8.1 Span of a Set of Vectors

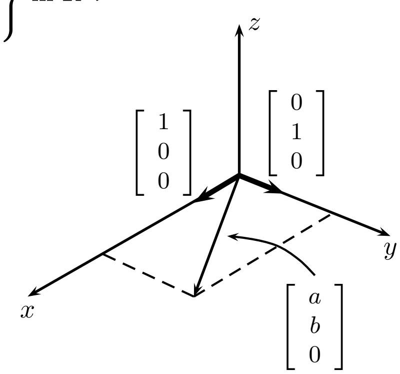
Performance Criteria:

8. (a) Describe the span of a set of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ as a line or plane containing a given set of points.

   (b) Determine whether a vector $\mathbf{w}$ is in the span of a set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ of vectors. If it is, write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$.

DEFINITION 8.1.1: The **span of a set $\mathcal{S}$ of vectors**, denoted span$(\mathcal{S})$ is the set of all linear combinations of those vectors.

◇ **Example 8.1(a):** Describe the span of the set $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.



Note that ANY vector with a zero third component can be written as a linear combination of these two vectors:

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

All the vectors with $x_3 = 0$ (or $z = 0$) are the xy plane in $\mathbb{R}^3$, so the span of this set is the xy plane. Geometrically we can see the same thing in the picture to the right. ♠

◇ **Example 8.1(b):** Describe span$\left(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right)$.

By definition, the span of this set is all vectors $\mathbf{v}$ of the form

$$\mathbf{v} = c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix},$$

which, because the two vectors are not scalar multiples of each other, we recognize as being a plane through the origin. It should be clear that all vectors created by such a linear combination will have a third component of zero, so the particular plane that is the span of the two vectors is the xy-plane. Algebraically we see that any vector $[a, b, 0]$ in the xy-plane can be created by

$$\left(\frac{a-3b}{7}\right) \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \left(\frac{2a+b}{7}\right) \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{a-3b}{7} \\ \frac{-2a+6b}{7} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{6a+3b}{7} \\ \frac{2a+b}{7} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{7a}{7} \\ \frac{7b}{7} \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \quad ♠$$

You might wonder how one would determine the scalars $\frac{a-3b}{7}$ and $\frac{2a+b}{7}$. You will see how this is done in the exercises!

At this point we should make a comment and a couple observations:

- First, some language: we can say that the span of the two vectors in Example 8.1(b) is the xy-plane, but we also say that the two vectors span the xy-plane. That is, the word span is used as either a noun or a verb, depending on how it is used.
- Note that in the two examples above we considered two different sets of two vectors, but in each case the span was the same. This illustrates that *different sets of vectors can have the same span.*
- Consider also the fact that if we were to include in either of the two sets additional vectors that are also in the xy-plane, it would not change the span. However, if we were to add another vector not in the xy-plane, the span would increase to all of $\mathbb{R}^3$.
- In either of the preceding examples, removing either of the two given vectors would reduce the span to a linear combination of a single vector, which is a line rather than a plane. But in some cases, removing a vector from a set does not change its span.
- The last two bullet items tell us that *adding or removing vectors from a set of vectors may or may not change its span.* This is a somewhat undesirable situation that we will remedy in the next chapter.
- It may be obvious, but it is worth emphasizing that (in this course) we will consider spans of finite (and usually rather small) sets of vectors, but a span itself always contains infinitely many vectors (unless the set $\mathcal{S}$ consists of only the zero vector).

It is often of interest to know whether a particular vector is in the span of a certain set of vectors. The next examples show how we do this.

$\diamond$ **Example 8.1(c):** Is $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ -4 \\ 1 \end{bmatrix}$ in the span of $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$?

The question is, "can we find scalars c_1, c_2 and c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -4 \\ 1 \end{bmatrix} \text{?"} \tag{1}$$

We should recognize this as the linear combination form of the system of equations below and to the left. The augmented matrix for the system row reduces to the matrix below and to the right.

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 3 \\ 2c_1 - c_2 &= -2 \\ 3c_1 + c_2 - 3c_3 &= -4 \\ 4c_1 - c_2 + c_3 &= 1 \end{aligned} \qquad\qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This tells us that the system above and to the left has no solution, so there are no scalars c_1, c_2 and c_3 for which equation (1) holds. Thus $\mathbf{v}$ is not in the span of $\mathcal{S}$. ♠

$\diamond$ **Example 8.1(d):** Is $\mathbf{v} = \begin{bmatrix} 19 \\ 10 \\ -1 \end{bmatrix}$ in span($\mathcal{S}$), where $\mathcal{S} = \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix} \right\}$?

Here we are trying to find scalars c_1, c_2 and c_3 such that

$$c_1 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 19 \\ 10 \\ -1 \end{bmatrix} \tag{2}$$

We should recognize this as the linear combination form of the system of equations below and to the left. The augmented matrix for the system row reduces to the matrix below and to the right.

$$\begin{aligned} 3c_1 - 5c_2 + \ \ c_3 &= \ \ 19 \\ -c_1 \qquad\ \ + 7c_3 &= \ \ 10 \\ 2c_1 + \ c_2 - \ 4c_3 &= -1 \end{aligned} \qquad\qquad \left[\begin{array}{rrrr} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

This tells us that (2) holds for $c_1 = 4$, $c_2 = -1$ and $c_3 = 2$, so $\mathbf{v}$ is in span($\mathcal{S}$). ♠

Sometimes, with a little thought, no computations are necessary to answer such questions, as the next examples show.

◇ **Example 8.1(e):** Is $\mathbf{v} = \left[\begin{array}{r} -4 \\ 2 \\ 5 \end{array}\right]$ in the span of $\mathcal{S} = \left\{ \left[\begin{array}{r} -3 \\ 0 \\ 2 \end{array}\right], \left[\begin{array}{r} -5 \\ 0 \\ 1 \end{array}\right] \right\}$?

One can see that any linear combination of the two vectors in S will have zero as its second component:

$$c_1 \left[\begin{array}{r} -3 \\ 0 \\ 2 \end{array}\right] + c_2 \left[\begin{array}{r} -5 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} -3c_1 \\ 0 \\ 2c_1 \end{array}\right] + \left[\begin{array}{c} -5c_2 \\ 0 \\ 1c_2 \end{array}\right] = \left[\begin{array}{c} -3c_1 - 5c_2 \\ 0 \\ 2c_1 + c_2 \end{array}\right]$$

Since the second component of $\mathbf{v}$ is not zero, $\mathbf{v}$ is not in the span of the set $\mathcal{S}$. ♠

◇ **Example 8.1(f):** Is $\mathbf{v} = \left[\begin{array}{r} 4 \\ 7 \\ -1 \end{array}\right]$ in span$\left(\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] \right)$?

Here we can see that if we multiply the three vectors in $\mathcal{S}$ by 4, 7 and -1, respectively, and add them, the result will be $\mathbf{v}$:

$$4 \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] + 7 \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right] - 1 \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} 4 \\ 0 \\ 0 \end{array}\right] + \left[\begin{array}{c} 0 \\ 7 \\ 0 \end{array}\right] + \left[\begin{array}{r} 0 \\ 0 \\ -1 \end{array}\right] = \left[\begin{array}{r} 4 \\ 7 \\ -1 \end{array}\right]$$

Therefore $\mathbf{v}$ is in span$\left(\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] \right)$. ♠

Sometimes we will be given an infinite set of vectors, and we'll ask whether a particular finite set of vectors *spans the infinite set*. By this we are asking whether the span of the finite set is the infinite set. For example, we might ask whether the vector $\mathbf{v} = [2, 3]$ spans $\mathbb{R}^2$. Because the span of the single vector $\mathbf{v}$ is just a line, $\mathbf{v}$ does not span $\mathbb{R}^2$. With the knowledge we have at this point, it can sometimes be difficult to tell whether a finite set of vectors spans a particular infinite set. The next chapter will give us a means for making such a judgement a bit easier.

We conclude with a few more observations. With a little thought, the following can be seen to be true. (Assume all vectors are non-zero.)

- The span of a single vector is all scalar multiples of that vector. In $\mathbb{R}^2$ or $\mathbb{R}^3$ the span of a single vector is a line through the origin.
- The span of a set of two non-parallel vectors in $\mathbb{R}^2$ is all of $\mathbb{R}^2$. In $\mathbb{R}^3$ it is a plane through the origin.
- The span of three vectors in $\mathbb{R}^3$ that do not lie in the same plane is all of $\mathbb{R}^3$.

Section 8.1 Exercises

1. Describe the span of each set of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ by telling what it is geometrically and, if it is a standard set like one of the coordinate axes or planes, specifically what it is. If it is a line that is not one of the axes, give two points on the line. If it is a plane that is not one of the coordinate planes, give three points on the plane.

   (a) The vector $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ in $\mathbb{R}^2$.

   (b) The set of vectors $\left\{ \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

   (c) The vectors $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ in $\mathbb{R}^2$.

   (d) The set $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

   (e) The vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ in $\mathbb{R}^3$.

2. For each of the following, determine whether the vector $\mathbf{w}$ is in the span of the set S. If it is, write it as a linear combination of the vectors in S.

   (a) $\mathbf{w} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}, \qquad S = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -11 \\ -1 \end{bmatrix} \right\}$

   (b) $\mathbf{w} = \begin{bmatrix} -5 \\ -23 \\ 12 \\ 8 \end{bmatrix}, \qquad S = \left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -4 \\ 5 \end{bmatrix} \right\}$

   (c) $\mathbf{w} = \begin{bmatrix} 8 \\ 38 \\ -14 \\ 11 \end{bmatrix}, \qquad S = \left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -4 \\ 5 \end{bmatrix} \right\}$

   (d) $\mathbf{w} = \begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix}, \qquad S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$