Functions and Their Graphs

2

Outline

2.1 Functions
2.2 The Graph of a Function
2.3 Properties of Functions
2.4 Library of Functions; Piecewise-defined Functions
2.5 Graphing Techniques: Transformations
2.6 Mathematical Models: Building Functions
- Chapter Review
- Chapter Test
- Cumulative Review
- Chapter Projects

2.3 Properties of Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

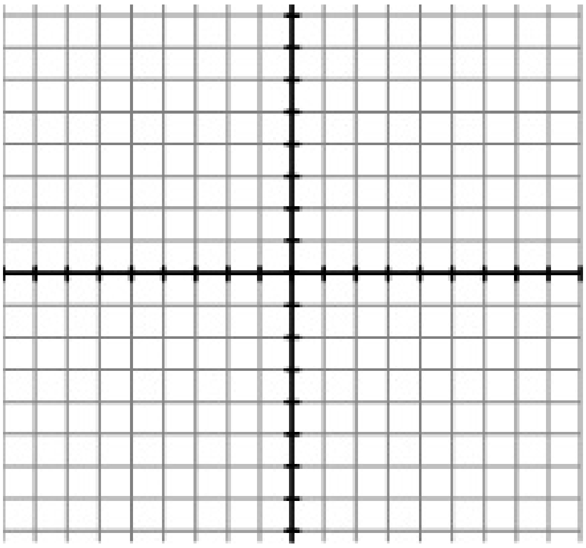
- Interval Notation (Appendix A, Section A.9, pp. A72–A73)
- Intercepts (Section 1.2, pp. 11–12)
- Slope of a Line (Section 1.3, pp. 19–21)
- Point–Slope Form of a Line (Section 1.3, p. 23)
- Symmetry (Section 1.2, pp. 12–14)

Now Work the 'Are You Prepared?' problems on page 76.

OBJECTIVES

1. Determine Even and Odd Functions from a Graph (p. 69)
2. Identify Even and Odd Functions from the Equation (p. 70)
3. Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant (p. 70)
4. Use a Graph to Locate Local Maxima and Local Minima (p. 71)
5. Use a Graph to Locate the Absolute Maximum and the Absolute Minimum (p. 72)
6. Use a Graphing Utility to Approximate Local Maxima and Local Minima and to Determine Where a Function Is Increasing or Decreasing (p. 74)
7. Find the Average Rate of Change of a Function (p. 74)

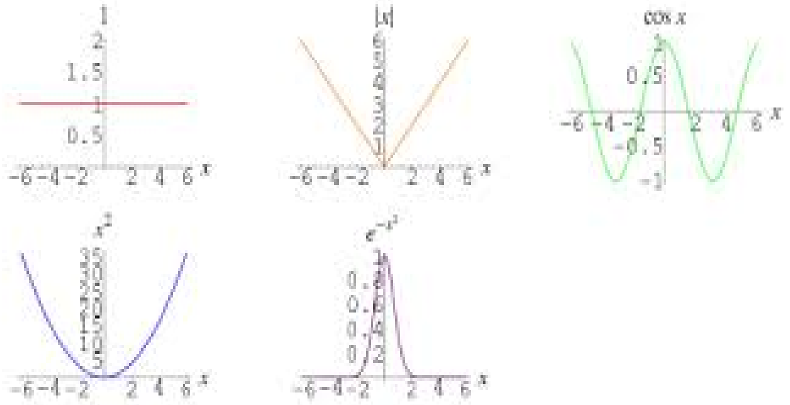
A function f is even, if and only if, whenever the point (x, y) is on the graph of f then the point $(-x, y)$ is also on the graph. Using function notation, we define an even function as follows:



DEFINITION

A function f is **even** if, for every number x in its domain, the number $-x$ is also in the domain and

$$f(-x) = f(x)$$



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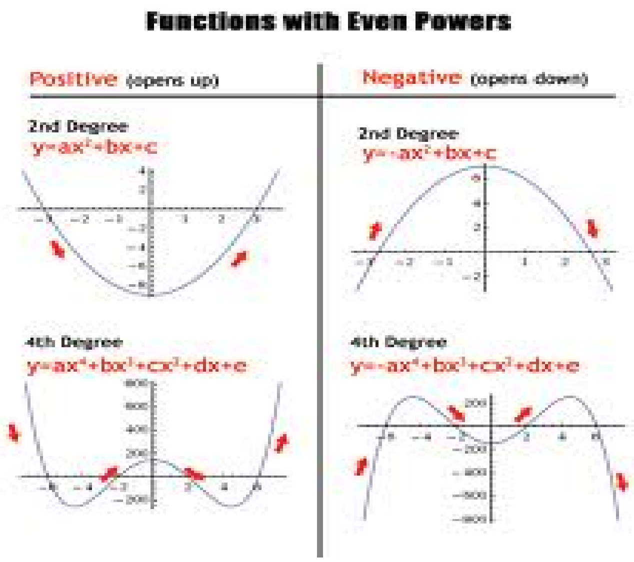

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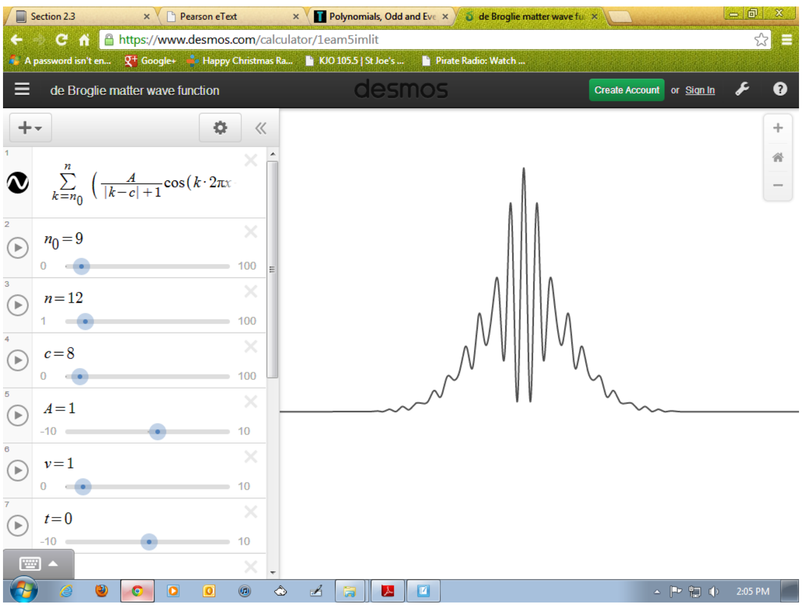
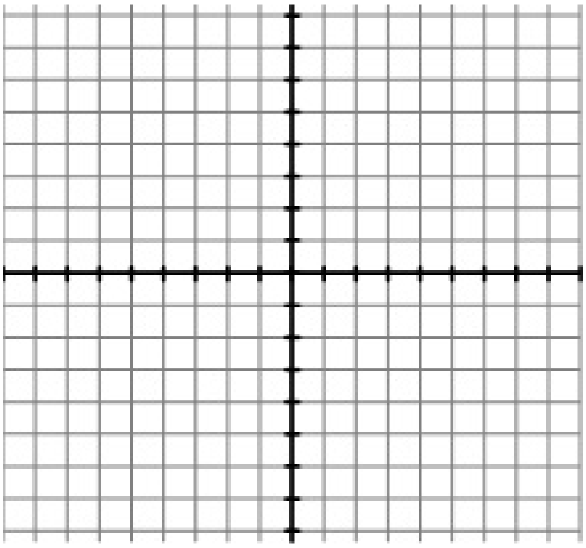


https://www.desmos.com/calculator/1eam5imlit

Note the Symmetry about the y-axis if this has not been mentioned previously.

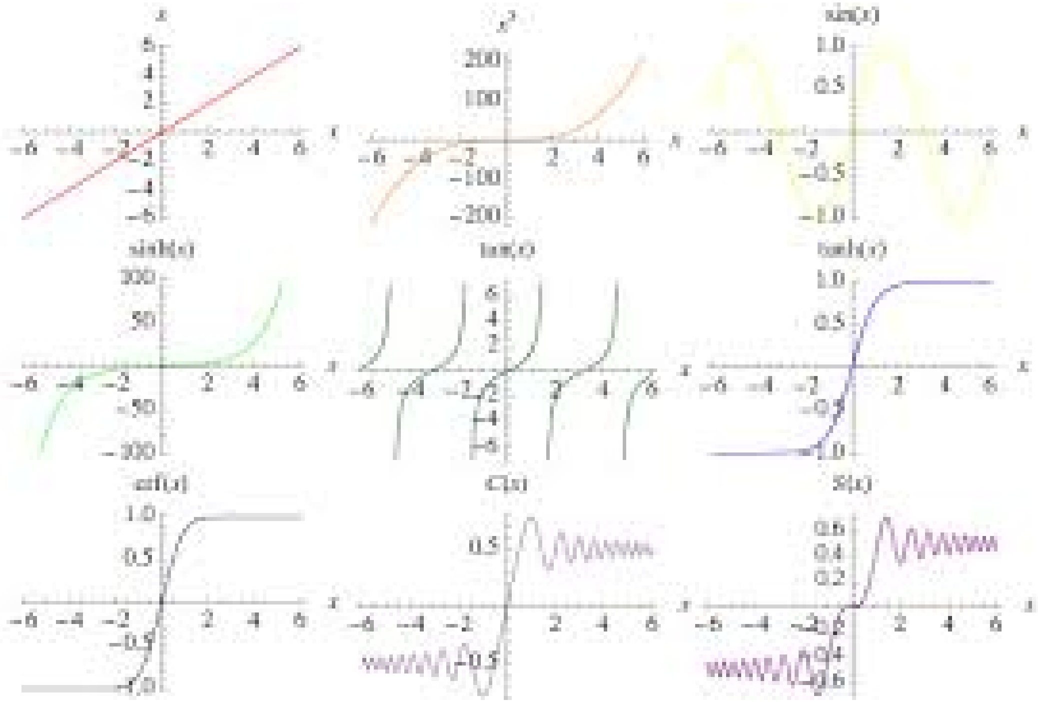
A function f is odd, if and only if, whenever the point (x, y) is on the graph of f then the point $(-x, -y)$ is also on the graph. Using function notation, we define an odd function as follows:



DEFINITION

A function f is **odd** if, for every number x in its domain, the number $-x$ is also in the domain and

$$f(-x) = -f(x)$$



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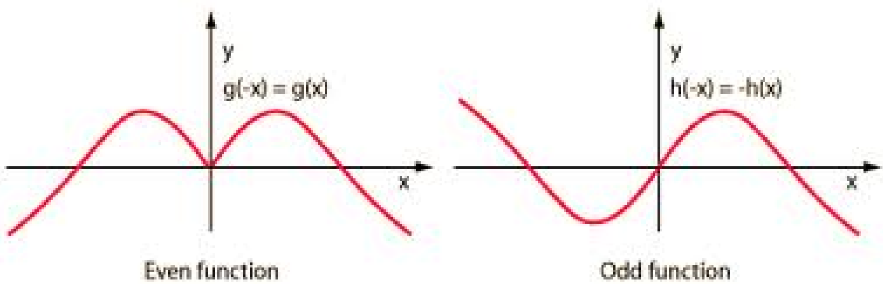


hyperphysics.phy-astr.gsu.edu

Note the symmetry about the origin if it has not been mentioned previously.

THEOREM

A function is even if and only if its graph is symmetric with respect to the y-axis. A function is odd if and only if its graph is symmetric with respect to the origin.

DEFINITIONS

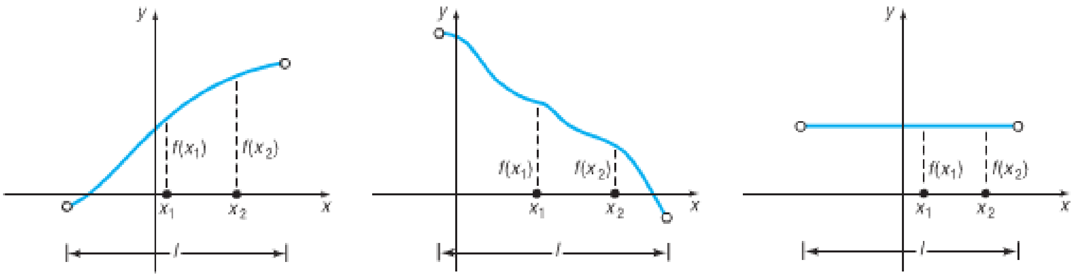
A function f is **increasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

A function f is **decreasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

A function f is **constant** on an open interval I if, for all choices of x in I, the values $f(x)$ are equal.

Figure 19 illustrates the definitions. The graph of an increasing function goes up from left to right, the graph of a decreasing function goes down from left to right, and the graph of a constant function remains at a fixed height.

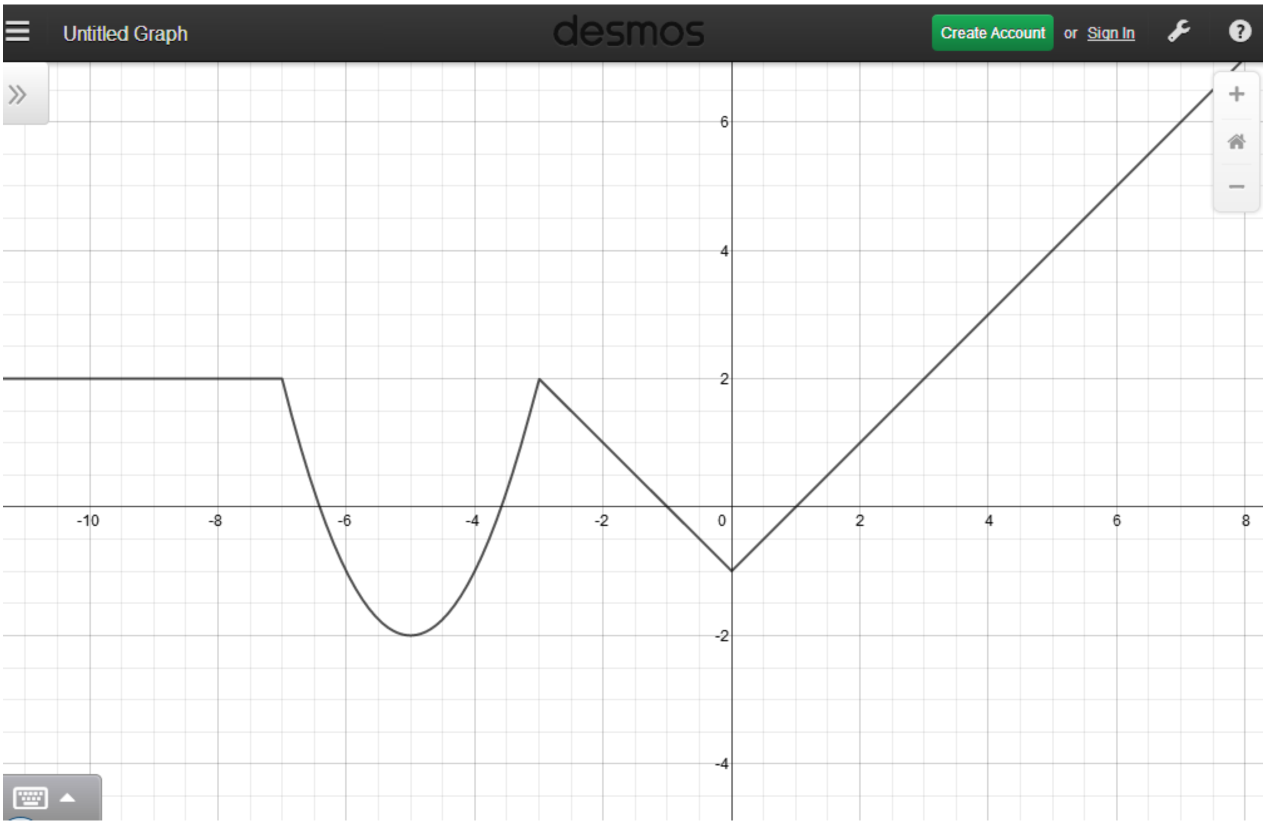
Figure 19



(a) For $x_1 < x_2$ in I, $f(x_1) < f(x_2)$; f is increasing on I

(b) For $x_1 < x_2$ in I, $f(x_1) > f(x_2)$; f is decreasing on I

(c) For all x in I, the values of f are equal; f is constant on I

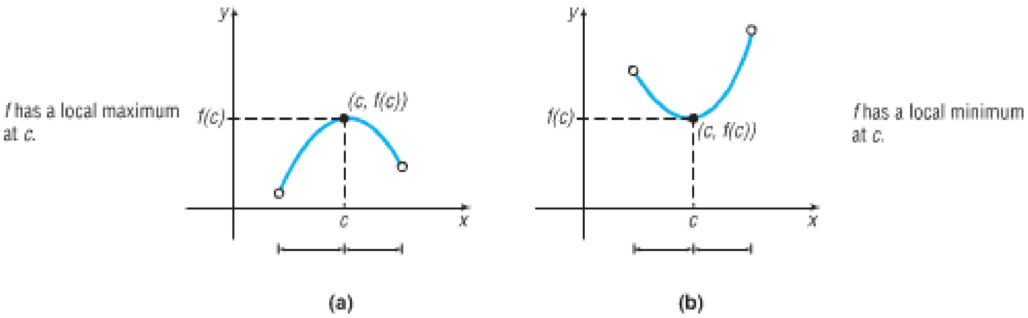


DEFINITIONS

A function f has a **local maximum** at c if there is an open interval I containing c so that for all x in I, $f(x) \le f(c)$. We call $f(c)$ a **local maximum value of f.**

A function f has a **local minimum** at c if there is an open interval I containing c so that, for all x in I, $f(x) \ge f(c)$. We call $f(c)$ a **local minimum value of f.**

Figure 20

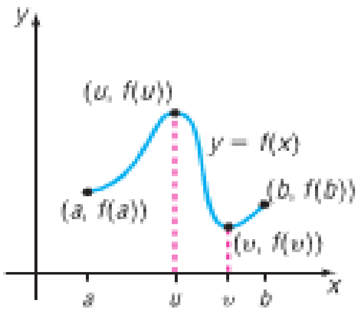


* The open interval (a, b) consists of all real numbers x for which $a < x < b$.
† Some texts use the term *relative* instead of *local*.

DEFINITION Let f denote a function defined on some interval I. If there is a number u in I for which $f(x) \leq f(u)$ for all x in I, then $f(u)$ is the **absolute maximum of f** on I and we say **the absolute maximum of f occurs at u**.

If there is a number v in I for which $f(x) \geq f(v)$ for all x in I, then $f(v)$ is the **absolute minimum of f** on I and we say **the absolute minimum of f occurs at v.**

Figure 22



domain: [a, b]
for all x in $[a, b]$, $f(x) \leq f(u)$
for all x in $[a, b]$, $f(x) \geq f(v)$
absolute maximum: $f(u)$
absolute minimum: $f(v)$

THEOREM

Extreme Value Theorem

If f is a continuous function* whose domain is a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

* Although it requires calculus for a precise definition, we'll agree for now that a continuous function is one whose graph has no gaps or holes and can be traced without lifting the pencil from the paper.

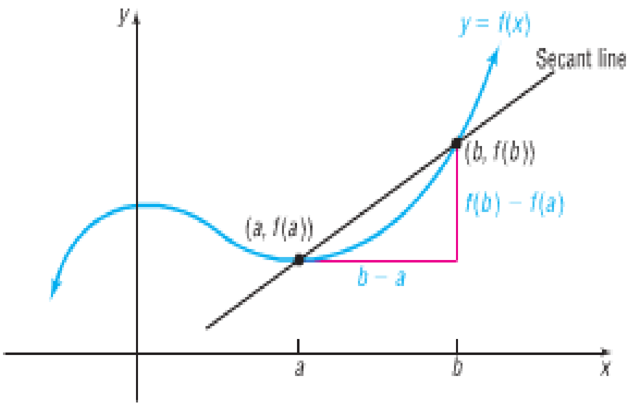
DEFINITION

If a and b, $a \neq b$, are in the domain of a function $y = f(x)$, the **average rate of change of f from a to b** is defined as

$$\boxed{\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} \qquad a \neq b \qquad \textbf{(1)}}$$

$$m_{sec} = \frac{f(b) - f(a)}{b - a}$$

Figure 26



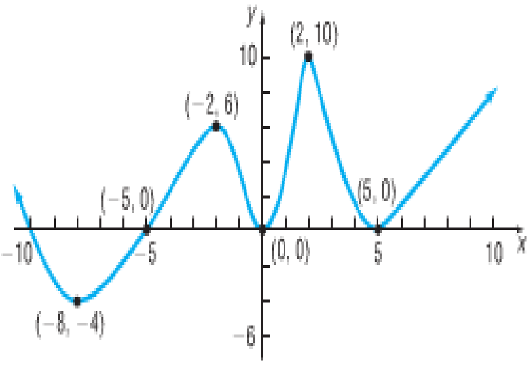
THEOREM

Slope of the Secant Line

The average rate of change of a function from a to b equals the slope of the secant line containing the two points $(a, f(a))$ and $(b, f(b))$ on its graph.

Skill Building

In Problems 11–20, use the graph of the function f given.



12. Is f decreasing on the interval $(-8, -4)$?

14. Is f decreasing on the interval $(2, 5)$?

16. List the interval(s) on which f is decreasing.

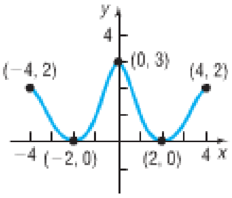
18. Is there a local maximum value at 5? If yes, what is it?

20. List the number(s) at which f has a local minimum. What are the local minimum values?

In Problems 21–28, the graph of a function is given. Use the graph to find:

(a) The intercepts, if any
(b) The domain and range
(c) The intervals on which it is increasing, decreasing, or constant
(d) Whether it is even, odd, or neither

21.



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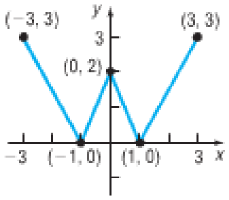

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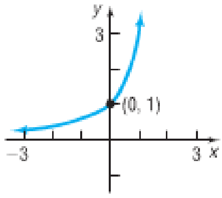

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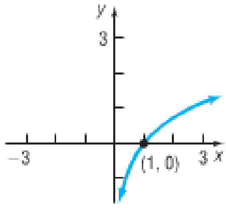

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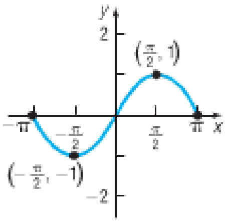

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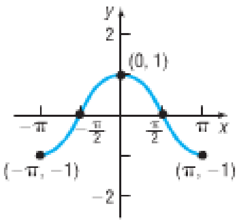

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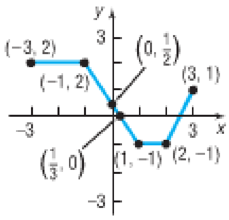

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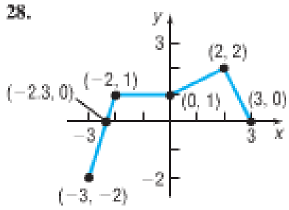

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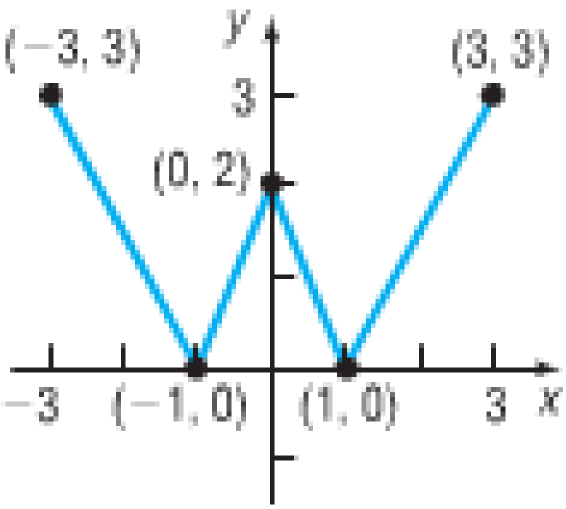


(a) The intercepts, if any

(b) The domain and range

(c) The intervals on which it is increasing, decreasing, or constant

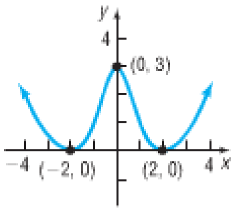
(d) Whether it is even, odd, or neither

In Problems 29–32, the graph of a function f is given. Use the graph to find:

(a) The numbers, if any, at which f has a local maximum value. What are the local maximum values?

(b) The numbers, if any, at which f has a local minimum value. What are the local minimum values?

29.



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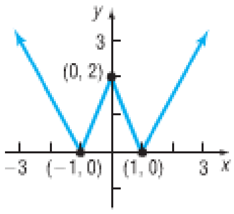

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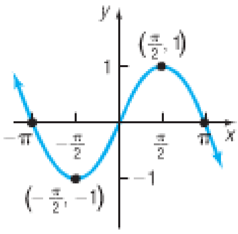

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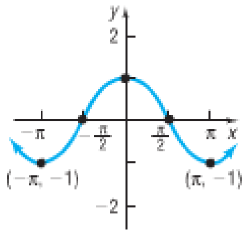

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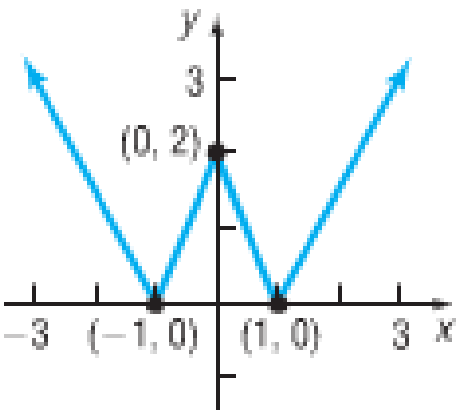


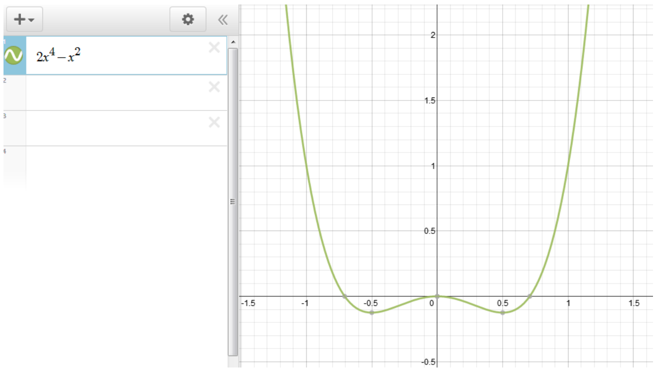
(a) The numbers, if any, at which f has a local maximum value. What are the local maximum values?

(b) The numbers, if any, at which f has a local minimum value. What are the local minimum values?

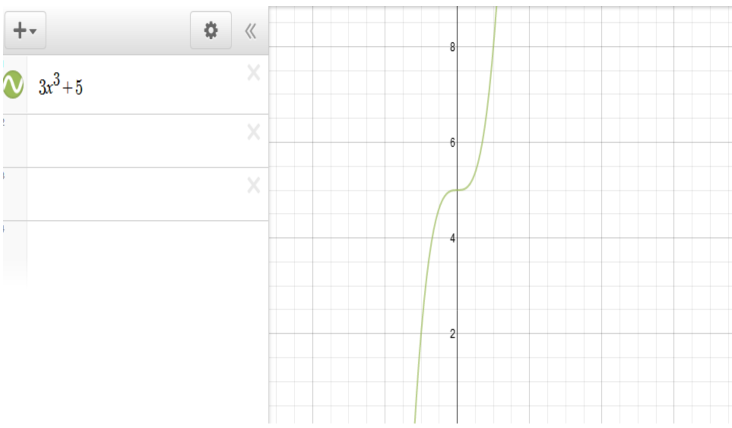
In Problems 33–44, determine algebraically whether each function is even, odd, or neither.

33. $f(x) = 4x^3$
34. $f(x) = 2x^4 - x^2$
35. $g(x) = -3x^2 - 5$
36. $h(x) = 3x^3 + 5$
37. $F(x) = \sqrt[3]{x}$
38. $G(x) = \sqrt{x}$
39. $f(x) = x + |x|$
40. $f(x) = \sqrt[3]{2x^2 + 1}$
41. $g(x) = \dfrac{1}{x^2}$
42. $h(x) = \dfrac{x}{x^2 - 1}$
43. $h(x) = \dfrac{-x^3}{3x^2 - 9}$
44. $F(x) = \dfrac{2x}{|x|}$

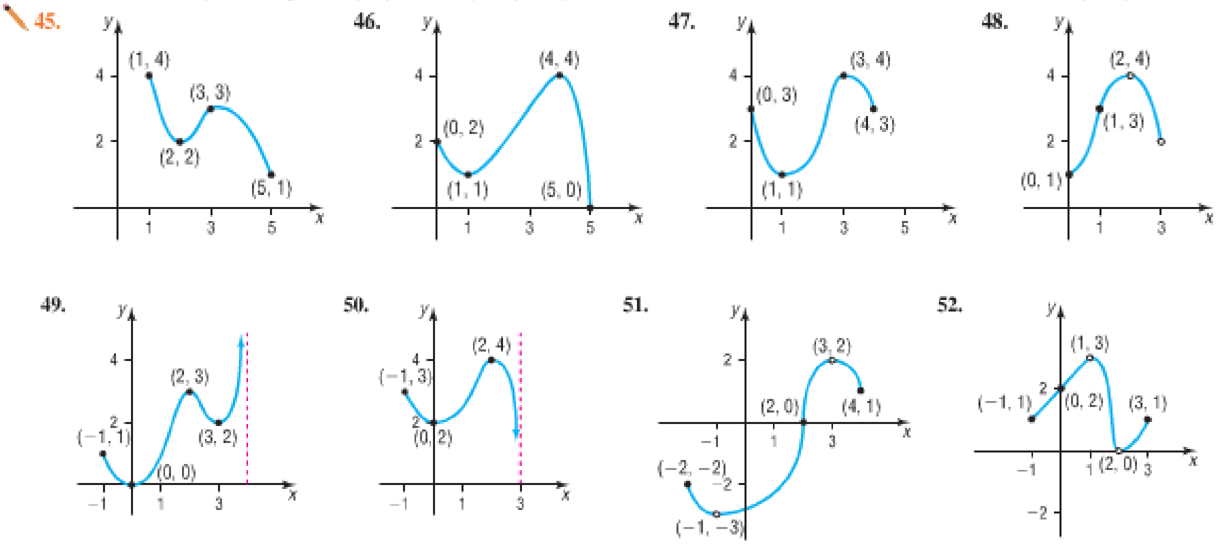
34. $f(x) = 2x^4 - x^2$



36. $h(x) = 3x^3 + 5$



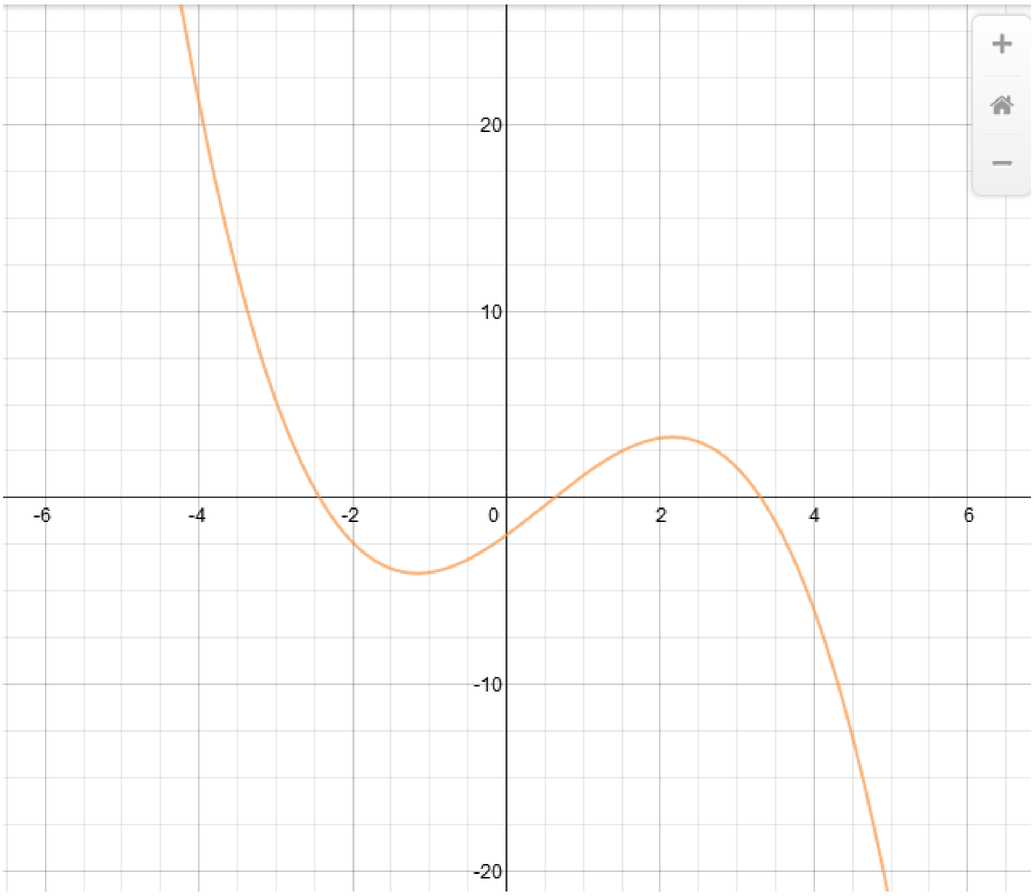
In Problems 45–52, for each graph of a function $y = f(x)$, find the absolute maximum and the absolute minimum, if they exist.



In Problems 53–60, use a graphing utility to graph each function over the indicated interval and approximate any local maximum values and local minimum values. Determine where the function is increasing and where it is decreasing. Round answers to two decimal places.

53. $f(x) = x^3 - 3x + 2 \quad (-2, 2)$
54. $f(x) = x^3 - 3x^2 + 5 \quad (-1, 3)$
55. $f(x) = x^5 - x^3 \quad (-2, 2)$
56. $f(x) = x^4 - x^2 \quad (-2, 2)$
57. $f(x) = -0.2x^3 - 0.6x^2 + 4x - 6 \quad (-6, 4)$
58. $f(x) = -0.4x^3 + 0.6x^2 + 3x - 2 \quad (-4, 5)$
59. $f(x) = 0.25x^4 + 0.3x^3 - 0.9x^2 + 3 \quad (-3, 2)$
60. $f(x) = -0.4x^4 - 0.5x^3 + 0.8x^2 - 2 \quad (-3, 2)$

58. $f(x) = -0.4x^3 + 0.6x^2 + 3x - 2 \quad (-4, 5)$



https://www.desmos.com/calculator

-.4x^3+.6x^2+3x-2

61. Find the average rate of change of $f(x) = -2x^2 + 4$
(a) From 0 to 2
(b) From 1 to 3
(c) From 1 to 4

62. Find the average rate of change of $f(x) = -x^3 + 1$
(a) From 0 to 2
(b) From 1 to 3
(c) From -1 to 1

63. Find the average rate of change of $g(x) = x^3 - 2x + 1$
(a) From -3 to -2
(b) From -1 to 1
(c) From 1 to 3

64. Find the average rate of change of $h(x) = x^2 - 2x + 3$
(a) From -1 to 1
(b) From 0 to 2
(c) From 2 to 5

65. $f(x) = 5x - 2$
(a) Find the average rate of change from 1 to 3.
(b) Find an equation of the secant line containing $(1, f(1))$ and $(3, f(3))$.

66. $f(x) = -4x + 1$
(a) Find the average rate of change from 2 to 5.
(b) Find an equation of the secant line containing $(2, f(2))$ and $(5, f(5))$.

67. $g(x) = x^2 - 2$
(a) Find the average rate of change from -2 to 1.
(b) Find an equation of the secant line containing $(-2, g(-2))$ and $(1, g(1))$.

68. $g(x) = x^2 + 1$
(a) Find the average rate of change from -1 to 2.
(b) Find an equation of the secant line containing $(-1, g(-1))$ and $(2, g(2))$.

69. $h(x) = x^2 - 2x$
(a) Find the average rate of change from 2 to 4.
(b) Find an equation of the secant line containing $(2, h(2))$ and $(4, h(4))$.

70. $h(x) = -2x^2 + x$
(a) Find the average rate of change from 0 to 3.
(b) Find an equation of the secant line containing $(0, h(0))$ and $(3, h(3))$.

DEFINITION

If a and b, $a \neq b$, are in the domain of a function $y = f(x)$, the **average rate of change of f** from a to b is defined as

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} \qquad a \neq b \qquad \textbf{(1)}$$

THEOREM

Slope of the Secant Line

The average rate of change of a function from a to b equals the slope of the secant line containing the two points $(a, f(a))$ and $(b, f(b))$ on its graph.

70. $h(x) = -2x^2 + x$
(a) Find the average rate of change from 0 to 3.
(b) Find an equation of the secant line containing $(0, h(0))$ and $(3, h(3))$.