ME630 Assignment 3

Aman Parekh (180073)

November 5, 2021

Burger's Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0$$

Explicit Method

For solving using the explicit method, we will apply the following discretization scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + u_j^n \frac{u_j^n - u_{j-1}^n}{\Delta x} - \nu\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = 0$$

$$\implies \boxed{u_j^{n+1} = u_j^n - \left(\frac{\Delta t}{\Delta x}\right) u_j^n (u_j^n - u_{j-1}^n) + \left(\frac{\nu \Delta t}{\Delta x^2}\right)(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}$$

The above is an explicit method as all terms on the RHS are known. This can be coded to arrive at a solution, shown ahead.

Note: In the above method, backward difference in space was used to discretize the $\frac{\partial u}{\partial x}$ term. Similarly, central difference could be used to arrive at the following scheme:

$$\implies \boxed{u_j^{n+1} = u_j^n - \left(\frac{\Delta t}{2\Delta x}\right) u_j^n (u_{j+1}^n - u_{j-1}^n) + \left(\frac{\nu \Delta t}{\Delta x^2}\right)(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}$$

or, forward difference could be used to arrive at the following scheme:

$$\implies \boxed{u_j^{n+1} = u_j^n - \left(\frac{\Delta t}{\Delta x}\right) u_j^n (u_{j+1}^n - u_j^n) + \left(\frac{\nu \Delta t}{\Delta x^2}\right)(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}$$

Plots for all three of the above methods are shown ahead.

In the lectures, the central difference scheme failed because the viscosity was zero, and the value of k_{num} was negative. That is why Lax-Wendroff was used to introduce numerical diffusion.

However, in the above Burger's equation, the viscosity is non-zero and diffusion exists naturally, therefore the central difference scheme does not fail. The upwind schemes(forward and backward difference) also work.

Implicit Method

The implicit method leads to the formation of the following finite difference stencil:

$$\boxed{u_j^{n+1} + 0.5\Delta t[L_x(u_j^n u_j^{n+1}) - \nu L_{xx}(u_j^{n+1})] = u_j^n + 0.5\nu\Delta t L_{xx}(u_j^n)}$$

This will give a Tri-Diagonal System with the following array entries:

$$a_j^n = -0.25\left(\frac{\Delta t}{\Delta x}\right)u_{j-1}^n - 0.5s$$

$$b_j^n = 1 + s$$

$$c_j^n = 0.25\left(\frac{\Delta t}{\Delta x}\right)u_{j+1}^n - 0.5s$$

$$d_j^n = 0.5u_{j-1}^n + (1-s)u_j^n + 0.5su_j^{n+1}$$

This can be solved using TDMA. Plots shown ahead.

Exact Solution

$$\bar{u} = \frac{\int_{-\infty}^{\infty}\left[\frac{x-\xi}{t}\right]e^{-0.5ReG}d\xi}{\int_{-\infty}^{\infty}e^{-0.5ReG}d\xi}$$

where G is given as,

$$G = \int_0^{\xi} u_0(\xi')d\xi' + \frac{0.5(x-\xi)^2}{t}$$

where u_0 is the initial condition and $Re = {}^1/_{\nu}$.

The integral is broken up in the following manner to facilitate solving it.

$$\bar{u} = \frac{\int_{-\infty}^{0}\left[\frac{x-\xi}{t}\right]e^{-0.5Re\left(\xi+\frac{0.5(x-\xi)^2}{t}\right)}d\xi + \int_0^{\infty}\left[\frac{x-\xi}{t}\right]e^{-0.5Re\left(\frac{0.5(x-\xi)^2}{t}\right)}d\xi}{\int_{-\infty}^{0}e^{-0.5Re\left(\xi+\frac{0.5(x-\xi)^2}{t}\right)}d\xi + \int_0^{\infty}e^{-0.5Re\left(\frac{0.5(x-\xi)^2}{t}\right)}d\xi}$$

This is possible as when $\xi < 0$

$$\int_0^{\xi} u_0(\xi')d\xi' = \int_0^{\xi} 1d\xi' = \xi$$

and when $\xi > 0$

$$\int_0^{\xi} u_0(\xi')d\xi' = \int_0^{\xi} 0d\xi' = 0$$

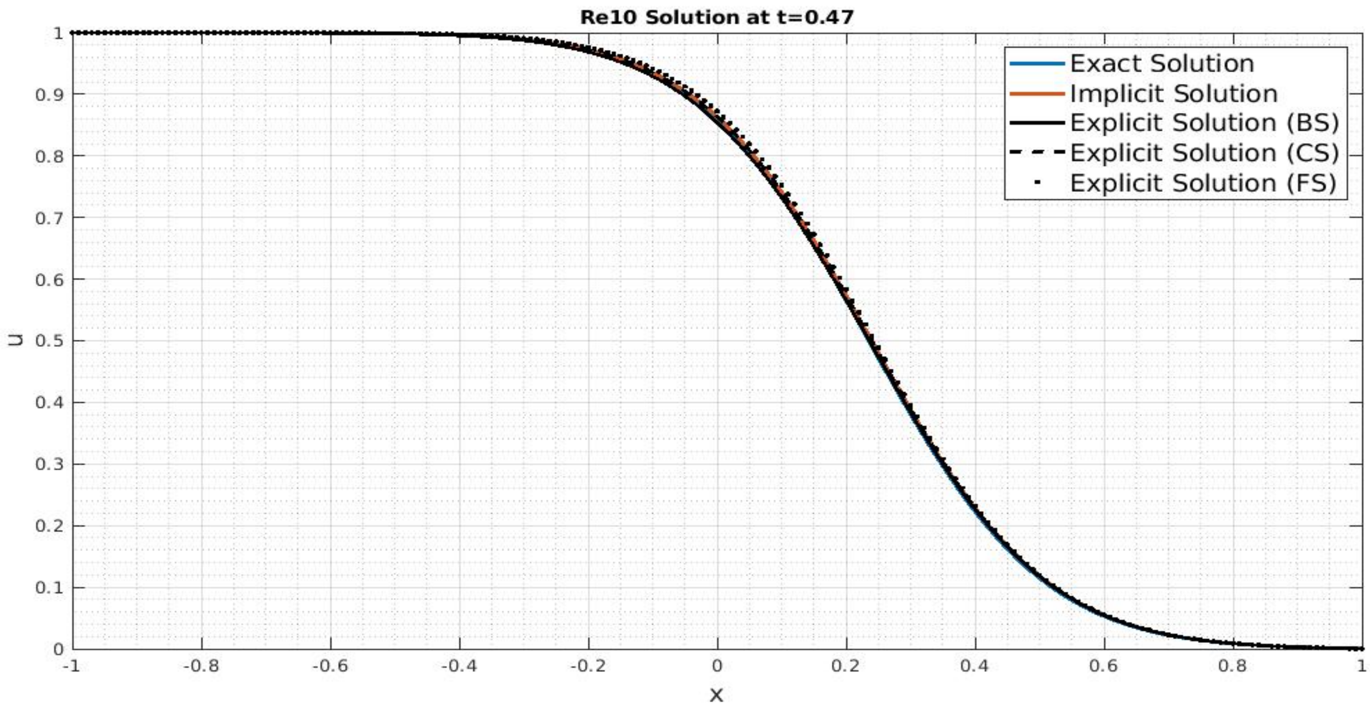

Figure 1: Plot for Re10

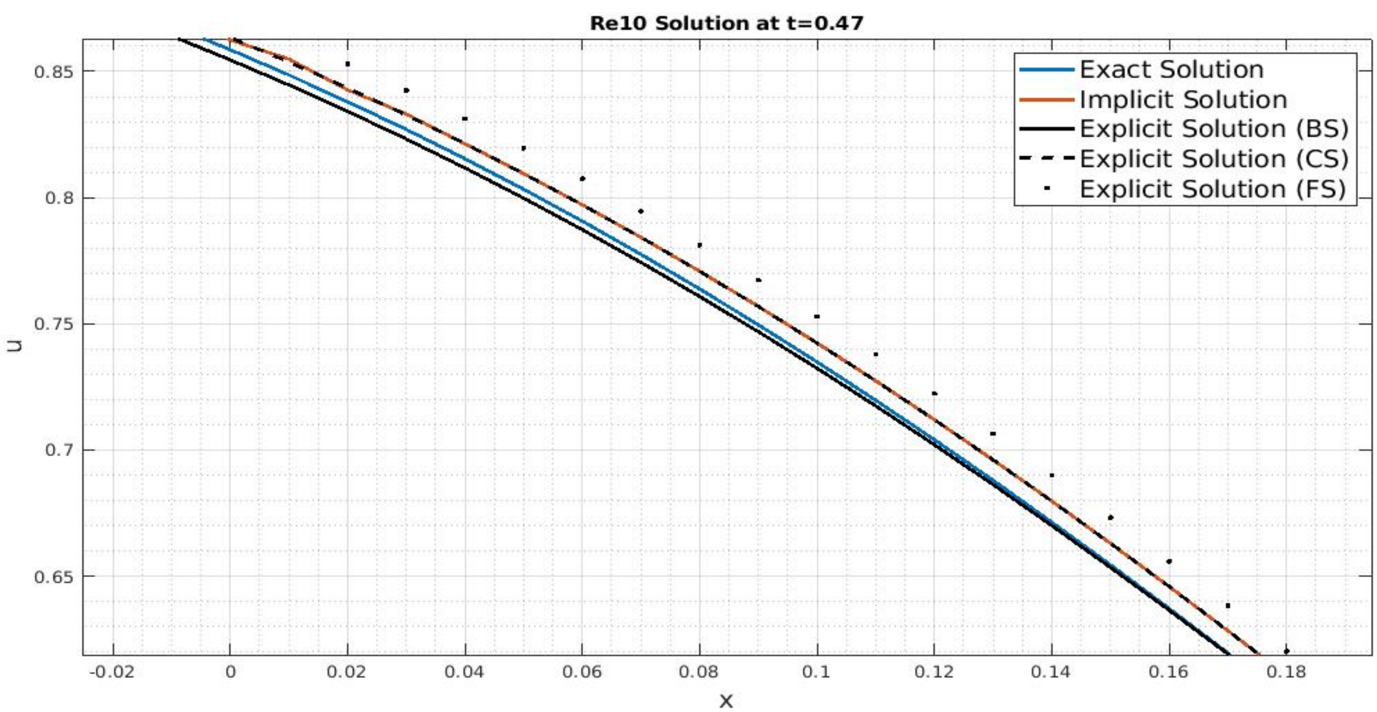

Figure 2: Plot for Re10 [Zoomed]

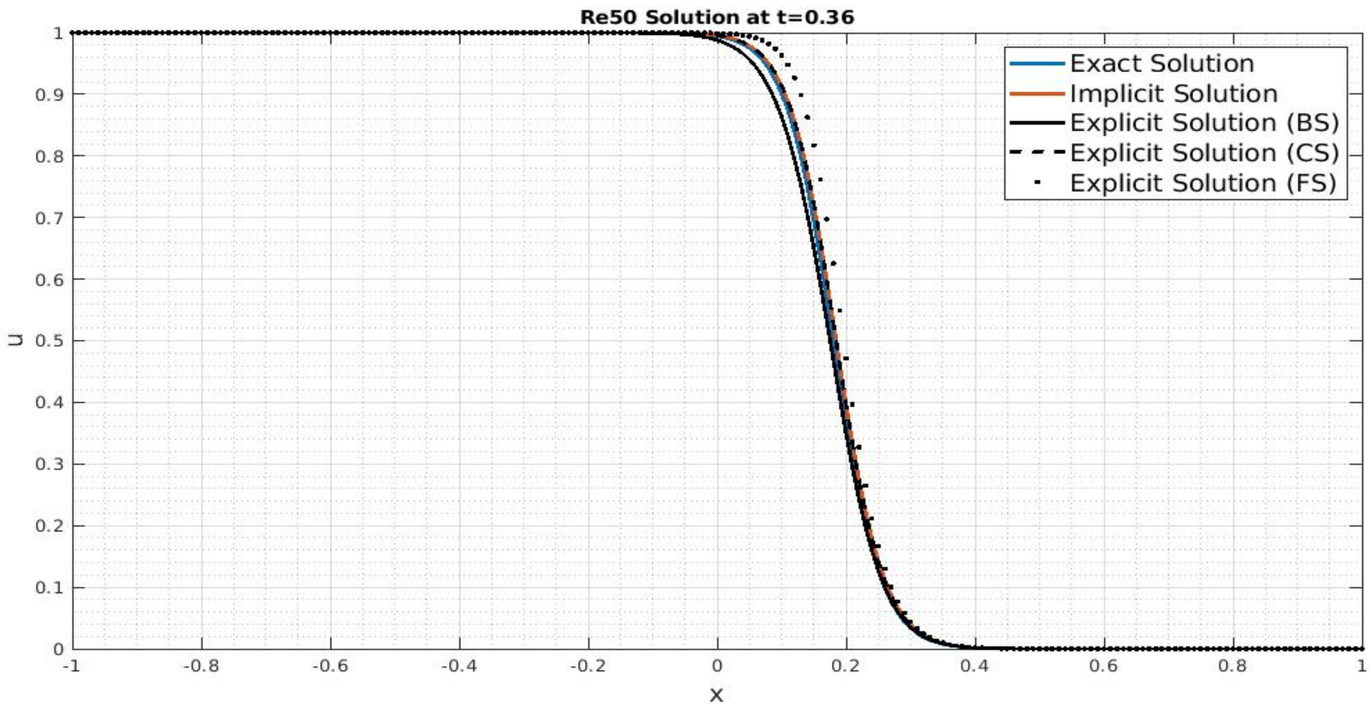

Figure 3: Plot for Re50

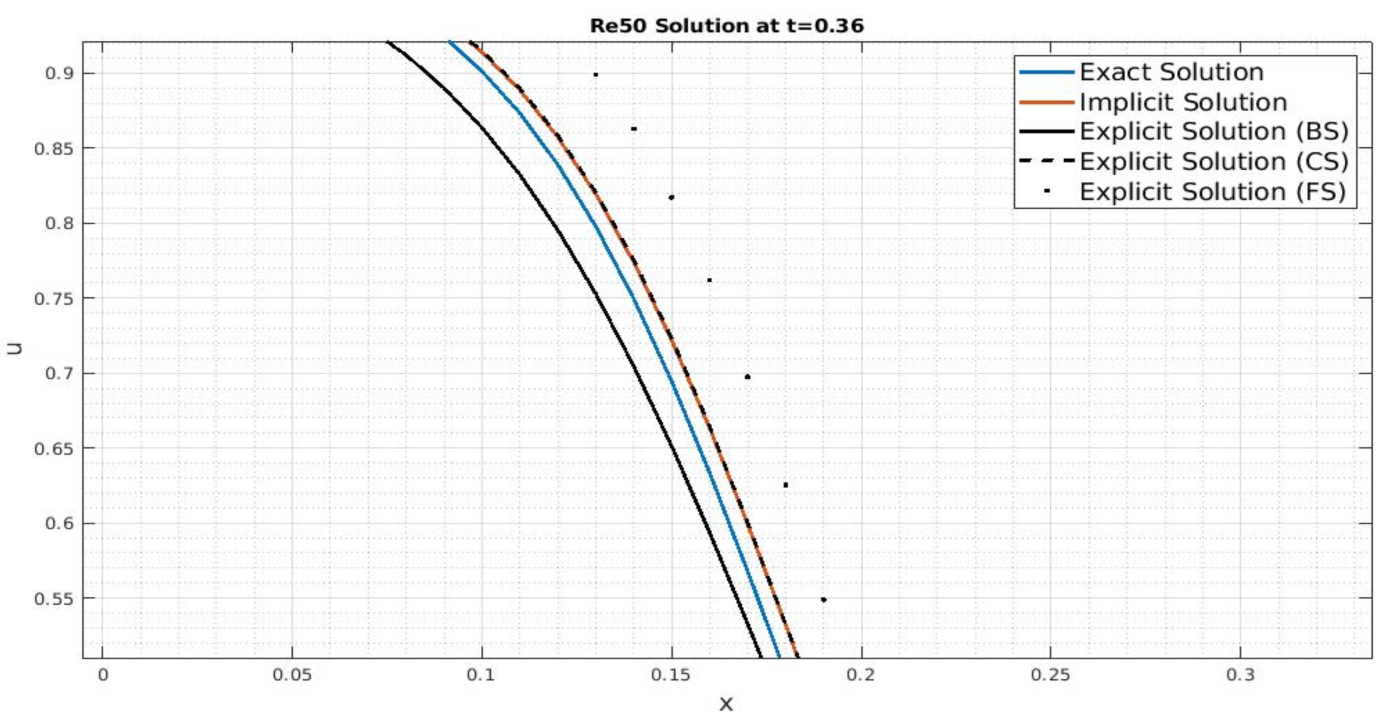

Figure 4: Plot for Re50 [Zoomed]

Observations

1. Both, implicit and explicit, solutions agree with the exact solution to a large extent.
2. The explicit scheme only shows diffusion, and no dispersion. Thus, the truncation terms would not have odd-ordered derivatives of space, but only even-ordered derivatives.
3. With increase in Reynolds Number, the upwind schemes(BS and FS) are getting less accurate. This is expected as by increasing Re, we are heading towards an inviscid burger's equation, which, as derived in the lectures is only 1^{st} order accurate in time.
4. The explicit solution required a smaller timestep compared to the implicit solution.
5. Even for $u > 0$, the FTFS scheme works as there is natural diffusion in the original equation.

(**Note:** For the datafiles generated in the codes. The first row is that of the x-discretization, and the last row is the final solution at the specified time.)