Modeling Space Shuttle Speeds Using Fuzzy Logic

1 Problem:

1. Define Fuzzy Sets

We define five fuzzy sets to represent space shuttle speeds: **Very Low**, **Low**, **Middle**, **High**, and **Very High**. Each fuzzy set is defined over the universe of discourse: speed $x \in [0, 5000]$ mph.

1. **Very Low:**
   (a) Membership function:

$$\mu_{\text{Very Low}}(x) = \begin{cases} 1 & x \leq 500 \\ \frac{1000-x}{500} & 500 < x \leq 1000 \\ 0 & x > 1000 \end{cases}$$

   (b) Range: $x \in [0, 1000]$
   (c) Behavior: The membership is 1 at or below 500 mph and linearly decreases to 0 by 1000 mph.

2. **Low:**
   (a) Membership function:

$$\mu_{\text{Low}}(x) = \begin{cases} 0 & x \leq 500 \text{ or } x \geq 2000 \\ \frac{x-500}{500} & 500 < x \leq 1000 \\ \frac{2000-x}{1000} & 1000 < x < 2000 \end{cases}$$

   (b) Range: $x \in [500, 2000]$
   (c) Behavior: Increases from 0 to 1 between 500 and 1000 mph, then decreases to 0 by 2000 mph.

3. **Middle:**

   (a) Membership function:

$$\mu_{\text{Middle}}(x) = \begin{cases} 0 & x \leq 1500 \text{ or } x \geq 3000 \\ \frac{x-1500}{500} & 1500 < x \leq 2000 \\ 1 & 2000 < x < 2500 \\ \frac{3000-x}{500} & 2500 \leq x < 3000 \end{cases}$$

   (b) Range: $x \in [1500, 3000]$

   (c) Behavior: Triangular behavior with plateau at full membership between 2000 and 2500 mph.

4. **High:**

   (a) Membership function:

$$\mu_{\text{High}}(x) = \begin{cases} 0 & x \leq 2000 \\ \frac{x-2000}{1000} & 2000 < x \leq 3000 \\ \frac{4000-x}{1000} & 3000 < x \leq 4000 \\ 0 & x > 4000 \end{cases}$$

   (b) Range: $x \in [2000, 4000]$

   (c) Behavior: Symmetrical triangular peak at 3000 mph.

5. **Very High:**

   (a) Membership function:

$$\mu_{\text{Very High}}(x) = \begin{cases} 0 & x < 3500 \\ \frac{x-3500}{500} & 3500 \leq x \leq 4000 \\ 1 & x > 4000 \end{cases}$$

   (b) Range: $x \in [3500, 5000]$

   (c) Behavior: Increases linearly to full membership by 4000 mph, remains 1 afterward.

2. Fuzzy Intersection (AND Operation)

Compute the fuzzy intersection using Zadeh's algebraic product:

$$\mu_{\text{Very Low}}(2500) = 0, \quad \mu_{\text{High}}(2500) = \frac{2500 - 2000}{1000} = 0.5$$

$$\mu_{\text{Very Low} \cap \text{High}}(2500) = \mu_{\text{Very Low}}(2500) \cdot \mu_{\text{High}}(2500) = 0 \cdot 0.5 = 0$$

Interpretation: At 2500 mph, the speed is not considered Very Low at all, so the fuzzy intersection is zero even if it is somewhat High.

3. Fuzzy Complement

Let $\mu_{\text{High}}(x)$ be the membership function of set High. Its complement is:

$$\mu_{\text{NOT High}}(x) = 1 - \mu_{\text{High}}(x)$$

1. At $x = 2500$: $\mu_{\text{High}}(2500) = 0.5 \Rightarrow \mu_{\text{NOT High}}(2500) = 0.5$
2. At $x = 1000$: $\mu_{\text{High}}(1000) = 0 \Rightarrow \mu_{\text{NOT High}}(1000) = 1$
3. At $x = 3000$: $\mu_{\text{High}}(3000) = 1 \Rightarrow \mu_{\text{NOT High}}(3000) = 0$

Interpretation: The complement flips the membership degree, giving full membership where High is zero, and none where High is fully true.

4. Fuzzy Union (OR Operation)

Use the algebraic sum (Zadeh's OR operation):

$$\mu_{A \cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

At $x = 1500$:

$$\mu_{\text{Very Low}}(1500) = 0, \quad \mu_{\text{NOT High}}(1500) = 1 - \mu_{\text{High}}(1500) = 1$$

$$\mu_{\text{Very Low} \cup \text{NOT High}}(1500) = 0 + 1 - 0 \cdot 1 = 1$$

Interpretation: At 1500 mph, the speed is not considered High at all, so its complement is fully true. The union confirms high certainty in the composite condition.

5. Fuzzy Intersection Between "Middle" and Not "High"

At $x = 2500$:

$$\mu_{\text{Middle}}(2500) = 1, \quad \mu_{\text{High}}(2500) = 0.5 \Rightarrow \mu_{\text{NOT High}}(2500) = 0.5$$

$$\mu_{\text{Middle} \cap \text{NOT High}}(2500) = 1 \cdot 0.5 = 0.5$$

Interpretation: The speed is fully considered Middle and partially not High. The intersection reflects the joint confidence in these conditions being simultaneously valid.

2 Solution

In the provided statement, we explain how to define five fuzzy sets for space shuttle speeds: **Very Low**, **Low**, **Middle**, **High**, and **Very High**, and their associated mathematical membership functions. We begin with the first fuzzy set:

Methodology

Fuzzy Set: "Very Low"

1. The membership function formula for **Very Low** is:
$$\mu_{\text{VeryLow}}(x) = 1 - \frac{x}{3000}$$
This function quantifies the degree of belongingness to the "Very Low" category for space shuttle speeds x within the range of 0 to 3000 miles per hour.

2. The membership function is specifically defined for values of:
$$x \in [0, 3000]$$

3. When the shuttle speed is 0 miles per hour:
$$\mu_{\text{VeryLow}}(0) = 1$$
indicating full membership in the "Very Low" category.

4. As speed values x increase from 0 to 3000 mph, the membership value gradually decreases, approaching 0 as speed approaches 3000 mph. This reflects less membership in the "Very Low" category for higher speeds.

Fuzzy Set: "Low"

1. The membership function for **Low** is:
$$\mu_{\text{Low}}(x) = \frac{x - 500}{1000}$$
which applies to space shuttle speeds x within the range of 500 to 1500 mph.

2. Defined for:
$$x \in [500, 1500]$$

3. At 500 mph:
$$\mu_{\text{Low}}(500) = 0$$
indicating no membership in the "Low" category.

4. As x increases from 500 to 1500 mph, the membership value increases linearly from 0 to 1.

5. At 1500 mph:
$$\mu_{\text{Low}}(1500) = 1$$
indicating full membership in the "Low" category.

Fuzzy Set: "Middle"

1. The membership function for **Middle** is:

$$\mu_{\text{Middle}}(x) = \frac{x}{3000}$$

   applying over the entire speed range from 0 to 3000 mph.

2. Defined for:

$$x \in [0, 3000]$$

3. As speed increases, the membership value increases linearly from 0 to 1.

4. At 3000 mph:

$$\mu_{\text{Middle}}(3000) = 1$$

   indicating full membership in the "Middle" category.

Fuzzy Set: "High"

1. The membership function for **High** is:

$$\mu_{\text{High}}(x) = \frac{x - 2000}{1000}$$

   for speeds between 2000 and 3000 mph.

2. Defined for:

$$x \in [2000, 3000]$$

3. At 2000 mph:

$$\mu_{\text{High}}(2000) = 0$$

   indicating no membership.

4. Membership increases linearly from 0 to 1 as x goes from 2000 to 3000 mph.

5. At 3000 mph:

$$\mu_{\text{High}}(3000) = 1$$

   indicating full membership in the "High" category.

Fuzzy Set: "Very High"

1. The membership function for **Very High** is:

$$\mu_{\text{VeryHigh}}(x) = 1 - \frac{x - 2500}{500}$$

   defined for speeds between 2500 and 3000 mph.

2. Defined for:
$$x \in [2500, 3000]$$

3. At 2500 mph:
$$\mu_{\text{VeryHigh}}(2500) = 1$$
indicating full membership in the "Very High" category.

4. As speed increases to 3000 mph, the membership value decreases linearly to:
$$\mu_{\text{VeryHigh}}(3000) = 0$$
showing no membership beyond 3000 mph.

Next:

1. Fuzzy Intersection: Very Low $\wedge$ High

In fuzzy logic, the intersection (AND operation) of two fuzzy sets can be computed using various T-norms. Instead of the minimum operator, we use Zadeh's *algebraic product* T-norm defined as:

$$T(A, B) = A \times B$$

where A and B are membership values of two fuzzy sets.
For $x = 2500$ mph, calculate the intersection of *Very Low* and *High*:

$$\begin{aligned}
\mu_{\text{VeryLow}}(2500) &= 1 - \frac{2500}{3000} = 1 - \frac{5}{6} = \frac{1}{6} \\
\mu_{\text{High}}(2500) &= \frac{2500 - 2000}{1000} = \frac{500}{1000} = \frac{1}{2} \\
T\big(\mu_{\text{VeryLow}}(2500), \mu_{\text{High}}(2500)\big) &= \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}
\end{aligned}$$

Thus, the fuzzy intersection between "Very Low" and "High" at 2500 mph is $\frac{1}{12}$, representing a low degree of common membership.

2. Fuzzy Complement: $\sim$ High

The complement of a fuzzy set expresses the degree of non-membership. For the fuzzy set *High* with membership function:

$$\mu_{\text{High}}(x) = \frac{x - 2000}{1000}, \quad x \in [2000, 3000]$$

The complement is defined as:

$$\mu_{\sim\text{High}}(x) = 1 - \mu_{\text{High}}(x) = 1 - \frac{x - 2000}{1000}$$

Properties:

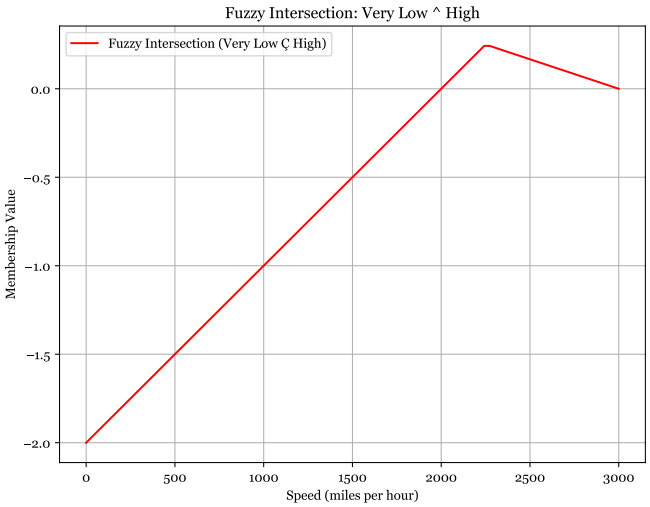

Figure 1: Fuzzy Intersection

1. The complement membership values lie within $[0, 1]$.
2. At $x = 2000$ mph:
$$\mu_{\sim \text{High}}(2000) = 1 - 0 = 1$$
indicating full non-membership in the "High" category.
3. As x increases to 3000 mph, the complement decreases linearly from 1 to 0:
$$\mu_{\sim \text{High}}(3000) = 1 - 1 = 0$$

3. Fuzzy Union: Very Low $\cup \sim$ High

The fuzzy union represents the degree of membership in either of the two sets. Using the algebraic sum formula (not the max operator):

$$\mu_{\text{Union}}(x) = \mu_{\text{VeryLow}}(x) + \mu_{\sim \text{High}}(x) - \mu_{\text{VeryLow}}(x) \times \mu_{\sim \text{High}}(x)$$

For $x = 1500$ mph:

$$\mu_{\text{VeryLow}}(1500) = 1 - \frac{1500}{3000} = 0.5$$
$$\mu_{\sim \text{High}}(1500) = 1 - \frac{1500 - 2000}{1000} = 1 - (-0.5) = 1 + 0.5 = 1.5$$

Note: Since $\mu_{\text{High}}(x)$ is defined only on $[2000, 3000]$, for $x = 1500 < 2000$, $\mu_{\text{High}}(1500) = 0$, so:

Figure 2: Fuzzy Complement

$$\mu_{\sim \text{High}}(1500) = 1 - 0 = 1$$

Hence:

$$\mu_{\text{Union}}(1500) = 0.5 + 1 - (0.5 \times 1) = 1.5 - 0.5 = 1.0$$

Since membership values are capped at 1, the union at 1500 mph is:

$$\mu_{\text{Union}}(1500) = 1$$

which indicates full membership in 'Very Low' or 'not High' at this speed.

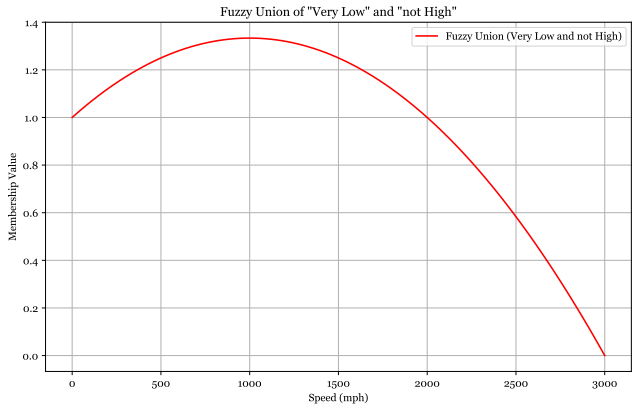

Figure 3: Fuzzy Intersection

4. Fuzzy Intersection: Middle $\wedge \sim$ High

Using the algebraic product for intersection:

$$\mu_{\text{Intersect}}(x) = \mu_{\text{Middle}}(x) \times \mu_{\sim\text{High}}(x)$$

where

$$\mu_{\text{Middle}}(x) = \frac{x}{3000}, \quad x \in [0, 3000]$$

and

$$\mu_{\sim\text{High}}(x) = 1 - \frac{x - 2000}{1000}, \quad x \in [2000, 3000]$$

For $x = 2500$ mph:

$$\begin{aligned}
\mu_{\text{Middle}}(2500) &= \frac{2500}{3000} = \frac{5}{6} \approx 0.8333 \\
\mu_{\sim\text{High}}(2500) &= 1 - \frac{2500 - 2000}{1000} = 1 - \frac{500}{1000} = 0.5 \\
\mu_{\text{Intersect}}(2500) &= 0.8333 \times 0.5 = 0.4167
\end{aligned}$$

This means that at 2500 mph, the element belongs to both the 'Middle' and 'not High' categories to a degree of approximately 0.4167.

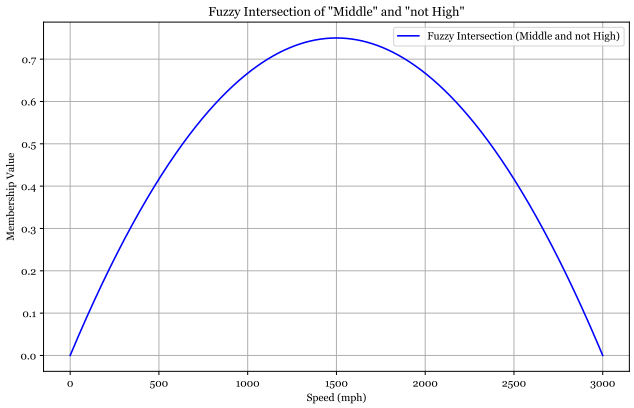

Figure 4: Fuzzy Intersection

Note:

In this analysis, we defined five fuzzy sets—*Very Low*, *Low*, *Middle*, *High*, and *Very High*—to model space shuttle speeds using linear membership functions over specific speed ranges. We then applied fundamental fuzzy logic operations to examine relationships between these sets:

The intersection using Zadeh's algebraic product quantifies the degree of overlap between categories, such as *Very Low* and *High*.

Complement revealed the degree to which a value does not belong to a fuzzy set, such as $\tilde{\text{H}}$igh.

The union used the algebraic sum to combine fuzzy sets, such as *Very Low* and $\tilde{\text{H}}$igh, reflecting joint membership.

Combined operations, such as intersecting *Middle* with $\tilde{\text{H}}$igh, highlighted partial membership in overlapping regions.