5 Exponential and Logarithmic Functions

Outline

5.5 Properties of Logarithms

OBJECTIVES

1. Work with the Properties of Logarithms (p. 296)
2. Write a Logarithmic Expression as a Sum or Difference of Logarithms (p. 298)
3. Write a Logarithmic Expression as a Single Logarithm (p. 299)
4. Evaluate Logarithms Whose Base Is Neither 10 Nor *e* (p. 301)

Establishing Properties of Logarithms

Consider the following properties from the world of exponents.

$$a^0 = 1 \qquad\qquad a^1 = a$$

Convert them to logarithmic equations.

$$\boxed{\log_a 1 = 0 \qquad \log_a a = 1}$$

EXAMPLE 1 **Establishing Properties of Logarithms**

(a) Show that $\log_a 1 = 0$. (b) Show that $\log_a a = 1$.

Solution (a) This fact was established when we graphed $y = \log_a x$ (see Figure 30 on page 286). To show the result algebraically, let $y = \log_a 1$. Then

$$\begin{aligned} y &= \log_a 1 \\ a^y &= 1 && \text{Change to an exponential statement.} \\ a^y &= a^0 && a^0 = 1 \text{ since } a > 0, a \neq 1 \\ y &= 0 && \text{Solve for } y. \\ \log_a 1 &= 0 && y = \log_a 1 \end{aligned}$$

(b) Let $y = \log_a a$. Then

$$\begin{aligned} y &= \log_a a \\ a^y &= a && \text{Change to an exponential statement.} \\ a^y &= a^1 && a = a^1 \\ y &= 1 && \text{Solve for } y. \\ \log_a a &= 1 && y = \log_a a \end{aligned}$$

To summarize:

$$\boxed{\log_a 1 = 0 \qquad \log_a a = 1}$$

Recall: $y = a^x$ and $y = \log_a x$ are inverses.

And: $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}

So:

$$y = a^{\log_a x} \qquad\qquad y = \log_a a^x$$

THEOREM

Properties of Logarithms

In the properties given next, M and a are positive real numbers, $a \neq 1$, and r is any real number.

The number $\log_a M$ is the exponent to which a must be raised to obtain M. That is,

$$a^{\log_a M} = M \qquad (1)$$

The logarithm to the base a of a raised to a power equals that power. That is,

$$\log_a a^r = r \qquad (2)$$

THEOREM

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, $a \neq 1$, and r is any real number.

The Log of a Product Equals the Sum of the Logs

$$\log_a(MN) = \log_a M + \log_a N \qquad (3)$$

The Log of a Quotient Equals the Difference of the Logs

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N \qquad (4)$$

The Log of a Power Equals the Product of the Power and the Log

$$\log_a M^r = r \log_a M \qquad (5)$$

$$a^x = e^{x \ln a} \qquad (6)$$

We shall derive properties (3), (5), and (6) and leave the derivation of property (4) as an exercise (see Problem 109).

Proof of Property (3) Let $A = \log_a M$ and let $B = \log_a N$. These expressions are equivalent to the exponential expressions

$$a^A = M \quad \text{and} \quad a^B = N$$

Now

$$\begin{aligned} \log_a(MN) = \log_a(a^A a^B) &= \log_a a^{A+B} && \text{Law of Exponents} \\ &= A + B && \text{Property (2) of logarithms} \\ &= \log_a M + \log_a N \end{aligned}$$

■

Proof of Property (5) Let $A = \log_a M$. This expression is equivalent to

$$a^A = M$$

Now

$$\begin{aligned} \log_a M^r = \log_a(a^A)^r &= \log_a a^{rA} && \text{Law of Exponents} \\ &= rA && \text{Property (2) of logarithms} \\ &= r \log_a M \end{aligned}$$

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Proof of Property (6) From property (1), with $a = e$, we have

$$e^{\ln M} = M$$

Now let $M = a^x$ and apply property (5).

$$e^{\ln a^x} = e^{x \ln a} = a^x$$

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THEOREM

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, $a \neq 1$.

If $M = N$, then $\log_a M = \log_a N$. **(7)**

If $\log_a M = \log_a N$, then $M = N$. **(8)**

THEOREM

Change-of-Base Formula

If $a \neq 1$, $b \neq 1$, and M are positive real numbers, then

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad (9)$$

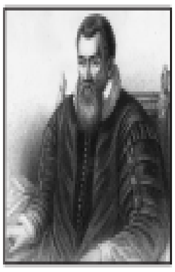
$$\log_a M = \frac{\log M}{\log a} \quad \text{and} \quad \log_a M = \frac{\ln M}{\ln a} \qquad (10)$$

SUMMARY Properties of Logarithms

In the list that follows, a, b, M, N, and r are real numbers. Also, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$, $M > 0$, and $N > 0$.

Definition	$y = \log_a x$ means $x = a^y$	
Properties of logarithms	$\log_a 1 = 0$; $\log_a a = 1$	$\log_a M^r = r \log_a M$
	$a^{\log_a M} = M$; $\log_a a^r = r$	$a^x = e^{x \ln a}$
	$\log_a(MN) = \log_a M + \log_a N$	If $M = N$, then $\log_a M = \log_a N$.
	$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$	If $\log_a M = \log_a N$, then $M = N$.
Change-of-Base Formula	$\log_a M = \dfrac{\log_b M}{\log_b a}$	

Historical Feature



John Napier (1550–1617)

Logarithms were invented about 1590 by John Napier (1550–1617) and Joost Bürgi (1552–1632), working independently. Napier, whose work had the greater influence, was a Scottish lord, a secretive man whose neighbors were inclined to believe him to be in league with the devil. His approach to logarithms was very different from ours; it was based on the relationship between arithmetic and geometric sequences, discussed in a later chapter, and not on the inverse function relationship of logarithms to exponential functions (described in Section 5.4). Napier's tables, published in 1614, listed what would now be called *natural logarithms* of sines and were rather difficult to use. A London professor, Henry Briggs, became interested in the tables and visited Napier. In their conversations, they developed the idea of common logarithms, which were published in 1617. Their importance for calculation was immediately recognized, and by 1650 they were being printed as far away as China. They remained an important calculation tool until the advent of the inexpensive handheld calculator about 1972, which has decreased their calculational, but not their theoretical, importance.

A side effect of the invention of logarithms was the popularization of the decimal system of notation for real numbers.

In Problems 13–28, use properties of logarithms to find the exact value of each expression. Do not use a calculator.

14. $\log_2 2^{-13}$

18. $e^{\ln 8}$

20. $\log_6 9 + \log_6 4$

22. $\log_8 16 - \log_8 2$

In Problems 13–28, use properties of logarithms to find the exact value of each expression. Do not use a calculator.

24. $\log_3 8 \cdot \log_8 9$ Hint: Use change of base.

26. $5^{\log_5 6 + \log_5 7}$

28. $e^{\log_{e^2} 9}$

In Problems 29–36, suppose that $\ln 2 = a$ *and* $\ln 3 = b$. *Use properties of logarithms to write each logarithm in terms of a and b.*

30. $\ln \frac{2}{3}$

36. $\ln \sqrt[4]{\frac{2}{3}}$

In Problems 37–56, write each expression as a sum and/or difference of logarithms. Express powers as factors.

38. $\log_3 \frac{x}{9}$

44. $\ln(xe^x)$

48. $\ln\left(x\sqrt{1+x^2}\right) \quad x > 0$

56. $\ln\left[\frac{5x^2\sqrt[3]{1-x}}{4(x+1)^2}\right] \quad 0 < x < 1$

In Problems 57–70, write each expression as a single logarithm.

58. $2\log_3 u - \log_3 v$

62. $\log(x^2 + 3x + 2) - 2\log(x + 1)$

68. $\frac{1}{3}\log(x^3 + 1) + \frac{1}{2}\log(x^2 + 1)$

70. $3\log_5(3x + 1) - 2\log_5(2x - 1) - \log_5 x$

In Problems 71–78, use the Change-of-Base Formula and a calculator to evaluate each logarithm. Round your answer to three decimal places.

72. $\log_5 18$

76. $\log_{\sqrt{5}} 8$

Applications and Extensions

In Problems 87–96, express y as a function of x. The constant C is a positive number.

88. $\ln y = \ln(x + C)$

94. $\ln(y + 4) = 5x + \ln C$

98. Find the value of $\log_2 4 \cdot \log_4 6 \cdot \log_6 8$.

110. Show that $\log_a\left(\frac{1}{N}\right) = -\log_a N$, where a and N are positive real numbers and $a \neq 1$.