Robust Methods in Computer Vision

CS 763

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Outline

- Least squares estimates
- Limitations of least squares and need for robustness
- Least median of squares method
- RANSAC method
- Application scenarios

Least squares Estimates

- Consider quantity *y* related to quantity *x* in the form $y = f(x;\mathbf{a})$.
- Here **a** is a vector of parameters for the function *f*. For example, $y = mx + c$, where $\mathbf{a} = (m,c)$.
- Now consider, we have *N* data points (x_i,y_i) where the y_i could be possibly corrupted by noise, and want to estimate **a**.
- This is done by minimizing the following w. r. t. **a**:

$$\sum_{i=1}^{N}(y_i - f(x_i;\mathbf{a}))^2$$

Least squares Estimates

- Why did we minimize the squared error loss? What would happen if we changed the power to 4? Or 1 or 3 (with absolute value)?

- Let us assume that the noise affecting y_i is (1) additive, and (2) Gaussian with mean zero and some known standard deviation σ. Then:

$$y_i = f(x_i;\mathbf{a}) + \eta_i, \eta_i \sim N(0,\sigma)$$

Least squares Estimates

- Let us also assume that the noise values affecting the different samples are independent of each other.
- Now given some value of **a** and some *x*$_i$, the probability density of *y*$_i$ is:

Likelihood of *y*$_i$

Likelihood of {*y*$_i$ }, i=1 to N

$$p(y_i \mid x_i, \mathbf{a}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_i - f(x_i;\mathbf{a}))^2}{2\sigma^2}}$$

$$p(\{y_i\}_{i=1}^{N} \mid \{x_i\}_{i=1}^{N}, \mathbf{a}) = \prod_{i=1}^{N} p(y_i \mid x_i, \mathbf{a}) \text{ by independence assumption on noise}$$

Least squares Estimates

- The probability density of a random variable Y at a value y is defined as follows:

$$p_Y(y) = \lim_{\delta \to 0} \frac{P(y \le Y \le y + \delta)}{\delta}$$

Least Squares Estimates

- We want to find a value of **a** which **maximizes** this probability density. This is called the **maximum likelihood estimate** of **a**.
- ..equivalent to **maximizing** the log of this probability density (why?)
- ..equivalent to **minimizing** the **negative** log of this probability density, i.e.

$$J(\mathbf{a}) = \sum_{i=1}^{N} \frac{(y_i - f(x_i;\mathbf{a}))^2}{2\sigma^2} + \log\left(\sigma\sqrt{2\pi}\right)$$

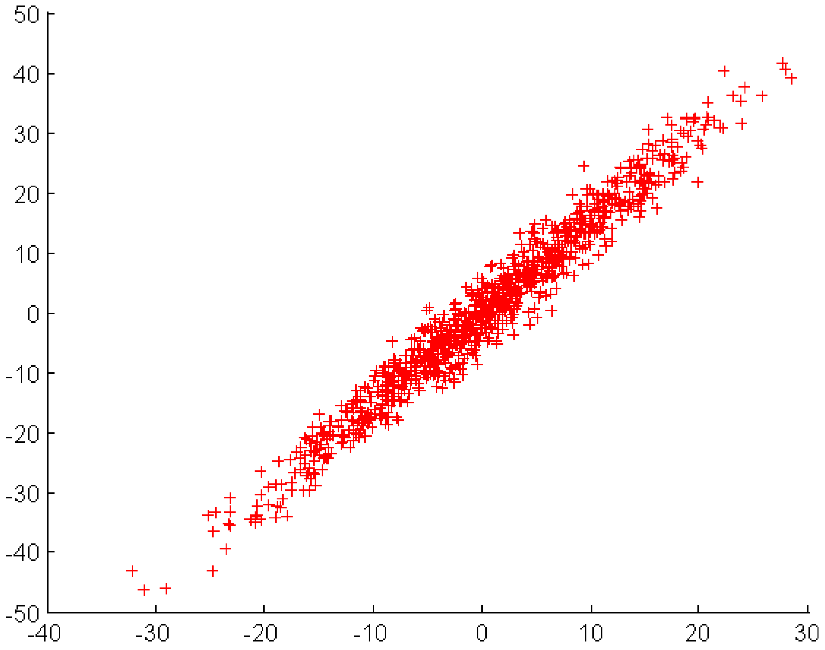
Least Squares Estimates

- This shows us that the least squares estimate is the same as the **maximum likelihood** estimate under the assumption that the noise affecting different samples was ***independent*** and ***Gaussian*** distributed with ***fixed*** variance and mean 0.
- Why maximum likelihood estimate of **a**? Intuitively, it is the value of **a** that best agrees with or supports the observations $\{y_\mathrm{i}\}$, $1 \leq i \ \leq N$.

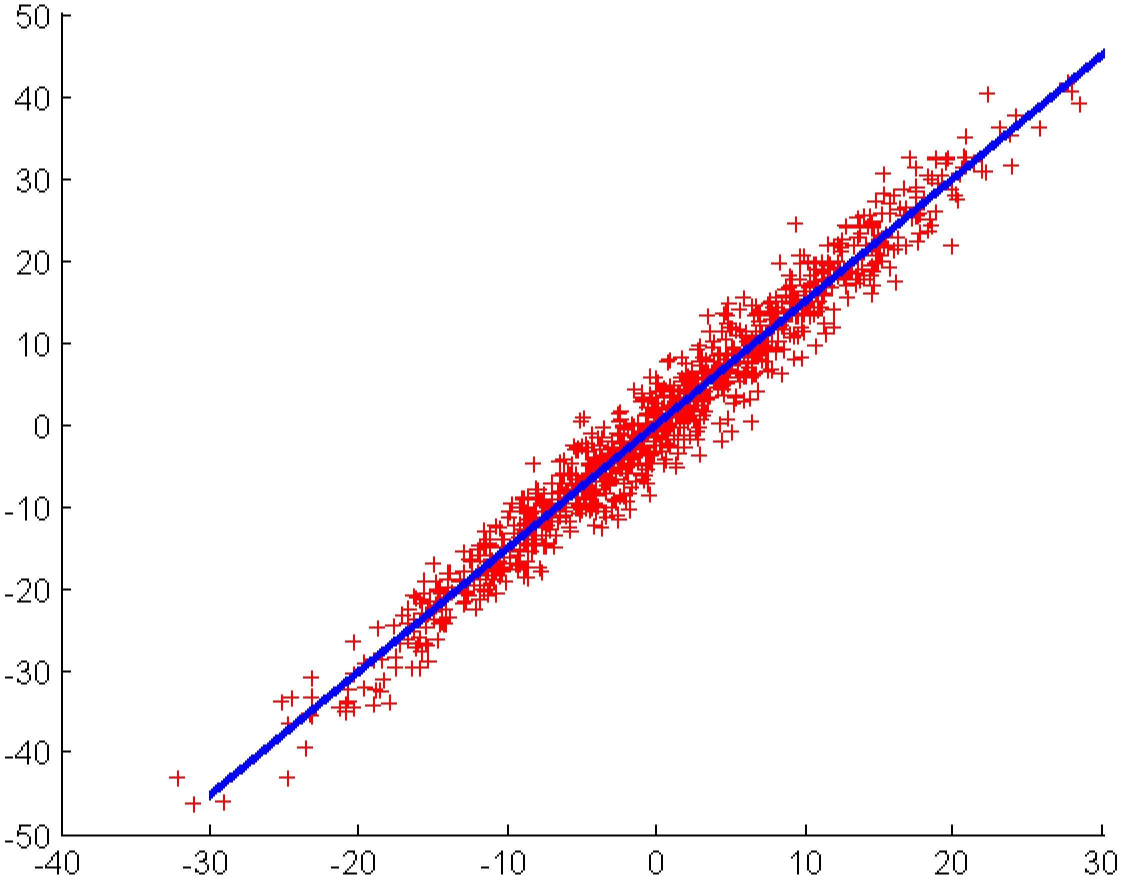
Least squares fit of a line

$$(m^*, c^*) = \arg\min J(m,c) = \sum_{i=1}^{N}(y_i - mx_i - c)^2$$

$$\therefore \begin{pmatrix} \sum_i x_i^2 & \sum_i x_i \\ \sum_i x_i & N \end{pmatrix}\begin{pmatrix} m \\ c \end{pmatrix} = \begin{pmatrix} \sum_i x_i y_i \\ \sum_i y_i \end{pmatrix}$$



```
N = 1000;x = 10*randn(N,1);
y = 1.5*x + randn(N,1)*3;
 scatter(x,y,'r+')
```



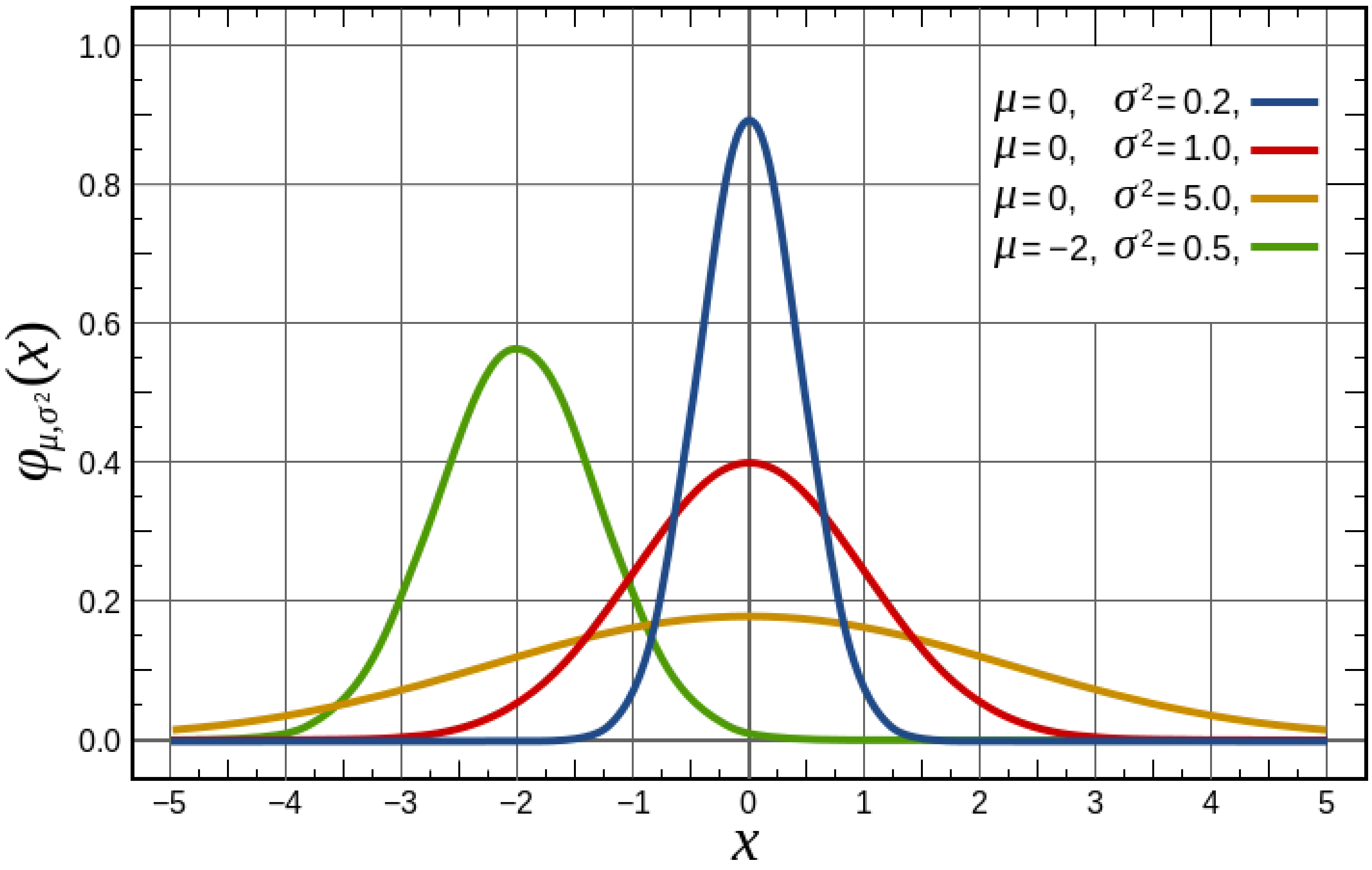
Result of a least-squares estimate under Gaussian noise:

Estimated slope: 1.5015 (versus 1.5)
Estimated intercept: 0.088 (versus 0)

Other Least-squares solutions in computer vision

- Camera calibration – SVD
- Parametric motion estimation – SVD or pseudo-inverse (affine, rotation, homography, etc)
- Fundamental/essential matrix estimation – SVD *(we will study this later in stereo-vision)*
- Optical Flow (Horn-Shunck as well as Lucas-Kanade) *(we will study this later)*

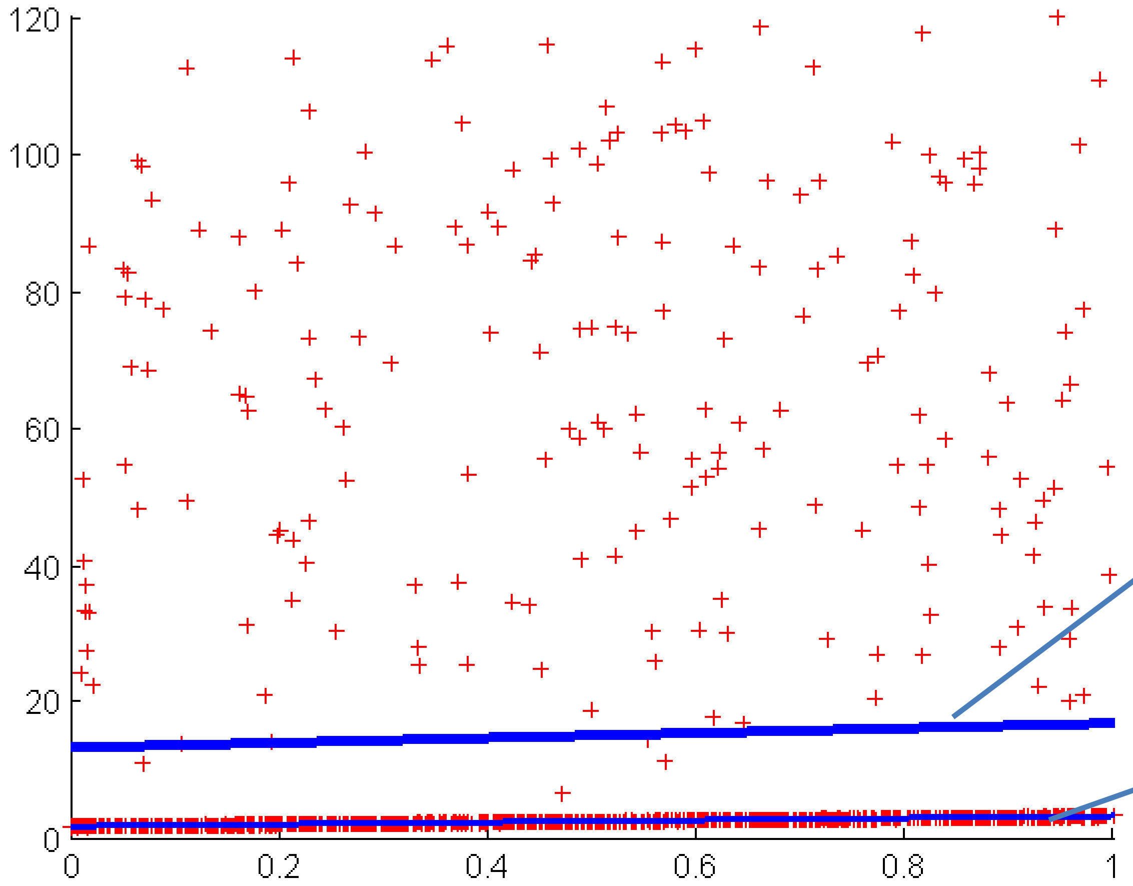
Outliers and Least-squares estimates



Observation: Let *x* be a random variable with a Gaussian distribution. Then the probability that *x* takes on any value in a small range far away from the mean (typically at a distance of more than +/- 3σ) is **very low**. See diagram above.

Outliers and Least-squares

- The upshot of the previous observation is this: the least squares estimate assumes that most points will lie ***close*** to the true (unknown) model – else their probability would be very low.
- Now, suppose the given dataset contains wild outliers, i.e. stray points that simply do not obey the model.
- These outliers will skew the least squares estimate – as it tries to force a solution which maximizes the likelihood of the ***outliers*** as well.
- Since outliers were extremely unlikely under the Gaussian probability density, the model (during maximum likelihood estimation) has to change itself to make the outliers more likely.



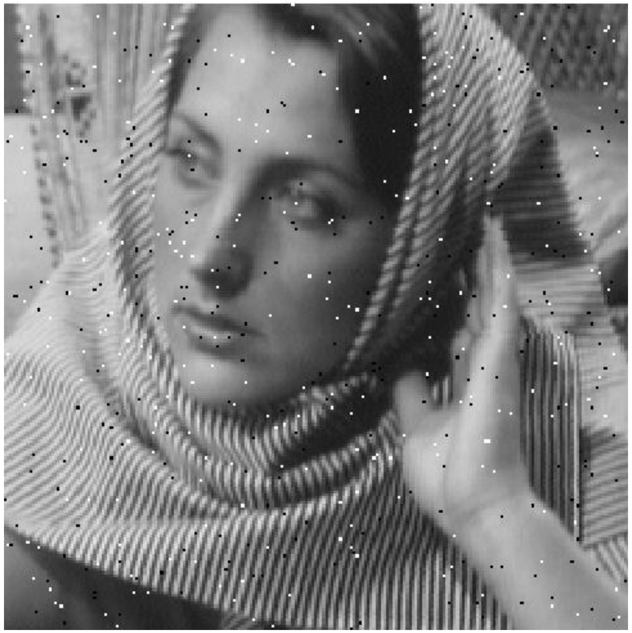
Skewed estimate

Actual line

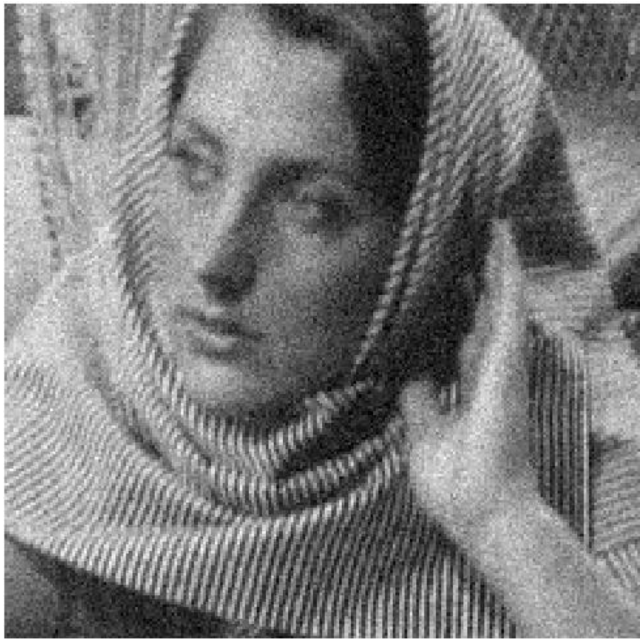
20% of the points are outliers. They have skewed the estimate of the slope from 1.5 to 3.6 and the intercept from 2 to 13.5.

Examples of outliers: (1)

- Salt and pepper noise in images



Salt and pepper noise (a special case of impulse noise)



Gaussian noise

Don’t worry about this example right now – we will encounter it when we study shape from shading and photometric stereo

Examples of outliers: (2)

- Shadows and specularities act as outliers in photometric stereo!

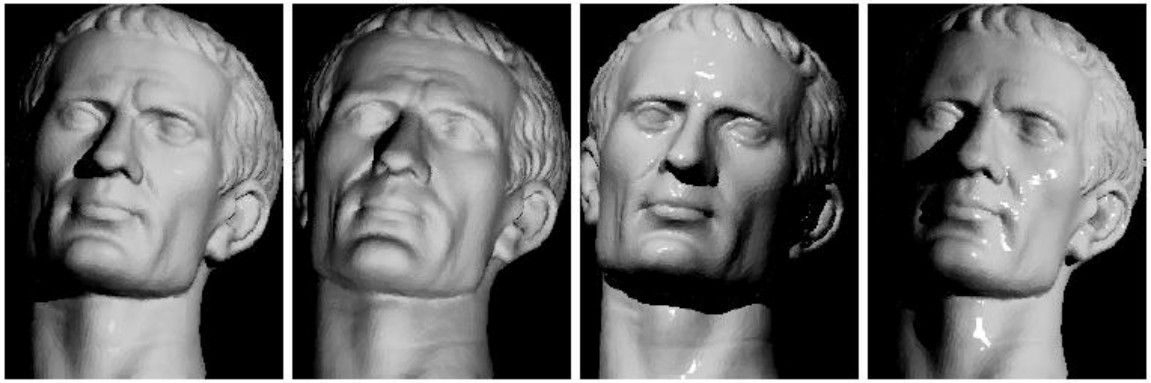
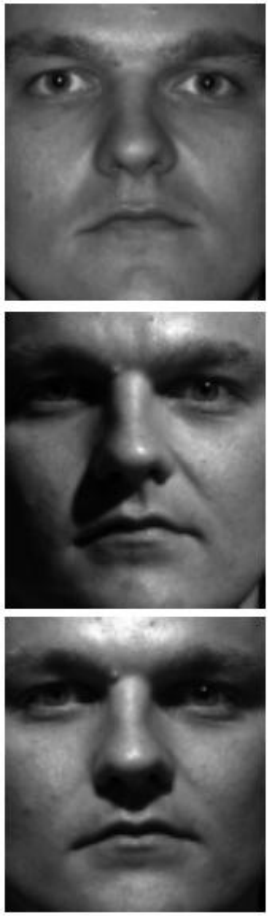


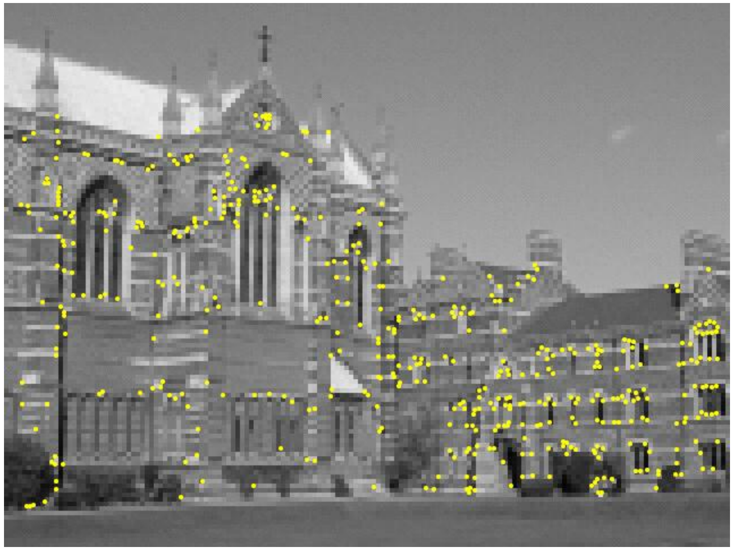
Fig. 7. **Input images (4 out of 40).** With all kinds of corruptions: Specularity, cast shadow, attached shadow.

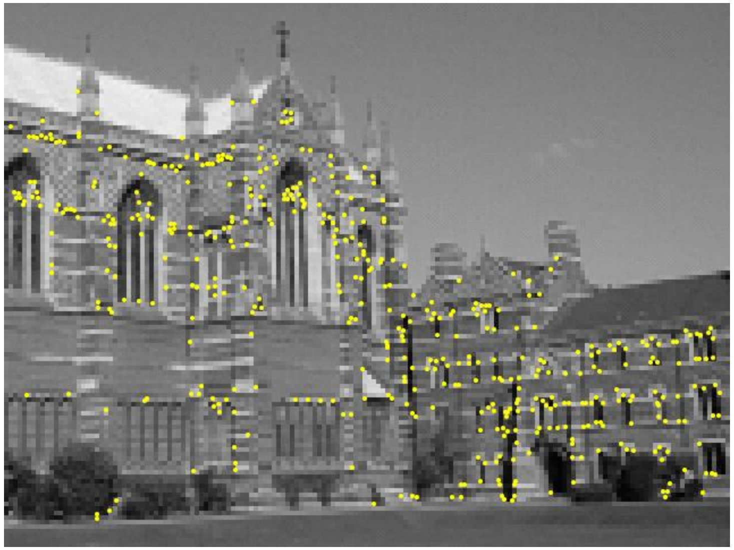


Images taken from paper: perception.csl.illinois.edu/matrix-rank/Files/**robust**_v19.pdf

Examples of outliers: (3)

- Estimation of the spatial transformation between images (could be translation, rotation, affine or homography) requires *N*+ pairs of corresponding points. Some of these correspondences can be faulty.





Examples of outliers: (3)

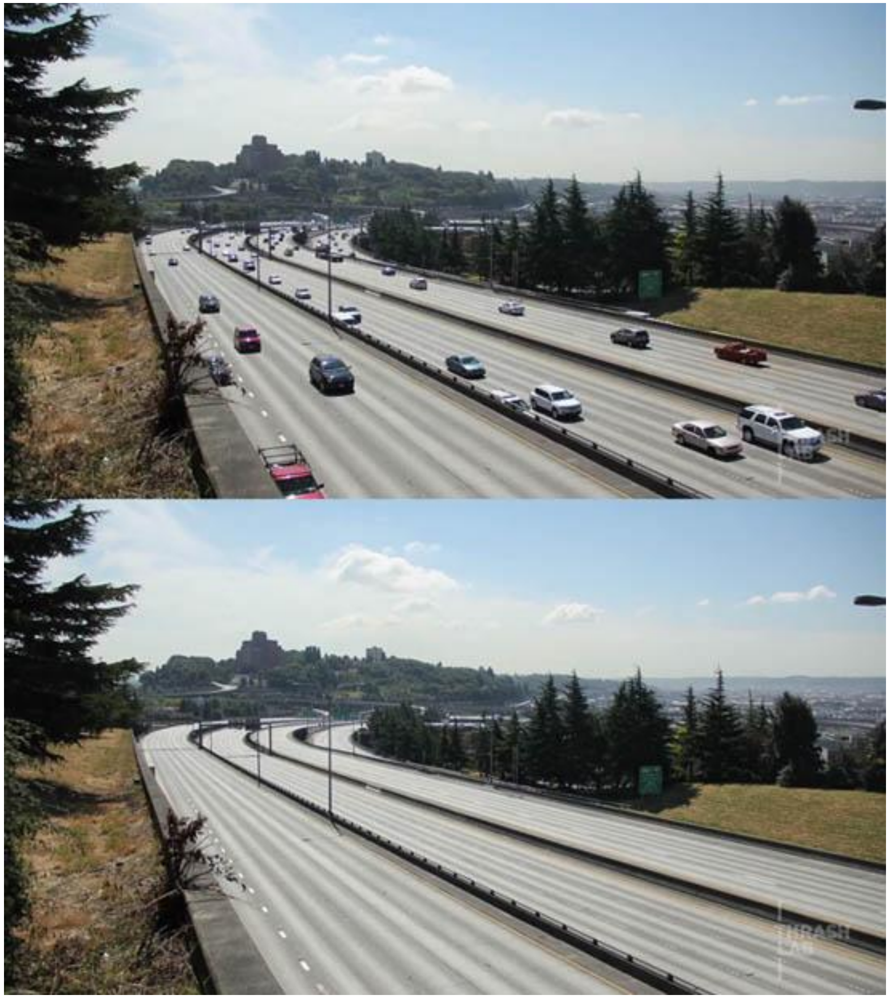
- Some of these correspondences can be faulty for various reasons

❑ occlusions/ difference in field of view / shadows

❑ algorithm errors

❑ identical objects in the scene

❑ change in the position of some objects in the scene (even though the global motion is homography, these objects will not conform to that same motion model).



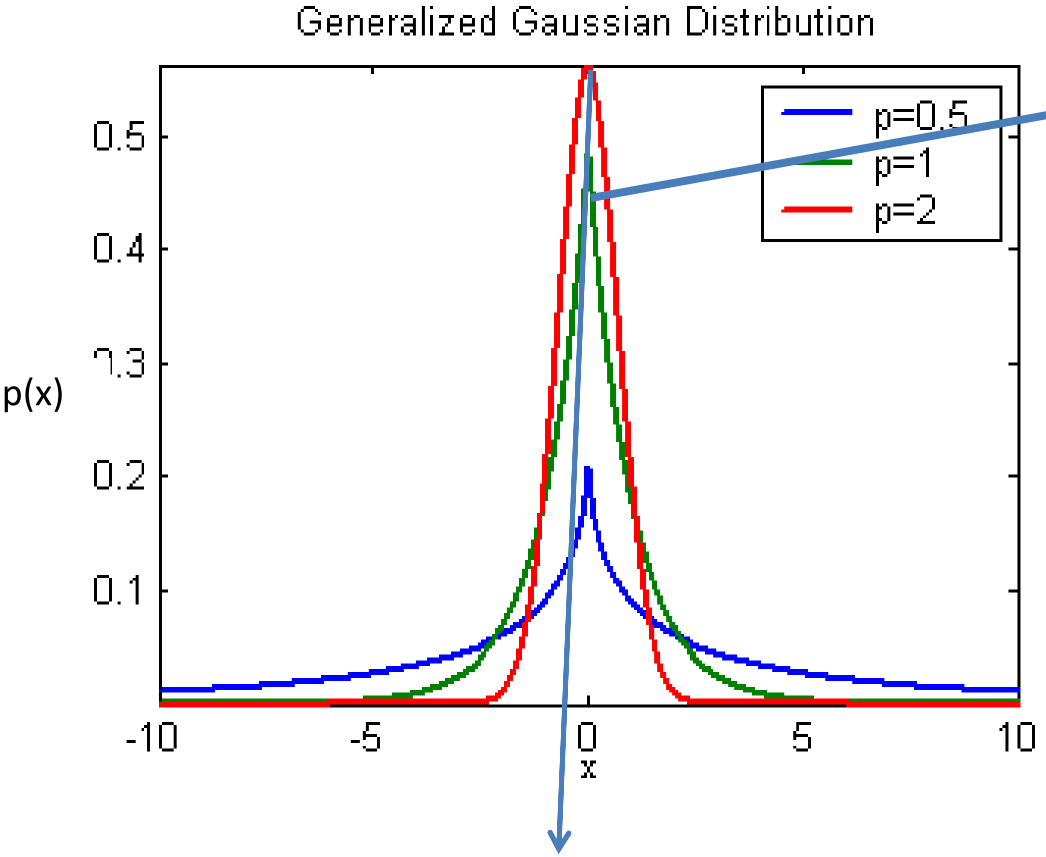
http://petapixel.com/2012/10/22/an-eerie-time-lapse-of-seattle-minus-all-the-humans/

Examples of outliers: (4)

- The motion between consecutive frames of a video in the following link may be modeled as affine. But some corresponding pairs of points (example: on independently moving objects) don’t conform to that model – those are outliers.

  https://www.youtube.com/watch?v=17VAuBL1Lxc

Dealing with Outliers: (1)



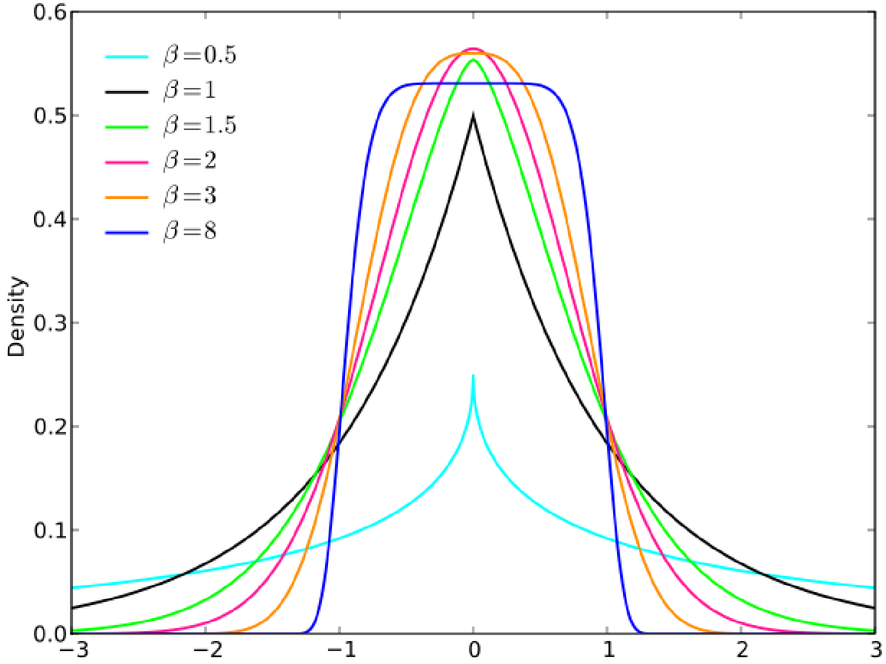
Gaussian

Laplacian probability density function: a distribution which has **heavier** tails than the Gaussian. This means that if a random variable is Laplacian distributed, the probability that it can take values far away from the mean, is higher than what it would be if the variable were Gaussian distributed.

$$p(x) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}$$

The Laplacian (or Laplace) pdf is **not** to be confused with the Laplacian of a function *f*(*x*,*y*), given as $f_{xx}(x,y) + f_{yy}(x,y)$ that we had studied in image processing.

Dealing with Outliers: (1)

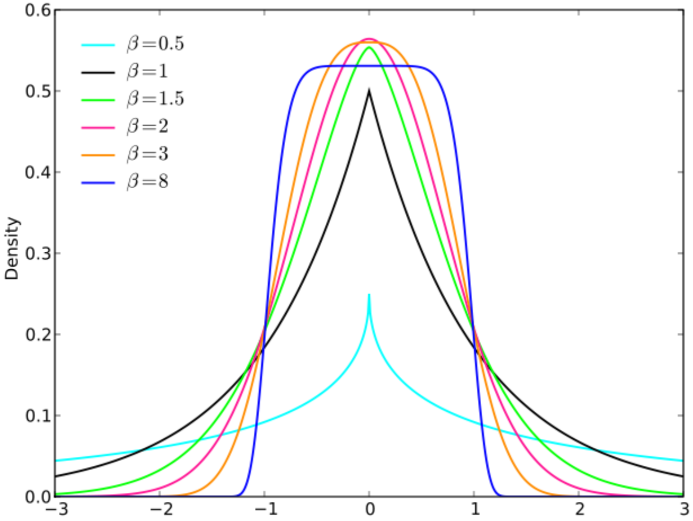


$$p(x) = \frac{\beta}{2\alpha\Gamma(1/\beta)} e^{-\left(\frac{|x-\mu|}{\alpha}\right)^{\beta}}$$

The Laplacian probability density function is a special case of the family of **Generalized Gaussian probability density functions** with **shape parameter** β and **scale parameter** α. As β reduces below 1, the density function becomes heavier tailed.

Question: Why do we care for heavier tails? Because they ensure that the wild outliers are more likely to occur (than the Gaussian pdf). Consequently, a maximum likelihood estimation assuming heavy-tailed noise models will be less affected by outliers.

Dealing with Outliers: (1)

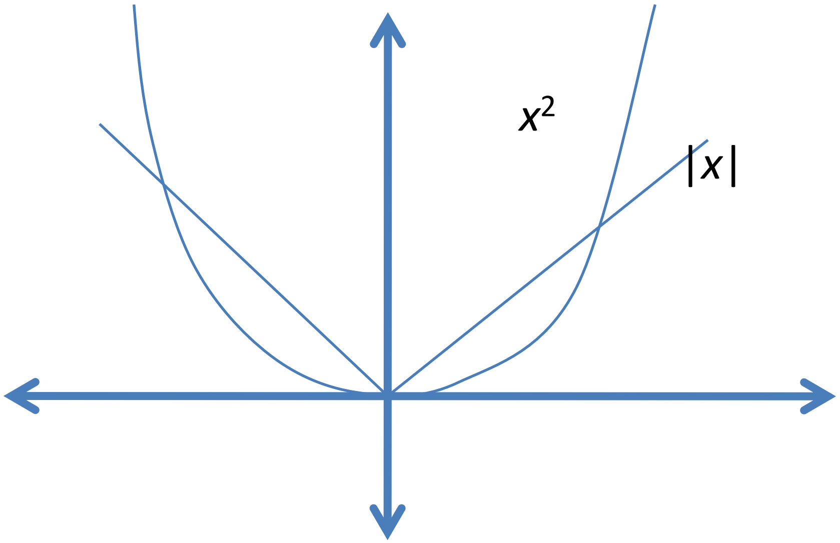


$$p(x) = \frac{\beta}{2\alpha\Gamma(1/\beta)} e^{-\left(\frac{|x-\mu|}{\alpha}\right)^{\beta}}$$

Why do these Generalized Gaussians with shape parameter β < 2 have heavier tails than the usual Gaussian (i.e. β = 2)?

Consider a zero-mean Gaussian and a zero-mean Laplacian (without loss of generality), i.e. β = 1.

The term inside the exponential in a Gaussian is x^2, where it is $|x|$ for a Laplacian.



Note that x^p grows faster than x^q beyond |x| = 1, for p > q

Dealing with Outliers: (1)

- Assume the noise has a Laplacian distribution.
- The maximum likelihood estimate of **a** is then given by minimizing the following:

$$J(\mathbf{a}) = \sum_{i=1}^{N} \frac{|y_i - f(x_i;\mathbf{a})|}{b} - \log(2b) \approx \sum_{i=1}^{N} \frac{\sqrt{(y_i - f(x_i;\mathbf{a})^2 + \varepsilon}}{b} - \log(2b)$$

- Unfortunately, there is no closed-form solution (based on inverse/pseudo-inverse) in this case – unlike the case for squared error!
- One will need iterative methods like adaptive gradient descent.

Dealing with Outliers: (1)

- One will need iterative methods like adaptive gradient descent.

```
Repeat till there is no change in $\mathbf{a}$
{
    $\alpha = \alpha_{\max}$
    while $(\alpha > \alpha_{\min})$
    {
        $\mathbf{a}' = \mathbf{a} - \alpha \frac{dJ(\mathbf{a})}{d\mathbf{a}}, 0 < \alpha << 1$
        if $J(\mathbf{a}') > J(\mathbf{a}), \alpha = \alpha_{\max} / 2$
        else $\{\mathbf{a} = \mathbf{a}';$ break$\}$
    }
}
```

α = **step size** of gradient descent

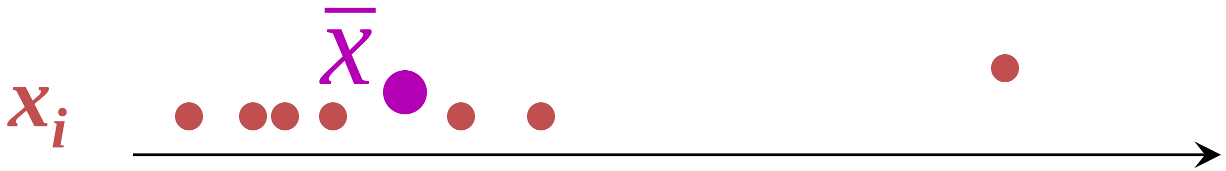
Gradient descent converges to a local minimum of the energy function (or objective function), i.e. J(**a**) in this slide, if the step size α is "small enough" to never.

Unfortunately, too small a step size is too expensive. A large step size may lead to increase in the energy function across iterations.

So we pick the largest possible step-size (within a given range) that reduces the energy – this is called **gradient descent with adaptive step-size** or **adaptive gradient descent**.

Robust statistics – simple example

Find "best" representative for the set of numbers



L2:

$$J(\bar{x}) = \sum_i \left|\bar{x} - x_i\right|^2 \rightarrow \min$$

L1:

$$J(\bar{x}) = \sum_i \left|\bar{x} - x_i\right| \rightarrow \min$$

Influence of x_i on E: $x_i \rightarrow x_i + \Delta$

$$J_{new} \cong J_{old} + 2(x_i - \bar{x}) \cdot \Delta + \Delta^2$$

proportional to $\left|\bar{x} - x_i\right|$

$$J_{new} \cong J_{old} \pm \Delta$$

equal for all x_i

Outliers influence the most

$$\bar{x} = \mathrm{mean}(x_i)$$

Majority decides

$$\bar{x} = \mathrm{median}(x_i)$$

Slide: Courtesy - Darya Frolova, Recent progress in optical flow

Elections and Robust statistics

many ordinary people

a very rich man

wealth

Oligarchy

Votes proportional to the wealth

like in L2 norm minimization

Democracy

One vote per person

like in L1 norm minimization

Slide: Courtesy - Darya Frolova, Recent progress in optical flow

New ways of defining the mean

- We know the mean as the one that minimizes the following quantity:

$$E(\mu) = \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n} \rightarrow \mu = \frac{\sum_{i=1}^{n} x_i}{n}$$

- Changing the error to sum of absolute values, we get:

$$E(\mu) = \frac{\sum_{i=1}^{n} | x_i - \mu |}{n} \rightarrow \mu = median(\{x_i\}_{i=1}^{n})$$

We will prove this in class!

New ways of defining the mean

- We can also use errors of the following type with 0 < q ≤ 1:

$$E(\mu) = \frac{\left(\sum_{i=1}^{n} | x_i - \mu |^q \right)^{1/q}}{n}$$

- Optimizing the above requires iterative methods (no closed form solutions).
- The mean computed using 0 < q ≤ 1 is quite robust to outliers – with q greater than or equal to 2, the mean is susceptible to outliers.

New ways of defining the mean

- The earlier definitions of the mean were for scalars. They can be extended for vectors in some *d* > 1 dimensions as well.

$$E(\boldsymbol{\mu}) = \frac{\sum_{i=1}^{n} (\mathbf{x_i} - \boldsymbol{\mu})^2}{n} \rightarrow \boldsymbol{\mu} = \frac{\sum_{i=1}^{n} \mathbf{x_i}}{n}$$

- For other q-norms (0 <= q < 1), we have:

$$E(\boldsymbol{\mu}) = \frac{\left(\sum_{i=1}^{n} |\mathbf{x_i} - \boldsymbol{\mu}|^q\right)^{1/q}}{n}$$

Dealing with outliers: (2) LMedS

- LMedS = Least Median of Squares
- It works as follows:

$$J(\mathbf{a}) = \text{median}_{i=1:N}\ (y_i - f(x_i;\mathbf{a}))^2;$$

select $\mathbf{a}$ for which $J(\mathbf{a})$ is minimum.

- This has no closed form solution either and you can’t do gradient descent type of techniques as the median is not differentiable.
- But it has an “algorithmic” solution.

Dealing with outliers: (2) LMedS

- **Step 1:** Arbitrarily choose *k* out of *N* points where *k* is the smallest number of points required to determine **a**. Call this set of *k* points as $\mathbf{C}$.

o Eg: If you had to do line fitting, *k* = ?

o Eg: If you were doing circle fitting, *k* = ?

o Eg: If you have to find the affine transformation between two point sets in 2D, you need *k* = ? correspondences.

Dealing with outliers: (2) LMedS

- **Step 2:** Determine **a** using an inverse (say) from $\mathbf{C}$.
- **Step 3:** Determine the squared residual errors for all the other *N*-*k* points, i.e. compute
  $$\{e_i = (y_i - f(x_i;\mathbf{a}))^2\}_{i \notin C}$$
- **Step 4:** Compute med_C = median of $\{e_i\}$.
- Repeat all these four steps for *S* different subsets of *k* points each.
- Pick the estimate of **a** corresponding to the $\mathbf{C}$ that has the least value of med_C .
- What's the time complexity of this algorithm?

Dealing with outliers: (2) LMedS

- S = number of subsets. What should be the minimum value of S?
- Let's say that some fraction p ($0 < p < 1$) of the N points are inliers ("good points").
- Then the probability that at least one of the S different subsets contains all inliers (i.e. yields good estimate of **a**) is: $P = 1-(1- p^k)^S$.
- Fix P to 0.99 (say) and compute S assuming you know p.
- Clearly S will increase hugely if either k is large (more parameters to determine) and/or if p is small (fewer inliers).

Dealing with outliers: (3) RanSaC

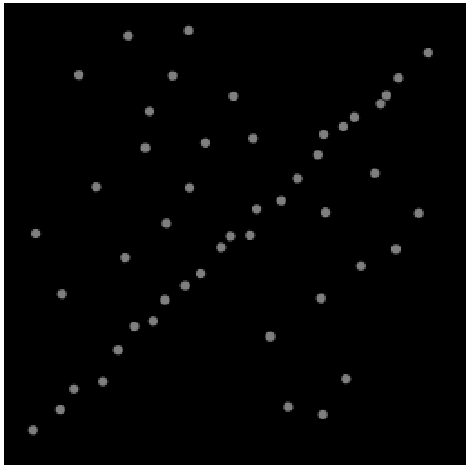
- RanSaC = **Ran**dom **Sa**mple **C**onsensus.
- Similar in spirit to LMedS.
- **Step 1:** Arbitrarily choose *k* out of *N* points where *k* is the smallest number of points required to determine **a**. Call this set of *k* points as $\mathbf{C}$.
- **Step 2:** Determine **a** using an inverse (say) from $\mathbf{C}$.
- **Step 3:** Determine the squared residual errors for all the other *N*-*k* points, i.e. compute

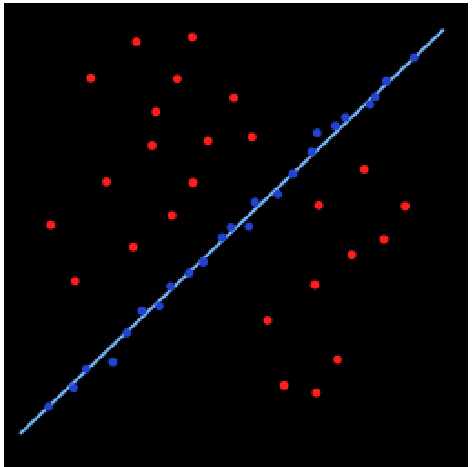
$$\{e_i = (y_i - f(x_i;\mathbf{a}))^2\}_{i \notin C}$$

Dealing with outliers: (3) RanSaC

- **Step 4:** Count the number of points for which e_i < threshold λ. These points form the "consensus set" for the chosen model.
- Repeat all 4 steps for multiple subsets and pick the subset which has maximum number of inliers and its corresponding estimate of **a**.
- Choice of *S* – same as LMedS.

Sample result with RanSaC for line fitting.

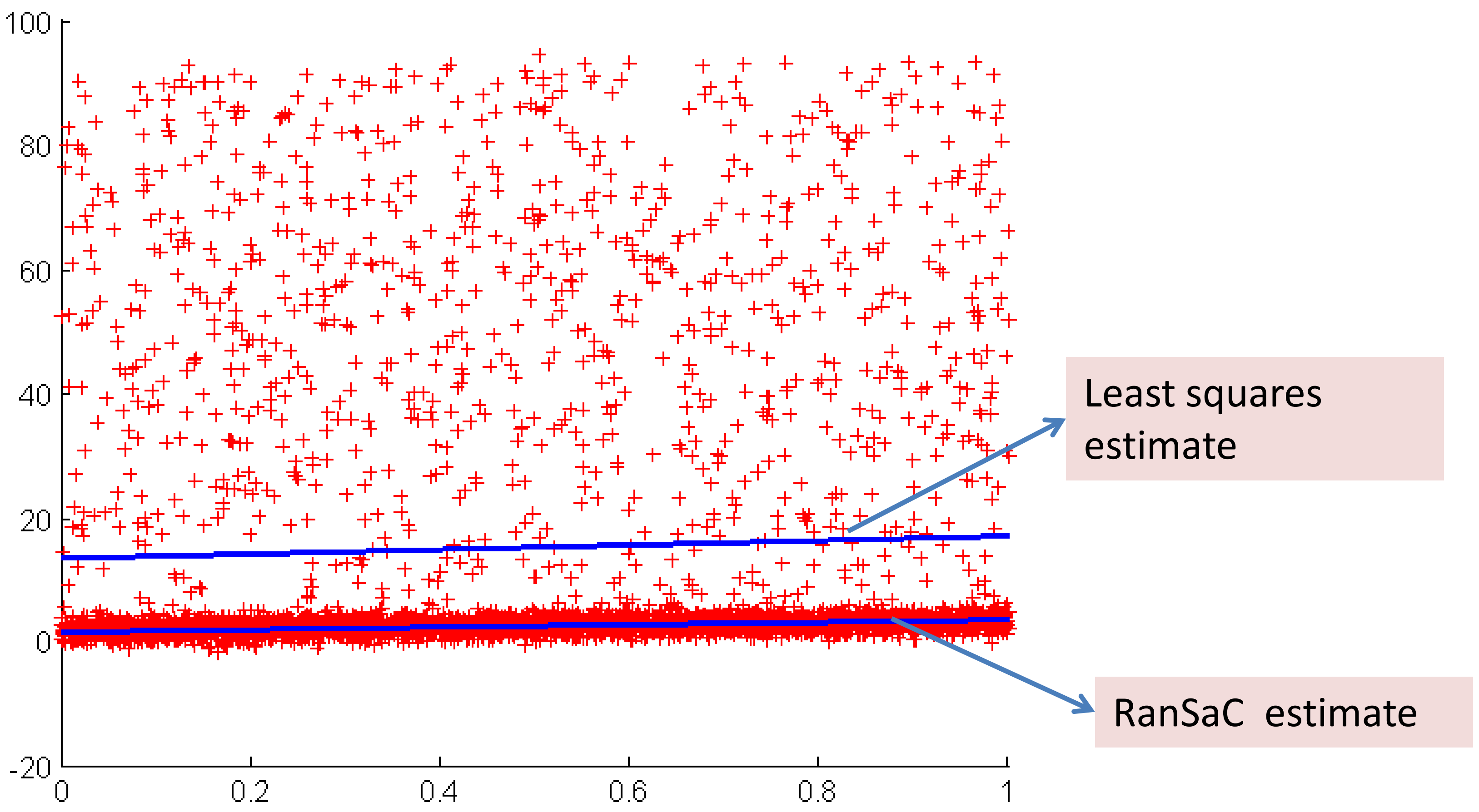




Dealing with outliers: (3) RanSaC

- **Step 4:** Count the number of points for which e_i < threshold λ. These points form the "consensus set" for the chosen model.
- Alternatively:

1. Repeat all 4 steps for multiple subsets and pick the subset $\mathbf{C}$ which has maximum number of inliers.
2. Estimate **a** using *all the points which were inliers for $\mathbf{C}$*.



Least squares estimate

RanSaC estimate

RANSAC versus LMedS

- LMedS needs no threshold to determine what is an inlier unlike RanSaC.
- But RanSaC has one advantage. What?

o LMedS will need at least 50% inliers (by definition of median).

o RanSaC can tolerate a smaller percentage of inliers (i.e. larger percentage of outliers).

Expected number of RanSaC iterations

- The probability that at least one point in a chosen set of k points is an outlier = $1\text{-}p^k$.

- The probability that the i-th set is the *first* set that contains no outliers = $(1\text{-}p^k)^{\mathrm{i}\text{-}1}p^k$ to be denoted as $Q(i)$.

Expected number of RanSaC iterations

- The expected number of sets to be drawn required to find the first no-outlier set =

$$= \sum_{i=1} iQ(i) = \sum_{i=1} i(1-p^k)^{i-1} p^k = p^k \sum_{i=1} i(1-p^k)^{i-1}$$

$$= p^k \sum_{i=1} \frac{d}{d(1-p^k)} (1-p^k)^i = p^k \frac{d}{d(1-p^k)} \left(\sum_{i=1} (1-p^k)^i \right)$$

$$= p^k \frac{d}{d(1-p^k)} \left(\frac{1}{1-(1-p^k)} \right) = \frac{p^k}{(p^k)^2} = p^{-k}$$

RanSaC Variant 1

- RanSaC picks the subset **C** with largest number of inliers (i.e. least number of outliers), which is equivalent to picking the subset that minimizes the following cost

$$J(C) = \sum_{i \notin C} \rho(e_i);$$

$$\rho(e_i) = 1, e_i^2 \geq T$$

$$= 0, e_i^2 < T$$

RanSaC Variant 1: MSAC

- One could instead minimize a cost function that gives weights to inliers to see how well they fit the model:

$$\hat{J}(C) = \sum_{i \notin C} \hat{\rho}(e_i);$$
$$\hat{\rho}(e_i) = T, e_i^2 \geq T$$
$$= e_i^2, e_i^2 < T$$

M-estimator: an estimator that weighs inliers by their “quality”, and outliers by a fixed constant

- This variant is called MSaC (M-estimator sample consensus).

Reminder: Planar Homography

- Given two images of a coplanar scene taken from two different cameras, how will you determine the planar homography matrix **H**?
- How many point correspondences will you require?

$$\mathbf{p}_{2,im} = \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} = \hat{\mathbf{H}}\mathbf{p}_{1,im} = \begin{pmatrix} \hat{H}_{11} & \hat{H}_{12} & \hat{H}_{13} \\ \hat{H}_{21} & \hat{H}_{22} & \hat{H}_{23} \\ \hat{H}_{31} & \hat{H}_{32} & \hat{H}_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}$$

$$x_{2,im} = \frac{\hat{H}_{11}u_1 + \hat{H}_{12}v_1 + \hat{H}_{13}w_1}{\hat{H}_{31}u_1 + \hat{H}_{32}v_1 + \hat{H}_{33}w_1} = \frac{\hat{H}_{11}x_1 + \hat{H}_{12}y_1 + \hat{H}_{13}}{\hat{H}_{31}x_1 + \hat{H}_{32}y_1 + \hat{H}_{33}}, x_{1,im} = \frac{u_1}{w_1}, y_{1,im} = \frac{v_1}{w_1}$$

$$y_{2,im} = \frac{\hat{H}_{21}u_1 + \hat{H}_{22}v_1 + \hat{H}_{23}w_1}{\hat{H}_{31}u_1 + \hat{H}_{32}v_1 + \hat{H}_{33}w_1} = \frac{\hat{H}_{21}x_1 + \hat{H}_{22}y_1 + \hat{H}_{23}}{\hat{H}_{31}x_1 + \hat{H}_{32}y_1 + \hat{H}_{33}}$$

$$x_{2i}x_{1i}\hat{H}_{31} + x_{2i}y_{1i}\hat{H}_{32} + x_{2i}\hat{H}_{33} - x_{1i}\hat{H}_{11} - y_{1i}\hat{H}_{12} - \hat{H}_{13} = 0$$

$$y_{2i}x_{1i}\hat{H}_{31} + y_{2i}y_{1i}\hat{H}_{32} + y_{2i}\hat{H}_{33} - x_{1i}\hat{H}_{21} - y_{1i}\hat{H}_{22} - \hat{H}_{23} = 0$$

$$\begin{pmatrix} -x_{1i} & -y_{1i} & -1 & 0 & 0 & 0 & x_{2i}x_{1i} & x_{2i}y_{1i} & x_{2i} \\ 0 & 0 & 0 & -x_{1i} & -y_{1i} & -1 & y_{2i}x_{1i} & y_{2i}y_{1i} & y_{2i} \\ & & & & & & & & \end{pmatrix} \begin{pmatrix} \hat{H}_{11} \\ \hat{H}_{12} \\ \hat{H}_{13} \\ \hat{H}_{21} \\ \hat{H}_{22} \\ \hat{H}_{23} \\ \hat{H}_{31} \\ \hat{H}_{32} \\ \hat{H}_{33} \end{pmatrix} = \mathbf{0}$$

$$\mathbf{Ah} = \mathbf{0}, \mathbf{A} \text{ has size } 2N \times 9, \mathbf{h} \text{ has size } 9 \times 1$$

There will be *N* such pairs of equations (i.e. totally 2*N* equations), given *N* pairs of corresponding points in the two images

The equation **Ah** = **0** will be solved by computing the SVD of **A**, i.e. **A** = **USV**^{T}. The vector **h** will be given by the singular vector in corresponding to the null singular value (in the ideal case) or the null singular value.

Application: RANSAC to determine Homography between two Images

- Determine sets Q_1 and Q_2 of salient feature points in both images, using the SIFT algorithm.
- Q_1 and Q_2 may have different sizes! Determine the matching points between Q_1 and Q_2 using methods such as matching of SIFT descriptors.
- Many of these matches will be near-accurate, but there will be outliers too!

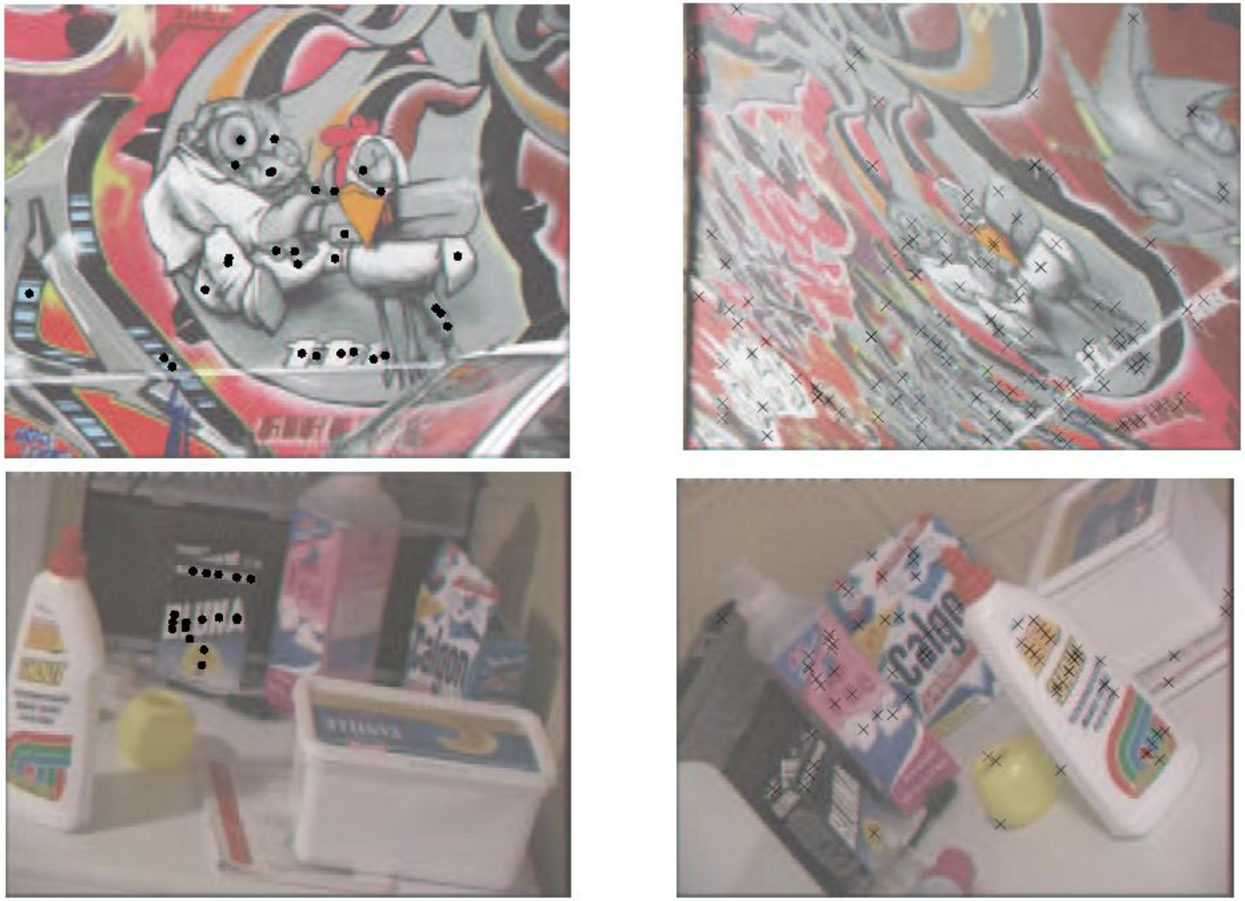


Figure 5.4.: Image pairs and detected points used in homography experiments (D and E). Inliers are marked as dots in left images and outliers as crosses in right images.

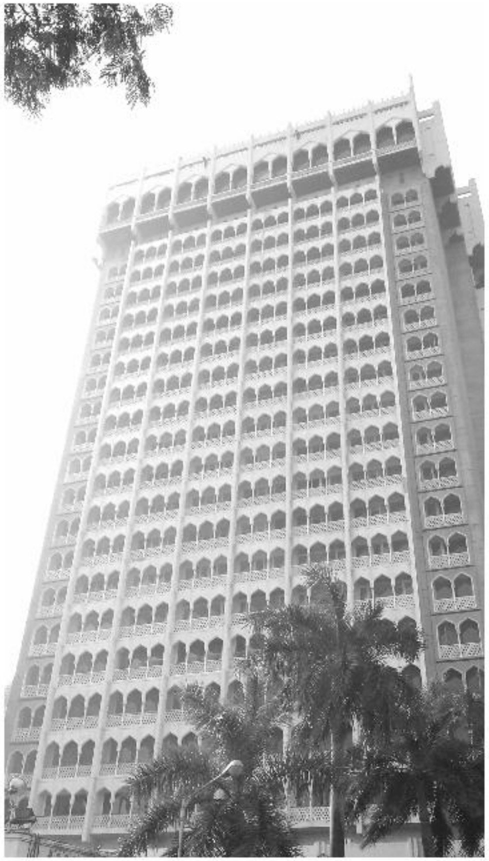
Application: RANSAC to determine Homography between two Images

- Pick a set of any *k* = 4 pairs of points and determine homography matrix **H** using SVD based method.
- Determine the number of inliers – i.e. those point pairs for which:

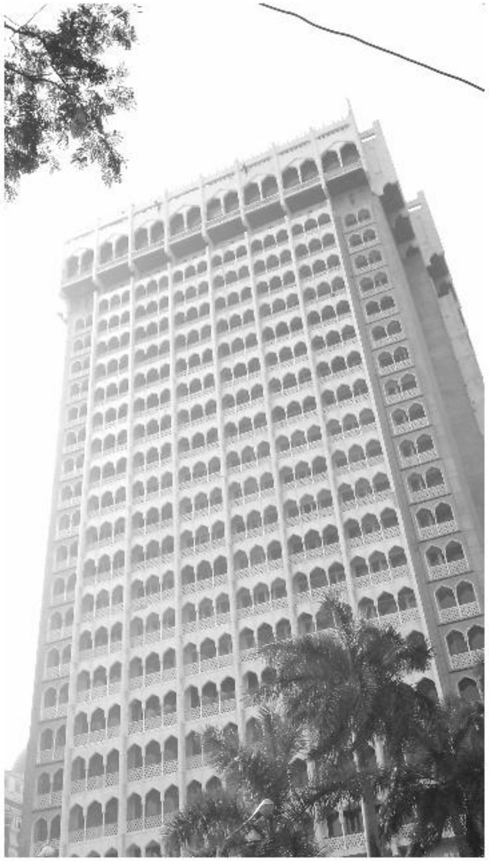
$$\left\| \mathbf{q}_{1i} - \mathbf{H}\mathbf{q}_{2i} \right\|_2^2 \leq \varepsilon$$

- Select the estimate of **H** corresponding to the set that yielded maximum number of inliers!

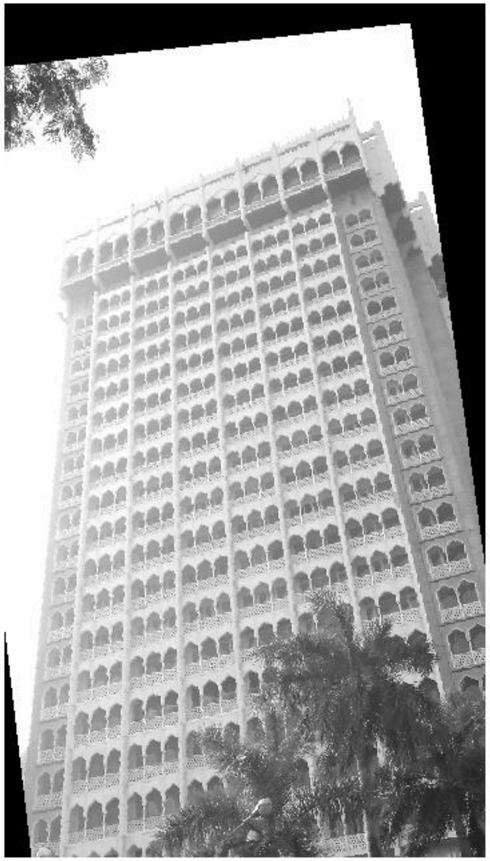
1st image



2nd image



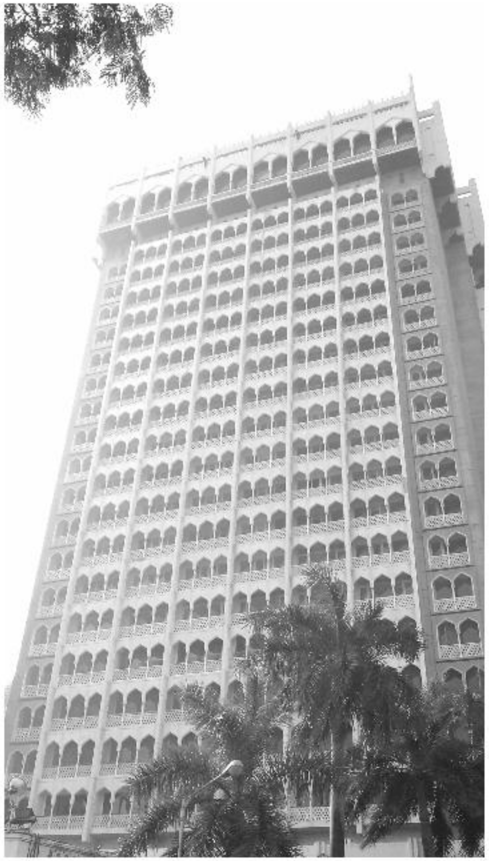
1st image: warped using estimated **H**



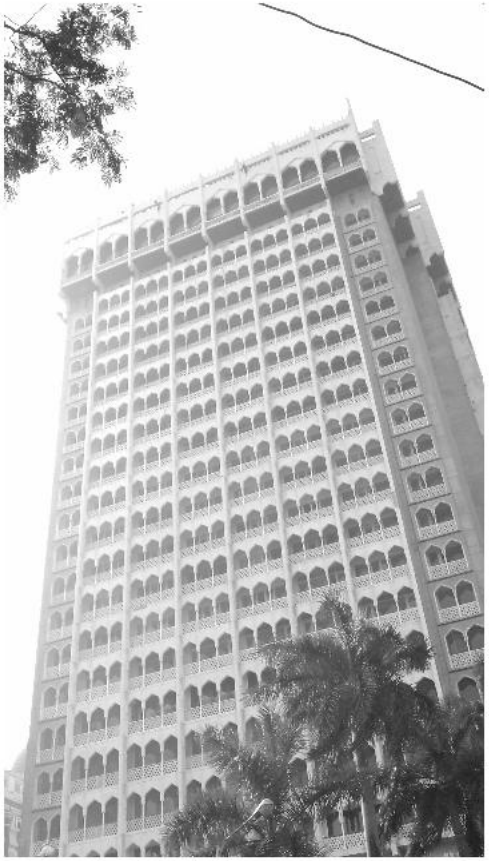
H =

```
   0.57882301155793   0.06780863137907 -28.33314842189324
  -0.06084045669542   0.56283594396435  30.61319941910327
   0.00002958152711  -0.00003144483692   0.58195535780312
```

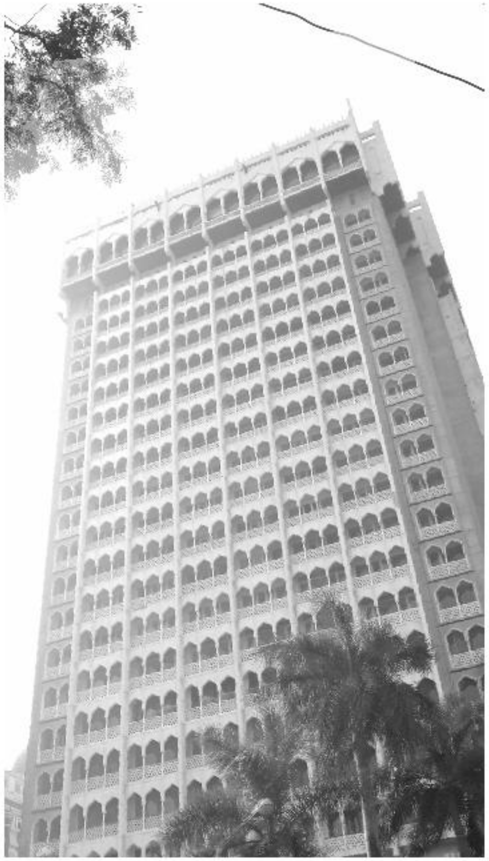
RANSAC result with 41% inliers (threshold on squared distance was 0.1) – point matching done using minor-Eigenvalue method with SSD based matching of 9 x 9 windows in a 50 x 50 neighborhood



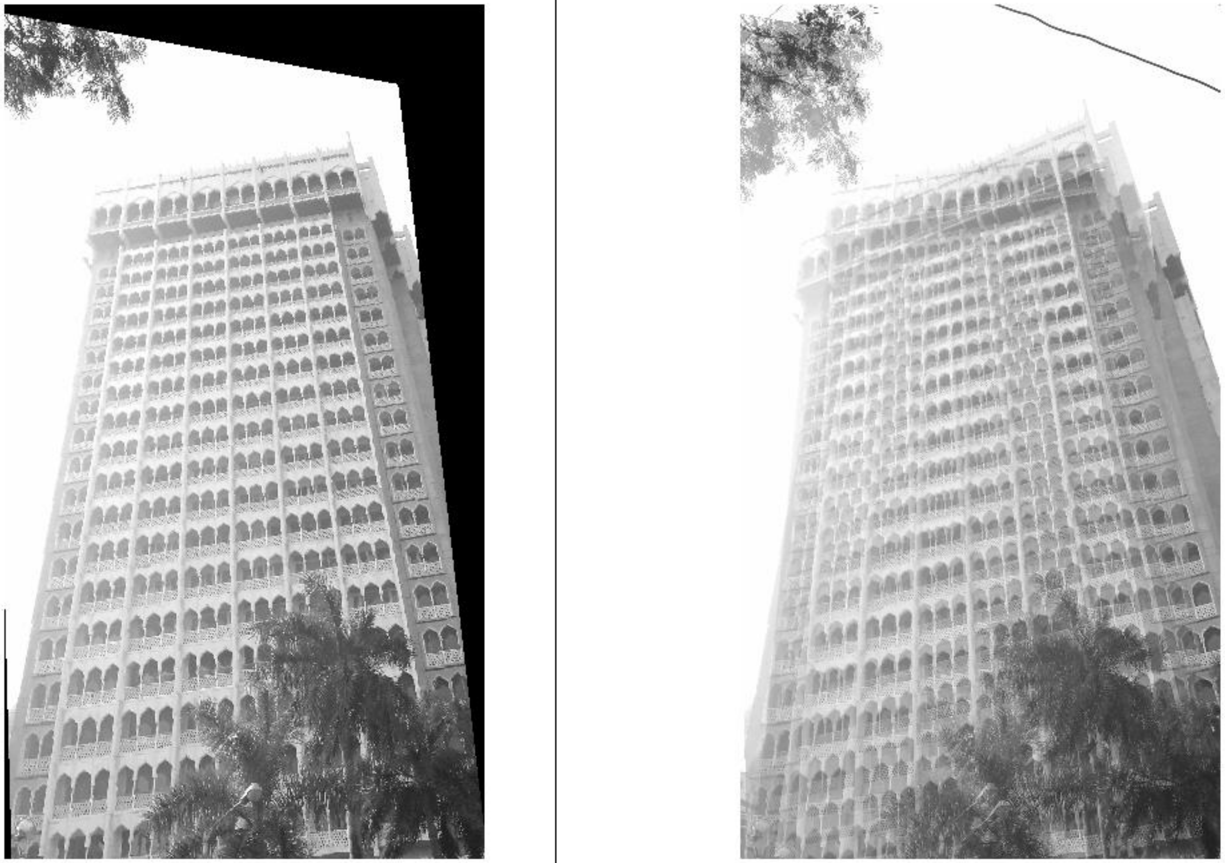
1st image



2nd image



1st image: warped using estimated **H** and overlapped/merged with 2nd image – to show accuracy of alignment



Left: Result of warping 1st image using **H** estimated with simple least-squares on the matching points (No RANSAC).
Right: Result merged with 2nd image.
Notice that the estimation is quite poor.

Some cautions with RANSAC

- Consider a dataset with a cloud of points all close to each other (degenerate set). A model created from an outlier point and any point from a degenerate set will have a large consensus set!

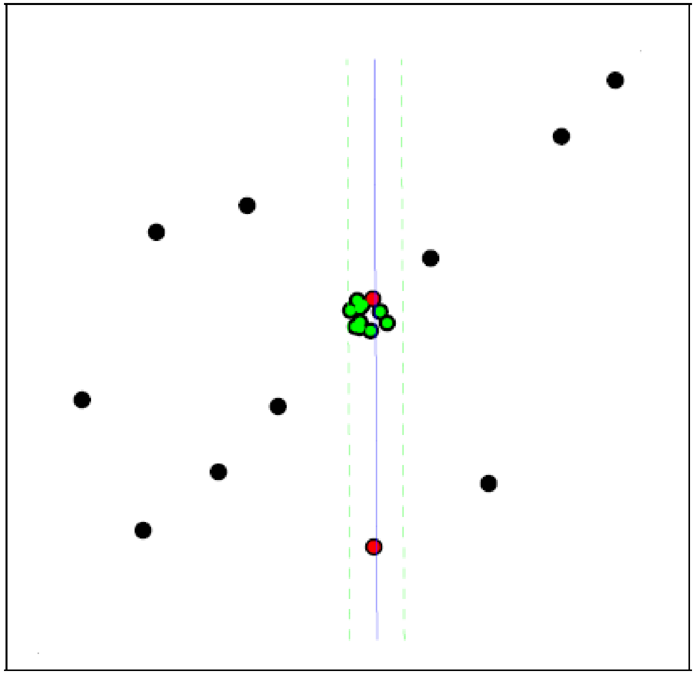


Image taken from Ph.D. thesis of Ondrej Chum, Czech Technical University, Prague

Some cautions with RANSAC

- A model created from a set of inliers need not always be optimal, i.e. it may have a very small consensus set.

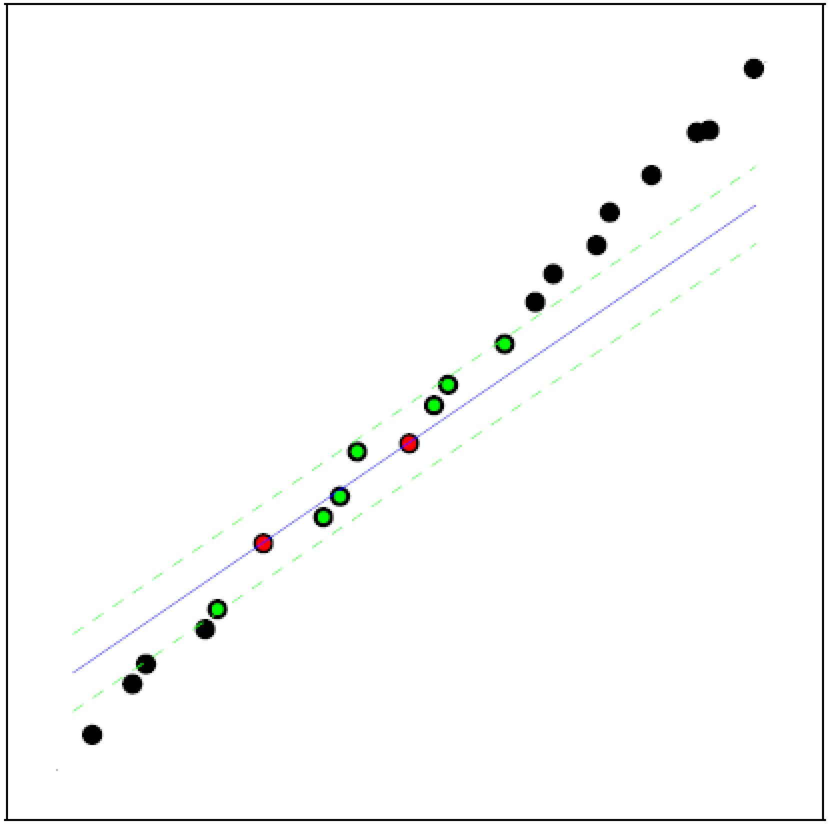


Image taken from Ph.D. thesis of Ondrej Chum, Czech Technical University, Prague

References

- Appendix A.7 of Trucco and Verri
- Article on Robust statistics by Chuck Stewart
- Original article on RanSaC by Fischler and Bolles
- Article on RanSaC variants by Torr and Zisserman