# Expected Number of Coin Tosses to Get Two Consecutive Heads

## 1. Problem

We are given an unbiased coin and want to estimate the expected number of tosses required to observe two consecutive heads for the first time. The coin is fair, i.e., the probability of heads (H) or tails (T) in any toss is:

$$P(H) = \frac{1}{2}, \quad P(T) = \frac{1}{2}$$

We aim to either derive a formula analytically or simulate the problem using a Monte Carlo algorithm. Here, we first present an analytical method using expected value equations.

## 2. Solution

Let $X$ be the expected number of tosses required to obtain two consecutive heads. We model the process by considering the different possible outcomes of the initial tosses.

We define three states:

1. $S_0$: Initial state, no heads yet.
2. $S_1$: Last toss was a head, but not two in a row.
3. $S_2$: Terminal state, two heads in a row (stop here).

Let $E_0$ be the expected number of tosses starting from $S_0$, and $E_1$ be the expected number from $S_1$.

We construct the following recurrence relations:

### From State $S_0$:

From the initial state, we toss a coin:

1. With probability $\frac{1}{2}$, we get a tail and stay in $S_0$ (wasted one flip): $\Rightarrow E_0 + 1$
2. With probability $\frac{1}{2}$, we get a head and move to $S_1$ (one flip used): $\Rightarrow E_1 + 1$

$$E_0 = \frac{1}{2}(E_0 + 1) + \frac{1}{2}(E_1 + 1)$$

### From State $S_1$:

Now the last toss was a head:

1. With probability $\frac{1}{2}$, we get a head again and reach the terminal state (two flips total): $\Rightarrow 2$
2. With probability $\frac{1}{2}$, we get a tail and return to $S_0$: $\Rightarrow E_0 + 2$

$$E_1 = \frac{1}{2}(2) + \frac{1}{2}(E_0 + 2)$$

### Solving the Equations

Start with the second equation:

$$E_1 = \frac{1}{2}(2) + \frac{1}{2}(E_0 + 2) = 1 + \frac{1}{2}E_0 + 1 = \frac{1}{2}E_0 + 2$$

Now substitute $E_1$ into the first equation:

$$E_0 = \frac{1}{2}(E_0 + 1) + \frac{1}{2}\left(\frac{1}{2}E_0 + 2 + 1\right) = \frac{1}{2}E_0 + \frac{1}{2} + \frac{1}{4}E_0 + \frac{3}{2}$$

$$E_0 = \left(\frac{3}{4}E_0\right) + 2 \Rightarrow E_0 - \frac{3}{4}E_0 = 2 \Rightarrow \frac{1}{4}E_0 = 2 \Rightarrow E_0 = 8$$

Thus, the expected number of tosses required to get two consecutive heads is:

$$\boxed{8}$$

## 3. Alternative Monte Carlo Simulation

A Monte Carlo algorithm can also estimate the expected number of tosses empirically. Here's a high-level description:

1. Repeat the following simulation $N$ times (e.g., $N = 10^6$):
   (a) Initialize a counter for tosses and a memory of the last toss.
   (b) Toss the coin until two heads appear in a row.
   (c) Record the number of tosses.
2. Compute the average number of tosses over all trials.

**Pseudocode:**

```
total_flips = 0
N = 1000000

for i in range(N):
    flips = 0
```

```
    last = None
    while True:
        flip = random.choice(['H', 'T'])
        flips += 1
        if last == 'H' and flip == 'H':
            break
        last = flip
    total_flips += flips

expected_flips = total_flips / N
```

**Output (approx.):** $\boxed{8}$

# 4. Note

Both the analytical and simulation-based approaches confirm that the expected number of coin tosses required to observe two consecutive heads using an unbiased coin is:

$$\boxed{8}$$