Functions and Their Graphs

2

Outline

2.1 Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Interval Notation (Appendix A, Section A.9, pp. A72–A73)
- Solving Inequalities (Appendix A, Section A.9, pp. A75–A78)
- Evaluating Algebraic Expressions, Domain of a Variable (Appendix A, Section A.1, pp. A6–A7)

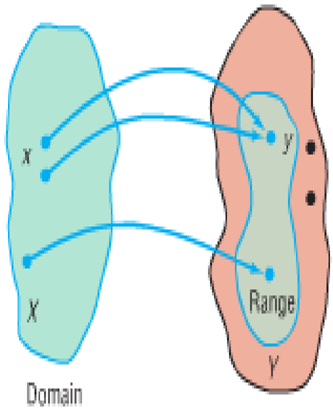
Now Work the 'Are You Prepared?' problems on page 56.

OBJECTIVES
1. Determine Whether a Relation Represents a Function (p. 46)
2. Find the Value of a Function (p. 49)
3. Find the Domain of a Function Defined by an Equation (p. 52)
4. Form the Sum, Difference, Product, and Quotient of Two Functions (p. 54)

The Function Game

http://www.functiongame.com/

Figure 5



DEFINITION

Let X and Y be two nonempty sets.* A **function** from X into Y is a relation that associates with each element of X exactly one element of Y.

The set X is called the **domain** of the function. For each element x in X, the corresponding element y in Y is called the **value** of the function at x, or the **image** of x. The set of all images of the elements in the domain is called the **range** of the function. See Figure 5.

Since there may be some elements in Y that are not the image of some x in X, it follows that the range of a function may be a subset of Y, as shown in Figure 5.

Not all relations between two sets are functions. The next example shows how to determine whether a relation is a function.

In Words

For a function, no input has more than one output. The domain of a function is the set of all inputs; the range is the set of all outputs.

EXAMPLE 4 **Determining Whether an Equation Is a Function**

Determine if the equation $y = 2x - 5$ defines y as a function of x.

Solution The equation tells us to take an input x, multiply it by 2, and then subtract 5. For any input x, these operations yield only one output y. For example, if $x = 1$, then $y = 2(1) - 5 = -3$. If $x = 3$, then $y = 2(3) - 5 = 1$. For this reason, the equation is a function.

Function Notation

$$f(x) = 2x - 5$$

DEFINITION

If f and g are functions:
The **sum** $f + g$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

REMEMBER The symbol $\cap$ stands for intersection. It means you should find the elements that are common to two sets.

The domain of $f + g$ consists of the numbers x that are in the domains of both f and g. That is, domain of $f + g$ = domain of $f \cap$ domain of g.

DEFINITION

The **difference** $f - g$ is the function defined by

$$(f - g)(x) = f(x) - g(x)$$

The domain of $f - g$ consists of the numbers x that are in the domains of both f and g. That is, domain of $f - g$ = domain of $f \cap$ domain of g.

DEFINITION

The **product** $f \cdot g$ is the function defined by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

The domain of $f \cdot g$ consists of the numbers x that are in the domains of both f and g. That is, domain of $f \cdot g$ = domain of $f \cap$ domain of g.

DEFINITION

The **quotient** $\frac{f}{g}$ is the function defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad g(x) \neq 0$$

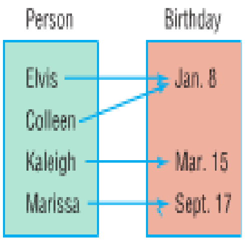
The domain of $\frac{f}{g}$ consists of the numbers x for which $g(x) \neq 0$ and that are in the domains of both f and g. That is,

$$\text{domain of } \frac{f}{g} = \{x \mid g(x) \neq 0\} \cap \text{domain of } f \cap \text{domain of } g$$

Skill Building

In Problems 15–26, determine whether each relation represents a function. For each function, state the domain and range.

15.



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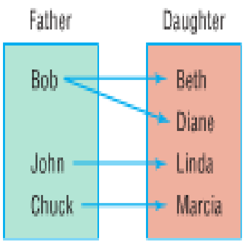

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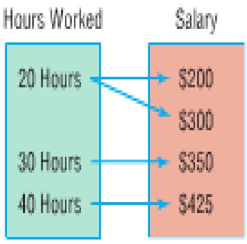

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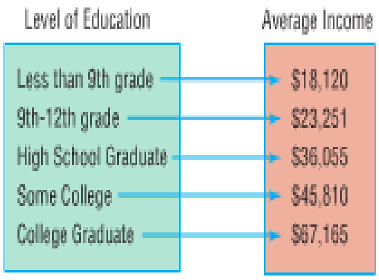


19. $\{(2,6),(-3,6),(4,9),(2,10)\}$
20. $\{(-2,5),(-1,3),(3,7),(4,12)\}$
21. $\{(1,3),(2,3),(3,3),(4,3)\}$
22. $\{(0,-2),(1,3),(2,3),(3,7)\}$
23. $\{(-2,4),(-2,6),(0,3),(3,7)\}$
24. $\{(-4,4),(-3,3),(-2,2),(-1,1),(-4,0)\}$
25. $\{(-2,4),(-1,1),(0,0),(1,1)\}$
26. $\{(-2,16),(-1,4),(0,3),(1,4)\}$

20. $\{(-2,5),(-1,3),(3,7),(4,12)\}$

24. $\{(-4,4),(-3,3),(-2,2),(-1,1),(-4,0)\}$

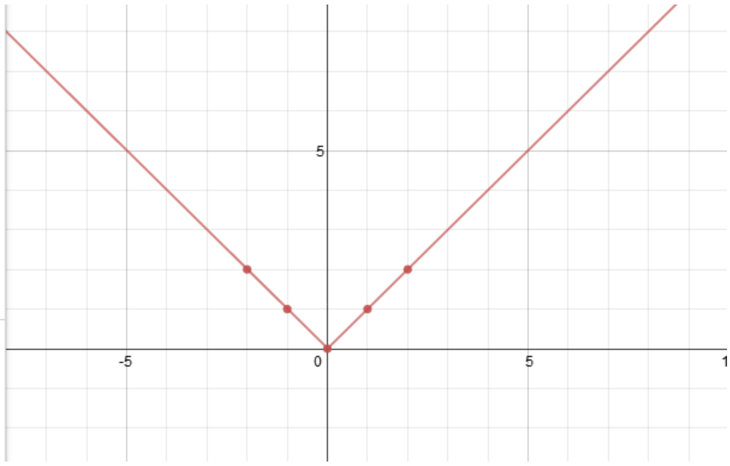
26. $\{(-2,16),(-1,4),(0,3),(1,4)\}$

In Problems 27–38, determine whether the equation defines y as a function of x.

27. $y = x^2$
28. $y = x^3$
29. $y = \frac{1}{x}$
30. $y = |x|$
31. $y^2 = 4 - x^2$
32. $y = \pm\sqrt{1 - 2x}$
33. $x = y^2$
34. $x + y^2 = 1$
35. $y = 2x^2 - 3x + 4$
36. $y = \frac{3x - 1}{x + 2}$
37. $2x^2 + 3y^2 = 1$
38. $x^2 - 4y^2 = 1$

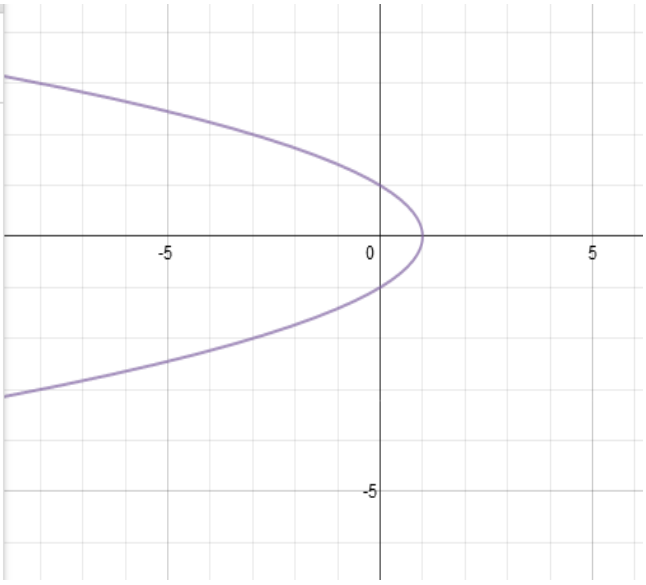
30. $y = |x|$

x	abs(x)
−2	2
−1	1
0	0
1	1
2	2



34. $x + y^2 = 1$

$x + y^2 = 1$



In Problems 39–46, find the following for each function:

(a) $f(0)$ (b) $f(1)$ (c) $f(-1)$ (d) $f(-x)$ (e) $-f(x)$ (f) $f(x+1)$ (g) $f(2x)$ (h) $f(x+h)$

39. $f(x) = 3x^2 + 2x - 4$
40. $f(x) = -2x^2 + x - 1$
41. $f(x) = \dfrac{x}{x^2+1}$
42. $f(x) = \dfrac{x^2-1}{x+4}$
43. $f(x) = |x| + 4$
44. $f(x) = \sqrt{x^2 + x}$
45. $f(x) = \dfrac{2x+1}{3x-5}$
46. $f(x) = 1 - \dfrac{1}{(x+2)^2}$

42. $f(x) = \dfrac{x^2 - 1}{x + 4}$

(a) $f(0)$

(b) $f(1)$

(c) $f(-1)$

(d) $f(-x)$

(e) $-f(x)$

(f) $f(x + 1)$

(g) $f(2x)$

(h) $f(x + h)$

And of course.

f ()

In Problems 47–62, find the domain of each function.

47. $f(x) = -5x + 4$

48. $f(x) = x^2 + 2$

49. $f(x) = \dfrac{x}{x^2 + 1}$

50. $f(x) = \dfrac{x^2}{x^2 + 1}$

51. $g(x) = \dfrac{x}{x^2 - 16}$

52. $h(x) = \dfrac{2x}{x^2 - 4}$

53. $F(x) = \dfrac{x - 2}{x^3 + x}$

54. $G(x) = \dfrac{x + 4}{x^3 - 4x}$

55. $h(x) = \sqrt{3x - 12}$

56. $G(x) = \sqrt{1 - x}$

57. $f(x) = \dfrac{4}{\sqrt{x - 9}}$

58. $f(x) = \dfrac{x}{\sqrt{x - 4}}$

59. $p(x) = \sqrt{\dfrac{2}{x - 1}}$

60. $q(x) = \sqrt{-x - 2}$

61. $P(t) = \dfrac{\sqrt{t - 4}}{3t - 21}$

62. $h(z) = \dfrac{\sqrt{z + 3}}{z - 2}$

Finding the Domain of a Function Defined by an Equation

1. Start with the domain as the set of real numbers.
2. If the equation has a denominator, exclude any numbers that give a zero denominator.
3. If the equation has a radical of even index, exclude any numbers that cause the expression inside the radical to be negative.

48. $f(x) = x^2 + 2$

54. $G(x) = \dfrac{x + 4}{x^3 - 4x}$

56. $G(x) = \sqrt{1 - x}$

In Problems 63–72, for the given functions f and g, find the following. For parts (a)–(d), also find the domain.

(a) $(f+g)(x)$ (b) $(f-g)(x)$ (c) $(f \cdot g)(x)$ (d) $\left(\frac{f}{g}\right)(x)$

(e) $(f+g)(3)$ (f) $(f-g)(4)$ (g) $(f \cdot g)(2)$ (h) $\left(\frac{f}{g}\right)(1)$

63. $f(x) = 3x + 4; \quad g(x) = 2x - 3$

64. $f(x) = 2x + 1; \quad g(x) = 3x - 2$

65. $f(x) = x - 1; \quad g(x) = 2x^2$

66. $f(x) = 2x^2 + 3; \quad g(x) = 4x^3 + 1$

67. $f(x) = \sqrt{x}; \quad g(x) = 3x - 5$

68. $f(x) = |x|; \quad g(x) = x$

69. $f(x) = 1 + \frac{1}{x}; \quad g(x) = \frac{1}{x}$

70. $f(x) = \sqrt{x-1}; \quad g(x) = \sqrt{4-x}$

71. $f(x) = \frac{2x+3}{3x-2}; \quad g(x) = \frac{4x}{3x-2}$

72. $f(x) = \sqrt{x+1}; \quad g(x) = \frac{2}{x}$

72. $f(x) = \sqrt{x+1}; \quad g(x) = \frac{2}{x}$

(a) $(f+g)(x)$

(e) $(f+g)(3)$

(c) $(f \cdot g)(x)$

(g) $(f \cdot g)(2)$

In Problems 75–82, find the difference quotient of f; that is, find $\frac{f(x+h)-f(x)}{h}$, $h \neq 0$, for each function. Be sure to simplify.

75. $f(x) = 4x + 3$

76. $f(x) = -3x + 1$

77. $f(x) = x^2 - x + 4$

78. $f(x) = 3x^2 - 2x + 6$

79. $f(x) = \frac{1}{x^2}$

80. $f(x) = \frac{1}{x+3}$

81. $f(x) = \sqrt{x}$ [**Hint:** Rationalize the numerator.]

82. $f(x) = \sqrt{x+1}$

76. $f(x) = -3x + 1$

Applications and Extensions

83. If $f(x) = 2x^3 + Ax^2 + 4x - 5$ and $f(2) = 5$, what is the value of A?

84. If $f(x) = 3x^2 - Bx + 4$ and $f(-1) = 12$, what is the value of B?

85. If $f(x) = \dfrac{3x + 8}{2x - A}$ and $f(0) = 2$, what is the value of A?

86. If $f(x) = \dfrac{2x - B}{3x + 4}$ and $f(2) = \dfrac{1}{2}$, what is the value of B?

87. If $f(x) = \dfrac{2x - A}{x - 3}$ and $f(4) = 0$, what is the value of A? Where is f not defined?

88. If $f(x) = \dfrac{x - B}{x - A}$, $f(2) = 0$ and $f(1)$ is undefined, what are the values of A and B?

89. **Geometry** Express the area A of a rectangle as a function of the length x if the length of the rectangle is twice its width.

90. **Geometry** Express the area A of an isosceles right triangle as a function of the length x of one of the two equal sides.

91. **Constructing Functions** Express the gross salary G of a person who earns \$10 per hour as a function of the number x of hours worked.

92. **Constructing Functions** Tiffany, a commissioned salesperson, earns \$100 base pay plus \$10 per item sold. Express her gross salary G as a function of the number x of items sold.

94. **Number of Rooms** The function

$$N(r) = -1.44r^2 + 14.52r - 14.96$$

represents the number N of housing units (in millions) that have r rooms, where r is an integer and $2 \le r \le 9$.

(a) Identify the dependent and independent variables.
(b) Evaluate $N(3)$. Provide a verbal explanation of the meaning of $N(3)$.

95. **Effect of Gravity on Earth** If a rock falls from a height of 20 meters on Earth, the height H (in meters) after x seconds is approximately

$$H(x) = 20 - 4.9x^2$$

(a) What is the height of the rock when $x = 1$ second? $x = 1.1$ seconds? $x = 1.2$ seconds? $x = 1.3$ seconds?
(b) When is the height of the rock 15 meters? When is it 10 meters? When is it 5 meters?
(c) When does the rock strike the ground?

96. **Effect of Gravity on Jupiter** If a rock falls from a height of 20 meters on the planet Jupiter, its height H (in meters) after x seconds is approximately

$$H(x) = 20 - 13x^2$$

(a) What is the height of the rock when $x = 1$ second? $x = 1.1$ seconds? $x = 1.2$ seconds?
(b) When is the height of the rock 15 meters? When is it 10 meters? When is it 5 meters?
(c) When does the rock strike the ground?

84. If $f(x) = 3x^2 - Bx + 4$ and $f(-1) = 12$, what is the value of B?