Principles of Programming Languages
Assignment 3

18CS30047 SOMNATH JENA

1a) Assuming true : Bool $\in \mathcal{E}_C$

Given $\mathcal{E}_0 \cup \{y : \text{Ref Bool}\}$

$\mathcal{E}_0 \cup \{y : \text{Ref Bool}\} \vdash y : \text{Ref Bool}$ — ① By Identifier rule

$\mathcal{E}_0 \cup \{y : \text{Ref Bool}\} \vdash \text{true} : \text{Bool}$ — ② By Constant rule

$\boxed{\mathcal{E}_0 \cup \{y : \text{Ref Bool}\} \vdash y := \text{true} : \text{Command}}$ — ③ By Assignment rule

$$\left[\frac{\mathcal{E} \vdash N : \text{Ref } T, \quad \mathcal{E} \vdash M : T}{\mathcal{E} \vdash N := M : \text{Command}}\right]$$

$$\frac{\mathcal{E}_0 \cup \{y : \text{Ref Bool}\} \vdash y : \text{Ref Bool} \;(1) \qquad \mathcal{E}_0 \cup \{y : \text{Ref Bool}\} \vdash \text{true} : \text{Bool} \;(2)}{\therefore \mathcal{E}_0 \cup \{y : \text{Ref Bool}\} \vdash y := \text{true} : \text{Command}}$$

1b) Given func1 : $A \to B$, func2 : $C \to B$

$\mathcal{E}_0 \cup \{x : A\} \vdash \text{func1} : A \to B$ — ① By Constant rule

$\mathcal{E}_0 \cup \{x : A\} \vdash x : A$ — ② By Identifier rule

$\mathcal{E}_0 \cup \{x : A\} \vdash \text{func1 } x : B$ — ③ By Application rule, ① & ②

$$\left[\frac{\mathcal{E} \vdash M : S \to T, \quad \mathcal{E} \vdash N : S}{\mathcal{E} \vdash M\,N : T}\right]$$

$\mathcal{E}_0 \cup \{x : A\} \vdash (\text{func1 } x) : B$ — ④ By Paren rule & ③

$\mathcal{E}_0 \vdash \lambda(x : A).(\text{func1 } x) : A \to B$ — ⑤ By Function rule & ④

$$\left[\frac{\mathcal{E} \cup \{x : S\} \vdash M : T}{\mathcal{E} \vdash \lambda(x : S).M : S \to T}\right]$$

$\mathcal{E}_0 \cup \{q : C\} \vdash \text{func2} : C \to B$ — ⑥ By Constant rule

$\mathcal{E}_0 \cup \{q : C\} \vdash q : C$ — ⑦ By Identifier rule

$\mathcal{E}_0 \cup \{q : C\} \vdash \text{func2 } q : B$ — ⑧ By Application rule, ⑥ & ⑦

$$\left[\frac{\mathcal{E} \vdash M : S \to T, \quad \mathcal{E} \vdash N : S}{\mathcal{E} \vdash M\,N : T}\right]$$

$\mathcal{E}_0 \cup \{q : C\} \vdash (\text{func2 } q) : B$ — ⑨ By Paren rule & ⑧

$\mathcal{E}_0 \vdash \lambda(q : C).(\text{func2 } q) : C \to B$ — ⑩ By Function rule & ⑨

$$\left[\frac{\mathcal{E} \cup \{x : S\} \vdash M : T}{\mathcal{E} \vdash \lambda(x : S).M : S \to T}\right]$$

$$\boxed{\begin{gathered}\mathcal{E}_0 \vdash \lambda(x : A).(\text{func1 } x) \cancel{: A \to B}; \\ \lambda(q : C).(\text{func2 } q) \cancel{: C \to B} : C \to B\end{gathered}}$$ — ⑪ By Sequencing rule ⑤ & ⑩

$$\left[\frac{\mathcal{E} \vdash M : S, \quad \mathcal{E} \vdash N : T}{\mathcal{E} \vdash M; N : T}\right]$$

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$$\dfrac{\dfrac{\dfrac{\xi_0 \cup \{x:A\} \vdash func1 : A \to B \quad ① \qquad \xi_0 \cup \{x:A\} \vdash x : A \quad ②}{\xi_0 \cup \{x:A\} \vdash (func1\ x) : B} ③\&④}{\xi_0 \vdash \lambda(x:A).(func1\ x) : A \to B} ⑤}{}$$

$$\dfrac{\dfrac{\xi_0 \cup \{q:C\} \vdash func2 : C \to B \quad ⑥ \qquad \xi_0 \cup \{q:C\} \vdash q : C \quad ⑦}{\xi_0 \cup \{q:C\} \vdash (func2\ q) : B} ⑧\&⑨}{\xi_0 \vdash \lambda(q:C).(func2\ q) : C \to B} ⑪$$

$\therefore \xi_0 \vdash \lambda(x:A).(func1\ x) : \cancel{A \to B} ; \lambda(q:C).(func2\ q) : C \to B$

c) Given $1 : Bool \to Bool \to Bool$, $true : Bool$

Let $\xi_1 = \xi_0 \cup \{w : Bool \to \pi\}$

$\xi_1 \cup \{x : Bool\} \vdash x : Bool$ —① By Identifier rule

$\xi_1 \cup \{x : Bool\} \vdash 1 : Bool \to Bool \to Bool$ —② By Constant rule

$\xi_1 \cup \{x : Bool\} \vdash 1x : Bool \to Bool$ —③ By Application rule, ① & ②

$$\left[\frac{\mathcal{E} \vdash M : S \to T,\ \mathcal{E} \vdash N : S}{\mathcal{E} \vdash M\ N : T}\right]$$

$\xi_1 \cup \{x : Bool\} \vdash true : Bool$ —④ By Constant rule

$\xi_1 \cup \{x : Bool\} \vdash 1x\ true : Bool$ —⑤ By Application rule, ③ & ④

$$\left[\frac{\mathcal{E} \vdash M : S \to T,\ \mathcal{E} \vdash N : S}{\mathcal{E} \vdash M\ N : T}\right]$$

$\xi_1 \cup \{x : Bool\} \vdash w : Bool \to \pi$ —⑥ By Identifier rule

$\xi_1 \cup \{x : Bool\} \vdash (1x\ true) : Bool$ —⑦ By Paren rule & ⑤

$\xi_1 \cup \{x : Bool\} \vdash w\ (1x\ true) : \pi$ —⑧ By Application rule ⑥ & ⑦

$$\left[\frac{\mathcal{E} \vdash M : S \to T,\ \mathcal{E} \vdash N : S}{\mathcal{E} \vdash M\ N : T}\right]$$

$\xi_1 \cup \{x : Bool\} \vdash (w\ (1x\ true)) : \pi$ —⑨ By Paren rule & ⑧

$\xi_1 \vdash \lambda(x : Bool).(w(1x\ true)) : Bool \to \pi$ —⑩ By Function rule & ⑨

$$\left[\frac{\mathcal{E} \cup \{x : S\} \vdash M : T}{\mathcal{E} \vdash \lambda(x:S).M : S \to T}\right]$$

$$\boxed{\xi_0 \vdash \lambda(w : Bool \to \pi).\lambda(x : Bool).(w(1x\ true)) : Bool \to \pi \to Bool \to \pi}$$

—⑪ By Function rule & ⑩

$$\left[\frac{\mathcal{E} \cup \{x : S\} \vdash M : T}{\mathcal{E} \vdash \lambda(x:S).M : S \to T}\right]$$

$$\dfrac{\dfrac{\xi_1 \cup \{x:Bool\} \vdash 1 : Bool \to Bool \to Bool \quad ② \qquad \xi_1 \cup \{x:Bool\} \vdash x : Bool \quad ①}{\xi_1 \cup \{x:Bool\} \vdash 1x : Bool \to Bool} ③ \qquad \xi_1 \cup \{x:Bool\} \vdash true : Bool \quad ④}{\xi_1 \cup \{x:Bool\} \vdash (1x\ true)\ Bool} ⑤\&⑦$$

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$$\dfrac{\dfrac{\dfrac{\mathcal{E}_1 \cup \{x: Bool\} \vdash 1 : Bool \to Bool \to Bool \quad (2) \qquad \mathcal{E}_1 \cup \{x: Bool\} \vdash x : Bool \quad (1)}{\mathcal{E}_1 \cup \{x : Bool\} \vdash 1x : Bool \to Bool \qquad \mathcal{E}_1 \cup \{x : Bool\} \vdash true : Bool \quad (3)}}{\mathcal{E}_1 \cup \{x: Bool\} \vdash \omega : Bool \to \pi \quad (6) \qquad \mathcal{E}_1 \cup \{x : Bool\} \vdash (1\, x\, true) : Bool \quad (5)\,\&\,(7)}}{\dfrac{\mathcal{E}_1 \cup \{x : Bool\} \vdash (\omega(1\, x\, true)) : \cancel{Bool \to}\, \pi \quad (8)\,\&\,(9)}{\dfrac{\mathcal{E}_1 \vdash \lambda(x : Bool).(\omega(1\, x\, true)) : Bool \to \pi \quad (10)}{\mathcal{E}_0 \vdash \lambda(\omega : Bool \to \pi).\lambda(x : Bool).(\omega(1\, x\, true)) : Bool \to \pi \to Bool \to \pi \quad (11)}}}$$

d) Given $+ : S \to S$

Let $\mathcal{E}_1 = \mathcal{E}_0 \cup \{f : S \to C\}$

$\mathcal{E}_1 \cup \{x : S\} \vdash + : S \to S$ — (1) By Constant rule

$\mathcal{E}_1 \cup \{x : S\} \vdash x : S$ — (2) By Identifier rule

$\mathcal{E}_1 \cup \{x : S\} \vdash + x : S$ — (3) By Application rule, (1) & (2) $\left[\dfrac{\mathcal{E} \vdash M : S \to T,\ \mathcal{E} \vdash N : S}{\mathcal{E} \vdash M N : T}\right]$

$\mathcal{E}_1 \cup \{x : S\} \vdash (+x) : S$ — (4) By Paren rule & (3)

$\mathcal{E}_1 \cup \{x : S\} \vdash f : S \to C$ — (5) By Identifier rule

$\mathcal{E}_1 \cup \{x : S\} \vdash f(+x) : C$ — (6) By Application rule, (4) & (5) $\left[\dfrac{\mathcal{E} \vdash M : S \to T,\ \mathcal{E} \vdash N : S}{\mathcal{E} \vdash M N : T}\right]$

$\mathcal{E}_1 \vdash \lambda(x : S).f(+x) : S \to C$ — (7) By Function rule & (6) $\left[\dfrac{\mathcal{E} \cup \{x : S\} \vdash M : T}{\mathcal{E} \vdash \lambda(x : S).M : S \to T}\right]$

$$\boxed{\mathcal{E}_0 \vdash \lambda(f : S \to C).\lambda(x : S).f(+x) : S \to C \to S \to C}$$

— (8) By Function rule & (7) $\left[\dfrac{\mathcal{E} \cup \{x : S\} \vdash M : T}{\mathcal{E} \vdash \lambda(x : S).M : S \to T}\right]$

$$\dfrac{\mathcal{E}_1 \cup \{x : S\} \vdash f : S \to C \quad (5) \qquad \dfrac{\mathcal{E}_1 \cup \{x : S\} \vdash + : S \to S \quad (1) \qquad \mathcal{E}_1 \cup \{x : S\} \vdash + x : S \quad (2)}{\mathcal{E}_1 \cup \{x : S\} \vdash (+x) : S \quad (3)\,\&\,(4)}}{\dfrac{\mathcal{E}_1 \cup \{x : S\} \vdash f(+x) : C \quad (6)}{\dfrac{\mathcal{E}_1 \vdash \lambda(x : S).f(+x) : S \to C \quad (7)}{\mathcal{E}_0 \vdash \lambda(f : S \to C).\lambda(x : S).f(+x) : S \to C \to S \to C \quad (8)}}}$$

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c) Given $\xi = \{x : Ref\ Bool, y : Bool\}$

$succ : Int \rightarrow Int$, $true : Bool$, $4 : Int$

$\xi_0 \vdash succ : Int \rightarrow Int$ — ① By Constant rule

$\xi_0 \vdash 4 : Int$ — ② By Constant rule

$\xi_0 \vdash succ\ 4 : Int$ — ③ By Application rule ① & ②

$$\left[\frac{\varepsilon \vdash M : S \rightarrow T, \varepsilon \vdash N : S}{\varepsilon \vdash M N : T}\right]$$

$\xi_0 \vdash x : Ref\ Bool$ — ④ By Identifier rule

$\xi_0 \vdash true : Bool$ — ⑤ By Constant rule

$\xi_0 \vdash x := true : Command$ — ⑥ By Assignment rule, ④ & ⑤

$$\left[\frac{\varepsilon \vdash N : Ref\ T, \varepsilon \vdash M : T}{\varepsilon \vdash N := M : Command}\right]$$

$\boxed{\xi_0 \vdash succ\ 4; x := true : Command}$ — ⑦ By Sequencing rule ③ & ⑥

$$\left[\frac{\varepsilon \vdash M : S, \varepsilon \vdash N : T}{\varepsilon \vdash M ; N : T}\right]$$

$$\cfrac{\cfrac{\cfrac{}{\xi_0 \vdash succ : Int \rightarrow Int}① \quad \cfrac{}{\xi_0 \vdash 4 : Int}②}{\xi_0 \vdash succ\ 4 : Int}③ \quad \cfrac{\cfrac{}{\xi_0 \vdash x : Ref\ Bool}④ \quad \cfrac{}{\xi_0 \vdash true : Bool}⑤}{\xi_0 \vdash x := true : Command}⑥}{\xi_0 \vdash succ\ 4; x := true : Command}⑦$$

2.a) Given $\phi : Float \rightarrow Integer$

Let $\xi_1 = \xi_0 \cup \{p : Float \rightarrow Integer\}$

$\xi_2 = \xi_1 \cup \{f : Float \rightarrow Float\}$

$\varepsilon_3 = \xi_2 \cup \{y : Float\}$

$\xi_3 \vdash f : Float \rightarrow Float$ — ① By Identifier rule

$\xi_3 \vdash y : Float$ — ② By Identifier rule

$\xi_3 \vdash f y : Float$ — ③ By Application rule ① & ②

$$\left[\frac{\varepsilon \vdash M : S \rightarrow T, \varepsilon \vdash N : S}{\varepsilon \vdash M N : T}\right]$$

$\xi_3 \vdash (f y) : Float$ — ④ By Paren rule & ③

$\xi_3 \vdash f (f y) : Float$ — ⑤ By Application rule ① & ④

$$\left[\frac{\varepsilon \vdash M : S \rightarrow T, \varepsilon \vdash N : S}{\varepsilon \vdash M N : T}\right]$$

$\xi_3 \vdash (f (f y)) : Float$ — ⑥ By Paren rule & ⑤

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$\xi_3 \vdash p : Float \rightarrow Integer$ — ⑦ By Identifier rule

$\xi_3 \vdash p(f(f\,y)) : Integer$ — ⑧ By Application rule, ⑦ & ⑥ $\left[\dfrac{\xi \vdash M : S \rightarrow T,\ \xi \vdash N : S}{\xi \vdash M\,N : T}\right]$

$\xi_2 \vdash \lambda(y : Float).\,p(f(f\,y)) : Float \rightarrow Integer$ — ⑨ By Function rule & ⑧ $\left[\dfrac{\xi \cup \{x : S\} \vdash M : T}{\xi \vdash \lambda(x : S).\,M : S \rightarrow T}\right]$

$\xi_1 \vdash \lambda(f : Float \rightarrow Float).\,\lambda(y : Float).\,p(f(f\,y)) : Float \rightarrow Float \rightarrow Float \rightarrow Integer$ — ⑩ By Function rule & ⑨ $\left[\dfrac{\xi \cup \{x : S\} \vdash M : T}{\xi \vdash \lambda(x : S).\,M : S \rightarrow T}\right]$

$\xi_0 \vdash \lambda(p : Float \rightarrow Integer).\,\lambda(f : Float \rightarrow Float).\,\lambda(y : Float).\,p(f(f\,y)) : Float \rightarrow Integer \rightarrow Float \rightarrow Float \rightarrow Float \rightarrow Integer$ — ⑪ By Function rule & ⑩ $\left[\dfrac{\xi \cup \{x : S\} \vdash M : T}{\xi \vdash \lambda(x : S).\,M : S \rightarrow T}\right]$

$\xi_0 \vdash (\lambda(p : Float \rightarrow Integer).\,\lambda(f : Float \rightarrow Float).\,\lambda(y : Float).\,p(f(f\,y))) : Float \rightarrow Integer \rightarrow Float \rightarrow Float \rightarrow Float \rightarrow Integer$ — ⑫ By Paren rule & ⑪

$\xi_0 \vdash \phi : Float \rightarrow Integer$ — ⑬ By Constant rule

$\xi_0 \vdash (\lambda(p : Float \rightarrow Integer).\,\lambda(f : Float \rightarrow Float).\,\lambda(y : Float).\,p(f(f\,y)))\,\phi : Float \rightarrow Float \rightarrow Float \rightarrow Integer$ — ⑭ By Application rule

$$\boxed{\therefore\ \xi_0 \vdash (\lambda(p : Float \rightarrow Integer).\,\lambda(f : Float \rightarrow Float).\,\lambda(y : Float).\,p(f(f\,y)))\,\phi : Float \rightarrow Float \rightarrow Float \rightarrow Integer}$$

(~~Repla~~ Using Int in place of Integer):

$$\dfrac{\xi_3 \vdash f : Float \rightarrow Float\ ①\qquad \xi_3 \vdash y : Float\ ②}{\xi_3 \vdash f\,y : Float}\ ③$$

$$\dfrac{\xi_3 \vdash f : Float \rightarrow Float\ ④\qquad \xi_3 \vdash f\,y : Float}{\xi_3 \vdash (f(f\,y)) : Float}\ ⑤\,\&\,⑥$$

$$\dfrac{\xi_3 \vdash p : Float \rightarrow Int\ ⑦\qquad \xi_3 \vdash (f(f\,y)) : Float}{\xi_3 \vdash p(f(f\,y)) : Int}\ ⑧$$

$$\dfrac{\xi_3 \vdash p(f(f\,y)) : Int}{\xi_2 \vdash \lambda(y : Float).\,p(f(f\,y)) : Float \rightarrow Int}\ ⑨$$

$$\dfrac{\xi_2 \vdash \lambda(y : Float).\,p(f(f\,y)) : Float \rightarrow Int}{\xi_1 \vdash \lambda(f : Float \rightarrow Float).\,\lambda(y : Float).\,p(f(f\,y)) : Float \rightarrow Float \rightarrow Float \rightarrow Int}\ ⑩$$

$$\dfrac{\xi_1 \vdash \lambda(f : Float \rightarrow Float).\,\lambda(y : Float).\,p(f(f\,y)) : Float \rightarrow Float \rightarrow Float \rightarrow Int}{\xi_0 \vdash (\lambda(p : Float \rightarrow Int).\,\lambda(f : Float \rightarrow Float).\,\lambda(y : Float).\,p(f(f\,y))) : Float \rightarrow Int \rightarrow Float \rightarrow Float \rightarrow Float \rightarrow Int}\ ⑪\,\&\,⑫$$

$$\dfrac{\xi_0 \vdash (\lambda(p : Float \rightarrow Int).\,\lambda(f : Float \rightarrow Float).\,\lambda(y : Float).\,p(f(f\,y))) : Float \rightarrow Int \rightarrow Float \rightarrow Float \rightarrow Float \rightarrow Int \qquad \xi_0 \vdash \phi : Float \rightarrow Int\ ⑬}{\xi_0 \vdash (\lambda(p : Float \rightarrow Int).\,\lambda(f : Float \rightarrow Float).\,\lambda(y : Float).\,p(f(f\,y)))\,\phi : Float \rightarrow Float \rightarrow Float \rightarrow Int}\ ⑭$$

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b) Given $\phi : Bool \to Bool \to Bool$, $true : Bool$

Let $\xi_1 = \xi_0 \cup \{func1 : Bool \to Char\}$

$\xi_2 = \xi_1 \cup \{\tau : Bool\}$

$\xi_2 \vdash \tau : Bool$ — (1) By Identifier rule

$\xi_2 \vdash \phi : Bool \to Bool \to Bool$ — (2) By Constant rule

$\xi_2 \vdash \phi\,\tau : Bool \to Bool$ — (3) By Application rule (1) & (2) $\left[\dfrac{\xi \vdash M : S \to T,\ \xi \vdash N : S}{\xi \vdash M\,N : T}\right]$

$\xi_2 \vdash true : Bool$ — (4) By Constant rule

$\xi_2 \vdash \phi\,\tau\,true : Bool$ — (5) By Application rule, (3) & (4) $\left[\dfrac{\xi \vdash M : S \to T,\ \xi \vdash N : S}{\xi \vdash M\,N : T}\right]$

$\xi_2 \vdash (\phi\,\tau\,true) : Bool$ — (6) By Paren rule & (5)

$\xi_2 \vdash func1 : Bool \to Char$ — (7) By Identifier rule

$\xi_2 \vdash func1(\phi\,\tau\,true) : Char$ — (8) By Application rule (7) & (6) $\left[\dfrac{\xi \vdash M : S \to T,\ \xi \vdash N : S}{\xi \vdash M\,N : T}\right]$

$\xi_1 \vdash \lambda(\tau : Bool).\,func1(\phi\,\tau\,true) : Bool \to Char$ — (9) By Function rule & (8) $\left[\dfrac{\xi \cup \{x : S\} \vdash M : T}{\xi \vdash \lambda(x : S).M : S \to T}\right]$

$\boxed{\xi \vdash \lambda(func1 : Bool \to Char).\,\lambda(\tau : Bool).\,func1(\phi\,\tau\,true) : Bool \to Char \to Bool \to Char}$ — (10) By Function rule & (9) $\left[\dfrac{\xi \cup \{x : S\} \vdash M : T}{\xi \vdash \lambda(x : S).M : S \to T}\right]$

$$\dfrac{\xi_2 \vdash func1 : Bool \to Char \ (7) \qquad \dfrac{\dfrac{\xi_2 \vdash \phi : Bool \to Bool \to Bool \ (2) \qquad \xi_2 \vdash \tau : Bool \ (1)}{\xi_2 \vdash \phi\,\tau : Bool \to Bool}(3) \qquad \xi_2 \vdash true : Bool \ (4)}{\xi_2 \vdash (\phi\,\tau\,true) : Bool}(5)\&(6)}{\xi_1 \vdash \lambda(\tau : Bool)\,\cancel{func1(\phi\,\tau\,true) : Bool} \to Char}(8)$$

$$\dfrac{\dfrac{\xi_2 \vdash func1(\phi\,\tau\,true) : Char}{\xi_1 \vdash \lambda(\tau : Bool).\,func1(\phi\,\tau\,true) : Bool \to Char}(9)}{\xi \vdash \lambda(func1 : Bool \to Char).\,\lambda(\tau : Bool).\,func1(\phi\,\tau\,true) : Bool \to Char \to Bool \to Char}(10)$$