Appendix A. Commutation Functions

A.1 Introduction

In this appendix we give an introduction to the use of commutation functions. These functions were invented in the 18th century and achieved great popularity, which can be ascribed to two reasons:

Reason 1

Tables of commutation functions simplify the calculation of numerical values for many actuarial functions.

Reason 2

Expected values such as net single premiums may be derived within a deterministic model closely related to commutation functions.

Both reasons have lost their significance, the first with the advent of powerful computers, the second with the growing acceptance of models based on probability theory, which allows a more complete understanding of the essentials of insurance. It may therefore be taken for granted that the days of glory for the commutation functions now belong to the past.

A.2 The Deterministic Model

Imagine a cohort of lives, all of the same age, observed over time, and denote by l_x the number still living at age x. Thus $d_x = l_x - l_{x+1}$ is the number of deaths between the ages of x and $x+1$.

Probabilities and expected values may now be derived from simple proportions and averages. So is, for instance,

$$ {}_tp_x = l_{x+t}/l_x \tag{A.2.1}$$

the proportion of persons alive at age $x+t$, relative to the number of persons alive at age x, and the probability that a life aged x will die within a year is

$$q_x = d_x / l_x \,. \tag{A.2.2}$$

In Chapter 2 we introduced the expected curtate future lifetime of a life aged x. Replacing ${}_kp_x$ by l_{x+k}/l_x in (2.4.3), we obtain

$$e_x = \frac{l_{x+1} + l_{x+2} + \cdots}{l_x} \,. \tag{A.2.3}$$

The numerator in this expression is the total number of complete future years to be "lived" by the l_x lives (x), so that e_x is the average number of completed years left.

A.3 Life Annuities

We first consider a life annuity-due with annual payments of 1 unit, as introduced in Section 4.2, the net single premium of which annuity was denoted by $\ddot{a}_x$. Replacing ${}_kp_x$ in (4.2.5) by l_{x+k}/l_x, we obtain

$$\ddot{a}_x = \frac{l_x + v l_{x+1} + v^2 l_{x+2} + \cdots}{l_x} \,, \tag{A.3.1}$$

or

$$l_x \, \ddot{a}_x = l_x + v l_{x+1} + v^2 l_{x+2} + \cdots \,. \tag{A.3.2}$$

This result is often referred to as the *equivalence principle*, and its interpretation within the deterministic model is evident: if each of the l_x persons living at age x were to buy an annuity of the given type, the sum of net single premiums (the left hand side of (A.3.2)) would equal the present value of the benefits (the right hand side of (A.3.2)).

Multiplying both numerator and denominator in (A.3.1) by v^x, we find

$$\ddot{a}_x = \frac{v^x l_x + v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + \cdots}{v^x l_x} \,. \tag{A.3.3}$$

With the abbreviations

$$D_x = v^x l_x \,, \quad N_x = D_x + D_{x+1} + D_{x+2} + \cdots \tag{A.3.4}$$

we then obtain the simple formula

$$\ddot{a}_x = \frac{N_x}{D_x} \,. \tag{A.3.5}$$

Thus the manual calculation of $\ddot{a}_x$ is extremely easy if tables of the commutation functions D_x and N_x are available. The function D_x is called the *"discounted number of survivors"*.

Similarly one may obtain formulas for the net single premium of a temporary life annuity,

$$\ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}, \tag{A.3.6}$$

immediate life annuities,

$$a_x = \frac{N_{x+1}}{D_x}, \tag{A.3.7}$$

and general annuities with annual payments: formula (4.4.2) may naturally be translated to

$$\mathrm{E}(Y) = \frac{r_0 D_x + r_1 D_{x+1} + r_2 D_{x+2} + \cdots}{D_x}. \tag{A.3.8}$$

For the special case $r_k = k + 1$ we obtain the formula

$$(I\ddot{a})_x = \frac{S_x}{D_x}; \tag{A.3.9}$$

here the commutation function S_x is defined by

$$\begin{aligned} S_x &= D_x + 2D_{x+1} + 3D_{x+2} + \cdots \\ &= N_x + N_{x+1} + N_{x+2} + \cdots . \end{aligned} \tag{A.3.10}$$

A.4 Life Insurance

In addition to (A.3.4) and (A.3.10) we now define the commutation functions

$$\begin{aligned} C_x &= v^{x+1} d_x , \\ M_x &= C_x + C_{x+1} + C_{x+2} + \cdots , \\ R_x &= C_x + 2C_{x+1} + 3C_{x+2} + \cdots \\ &= M_x + M_{x+1} + M_{x+2} + \cdots . \end{aligned} \tag{A.4.1}$$

Replacing ${}_k p_x q_{x+k}$ in equation (3.2.3) by d_{x+k}/l_x, we obtain

$$\begin{aligned} A_x &= \frac{v d_x + v^2 d_{x+1} + v^3 d_{x+2} + \cdots}{l_x} \\ &= \frac{C_x + C_{x+1} + C_{x+2} + \cdots}{D_x} \\ &= \frac{M_x}{D_x} . \end{aligned} \tag{A.4.2}$$

Similarly one obtains

$$\begin{aligned} (IA)_x &= \frac{v d_x + 2v^2 d_{x+1} + 3v^3 d_{x+2} + \cdots}{l_x} \\ &= \frac{C_x + 2C_{x+1} + 3C_{x+2} + \cdots}{D_x} \\ &= \frac{R_x}{D_x} . \end{aligned} \tag{A.4.3}$$

Obviously these formulae may be derived within the deterministic model by means of the equivalence principle. In order to determine A_x one would start with

$$l_x A_x = v d_x + v^2 d_{x+1} + v^3 d_{x+2} + \cdots \tag{A.4.4}$$

by imagining that l_x persons buy a whole life insurance of 1 unit each, payable at the end of the year of death, in return for a net single premium.

Corresponding formulae for term and endowment insurances are

$$\begin{aligned} A^1_{x:\overline{n}|} &= \frac{M_x - M_{x+n}}{D_x}, \\ A_{x:\overline{n}|} &= \frac{M_x - M_{x+n} + D_{x+n}}{D_x}, \\ (IA)^1_{x:\overline{n}|} &= \frac{C_x + 2C_{x+1} + 3C_{x+2} + \cdots + nC_{x+n-1}}{D_x} \\ &= \frac{M_x + M_{x+1} + M_{x+2} + \cdots + M_{x+n-1} - nM_{x+n}}{D_x} \\ &= \frac{R_x - R_{x+n} - nM_{x+n}}{D_x}, \end{aligned} \tag{A.4.5}$$

which speak for themselves.

The commutation functions defined in (A.4.1) can be expressed in terms of the commutation functions defined in Section 3. From $d_x = l_x - l_{x+1}$ follows

$$C_x = vD_x - D_{x+1}\,. \tag{A.4.6}$$

Summation yields the identities

$$M_x = vN_x - (N_x - D_x) = D_x - dN_x \tag{A.4.7}$$

and

$$R_x = N_x - dS_x\,. \tag{A.4.8}$$

Dividing both equations by D_x, we retrieve the identities

$$\begin{aligned} A_x &= 1 - d\,\ddot{a}_x\,, \\ (IA)_x &= \ddot{a}_x - d(I\ddot{a})_x\,, \end{aligned} \tag{A.4.9}$$

see equations (4.2.8) and (4.5.2).

A.5 Net Annual Premiums and Premium Reserves

Consider a whole life insurance with 1 unit payable at the end of the year of death, and payable by net annual premiums. Using (A.3.5) and (A.4.2) we find

$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{M_x}{N_x}\,. \tag{A.5.1}$$

Of course, the deterministic approach, i.e. the condition

$$P_x l_x + vP_x l_{x+1} + v^2 P_x l_{x+2} + \cdots = vd_x + v^2 d_{x+1} + v^3 d_{x+2} + \cdots \tag{A.5.2}$$

leads to the same result.

The net premium reserve at the end of year k then becomes

$${}_kV_x = A_{x+k} - P_x \ddot{a}_{x+k} = \frac{M_{x+k} - P_x N_{x+k}}{D_{x+k}}\,. \tag{A.5.3}$$

This result may also be obtained by the deterministic condition

$$\begin{aligned} {}_kV_x l_{x+k} + P_x l_{x+k} + vP_x l_{x+k+1} + v^2 P_x l_{x+k+2} + \cdots \\ = vd_{x+k} + v^2 d_{x+k+1} + v^3 d_{x+k+2} + \cdots\,. \end{aligned} \tag{A.5.4}$$

Here one imagines that each person alive at time k is allotted the amount ${}_kV_x$; the condition (A.5.4) states that the sum of the net premium reserve and the present value of future premiums must equal the present value of all future benefit payments.

The interested reader should be able to apply this technique to other, more general situations.

Appendix B. Simple Interest

In practice, the accumulation factor for a time interval of length h is occasionally approximated by

$$(1+i)^h \approx 1 + hi\,. \tag{B.1}$$

This approximation is obtained by neglecting all but the linear terms in the Taylor expansion of the left hand side above; alternatively the right hand side may be obtained by linear interpolation between $h = 0$ and $h = 1$. Similarly an approximation for the discount factor for an interval of length h is

$$v^h = (1-d)^h \approx 1 - hd\,. \tag{B.2}$$

The approximations (B.1) and (B.2) have little practical importance since the advent of pocket calculators.

Interest on transactions with a savings account is sometimes calculated according to the following rule: If an amount of r is deposited (drawn) at time u $(0 < u < 1)$, it is valued at time 0 as

$$rv^u \approx r(1-ud)\,. \tag{B.3}$$

At the end of the year (time 1) the amount is valued as

$$\begin{aligned} r(1+i)^{1-u} &= r(1+i)v^u \approx r(1+i)(1-ud) \\ &= r\{1+(1-u)i\}\,. \end{aligned} \tag{B.4}$$

This technique amounts to accumulation from time u to time 1 according to (B.1) or discounting from u to 0 according to (B.2). With a suitably chosen variable force of interest the rule is exact; this variable force of interest is determined by equating the accumulation factors:

$$1 + (1-u)i = \exp\left(\int_u^1 \delta(t)dt\right)\,. \tag{B.5}$$

Differentiating the logarithms gives the expression

$$\delta(u) = \frac{i}{1+(1-u)i} = \frac{d}{1-ud} \tag{B.6}$$

for $0 < u < 1$. The force of interest thus increases from $\delta(0) = d$ to $\delta(1) = i$ during the year.

The technique sketched above is based on the assumption that the accumulation factor for the time interval from u to 1 is a linear function of u; this assumption is analogous to *Assumption c* of Section 2.6, concerning mortality for fractional durations. The similarity between (B.6) and (2.6.10) is evident.

Appendix C

Exercises

C.0 Introduction

These exercises provide two types of practice. The first type consists of theoretical exercises, some demonstrations, and manipulation of symbols. Some of these problems of the first kind are based on Society of Actuaries questions from examinations prior to May 1990. The second type of practice involves using a spreadsheet program. Many exercises are solved in Appendix D. For the spreadsheet exercises, we give a guide to follow in writing your own program. For the theoretical exercises, we usually give a complete description. We provide guides for solving the spreadsheet problems, rather than computer codes. The student should write a program and use the guide to verify it. We use the terminology of *Excel* in the guides. The terminology of other programs is analogous.

I would like to thank Hans Gerber for allowing me to contribute these exercises to his textbook. It is a pleasure to acknowledge the assistance of Georgia State University graduate students, Masa Ozeki and Javier Suarez who helped by checking solutions and proofreading the exercises.

I hope that students will find these exercises challenging and enlightening.

Atlanta, June 1995

Samuel H. Cox

C.1 Mathematics of Compound Interest: Exercises

A *bond* is a contract obligating one party, the borrower or bond issuer, to pay to the other party, the lender or bondholder, a series of future payments defined by the face value, F, and the coupon rate, c. At the end of each future period the borrower pays cF to the lender. The bond matures after N periods with a final coupon payment and a simultaneous payment of the redemption value C. Usually C is equal to F. Investors (lenders) require a yield to maturity of $i \geq 0$ effective per period. The price, P, is the present value of future cash flows paid to the bondholder. The five values are related by the following equation.

$$P = cF\frac{1-v^N}{i} + Cv^N$$

where $v = 1/(1+i)$.

C.1.1 Theory Exercises

1. Show that

$$i^{(m)} - d^{(m)} = \frac{i^{(m)}d^{(m)}}{m}.$$

2. Show that $d < d^{(2)} < d^{(3)} < \ldots < \delta < \ldots < i^{(3)} < i^{(2)} < i$ and

$$i^{(m)} - d^{(n)} \leq \frac{i^2}{min(m,n)}.$$

3. A company must retire a bond issue with five annual payments of 15,000. The first payment is due on December 31, 1999. In order to accumulate the funds, the company begins making annual payments of X on January 1, 1990 into an account paying effective annual interest of 6%. The last payment is to be made on January 1, 1999. Calculate X.

4. At a nominal annual rate of interest j, convertible semiannually, the present value of a series of payments of 1 at the end of every 2 years, which continue forever, is 5.89. Calculate j.

5. A perpetuity consists of yearly increasing payments of $(1+k)$, $(1+k)^2$, $(1+k)^3$, etc., commencing at the end of the first year. At an annual effective interest rate of 4%, the present value one year before the first payment is 51. Determine k.

6. Six months before the first coupon is due a ten-year semi-annual coupon bond sells for 94 per 100 of face value. The rate of payment of coupons is 10% per year. The yield to maturity for a zero-coupon ten-year bond is 12%. Calculate the yield to maturity of the coupon payments.

7. A loan of 1000 at a nominal rate of 12% convertible monthly is to be repaid by six monthly payments with the first payment due at the end of one month. The first three payments are x each, and the final three payments are $3x$ each. Calculate x.

8. A loan of 4000 is being repaid by a 30-year increasing annuity immediate. The initial payment is k, each subsequent payment is k larger than the preceding payment. The annual effective interest rate is 4%. Calculate the principal outstanding immediately after the ninth payment.

9. John pays 98.51 for a bond that is due to mature for 100 in one year. It has coupons at 4% convertible semiannually. Calculate the annual yield rate convertible semiannually.

10. The death benefit on a life insurance policy can be paid in four ways. All have the same present value:

(i) A perpetuity of 120 at the end of each month, first payment one month after the moment of death;

(ii) Payments of 365.47 at the end of each month for n years, first payment one month after the moment of death;

(iii) A payment of 17,866.32 at the end of n years after the moment of death; and

(iv) A payment of X at the moment of death.

Calculate X.

C.1.2 Spreadsheet Exercises

1. A serial bond with a face amount of 1000 is priced at 1145. The owner of the bond receives annual coupons of 12% of the outstanding principal. The principal is repaid by the following schedule:

(i) 100 at the end of each years 10 through 14, and

(ii) 500 at the end of year 15.

(a) Calculate the investment yield using the built-in Goal Seek procedure.

(b) Use the graphic capability of the spreadsheet to illustrate the investment yield graphically. To do this, construct a Data Table showing various investment yield values and the corresponding bond prices. From the graph, determine which yield corresponds to a price of 1,145.

2. A deposit of 100,000 is made into a newly established fund. The fund pays nominal interest of 12% convertible quarterly. At the end of each six months a withdrawal is made from the fund. The first withdrawal is X, the second is $2X$, the third is $3X$, and so on. The last is the sixth withdrawal which exactly exhausts the fund. Calculate X.

3. A loan is to be repaid by annual installments 100, 200, 300, 300, 200 and 100. In the fifth installment, the amount of principal repayment is equal to six times the amount of interest. Calculate the annual effective interest rate.

4. A company borrows 10,000. Interest of 350 is paid semiannually, but no principal is paid until the entire loan is repaid at the end of 5 years. In order to accumulate the principal of the loan at the end of five years, the company makes equal semiannual deposits, the first due in six months, into a fund that credits interest at a nominal annual rate of 6% compounded semiannually. Calculate the internal rate of return effective per year for the company on the entire transaction.

5. Deposits of 100 are made into a fund at the beginning of each year for 10 years. Beginning ten years after the last deposit, X is withdrawn each year from the fund in perpetuity.
(a) $i = 10\%$. Calculate X.
(b) Draw the graph of X as a function of i for i varying from 1% to 21% in increments of 2%.

6. A bank credits savings accounts with 8% annual effective interest on the first 100,000 of beginning year account value and 9% on the excess over 100,000. An initial deposit of 300,000 is made. One year later level annual withdrawals of X begin and run until the account is exactly exhausted with the tenth withdrawal. Calculate X.

7. In order to settle a wrongful injury claim, an annuity is purchased from an insurance company. According to the annuity contract, the insurer is obliged to make the following future payments on July 1 of each year indicated:

Year	Amount
1995	50,000
1996	60,000
1997	75,000
1998	100,000
1999	125,000
2000	200,000

The insurer is considering hedging its future liability under the annuity contract by purchasing government bonds. The financial press publishes the market prices for the following government bonds available for sale on July 1, 1994. Each bond has a face amount of 10,000, each pays annual coupons on July 1, and the first coupon payment is due in one year.

Maturity	Coupon Rate	Price
1995	4.250%	9,870
1996	7.875%	10,180
1997	5.500%	9,600
1998	5.250%	9,210
1999	6.875%	9,740
2000	7.875%	10,120

Determine how many bonds of each maturity the insurer should buy on July 1, 1994 so that the aggregate cash flow from the bonds will exactly match the insurer's obligation under the terms of the claim settlement. Assume that fractions of bonds may be purchased.

8. A loan of 100,000 is repayable over 20 years by semiannual payments of 2500, plus 5% interest (per year convertible twice per year) on the outstanding balance. Immediately after the tenth payment the lender sells the loan for 65,000. Calculate the corresponding market yield to maturity of the loan (per year convertible twice per year).

9. A bond with face value 1000 has 9% annual coupons. The borrower may call the bond at the end of years 10 though 15 by paying the face amount plus a call premium, according to the schedule:

Year	10	11	12	13	14	15
Premium	100	80	60	40	20	0

For example, if the borrower elects to repay the debt at the end of year 11 (11 years from now), a payment of $1000 + 80 = 1080$ plus the coupon then due of 90 would be paid to the lender. The debt is paid; no further payments would be made. Calculate the price now, one year before the next coupon payment, to be certain of a yield of at least 8% to the call date.

10. Equal deposits of 200 are made to a bank account at the beginning of each quarter of a year for five years. The bank pays interest from the date of deposit at an annual effective rate of i. One quarter year after the last deposit the account balance is 5000. Calculate i.

C.2 The Future Lifetime of a Life Aged x: Exercises

These exercises sometimes use the commutation function notation introduced in Appendix A and the following notation with regard to mortality tables. The Illustrative Life Table is given in Appendix E. It is required for some exercises.

A mortality table covering the range of ages x $(0 \leq x < \omega)$ is denoted by l_x, which represents the number l_0 of the new-born lives who survive to age x. The probability of surviving to age x is $s(x) = l_x/l_0$. The rule for calculating conditional probabilities establishes this relationship to ${}_tp_x$:

$$ {}_tp_x = \Pr(T(0) > x + t | T(0) > x) = \frac{s(x+t)}{s(x)} = \frac{l_{x+t}}{l_x}. $$

In the case that the conditioning involves more information than mere survival, the notation ${}_tp_{[x]}$ is used. Thus if a person age x applies for insurance and is found to be in good health, the mortality function is denoted ${}_tp_{[x]}$ rather than ${}_tp_x$. The notation $[x]$ tells us that some information in addition to $T(0) > x$ was used in preparing the survival distribution. This gives rise to the select and ultimate mortality table discussed in the text.

Here are some additional mortality functions:

$$
\begin{aligned}
m_x &= \text{central death rate} = \frac{d_x}{L_x} \\
L_x &= \text{average number of survivors to } (x, x+1) \\
&= \int_x^{x+1} l_y dy = \int_0^1 l_{x+t} dt \\
d_x &= \text{number of deaths in } (x, x+1) = l_x - l_{x+1}.
\end{aligned}
$$

Since ${}_tp_x \mu_{x+t} = -\frac{d}{dt} {}_tp_x$, then in terms of l_x we have $l_{x+t}\mu_{x+t} = -\frac{d}{dt} l_{x+t}$ or, letting $y = x + t$, we have $l_y \mu_y = -\frac{d}{dy} l_y$ for all y. The following are useful for calculating $\mathrm{Var}(T)$ and $\mathrm{Var}(K)$:

$$
\begin{aligned}
\mathrm{E}[T^2] &= \int_0^\infty t^2_t p_x \mu_{x+t} dt \\
&= \int_0^\infty 2t_t p_x dt \\
\mathrm{E}[K^2] &= \sum_{k=1}^\infty k^2_{k-1} p_x q_{x+k-1} \\
&= \sum_{k=0}^\infty (2k+1)_{k+1} p_x.
\end{aligned}
$$

C.2.1 Theory Exercises

1. Given:

$$_tp_x = \frac{100 - x - t}{100 - x}$$

for $0 \leq x < 100$ and $0 \leq t \leq 100 - x$. Calculate μ_{45}.

2. Given:

$$_tp_x = 1 - \left(\frac{t}{100}\right)^{1.5}$$

for $x = 60$ and $0 < t < 100$. Calculate $\mathrm{E}[T(x)]$.

3. Given: $\mu_{x+t} = \dfrac{1}{85 - t} + \dfrac{3}{105 - t}$ for $0 \leq t < 85$. Calculate $_{20}p_x$.

4. Given: $_tp_x = \left(\dfrac{1 + x}{1 + x + t}\right)^3$ for $t \geq 0$. Calculate the complete life expectancy of a person age $x = 41$.

5. Given: $q_x = 0.200$. Calculate $m_x = \dfrac{q_x}{\int_0^1 {}_tp_x dt}$ using *assumption c*, the Balducci assumption.

6. Given:

(i) μ_{x+t} is constant for $0 \leq t < 1$ and

(ii) $q_x = 0.16$.

Calculate the value of t for which $_tp_x = 0.95$.

7. Given:

(i) The curve of death $l_x\mu_x$ is constant for $0 \leq x < \omega$.

(ii) $\omega = 100$.

Calculate the variance of the remaining lifetime random variable $T(x)$ at $x = 88$.

8. Given:

(i) When the force of mortality is μ_{x+t}, $0 < t < 1$, then $q_x = 0.05$.

(ii) When the force of mortality is $\mu_{x+t} - c$, $0 < t < 1$, then $q_x = 0.07$.

Calculate c.

9. Prove:

(i) $_tp_x = \exp\left(-\int_x^{x+t} \mu_s ds\right)$ and

(ii) $\frac{\partial}{\partial x} {}_tp_x = (\mu_x - \mu_{x+t})_tp_x$.

10. You are given the following excerpt from a select and ultimate mortality table with a two-year select period.

x	$100q_{[x]}$	$100q_{[x]+1}$	$100q_{x+2}$
30	0.438	0.574	0.699
31	0.453	0.599	0.734
32	0.472	0.634	0.790
33	0.510	0.680	0.856
34	0.551	0.737	0.937

Calculate $100({}_{1|}q_{[30]+1})$.

11. Given:

$$l_x = (121 - x)^{1/2}$$

for $0 \leq x \leq 121$. Calculate the probability that a life age 21 will die after attaining age 40, but before attaining age 57.

12. Given the following table of values of e_x:

Age x	e_x
75	10.5
76	10.0
77	9.5

Calculate the probability that a life age 75 will survive to age 77. Hint: Use the recursion relation $e_x = p_x(1 + e_{x+1})$.

13. Mortality follows de Moivre's law and $\mathrm{E}[T(16)] = 36$. Calculate $\mathrm{Var}(T(16))$.

14. Given:

$${}_tp_{30} = \frac{7800 - 70t - t^2}{7800}$$

for $0 \leq t \leq 60$. Calculate the exact value of $q_{50} - \mu_{50}$.

15. Given:

$${}_tp_x = \left(\frac{100 - x - t}{100 - x}\right)^2$$

for $0 \leq t \leq 100 - x$. Calculate $\mathrm{Var}(T(x))$.

16. Given: $q_x = 0.420$ and *assumption b* applies to the year of age x to $x + 1$. Calculate m_x, the central death rate exactly. (See exercise 5.)

17. Consider two independent lives, which are identical except that one is a smoker and the other is a non-smoker. Given:

(i) μ_x is the force of mortality for non-smokers for $0 \leq x < \omega$.

(ii) $c\mu_x$ is the force of mortality for smokers for $0 \leq x < \omega$, where c is a constant, $c > 1$.

Calculate the probability that the remaining lifetime of the smoker exceeds that of the non-smoker.

18. Derive an expression for the derivative of q_x with respect to x in terms of the force of mortality.

19. Given: $\mu_x = kx$ for all $x > 0$ where k is a positive constant and ${}_{10}p_{35} = 0.81$. Calculate ${}_{20}p_{40}$.

20. Given:

(i) $l_x = 1000(\omega^3 - x^3)$ for $0 \leq x \leq \omega$ and

(ii) $\mathrm{E}[T(0)] = 3\omega/4$.

Calculate $\mathrm{Var}(T(0))$.

C.2.2 Spreadsheet Exercises

1. Put the Illustrative Life Table l_x values into a spreadsheet. Calculate d_x and $1000q_x$ for $x = 0, 1, \ldots, 99$.

2. Calculate $e_x, x = 0, 1, 2, \ldots, 99$ for the Illustrative Life Table. Hint: Use formula (2.4.3) to get $e_{99} = p_{99} = 0$ for this table. The recursive formula $e_x = p_x(1 + e_{x+1})$ follows from (2.4.3). Use it to calculate from the higher age to the lower.

3. A sub-standard mortality table is obtained from a standard table by adding a constant c to the force of mortality. This results in sub-standard mortality rates q_x^s which are related to the standard rates q_x by $q_x^s = 1 - e^{-c}(1 - q_x)$. Use the Illustrative Life Table for the standard mortality. A physician examines a life age $x = 40$ and determines that the expectation of remaining lifetime is 10 years. Determine the constant c, and the resulting substandard table. Prepare a table and graph of the mortality ratio (sub-standard q_x^s to standard q_x) by year of age, beginning at age 40.

4. Draw the graph of $\mu_x = Bc^x, x = 0, 1, 2, \ldots, 110$ for $B = 0.0001$ and each value of $c = 1.01, 1.05, 1.10, 1.20$. Calculate the corresponding values of l_x and draw the graphs. Use $l_0 = 100,000$ and round to an integer.

5. Let $q_x = 0.10$. Draw the graphs of μ_{x+u} for u running from 0 to 1 increments of 0.05 for each of the interpolation formulas given by *assumptions a, b,* and *c.*

6. Substitute ${}_uq_x$ for μ_{x+u} in Exercise 5 and rework.

7. Use the method of least squares (and the spreadsheet Solver feature) to fit a Gompertz distribution to the Illustrative Life Table values of ${}_tp_x$ for $x = 50$ and $t = 1, 2, \ldots, 50$. Draw the graph of the table values and the Gompertz values on the same axes.

8. A sub-standard mortality table is obtained from a standard table by multiplying the standard q_x by a constant $k \geq 1$, subject to an upper bound of 1.

Thus the substandard q_x^s mortality rates are related to the standard rates q_x by $q_x^s = \min(kq_x, 1)$.
(a) For values of k ranging from 1 to 10 in increments of 0.5, calculate points on the graph of ${}_tp_x^s$ for age $x = 45$ and t running from 0 to the end of the table in increments of one year. Draw the graphs in a single chart.
(b) Calculate the sub-standard life expectancy at age $x = 45$ for each value of k in (a).

C.3 Life Insurance

C.3.1 Theory Exercises

1. Given:

(i) The survival function is $s(x) = 1 - x/100$ for $0 \le x \le 100$.

(ii) The force of interest is $\delta = 0.10$.

Calculate $50{,}000\bar{A}_{30}$.

2. Show that

$$\frac{(IA)_x - A^{1}_{x:\overline{1}|}}{(IA)_{x+1} + A_{x+1}}$$

simplifies to vp_x.

3. Z_1 is the present value random variable for an n-year continuous endowment insurance of 1 issued to (x). Z_2 is the present value random variable for an n-year continuous term insurance of 1 issued to x. Given:

(i) $\mathrm{Var}(Z_2) = 0.01$

(ii) $v^n = 0.30$

(iii) ${}_np_x = 0.8$

(iv) $\mathrm{E}[Z_2] = 0.04$.

Calculate $\mathrm{Var}(Z_1)$.

4. Use the Illustrative Life Table and $i = 5\%$ to calculate $A_{45:\overline{20}|}$.

5. Given:

(i) $A_{x:\overline{n}|} = u$

(ii) $A^{1}_{x:\overline{n}|} = y$

(iii) $A_{x+n} = z$.

Determine the value of A_x in terms of u, y, and z.

6. A continuous whole life insurance is issued to (50). Given:

(i) Mortality follows de Moivre's law with $\omega = 100$.

(ii) Simple interest with $i = 0.01$.

(iii) $b_t = 1000 - 0.1t^2$.

Calculate the expected value of the present value random variable for this insurance.

7. Assume that the forces of mortality and interest are each constant and denoted by μ and δ, respectively. Determine $\mathrm{Var}(v^T)$ in terms of μ and δ.

8. For a select and ultimate mortality table with a one-year select period, $q_{[x]} = 0.5q_x$ for all $x \geq 0$. Show that $A_x - A_{[x]} = 0.5vq_x(1 - A_{x+1})$.

9. A single premium whole life insurance issued to (x) provides 10,000 of insurance during the first 20 years and 20,000 of insurance thereafter, plus a return without interest of the net single premium if the insured dies during the first 20 years. The net single premium is paid at the beginning of the first year. The death benefit is paid at the end of the year of death. Express the net single premium using commutation functions.

10. A ten-year term insurance policy issued to (x) provides the following death benefits payable at the end of the year of death.

Year of Death	Death Benefit
1	10
2	10
3	9
4	9
5	9
6	8
7	8
8	8
9	8
10	7

Express the net single premium for this policy using commutation functions.

11. Given:

(i) The survival function is $s(x) = 1 - x/100$ for $0 \leq x \leq 100$.

(ii) The force of interest is $\delta = 0.10$.

(iii) The death benefit is paid at the moment of death.

Calculate the net single premium for a 10-year endowment insurance of 50,000 for a person age $x = 50$.

12. Given:

(i) $s(x) = e^{-0.02x}$ for $x \geq 0$

(ii) $\delta = 0.04$.

Calculate the median of the present value random variable $Z = v^T$ for a whole life policy issued to (y).

13. A 2-year term insurance policy issued to (x) pays a death benefit of 1 at the end of the year of death. Given:

(i) $q_x = 0.50$

(ii) $i = 0$

(iii) $\mathrm{Var}(Z) = 0.1771$

where Z is the present value of future benefits. Calculate q_{x+1}.

14. A 3-year term life insurance to (x) is defined by the following table:

Year t	Death Benefit	q_{x+t}
0	3	0.20
1	2	0.25
2	1	0.50

Given: $v = 0.9$, the death benefits are payable at the end of the year of death and the expected present value of the death benefit is Π. Calculate the probability that the present value of the benefit payment that is actually made will exceed Π.

15. Given:

(i) $A_{76} = 0.800$

(ii) $D_{76} = 400$

(iii) $D_{77} = 360$

(iv)] $i = 0.03$.

Calculate A_{77} by use of the recursion formula (3.6.1).

16. A whole life insurance of 50 is issued to (x). The benefit is payable at the moment of death. The probability density function of the future lifetime, T, is

$$g(t) = \begin{cases} t/5000 & \text{for } 0 \le t \le 100 \\ 0 & \text{elsewhere.} \end{cases}$$

The force of interest is constant: $\delta = 0.10$. Calculate the net single premium.

17. For a continuous whole life insurance, $\mathrm{E}[v^{2T}] = 0.25$. Assume the forces of mortality and interest are each constant. Calculate $\mathrm{E}[v^T]$.

18. There are 100 club members age x who each contribute an amount w to a fund. The fund earns interest at $i = 10\%$ per year. The fund is obligated to pay 1000 at the moment of death of each member. The probability is 0.95 that the fund will meet its benefit obligations. Given the following values calculated at $i = 10\%$: $\bar{A}_x = 0.06$ and ${}^2\bar{A}_x = 0.01$. Calculate w. Assume that the future lifetimes are independent and that a normal distribution may be used.

19. An insurance is issued to (x) that

(i) pays 10,000 at the end of 20 years if x is alive and

(ii) returns the net single premium Π at the end of the year of death if (x) dies during the first 20 years.

Express Π using commutation functions.

20. A whole life insurance policy issued to (x) provides the following death benefits payable at the year of the year of death.

Year of Death	Death Benefit
1	10
2	10
3	9
4	9
5	9
6	8
7	8
8	8
9	8
10	7
each other year	7

Calculate the net single premium for this policy.

C.3.2 Spreadsheet Exercises

1. Calculate the A_x column of the Illustrative Life Table at $i = 5\%$. Use the recursive method suggested by formula (3.6.1). Construct a graph showing the values of A_x for $i = 0, 2.5\%, 5\%, 7.5\%, 10\%$ and $x = 0, 1, 2, \ldots, 99$.

2. The formula for increasing life insurances, in analogy to (3.6.1), is $(IA)_x = vq_x + vp_x(A_{x+1} + IA_{x+1})$. Use this (and (3.6.1)) to calculate a table of values of $(IA)_x$ for the Illustrative Life Table and $i = 5\%$.

3. Calculate the net single premium of an increasing 20 year term insurance for issue age $x = 25$, assuming that the benefit is 1 the first year, $1 + g$ the second year, $(1 + g)^2$ the third year and so on. Use the Illustrative Life Table at $i = 5\%$ and $g = 6\%$. Try to generalize to a table of premiums for all issue ages $x = 0, 1, 2, \ldots, 99$.

4. Calculate the net single premium for a decreasing whole life insurance with an initial benefit of $100 - x$ at age x, decreasing by 1 per year. The benefit is paid at the moment of death. Use the Illustrative Life Table at $i = 5\%$ and $x = 50$. Generalize so that x and i are input cell values, and your spreadsheet calculates the premium for reasonable interest rates and ages.

5. For a life age $x = 35$, calculate the variance of the present value random variable for a whole life insurance of 1000. The interest rate i varies from 0 to 25% by increments of 0.5%. Mortality follows the Illustrative Life Table. Draw the graph.

C.4 Life Annuities

C.4.1 Theory Exercises

1. Using *assumption a* and the Illustrative Life Table with interest at the effective annual rate of 5%, calculate $\ddot{a}^{(2)}_{40:\overline{30}|}$.

2. Demonstrate that

$$\frac{(I\ddot{a})_x - \ddot{a}_{x:\overline{1}|}}{(I\ddot{a})_{x+1} + \ddot{a}_{x+1}}$$

simplifies to $a_{x:\overline{1}|}$.

3. $\left(\bar{I}_{\overline{n}|}\bar{a}\right)_x$ is equal to E$[Y]$ where

$$Y = \begin{cases} \left(\bar{I}\bar{a}\right)_{\overline{T}|} & \text{if } 0 \leq T < n \text{ and} \\ \left(\bar{I}\bar{a}\right)_{\overline{n}|} + n\left({}_{n|}\bar{a}_{\overline{T-n}|}\right) & \text{if } T \geq n \end{cases}$$

The force of mortality is constant, $\mu_x = 0.04$ for all x, and the force of interest is constant, $\delta = 0.06$. Calculate $\frac{\partial}{\partial n}\left(\bar{I}_{\overline{n}|}\bar{a}\right)_x$.

4. Given the following information for a 3-year temporary life annuity due, contingent on the life of (x):

t	Payment	p_{x+t}
0	2	0.80
1	3	0.75
2	4	0.50

and $v = 0.9$. Calculate the variance of the present value of the indicated payments.

5. Given:

(i) $l_x = 100,000(100 - x)$, $0 \leq x \leq 100$ and

(ii) $i = 0$.

Calculate $(I\bar{a})_{95}$ exactly.

6. Calculate ${}_{10|}\ddot{a}^{(12)}_{25:\overline{10}|}$ using the Illustrative Life Table, *assumption a* and $i = 5\%$. (The symbol denotes an annuity issued on a life age 25, the first payment deferred 10 years, paid in level monthly payments at a rate of 1 per year during the lifetime of the annuitant but not more than 10 years.)

7. Given:

(i)

x	69	70	71	72	$\cdots$	79	80	81	82
S_x	77,938	67,117	57,520	49,043	$\cdots$	13,483	10,875	8,691	6,875

(ii) $\alpha(12) = 1.00028$ and $\beta(12) = 0.46812$

(iii) ***assumption a*** applies: deaths are distributed uniformly over each year of age.

Calculate $(I\ddot{a})^{(12)}_{75:\overline{10}|}$.

8. Show:

$$ {}_np_x d\ddot{a}_{\overline{n}|} + \sum_{k=0}^{n-1}(1 - v^{k+1})_k p_x q_{x+k} = 1 - A_{x:\overline{n}|} $$

9. Y is the present value random variable of a whole life annuity due of 1 per year issued to (x). Given: $\ddot{a}_x = 10$, evaluated with $i = 1/24 = e^\delta - 1$, and $\ddot{a}_x = 6$, evaluated with $i = e^{2\delta} - 1$. Calculate the variance of Y.

10. $\ddot{a}_{x:\overline{n}|}$ is equal to $\mathrm{E}[Y]$ where

$$ Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & \text{if } 0 \le K < n \text{ and} \\ \ddot{a}_{\overline{n}|} & \text{if } K \ge n. \end{cases} $$

Show that

$$ \mathrm{Var}[Y] = \frac{\mathrm{M}(-2\delta) - \mathrm{M}(-\delta)^2}{d^2} $$

where $\mathrm{M}(u) = \mathrm{E}(e^{u \min(K+1,n)})$ is the moment generating function of the random variable $\min(K+1, n)$.

11. Given $i = 0.03$ and commutation function values:

x	27	28	29	30	31
S_x	1,868	1,767	1,670	1,577	1,488

Calculate the commutation M_{28}.

12. Given the following functions valued at $i = 0.03$:

x	$\ddot{a}_x$
72	8.06
73	7.73
74	7.43
75	7.15

Calculate p_{73}.

13. Given the following information for a 3-year life annuity due, contingent on the life of (x):

t	Payment	p_{x+t}
0	2	0.80
1	3	0.75
2	4	0.50

Assume that $i = 0.10$. Calculate the probability that the present value of the indicated payments exceeds 4.

14. Given $l_x = 100,000(100 - x)$, $0 \le x \le 100$ and $i = 0$. Calculate the present value of a whole life annuity issued to (80). The annuity is paid continuously at an annual rate of 1 per year the first year and 2 per year thereafter.

15. As in exercise 14, $l_x = 100,000(100 - x)$, $0 \le x \le 100$ and $i = 0$. Calculate the present value of a temporary 5-year life annuity issued to (80). The annuity is paid continuously at an annual rate of 1 per year the first year and 2 per year for four years thereafter.

16. Given $\delta = 0$, $\int_0^\infty t\,{}_tp_x dt = g$, and $\mathrm{Var}(\bar{a}_{\overline{T|}}) = h$, where T is the future lifetime random variable for (x). Express $\mathrm{E}[T]$ in terms of g and h.

17. Given:

x	69	70	71	72	$\cdots$	79	80	81	82
S_x	77,938	67,117	57,520	49,043	$\cdots$	13,483	10,875	8,691	6,875

Calculate $(D\ddot{a})_{70:\overline{10|}}$ which denotes the present value of a decreasing annuity. The first payment of 10 is at age 70, the second of 9 is scheduled for age 71, and so on. The last payment of 1 is scheduled for age 79.

18. Show that

$$\frac{a_{\overline{1|}}S_x - \ddot{a}_{\overline{2|}}S_{x+1} + \ddot{a}_{\overline{1|}}S_{x+2}}{D_x}$$

simplifies to A_x.

19. For a force of interest of $\delta > 0$, the value of $\mathrm{E}\left(\bar{a}_{\overline{T|}}\right)$ is equal to 10. With the same mortality, but a force of interest of 2δ, the value of $\mathrm{E}\left(\bar{a}_{\overline{T|}}\right)$ is 7.375. Also $\mathrm{Var}(\bar{a}_{\overline{T|}}) = 50$. Calculate $\bar{A}_x$.

20. Calculate $\bar{a}_{x+u}$ using the Illustrative Life Table at 5% for age $x+u = 35.75$. *Assumption a* applies.

C.4.2 Spreadsheet Exercises

1. Calculate $\ddot{a}_x$ based on the Illustrative Life Table at $i = 5\%$. Use the recursion formula (4.6.1). Construct a graph showing the values of $\ddot{a}_x$ for $i = 0, 2.5\%, 5\%, 7.5\%, 10\%$ and $x = 0, 1, 2, \ldots, 99$.

2. Consider again the structured settlement annuity mentioned in exercise 7 of Section C.1. In addition to the financial data and the scheduled payments, include now the information that the payments are contingent upon the survival of a life subject to the mortality described in exercise 3 of Section C.2. Calculate the sum of market values of bonds required to hedge the expected value of the annuity payments.

3. A life age $x = 50$ is subject to a force of mortality ν_{50+t} obtained from the force of mortality standard as follows:

$$\nu_{50+t} = \begin{cases} \mu_{50+t} + c & \text{for } 0 \leq t \leq 15 \\ \mu_{50+t} & \text{otherwise} \end{cases}$$

where μ_{50+t} denotes the force of mortality underlying the Illustrative Life Table. The force of interest is constant $\delta = 4\%$. Calculate the variance of the present value of an annuity immediate of one per annum issued to (50) for values of $c = -0.01, -0.005, 0, 0.005,$ and 0.01. Draw the graph.

4. Create a spreadsheet which calculates $\ddot{a}^{(m)}_{x+u}$ and A_{x+u} for a given age, $x+u$, with x an integer and $0 \leq u \leq 1$, and a given interest rate i. Assume that mortality follows the Illustrative Life Table. Use formulas (4.8.5) and (4.3.5) (or (4.8.6) and (4.3.5) if you like.) for the annuity and analogous ones for the life insurance.

5. Use your spreadsheet's built-in random number feature to simulate 200 values of $Y = 1 + v + \cdots + v^K = \ddot{a}_{\overline{K+1|}}$ where $K = K(40)$. Use $i = 5\%$ and assume mortality follows the Illustrative Life Table. Compare the sample mean and variance to the values given by formulas (4.2.7) and (4.2.9).

C.5 Net Premiums

C.5.1 Notes

The exercises sometimes use the notation based on the system of International Actuarial Notation. Appendix 4 of *Actuarial Mathematics* by Bowers *et al.* describes the system. Here are the premium symbols and definitions used in these exercises.

$\overline{P}\left(\overline{A}_x\right)$ denotes the annual rate of payment of net premium, paid continuously, for a whole life insurance of 1 issued on the life of (x), benefit paid at the moment of death.

$\overline{P}\left(\overline{A}_{x:\overline{n}|}\right)$ denotes the annual rate of payment of net premium for an endowment insurance of 1 issued on the life of (x). The death benefit is paid at the moment of death.

A life insurance policy is fully continuous if the death benefit is paid at the moment of death, and the premiums are paid continuously over the premium payment period.

Policies with limited premium payment periods can be described symbolically with a pre-subscript. For example, ${}_nP\left(\overline{A}_x\right)$ denotes the annual rate of payment of premium, paid once per year, for a whole life insurance of 1 issued on the life of (x), benefit paid at the moment of death. For a policy with the death benefit paid at the end of the year of death the symbol is simplified to ${}_nP_x$.

C.5.2 Theory Exercises

1. Given: ${}_{20}P_{25} = 0.046$, $P_{25:\overline{20}|} = 0.064$, and $A_{45} = 0.640$. Calculate $P^{1}_{25:\overline{20}|}$.

2. A level premium whole life insurance of 1, payable at the end of the year of death, is issued to (x). A premium of G is due at the beginning of each year, provided (x) survives. Given:

(i) $L =$ the insurer's loss when $G = P_x$

(ii) $L^* =$ the insurer's loss when G is chosen such that $\mathrm{E}[L^*] = -0.20$

(iii) $\mathrm{Var}[L] = 0.30$

Calculate $\mathrm{Var}[L^*]$.

3. Use the Illustrative Life Table and $i = 5\%$ to calculate the level net annual premium payable for ten years for a whole life insurance issued to a person age 25. The death benefit is 50,000 initially, and increases by 5,000 at ages 30, 35, 40, 45 and 50 to an ultimate value of 75,000. Premiums are paid at the beginning of the year and the death benefits are paid at the end of the year.

4. Given the following values calculated at $d = 0.08$ for two whole life policies issued to (x):

	Death Benefit	Premium	Variance of Loss
Policy A	4	0.18	3.25
Policy B	6	0.22	

Premiums are paid at the beginning of the year and the death benefit sare paid at the end of the year. Calculate the variance of the loss for policy B.

5. A whole life insurance issued to (x) provides 10,000 of insurance. Annual premiums are paid at the beginning of the year for 20 years. Death claims are paid at the end of the year of death. A premium refund feature is in effect during the premium payment period which provides that one half of the last premium paid to the company is refunded as an additional death benefit. Show that the net annual premium is equal to

$$\frac{10,000A_x}{(1+d/2)\ddot{a}_{x:\overline{20}|} - \left(1 - v^{20}{}_{20}p_x\right)/2}.$$

6. Obtain an expression for the annual premium ${}_nP_x$ in terms of net single premiums and the rate of discount d. (${}_nP_x$ denotes the net annual premium payable for n years for a whole life insurance issued to x.)

7. A whole life insurance issued to (x) provides a death benefit in year j of $b_j = 1,000(1.06)^j$ payable at the end of the year. Level annual premiums are payable for life. Given: $1,000P_x = 10$ and $i = 0.06$ per year. Calculate the net annual premium.

8. Given:

(i) $A_x = 0.25$

(ii) $A_{x=20} = 0.40$

(iii) $A_{x:\overline{20}|} = 0.55$

(iv) i = 0.03

(v) ***assumption a*** applies.

Calculate $1000P\left(\bar{A}_{x:\overline{20}|}\right)$.

9. A fully continuous whole life insurance of 1 is issued to (x). Given:

(i) The insurer's loss random variable is $L = v^T - \bar{P}\left(\bar{A}_x\right)\bar{a}_{\overline{T}|}$.

(ii) The force of interest δ is constant.

(iii) The force of mortality is constant: $\mu_{x+t} = \mu, t \geq 0$.

Show that $\mathrm{Var}(L) = \mu/(2\delta + \mu)$.

10. A fully-continuous level premium 10-year term insurance issued to (x) pays a benefit at death of 1 plus the return of all premiums paid accumulated with interest. The interest rate used in calculating the death benefit is the same as

that used to determine the present value of the insurer's loss. Let G denote the rate of annual premium paid continuously.
(a) Write an expression for the insurer's loss random variable L.
(b) Derive an expression for $\text{Var}[L]$.
(c) Show that, if G is determined by the equivalence principle, then

$$\text{Var}[L] = {}^2\bar{A}^1_{x:\overline{10}|} + \frac{\left(\bar{A}^1_{x:\overline{10}|}\right)^2}{{}_{10}p_x}.$$

The pre-superscript indicates that the symbol is evaluated at a force of interest of 2δ, where δ is the force of interest underlying the usual symbols.

11. Given:

(i) $i = 0.10$

(ii) $a_{30:\overline{9}|} = 5.6$

(iii) $v^{10}{}_{10}p_{30} = 0.35$

Calculate $1000\, P^1_{30:\overline{10}|}$

12. Given:

(i) $i = 0.05$

(ii) $10,000 A_x = 2,000.$

Apply *assumption a* and calculate $10,000\bar{P}\left(\bar{A}_x\right) - 10,000P\left(\bar{A}_x\right)$.

13. Show that

$$1 - \frac{\left(P_{30:\overline{15}|} - {}_{15}P_{30}\right)\ddot{a}_{30:\overline{15}|}}{v^{15}{}_{15}p_{30}}$$

simplifies to A_{45}.

14. Given:

(i) $\bar{A}_x = 0.3$

(ii) $\delta = 0.07.$

A whole life policy issued to (x) has a death benefit of 1,000 paid at the moment of death. Premiums are paid twice per year. Calculate the semi-annual net premium using *assumption a*.

15. Given the following information about a fully continuous whole life insurance policy with death benefit 1 issued to (x):

(i) The net single premium is $\bar{A}_x = 0.4$.

(ii) $\delta = 0.06$

(iii) $\text{Var}[L] = 0.25$ where L denotes the insurer's loss associated with the net annual premium $\bar{P}\left(\bar{A}_x\right)$.

Under the same conditions, except that the insurer requires a premium rate of $G = 0.05$ per year paid continuously, the insurer's loss random variable is L^*. Calculate $\mathrm{Var}[L^*]$.

16. A fully discrete 20-year endowment insurance of 1 is issued to (40). The insurance also provides for the refund of all net premiums paid accumulated at the interest rate i if death occurs within 10 years of issue. Present values are calculated at the same interest rate i. Using the equivalence principle, the net annual premium payable for 20 years for this policy can be written in the form:

$$\frac{A_{40:\overline{20}|}}{k}$$

Determine k.

17. L is the loss random variable for a fully discrete, 2-year term insurance of 1 issued to (x). The net level annual premium is calculated using the equivalence principle. Given:

(i) $q_x = 0.1$,

(ii) $q_{x+1} = 0.2$ and

(iii) $v = 0.9$.

Calculate $\mathrm{Var}(L)$.

18. Given:

(i) $\bar{A}^{1}_{x:\overline{n}|} = 0.4275$

(ii) $\delta = 0.055$, and

(iii) $\mu_{x+t} = 0.045, t \geq 0$

Calculate $1,000\bar{P}\left(\bar{A}_{x:\overline{n}|}\right)$.

19. A 4-year automobile loan issued to (25) is to be repaid with equal annual payments at the end of each year. A four-year term insurance has a death benefit which will pay off the loan at the end of the year of death, including the payment then due. Given:

(i) $i = 0.06$ for both the actuarial calculations and the loan,

(ii) $\ddot{a}_{25:\overline{4}|} = 3.667$, and

(iii) ${}_4q_{25} = 0.005$.

(a) Express the insurer's loss random variable in terms of K, the curtate future lifetime of (25), for a loan of 1,000 assuming that the insurance is purchased with a single premium of G.
(b) Calculate G, the net single premium rate per 1,000 of loan value for this insurance.
(c) The automobile loan is 10,000. The buyer borrows an additional amount to pay for the term insurance. Calculate the total annual payment for the loan.

20. A level premium whole life insurance issued to (x) pays a benefit of 1 at the end of the year of death. Given:

(i) $A_x = 0.19$

(ii) ${}^2A_x = 0.064$, and

(iii) $d = 0.057$

Let G denote the rate of annual premium to be paid at the beginning of each year while (x) is alive.
(a) Write an expression for the insurer's loss random variable L.
(b) Calculate E[L] and Var[L], assuming $G = 0.019$.
(c) Assume that the insurer issues n independent policies, each having $G = 0.019$. Determine the minimum value of n for which the probability of a loss on the entire portfolio of policies is less than or equal to 5%. Use the normal approximation.

C.5.3 Spreadsheet Exercises

1. Reproduce the Illustrative Life Table values of C_x. Calculate M_x recursively, from the end of the table where $M_{99} = C_{99}$, using the relation $M_x = C_x + M_{x-1}$. Calculate the values of $S_x = M_x + M_{x+1} + \ldots$ using the same technique.

2. Use the Illustrative Life Table to calculate the initial net annual premium for a whole life insurance policy issued at age $x = 30$. The benefit is inflation protected: each year the death benefit and the annual premium increase by a factor of $1 + j$, where $j = 0.06$. Calculate the initial premium for interest rates of $i = 0.05, 0.06, 0.07$ and 0.08. Draw the graph of the initial premium as a function of i .

3. Use $i = 4\%$, the Illustrative Life Table, and the utility function $u(x) = (1-e^{-ax})/a, a = 10^{-6}$, to calculate annual premiums for 10-year term insurance, issue age 40, using formula (5.2.9) : $E[u(-L)] = u(0)$. Display your results in a table with the sum insured C, the calculated premium, and the ratio to the net premium (loading). Draw the graph of the loading as a function of the sum insured. Do the same for premiums based on $a = 10^{-4}, 10^{-5}, 10^{-7}$ and 10^{-8} also. Show all the graphs on the same chart.

4. A whole life policy is issued at age 10 with premiums payable for life. If death occurs before age 15, the death benefit is the return of net premiums paid with interest to the end of the year of death. If death occurs after age 15, the death benefit is 1000. Calculate the net annual premium. Use the Illustrative Life Table and $i = 5\%$. Convince yourself that the net premium is independent of q_x for $x < 15$. (This problem is based on problem 21 at the end of Chapter 4 of *Life Contingencies* by C. W. Jordan.)

5. A 20-year term insurance is issued at age 45 with a face amount of 100,000. The net premium is determined using $i = 5\%$ and the Illustrative Life Table. The benefit is paid at the end of the year. Net premiums are invested in a fund earning j per annum and returned at age 65 if the insured survives. Calculate the net premium for values of j running from 5% to 9% in increments of 1%.

6. Determine the percentage z of annual salary a person must save each year in order to provide a retirement income which replaces 50% of final salary. Assume that the person is age 30, that savings earn 5% per annum, that salary increases at a rate of $j = 6\%$ per year, and that mortality follows the Illustrative Life Table. Draw the graph of z as a function of j running from 3% to 7% in increments of 0.5%

7. Mortality historically has improved with time. Let q_x denote the mortality table when a policy is issued. Suppose that the improvement (decreasing q_x) is described by $k^t q_x$ where t is the number of years since the policy was issued and k is a constant, $0 < k < 1$. Calculate the ratio of net premiums on the initial mortality basis to net premiums adjusted for $t = 10$ years of mortality improvement. Use $x = 30, k = 0.99$, the Illustrative Life Table for the initial mortality and $i = 5\%$.

C.6 Net Premium Reserves

Here are the additional symbols and definitions for reserves used in these exercises. Policies with premiums paid at the beginning of the year and death benefits paid at the end of the year of death are called fully discrete policies. Policies with premiums paid continuously and death benefits paid at the moment of death are called fully continuous.

$_k\overline{V}\left(\bar{A}_x\right)$ denotes the net premium reserve at the end of year k for a fully continuous whole life policy issued to (x).

$_k\overline{V}\left(\bar{A}_{x:\overline{n}|}\right)$ denotes the net premium reserve at the end of year k for a fully continuous n-year endowment insurance policy issued to (x).

Policies with limited premium payment periods can be described symbolically with a pre-superscript. For example, $^n_kV(\bar{A}_x)$ denotes the net premium reserve at the end of year k for an n-payment whole life policy issued to (x) with the benefit of 1 paid at the moment of death. Note that the corresponding net premium is denoted $_nP(\bar{A}_x)$.

C.6.1 Theory Exercises

1. A 20-year fully discrete endowment policy of 1000 is issued at age 35 on the basis of the Illustrative Life Table and $i = 5\%$. Calculate the amount of reduced paid-up insurance available at the end of year 5, just before the sixth premium is due. Assume that the entire reserve is available to fund the paid-up policy as described in section 6.8.

2. Given: $_{10}V_{25} = 0.1$ and $_{10}V_{35} = 0.2$. Calculate $_{20}V_{25}$.

3. Given: $_{20}V(\bar{A}_{40}) = 0.3847$, $\bar{a}_{40} = 20.00$, and $\bar{a}_{60} = 12.25$. Calculate $_{20}\overline{V}\left(\bar{A}_{40}\right) - {}_{20}V\left(\bar{A}_{40}\right)$.

4. Given the following information for a fully discrete 3-year special endowment insurance issued to (x):

k	c_{k+1}	q_{x+k}
0	2	0.20
1	3	0.25
2	4	0.50

Level annual net premiums of 1 are paid at the beginning of each year while (x) is alive. The special endowment amount is equal to the net premium reserve for year 3. The effective annual interest rate is $i = 1/9$. Calculate the end of policy year reserves recursively using formula (6.3.4) from year one with $_0V = 0$.

5. Given: $i = 0.06, q_x = 0.65, q_{x+1} = 0.85$, and $q_{x+2} = 1.00$. Calculate $_1V_x$. (Hint due to George Carr 1989: Calculate the annuity values recursively from $\ddot{a}_{x+2}$ back to $\ddot{a}_x$. Use (6.5.3).)

6. A whole life policy for 1000 is issued on May 1, 1978 to (60). Given:

(i) $i = 6\%$

(ii) $q_{70} = 0.033$

(iii) $1000_{10}V_{60} = 231.14$

(iv) $1000P_{60} = 33.00$, and

(v) $1000_{11}V_{60} = 255.40$

A simple method widely used in practice is used to approximate the reserve on December 31, 1988. Calculate the approximate value.

7. Given: For $k = 0, 1, 2, \ldots\ {}_{k|}q_x = (0.5)^{k+1}$. Show that the variance of the loss random variable L for a fully discrete whole life insurance for (x) is

$$\frac{1}{2}\left(\frac{v^2}{2 - v^2}\right)$$

where $v = (1 + i)^{-1}$.

8. Given, for a fully discrete 20-year deferred life annuity of 1 per year issued to (35):

(i) Mortality follows the Illustrative Life Table.

(ii) $i = 0.05$

(iii) Level annual net premiums are payable for 20 years.

Calculate the net premium reserve at the end of 10 years for this annuity.

9. A special fully discrete 2-year endowment insurance with a maturity value of 1000 is issued to (x). The death benefit in each year is 1000 plus the net level premium reserve at the end of that year. Given $i = 0.10$ and the following data:

k	q_{x+k}	c_{k+1}	${}_kV$
0	0.10	$1000 + {}_1V$	0
1	0.11	2000	?
2			1000

Calculate the net level premium reserve ${}_1V$.

10. Use the Illustrative Life Tables and $i = 0.05$ to calculate $1000_{15}V_{45:\overline{20|}}$.

11. Use the Illustrative Life Tables and $i = 0.05$ to calculate $1000\ {}_{15}V^{1}_{45:\overline{20|}}$.

12. Given the data in exercise 4, calculate the variance of the loss Λ_1 allocated to policy year two.

13. Given:

k	$\ddot{a}_{\overline{k|}}$	${}_{k-1|}q_x$
1	1.000	0.33
2	1.930	0.24
3	2.795	0.16
4	3.600	0.11

Calculate ${}_2V_{x:\overline{4}|}$.

14. A fully discrete whole life policy with a death benefit of 1000 is issued to (40). Use the Illustrative Life Table and $i = 0.05$ to calculate the variance of the loss allocated to policy year 10.

15. At an interest rate of $i = 4\%$, ${}^{20}_{23}V_{15} = 0.585$ and ${}^{20}_{24}V_{15} = 0.600$. Calculate p_{38}.

16. A fully discrete whole life insurance is issued to (x). Given: $P_x = 4/11$, ${}_tV_x = 0.5$, and $\ddot{a}_{x+t} = 1.1$. Calculate i.

17. For a special fully discrete whole life insurance of 1000 issued on the life of (75), increasing net premiums, Π_k, are payable at time k, for $k = 0, 1, 2, \ldots$. Given:

(i) $\Pi_k = \Pi_0(1.05)^k$ for $k = 0, 1, 2, \ldots$

(ii) Mortality follows de Moivre's law with $\omega = 105$.

(iii) $i = 5\%$

Calculate the net premium reserve at the end of policy year five.

18. Given for a fully discrete whole life insurance for 1500 with level annual premiums on the life of (x):

(i) $i = 0.05$

(ii) The reserve at the end of policy year h is 205.

(iii) The reserve at the end of policy year $h - 1$ is 179.

(iv) $\ddot{a}_x = 16.2$

Calculate $1000q_{x+h-1}$

19. Given:

(i) $1 + i = (1.03)^2$

(ii) $q_{x+10} = 0.08$

(iii) $1000{}_{10}V_x = 311.00$

(iv) $1000P_x = 60.00$

(v) $1000{}_{11}V_x = 340.86$

(a) Approximate $1000{}_{10.5}V_x$ by use of the traditional rule: interpolate between reserves at integral durations and add the unearned premium.
(b) *Assumption a* applies. Calculate the exact value of $1000{}_{10.5}V_x$.

20. Given: $q_{31} = 0.002$, $\ddot{a}_{32:\overline{13}|} = 9$, and $i = 0.05$. Calculate ${}_1V_{31:\overline{14}|}$.

C.6.2 Spreadsheet Exercises

1. Calculate a table of values of ${}_tV_{30}$ for $t = 0, 1, 2, \dots, 69$, using the Illustrative Life Table and $i = 4\%$. Recalculate for $i = 6\%$ and 8%. Draw the three graphs of ${}_tV_{30}$ as a function of t, corresponding to $i = 4\%, 6\%$, and 8%. Put the graphs on a single chart.

2. A 10-year endowment insurance with a face amount of 1000 is issued to (50). Calculate the savings Π_k^r and risk Π_k^s components of the net annual premium $1000P_{50:\overline{10|}}$ (formulas 6.3.6 and 6.3.7) over the life of the policy. Use the Illustrative Life Table and $i = 4\%$. Draw the graph of Π_k^s as a function of the policy year k. Investigate its sensitivity to changes in i by calculating the graphs for $i = 1\%$ and 7% and showing all three on a single chart.

3. A 10,000 whole life policy is issued to (30) on the basis of the Illustrative Life Table at 5%. The actual interest earned in policy years 1 - 5 is $i' = 6\%$. Assume the policyholder is alive at age 35 and the policy is in force.
(a) Calculate the technical gain realized in each year using method 2 (page 69).
(b) Calculate the accumulated value of the gains (using $i' = 6\%$) at age 35.
(c) Determine the value of i' (level over five years) for which the accumulated gains are equal to 400.

4. This exercise concerns a flexible life policy as described in section 6.8. The policyholder chooses the benefit level c_{k+1} and the annual premium Π_k at the beginning of each policy year $k+1$. The choices are subject to these constraints:

(i) $\Pi_0 = 100,000P_x$, the net level annual premium for whole life in the amount of 100,000.

(ii) $0 \le \Pi_{k+1} \le 1.2\Pi_k$ for $k = 0, 1, \dots$

(iii) $c_1 \le c_{k+1} \le 1.2c_k$ for $k = 1, 2, \dots$

(iv)] ${}_{k+1}V \ge 0$ for $k = 0, 1, 2, \dots$

The initial policyholder's account value is ${}_0V = 0$. Thereafter the policyholder's account values accumulate according to the recursion relation (6.3.4) with the interest rate specified in the policy as $i = 5\%$. Investigate the insurer's cumulative gain on the policy under two scenarios:
(S_1) The policyholder attempts to maximize insurance coverage at minimal costs over the first five policy years. The strategy is implemented by choosing $c_{k+1} = 1.2c_k$ for $k = 1, 2, \dots$ and choosing the level premium rate which meets the constraints but has ${}_5V = 0$. Calculate the insurer's annual gains assuming $i' = 5.5\%$ and the policyholder dies during year 5.
(S_2) The policyholder elects to maximize savings by choosing minimum coverage and maximum premiums. Calculate the present value of the insurer's annual gains assuming $i' = 5.5\%$ and the policyholder survives to the end of year 5.

C.7 Multiple Decrements: Exercises

C.7.1 Theory Exercises

1. In a double decrement table $\mu_{1,x+t} = 0.01$ for all $t \geq 0$ and $\mu_{2,x+t} = 0.02$ for all $t \geq 0$. Calculate $q_{1,x}$.

2. Given $\mu_{j,x+t} = j/150$ for $j = 1, 2, 3$ and $t > 0$. Calculate $E[T \mid J = 3]$.

3. A whole life insurance policy provides that upon accidental death as a passenger on public transportation a benefit of 3000 will be paid. If death occurs from other accidental causes, a death benefit of 2000 will be paid. Death from causes other than accidents carries a benefit of 1000. Given, for all $t \geq 0$:

(i) $\mu_{j,x+t} = 0.01$ where $j = 1$ indicates accidental death as a passenger on public transportation.

(ii) $\mu_{j,x+t} = 0.03$ where $j = 2$ indicates accidental death other than as a passenger on public transportation.

(iii) $\mu_{j,x+t} = 0.03$ where $j = 3$ indicates non-accidental death.

(iv) $\delta = 0.03$.

Calculate the net annual premium assuming continuous premiums and immediate payment of claims.

4. In a double decrement table, $\mu_{1,x+t} = 1$ and $\mu_{2,x+t} = \frac{t}{t+1}$ for all $t \geq 0$. Calculate

$$m_x = \frac{q_x}{\int_0^1 {}_tp_x dt}.$$

5. A two year term policy on (x) provides a benefit of 2 if death occurs by accidental means and 1 if death occurs by other means. Benefits are paid at the moment of death. Given for all $t \geq 0$:

(i) $\mu_{1,x+t} = t/20$ where 1 indicates accidental death.

(ii) $\mu_{2,x+t} = t/10$ where 2 indicates other than accidental death.

(iii) $\delta = 0$

Calculate the net single premium.

6. A multiple decrement model has 3 causes of decrement. Each of the decrements has a uniform distribution over each year of age so that the equation (7.3.4) holds for at all ages and durations. Given:

(i) $\mu_{1,30+0.2} = 0.20$

(ii) $\mu_{2,30+0.4} = 0.10$

(iii) $\mu_{3,30+0.8} = 0.15$

Calculate q_{30}.

7. Given for a double decrement table:

x	$q_{1,x}$	$q_{2,x}$	p_x
25	0.01	0.15	0.84
26	0.02	0.10	0.89

(a) For a group of 10,000 lives aged $x = 25$, calculate the expected number of lives who survive one year and fail due to decrement $j = 1$ in the following year.
(b) Calculate the effect on the answer for (a) if $q_{2,25}$ changes from 0.15 to 0.25.

8. Given the following data from a double decrement table:

(i) $q_{1,63} = 0.050$

(ii) $q_{2,63} = 0.500$

(iii) ${}_{1|}q_{63} = 0.070$

(iv) ${}_{2|}q_{1,63} = 0.042$

(v) ${}_3p_{63} = 0.$

For a group of 500 lives aged $x = 63$, calculate the expected number of lives who will fail due to decrement $j = 2$ between ages 65 and 66.

9. Given the following for a double decrement table:

(i) $\mu_{1,x+0.5} = 0.02$

(ii) $q_{2,x} = 0.01$

(iii) Each decrement is uniformly distributed over each year of age, thus (7.3.4) holds for each decrement.

Calculate $1000q_{1,x}$.

10. A multiple decrement table has two causes of decrement: (1) accident and (2) other than accident. A fully continuous whole life insurance issued to (x) pays c_1 if death results by accident and c_2 if death results other than by accident. The force of decrement 1 is a positive constant μ_1. Show that the net annual premium for this insurance is $c_2\bar{P}_x + (c_1 - c_2)\mu_1$.

C.8 Multiple Life Insurance: Exercises

C.8.1 Theory Exercises

1. The following excerpt from a mortality table applied to each of two independent lives (80) and (81):

x	q_x
80	0.50
81	0.75
82	1.00

Assumption a applies. Calculate $q^{1}_{80:81}$, $q_{80:\overset{2}{81}}$, $q_{80:81}$ and $q_{\overline{80:81}}$.

2. Given:

(i) $\delta = 0.055$

(ii) $\mu_{x+t} = 0.045, t \geq 0$

(iii) $\mu_{y+t} = 0.035, t \geq 0$

Calculate $\bar{A}^{2}_{xy}$ as defined by formula (8.8.8).

3. In a certain population, smokers have a force of mortality twice that of non-smokers. For non-smokers, $s(x) = 1 - x/75$, $0 \leq x \leq 75$. Calculate $\overset{\circ}{e}_{55:65}$ for a smoker (55) and a non-smoker (65) .

4. A fully-continuous insurance policy is issued to (x) and (y). A death benefit of 10,000 is payable upon the second death. The premium is payable continuously until the last death. The annual rate of payment of premium is c while (x) is alive and reduces to $0.5c$ upon the death of (x) if (x) dies before (y). The equivalence principle is used to determine c. Given:

(i) $\delta = 0.05$

(ii) $\bar{a}_x = 12$

(iii) $\bar{a}_y = 15$

(iv) $\bar{a}_{xy} = 10$

Calculate c.

5. A fully discrete last-survivor insurance of 1 is issued on two independent lives each age x. Level net annual premiums are paid until the first death. Given:

(i) $A_x = 0.4$

(ii) $A_{xx} = 0.55$

(iii) $a_x = 9.0$

Calculate the net annual premium.

6. A whole life insurance pays a death benefit of 1 upon the second death of (x) and (y). In addition, if (x) dies before (y), a payment of 0.5 is payable at the time of death. Mortality for each life follows the Gompertz law with a force of mortality given by $\mu_z = Bc^z, z \geq 0$. Show that the net single premium for this insurance is equal to

$$\bar{A}_x + \bar{A}_y - \bar{A}_w\left(1 - 0.5c^{x-w}\right)$$

where $c^w = c^x + c^y$.

7. Given:

(i) Male mortality has a constant force of mortality $\mu = 0.04$.

(ii) Female mortality follows de Moivre's law with $\omega = 100$.

Calculate the probability that a male age 50 dies after a female age 50.

8. Given:

(i) Z is the present-value random variable for an insurance on the independent lives of (x) and (y) where

$$Z = \begin{cases} v^{T(y)} & \text{if } T(y) > T(x) \\ 0 & \text{otherwise} \end{cases}$$

(ii) (x) is subject to a constant force of mortality of 0.07.

(iii) (y) is subject to a constant force of mortality of 0.09.

(iv) The force of interest is a constant $\delta = 0.06$.

Calculate $\mathrm{Var}[Z]$.

9. A fully discrete last-survivor insurance of 1000 is issued on two independent lives each age 25. Net annual premiums are reduced by 40% after the first death. Use the Illustrative Life Table and $i = 0.05$ to calculate the initial net annual premium.

10. A life insurance on John and Paul pays death benefits at the end of the year of death as follows:

(i) 1 at the death of John if Paul is alive,

(ii) 2 at the death of Paul if John is alive,

(iii) 3 at the death of John if Paul is dead and

(iv) 4 at the death of Paul if John is dead.

The joint distribution of the lifetimes of John and Paul is equivalent to the joint distribution of two independent lifetimes each age x. Show that the net single premium of this life insurance is equal to $7\bar{A}_x - 2\bar{A}_{xx}$.

C.8.2 Spreadsheet Exercises

1. Use the Illustrative Life Table and $i = 5\%$ to calculate the joint life annuity, $a_{x:y}$, the joint-and-survivor annuity, $a_{\overline{x:y}}$, and the reversionary annuity, $a_{x/y}$, for independent lives lives age $x = 65$ and $y = 60$.

2. (8.4.8) A joint-and-survivor annuity is payable at the rate of 10 per year at the end of each year while either of two independent lives (60) and (50) is alive. Given:

(i) The Illustrative Life Table applies to each life.

(ii) $i = 0.05$

Calculate a table of survival probabilities for the joint-and-survivor status. Use it to calculate the variance of the annuity's present value random variable.

3. Use the Illustrative Life Table and $i = 5\%$ to calculate the net level annual premium for a second-to-die life insurance on two independent lives age (35) and (40). Assume that premiums are paid at the beginning of the year as long as both insured lives survive. The death benefit is paid at the end of the year of the second death.

4. Calculate the net premium reserve at the end of years 1 through 10 for the policy in exercise 3. Assume that the younger life survives 10 years and that the older life dies in the sixth policy year.

5. Given:

(i) $\mu_x = A + Bc^x$ for $x \geq 0$ where $A = 0.004$, $B = 0.0001$, $c = 1.15$, and

(ii) $\delta = 5\%$.

Approximate $\bar{a}_{30:40}$ and $\bar{A}^{1}_{30:40}$.

C.9 The Total Claim Amount in a Portfolio

C.9.1 Theory Exercises

1. The claim made in respect of policy h is denoted S_h. The three possible values of S_h are as follows:

$$S_h = \begin{cases} 0 & \text{if the insured life } (x) \text{ survives,} \\ 100 & \text{if the insured surrenders the policy, and} \\ 1000 & \text{if the insured dies.} \end{cases}$$

The probability of death is $q_{1,x} = 0.001$, the probability of surrender is $q_{2,x} = 0.15$, and the probability of survival is $p_x = 1 - q_{1,x} - q_{2,x}$. Use the normal approximation to calculate the probability that the aggregate claims of five identically distributed policies $S = S_1 + \cdots + S_5$ exceeds 200.

2. The aggregate claims S are approximately normally distributed with mean μ and variance σ^2. Show that the stop-loss reinsurance net premium $\rho(\beta) = E[(X - \beta)^+]$ is given by

$$\rho(\beta) = (\mu - \beta)\Phi\left(\frac{\mu - \beta}{\sigma}\right) + \sigma\phi\left(\frac{\mu - \beta}{\sigma}\right)$$

where Φ and ϕ are the standard normal distribution and density functions.

3. Consider the compound model described by formula (9.4.6): $S = X + .. + X_N$ where N, X_i are independent, and X_i are identically distributed. Show that the moment generating function of S is $M_S(t) = M_N(\log(M_X(t)))$ where $M_N(t)$ and $M_X(t)$ are the moment generating functions of N and X. This provides a means of estimating moments of S from estimates of moments of X and N. For example, $\mathrm{E}[S] = \mathrm{E}[N]\mathrm{E}[X]$ and

$$\mathrm{E}[S^2] = \mathrm{E}[N^2]\mathrm{E}[X]^2 + \mathrm{E}[N]\left(\mathrm{E}[X^2] - \mathrm{E}[X]^2\right).$$

4. A reinsurance contract provides a payment of

$$R = \begin{cases} S - \beta & \text{if } \beta < S < \gamma \\ \gamma - \beta & \text{if } S \geq \gamma \end{cases}$$

Express $E[R]$ in terms of the cummulative distribution function of S.

5. (a) Express $F(x)$ in terms of the function $\rho(\beta)$.

(b) Given that $\rho(\beta) = \left(2 + \beta + \frac{1}{4}\beta^2\right), \beta \geq 0$, find $F(x)$ and $f(x)$.

6. Suppose that $f(0), f(1), f(2), \ldots$ are probabilities for which the following holds:

$$\begin{aligned} f(1) &= 3f(0), f(2) = 2f(0) + 1.5f(1), \\ f(x) &= \frac{1}{x}\left(3f(x-3) + 4f(x-2) + 3f(x-1)\right) \text{ for } x = 3, 4, \ldots \end{aligned}$$

What is the value of $f(0)$?

7. Suppose that $\log S$ is normally distributed with parameters, μ and σ. Calculate the net stop-loss premium $\rho(\beta) = \mathrm{E}[(S - \beta)^+]$ for a deductible β.

8. (a) For the portfolio defined by (9.3.5), calculate the distribution of aggregate claims by applying the method of dispersion with a span of 0.5.

(b) Apply the compound Poisson approximation with the same disceti- zation.

C.10 Expense Loadings

C.10.1 Theory Exercises

1. Consider the endowment policy of section 6.2, restated here for convenience: sum insured = 1000, duration $n = 10$, initial age $x = 40$, De Moivre mortality with $\omega = 100$, and $i = 4\%$.

(i) The acquisition expense is 50. No other expenses are recognized ($\beta = \gamma = 0$). Calculate the expense-loaded annual premium and the expense-loaded premium reserves for each policy year.

(ii) Determine the maximum value of acquisition expense if negative expense-loaded reserves are to be avoided.

2. Give a verbal interpretation of $-{}_kV^{\alpha}$.

3. Consider the term insurance policy of section 6.2, restated here for convenience: sum insured = 1000, duration $n = 10$, initial age $x = 40$, De Moivre mortality with $\omega = 100$, and $i = 4\%$.

(i) The acquisition expense is 40. No other expenses are recognized ($\beta = \gamma = 0$). Calculate the expense-loaded annual premium and the expense-loaded premium reserves for each policy year.

(ii) If the expense-loaded premium reserves are not allowed to be negative, what is the insurer's initial investment for selling such a policy?

4. Calculate the components $1000P$, $1000P^{\alpha}$, $1000P^{\beta}$ and $1000P^{\gamma}$ of the expense loaded premium $1000P^{a}$ for a whole life insurance of 1000 issued to a life age 35. The policy has level annual premiums for 30 years, becoming paid-up at age 65. The company has expenses as follows:

acquisition expense	12 at the time of issue,
collection expenses	15% of each expense loaded premium, and
administration expens	1 at the beginning of each policy year.

Use the Illustrative Life Table and $i = 5\%$.

5. For the policy described in exercise 4, calculate components $1000{}_kV$, $1000{}_kV^{\alpha}$, and $1000{}_kV^{\gamma}$ of the expense-loaded premium reserve $1000{}_kV^{a}$ for year $k = 10$.

C.10.2 Spreadsheet Exercises

1. Develop a spreadsheet to calculate the expense-loaded premium components and the expense-loaded premium reserve components for each policy year of a 20-year endowment insurance issued to a life age 40. Use the Illustrative Life Table and $i = 6\%$. Assume that acquisition expense is 20 per 1000 of insurance, collection expense is 5% of the expense-loaded premium, and administration expense is 3 at the beginning of each policy year.

C.11 Estimating Probabilities of Death

C.11.1 Theory Exercises

1. Consider the following two sets of data:

(a) $D_x = 36$ $E_x = 4820$
(b) $D_x = 360$ $E_x = 48200$

For each set, calculate a 90% confidence interval for q_x.

2. We model the uncertainty about θ (the unknown value of $\mu_{x+1/2}$) by a gamma distribution such that $\mathrm{E}[\theta] = 0.007$ and $\mathrm{Var}(\theta) = 0.000007$. An additional 36 deaths are observed for an additional exposure of 4820. Calculate our posterior expectation and standard deviation of θ, and our estimate for q_x.

3. Write down the equations from which

(i) λ^l and

(ii) λ^u are obtained.

(iii) Rewrite these equations in terms of integrals over $f(x;n)$, the probability density function of the gamma distribution with shape parameter n and scale parameter 1.

4. In a clinical experiment, a group of 50 rats is under observation until the 20th rat dies. At that time the group has lived a total of 27.3 rat years. Estimate the force of mortality (assumed to be constant) of this group of rats. What is their life expectancy?

5. A certain group of lives has a total exposure of 9758.4 years between ages x and $x+1$. There were 357 deaths by cause one, 218 deaths by cause two, and 528 deaths by all other causes combined. Estimate the probability that a life age x will die by cause one within a year.

6. There are 100 life insurance policies in force, insuring lives age x. An additional 60 polices are issued at age $x + \frac{1}{4}$. Four deaths are observed between age x and $x+1$. Calculate the classical estimator given by formula (11.2.4), and the maximum likelihood estimator based on the *assumption b*, a constant force of mortality (11.4.2).

7. The force of mortality is constant over the year age $(x, x+1]$. Ten lives enter observation at age x. Two lives enter observation at age $x+0.4$. Two lives leave observation at age $x+0.8$, one leaves at age $x+0.2$ and one leaves at age $x+0.5$. There is one death at age $x+0.06$. Calculate the maximum likelihood estimate of the force of mortality.

8. A double decrement model is used to study two causes of death in the interval of age $(x, x+1]$.

The forces of each cause are constant.

1,000 lives enter the study at age x.

40 deaths occur due to cause 1 in $(x, x+1]$.

50 deaths occur due to cause 2 in $(x, x+1]$.

Calculate the maximum likelihood estimators of the forces of decrement.

9. The Illustrative Life Table is used for a standard table in a mortality study.

The study results in the following values of exposures E_x and deaths D_x over $[40, 45)$

x	E_x	D_x
40	1150	6
41	900	5
42	1200	12
43	1400	9
44	1300	13

Calculate the mortality ratio $\hat{f}$ and the 90% confidence interval for f. Calculate the estimates of $\hat{q}_{40}, \hat{q}_{41}, \ldots \hat{q}_{45}$ corresponding to $\hat{f}$.

Appendix D

Solutions

D.0 Introduction

We offer solutions to most theory exercises which we hope students will find useful. When the solution is straightforward we simply give the answer. For the spreadsheet exercises we describe the solution and give some values to use to verify your work. We leave the joy of writing the program to the student.

We have tried hard to avoid errors. We hope that students and other users who discover errors will inform us promptly. We are also interested in seeing elegant or insightful solutions and new problems.

The solutions occasionally refer to the Illustrated Life Table and its functions. They are in Appendix E.

D.1 Mathematics of Compound Interest

D.1.1 Solutions to Theory Exercises

1. This follows easily from equation (1.5.8).

2. Fix $i > 0$ and consider the function $f(x) = [(1+i)^x - 1]x^{-1} = (e^{\delta x} - 1)/x$. From the power series expansion

$$f(x) = \delta + \frac{1}{2}\delta^2 x + \frac{1}{3!}\delta^3 x^2 + \dots,$$

it is easy to see that $f'(x) > 0$ for all $x > 0$. It follows that $f(x)$ increases from $f(0+) = \delta$ to $f(1) = i$. Therefore $g(x) = f(x^{-1})$ decreases on $[1, \infty)$ from i to δ. Thus, $i^{(m)} = g(m)$ decreases to δ as m increases. Similarly, $d^{(m)}$ increases to δ as m increases.

3. The accumulated value of the deposits as of January 1, 1999 is $Xs_{\overline{10}|0.06}$. The present value of the bond payments as of January 1,1999 is $15,000a_{\overline{5}|0.06}$. Equate the two values and solve for $X = 4794$.

4. Let i be the effective annual interest rate. Then $1 + i = (1 + j/2)^2$. The equation of value is

$$\begin{aligned} 5.89 &= v^2 + v^4 + \dots \\ &= \frac{v^2}{1 - v^2}. \end{aligned}$$

Hence $(1 + j/2)^4 = 1 + 1/5.89$ and so $j = 8\%$.

5. Let $u = \frac{1+k}{1.04}$ and write the equation of value as follows:

$$\begin{aligned} 51 &= \frac{1+k}{1.04} + \left(\frac{1+k}{1.04}\right)^2 + \dots \\ &= u + u^2 + \dots \\ &= \frac{u}{1-u} \\ &= \frac{1+k}{0.04 - k}. \end{aligned}$$

Solve for $k = 2\%$.

6. Use equation (1.9.8) with a starting value of $\delta = 12\%$. The price of the coupon payments is $p = 94 - 100(1.12)^{-10} = 61.80$. The sum of the payments is $r = 100$ and

$$a(\delta) = 5\frac{1 - e^{-10\delta}}{e^{\delta/2} - 1}.$$

The solution is $\delta = 9.94\%$. This is equivalent to 10.19% per year convertible semiannually.

7. The equation of value is

$$\begin{aligned} 1000 &= x(v+v^2+v^3)+3x(v^4+v^5+v^6) \\ &= x(3a_{\overline{6}|}-2a_{\overline{3}|}) \\ &= x(11.504459) \end{aligned}$$

where the symbols correspond to a values of $i = 1\%$. So $x = 86.92$.

8. At the time of the loan,

$$\begin{aligned} 4000 &= kv+2kv^2+3kv^3+\ldots+30kv^{30} \\ &= k\,(Ia)_{\overline{30}|} \\ &= k\frac{\ddot{a}_{\overline{30}|}-30v^{30}}{0.04} \end{aligned}$$

so $k = 18.32$. Note that the initial payment is less than the interest (160) required on the loan so the loan balance increases. Immediately after the ninth payment the outstanding payments, valued at the original loan interest rate, is found as follows:

$$\begin{aligned} 10kv+11kv^2+\ldots+30kv^{21} &= 9ka_{\overline{21}|}+k\,(Ia)_{\overline{21}|} \\ &= (18.32)(9(14.02916)+134.37051) \\ &= 4774.80. \end{aligned}$$

9. Let $j = i^{(2)}/2$ and solve $98.51 = 2(1+j)^{-1}+102(1+j)^{-2}$ for $(1+j)^{-1} = 0.9729882$. The corresponds to $i^{(2)} = 5.55\%$.

10. From (i) and (ii) we get $12(120)\,a^{(12)}_{\overline{\infty}|} = 12(365.47)\,a^{(12)}_{\overline{n}|}$ and solve for $v^n = 0.6716557$. Now use (iii) and (iv) to solve for $X = 12000$.

D.1.2 Solutions to Spreadsheet Exercises

1. (a) The investment yield is 9.986%.

2. *Guide*: Set up a spreadsheet with a trial value of X. Since a total of $X+2X+3X+\cdots+6X = 21X$ is withdrawn, a good trial value is about $100,000/20 = 5,000$. Use the fundamental formula (1.2.1) to calculate the fund balance at the end of each half-year. Then experiment with X until an end-of-year six balance of zero is found. $X = 6,128$.(Alternatively, in the last stage, use Goal Seek to determine the value of X which makes the target balance zero.) Note that the end-of-year six balance is the fund balance beginning the thirteenth half-year. Adapting notation of the text to half-years we have $F_0 = 100,000, F_1 = F_0 + (1.03)^2 - X, F_2 = F_1 + (1.03)^2 F_1 - 2X$, and so on.

3. *Guide*: Set up an amortization table using a spreadsheet and a trial value of $i = 0.03$. In a cell apart from the table, calculate the target $P - 6I$ for the fifth year. Then use Goal Seek to determine i so that the target cell is zero. $i = 10.93\%$.

4. *Guide*: The fund deposit X satisfies $Xs_{\overline{10}|:0.03} = 10,000$. In effect, the company accepts 10,000 now in exchange for 10 semiannual payments of 300+X. Calculate X using the spreadsheet financial functions. The internal rate of return j equates the future cash flows 300+X to 10,000. Set up your spreadsheet with a trial value of j. Use the Goal Seek feature to detemine the value of j. $j = 7.80\%$

5. *Guide*: Put $i = 10\%$ and a trial value of X into cells. Calculate the net present value of the payments of 100 minus the payments of X in another cell as follows:

$$100\ddot{a}_{\overline{10}|} - Xv^{19}\ddot{a}_{\overline{\infty}|} = \frac{100 - v_{10}100 - v_{19}X}{d}$$

Use the Goal Seek feature to determine the valaue of X for which the resent value is zero. $X = 375.80$

6. *Guide*: Set up a spreadsheet to amortize the loan using a trial value of $X = 30,000$. The interest credited in year k is

$$0.08\min(100000, B) + 0.09\max(0, B - 100000)$$

where B_k is the beginning year balance. $B_0 = 300,000$, $B_1 = 300,000 + 8,000 + 18,000 - X$, and so on recursively. Use the Goal Seek feature to determine X so that $B_{11} = 0$ (beginning year 11 = end of year 10). $X = 45,797.09$

7. *Guide*: Work from the last year back to the present. The required cash flow for the last year is known and so is the coupon, so you can calculate the number of longest maturity bonds to buy. Then work on the next to the last year, knowing the required cash flow and the number of bonds paying a coupon (but maturing in the following year). And so on.

The total market value is 450,179. You need 1.87 bonds maturing in 1995, etc.

8. *Guide*: Use the Goal Seek feature. The market yield is 7.46

9. *Guide*: Use the Goak Seek feature to find the price for each call date to yield 8%. The price is the minimum of these: 1,085.59.

10. *Guide*: With a trial value for the interest rate, use the future value function (FV) to find the balance after 20 quarters. Use Goal Seek to set the future value to 5000. The solution is $i = 8.58\%$

D.2 The Future Lifetime of a Life Aged x

D.2.1 Solutions to Theory Exercise

1. Use equation (2.2.5). $\mu_{45} = -\frac{d}{dt}\ln({}_tp_x)$ evaluated at $t = 45 - x$. Thus

$$\begin{aligned}\mu_{45} &= -\frac{d}{dt}\ln\left(\frac{100-x-t}{100-x}\right)\\ &= \left.\frac{1}{100-x-t}\right|_{t=45-x}\\ &= \frac{1}{55}.\end{aligned}$$

2. Use equation (2.1.11).

$$\begin{aligned}E[T(x)] &= \int_0^\infty {}_tp_x dt\\ &= \int_0^{100}\left(1-\left(\frac{t}{100}\right)^{1.5}\right)dt\\ &= t - \frac{1}{1000}\left(\frac{t^{2.5}}{2.5}\right)\Big|_0^{100}\\ &= 60.\end{aligned}$$

3. Use (2.2.6). First: $\int_0^{20}\mu_{x+t}dt = -\ln(85-t) - 3\ln(105-t)|_0^{20} = -\ln\left(\frac{60}{85}\left(\frac{85}{105}\right)^3\right)$. Then ${}_{20}p_x = \frac{60}{85}\left(\frac{85}{105}\right)^3 = 0.3745.$

4. Use (2.1.11).

$$\begin{aligned}\mathring{e}_{41} &= \int_0^\infty {}_tp_{41}dt\\ &= \int_0^\infty\left(\frac{42}{42+t}\right)^3\\ &= (42)^3\frac{(42+t)^{-2}}{-2}\Big|_0^\infty\\ &= 21.\end{aligned}$$

5. The symbol m_x denotes the central death rate: Deaths $d_x = l_x - l_{x+1}$ and average population $= \displaystyle\int_x^{x+1} l_y dy = l_x\int_0^1 \frac{l_{x+t}}{l_x}dt$. Divide each of deaths and average population by l_x to obtain $m_x = \dfrac{q_x}{\int_0^1 {}_tp_x}dt$. Use (2.6.9).

$$\int_0^1 {}_tp_x dt = \int_0^1 \frac{1-q_x}{1-(1-t)q_x}dt$$

$$\begin{aligned} &= \frac{p_x}{q_x} \ln[1-(1-t)q_x]\Big|_0^1 \\ &= \frac{-p_x \ln(p_x)}{q_x}. \end{aligned}$$

The formula for m_x is the reciprocal of this quantity multiplied by q_x. To work the exercise substitute $q_x = 0.2$ and $p_x = 0.8$. The answer is $m_x = 0.224$.

6. Use equation (2.2.6). $e^{-\mu} = p_x = 1 - 0.16 = 0.84$ so ${}_tp_x = e^{-t\mu} = (0.84)^t = 0.95$ and, $t = \ln(0.95)/\ln(0.84) = 0.294$.

7. Since $l_x\mu_x$ is constant, l_x is linear. Thus $T(88)$ is uniform on (0,12). Therefore $\mathrm{Var}(T) = (12)^2/12 = 12$.

8. Before: $0.95 = \exp\left(-\int_0^1 \mu_{x+t}dt\right) = p_x$. After: $0.93 = \exp\left(-\int_0^1 (\mu_{x+t} - c)dt\right)$ $= p_x e^c = 0.95e^c$. Therefore $e^c = 93/95$, $c = \log(93/95) = -0.0213$.

9. Make the change of variables $x + s = y$ in equation (2.2.6) to prove (i). Use the rules for differentiating integrals to prove (ii).

10.

$$\begin{aligned} 100_{1|}q_{[30]+1} &= 100\left(p_{[30]+1}\right)\left(q_{[30]+2}\right) \\ &= \left(1 - q_{[30]+1}\right)\left(100q_{32}\right) \\ &= (1 - 0.00574)(0.699) \\ &= 0.695 \end{aligned}$$

11.

$$\frac{l_{40} - l_{57}}{l_{21}} = \frac{(81)^{1/2} - (64)^{1/2}}{(100)^{1/2}} = 0.10$$

12. Use this relation:

$$e_x = E[K(x)] = p_x E[K(x)|T(x) \ge 1] + q_x E[K(x)|T(x) < 1] = p_x(1 + e_{x+1}).$$

Thus $p_x = e_x/(1 + e_{x+1})$. ${}_2p_{75} = p_{75}p_{76} = \frac{10}{1+9.5}\frac{10.5}{1+10.0} = 0.909$.

13. $T(16)$ is uniform on $(0, \omega - 16)$ since we have a de Moivre mortality table. Hence $\mathrm{E}[T(16)] = (\omega - 16)/2$ and $\mathrm{Var}[T(16)] = (\omega-16)^2/12$. Therefore $\omega - 16 = 2(36) = 72$ and $\mathrm{Var}[T] = (72)^2/12 = (72)(6) = 432$.

14. $q_{50} = 1 - {}_{21}p_{30}/{}_{20}p_{30} = 111/6000$. And $\mu_{30+t} = -\frac{1}{{}_tp_{30}}\frac{\partial}{\partial t}{}_tp_{30} = \frac{70+2t}{7800-70t-t^2}$. Therefore, $q_{50} - \mu_{50} = 1/6000$.

15. $E[T] = \int_0^{100-x} {}_tp_x dt = \int_0^a \left(\frac{a-t}{a}\right)^2 dt = \frac{a}{3}$ where $a = 100 - x$. $E[T^2] = \int_0^{100-x} 2t{}_tp_x dt = 2\int_0^a t{}_tp_x dt$. Use integration by parts to obtain $E[T^2] = \frac{a^2}{6}$. Hence $\mathrm{Var}(T) = \mathrm{E}(T^2) - \mathrm{E}(T)^2 = a^2(1/6 - 1/9) = (100 - x)^2/18$.

16. $m_x = \frac{q_x}{\int_0^1 {}_tp_x dt}$ and, because of the constant force of mortality, ${}_tp_x = e^{-\mu t}$ where $\mu = -\ln(p_x)$. Hence, $\int_0^1 {}_tp_x dt = q_x/\mu$ and $m_x = \mu = 0.545$.

17. Let $T = T(x)$ be the lifetime of the non-smoker and $T^s = T^s(x)$ be the lifetime of the smoker. Use formula (2.2.6): $\Pr(T^s > t) = \exp\left(-\int_0^t c\mu_{x+u}du\right) = ({}_tp_x)^c$ where ${}_tp_x = \Pr(T > t)$. Hence $\Pr(T^s > T) = \int_0^\infty \Pr(T^s > t)g(t)dt = \int_0^\infty ({}_tp_x)^c g(t)dt = -\int_0^\infty w(t)^c w'(t)dt$ where $w(t) = {}_tp_x$. Hence, $\Pr(T^s > T) = \left.\frac{[w(t)]^{c+1}}{c+1}\right|_0^\infty = 1/(c+1)$.

18. See exercise 9. $q_x = 1 - p_x = 1 - \exp\left(-\int_x^{x+1} \mu_y dy\right)$ which we get by a change of variable of integration in formula (2.2.6). Now apply the rules for differentiation of integrals:

$$\frac{dq_x}{dx} = -\exp\left(-\int_x^{x+1} \mu_y dy\right)(-\mu_{x+1} + \mu_x) = p_x(\mu_{x+1} - \mu_x).$$

19. $\int_{35}^{45} \mu_x dx = 400k$ and so $0.81 = {}_{10}p_{35} = \exp(-\int_{35}^{45} \mu_x dx) = \exp(-400k)$. Similarly ${}_{20}p_{40} = \exp(-\int_{40}^{60} \mu_x dx) = e^{-1000k} = ((0.81)^{1/400})^{1000} = (0.81)^{5/2} = (0.9)^5 = 0.59$

20. $E[X^2] = 2\int_0^\omega x_x p_0 dx = 3\omega^2/5$. $\text{Var}[X] = (3\omega^2/5) - (3\omega/4)^2 = \omega^2(3/5 - 9/16) = 3\omega^2/80$.

D.2.2 Solutions to Spreadsheet Exercises

1. See appendix E.

2. Check value: $e_0 = 71.29$.

3. $c = 0.09226$. Assume that "expectation of remaining life" refers to complete life expectation and that *assumption a* applies, so that $\mathring{e}_x = e_x + 0.5$.

4. Use formula (2.3.4) with $A = 0$. Check values: $\ell_{40} = 99,510$ when $c = 1.01$ and $\ell_{50} = 680$ when $c = 1.20$.

5. Under *assumption a*, $\mu_{x+0.6} = 0.10638$ for example.

6. Under *assumption b*, ${}_{0.4}q_x = 0.04127$ for example.

7. Use trial values such as $B = 0.0001$ and $c = 1$ to calculate Gompertz values, and the sum of their squared differences from the table values. Use the optimization feature to determine values of B and c which minimize the sum. Solution: $B = 2.69210^{-5}$ and $c = 1.105261$.

8. For $k = 7.5$, $e_{45} = 12.924$. For $k = 1$, $e_{45} = 30.890$.

D.3 Life Insurance

D.3.1 Solutions to Theory Exercises

1. The issue age is $x = 30$. From (i), $T = T(30)$ is uniformly distributed on $(0, 70)$. The present value random variable is $Z = 50,000v^T$. Hence, $\bar{A}_{30} = E[Z] = 50,000 \int_0^{70} v^t \frac{1}{70} dt = (50,000/70)(1 - e^{-7})/(0.10) = 7,136$.

2. Use the recursion relation:

$$(IA)_x = A^1_{x:\overline{1}|} + vp_x(A_{x+1} + (IA)_{x+1}).$$

An alternative solution in terms of commutation functions goes like this: The numerator can be written as follows:

$$\frac{R_x - C_x}{D_x} = \frac{M_x + R_{x+1} - C_x}{D_x} = \frac{M_{x+1} + R_{x+1}}{D_x}.$$

The denominator is $\frac{R_{x+1}+M_{x+1}}{D_{x+1}}$. Hence the ratio is $D_{x+1}/D_x = vp_x$.

3. Let Z_3 be the present value random variable for the pure endowment, so $Z_1 = Z_2 + Z_3$. It follows that $\mathrm{Var}(Z_1) = \mathrm{Var}(Z_2) + 2\mathrm{Cov}(Z_2, Z_3) + \mathrm{Var}(Z_3)$. Now use the fact that $Z_2 Z_3 = 0$ to obtain $\mathrm{Cov}(Z_2, Z_3) = -\mathrm{E}[Z_2]\mathrm{E}[Z_3]$. Z_3 is v^n times the Bernoulli random variable which is 1 with probability ${}_np_x$, zero otherwise. Hence $\mathrm{Var}(Z_1) = 0.01 + 2(-\mathrm{E}[Z_2]\mathrm{E}[Z_3]) + \mathrm{Var}(Z_3) = 0.01 - 2(0.04)(0.24) + (0.30)^2(0.8)(1 - 0.8) = 0.0052$.

4. $A_{45:\overline{20}|} = (M_{45} - M_{65} + D_{65})/D_{45} = 0.40822$.

5. Use the recursion relation $A_x = A^1_{x:\overline{n}|} + v^n {}_np_x A_{x+n}$ and the relation $A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + v^n {}_np_x$. In terms of the given relations these are $A_x = y + v^n {}_np_x z$ and $u = y + v^n {}_np_x$. Hence $A_x = y + (u - y)z = (1 - z)y + uz$.

6. From (ii), the discount function is $v_t = 1/(1 + 0.01t) = 100/(100 + t)$. The benefit function is: $b_t = (10,000 - t^2)/10 = (100 + t)(100 - t)/10$. Hence $Z = v_T b_T = 10(100 - T)$ and so $\mathrm{E}[Z] = 10(100 - \mathrm{E}[T])$. Now use item (i): $T = T(50)$ is uniform on (0, 50] so $\mathrm{E}[T] = 25$ and $\mathrm{E}[Z] = 750$.

7. $\mathrm{E}[v^T] = \displaystyle\int_0^\infty e^{\delta t} e^{-\mu t} \mu dt = A_x = \frac{\mu}{\mu + \delta}$ and $\mathrm{E}[(v^T)^2] = {}^2A_x = \dfrac{\mu}{\mu + 2\delta}$.
Therefore, $\mathrm{Var}[v^T] = {}^2A_x - A_x^2 = \ldots = \dfrac{\mu\delta^2}{(\mu + 2\delta)(\mu + \delta)^2}$.

8. Consider the recursion relation $A_x = vq_x(1 - A_{x+1}) + vA_{x+1}$. The analog for select mortality with a one year select period goes like this: Since the select period is one year, $K([x] + 1)$ and $K(x + 1)$ are identically distributed. Hence, using the theorem on conditional expectations, we have $A_{[x]} = E[v^{K[x]+1}] = vq_{[x]} + E[v^{K[x+1]+1+1}](1 - q_{[x]}) = vq_{[x]} + E[v^{K(x+1)+1}]v(1 - q_{[x]}) = vq_{[x]} + A_{x+1}v(1 - q_{[x]})$. Hence, $A_{[x]} = vq_{[x]}(1 - A_{x+1}) + vA_{x+1}$. By combining the two recursion relations, we see that $A_x - A_{[x]} = v(q_x - q_{[x]})(1 - A_{x+1}) = 0.5vq_x(1 - A_{x+1})$.

9. Let P be the single premium. Use formula (3.4.3). The benefit function is $c_{k+1} = 10,000 + P$ for $k = 0, 1, \ldots, 19$ and $c_{k+1} = 20,000$ for $k = 20, 21, \ldots$. Hence $PD_x = (10,000 + P)M_x + (10,000 - P)M_{x+20}$. Now solve for $P = 10,000\dfrac{M_x + M_{x+20}}{D_x - (M_x - M_{x+20})}$.

10. Use formula (3.4.3). Make a column of differences of the death benefit column c_k. In calculating the differences, use a death benefit of 0 at age $x - 1$ and age $x + 11$. Also put in the ages to avoid confusion about which age to use in the solution. You will obtain a table like this:

Age at Death	Year of Death	c_k	$c_k - c_{k-1}$
x	1	10	10
$x+1$	2	10	0
$x+2$	3	9	-1
$x+3$	4	9	0
$x+4$	5	9	0
$x+5$	6	8	-1
$x+6$	7	8	0
$x+7$	8	8	0
$x+8$	9	8	0
$x+9$	10	7	-1
$x+10$	11	0	-7

From the table, we see that the net single premium is written in terms of commutation functions as follows:

$$\frac{10M_x - M_{x+2} - M_{x+5} - M_{x+9} - 7M_{x+10}}{D_x}.$$

11.

$$Z = \begin{cases} v^T & \text{if } T < 10 \\ v^{10} & \text{otherwise} \end{cases}$$

T is uniform on $(0, 50)$. Hence, the net single premium is

$$\begin{aligned} 50,000\left[v^{10}{}_{10}p_{50} + \int_0^{10} v^t g(t)dt\right] &= 50,000\left[\frac{40}{50}e^{-(0.10)10} + \int_0^{10} v^t \frac{1}{50}dt\right] \\ &= 10,000[1 + 3e^{-1}] = 21,036. \end{aligned}$$

12. Let m be the answer. $m = v^T$, where ${}_tp_y = 0.5$. Since ${}_tp_y = \frac{s(y+t)}{s(y)} = e^{-0.02t} = 0.5$, then $m = e^{0.04t} = (e^{-0.02t})^2 = (v.5)^2 = 0.25$.

13. Since $i = 0$, then $Z = 0$ or 1. $Z = 1$ if $K = 0$ or $K = 1$ which occurs with probability ${}_2q_x = q_x + p_xq_{x+1} = 1/2 + 1/2q_{x+1}$. And $Z = 0$ if $K > 1$, which occurs with probability $1 - (q_x + p_xq_{x+1}) = 1/2 - 1/2q_{x+1}$. Hence Z is Bernoulli;

its variance is $(1/2 + 1/2q_{x+1})(1/2 - 1/2q_{x+1}) = 1/4(1 - q_{x+1}^2)$. Set this equal to 0.1771 and solve. The result is $q_{x+1} = 0.54$.

14. Calculate values of Z and its density function $f(t)$ in the given table. Obtain the following:

t	$c(t)$	q_{x+t}	Z	$f(t)$	$Zf(t)$
0	3	0.20	2.700	0.2	0.5400
1	2	0.25	1.620	0.2	0.3240
2	1	0.50	0.729	0.3	0.2187
≥ 3	0		0	0.3	0

$\Pi = 1.0827$. The values of Z which are greater than $\Pi = 1.0827$ are $Z = 2.7$ and $Z = 1.620$. Hence $\Pr[Z > \Pi] = 0.2 + 0.2 = 0.4$.

15. $A_x = vq_x + vp_x A_{x+1}$ so $A_{76} = vq_{76} + (D_{77}/D_{76})A_{77}$ since $D_{77}/D_{76} = vp_{76}$. Since $p_{76} = (1+i)(D_{77}/D_{76}) = (1.03)(360/400) = (1.03)(0.9) = 0.927$ then $0.8 = (1.03)^{-1}(1 - 0.927) + (0.9)A_{77}$. Now solve for $A_{77} = 0.810$.

16. The net single premium is $50\int_0^{100} v^t g(t)dt$. Integrate by parts to get a net single premium of $1 - 11e^{-10} = 0.999501$.

17. See Exercise 7. $\mathrm{E}[v^{2T}] = 0.25$ implies that $\frac{\mu}{\mu+2\delta} = 0.25$ so $3\mu = 2\delta$. $\mathrm{E}[v^T] = \frac{\mu}{\mu+\delta} = \frac{\mu}{\mu+1.5\mu} = 0.4$.

18. $0.95 = \Pr[(Z_1 + Z_2 + \cdots + Z_{100})1000 \leq 100w]$ where the random variables Z_i are independent and identically distributed like v^T. Now $Y = (1/100)\sum_{k=1}^{100} Z_k$ is approximately normal with mean $E[Z] = 0.06$ and variance equal to $(1/100)\mathrm{Var}(Z) = (1/100)[0.01 - (0.06)^2] = 0.64(10^{-6})$. Thus the mean and standard deviation of Y are 0.06 and 0.008. Therefore $0.95 = \Pr(Y \leq w/1000)$ implies that $w = 1000(0.06 + (1.645)(0.008)) = 73.16$.

19. Use $\Pi = \Pi\, A^{1}_{x:\overline{20}|} + 10,000v^{20}{}_{20}p_x$ or $\Pi D_x = \Pi(M_x - M_{x+20}) + 10,000D_{x+20}$ and solve for Π.

20. Use formula (3.4.3). Make a column of differences of the death benefit column c_k. Use a death benefit of 0 at age $x-1$.

Age at Death	Year of Death	c_k	$c_k - c_{k-1}$
x	1	10	10
$x+1$	2	10	0
$x+2$	3	9	-1
$x+3$	4	9	0
$x+4$	5	9	0
$x+5$	6	8	-1
$x+6$	7	8	0
$x+7$	8	8	0
$x+8$	9	8	0
$x+9$	10	7	-1
$\geq x+10$	≥ 11	7	0

Then the net single premium is given as follows:

$$\frac{10M_x - M_{x+2} - M_{x+5} - M_{x+9}}{D_x}.$$

D.3.2 Solution to Spreadsheet Exercises

1. For $i = 2.5\%, 5.0\%$, and 7.5%, the single premium life insurance at age zero is $A_0 = 0.19629, 0.06463$ and 0.03717.

2. At $i = 5\%$, $(IA)_0 = 2.18345$.

3. Guide: Set up a table with benefits and probabilities of survival to get them. The net single premium is 0.0445.

4. Guide: Use the VLOOKUP() function to construct the array of survial probabilites for a given issue age. Check values: $(DA)_{50:\overline{50}|} = 9.0023$ at $i = 5\%$. $(DA)_{25:\overline{75}|} = 3.3947$ at $i = 6\%$.

5. Guide: Set up a spreadsheet to calculate the values of A_x and ${}^2A_x$. According to formula (3.2.4), the second moment can be calculated by changing the force of interest from δ to 2δ. Put δ in a cell and let it drive the interest calculations. Use the Data Table (or What if?) feature to find the two values of $\mathrm{E}[v^{K+1}]$, corresponding to δ and 2δ. Check values: $\mathrm{Var}(v^{K+1}) = 20,190$ when $i = 5\%$, and $\mathrm{Var}(v^{K+1}) = 17,175$ when $i = 2\%$.

D.4 Life Annuities

D.4.1 Solutions to Theory Exercises

1. Use formula (4.3.9) with $m = 2, x = 40, n = 30$. $\ddot{a}_{40:\overline{30}|} = D_{40}^{-1}(N_{40} - N_{70}) = 15.1404$, $D_{70}/D_{40} = v^{30}p_{40} = 0.1644$, $\alpha(2) = \frac{id}{i^{(2)}d^{(2)}} = 1.00015$, and $\beta(2) = 0.25617$. The answer is $\ddot{a}^{(2)}_{40:\overline{30}|} = 14.9286$.

2. Use the recursion formula (4.6.2) for $a_{x:\overline{1}|}$ to develop the following recursion formula: $(I\ddot{a})_x = \ddot{a}_{x:\overline{1}|} + vp_x\left(\ddot{a}_{x+1} + (I\ddot{a})_{x+1}\right)$. The ratio simplifies consequently to $vp_x = a_{x:\overline{1}|}$.

3. $\left(\bar{I}_{\overline{n}|}\bar{a}\right)_x = \int_0^n tv^t{}_tp_x dt + \int_n^\infty nv^t{}_tp_x dt$. Differentiate before making the substitutions $v_t = e^{-0.06t}$ and ${}_tp_x = e^{-0.04t}$. Use Liebnitz's rule for differentiating integrals:

$$\begin{aligned}\frac{\partial}{\partial n}\left(\bar{I}_{\overline{n}|}\bar{a}\right)_x &= nv^n{}_np_x + \int_n^\infty v^t{}_tp_x dt - nv^n{}_np_x \\ &= \int_n^\infty v^t{}_tp_x dt = \int_n^\infty e^{-0.10t}dt = 10e^{-0.10n}.\end{aligned}$$

4. Arrange the calculations in a table:

Event	Pr[Event]	Present Value (PV)	(PV)Pr[Event]	$(PV)^2$Pr[Event]
$K=0$	0.2	2.00	0.400	0.800
$K=1$	0.2	4.70	0.940	4.418
$K\geq 2$	0.6	7.94	4.764	37.826

$\mathrm{E}[PV] = 6.104$ and $\mathrm{E}[PV^2] = 43.044$. Hence, the variance is $43.044-(6.104)^2 = 5.785$.

5. $(I\bar{a})_{95} = \bar{a}_{95} + {}_{1|}\bar{a}_{95} + {}_{2|}\bar{a}_{95} + {}_{3|}\bar{a}_{95} + \cdots$. Since $\omega = 100$, $i = 0$ and $T(95)$ is uniform on $(0, 5)$, then the five non-zero terms are $\bar{a}_{95} = E[T(95)] = 2.5, {}_{1|}\bar{a}_{95} = p_{95}\bar{a}_{96} = (0.8)(2) = 1.6, {}_{2|}\bar{a}_{95} = {}_2p_{95}\bar{a}_{97} = (0.6)(1.5) = 0.9, {}_{3|}\bar{a}_{95} = {}_3p_{95}\bar{a}_{98} = (0.4)(1) = 0.4$ and ${}_{4|}\bar{a}_{95} = {}_4p_{95}\bar{a}_{99} = (0.2)(0.5) = 0.1$. Hence, the answer is 5.5. Alternatively, we can calculate expected present values conditionally on the year of death. There are five years of interest and they are equally likely. This yields $(0.5 + 2.0 + 4.5 + 8.0 + 12.5)/5 = 5.5$.

6. Use formula (4.3.9) or its equivalent in terms of commutation functions. ${}_{10|}\ddot{a}_{25:\overline{10}|} = D_{25}{}^{-1}(N_{35} - N_{45}) = 4.85456$, $(D_{35}-D_{45})/D_{25} = 0.24355$, $\alpha(12) = 1.00020$, and $\beta(12) = 0.46651$. Hence,

$${}_{10|}\ddot{a}^{(12)}_{25:\overline{10}|} = \alpha(12)\,{}_{10|}\ddot{a}_{25:\overline{10}|} - \beta(12)({}_{10}p_{25}v^{10} - {}_{20}p_{25}v^{20}) = 4.74191.$$

Another solution is based on formula (4.3.2) and (3.3.10), adjusted for temporary rather than whole life contracts:

$$\begin{aligned}A^{(12)}_{35:\overline{10}|} &= \frac{i}{i^{(12)}} A^{1}_{35:\overline{10}|} + {}_{10}p_{35}v^{10} \\ &= \frac{i}{i^{(12)}} (M_{35} - M_{45}) / D_{35} + D_{45}/D_{35} = 0.61814.\end{aligned}$$

Hence,

$$_{10|}\ddot{a}^{(12)}_{25:\overline{10}|} = (D_{35}/D_{25}) \left(1 - A^{(12)}_{35:\overline{10}|}\right) / d^{(12)} = 4.74200.$$

7. Use formula (4.3.9) to derive a formula analogous to formula (4.5.4) for temporary annuities. Then use the formulas analogous to (A.3.6) and (A.3.9) to write the result in terms of commutation functions:

$$\begin{aligned}(I\ddot{a})^{(m)}_{x:\overline{n}|} &= \alpha(m) \left((I\ddot{a})_{x:\overline{n}|}\right) - \beta(m) \left(\ddot{a}_{x:\overline{n}|} - nv^n {}_np_x\right) \\ &= \alpha(m) \frac{S_x - S_{x+n} - nN_{x+n}}{D_x} - \beta(m) \frac{N_x - N_{x+n} - nD_{x+n}}{D_x}\end{aligned}$$

Now calculate the N and D values by differencing the successive values of the given values of S. We need $N_{70} = S_{70} - S_{71} = 9597$, $N_{80} = S_{80} - S_{81} = 2184$, $D_{70} = N_{70} - N_{71} = 9597 - 8477 = 1120$, and $D_{80} = N_{80} - N_{81} = 368$. We get $(I\ddot{a})^{(12)}_{70:\overline{10}|} = 29.16$.

8.

$$\begin{aligned}&{}_np_x d\ddot{a}_{\overline{n}|} + d\sum_{k=0}^{n-1} \frac{(1 - v^{k+1})}{d} {}_kp_x q_{x+k} \\ &= d\left(\ddot{a}_{\overline{n}|} \Pr(K \geq n) + \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} \Pr(K = k)\right) \\ &= d\mathrm{E}\left(\ddot{a}_{\overline{\min(K+1,n)}|}\right) \\ &= d\ddot{a}_{x:\overline{n}|} = 1 - A_{x:\overline{n}|}\end{aligned}$$

9. Use formulas (4.2.9), (3.2.4) and (3.2.5). $\mathrm{Var}(Y) = d^{-2}\left(\mathrm{E}[e^{-2\delta(K+1)}] - A_x^2\right)$. Now use $A_x = 1 - d\ddot{a}_x$ twice. A_x evaluated at δ is $1 - (0.04)10 = 0.6$. The discount corresponding to 2δ is $1 - v^2 = 1 - (0.96)^2 = 0.0784$ so $\mathrm{E}[e^{-2\delta(K+1)}] =$ "A_x evaluated at 2δ," is $1 - (0.0784)(6) = 0.5296$. Therefore and $\mathrm{Var}(Y) = (0.5296 - (0.6)^2)/(0.04)^2 = 106$.

10. Use formulas (4.2.13) and (3.2.12).

11. Use formula (A.4.7). $N_{28} = S_{28} - S_{29} = 97$, $N_{29} = S_{29} - S_{30} = 93$, and $D_{28} = N_{28} - N_{29} = 4$. Hence, $M_{28} = 4 - (3/103)97 = 1.1748$ where we used the commutation function version of formula (4.2.8): $D_x = dN_x + M_x$.

12. Use the recursion relation $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$. $7.73 = 1 + (1.03)^{-1}p_{73}(7.43)$ so $p_{73} = (1.03)(6.73)/7.43 = 0.93$.

13. The values of the present value random variable Y are 2, $2 + 3v = 4.7273$, and $2 + 3v + 4v^2 = 8.0331$. Hence, $\Pr(Y > 4) = \Pr(K > 0) = 1 - 0.80 = 0.2$

14. Use formula (4.4.8) with $r(t) = 1$ if $0 \leq t < 1$ and $r(t) = 2$ for $t \geq 1$. Integration by parts applied to $\mathrm{E}(Y) = \int_0^{20} r(t)(1 - 0.05t)dt$ with $w(t) = \int_0^t r(s)ds$ yields $\mathrm{E}(Y) = 0.05\int_0^{20} w(t)dt = 19.025$. Alternatively, the annuity can be viewed as the sum of two annuities each having constant rate of payment of 1 per year. The first begins paying at age 80, the second at age 81. Using this approach we have $E(Y) = \bar{a}_{80} + vp_{80}\bar{a}_{81} = \mathrm{E}(T(80)) + 0.95\mathrm{E}(T(81))$ where $T(80)$ and $T(81)$ are uniformly distributed on $(0, 20)$ and $(0, 19)$, respectively. Again we get $\mathrm{E}(Y) = 10 + 0.95(9.5) = 19.025$.

15. Consider the sum of two annuities approach, as in exercise 14: $\mathrm{E}(Y) = \bar{a}_{80:\overline{5}|} + vp_{80}\bar{a}_{81:\overline{4}|} = \mathrm{E}[\min(T(80), 5)] + 0.95\mathrm{E}[\min(T(81), 4)] = (2.5)(0.25) + (5)(0.75) + (0.95)[(2)(4/19) + 4(15/19)] = 7.775$.

16. Since $\delta = 0$, $h = \mathrm{Var}(T) = \mathrm{E}(T^2) - (\mathrm{E}(T))^2$. Also $\mathrm{E}(T^2) = \int_0^\infty t^2 g(t)dt = 2\int_0^\infty t(1 - G(t))dt = 2g$ by parts integration. Hence, $\mathrm{E}(T) = \sqrt{2g - h}$.

17. $(D\ddot{a})_{70:\overline{10}|} = D_{70}^{-1}(10N_{70} - S_{71} + S_{81}) = 42.09$.

18.

$$
\begin{aligned}
\frac{a_{\overline{1}|}S_x - \ddot{a}_{\overline{2}|}S_{x+1} + \ddot{a}_{\overline{1}|}S_{x+2}}{D_x} &= \frac{vS_x - (1+v)S_{x+1} + S_{x+2}}{D_x} \\
&= \frac{vN_x - N_{x+1}}{D_x}.
\end{aligned}
$$

The formula (A.4.6) $C_y = vD_y - D_{y+1}$, summed over y running from x to the end of the table, gives $M_x = vN_x - N_{x+1}$, from which we see the simplification to A_x.

19. Let $Z = e^{-\delta T}$ and $Y = \bar{a}_{\overline{T}|} = \delta^{-1}(1 - Z)$. From the given data, we find that $\mathrm{E}(Z) = 1 - 10\delta$, $\mathrm{E}(Z^2) = 1 - 14.75\delta$, and $50 = \mathrm{Var}(Y) = \delta^{-2}\left(\mathrm{E}(Z^2) - \mathrm{E}(Z)^2\right)$. First solve for $\delta = 3.5\%$. Then $\bar{A}_x = 1 - \delta\bar{a}_x = 0.65$.

20. Apply formula (4.8.9) to obtain $A_{35.75} = 0.17509$. Apply formula (3.3.5) to obtain $\bar{A}_{35.75} = 0.18046$. Then $\bar{a}_{35.75} = (1 - \bar{A}_{35.75})/\delta = 16.79725$.

D.4.2 Solutions to Spreadsheet Exercises

1. Set up your spreadsheet to calculate the required annuity value with reference to a single age and interest rate. Use VLOOKUP() references to the mortality values, which may be on a separate sheet. Then use the Data Table feature to calculate the array of values for x running down a column and i accross a row. Check values: $\ddot{a}_{30} = 18.058$ at $i = 5\%$ and $\ddot{a}_{20} = 13.753$ at $i = 7.5\%$.

2. The expected market value is 309,153.

3. Guide: Set up a table to calculate A_x and $\mathrm{E}[Z^2]$ with a reference to a single value of c. Use formula (3.2.4). Then use the Table feature to allow for different values of c.

4. For $i = 5\%$, $\ddot{a}^{(2)}_{24.5} = 18.3831$ and $A_{24.5} = 0.11255$. For $i = 6\%$, $\ddot{a}^{(12)}_{30.25} = 15.37108$ and $A_{30.25} = 0.10369$.

5. Guide: From the Illustrative Life Table set up a table with the cummulative distribution function of K. Fill a column with 200 random numbers from the interval [0, 1] using RAND(). Use the VLOOKUP() function to find the corresponding value of K. Evaluate Y for each value of K, then calculate the sample mean and variance using the built-in functions. The theoretical answers are $\ddot{a}_{40} = 16.632$ and $\mathrm{Var}(Y) = 10.65022$.

D.5 Net Premiums:Solutions

D.5.1 Theory Exercises

1. Use formula (5.3.15): $P_{25:\overline{20}|} = P^1_{25:\overline{20}|} + P_{25:\overline{20}|}^{\ \ \ 1} = 0.064$. Now use the relation ${}_{20}P_{25} = P^1_{25:\overline{20}|} + P_{25:\overline{20}|}^{\ \ \ 1} A_{45} = 0.046$ to obtain

$$P_{25:\overline{20}|}^{\ \ \ 1} = (0.064 - 0.046)/(1 - 0.64) = 0.05.$$

Therefore $P^1_{25:\overline{20}|} = 0.064 - 0.05 = 0.014$.

2. $L^* = v^{K+1} - G\ddot{a}_{\overline{K+1}|} = -G/d + (1 + G/d)v^{K+1}$ and hence

$$\mathrm{Var}[L^*] = (1 + G/d)^2\mathrm{Var}[v^{K+1}] = (d + G)^2 d^{-2}\mathrm{Var}[v^{K+1}].$$

Similarly,

$$L = v^{K+1} - P_x\ddot{a}_{\overline{K+1}|} = -P_x/d + (1 + P_x/d)v^{K+1}$$

and

$$\mathrm{Var}[L] = (d + P_x)^2 d^{-2}\mathrm{Var}[v^{K+1}] = 0.30.$$

Now use $\mathrm{E}[L] = 0$ and $\mathrm{E}[L^*] = -0.20$ to find that $0 = A_x + P_x\ddot{a}_x = 1 - (d + P_x)\ddot{a}_x$ and $-0.2 = 1 - (d + G)\ddot{a}_x$. Hence

$$\begin{aligned}\mathrm{Var}[L^*] &= (d + G)^2 d^{-2}\mathrm{Var}[v^{K+1}] = 0.30(d + G)^2/(d + P_x)^2\\ &= 0.30[(d + G)/(d + P_x)]^2 = 0.432.\end{aligned}$$

3. Let P denote the net annual premium.

$$\begin{aligned}P &= \frac{5.000(10A_{25} + {}_{5|}A_{25} + {}_{10|}A_{25} + {}_{15|}A_{25} + {}_{20|}A_{25} + {}_{25|}A_{25}}{\ddot{a}_{25\,:\,\overline{10}|}}\\ &= \frac{5.000(10M_{25} + M_{30} + M_{35} + M_{40} + M_{45} + M_{50}}{N_{25} - N_{35}}\\ &= 1012.33.\end{aligned}$$

4.

$$\begin{aligned}\text{Loss A} &= 4v^{K+1} - 0.18\ddot{a}_{\overline{K+1}|}\\ &= -0.18/d + (4 + 0.18/d)v^{K+1} = -2.25 + 6.25v^{K+1}\end{aligned}$$

Using the table we find that Var[Loss A] $= 3.25 = (6.25)^2\mathrm{Var}[v^{K+1}]$. Similarly, Loss B $= 6v^{K+1} - 0.22\ddot{a}_{\overline{K+1}|} = -2.75 + 8.75v^{K+1}$ and Var[Loss B] $= (8.75)^2\mathrm{Var}[v^{K+1}] = (8.75)^2(3.25)/(6.25)^2 = 6.37$.

5. Let Q be the answer. The death benefit is $10,000 + Q/2$ initially. After the premiums are paid up it reduces to 10,000. Hence, the expected value of the present value of benefits is $10,000A_x + (Q/2)\, A^{1}_{x:\overline{20}|}$. The expected present value of premiums is $Q\ddot{a}_{x:\overline{20}|}$. The equivalence principle implies that

$$Q = \frac{10,000A_x}{\ddot{a}_{x:\overline{20}|} - (\,A^{1}_{x:\overline{20}|})/2}.$$

Now rearrange the denominator.

$$\begin{aligned}\ddot{a}_{x:\overline{20}|} - (\,A^{1}_{x:\overline{20}|})/2 &= \ddot{a}_{x:\overline{20}|} - (1 - d\ddot{a}_{x:\overline{20}|} - A_{x:\overset{1}{\overline{20}|}})/2\\ &= (1 + d/2)\ddot{a}_{x:\overline{20}|} - (1 - v^{20}{}_{20}P_x)/2.\end{aligned}$$

Hence,

$$Q = \frac{10,000A_x}{(1 + d/2)\ddot{a}_{x:\overline{20}|} - (1 - v^{20}{}_{20}P_x)/2}.$$

6. $A_x = {}_nP_x\ddot{a}_{x:\overline{n}|} = {}_nP_x(1 - A_{x:\overline{n}|})/d$. Hence ${}_nP_x = dA_x/(1 - A_{x:\overline{n}|})$.

7. The present value of death benefits is $1000(1.06)v$ in year $1, 10000(1.06)^2v^2$ in year 2, *etc.* Since $v = 1/1.06$, the present value of death benefits is 1000, independent of the year of death. Let Q be the net annual premium. The expected present value of net annual premiums is $Q\ddot{a}_x$. Hence $Q = 1,000/\ddot{a}_x = 1000(d + P_x) = 1000(0.06/1.06) + 1000P_x = 66.60$.

8. Solve the two equations:

$A_x = A^{1}_{x:\overline{20}|} + v^n{}_nP_xA_{x+20}$ and $A_{x:\overline{20}|} = A^{1}_{x:\overline{20}|} = v^n{}_mP_x$ for $v^n{}_mP_x = 0.5$ and $A^{1}_{x:\overline{20}|} = 0.05$.

Use $A_{x:\overline{20}|} = 1 - d\ddot{a}_{x:\overline{20}|}$ to find $\ddot{a}_{x:\overline{20}|} = 15.45$. This yields $1000P\left(\bar{A}_{x:\overline{20}|}\right) = 1000\left(\bar{A}^{1}_{x:\overline{20}|} = v^n{}_mP_x\right)/\ddot{a}_{x:\overline{20}|} = 1000\left[(\frac{i}{\delta})A^{1}_{x:\overline{20}|} + v^n{}_mP_x\right]/\ddot{a}_{x:\overline{20}|} = 35.65$

9. $L = v^T - \bar{P}(\bar{A}_x) = -\bar{P}(\bar{A}_x)/\delta + (\delta + \bar{P}(\bar{A}_x))(\delta)^{-1}v^T$. By the continuous payment analog of formula (5.3.5), we have $\bar{a}_x(\delta + \bar{P}(\bar{A}_x)) = 1$. Hence, $\mathrm{Var}(L) = \mathrm{Var}(v^T)/(\delta\bar{a}_x)^2$. By exercise 7 of chapter 3,

$$\mathrm{Var}[v^T] = \mathrm{E}[v^{2T}] - (\mathrm{E}[v^T])^2 = \frac{\mu\delta^2}{(2\delta + \mu)(\delta + \mu)^2}.$$

Finally, $\mathrm{Var}(L) = \mu/(2\delta + \mu)$ since $\bar{a}_x = 1/(\delta + \mu)$.

10. (a) The insurer's loss random variable is the present value of benefits less the present value of premiums. The death benefit payable at the moment of death is $1 + G\bar{s}_{\overline{T}|}$, provided $T < 10$. The present value of death benefits is

$v^T(1+G\bar{s}_{\overline{T}|}) = v^T + G\bar{a}_{\overline{T}|}$ if $T < 10$ and 0 if $T \geq 10$. The present value of premiums is $G\bar{a}_{\overline{T}|}$ if $T < 10$ and $G\bar{a}_{\overline{10}|}$ if $T \geq 10$. Therefore, the insurer's loss is

$$L = \begin{cases} v^T & \text{if } T < 10 \\ -G\bar{a}_{\overline{10}|} & \text{otherwise.} \end{cases}$$

(b) $\mathrm{E}[L] = \mathrm{E}[v^T|T < 10]{}_{10}q_x + (-G\bar{a}_{\overline{10}|}){}_{10}p_x = \bar{A}^1_{x:\overline{10}|} - G\bar{a}_{\overline{10}|}\,{}_{10}p_x$.

$E[L^2] = E[v^{2T}|T < 10]{}_{10}q_x + (-G\bar{a}_{\overline{10}|})^2{}_{10}p_x = {}^2\bar{A}^1_{x:\overline{10}|} - G^2(\bar{a}_{\overline{10}|})^2{}_{10}p_x$.

$\mathrm{Var}[L] = {}^2\bar{A}^1_{x:\overline{10}|} + (G\bar{a}_{\overline{10}|})^2{}_{10}p_x - (\bar{A}^1_{x:\overline{10}|} - G\bar{a}_{\overline{10}|}\,{}_{10}p_x)^2$.

(c) If G is determined by the equivalence principle, then $G = \bar{A}^1_{x:\overline{10}|}/(\bar{a}_{\overline{10}|}\,{}_{10}p_x)$. This can be substituted into the last expression to find

$\mathrm{Var}[L] = {}^2\bar{A}^1_{x:\overline{10}|} + (\bar{A}^1_{x:\overline{10}|})^2/{}_{10}p_x$.

11. $\ddot{a}_{30:\overline{10}|} = 1 + a_{30:\overline{9}|} = 6.6$. Hence $A_{30:\overline{10}|} = 1 - d\ddot{a}_{30:\overline{10}|} = 0.4$ Therefore $A^1_{30:\overline{10}|} = A_{30:\overline{10}|} - v^{10}{}_{10}P_{30} = 0.4 - 0.035 = 0.05$. Hence, $1000P^1_{30:\overline{10}|} = 1000A^1_{30:\overline{10}|}/\ddot{a}_{30:\overline{10}|} = 7.58$.

12. $A_x = 0.2$ and $\bar{A}_x = (i/\delta)Ax = 0.2049593$ by *assumption a*. Hence, $\ddot{a}_x = (1-A_x)/d = 16.8$ and hence $10000P(\bar{A}_x) = 10000\bar{A}_x/\ddot{a}_x = 2{,}049.59/16.8 = 122$. Also, $\bar{a}_x = (1-\bar{A}_x)/\delta = 16.2951$. Hence, $10000\bar{P}(\bar{A}_x) = 2049.59/16.2951 = 125.78$ and the answer is 3.78.

13.

$$\begin{aligned} 1 - \frac{\left(P_{30:\overline{15}|} - {}_{15}P_{30}\right)\ddot{a}_{30:\overline{15}|}}{v^{15}({}_{15}P_{30})} &= 1 - \frac{A_{30:\overline{15}|} - A_{30}}{{}_{15}E_{30}} \\ &= 1 - \frac{A_{30:\overline{15}|} - A_{30}}{A_{30:\overset{1}{\overline{15}|}}} \\ &= \frac{A_{30:\overset{1}{\overline{15}|}} - A_{30:\overline{15}|} + A_{30}}{A_{30:\overset{1}{\overline{15}|}}} \\ &= \frac{A_{30} - A^1_{30:\overline{15}|}}{A_{30:\overset{1}{\overline{15}|}}} \\ &= \frac{M_{30} - (M_{30} - M_{45})}{D_{45}} = A_{45}. \end{aligned}$$

14. Under *assumption a*, $A_x = \frac{\delta}{i}\bar{A}_x = \frac{0.07}{e^{0.07}-1}0.03 = 0.289622$.

$\ddot{a}_x = \dfrac{1 - Ax}{d} = 10.507587$

Using 14.3.47, $\ddot{a}_x^{(2)} = d^{(2)}\ddot{a}_x - \beta^{(2)} = 10.252457$.

Therefore, $1000\left(0.5P^{(2)}(\bar{A}_x)\right) = \dfrac{500\bar{A}_x}{\ddot{a}_x^{(2)}} = 14.63$.

15. $\mathrm{Var}[L] = 0.25$ where $L = v^T - \bar{P}\left(\bar{A}_x\right)\bar{a}_{\overline{T}|} = \left(\delta\bar{a}_x\right)^{-1}v^T - \bar{P}\left(\bar{A}_x\right)/\delta$. Therefore $\mathrm{Var}[L] = \mathrm{Var}[v^T]/\left(\delta\bar{a}_x\right)^2 = (100/36)\mathrm{Var}[v^T]$. Hence $0.25 = \mathrm{Var}[L] = (100/36)\mathrm{Var}[v^T]$ and $\mathrm{Var}[v^T] = 0.09$. $L^* = v^T - G\bar{a}_{\overline{T}|} = (1 + G/\delta)v^T - G/\delta$ and $\mathrm{Var}[L^*] = (11/6)^2\mathrm{Var}[v^T]$ and hence $\mathrm{Var}[L^*] = (11/6)^2\mathrm{Var}[v^T] = 0.3025$.

16. Let $P = \dfrac{A_{40:\overline{20}|}}{k}$ = net annual premium. Then $P\ddot{a}_{40:\overline{20}|} = A_{40:\overline{20}|} + E[W]$ where $W = Pv^{K+1}\ddot{s}_{\overline{K+1}|}$ if $K < 10$ and $W = 0$ if $K \geq 10$. Since $E[W] = P\left(\ddot{a}_{40:\overline{10}|} - {}_{10}E_{40}\ddot{s}_{\overline{10}|}\right)$, $P\left(\ddot{a}_{40:\overline{20}|} - \ddot{a}_{40:\overline{10}|} + {}_{10}E_{40}\ddot{s}_{\overline{10}|}\right) = A_{40:\overline{20}|}$ and therefore

$$\begin{aligned} k &= \ddot{a}_{40:\overline{20}|} - \ddot{a}_{40:\overline{10}|} + {}_{10}E_{40}\ddot{s}_{\overline{10}|} \\ &= \left(\ddot{a}_{50:\overline{10}|} + \ddot{s}_{\overline{10}|}\right){}_{10}E_{40}. \end{aligned}$$

17. Let P denote the annual premium. Then $L = -P - Pv$ with probability $(0.9)(0.8) = 0.72, L = v^2 - P - Pv$ with probability $(0.9)(0.2) = 0.18$ and $L = v - P$ with probability 0.1. By the equivalence principle, $P = \left(vq_x + v^2 p_x q_{x+1}\right)/\left(1 + vp_x\right) = 0.13027621$. Thus the values of L are $L = -0.2475$ with probability 0.72. $L = 0.5625$ with probability 0.18 and $L = 0.7697$ with probability 0.1. Hence $\mathrm{Var}[L] = E[L^2] = (-0.2475)^2(0.72) + (0.5625)^2(0.18) + (0.7697)^2(0.1) = 0.160$

18. $1{,}000\bar{P}\left(\bar{A}_{x:\overline{n}|}\right) = 1{,}000\bar{A}_{x:\overline{n}|}/\bar{a}_{x:\overline{n}|}$. $0.4275 = \bar{A}^{1}_{x:\overline{n}|} = \int_0^n \mu e^{-(\delta+\mu)t}dt = 0.45\left(1 - e^{-0.1n}\right)$ and hence $e^{-0.1n} = 0.05$. Therefore $\bar{A}_{x:\overline{n}|} = 0.05 + 0.4275 = 0.4775$ and $\bar{a}_{x:\overline{n}|} = (1-0.4775)/0.055 = 9.5$. $1{,}000\bar{P}\left(\bar{A}_{x:\overline{n}|}\right) = 1{,}000(0.4775)/9.5 = 50.26$.

19. The loan payment for a loan of 1,000 is $P = 1{,}000/a_{\overline{4}|} = 288.60$. The death benefit paid at $K+1$ is $b_{K+1} = P\ddot{a}_{\overline{4-K}|}$ if $K < 4$ and $b_{K+1} = 0$ otherwise. The present value of the death benefit is $Z = v^{K+1}b_{K+1} = Pv^{K+1}\left(1 - v^{4-K}\right)/d$ if $K < 4$ and $Z = 0$ if $K \geq 4$.

(a) $L = Z - G = P\left(v^{K+1} - v^5\right)/d - G$ if $K < 4$ and $L = -G$ if $K \geq 4$.

(b) $0 = E[L] = E[Z] - G = P\left(A^{1}_{25:\overline{4}|} - v^5{}_4q_{25}\right)/d - G$.

So now calculate as follows: $A^{1}_{25:\overline{4}|} = A_{25:\overline{4}|} - v^4{}_4p_{25} = 1 - d\ddot{a}_{25:\overline{4}|} - v^4{}_4p_{25} = 0.0043$. $G = 288.6(0.0043 - 0.00373629)/d = 2.87$.

(c) Let G^* denote the additional amount of the borrowing to pay for term insurance and L^* denote the loss random variable in this case.

$$L^* = \begin{cases} \frac{10{,}000+G^*}{a_{\overline{4}|}}(v^{k+1} - v^5)\frac{1}{d} - G^* & (k < 4) \\ -G^* & \text{(otherwise)} \end{cases}$$

$$E[L^*] = \frac{10,000 + G^*}{a_{\overline{4}|}d}(A^1_{25:\overline{4}|} - v^5{}_4q_{25}) - G^* = 0$$

Using the result in (ii), $A^1_{25:\overline{4}|} - v^5{}_4q_{25} = 0.0005638$. Therefore $(10{,}000 + G^*)(0.0028745) - G^* = 0$, and $G^* = 28.827866$. The annual payments is $(10,000{+}28,827866)/a_{\overline{4}|} = 2,894.23$.

20. (a) $L = v^{K+1} - G\ddot{a}_{\overline{K+1}|} = (1 + G/d)v^{K+1} - G/d$ where K is the curtate lifetime of (x).

(b) $\mathrm{E}[L] = A_x - G\ddot{a}_x = (1 + G/d)A_x - G/d = -0.08$ and
$\mathrm{Var}[L] = (1 + G/d)^2({}^2A_x - A_x^2) = 0.0496$.

(c) 0.05 = probability of loss $= \Pr[S > 0]$ where $S = L_1 + \ldots + L_N$ and the L_i are independent and distributed like L. Thus $E[S] = NE[L] = N(-0.08)$ and $\mathrm{Var}[S] = N\mathrm{Var}[L] = N(0.0496)$ and, using the normal approximation, we have $0.95 = Pr[S \le 0] = Pr(Z \le (0 - N(-0.08))/(0.0496N)^{1/2}) = Pr(Z \le N^{1/2}(0.3592))$. This gives $1.645 = N^{1/2}(0.3592)$ and $N = 20.97$. So a portfolio of $N = 21$ would have a probability of a loss of a little less than 0.05.

D.5.2 Solutions to Spreadsheet Exercises

1. Check value: $S_0 = 21,834,463$.

2. For $i = 5\%$, the premium is 0.0253. For $i = 8\%$, it is 0.0138.

Guide: Set up a spreadsheet with survival probabilities, increasing benefits and increasing premiums. Calculate the expected present value of benefits and premiums and use the solver feature to find the initial premium so that the difference is 0.

3. Some values of the premium P are as follows:

$P = 36675.49$ with $C = 500,000, a = 10^{-6}$, and

$P = 43920.03$ with $C = 500,000, a = 10^{-5}$,

$P = 375046$ with $C = 1,000,000, a = 10^{-5}$.

Guide: Set up a table with columns for the present value of benefits and present value of premiums for each of the ten years. Find L and $U(-L)$ and calculate $\mathrm{E}[U(-L)]$. Use the solver feature to find the premium so that formula (5.2.9) holds.

4. P=3.0807.

Guide: Set up a table with probabilities of survival, death benefits and premiums. Set the death benefits according to the refund condition. Set the expected present value of premiums and benefits equal by using the solver function and letting the premium vary. Try it with different death probabilities for the first five years.

5. Guide: Set up a spreadsheet with survival probabilities and present value of premiums. Find the balance and present the value of benefits. Use the solver feature to find the premium that equates the present value of premiums to the present value of benefits.

6. $z = 8.5\%$ for $j = 3\%$ and $z = 14.3\%$ for $j = 6\%$

Guide: Set up a spreadsheet with survival probabilities and cash flows. Cash flows consist of savings during the pre-retirement period and payments during the retirement period. Use the solver feature to find z so that the expected NPV of the cash flows at 5% is 0.

7. The ratio is 1.0601.

D.6 Net Premium Reserves:Solutions

D.6.1 Theory Exercises

1. $P_{35:\overline{20}|} = 0.03067, P_{40:\overline{15}|} = 0.04631$, Reduced Paid-up $= 337.84$.

2. Use the formula

$$\begin{aligned} 1 - {}_{20}V_{25} &= \frac{\ddot{a}_{45}}{\ddot{a}_{25}} \\ &= \frac{\ddot{a}_{35}}{\ddot{a}_{25}}\frac{\ddot{a}_{45}}{\ddot{a}_{35}} \\ &= (1 - {}_{10}V_{25})(1 - {}_{10}V_{35}) = 0.72 \end{aligned}$$

and ${}_{20}V_{25} = 0.28$.

3. ${}_{20}\bar{V}\left(\bar{A}_{40}\right) - {}_{20}V\left(\bar{A}_{40}\right) = 1 - \frac{\bar{a}_{60}}{\bar{a}_{40}} - {}_{20}V\left(\bar{A}_{40}\right) = 1 - \frac{12.25}{20} - 0.3847 = 0.0028.$

4.

k	0	1	2
${}_{k+1}V$	0.8889	1.7984	2.2186.

5. $\ddot{a}_{x+1} = 1.14151, \ddot{a}_x = 1.37691$, Answer: 0.171.

6. Answer: 258.31

7. Use

$$\begin{aligned} \mathrm{Var}(L) &= \mathrm{Var}\left[v^{K+1} - P_x\frac{1 - v^{K+1}}{d}\right] \\ &= (1 + P_x/d)^2 \,\mathrm{Var}\left[v^{K+1}\right] \\ &= \frac{1}{(1 - A_x)^2}\left(\mathrm{E}[e^{-(K+1)2\delta}] - \left(\mathrm{E}[e^{-(K+1)\delta}]\right)^2\right) \end{aligned}$$

Calculate the moment generating function of $K + 1$,

$$M(-s) = A_x = \sum_{k=0}^{\infty} e^{-(k+1)s}{}_{k|}q_x = \sum_{k=0}^{\infty} v^{k+1}(0.5)^{k+1} = \frac{v(0.5)}{1 - v(0.5)} = \frac{v}{2 - v}$$

where $v = e^{-s}$. Obtain $E\left[e^{-(K+1)\delta}\right]$ by setting $s = \delta$, and $E\left[e^{-(K+1)2\delta}\right]$ by setting $s = 2\delta$. Substitute and simplify. It is not easy.

8. Answer: 4.88.

9. Answer: 480.95.

10. Answer 644.50.

11. Premium = 7.92, Reserve = 33.72.

12. $\Lambda_1 = \begin{cases} 0 & \text{with probability } 0.23 \\ v - {}_1V - 1 & \text{with probability } 0.2 \\ {}_2Vv - {}_1V - 1 & \text{with probability } 0.6 \end{cases}$

$\mathrm{E}[\Lambda_1] = 0$, and $\mathrm{Var}[\Lambda_1] = 0.1754$

13. Answer: 0.2841.

14. Use formula (6.7.9): $\mathrm{Var}[\Lambda_9] = v^2(1.000)^2(1 - {}_{10}V_{40})^2{}_9p_{40}p_{49}q_{49} =$

$(\frac{1}{1.05})^2(1.000)^2(14.63606/16.632258)^2\, \frac{8,950,994}{9,313,144}0.00546 = 3,685.83$

15. The recursion relation between the two reserves is especially simple since we are beyond the premium paying period: $0.585(1.04) = 0.600p_38 + q_{38}$ and hence $p_{38} = [1 - (1.04)(0.585)] \div (0.4) = 0.979$.

16. Use ${}_tV_x = 1 - (P_x + d)\,\ddot{a}_{x+t}$ to solve for $d = \frac{1}{11}$ and $i = 0.10$.

17. Answer: 18.77.

18. Answer: 3.99.

19. (a) $1000_{10.5}V_x = 0.5(311 + 340.86) + 0.5(60) = 355.93$.

(b)

$$1000_{10.5}V_x = 1000(v^{0.5}{}_{0.5}p_{x+10+0.5}\,({}_{11}V_x)) + 1000v^{0.5}{}_{0.5}q_{x+10+0.5}$$

Use *assumption a* to obtain ${}_{0.5}q_{x+10+0.5} = \dfrac{0.5q_{x+10}}{1 - 0.5q_{x+10}} = \dfrac{4}{96}$. This yields $1000_{10.5}V_x = 1000(1.03)^{-1}\left(\frac{4}{96}\right) + (1.03)^{-1}\left(\frac{92}{96}\right)(340.86) = 357.80$.

20. Answer: 0.058

D.6.2 Solutions to Spreadsheet Exercises

1. ${}_{10}V_{30} = 0.09541$ with $i = 4\%$

${}_{15}V_{30} = 0.11002$ with $i = 6\%$

Guide: Use formula (6.5.4).

2. For $i = 6\%, \Pi_3^s = 0.0706$ and $\Pi_0^r = 0.0052$.

For $i = 4\%, \Pi_3^s = 0.0791$ and $\Pi_0^r = 0.0052$.

3. (a.) $G_1 = 15.6$

$G_5 = 20.1$

(b.) accumulated gain = 98.5

(c.) $i' = 23.24\%$

4. $(S_1)\pi_1, \ldots, \pi_4 = 231.09$ and $G_2 = 361.03$

(S_2) the present values of gains is 1,396.5

Guide: Set up a table with premiums, benefits, revenues and gains. Use formula (6.9.1) to calculate the gains. Use the solver feature to find Π, so that ${}_5V = 0$ with $\Pi_1 = \Pi_2 = \Pi_3 = \Pi_4 = \Pi$.

D.7 Multiple Decrements:Solutions

D.7.1 Theory Exercises

1. ${}_tp_x = \exp\left(-\int_0^t \mu_{x+s}ds\right) = e^{-0.03t}$ $q_{1,x} = \int_0^t g_1(t)dt = \int_0^1 {}_tp_x\mu_{1,x+t}dt =$ 0.00985.

2. Because T is exponential with parameter $\mu = 0.04$,

$$E[T \mid J=3] = \frac{1}{Pr(J=3)}\int_0^\infty t_tp_x\mu_{3,x+t}dt$$

$$= \frac{1}{Pr(J=3)}\frac{3}{150}\int_0^\infty t_tp_xdt$$

$$= \frac{1}{Pr(J=3)}\frac{3}{150}\frac{150}{6}\int_0^\infty t_tp_x\mu_{x+t}dt$$

$$= \frac{1}{Pr(J=3)}\frac{1}{2}E[T] = \frac{1}{Pr(J=3)}\frac{1}{2}\left(\frac{1}{0.04}\right).$$

Also $\Pr(J=3) = \mu_3/\mu = 0.5$, hence $E[T|J=3] = E[T] = 25$.

3. The present value of future benefits is

$$1000\int_0^\infty (3g_1(t) + 2g_2(t) + g_3(t))\, e^{-t\delta}dt = 1000\int_0^\infty 0.12e^{-0.07t}e^{-0.03t}dt = 1200$$

The net annual premium paid continuously, P, satisfies $P\bar{a}_x = 1200$. Since T is exponentially distributed with parameter 0.07, then $\bar{a}_x = 1/(0.07+0.03) = 10$. Hence, $P = 120$.

4. $\int_0^t \mu_{x+s}ds = 2t - \log(t+1)$ and so ${}_tp_x = (t+1)e^{-2t}$. $\int_0^1 {}_tp_xdt = 0.75 - 1.25e^{-2}$.

$$m_x = \frac{1-2e^{-2}}{0.75-1.25e^{-2}} = 1.25567.$$

5. The NSP is equal to

$$\int_0^2 2_tp_x\mu_{1,x+t}dt + \int_0^2 {}_tp_x\mu_{2,x+t}dt = \int_0^2 \left(\frac{2t}{20} + \frac{t}{10}\right){}_tp_xdt$$

Since ${}_tp_x = \exp\left(-\int_0^t \frac{3s}{20}ds\right) = \exp\left(\frac{-3t^2}{40}\right)$, then the NSP is given by

$$\frac{2}{10}\int_0^2 t\exp\left(\frac{-3t^2}{40}\right)dt = \frac{4}{3}\left(1-e^{-0.3}\right) = 0.3456.$$

6. Apply equation (7.3.6) three times. Use the relation $q_{1,30} + q_{2,30} + q_{3,30} = q_{30}$ to solve for $q_{30} = 0.375$.

7. (a) 10000(0.84)(0.02) = 168 (b) 10000(1 - 0.01 - 0.25)(0.02) = 148.

8. $500_{2|}q_{2,63} = 500p_{63}p_{64}q_{2,65}$ Now calculate in order:

(i) $p_{63} = 1 - q_{1,63} - q_{2,63} = 0.45$

(ii) $p_{63}q_{64} = {}_{1|}q_{63} = 0.07$

(iii $q_{64} = \frac{7}{45}$ and ${}_2p_{63} = 0.38$

(iv) ${}_2p_{63}q_{1,65} = 0.042$ so $q_{1,65} = \frac{0.042}{0.38}$

(v) $q_{65} = 1$ and so $q_{2,65} = \frac{0.338}{0.38}$

Hence, $500_{2|}q_{2,63} = 500(0.338) = 169.$

9. $0.02 = \dfrac{q_{1,x}}{1 - 0.5q_x}$ and $q_{1,x} + q_{2,x} = q_x$ so $1000q_x = 19.703$.

10. The NSP is given by

$$\begin{aligned}
\int_0^\infty c_1 v^t {}_tp_x \mu_{1,x+t} dt + \int_0^\infty c_2 v^t {}_tp_x \mu_{2,x+t} dt \\
&= c_1\mu_1\bar{a}_x + c_2 \int_0^\infty v^t {}_tp_x \left(\mu_{x+t} - \mu_{1,x+t}\right) dt \\
&= c_1\mu_1\bar{a}_x + c_2\bar{A}_x - c_2\mu_1\bar{a}_x.
\end{aligned}$$

Hence, the net annual premium is $\text{NSP}/\bar{a}_x = (c_1 - c_2)\mu_1 + c_2\bar{P}_x$.

D.8 Multiple Life Insurance: Solutions

D.8.1 Theory Exercises

1. Apply equations (2.6.3) and (2.6.4).

$$\begin{aligned} q^{1}_{80:81} &= \int_0^1 {}_tp_{80:81}\mu_{80+t}dt \\ &= \int_0^1 {}_tp_{80}{}_tp_{81}\left(\mu_{80+t}\right)dt \\ &= \int_0^1 {}_tp_{81}q_{80}dt \\ &= q_{80}(1-0.5q_{81}) = 0.3125. \end{aligned}$$

Similarly

$$\begin{aligned} q_{80:\overset{2}{81}} &= \int_0^1 (1-{}_tp_{80})\,{}_tp_{81}\left(\mu_{81+t}\right)dt \\ &= 0.5q_{80}q_{81} \\ &= 0.1875 \end{aligned}$$

$$\begin{aligned} q_{80:81} &= q^{1}_{80:81} + q_{80:\overset{1}{81}} \\ &= q_{80}(1-0.5q_{81}) + q_{81}(1-0.5q_{80}) \\ &= q_{80} + q_{81} - q_{80}q_{81} \\ &= 0.875 \end{aligned}$$

$$\begin{aligned} q_{\overline{80:81}} &= q_{80}q_{81} \\ &= 0.325. \end{aligned}$$

2.

$$\begin{aligned} \bar{A}^{2}_{xy} &= \int_0^\infty v^t(1-{}_tp_y){}_tp_x\mu_{x+t}dt \\ &= \int_0^\infty e^{-\delta t}(1-e^{-\mu_x t})e^{-\mu_y t}\mu_x dt \\ &= \mu_x\left(\frac{1}{\delta+\mu_x} - \frac{1}{\delta+\mu_x+\mu_y}\right) \\ &= 0.1167. \end{aligned}$$

3. For non-smokers,

$${}_tp_x = \frac{75-x-t}{75-x} \text{ for } 0 \le t \le 75-x.$$

Let $'$ denote the mortality functions for smokers. Since $\mu'_z = 2\mu_z, z \geq 0$, then

$$\begin{aligned} {}_tp'_x &= \exp(-\int_x^{x+1} \mu'_z dz) \\ &= \exp\left(-2\int_x^{x+1} \mu_z dz\right) \\ &= ({}_tp_x)^2 \end{aligned}$$

Hence

$$\begin{aligned} \mathring{e}_{65:55} &= \int_0^{10} {}_tp_{65}\,{}_tp'_{55}dt \\ &= \int_0^{10} {}_tp_{65}\left({}_tp_{55}\right)^2 dt \\ &= \int_0^{10} \left(\frac{10-t}{10}\right)\left(\frac{20-t}{20}\right)^2 dt \\ &= 3\frac{13}{24}. \end{aligned}$$

4. From the equation obtained from the equivalence principle, we have

$$10,000\bar{A}_{\overline{xy}} = c\bar{a}_x + 0.5c\left(\bar{a}_y - \bar{a}_{xy}\right).$$

Use $\bar{A}_{\overline{xy}} = 1 - \delta\bar{a}_{\overline{xy}}$ and $\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}$ together with the given values to determine $c = 103.45$.

5. Let Π denote the answer. By the equivalence principle,

$$\Pi\ddot{a}_{xx} = A_{\overline{xx}}.$$

Since $\ddot{a}_x = 1 + a_x = 10$ and $A_x = 1 - d\ddot{a}_x$, then $d = 0.06$. Since $A_{xx} = 1 - d\ddot{a}_{xx}$, then $\ddot{a}_{xx} = 7.5$. Since $A_{xx} + A_{\overline{xx}} = 2A_x$, $A_{\overline{xx}} = 0.25$. Hence $\Pi = \frac{1}{30}$.

6. The net single premium is

$$0.5\,\bar{A}^1_{xy} + \bar{A}_{\overline{xy}} \;=\; 0.5\,\bar{A}^1_{xy} + \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

Now use the following result from the discussion of Gompertz' Law in section 8.3:

$$\begin{aligned} \bar{A}^1_{xy} &= \int_0^{\infty} v^t{}_tp_{xy}\mu_{x+t}dt \\ &= \int_0^{\infty} v^t{}_tp_{xy}Bc^{x+t}dt \\ &= \left(\frac{c^x}{c^x+c^y}\right)\int_0^{\infty} v^t{}_tp_{xy}Bc^{w+t}dt \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{c^x}{c^x+c^y}\right)\int_0^\infty v^t\,{}_tp_{xy}\mu_{w+t}dt \\
&= \left(\frac{c^x}{c^x+c^y}\right)\bar{A}_w \\
&= c^{x-w}\bar{A}_w
\end{aligned}
$$

where $c^w = c^x + c^y$. The solution now follows easily by using $\bar{A}_{xy} = \bar{A}_w$.

7.

$$
\begin{aligned}
{}_\infty q^1_{50M:50F} &= \int_0^\infty {}_tp_{50M}\,{}_tp_{50F}\,\mu^F_{50+t}dt \\
&= \int_0^{50} e^{-0.04t}\frac{1}{50}dt \\
&= 0.4323.
\end{aligned}
$$

Therefore, ${}_\infty q^2_{50M:50F} = 1 - {}_\infty q^1_{50M:50F} = 0.5677.$

8.

$$
\begin{aligned}
\mathrm{E}[Z] &= \bar{A}^2_{y:x} \\
&= \int_0^\infty (1-{}_tp_x)\,{}_tp_y\mu_y dt \\
&= \frac{\mu_y}{\delta+\mu_y} - \frac{\mu_y}{\delta+\mu_y+\mu_x} \\
&= \frac{21}{110} \\
&= 0.1909091 \\
\mathrm{E}[Z^2] &= {}^2\bar{A}^2_{y:x} \\
&= \frac{\mu_y}{2\delta+\mu_y} - \frac{\mu_y}{2\delta+\mu_y+\mu_x} \\
&= \frac{6}{63} \\
&= 0.0952381 \\
\mathrm{Var}[Z] &= 0.0587918
\end{aligned}
$$

9. Let Π denote the initial premium.

$$
\begin{aligned}
1000A_{\overline{25:25}} &= 0.4\Pi\ddot{a}_{25:25} + 0.6\Pi\ddot{a}_{\overline{25:25}} \\
\Pi &= \frac{1000\,(2A_{25} - A_{25:25})}{1.2\ddot{a}_{25} - 0.2\ddot{a}_{25:25}} \\
&= 3.5349.
\end{aligned}
$$

10.

$$
\mathrm{E}[(i)] = \int_0^\infty v^t\left({}_tp_x\right)^{\mathrm{Paul}}\left({}_tp_x\mu_{x+t}\right)^{\mathrm{John}}dt
$$

$$
\begin{aligned}
&= \bar{A}^{1}_{xx} \\
\mathrm{E}[(ii)] &= 2\mathrm{E}[(i)] \\
\mathrm{E}[(iii)] &= 3\int_0^\infty v^t \left(1 - {}_tp_x\right)^{\text{Paul}} \left({}_tp_x \mu_{x+t}\right)^{\text{John}} dt \\
&= 3\left(\bar{A}_x - \bar{A}^{1}_{xx}\right) \\
\mathrm{E}[(iv)] &= 4\left(\bar{A}_x - \bar{A}^{1}_{xx}\right) \\
\text{Total} &= 7\bar{A}_x - 4\,\bar{A}^{1}_{xx} = 7\bar{A}_x - 2\bar{A}_{xx}.
\end{aligned}
$$

D.8.2 Solutions to Spreadsheet Exercises

1. Check values: $a_{65:60} = 7.9479$, $a_{\overline{65:60}} = 12.7011$, and $a_{65/60} = 3.10736$.

Guide: Set up a table with values for ${}_tp_{60}$ and ${}_tp_{65}$. Use formulas (8.2.3) and (8.7.3) to calculate the corresponding joint and last surviver probabilities. Calulate the annuities based on this probabilities. Use the discrete version of (8.7.6) to calculate the reversionary annuity.

2. The variance is 589.772.

Guide: From the Illustrative Life Table, calculate probabilities of survival for the last-survivor status. See the guide to exercise 1. Find the present value for the annuities certain and calculate the expected present value. Use formula (4.2.9).

3. $P = 0.0078$

Guide: Calculate probabilities of survival for the joint and last-survivor status. Find the insurance single premium for the last survivor status and the annuity for the joint status.

4. ${}_3V = 0.0258$ ${}_7V = 0.2247$

Guide: Use equation (6.3.4) to find the reserve for the first five years. Recognize that after the death of (40), the reserves are the net single premiums for (35).

5. $\bar{a}_{30:40} \approx 7.02017$ and $\bar{A}^{1}_{30:40} \approx 0.14558$

Guide: For a Makeham law, the formula (7.3.4) provides ${}_tp_x$. Integrate numerically to find $\bar{a}_x$ for typical values of x, A, B and c in four cells. Now use the fact that $\mu_{30:40}(t) = A' + Bc^{w+t} = \mu_{w+t}$ where $A' = 2A$ and $c^w = c^x + c^y$, so $\bar{a}_{30:40} = \bar{a}_w$ with these values in the appropriate cells: $x = \log\left(c^{30} + c^{40}\right) / \log c = 41.58$, $c = 1.15$, $A = 0.008$, and $B = 0.0001$. To obtain $\bar{A}^{1}_{30:40}$ either evaluate an integral numerically or use the relations:

$$\bar{A}^{1}_{xy} = \frac{c^x}{c^w}\left(\bar{A}_{xy} - A(1 - c^{y-x})\bar{a}_{xy}\right) \text{ and } \bar{A}_{xy} = 1 - \delta\bar{a}_{xy}$$

D.9 The Total Claim Amount in a Portfolio

D.9.1 Theory Exercises

1. $\mathrm{E}[S_1] = 1000(0.001) + 100(0.15) = 16$ and $\mathrm{Var}[S_1] = 1000^2(0.001) + 100^2(0.15) - 16^2 = 2244$. $\mathrm{E}[S] = 5\mathrm{E}[S^1] = 80$ and $\mathrm{Var}[S] = 5\mathrm{Var}[S_1] = 11,220$. $\Pr(S > 200) = 1 - \Phi((200-80)/\sqrt{(}11,220))) \approx 1 - \Phi(1.1329) \approx 0.1286$. The exact value is $\Pr(S > 200) = 1 - \Pr(S=0) - \Pr(S=100) - \Pr(S=200) = 1 - (0.849)^5 - 5(0.15)(0.849)^4 - 10(0.15)^2(0.849)^3 \approx 0.032$.

2. $\rho(\beta) = E[(S-\beta)^+] = \sigma\int_k^\infty (x-k)\phi(x)dx$ where $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ and $k = (\beta-\mu)/\sigma$. Hence $\rho(\beta) = \sigma\int_k^\infty x\phi(x)dx - \sigma k(1-\Phi(k))$. Now use the fact that $x\phi(x) = -\phi'(x)$ to obtain

$$\rho(\beta) = \sigma\phi(k) - \sigma k(1-\Phi(k)) = \sigma\phi(-k) + (\mu-\beta)\Phi(-k)$$

$$= \sigma\phi(\frac{\beta-\mu}{\sigma}) + (\mu-\beta)\Phi(\frac{\mu-\beta}{\sigma}).$$

3. $M_S(t) = \sum_{k=0}^{\infty} \mathrm{E}[\exp(t\,(X_1+\cdots+X_k)]\Pr(N=k) = \sum_{k=0}^{\infty}(M_X(t))^k \Pr(N=k)$

$= \sum_{k=0}^{\infty} \exp(k\log(M_X(t)))\Pr(N=k) = M_N(\log(M_X(t)))$

Differentiate to get the moment relations:

$M_S'(t) = M_N'(\log(M_X(t))\dfrac{M_X'(t)}{M_X(t)}$ and

$$M_S''(t) = M_N''(\log(M_X(t)))\left(\frac{M_X'(t)}{M_X(t)}\right)^2 + M_N'(\log(M_x(t)))\left[\frac{M_X''(t)M_X(t) - (M_X'(t))^2}{M_x(t)^2}\right]$$ Evaluate with $t=0$.

4. $\mathrm{E}[R] = \displaystyle\int_\beta^\gamma [1-F(x)]dx$

5. (a) $F(x) = 1 + \rho'(x), x \geq 0$

(b) $F(x) = 1 - \left(1 + \frac{1}{2}x + \frac{1}{4}x^2\right)e^{-x}$ and $f(x) = \left(\frac{1}{2} + \frac{1}{4}x^2\right)e^{-x}$

6. e^{-6}

7. $\rho(\beta) = e^{\mu+\sigma^2/2}\left[1 - \Phi\left(\frac{\log\beta - \mu - \sigma^2}{\sigma}\right)\right] - \beta\left[1 - \Phi\left(\frac{\log\beta-\mu}{\sigma}\right)\right]$

8. (a)

x	$Pr(S_1 + S_2 = x)$	$Pr(S_1 + S_2 + S_3 = x)$	$F(x)$
0	0.56	0.336	0.336
0.5	0.07	0.042	0.378
1.0	0.08	0.048	0.426
1.5	0.09	0.166	0.592
2.0	0.01	0.020	0.612
2.5	0.15	0.162	0.774
3.0	0.01	0.087	0.861
3.5	0.01	0.023	0.884
4.0	0.01	0.053	0.937
4.5		0.012	0.949
5.0	0.01	0.024	0.973
5.5		0.018	0.991
6.0		0.002	0.993
6.5		0.004	0.997
7.0		0.001	0.998
7.5		0.001	0.999
8.0		0.001	1.000

(b)

x	$f(x)$	$F(x)$
0	0.4066	0.4066
0.5	0.0407	0.4472
1.0	0.0427	0.4899
1.5	0.1261	0.6160
2.0	0.1364	0.7524
2.5	0.0659	0.8183
3.0	0.0365	0.8548
3.5	0.0446	0.8994
4.0	0.0372	0.9366
4.5	0.0214	0.9580
5.0	0.0126	0.9707
5.5	0.0101	0.9807
6.0	0.0075	0.9882
6.5	0.0045	0.9927
7.0	0.0026	0.9954
7.5	0.0018	0.9971
8.0	0.0012	0.9983

D.10 Expense Loadings

D.10.1 Theory Exercises

1.a.

Development of Endowment Reserves

k	Net Premium	Expense-Loaded
0	0	
1	77	31
2	158	116
3	244	206
4	335	302
5	431	403
6	532	509
7	639	621
8	752	740
9	873	867

1.b. 83.60

2. $-_kV^\alpha$ is the unamortized acquisition expense at the end of policy year k.

3.a. The expense-loaded premium is 22.32

Development of Term Insurance Reserves

k	Net Premium	Expense-Loaded
0	0.0	
1	1.3	-35.6
2	2.3	-31.3
3	3.1	-27.1
4	3.7	-22.9
5	4.0	-18.8
6	3.9	-14.8
7	3.6	-10.9
8	2.8	-7.10
9	1.6	-3.50

3.b. The one-year net cost of insurance is $1000vq_x = 16.03$. The first year loading is 22.32 - 16.03 = 6.29. The acquisition expense is 40, requiring an investment of 40 - 6.29 = 33.71.

4. $1000P = 11.06, 1000P^\alpha = 0.78, 1000P^\beta = 2.29$, and $1000P^\gamma = 1.127$. The expense-loaded premium is $1000P^a = 15.25$.

5. The first 10 years of reserves are given below. They can be developed easily using a spreadsheet program and the recursion relations:

k	q_{x+k}	$1000{}_kV$	$1000{}_kV^{\alpha}$	$1000{}_kV^{\gamma}$	$1000{}_kV^{a}$
0	0.00201	0.00	-12.00	0.00	-12.00
1	0.00214	9.63	-11.81	0.13	-2.05
2	0.00228	19.62	-11.61	0.27	8.29
3	0.00243	30.01	-11.40	0.42	19.03
4	0.00260	40.80	-11.18	0.58	30.19
5	0.00278	51.99	-10.95	0.74	41.78
6	0.00298	63.60	-10.71	0.91	53.80
7	0.00320	75.64	-10.47	1.10	66.27
8	0.00344	88.12	-10.21	1.29	79.20
9	0.00371	101.05	-9.94	1.49	92.61
10	0.00400	114.43	-9.65	1.71	106.49

$$
{}_kV^{\gamma} = \begin{cases} vp_{x+k}\ {}_{k+1}V^{\gamma} + \gamma - P^{\gamma} & \text{for } 0 \le k \le 29 \\ vp_{x+k}\ {}_{k+1}V^{\gamma} + \gamma & \text{for } 30 \le k \le 64 \\ \gamma & \text{for } k = 64 \end{cases}
$$

$$
{}_kV^{\alpha} = \begin{cases} vp_{x+k}\ {}_{k+1}V^{\alpha} - P^{\alpha} & \text{for } 0 \le k \le 29 \\ 0 & \text{for } k \ge 30 \end{cases}
$$

$$
{}_kV^{a} = \begin{cases} v\left(p_{x+k}\ {}_{k+1}V^{a} + 1000q_{x+k}\right) - (1-\beta)P^{a} + \gamma & \text{for } 0 \le k \le 29 \\ v\left(p_{x+k}\ {}_{k+1}V^{a} + 1000q_{x+k}\right) + \gamma & \text{for } 30 \le k \le 64 \\ \gamma + 1000v & \text{for } k = 64 \end{cases}
$$

D.10.2 Spreadsheet Exercises

1. Guide: Use the recursion formulas. Here are the results: $1000P = 28.42$, $1000P^{\alpha} = 1.70$, $1000P^{\beta} = 1.74$, and $1000P^{\gamma} = 3.0$. The expense-loaded premium is $1000P^{a} = 34.68$.

k	q_{x+k}	1000_kV	1000_kV^{α}	1000_kV^{γ}	1000_kV^{a}
0	0.00278	0.00	-20.00	0.00	-20.00
1	0.00298	27.42	-19.45	0.00	7.97
2	0.00320	56.38	-18.87	0.00	37.51
3	0.00344	86.97	-18.26	0.00	68.71
4	0.00371	119.28	-17.61	0.00	101.67
5	0.00400	153.42	-16.93	0.00	136.49
6	0.00431	189.51	-16.21	0.00	173.30
7	0.00466	227.68	-15.45	0.00	212.23
8	0.00504	268.06	-14.64	0.00	253.42
9	0.00546	310.79	-13.78	0.00	297.01
10	0.00592	356.05	-12.88	0.00	343.17
11	0.00642	404.01	-11.92	0.00	392.09
12	0.00697	454.88	-10.90	0.00	443.98
13	0.00758	508.87	-9.82	0.00	499.05
14	0.00824	566.24	-8.68	0.00	557.57
15	0.00896	627.27	-7.45	0.00	619.82
16	0.00975	692.28	-6.15	0.00	686.12
17	0.01062	761.62	-4.77	0.00	756.85
18	0.01158	835.69	-3.29	0.00	832.41
19	0.01262	914.98	-1.70	0.00	913.28

D.11 Estimating Probabilities of Death

D.11.1 Theory Exercises

1. (a) (0.00553, 0.00981)
(b) (0.00682, 0.00811)

2. 0.007388, 0.00112, 0.007360

3.a.

$$\sum_{k=n}^{\infty} \frac{(\lambda^l)^k}{k!} e^{-\lambda^l} = w$$

or

$$\sum_{k=0}^{n-1} \frac{(\lambda^l)^k}{k!} e^{-\lambda^l} = 1 - w$$

b.

$$\sum_{k=0}^{n} \frac{(\lambda^u)^k}{k!} e^{-\lambda^u} = w$$

c.

$$\int_{\lambda^l}^{\infty} f(x;n)dx = 1 - w$$

$$\int_{\lambda^u}^{\infty} f(x;n+1)dx = w$$

4. 0.7326, 1.365 years

5. 0.0346

6. The classical estimator is $4/147 \approx 0.02720$. The MLE is $1 - \exp(-4/145) \approx 0.02722$.

7. $\hat{\mu} = 1/9.1$

8. $\hat{\mu} = \frac{90}{1000}, \hat{q} = 1 - \bar{e}^{\hat{\mu}} = 0.0861.$

$$\hat{q}_1 = \frac{4}{9}\hat{q} = 0.0383$$

$$\hat{q}_2 = \frac{5}{9}\hat{q} = 0.0478$$

9. Observed deaths = 45. Expected deaths =19.4 from the Illustrative Life Table. From the table in section 11.5, we get a 90% confidence interval $34.56 \leq \lambda \leq 57.69$. Therefore $\hat{f} = 45/19.4 = 2.32$ and the 90% interval is (1.78, 2.97).

Appendix E

Tables

E.0 Illustrative Life Tables

Basic Functions and Net Single Premiums at $i = 5\%$ [1]

x	ℓ_x	d_x	$1000q_x$	$\ddot{a}_x$	$1000A_x$	$1000\left({}^2A_x\right)$	x
0	10,000,000	204,200	20.42	19.642724	64.63	28.72	0
1	9,795,800	13,126	1.34	19.982912	48.43	11.47	1
2	9,782,674	11,935	1.22	19.958801	49.58	11.33	2
3	9,770,739	10,943	1.12	19.931058	50.90	11.28	3
4	9,759,796	10,150	1.04	19.899898	52.39	11.33	4
5	9,749,646	9,555	0.98	19.865552	54.02	11.46	5
6	9,740,091	9,058	0.93	19.828262	55.80	11.67	6
7	9,731,033	8,661	0.89	19.788078	57.71	11.95	7
8	9,722,372	8,458	0.87	19.745056	59.76	12.29	8
9	9,713,914	8,257	0.85	19.699446	61.93	12.69	9
10	9,705,657	8,250	0.85	19.651122	64.23	13.15	10
11	9,697,407	8,243	0.85	19.600339	66.65	13.66	11
12	9,689,164	8,333	0.86	19.546971	69.19	14.23	12
13	9,680,831	8,422	0.87	19.491083	71.85	14.84	13
14	9,672,409	8,608	0.89	19.432543	74.64	15.50	14
15	9,663,801	8,794	0.91	19.371410	77.55	16.21	15
16	9,655,007	8,979	0.93	19.307550	80.59	16.98	16
17	9,646,028	9,164	0.95	19.240821	83.77	17.81	17
18	9,636,864	9,348	0.97	19.171075	87.09	18.70	18
19	9,627,516	9,628	1.00	19.098154	90.56	19.67	19
20	9,617,888	9,906	1.03	19.022085	94.19	20.71	20
21	9,607,982	10,184	1.06	18.942699	97.97	21.82	21
22	9,597,798	10,558	1.10	18.859825	101.91	23.02	22
23	9,587,240	10,929	1.14	18.773468	106.03	24.31	23
24	9,576,311	11,300	1.18	18.683440	110.31	25.69	24
25	9,565,011	11,669	1.22	18.589547	114.78	27.17	25
26	9,553,342	12,133	1.27	18.491584	119.45	28.77	26
27	9,541,209	12,690	1.33	18.389518	124.31	30.49	27
28	9,528,519	13,245	1.39	18.283311	129.37	32.33	28
29	9,515,274	13,892	1.46	18.172737	134.63	34.30	29
30	9,501,382	14,537	1.53	18.057738	140.11	36.41	30
31	9,486,845	15,274	1.61	17.938070	145.81	38.67	31
32	9,471,571	16,102	1.70	17.813654	151.73	41.09	32
33	9,455,469	16,925	1.79	17.684401	157.89	43.68	33
34	9,438,544	17,933	1.90	17.550034	164.28	46.45	34
35	9,420,611	18,935	2.01	17.410616	170.92	49.40	35
36	9,401,676	20,120	2.14	17.265850	177.82	52.56	36
37	9,381,556	21,390	2.28	17.115771	184.96	55.93	37
38	9,360,166	22,745	2.43	16.960229	192.37	59.52	38
39	9,337,421	24,277	2.60	16.799062	200.04	63.35	39
40	9,313,144	25,891	2.78	16.632259	207.99	67.41	40
41	9,287,253	27,676	2.98	16.459630	216.21	71.74	41
42	9,259,577	29,631	3.20	16.281129	224.71	76.34	42
43	9,229,946	31,751	3.44	16.096696	233.49	81.23	43
44	9,198,195	34,125	3.71	15.906249	242.56	86.41	44
45	9,164,070	36,656	4.00	15.709844	251.91	91.90	45
46	9,127,414	39,339	4.31	15.507365	261.55	97.71	46
47	9,088,075	42,350	4.66	15.298671	271.49	103.86	47
48	9,045,725	45,590	5.04	15.083893	281.72	110.36	48
49	9,000,135	49,141	5.46	14.862997	292.24	117.23	49

Basic Functions and Net Single Premiums at $i = 5\%$

x	ℓ_x	d_x	$1000q_x$	$\ddot{a}_x$	$1000A_x$	$1000\left({}^2A_x\right)$	x
50	8,950,994	52,990	5.92	14.636061	303.04	124.46	50
51	8,898,004	57,125	6.42	14.403131	314.14	132.08	51
52	8,840,879	61,621	6.97	14.164220	325.51	140.10	52
53	8,779,258	66,547	7.58	13.919449	337.17	148.53	53
54	8,712,711	71,793	8.24	13.669034	349.09	157.36	54
55	8,640,918	77,423	8.96	13.413011	361.29	166.63	55
56	8,563,495	83,494	9.75	13.151498	373.74	176.33	56
57	8,480,001	90,058	10.62	12.884699	386.44	186.47	57
58	8,389,943	97,156	11.58	12.612883	399.39	197.05	58
59	8,292,787	104,655	12.62	12.336384	412.55	208.08	59
60	8,188,132	112,669	13.76	12.055342	425.94	219.56	60
61	8,075,463	121,213	15.01	11.770064	439.52	231.49	61
62	7,954,250	130,291	16.38	11.480898	453.29	243.87	62
63	7,823,959	139,892	17.88	11.188205	467.23	256.69	63
64	7,684,067	149,993	19.52	10.892369	481.32	269.94	64
65	7,534,074	160,626	21.32	10.593780	495.53	283.63	65
66	7,373,448	171,728	23.29	10.292913	509.86	297.73	66
67	7,201,720	183,212	25.44	9.990230	524.27	312.23	67
68	7,018,508	195,044	27.79	9.686158	538.75	327.11	68
69	6,823,464	207,229	30.37	9.381167	553.28	342.37	69
70	6,616,235	219,527	33.18	9.075861	567.82	357.96	70
71	6,396,708	231,945	36.26	8.770664	582.35	373.88	71
72	6,164,763	244,248	39.62	8.466182	596.85	390.09	72
73	5,920,515	256,358	43.30	8.162908	611.29	406.56	73
74	5,664,157	267,971	47.31	7.861452	625.65	423.26	74
75	5,396,186	278,929	51.69	7.562297	639.89	440.15	75
76	5,117,257	288,972	56.47	7.265989	654.00	457.21	76
77	4,828,285	297,809	61.68	6.973063	667.95	474.38	77
78	4,530,476	305,218	67.37	6.683979	681.71	491.66	78
79	4,225,258	310,810	73.56	6.399299	695.27	508.97	79
80	3,914,448	314,330	80.30	6.119411	708.60	526.31	80
81	3,600,118	315,514	87.64	5.844708	721.68	543.60	81
82	3,284,604	314,041	95.61	5.575590	734.50	560.84	82
83	2,970,563	309,770	104.28	5.312277	747.03	577.99	83
84	2,660,793	302,506	113.69	5.055031	759.28	594.95	84
85	2,358,287	292,168	123.89	4.803941	771.24	611.84	85
86	2,066,119	278,802	134.94	4.558938	782.91	628.49	86
87	1,787,317	262,539	146.89	4.319797	794.29	645.05	87
88	1,524,778	243,675	159.81	4.085991	805.43	661.36	88
89	1,281,103	222,592	173.75	3.856599	816.35	677.76	89
90	1,058,511	199,815	188.77	3.630178	827.13	694.07	90
91	858,696	175,973	204.93	3.404320	837.89	710.54	91
92	682,723	151,749	222.27	3.175241	848.80	727.33	92
93	530,974	127,890	240.86	2.936743	860.15	745.41	93
94	403,084	105,096	260.73	2.678823	872.44	764.65	94
95	297,988	84,006	281.91	2.384461	886.46	787.86	95
96	213,982	74,894	350.00	2.024369	903.60	817.60	96
97	139,088	66,067	475.00	1.654770	921.21	847.96	97
98	73,021	49,289	675.00	1.309539	937.65	885.10	98
99	23,732	23,732	1000.00	1.000000	952.35	914.67	99

E.1 Commutation Columns

Illustrative Life Table and $i = 5\%$

x	D_x	N_x	C_x	M_x	x
0	10,000,000.0	196,427,244.2	194,476.190	646,321.725	0
1	9,329,333.3	186,427,244.2	11,905.669	451,845.535	1
2	8,873,173.7	177,097,910.9	10,309.902	439,939.866	2
3	8,440,331.7	168,224,737.2	9,002.833	429,629.964	3
4	8,029,408.3	159,784,405.5	7,952.791	420,627.131	4
5	7,639,102.8	151,754,997.2	7,130.088	412,674.340	5
6	7,268,205.9	144,115,894.4	6,437.351	405,544.252	6
7	6,915,663.5	136,847,688.5	5,862.106	399,106.901	7
8	6,580,484.1	129,932,025.0	5,452.102	393,244.795	8
9	6,261,675.6	123,351,540.9	5,069.082	387,792.693	9
10	5,958,431.5	117,089,865.3	4,823.604	382,723.611	10
11	5,669,873.0	111,131,433.8	4,590.011	377,900.007	11
12	5,395,289.1	105,461,560.8	4,419.168	373,309.996	12
13	5,133,951.4	100,066,271.7	4,253.682	368,890.828	13
14	4,885,223.8	94,932,320.3	4,140.595	364,637.146	14
15	4,648,453.5	90,047,096.5	4,028.633	360,496.551	15
16	4,423,070.0	85,398,643.0	3,917.508	356,467.918	16
17	4,208,530.1	80,975,573.0	3,807.831	352,550.410	17
18	4,004,316.0	76,767,042.9	3,699.321	348,742.579	18
19	3,809,935.0	72,762,726.9	3,628.692	345,043.258	19
20	3,624,880.8	68,952,791.9	3,555.683	341,414.566	20
21	3,448,711.8	65,327,911.1	3,481.399	337,858.883	21
22	3,281,006.0	61,879,199.3	3,437.382	334,377.484	22
23	3,121,330.2	58,598,193.3	3,388.732	330,940.102	23
24	2,969,306.7	55,476,863.1	3,336.921	327,551.370	24
25	2,824,574.3	52,507,556.4	3,281.798	324,214.449	25
26	2,686,788.9	49,682,982.1	3,249.804	320,932.651	26
27	2,555,596.8	46,996,193.2	3,237.138	317,682.847	27
28	2,430,664.6	44,440,596.4	3,217.824	314,445.709	28
29	2,311,700.8	42,009,931.8	3,214.296	311,227.885	29
30	2,198,405.5	39,698,231.0	3,203.366	308,013.589	30
31	2,090,516.2	37,499,825.5	3,205.496	304,810.223	31
32	1,987,762.3	35,409,309.3	3,218.348	301,604.727	32
33	1,889,888.6	33,421,547.0	3,221.755	298,386.379	33
34	1,796,672.2	31,531,658.4	3,251.079	295,164.624	34
35	1,707,865.3	29,734,986.2	3,269.268	291,913.545	35
36	1,623,269.1	28,027,120.9	3,308.445	288,644.277	36
37	1,542,662.1	26,403,851.8	3,349.789	285,335.832	37
38	1,465,852.2	24,861,189.7	3,392.370	281,986.043	38
39	1,392,657.4	23,395,337.5	3,448.443	278,593.673	39
40	1,322,891.9	22,002,680.1	3,502.576	275,145.230	40
41	1,256,394.5	20,679,788.2	3,565.765	271,642.654	41
42	1,193,000.4	19,423,393.7	3,635.854	268,076.889	42
43	1,132,555.0	18,230,393.3	3,710.464	264,441.035	43
44	1,074,913.3	17,097,838.3	3,797.993	260,730.571	44
45	1,019,929.0	16,022,925.0	3,885.414	256,932.578	45
46	967,475.5	15,002,996.0	3,971.241	253,047.164	46
47	917,434.0	14,035,520.5	4,071.618	249,075.923	47
48	869,675.1	13,118,086.5	4,174.399	245,004.305	48
49	824,087.6	12,248,411.4	4,285.278	240,829.906	49

Illustrative Life Table and $i = 5\%$

x	D_x	N_x	C_x	M_x	x
50	780,560.0	11,424,323.8	4,400.881	236,544.628	50
51	738,989.6	10,643,763.8	4,518.379	232,143.747	51
52	699,281.3	9,904,774.2	4,641.901	227,625.368	52
53	661,340.3	9,205,492.9	4,774.263	222,983.467	53
54	625,073.6	8,544,152.6	4,905.357	218,209.204	54
55	590,402.8	7,919,079.0	5,038.129	213,303.847	55
56	557,250.3	7,328,676.2	5,174.462	208,265.718	56
57	525,540.1	6,771,425.9	5,315.486	203,091.256	57
58	495,198.9	6,245,885.8	5,461.362	197,775.770	58
59	466,156.6	5,750,686.9	5,602.760	192,314.408	59
60	438,355.9	5,284,530.3	5,744.566	186,711.648	60
61	411,737.3	4,846,174.4	5,885.897	180,967.082	61
62	386,244.8	4,434,437.1	6,025.437	175,081.185	62
63	361,826.8	4,048,192.3	6,161.376	169,055.748	63
64	338,435.6	3,686,365.5	6,291.679	162,894.372	64
65	316,027.9	3,347,929.9	6,416.853	156,602.693	65
66	294,562.1	3,031,902.0	6,533.683	150,185.840	66
67	274,001.7	2,737,339.9	6,638.678	143,652.157	67
68	254,315.3	2,463,338.2	6,730.866	137,013.479	68
69	235,474.2	2,209,022.9	6,810.823	130,282.613	69
70	217,450.3	1,973,548.7	6,871.439	123,471.790	70
71	200,224.1	1,756,098.4	6,914.416	116,600.351	71
72	183,775.2	1,555,874.3	6,934.453	109,685.935	72
73	168,089.5	1,372,099.1	6,931.684	102,751.482	73
74	153,153.6	1,204,009.6	6,900.656	95,819.798	74
75	138,959.9	1,050,856.0	6,840.801	88,919.142	75
76	125,502.0	911,896.1	6,749.626	82,078.341	76
77	112,776.0	786,394.1	6,624.796	75,328.715	77
78	100,781.0	673,618.1	6,466.295	68,703.919	78
79	89,515.6	572,837.1	6,271.206	62,237.624	79
80	78,981.7	483,321.5	6,040.218	55,966.418	80
81	69,180.5	404,339.8	5,774.257	49,926.200	81
82	60,111.9	335,159.3	5,473.619	44,151.943	82
83	51,775.8	275,047.4	5,142.073	38,678.324	83
84	44,168.2	223,271.6	4,782.374	33,536.251	84
85	37,282.6	179,103.4	4,398.990	28,753.877	85
86	31,108.3	141,820.8	3,997.853	24,354.887	86
87	25,629.1	110,712.5	3,585.383	20,357.034	87
88	20,823.2	85,083.4	3,169.300	16,771.651	88
89	16,662.4	64,260.2	2,757.228	13,602.351	89
90	13,111.7	47,597.8	2,357.230	10,845.123	90
91	10,130.1	34,486.1	1,977.109	8,487.893	91
92	7,670.6	24,356.0	1,623.757	6,510.784	92
93	5,681.6	16,685.4	1,303.294	4,887.027	93
94	4,107.7	11,003.8	1,020.006	3,583.733	94
95	2,892.1	6,896.1	776.493	2,563.727	95
96	1,977.9	4,004.0	659.303	1,787.234	96
97	1,224.4	2,026.1	553.902	1,127.931	97
98	612.2	801.7	393.559	574.029	98
99	189.5	189.5	180.470	180.470	99

E.2 Multiple Decrement Tables

Illustrative Service Table

x	ℓ_x	$d_{1,x}$	$d_{2,x}$	$d_{3,x}$	$d_{4,x}$
30	100,000	100	19,990	0	0
31	79,910	80	14,376	0	0
32	65,454	72	9,858	0	0
33	55,524	61	5,702	0	0
34	49,761	60	3,971	0	0
35	45,730	64	2,693	46	0
36	42,927	64	1,927	43	0
37	40,893	65	1,431	45	0
38	39,352	71	1,181	47	0
39	38,053	72	989	49	0
40	36,943	78	813	52	0
41	36,000	83	720	54	0
42	35,143	91	633	56	0
43	34,363	96	550	58	0
44	33,659	104	505	61	0
45	32,989	112	462	66	0
46	32,349	123	421	71	0
47	31,734	133	413	79	0
48	31,109	143	373	87	0
49	30,506	156	336	95	0
50	29,919	168	299	102	0
51	29,350	182	293	112	0
52	28,763	198	259	121	0
53	28,185	209	251	132	0
54	27,593	226	218	143	0
55	27,006	240	213	157	0
56	26,396	259	182	169	0
57	25,786	276	178	183	0
58	25,149	297	148	199	0
59	24,505	316	120	213	0
60	23,856	313	0	0	3,552
61	19,991	298	0	0	1,587
62	18,106	284	0	0	2,692
63	15,130	271	0	0	1,350
64	13,509	257	0	0	2,006
65	11,246	204	0	0	4,448
66	6,594	147	0	0	1,302
67	5,145	119	0	0	1,522
68	3,504	83	0	0	1,381
69	2,040	49	0	0	1,004
70	987	17	0	0	970

References

A thorough introduction into the theory of compound interest is given in Butcher-Nesbitt [3]. The textbook by Bowers-Gerber-Hickman-Jones-Nesbitt [2] is the natural reference for Chapters 2–10; it contains numerous examples and exercises. The "classical method" of Chapter 11 is documented in Batten [1] and updated in Hoem [7]. In this respect the reader may also orient himself with the text of Elandt-Johnson [6].

An extensive bibliography is given by Wolthuis-van Hoek [14].

The classical texts in life insurance mathematics are those by Zwinggi [15], Saxer [12] and Jordan [9]; the newer book by Wolff [13] is of impressive completeness. The monographs by Isenbart-Münzner [8] and Neill [10] are written in the traditional style; however, the former book may appeal to non-mathematicians. The three volumes by Reichel [11] have an unconventional approach and will appeal to the mathematically minded reader. The books by De Vylder [4] and De Vylder-Jaumain [5] give a very elegant presentation of the subject.

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