MagNote (tz-dd-062907): The Operation and Describing Equations of the Differential Drive System for Mobile Robots

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***Abstract*— We provide a delightfully formal treatment of the differential drive system commonly used for mobile robots operating on smooth surfaces. The differential drive provides steering without the need for servo-actuation; steering occurs as a result of the speed difference between the left and right wheels.**

I. Introduction

This note is inspired by the differential drive description provided in [1], but is slightly more detailed than LaValle's treatment and should provide a reader with only superficial knowledge of mechanical engineering (i.e., the electrical or computer engineer) with a solid understanding of the differential drive system. Regardless of the reader's background in mechanics, we recommend LaValle's excellent book, which manages to reconcile several different fields related to robotics in a clear and thoughtful manner.

Consider a disc with uniform mass distribution which we will refer to simply as a "wheel"[1], rolling without slip on a flat surface, as shown in Figure 1. By "slip" we refer to the sliding of the wheel with respect to the surface[2]— we forbid this from happening here by assumption. Yes, controlled slip is the basis of nearly all motorsports and, therefore, the root of nearly all fun; it's also the cause of most headaches for modeling the motion of wheels. Practically speaking, to have no slip means that there is always ample traction available.

The wheel has rotational rate, ω, measured in radians per second, and radius, ρ, measured in meters. The center of the wheel translates at a speed of v meters per second. It's relatively easy to show (see the Appendix) that

$$v = \rho\,\omega \tag{1}$$

so that, correspondingly, a larger wheel provides more translational speed for a given rotational rate. Smaller (and lighter) wheels provide more acceleration, since they present a reduced inertial load on the drive motor. Recall by Newton's Second Law, $J\ddot{\theta} = \tau$; as the moment of inertia, J, decreases, acceleration, $\ddot{\theta}$, increases, for the same net torque, τ.

For our purposes, we will assume that the wheel is quite narrow, in fact, it has a cross-sectional width of *zero*. Thus, there is only a single point of the wheel in contact with the surface at any time. Note that, because it has zero width, the wheel may spin like a top on the surface without any slipping (since the single contact point does not slide with respect to the surface). This allows the differential drive system to pivot about a stationary wheel, describing a circular path of radius equal to the distance between the wheels, as we shall see below.

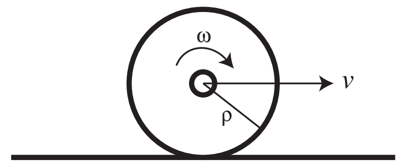

Fig. 1. Side view of a rolling wheel with radius ρ and zero cross-sectional width. We assume a single point of contact with the ground, and that the wheel never slides with respect to the ground.

Combining two such wheels spaced apart by λ and driven by separate motors provides the basis for the robot depicted in Figure 2, where all useful coordinate systems are also shown. This is a planar robot with three degrees of freedom described by the triple (X, Y, Θ); specifically, X and Y give the position of C with respect to O, and Θ provides the orientation of the body frame with respect to the global frame. Note that the robot's nose is always collinear with the body-frame $x_{L,1}$ axis.

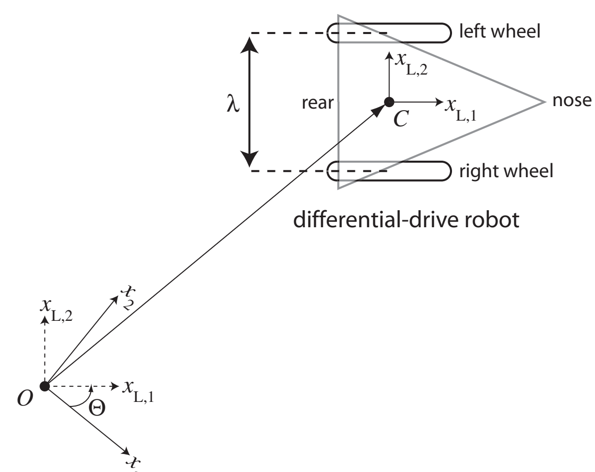

Fig. 2. Top view of differential-drive robot. The center of mass and body-frame origin is indicated by C and local coordinate system $x_{L,1}$ and $x_{L,2}$. The centerlines of the wheels pass through C. The global frame has origin O and coordinates x_1 and x_2.

[1]We shall use the term wheel to describe both the wheel and any mounted outer/removable rubber or plastic tire.

[2]Therefore, in terms of friction, the only resistance to movement that the wheel experiences is characterized by the static (opposed to kinetic) coefficient of friction.

The robot is assumed to be "magically balanced" on the two wheels; in an actual implementation, either an active balancing system or a rear-mounted caster wheel can be used.

II. Differential Drive is Your Friend

The differential drive system allows the rotational and translational rates of the robot's motion to be set independently—try that with a front-wheel steered, rear-wheel drive setup (for which there is no rotation without translation). As LaValle shows, the differential drive is even capable of *pure* rotation about the robot's center point, C, if the wheels are driven at the same speed but in opposite directions.

First, however, we wish to show the relatively innocuous nature of differential drive. That is, given any speed difference, Δ, between the left and right drive wheels (each driven at constant speed), there exists an arc (circle) about which the robot travels without wheel slip. Now, for a given path in space, yes, the wheels must be traveling at specific rates; the issue here, however, is, given arbitrary wheel rates, does there exist a slip-free path? If so, under our (assumed) no-slip conditions, this must be the path taken by the robot. The existence of such a path may be intuitively obvious to the reader; it was not obvious for me, which is why I bother with the following derivation.

Let us denote the constant left and right rotational wheel rates by ω_L and ω_R, respectively. Thus,

$$\Delta = \omega_R - \omega_L. \tag{2}$$

For convenience, we assume (without loss of generality) that $\Delta \geq 0$. Let us express the wheel rates in differential form:

$$\omega_R = \omega_C + \frac{\Delta}{2} \tag{3}$$

$$\omega_L = \omega_C - \frac{\Delta}{2} \tag{4}$$

where $\omega_C = \frac{\omega_R + \omega_L}{2}$. Consider a time interval T_a. In this time, the right wheel center travels a distance of $T_a \rho \omega_R$, the left wheel center, $T_a \rho \omega_L$. We can express these distances as:

$$s_R = \rho T_a \omega_C + \left(\frac{\Delta}{2}\right) \rho T_a \tag{5}$$

$$s_L = \rho T_a \omega_C - \left(\frac{\Delta}{2}\right) \rho T_a \tag{6}$$

For these two distances, we want to show that there exists a pair of concentric circles with a difference in radii of λ that describe the motions of the left and right wheels, as shown in Figure 3. This will prove that it is possible for a no-slip translation with rotation to occur. Specifically, we need to show that there exists a distance R_a, measured in meters, and an angle Φ, measured in radians, such that $s_R = (R_a + \lambda)\Phi$ and $s_L = R_a \Phi$. We therefore need to show that there exists a Φ such that

$$s_R - s_L = \lambda \Phi. \tag{7}$$

Subtracting (5) and (6) and factoring yields

$$s_R - s_L = \rho T_a \Delta = \lambda \left[\left(\frac{\rho}{\lambda}\right) T_a \Delta \right] \tag{8}$$

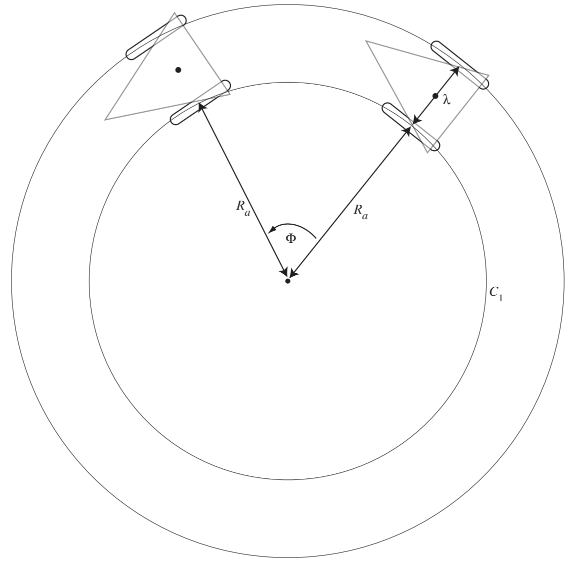

Fig. 3. Movement of robot along a curved path under no-slip conditions. The left and right wheel speeds have been chosen arbitrarily; we wish to show the existence of circles C_1 and C_2.

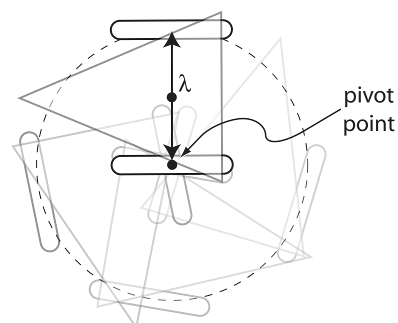

Fig. 4. No-slip path of robot with left-wheel pivot.

and we have

$$\Phi = \left(\frac{\rho}{\lambda}\right) T_a \Delta. \tag{9}$$

Does this make sense? Consider the case in which the left wheel doesn't move, i.e., $\omega_L = 0$. Intuitively, we'd expect the robot to move in a circle centered about the center of the left wheel, as shown in Figure 4. This circle has a radius of λ and a circumference of $2\pi\lambda$. The circle is being traversed at a speed of $v_R = \rho\Delta$, which, in a time T_a, corresponds to $T_a v_R$ meters or $2\pi\left(\frac{T_a v_R}{2\pi\lambda}\right) = \left(\frac{\rho}{\lambda}\right) T_a \Delta = \Phi$ radians.

Wow, it really works! In other words, the expression (9) is valid, and the circles C_1 and C_2 of Figure 3 exist *for any* speed difference. And, by the way, although we've assumed constant motor speeds in the foregoing, one should realize that these relations hold instantaneously, so we can speak of *instantaneous arcs/circles* of traversal that describe real-world motion.

OK, fantastic, and despite the redundancy, let me re-state the possibly obvious: *we now know that no matter how the left and right motor speeds are chosen, there is an instantaneous arc corresponding to some circle of radius R_a that describes the robot's precise slip-free motion at that instant*.

III. Make it So!

We will assume that, regardless of the nature of the drivetrain, we wish the robot to move at a translational speed of v_τ and rotational speed ω_ρ. It's important for these two speed components to occur simultaneously, at least in the applications we have in mind[3].

The translational velocity is a vector of magnitude $|v_\tau|$, collinear with the body frame's $x_{L,1}$ axis (as shown in Figure 2), and pointing in the direction from C to the nose of the robot (for positive velocities). The rotational speed is simply $\omega_\rho = \dot{\Theta}$ (recall that Θ is the rotation of the body frame with respect to the global frame as shown in Figure 2).

Our next problem is to derive expressions for the left and right wheel rotational rates in order to give us any v_τ and ω_ρ that we desire. For the time being, let us ignore the dynamics of the drive motors, and assume that whatever rotational speed we want is instantaneously manifested by the wheel. This is only to de-clutter our discussion.

Let's develop these relations intuitively, and then prove that they work[4]. Since this is a differential drive, it makes sense to express the rotational speeds differentially as in (3) and (4). Now, however, we can be more precise. Specifically, when the speed difference, Δ, is zero, the left and right wheel speeds should equal v_τ.

In other words,

$$\text{(for no rotation) } v_L = v_R = \rho\omega_C = v_\tau. \tag{10}$$

Therefore, $\omega_C = \left(\frac{1}{\rho}\right) v_\tau$. Correspondingly, when the robot is simply spinning on the spot, there is no translation, and we should have that $\omega_L = -\omega_R = -\left(\frac{\lambda}{2\rho}\right)\omega_\rho$. The quantity $\left(\frac{\lambda}{2\rho}\right)$ is a gear diameter ratio; it takes so many turns of the wheel "gear" to traverse the circular path "gear." These intuitive arguments lead to the proposed wheel rotational speed relations:

$$\omega_R = \left(\frac{1}{\rho}\right) v_\tau + \left(\frac{\lambda}{2\rho}\right)\omega_\rho \tag{11}$$

$$\omega_L = \left(\frac{1}{\rho}\right) v_\tau - \left(\frac{\lambda}{2\rho}\right)\omega_\rho \tag{12}$$

and the billion dollar question is? If you said "Are these really correct, Dr. Z?" you are well on your way to a billion dollars. Well, think of it as a necessary rather than sufficient condition for riches...

Let's prove it! Suppose that v_τ and ω_ρ are constants and given. What is the motion produced by the robot when driven by (11) and (12)? That is, suppose our wheels are driven according to (11) and (12), and let's find the resulting translational speed of C and rotational rate of the body frame with respect to O.

From our previous discussion, we know that there exist circles C_1 and C_2, as shown in Figure 3, corresponding to no-slip motion generated by (11) and (12). Consider a fixed amount of time, $T_a > 0$, that is given. Thus, from (9), $\Phi = \left(\frac{\rho}{\lambda}\right) T_a \left(\frac{\lambda}{\rho}\right)\omega_\rho = T_a\omega_\rho$. The point C traverses an arc length of

$$s_C = \left(R_a + \frac{\lambda}{2}\right)\Phi = \left(R_a + \frac{\lambda}{2}\right) T_a\omega_\rho \tag{13}$$

and is therefore traveling at a speed of

$$v_C = \frac{s_C}{T_a} = \left(R_a + \frac{\lambda}{2}\right)\omega_\rho. \tag{14}$$

Similarly, the speeds of the right and left wheel centers are

$$v_R = \left(\frac{1}{T_a}\right)(R_a + \lambda)\Phi = (R_a + \lambda)\omega_\rho \tag{15}$$

and

$$v_L = \left(\frac{1}{T_a}\right) R_a\Phi = R_a\omega_\rho, \tag{16}$$

respectively. Therefore, it is clear that $v_C = \frac{v_R + v_L}{2}$. We can also express v_C as

$$v_C = \frac{\rho\omega_R + \rho\omega_L}{2}, \tag{17}$$

which, after substituting from (11) and (12), yields $v_C = v_\tau$. This is great news, because it shows that (11) and (12) lead to the correct translational motion.

For rotational motion, suppose that $\Phi = 2\pi$. This means that the robot has made one full traversal of the circular path, and has gone through an orientation change of exactly 360 degrees. We know from (11) and (12) that $\Delta = \left(\frac{\lambda}{\rho}\right)\omega_\rho$, and using (9) to solve for the traversal time yields

$$T_a = \frac{2\pi}{\omega_\rho}. \tag{18}$$

Therefore the robot's body frame has gone through 2π radians in T_a seconds, leading to a rotational rate of $\frac{2\pi}{T_a} = 2\pi\left(\frac{\omega_\rho}{2\pi}\right) = \omega_\rho$, which is exactly the result we wanted.

In summary, the speed control relations (11) and (12) lead to translational and rotational motion components of exactly v_τ and ω_ρ.

IV. A Dose of Reality

We have made assumptions here of no slip and wheels of zero width! The reader probably realizes that an extremely narrow wheel will have limited traction, so these assumptions are in direct conflict with each other. They are simplifications to make things tractable. In reality, a robot that is pivoting about a stationary wheel will experience slip; abrupt accelerations will exceed the traction limits of the wheels and cause sliding. Nonetheless, the foregoing discussion should be approximately true for any real system in which both traction and acceleration are moderate.

Position estimation is an important feature for robots in diverse applications. If the wheels slip very little, speedometer measurements can yield accurate position estimates. This method, however, fails if the robot is subjected to an external

[3] Why? Because our target sensor provides a bearing and distance to a source; in effect, pointing to the source in polar fashion. It is natural, therefore, for the robot to perform a simultaneous re-orientation (to minimize bearing error) and translation (to reduce distance error). It may not be the only way to do it, but it seemed natural to me in the formulation of the control.

[4] This is not the way to write a paper, but, hey, this is edifying note, not some bit of novel research.

disturbance that creates motion. For example, suppose that the robot is picked up and carried to another location, or is bumped on its side by another robot operating in the environment. We would like some method of integrating these events, and may have to resort to accelerometers or gyroscopic sensors (which can be inaccurate, but are likely better than nothing). Although global positioning systems are a possibility, for hazardous or extra-terrestrial environments, GPS signals may not be continuously available (it could, however, be a dandy means of intermittent accelerometer calibration).

V. APPENDIX

Given that the wheel of Figure 1 has a circumference of $C_w = 2\pi\rho$, we have that $v = \left(\frac{C_w}{2\pi}\right)\omega = \rho\omega$. See? Easy.

REFERENCES

[1] S. M. LaValle, *Planning Algorithms*. Cambridge University Press, 2006.