5 Exponential and Logarithmic Functions

Outline

5.3 Exponential Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Exponents (Appendix A, Section A.1, pp. A7–A9, and Section A.10, pp. A81–A87)
- Graphing Techniques: Transformations (Section 2.5, pp. 90–99)
- Solving Equations (Appendix A, Section A.6, pp. A44–A51)
- Average Rate of Change (Section 2.3, pp. 74–76)
- Quadratic Functions (Section 3.3, pp. 134–142)
- Linear Functions (Section 3.1, pp. 118–121)
- Horizontal Asymptotes (Section 4.2, pp. 191–192)

Now Work the **'Are You Prepared?'** problems on page 278.

OBJECTIVES
1. Evaluate Exponential Functions (p. 267)
2. Graph Exponential Functions (p. 271)
3. Define the Number *e* (p. 274)
4. Solve Exponential Equations (p. 276)

THEOREM

Laws of Exponents

If s, t, a, and b are real numbers with $a > 0$ and $b > 0$, then

$$a^s \cdot a^t = a^{s+t} \qquad (a^s)^t = a^{st} \qquad (ab)^s = a^s \cdot b^s$$

$$1^s = 1 \qquad a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s \qquad a^0 = 1 \qquad \textbf{(1)}$$

DEFINITION

An **exponential function** is a function of the form

$$\boxed{f(x) = Ca^x}$$

where a is a positive real number ($a > 0$), $a \neq 1$, and $C \neq 0$ is a real number. The domain of f is the set of all real numbers. The base a is the **growth factor,** and because $f(0) = Ca^0 = C$, we call C the **initial value.**

WARNING It is important to distinguish a power function, $g(x) = ax^n$, $n \geq 2$, an integer, from an exponential function, $f(x) = C \cdot a^x$, $a \neq 1$, $a > 0$. In a power function, the base is a variable and the exponent is a constant. In an exponential function, the base is a constant and the exponent is a variable. ■

THEOREM

For an exponential function $f(x) = Ca^x$, where $a > 0$ and $a \neq 1$, if x is any real number, then

$$\frac{f(x+1)}{f(x)} = a \quad \text{or} \quad f(x+1) = af(x)$$

Proof

$$\frac{f(x+1)}{f(x)} = \frac{Ca^{x+1}}{Ca^x} = a^{x+1-x} = a^1 = a \quad \blacksquare$$

In Words

For 1-unit changes in the input x of an exponential function $f(x) = C \cdot a^x$, the ratio of consecutive outputs is the constant a.

x	y	Average Rate of Change	Ratio of Consecutive Outputs
-1	5		
		$\dfrac{\Delta y}{\Delta x} = \dfrac{2-5}{0-(-1)} = -3$	$\dfrac{2}{5}$
0	2		
		-3	$-\dfrac{1}{2}$
1	-1		
		-3	4
2	-4		
		-3	$\dfrac{7}{4}$
3	-7		

(a)

(a) See Table 2(a). The average rate of change for every 1-unit increase in x is -3. Therefore, the function is a linear function. In a linear function the average rate of change is the slope m, so $m = -3$. The y-intercept b is the value of the function at $x = 0$, so $b = 2$. The linear function that models the data is $f(x) = mx + b = -3x + 2$.

x	y	Average Rate of Change	Ratio of Consecutive Outputs
-1	32		
		$\frac{\Delta y}{\Delta x} = \frac{16 - 32}{0 - (-1)} = -16$	$\frac{16}{32} = \frac{1}{2}$
0	16		
		-8	$\frac{8}{16} = \frac{1}{2}$
1	8		
		-4	$\frac{4}{8} = \frac{1}{2}$
2	4		
		-2	$\frac{2}{4} = \frac{1}{2}$
3	2		

(b)

(b) See Table 2(b). For this function, the average rate of change from -1 to 0 is -16, and the average rate of change from 0 to 1 is -8. Because the average rate of change is not constant, the function is not a linear function. The ratio of consecutive outputs for a 1-unit increase in the inputs is a constant, $\frac{1}{2}$. Because the ratio of consecutive outputs is constant, the function is an exponential function with growth factor $a = \frac{1}{2}$. The initial value of the exponential function is $C = 16$. Therefore, the exponential function that models the data is $g(x) = Ca^x = 16 \cdot \left(\frac{1}{2}\right)^x$.

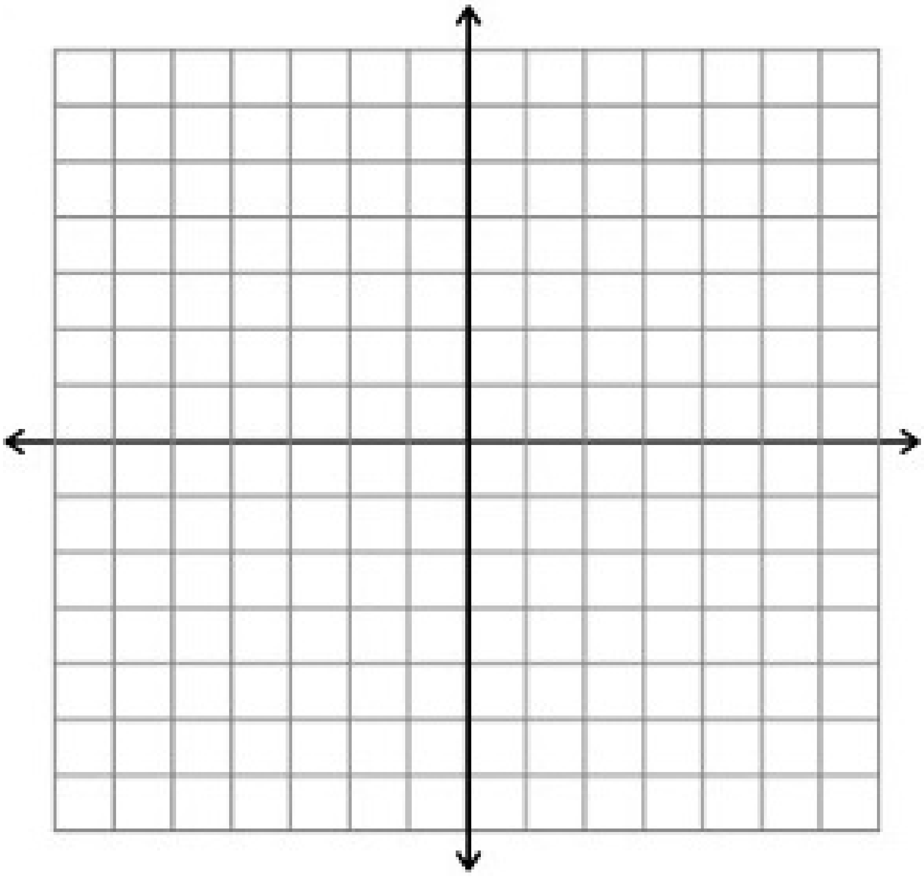
x	y	Average Rate of Change	Ratio of Consecutive Outputs
-1	2		
		$\frac{\Delta y}{\Delta x} = \frac{4-2}{0-(-1)} = 2$	2
0	4		
		3	$\frac{7}{4}$
1	7		
		4	$\frac{11}{7}$
2	11		
		5	$\frac{16}{11}$
3	16		

(c)

(c) See Table 2(c). For this function, the average rate of change from -1 to 0 is 2, and the average rate of change from 0 to 1 is 3. Because the average rate of change is not constant, the function is not a linear function. The ratio of consecutive outputs from -1 to 0 is 2, and the ratio of consecutive outputs from 0 to 1 is $\frac{7}{4}$. Because the ratio of consecutive outputs is not a constant, the function is not an exponential function.

$$f(x) = 2^x$$

x	y



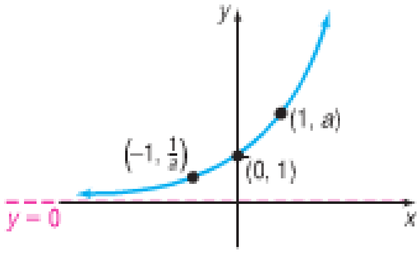
Exploring Exponential Functions

https://www.desmos.com/calculator/3namqjnhhf

Properties of the Exponential Function $f(x) = a^x$, $a > 1$

1. The domain is the set of all real numbers or $(-\infty, \infty)$ using interval notation; the range is the set of positive real numbers or $(0, \infty)$ using interval notation.
2. There are no x-intercepts; the y-intercept is 1.
3. The x-axis $(y = 0)$ is a horizontal asymptote as $x \to -\infty$ $\left[\lim_{x \to -\infty} a^x = 0\right]$.
4. $f(x) = a^x$, where $a > 1$, is an increasing function and is one-to-one.
5. The graph of f contains the points $(0, 1)$, $(1, a)$, and $\left(-1, \frac{1}{a}\right)$.
6. The graph of f is smooth and continuous, with no corners or gaps. See Figure 21.

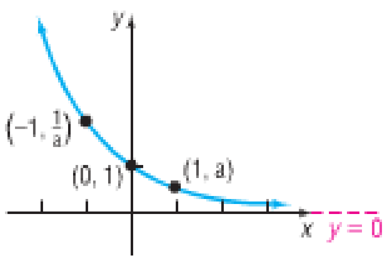
Figure 21



Properties of the Exponential Function $f(x) = a^x$, $0 < a < 1$

1. The domain is the set of all real numbers or $(-\infty, \infty)$ using interval notation; the range is the set of positive real numbers or $(0, \infty)$ using interval notation.
2. There are no x-intercepts; the y-intercept is 1.
3. The x-axis $(y = 0)$ is a horizontal asymptote as $x \to \infty$ $\left[\lim_{x \to \infty} a^x = 0\right]$.
4. $f(x) = a^x$, $0 < a < 1$, is a decreasing function and is one-to-one.
5. The graph of f contains the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.
6. The graph of f is smooth and continuous, with no corners or gaps. See Figure 25.

Figure 25



Define the Number *e*

https://twitter.com/e_scitalk

Consider the following summation formula built by the given pattern.

$$1+\frac{1}{1!}=1+\frac{1}{1}=$$

$$1+\frac{1}{1!}+\frac{1}{2!}=1+\frac{1}{1}+\frac{1}{1\cdot 2}=\left(1+\frac{1}{1}\right)+\frac{1}{2}=$$

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}=1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}=\left(1+\frac{1}{1}+\frac{1}{2}\right)+\frac{1}{6}=$$

Continuing this pattern on excel ...

n	1 + 1/1! + 1/2! + 1/3! + ...
1	2.000000000000000
2	2.500000000000000
3	2.666666666666670
4	2.708333333333330
5	2.716666666666670
6	2.718055555555560
7	2.718253968253970
8	2.718278769841270
9	2.718281525573190
10	2.718281801146380
11	2.718281826198490
12	2.718281828286170
13	2.718281828446760
14	2.718281828458230
15	2.718281828458990
16	2.718281828459040
17	2.718281828459050
18	2.718281828459050
19	2.718281828459050
20	2.718281828459050

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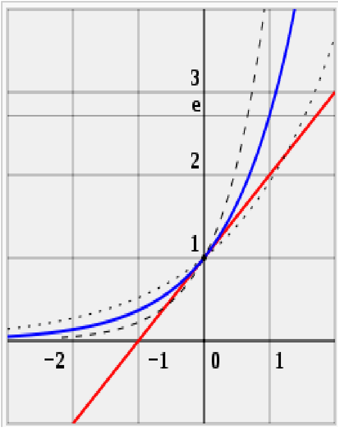
e (mathematical constant)

From Wikipedia, the free encyclopedia

"Euler's number" redirects here. For γ, a constant in number theory, see Euler–Mascheroni constant. For other uses, see List of things named after Leonhard Euler#Euler's numbers.

The number ***e*** is an important mathematical constant that is the base of the natural logarithm. It is approximately equal to 2.71828,[1] and is the limit of $(1 + 1/n)^n$ as n approaches infinity, an expression that arises in the study of compound interest. It can also be calculated as the sum of the infinite series[2]

$$e = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots$$



The constant can be defined in many ways; for example, ***e*** is the unique real number such that the value of the derivative (slope of the tangent line) of the function $f(x) = e^x$ at the point $x = 0$ is equal to 1.[3] The function e^x so defined is called the exponential function, and its inverse is the natural logarithm, or logarithm to base e. The natural logarithm of a positive number k can also be defined directly as the area under the curve $y = 1/x$ between $x = 1$ and $x = k$, in which case, ***e*** is the number whose natural logarithm is 1. There are also more alternative characterizations.

DEFINITION

The **number e** is defined as the number that the expression

$$\left(1+\frac{1}{n}\right)^n \qquad (2)$$

approaches as $n \to \infty$. In calculus, this is expressed using limit notation as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Table 5

n	$\frac{1}{n}$	$1+\frac{1}{n}$	$\left(1+\frac{1}{n}\right)^n$
1	1	2	2
2	0.5	1.5	2.25
5	0.2	1.2	2.48832
10	0.1	1.1	2.59374246
100	0.01	1.01	2.704813829
1,000	0.001	1.001	2.716923932
10,000	0.0001	1.0001	2.718145927
100,000	0.00001	1.00001	2.718268237
1,000,000	0.000001	1.000001	2.718280469
1,000,000,000	10^{-9}	$1+10^{-9}$	2.718281827

Solve Exponential Equations

If $a^u = a^v$, then $u = v$. (3)

In Words

When two exponential expressions with the same base are equal, then their exponents are equal.

SUMMARY Properties of the Exponential Function

$f(x) = a^x, \quad a > 1$	Domain: the interval $(-\infty, \infty)$; range: the interval $(0, \infty)$ x-intercepts: none; y-intercept: 1 Horizontal asymptote: x-axis $(y = 0)$ as $x \to -\infty$ Increasing; one-to-one; smooth; continuous See Figure 21 for a typical graph.
$f(x) = a^x, \quad 0 < a < 1$	Domain: the interval $(-\infty, \infty)$; range: the interval $(0, \infty)$ x-intercepts: none; y-intercept: 1 Horizontal asymptote: x-axis $(y = 0)$ as $x \to \infty$ Decreasing; one-to-one; smooth; continuous See Figure 25 for a typical graph.

If $a^u = a^v$, then $u = v$.

In Problems 15–24, approximate each number using a calculator. Express your answer rounded to three decimal places.

https://www.google.com/#q=calculator

http://www.wolframalpha.com/

16. (a) $5^{1.7}$ (b) $5^{1.73}$

(c) $5^{1.732}$ (d) $5^{\sqrt{3}}$

20. (a) $2.7^{3.1}$ (b) $2.71^{3.14}$

(c) $2.718^{3.141}$ (d) e^{π}

In Problems 25–32, determine whether the given function is linear, exponential, or neither. For those that are linear functions, find a linear function that models the data; for those that are exponential, find an exponential function that models the data.

26.

x	$g(x)$
−1	2
0	5
1	8
2	11
3	14

28.

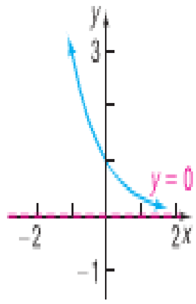
x	$F(x)$
−1	$\frac{2}{3}$
0	1
1	$\frac{3}{2}$
2	$\frac{9}{4}$
3	$\frac{27}{8}$

In Problems 33–40, the graph of an exponential function is given. Match each graph to one of the following functions.

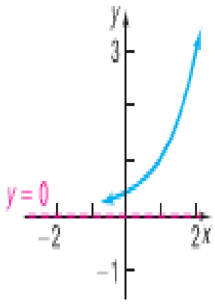
(a) $y = 3^x$ (b) $y = 3^{-x}$ (c) $y = -3^x$ (d) $y = -3^{-x}$

(e) $y = 3^x - 1$ (f) $y = 3^{x-1}$ (g) $y = 3^{1-x}$ (h) $y = 1 - 3^x$

33.



34.



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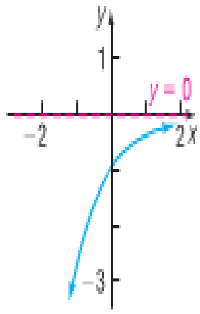

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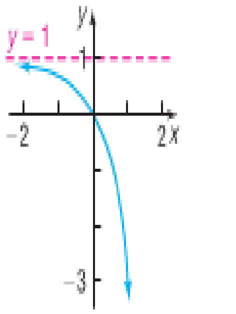

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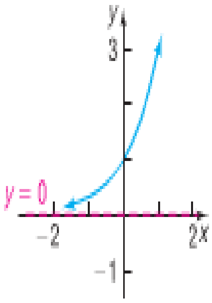

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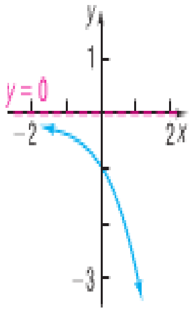

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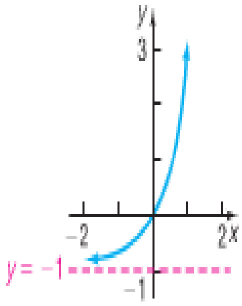

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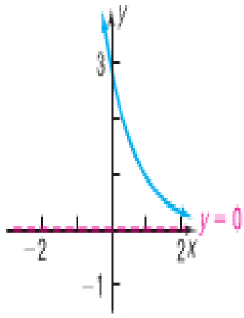
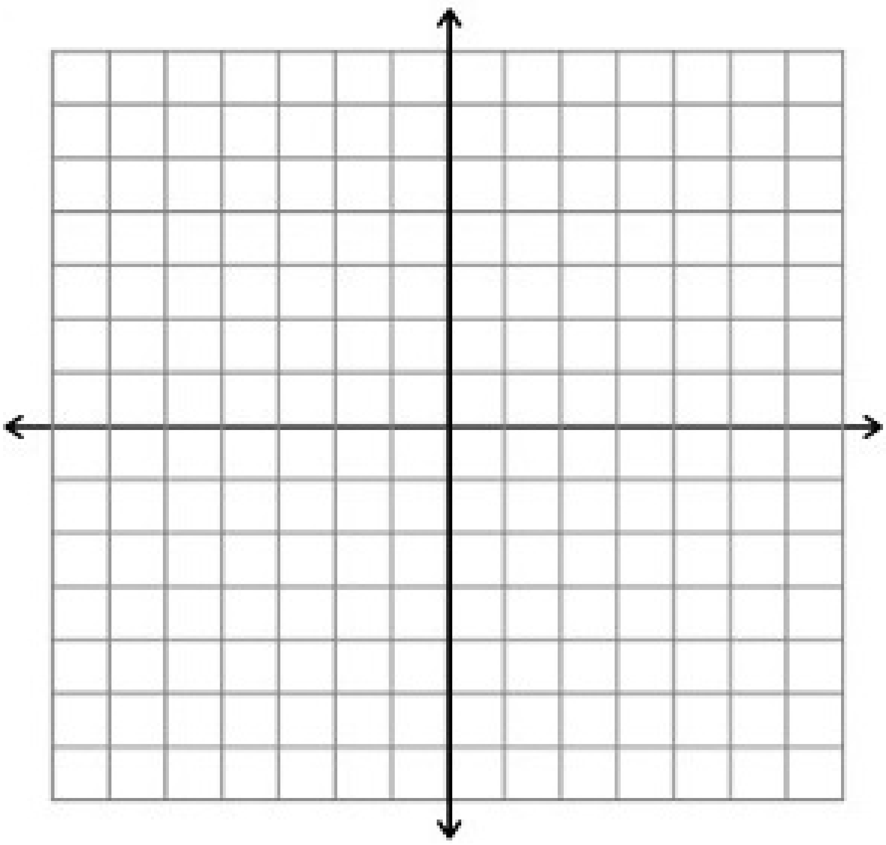


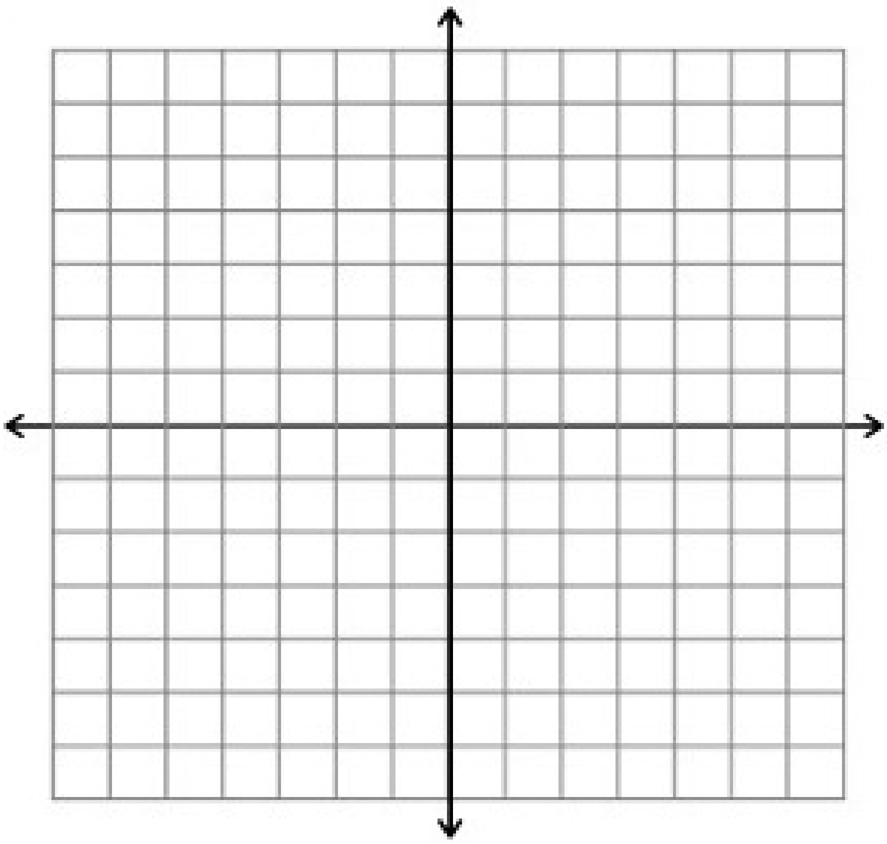
In Problems 41–52, use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

50. $f(x) = 1 - 2^{x+3}$



In Problems 53–60, begin with the graph of $y = e^x$ [Figure 27] and use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

58. $f(x) = 9 - 3e^{-x}$



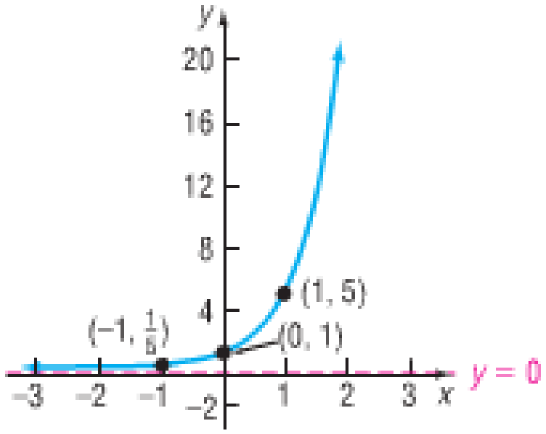
In Problems 61–80, solve each equation.

64. $3^{-x} = 81$

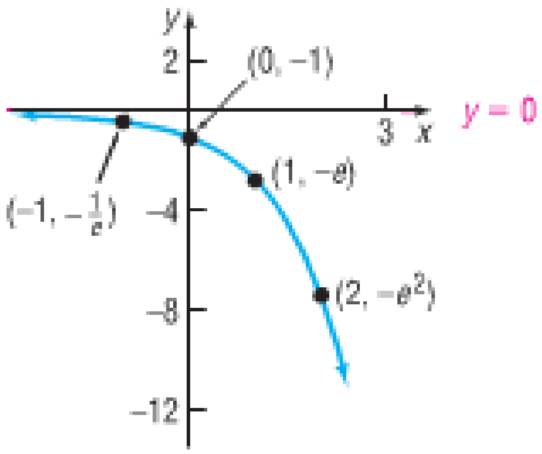
72. $9^{-x+15} = 27^{x}$

In Problems 85–88, determine the exponential function whose graph is given.

86.

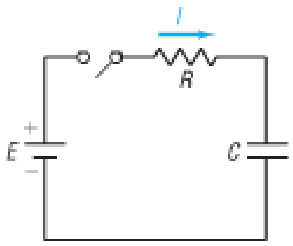


88.



114. Current in a *RC* Circuit The equation governing the amount of current I (in amperes) after time t (in microseconds) in a single RC circuit consisting of a resistance R (in ohms), a capacitance C (in microfarads), and an electromotive force E (in volts) is

$$I = \frac{E}{R}e^{-t/(RC)}$$



(a) If $E = 120$ volts, $R = 2000$ ohms, and $C = 1.0$ microfarad, how much current I_1 is flowing initially ($t = 0$)? After 1000 microseconds? After 3000 microseconds?
(b) What is the maximum current?
(c) Graph the function $I = I_1(t)$, measuring I along the y-axis and t along the x-axis.
(d) If $E = 120$ volts, $R = 1000$ ohms, and $C = 2.0$ microfarads, how much current I_2 is flowing initially? After 1000 microseconds? After 3000 microseconds?
(e) What is the maximum current?
(f) Graph the function $I = I_2(t)$ on the same coordinate axes as $I_1(t)$.