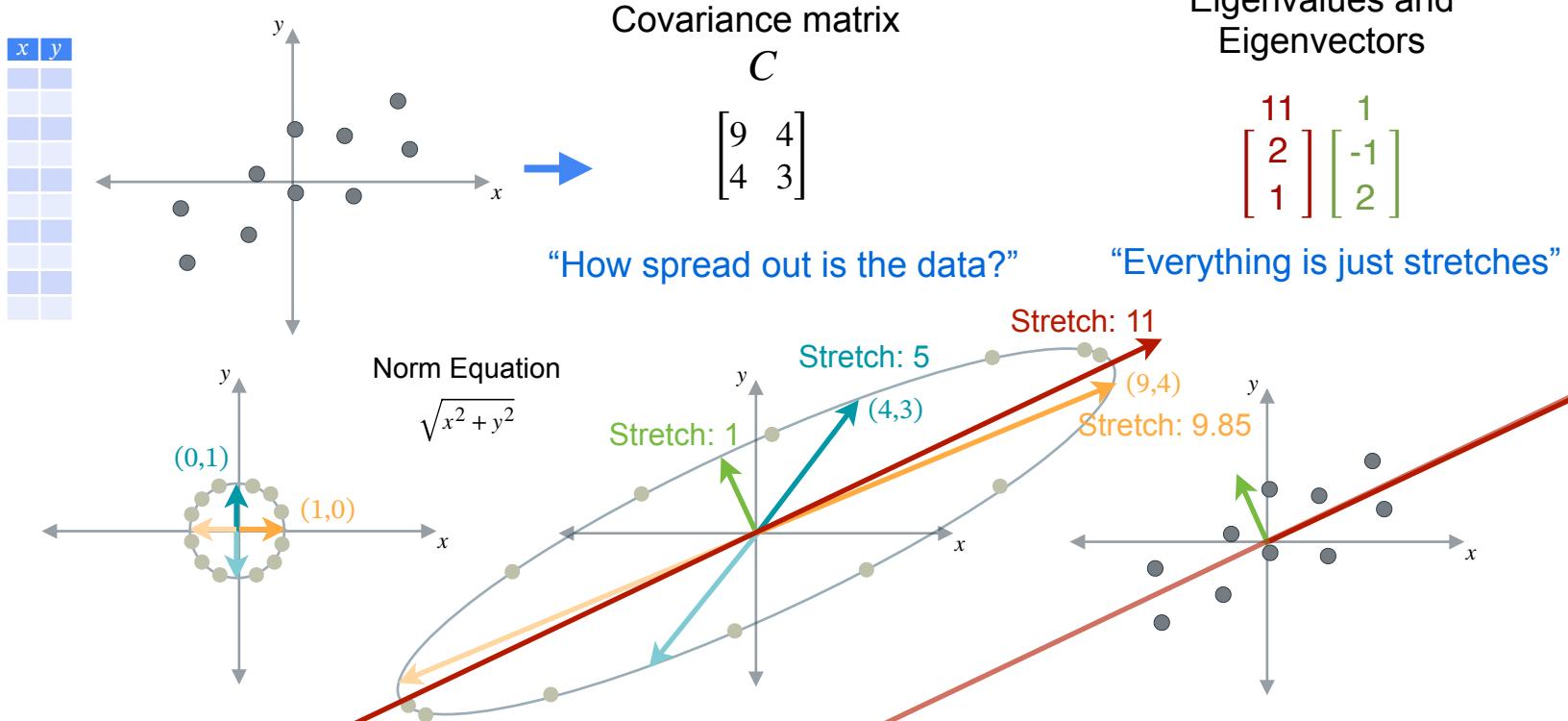
PCA: Why It Works



PCA: Why It Works



PCA: Why It Works





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Determinants and Eigenvectors

**PCA - Mathematical
formulation**

PCA Mathematical formulation

You have n observations of 5 variables (x_1, x_2, x_3, x_4, x_5)

Goal: Reduce to 2 variables

1 Create matrix

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{15} \\ x_{21} & x_{22} & \dots & x_{25} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{n5} \end{bmatrix}$$

5 variables

n Observations

2 Center the data

$$X - \mu = \begin{bmatrix} x_{11} - \mu_1 & x_{12} - \mu_2 & \dots & x_{15} - \mu_5 \\ x_{21} - \mu_1 & x_{22} - \mu_2 & \dots & x_{25} - \mu_5 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} - \mu_1 & x_{n2} - \mu_2 & \dots & x_{n5} - \mu_5 \end{bmatrix}$$

PCA Mathematical formulation

You have n observations of 5 variables $(x_1, x_2, x_3, x_4, x_5)$

Goal: Reduce to 2 variables

3

Calculate Covariance Matrix

$$C = \frac{1}{n-1}(X - \mu)^T(X - \mu) = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) & Cov(X_1, X_3) & Cov(X_1, X_4) & Cov(X_1, X_5) \\ Cov(X_1, X_2) & Var(X_2) & Cov(X_2, X_3) & Cov(X_2, X_4) & Cov(X_2, X_5) \\ Cov(X_1, X_3) & Cov(X_2, X_3) & Var(X_3) & Cov(X_3, X_4) & Cov(X_3, X_5) \\ Cov(X_1, X_4) & Cov(X_2, X_4) & Cov(X_3, X_4) & Var(X_4) & Cov(X_4, X_5) \\ Cov(X_1, X_5) & Cov(X_2, X_5) & Cov(X_3, X_5) & Cov(X_4, X_5) & Var(X_5) \end{bmatrix}$$

PCA Mathematical formulation

You have n observations of 5 variables (x_1, x_2, x_3, x_4, x_5)

Goal: Reduce to 2 variables

- 4 Calculate Eigenvectors and Eigenvalues

- 5 Create Projection Matrix

- 6 Project Centered Data

Big	λ_1	v_1
	λ_2	v_2
	λ_3	v_3
	λ_4	v_4
Small	λ_5	v_5

$$V = \begin{bmatrix} & \\ \frac{1}{\|v_1\|_2} & \frac{1}{\|v_2\|_2} \\ & \end{bmatrix}$$

$$X_{PCA} = (X - \mu)V$$



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Determinants and Eigenvectors

Conclusion