M22

Topic: Numbers & Algebra

What are the main characteristics of unit fractions and how can they be expressed as combinations of other unit fractions under different arithmetic operations?

Subject: Mathematics Analysis and Approaches

Extended Essay

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Introduction

Numbers are classified in different ways based on various characteristics. The most common way of classifying numbers is natural numbers, integers, rational numbers, algebraic numbers, real numbers, imaginary numbers and complex numbers. I have always been fascinated by the classification of numbers and I also continue to strive to find patterns in numbers such as prime numbers and perfect numbers.

However, because of the known difficulty of those topics, I decided to focus this essay on a more basic set of numbers: unit fractions. Reciprocals are “multiplicative inverses” of a number and a unit fraction is the reciprocal of an integer. To obtain the reciprocal of a number, we simply switch the numerator and denominator, keeping in mind that the denominator for an integer is 1.

Due to my interest and the extensive nature of this topic, I believe that the question “What are the main characteristics of unit fractions and how can they be expressed as combinations of other unit fractions under different arithmetic operations?” is worthy of an investigation.

The essay is split into two parts. In part 1, I explore and explain some of the most interesting characteristics of reciprocals. In part 2, I focus on the ways in which any one unit fraction can be expressed as the sum, difference, product and quotient of two other unit fractions.

Part 1

General Observations & Results

For a secure start to my investigation, I created a table of the first 20 unit fractions. The first column states the denominators of the unit fractions. The second column states the decimal forms of these fractions. The final column states the number of digits within the recurring cycle of that unit fraction, where such a recurring cycle exists.

Denominator or m	Decimal	Length of recurring cycle
1	1	0
2	.5	0
3	$.\overline{3}$	1
4	.25	0
5	.2	0
6	$.1\overline{6}$	1
7	$.\overline{142857}$	6
8	.125	0
9	$.\overline{1}$	1
10	.1	0

Denominator or m	Decimal	Length of recurring cycle
11	$.\overline{09}$	2
12	$.08\overline{3}$	1
13	$.\overline{076923}$	6
14	$.0\overline{714285}$	6
15	$.0\overline{6}$	1
16	.0625	0
17	$.\overline{05882352941}$ $\overline{17647}$	16
18	$0.0\overline{5}$	1
19	$.\overline{05263157894}$ $\overline{7368421}$	18
20	0.05	0

There are two distinct sets of unit fractions. One is the unit fractions, that as a decimal, have a fixed value. A unit fraction with a fixed decimal value has no recurring cycle. Examples are $\frac{1}{2} = 0.5$, $\frac{1}{4} = 0.25$, $\frac{1}{5} = 0.2$

The second set is the unit fractions that do not produce a fixed decimal value, but have a recurring cycle. Examples are $\frac{1}{3} = 0.\overline{3}$, $\frac{1}{6} = 0.1\overline{6}$, $\frac{1}{7} = 0.\overline{142857}$

All unit fractions with a fixed decimal value can be derived from $\frac{1}{2^m \cdot 5^n}$ {m, n ∈ N}

For example:

When m = 3 and n = 2, the expression will produce a fixed value: $\frac{1}{2^3 \cdot 5^2} = \frac{1}{8 \cdot 25} = \frac{1}{200} = 0.005$

When m = 7 and n = 1: $\frac{1}{2^7 \cdot 5^1} = \frac{1}{128 \cdot 5} = \frac{1}{640} = 0.0015625$

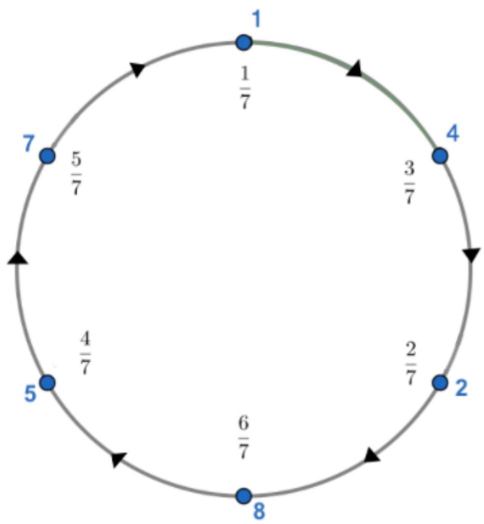
Another observation that can be made with the first 20 unit fractions, is that denominators that are multiples of 3, have a cycle of 1 digit. Also, denominators that are multiples of 7 OR 13 have a recurring cycle of 6 digits.

However, finding a rule for this conjecture is not simple as there are many exceptions in these cases. For example, $\frac{1}{21} = 0.\overline{047619}$, is considered a multiple of 7 rather than 3 as it has a recurring cycle of 6 digits instead of 1 digit. From this point, we conjecture that it follows the rule of its largest factor, however, $\frac{1}{81} = 0.\overline{012345679}$, has a recurring cycle of 9 digits even though the reciprocal of the highlargestest factor of the denominator $\frac{1}{27} = 0.\overline{037}$ has a recurring cycle of 3 digits.

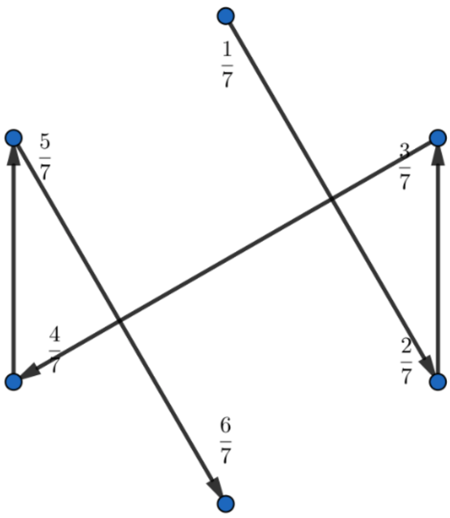
These complications arise as soon as the denominator becomes the product of two or more prime numbers, each of which may have their own recurring cycle. Thus finding a general pattern or formula for all unit fractions that have a recurring cycle is therefore impossible. The structure of the cycle, the number of digits in the cycle and their order are entirely dependent on the prime factors in the denominator. Since there are an infinite number of prime numbers and an infinite number of ways they can be combined, it seems almost certain that there will be an infinite number of different recurring cycles. It might still be the case however, that, for small numerators such as 15 and 18, we can still see 'traces' of their common factor 3 within their different recurring cycles.

Families Of Interest

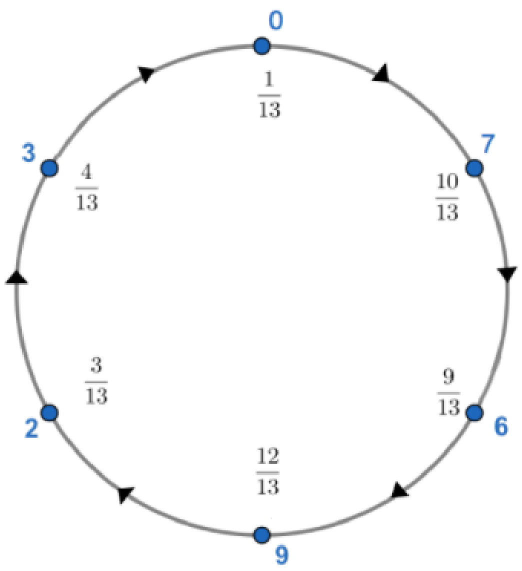
We can now consider what patterns might emerge with a changing numerator and a common denominator. As the numerator increases, the digits in the recurring cycle remain the same but their order changes. See the diagram below. Interestingly, while the family of sevenths can be shown on one diagram, the family of thirteenths needs two diagrams.

<table>
<tr><th>Denominator or m</th><th>Cycle number</th><th>Decimal</th><th>The left value is the numerator (n) and the right value is the starting value in the recurring cycle</th><th>Diagram</th></tr>
<tr><td rowspan="3">7</td><td rowspan="3">1 of 1</td><td rowspan="3">$.\overline{142857}$</td><td>1 - 1</td><td rowspan="3">

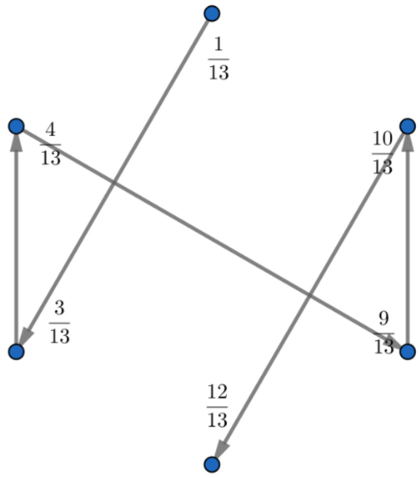
The numbers on the outside show at which point the decimal form of the fraction starts in the recurring (clockwise) cycle.
e.g. $\frac{1}{7} = 0.142857...$, is placed at the point 1 and $\frac{3}{7} = 0.428571...$, is placed at the point 4 as that is the value where the recurring cycle starts.</td></tr>
<tr><td>2 - 2</td></tr>
<tr><td>3 - 4</td></tr>
</table>

			4 - 5	 I ignored the circle and traced line segments that follow the fractions with increasing numerator, tracking the path of successive starting points. This was done to find a pattern between the recurring cycle of fractions in the same family.
			5 - 7	
			6 - 8	

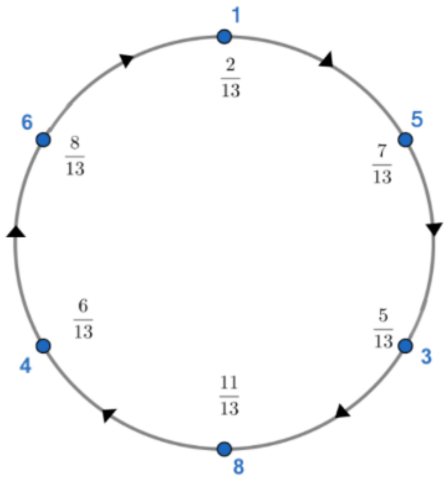
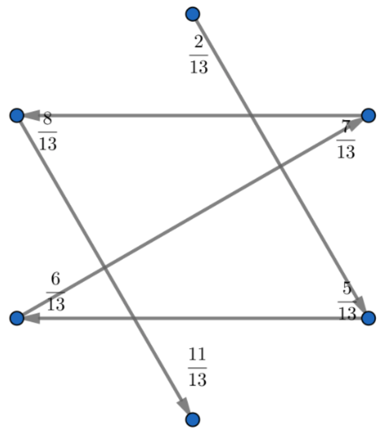
An example of a denominator that has 2 recurring cycles is 13. (Cycle 1: $\frac{1}{13}$ = 0.**0**76923... and $\frac{3}{13}$ = 0.**2**30769…/ Cycle 2: $\frac{2}{13}$ = 0.**1**53846... and $\frac{5}{13}$ = 0.**3**84615…).

<table>
<tr><th>Denominator or m</th><th>Cycle number</th><th>Decimal</th><th>The left value is the numerator (n) and the right value is the starting value in the recurring cycle</th><th>Diagram</th></tr>
<tr><td rowspan="3">13</td><td rowspan="3">1 of 2</td><td rowspan="3">$.\overline{076923}$</td><td>1 - 0</td><td rowspan="3">

Again, the numbers on the outside show at which point the decimal form of the fraction starts in the recurring (clockwise) cycle.</td></tr>
<tr><td>3 - 2</td></tr>
<tr><td>4 - 3</td></tr>
</table>

<table>
<tr><td></td><td></td><td></td><td>9 - 6</td><td rowspan="3">

Traced line segments that follow the fractions with increasing numerator to track the path of successive starting points</td></tr>
<tr><td></td><td></td><td></td><td>10 - 7</td></tr>
<tr><td></td><td></td><td></td><td>12 - 9</td></tr>
</table>

Denominator or *m*	Cycle number	Decimal	The left value is the numerator (n) and the right value is the starting value in the recurring cycle	Diagram
13	2 of 2	$.\overline{153846}$	2 - 1	 Fractions are shown by their starting points  Tracking the path of successive starting points
			5 - 3	
			6 - 4	
			7 - 5	
			8 - 6	
			11 - 8	

It is amazing to observe the various star-like patterns in diagrams that emerge from the recurrence of unit fractions. When combining several of these patterns, interesting shapes are formed. This shows unique patterns when combining unit fractions from the same family.

Recurring Decimals As Infinite Geometric Series

Any decimal that has a recurring cycle, can also be expressed as an infinite geometric series.

$0.\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots$

This is a geometric series with $U_1 = \frac{3}{10}$ and $r = \frac{1}{10}$ where U_1= first term and r = ratio.

For a geometric series $S_\infty = \frac{U_1}{1-r}$

So $S_\infty = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{1}{3}$

This technique can be used to find the fraction equivalent to any recurring decimal

Another example: $0.\overline{142857} = \frac{142857}{10^6} + \frac{142857}{10^{12}} + \frac{142857}{10^{18}} + \dots$

$U_1 = \frac{142857}{10^6}$ and $r = \frac{1}{10^6}$

So $S_\infty = \frac{\frac{142857}{10^6}}{1-\frac{1}{10^6}} = \frac{\frac{142857}{10^6}}{\frac{999999}{10^6}} = \frac{142857}{999999} = \frac{1}{7}$

It is fascinating to see how recurrence can be interpreted through one of the key topics on the analysis and approaches syllabus: sequences and series.

Part 2

Mathematical operations

In the remaining and major part of this essay I will examine how the unit fraction $\frac{1}{p}$ may be expressed as the sum, difference, product and quotient of two other unit fractions.

In each case I will start from the basic equation $\frac{1}{p} = \frac{1}{m} * \frac{1}{n}$ where * stands, in turn, for $+, \ -, \ \times \ and \ \div$. For this investigation, m and n are restricted to ℕ as we do not want negative values for m, n.

Early examples will be derived from putting p = 1, 2, 3, … with the objective of deriving a full picture of the various possible solutions pairs (m, n), where such pairs exist.

Addition

$\frac{1}{p} = \frac{1}{m} + \frac{1}{n}$

We require $\frac{1}{p} = \frac{1}{m} + \frac{1}{n}$

⇔ $mn = np + mp$

⇔ $p = \frac{mn}{m+n}$

<u>Let p = 1</u>

⇔ $\frac{mn}{m+n}$= 1

⇔ mn = m + n

⇔ mn - n = m

⇔ (m - 1) n = m

⇔ n =$\frac{m}{m-1}$

m must be > 1 to derive a real, positive value for n

Varying the value of m to derive n					
Value of m	2	3	4	5	6
Value of n	$\frac{2}{2-1} = 2$	$\frac{3}{3-1} = \frac{3}{2}$	$\frac{4}{4-1} = \frac{4}{3}$	$\frac{5}{5-1} = \frac{5}{4}$	$\frac{6}{6-1} = \frac{6}{5}$
n ∈ℕ ?	yes	no	no	no	no

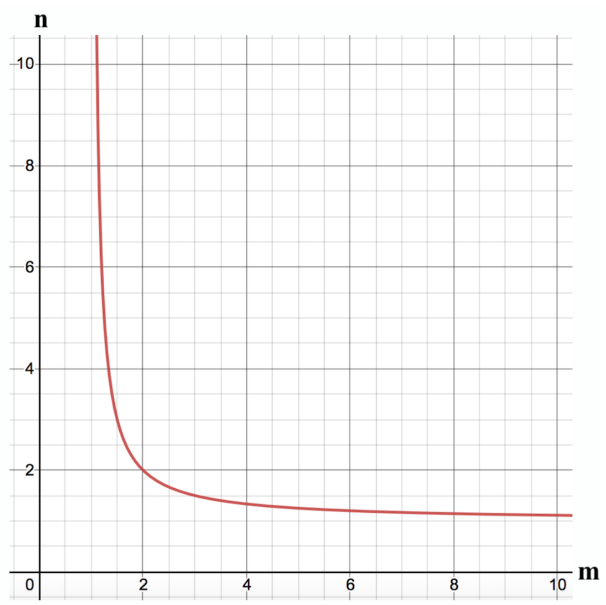
It appears that the only solution for p = 1 is m = 2, n = 2. To check this, we can draw the graph of

$n = \frac{m}{m-1}$ (for m > 1)

Only one point on the graph gives a pair of solutions that are both positive integers.

Final answer: for p = 1, there is only one solution, m = 2, n = 2

$n = \frac{m}{m-1}$, m ≠1, n ≠1



Let p = 2

$$\frac{mn}{m+n} = 2$$

$\Leftrightarrow mn = 2m + 2n$

$\Leftrightarrow mn - 2n = 2m$

$\Leftrightarrow n(m - 2) = 2m$

$\Leftrightarrow n = \frac{2m}{m-2}$

m must be > 2 to derive a real, positive value for n

Varying the value of m to derive n					
Value of m	3	4	5	6	7
Value of n	$\frac{2(3)}{3-2} = 6$	$\frac{2(4)}{4-2} = 4$	$\frac{2(5)}{5-2} = \frac{10}{3}$	$\frac{2(6)}{6-2} = 3$	$\frac{2(7)}{7-2} = \frac{14}{5}$
n ∈N ?	yes	yes	no	yes	no

As soon as m > n, then the same solutions are produced but in reverse order. This is because, for addition $\frac{1}{m} + \frac{1}{n} = \frac{1}{n} + \frac{1}{m}$.

So there are three solution pairs for (m, n). These are: (3,6), (4,4) , (6,3) .

Let p = 3

$\frac{mn}{m+n} = 3$

$\Leftrightarrow mn = 3m + 3n$

$\Leftrightarrow mn - 3n = 3m$

$\Leftrightarrow n(m - 3) = 3m$

$\Leftrightarrow n = \frac{3m}{m-3}$

m must be > 3 to derive a real, positive value for n

Varying the value of m to derive n					
Value of m	4	5	6	7	8
Value of n	$\frac{3(4)}{4-3} = 12$	$\frac{3(5)}{5-3} = \frac{15}{2}$	$\frac{3(6)}{6-3} = 6$	$\frac{3(7)}{7-3} = \frac{21}{4}$	$\frac{3(8)}{8-3} = \frac{24}{5}$
n ∈N ?	yes	no	yes	no	no

It was not necessary to extend the table further because after m = 6 (when m > n) then, once again, the same solutions will emerge, but in reverse order. In other words, a symmetry is observed, so values after and before m = n are essentially repeated as addition is commutative.

The full set of solutions for p = 3 will then be: (4,12) , (6,6) , (12, 4)

Let p = 4

$$\frac{mn}{m+n} = 4$$

$\Leftrightarrow mn = 4m + 4n$

$\Leftrightarrow mn - 4n = 4m$

$\Leftrightarrow n(m - 4) = 4m$

$\Leftrightarrow n = \frac{4m}{m-4}$

m must be > 4 to get a real, positive value for n

Varying the value of m to get n					
Value of m	5	6	7	8	9
Value of n	$\frac{4(5)}{5-4} = 20$	$\frac{4(6)}{6-4} = 12$	$\frac{4(7)}{7-4} = \frac{28}{3}$	$\frac{4(8)}{8-4} = 8$	$\frac{4(9)}{9-4} = \frac{36}{5}$
n ∈N ?	yes	yes	no	yes	no

Again, it is not necessary to extend the table further as only repetition will emerge after m > n.

The full set of solutions for p = 4 will be: (5,20), (6,12) , (8,8) , (12, 6) , (20, 5)

Two patterns can be observed from these results.

Pattern 1	Pattern 2
$\frac{1}{2} + \frac{1}{2} = \frac{1}{1}$ $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$ In this set of result, the identity being used is $\frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$ <u>Proof of identity</u> $\frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$ $LHS \equiv \frac{1}{2n} + \frac{1}{2n}$ $\equiv \frac{2n+2n}{(2n)\,(2n)}$ $\equiv \frac{4n}{4n^2}$ $\equiv \frac{1}{n} \equiv RHS$	$\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ $\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$ $\frac{1}{5} + \frac{1}{20} = \frac{1}{4}$ In this set of result, the identity being used is $\frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{1}{n}$ <u>Proof of identity</u> $\frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{1}{n}$ $LHS \equiv \frac{1}{n+1} + \frac{1}{n(n+1)}$ $\equiv \frac{(n+1)+(n(n+1))}{(n+1)\,(n(n+1))}$ $\equiv \frac{(n+1)+(n^2+n)}{(n+1)\,(n^2+n)}$ $\equiv \frac{n^2+2n+1}{n^3+2n^2+n}$ $\equiv \frac{n^2+2n+1}{n\,(n^2+2n+1)}$ $\equiv \frac{1}{n} \equiv RHS$

However, it is important to note that the two identities above do not generate all possible values for m and n for every value of p as $\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$, m = 6, n = 12 for p = 4, has not, so far, been obtained. This means that either there is another general identity that justifies these values or there is another identity that better contains the findings.

After carefully substituting other values for p, further examples and patterns are revealed below.

$\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$	$\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$	$\frac{1}{5} + \frac{1}{20} = \frac{1}{4}$
$\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$	$\frac{1}{8} + \frac{1}{24} = \frac{1}{6}$	$\frac{1}{10} + \frac{1}{40} = \frac{1}{8}$
$\frac{1}{9} + \frac{1}{18} = \frac{1}{6}$	$\frac{1}{12} + \frac{1}{36} = \frac{1}{9}$	$\frac{1}{15} + \frac{1}{60} = \frac{1}{12}$
$\frac{1}{3k} + \frac{1}{6k} = \frac{1}{2k}$	$\frac{1}{4k} + \frac{1}{12k} = \frac{1}{3k}$	$\frac{1}{5k} + \frac{1}{20k} = \frac{1}{4k}$

Using these values, my conjecture is that $\frac{1}{k(n+1)} + \frac{1}{kn(n+1)} = \frac{1}{kn}$ is a third identity which look similar to the second identity found earlier, but interestingly also covers all possibilities from identity 1, which is when the denominators are equal (m = n)

Proof of identity

$$\frac{1}{k(n+1)} + \frac{1}{kn(n+1)} = \frac{1}{kn}$$

$$LHS \equiv \frac{kn(n+1) + k(n+1)}{(k(n+1))(kn(n+1))}$$

$$\equiv \frac{kn^2 + kn + kn+k}{(kn+k)(kn^2+kn)}$$

$$\equiv \frac{kn^2 + 2kn + k}{k^2n^3 + 2\,k^2n^2 + k^2n}$$

$$\equiv \frac{kn^2 + 2kn + k}{kn(kn^2 + 2kn + k)}$$

$$\equiv \frac{1}{kn} \equiv RHS$$

$\frac{1}{k(n+1)} + \frac{1}{kn(n+1)} = \frac{1}{kn}$; This identity is valid for all n, k $\in\mathbb{Q}$, however, for this situation where we add unit fractions specifically, n, k $\in\mathbb{N}$. Note that negative values work as well!

Subtraction

After some trial and error, I found a pattern that may cover a fraction of the total possibilities.

$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

In these example, I am using the identity

$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

<u>Proof</u>

$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

$LHS \equiv \frac{1}{n} - \frac{1}{n+1}$

$\equiv \frac{n+1-n}{n\,(n+1)}$

$\equiv \frac{1}{n(n+1)} \equiv RHS$

Notice that this is a re-arranged version of adding fractions: $\frac{1}{n} = \frac{1}{n(n+1)} + \frac{1}{n+1}$

This shows that re-arranging the identity for adding two unit fractions, we can derive at least one identity for subtracting two unit fractions very easily:

$\frac{1}{k(n+1)} + \frac{1}{kn(n+1)} = \frac{1}{kn}$

$\Leftrightarrow \frac{1}{kn} - \frac{1}{k(n+1)} = \frac{1}{kn(n+1)}$

So I can now put $\frac{1}{p} = \frac{1}{kn(n+1)} = \frac{1}{kn} - \frac{1}{k(n+1)}$

I then noticed a difficulty. This identity does not always work in terms of providing solutions for p. Even if I put k = 1, then I am still requiring p = n(n+1). By definition, that cannot work whenever p is a prime number greater than two. In fact, it will only work at all (with k = 1) for numbers such as 2 where 2 = 1 x 2 or 6 where 6 = 2 x 3 or 12 where 12 = 3 x 4.

This can be seen in more detail as we examine the various possibilities for p in a methodical way.

p = 1 does not work because it requires, $\frac{1}{m} > 1$, which is not possible as m ∈N:

$$\frac{1}{p} = \frac{1}{m} - \frac{1}{n}$$

$$\Leftrightarrow 1 = \frac{1}{m} - \frac{1}{n}$$

$\frac{1}{m} - \frac{1}{n}$ will always be < 1 for all m, n ∈N

When p = 2, using the identity (with k = 1) we get $\frac{1}{2} = \frac{1}{1} - \frac{1}{2}$. The solution pair for p = 2 is (1, 2).

<u>p = 2</u>

$$\frac{1}{p} = \frac{1}{m} - \frac{1}{n}$$

$$\Leftrightarrow \frac{1}{2} = \frac{1}{m} - \frac{1}{n}$$

$$\Leftrightarrow 2 = \frac{nm}{n-m}$$

$$\Leftrightarrow 2n - 2m = nm$$

$$\Leftrightarrow -2m = n(m - 2)$$

$$\Leftrightarrow \frac{-2m}{m-2} = n$$

Varying the value of m to derive n				
Value of m	3	4	5	6
Value of n	$\frac{-2(3)}{3-2} = -6$	$\frac{-2(4)}{4-2} = -4$	$\frac{-2(5)}{5-2} = \frac{-10}{3}$	$\frac{-2(6)}{6-2} = -3$
n ∈N ?	no	no	no	no

The function does produce integers, however, not positive integers. If these are substituted into our original equation $\frac{1}{p} = \frac{1}{m} - \frac{1}{n}$, the negative values for n simplify the equation to addition. However, they interestingly produce solutions for other values of p.

For (3, -6): $\frac{1}{2} = \frac{1}{3} - \frac{1}{-6}$

$$\Leftrightarrow \frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\Leftrightarrow \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

<u>p = 3</u>

However, for p = 3, the identity fails to give a solution pair because kn(n+1) does not equal 3 for any integer values of k, n as 3 is a prime number. What about $\frac{1}{3} = \frac{1}{2} - \frac{1}{6}$? Solution pair (2, 6) for p = 3 is derived from the same identity when it is rearranged (when $\frac{1}{n+1}$ is the subject). We can now write:

$$\frac{1}{p} = \frac{1}{n+1} = \frac{1}{n} - \frac{1}{n(n+1)}$$

For p = 3, we put n = 2:

$$\frac{1}{3} = \frac{1}{2} - \frac{1}{2(2+1)}$$

$$\Leftrightarrow \frac{1}{3} = \frac{1}{2} - \frac{1}{6}$$

p = 4

Using the first identity $\frac{1}{kn(n+1)} = \frac{1}{kn} - \frac{1}{k(n+1)}$, when k = 2 and n = 1, we get the solution pair (2, 4):

$$\frac{1}{(2)(1)(1+1)} = \frac{1}{(2)(1)} - \frac{1}{(2)(1+1)}$$

$$\Leftrightarrow \frac{1}{4} = \frac{1}{2} - \frac{1}{4}$$

p = 5

Again we cannot use the identity used for p = 2 and p = 4 because kn(n+1) does not equal 5 for any integer values of k, n as 5 is a prime number. However, again, if we use the identity $\frac{1}{n+1} = \frac{1}{n} - \frac{1}{n(n+1)}$, we get the solution pair (4, 20):

For p = 5 and n = 4

$$\frac{1}{4+1} = \frac{1}{4} - \frac{1}{4(4+1)}$$

$$\Leftrightarrow \frac{1}{5} = \frac{1}{4} - \frac{1}{20}$$

Now we can always find at least one solution pair for a subtraction using either identity. Working above shows that both identities can be used for a composite number, but only the second identity, $\frac{1}{n+1} = \frac{1}{n} - \frac{1}{n(n+1)}$, can be used for all prime numbers.

Multiplication

Multiplying two unit fractions together will always give a unit fraction because the product of the numerators, 1 x 1, will always equal 1. But the interesting challenge is to investigate how many solution pairs (m, n) there are for $\frac{1}{p} = \frac{1}{m} \times \frac{1}{n}$ which, of course, requires that p = mn. As before I will work through the various possibilities for increasing values of p.

<u>p = 1</u>

$\frac{1}{p} = \frac{1}{m} \times \frac{1}{n}$

⇔ 1 = mn

⇔ $\frac{1}{m} = n$

Varying the value of m to derive n			
Value of m	1	2	3
Value of n	$\frac{1}{1} = 1$	$\frac{1}{2} = 0.5$	$\frac{1}{3} = 0.\overline{3}$
n ∈N ?	yes	no	no

It is not necessary to continue substituting values for m because n will continue to get smaller but never reach 0 due to its numerator. Only one solution pair (1, 1) is produced for p = 1.

p = 2

$\frac{1}{p} = \frac{1}{m} \times \frac{1}{n}$

⇔ 2 = mn

⇔ $\frac{2}{m} = n$

Varying the value of m to derive n				
Value of m	1	2	3	4
Value of n	$\frac{2}{1} = 2$	$\frac{2}{2} = 1$	$\frac{2}{3} = 0.\overline{6}$	$\frac{2}{4} = 0.5$
n ∈N ?	yes	yes	no	no

Again it is not necessary to continue increasing the value of m as n will continue to get smaller but never equal 0 because the numerator is 2. Substituting values for m is not necessary after n = 1. This shows two solution pairs for p = 2, (1, 2), (2, 1)

p = 3

$\frac{1}{p} = \frac{1}{m} \times \frac{1}{n}$

⇔ 3 = mn

⇔ $\frac{3}{m} = n$

Varying the value of m to derive n					
Value of m	1	2	3	4	5
Value of n	$\frac{3}{1} = 3$	$\frac{3}{2} = 1.5$	$\frac{3}{3} = 1$	$\frac{3}{4} = 0.75$	$\frac{3}{5} = 0.6$
n ∈N ?	yes	no	yes	no	no

This again shows that substituting values after n = 1 is not necessary. The final solution pairs for p = 3 are (1, 3), (3, 1)

<u>p = 4</u>

$\frac{1}{p} = \frac{1}{m} \times \frac{1}{n}$

⇔ 4 = mn

⇔ $\frac{4}{m} = n$

Varying the value of m to derive n				
Value of m	1	2	3	4
Value of n	$\frac{4}{1} = 4$	$\frac{4}{2} = 2$	$\frac{4}{3} = 1.\overline{3}$	$\frac{4}{4} = 1$
n ∈N ?	yes	yes	no	yes

The solution pairs for p = 4 are (1, 4), (2, 2), (4, 1)

The findings for p = 1, 2, 3, 4 show that the solution pairs (m, n) completely depend on how p factorises. For example, for p = 30, the solution pairs would be

(1, 30), (2, 15), (3, 10), (5, 6), (6, 5), (10, 3), (15, 2), (30, 1). Note that these are the factors of p. While there is no general formula to predict the solution pairs under multiplication, it is possible to say that: when p is a prime number, there will only be two solution pairs (1, p) and (p, 1). When p is a composite number, the number of solution pairs will depend on the number and different possible combinations of the prime factors of p.

Division

As before, I shall be looking for solution pairs (m, n) such that $\frac{1}{p} = \frac{1}{m} \div \frac{1}{n}$. This requires

$\frac{1}{m} \div \frac{1}{n} = \frac{1}{p}$

$\Leftrightarrow \frac{n}{m} = \frac{1}{p}$

$\Leftrightarrow \frac{m}{n} = p$ $\{m,\ n \in \mathbb{N}\}\,\{m,\ n \neq 0\}\,\{\frac{m}{n} \in \mathbb{N}\}$(so m = kn)

In this case, it is immediately possible for every value of p to have an infinite number of solutions.

For example, when p = 7, then

$\frac{1}{7} = \frac{1}{7} \div \frac{1}{1}$

or $\frac{1}{7} = \frac{1}{14} \div \frac{1}{2}$

or $\frac{1}{7} = \frac{1}{21} \div \frac{1}{3}$

or $\frac{1}{7} = \frac{1}{7k} \div \frac{1}{k}$

So the solution pairs for p = 7, will belong to the infinite set {(7, 1), (14, 2), (21, 3), …, (7k, k), …}.

So the solution pairs for p generally will belong to the infinite set {(p, 1), (2p, 2), …, (kp, k)}

Summary of findings

A unit fraction with a fixed decimal value can always be represented as $\frac{1}{2^m \cdot 5^n}$ {m, n ∈ N}

A unit fraction whose decimal equivalent is as recurring decimal as well as the respective families of such unit fractions can be represented diagrammatically in a way which shows the start and finish points of the cycles of every member of that family.

When p is prime, the length of the recurring cycle of its reciprocal will not exceed p-1. However, in other cases, such as p = 41, the cycle length can be remarkably short ($\frac{1}{41} = 0.\overline{02439}$)

All unit fractions with a recurring cycle can be expressed as infinite geometric series.

A given unit fraction can be represented as the sum of two other unit fractions through the use of the identity $\frac{1}{k(n+1)} + \frac{1}{kn(n+1)} = \frac{1}{kn}$.

A given unit fraction can only be represented as the difference of two other unit fractions by use of that same identity (with k = 1) re-arranged to become either $\frac{1}{p} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ or $\frac{1}{p} = \frac{1}{n+1} = \frac{1}{n} - \frac{1}{n(n+1)}$.

A given unit fraction $\frac{1}{p}$ can only be expressed as $\frac{1}{1} \times \frac{1}{p}$ or $\frac{1}{p} \times \frac{1}{1}$ when p is a prime number greater than 2, but in multiple ways when p is composite. The actual number of ways when p is composite is then determined by the number and powers of each of the prime factors of p.

A given unit fraction can be presented as the quotient of two other unit fractions in an infinite number of ways.

Further Research and The Basel problem

Towards the end of this essay, I began to research further, particularly about the sum of various groups of unit fractions such as $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{k}$ (where k is known) and even the sum to infinity:

$S_\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} + ...$ This is the well known harmonic series which never converges to a finite answer. However, what really caught my interest was the discovering that:

$$S_\infty = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...$$

all following unit fractions of the form $\frac{1}{n^2}$ to infinity do converge, and not only that, but it converges to $\frac{\pi^2}{6}$. Wow! This is known as the Basel problem.

The Basel problem is a problem posed in 1650, by Pietro Mengoli, and solved in 1734, by the famous mathematician Leonhard Euler. The Basel problem was to present a precise summation of the reciprocals of the squares of all natural numbers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...$$

Euler solved this problem using the following method.

This method has been informed from source Haese Mathematics HL Calculus (Quinn et al.) and Numbers (Taylor), but developed as simply as I can using my HL mathematics course.

He started with the Taylor series $sin(x)$

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

He then developed this to create a series for $\frac{sin(x)}{x}$

$$\frac{sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ...$$

He then evaluated the zeroes of x for $sin(x)$and then $\frac{sin(x)}{x}$

$sin(x) \ = \ 0$

$x \ = \ 0 \ + \ k\pi$

$x \ = \ k\pi \qquad\qquad\qquad k \ \in \ Z$

$\frac{sin(x)}{x} \ = \ 0$

$sin(x) \ = \ 0(x)$

$x \ = \ k\pi \qquad\qquad\qquad k \ \in \ Z, \ x \ \neq \ 0$

Euler then evaluated another equation separately, and also evaluated the zeroes for this equation:

$(1 \ - \ \frac{x}{\pi}) \ (1 \ + \ \frac{x}{\pi}) \ (1 \ - \ \frac{x}{2\pi}) \ (1 \ + \ \frac{x}{2\pi}) \ (1 \ - \ \frac{x}{3\pi}) \ (1 \ + \ \frac{x}{3\pi}) \ldots$

This factorised equation makes it easy to find all the zeroes:

$(1 \ - \ \frac{x}{\pi}) \ , \ x \ = \ \pi$

$(1 \ + \ \frac{x}{\pi}) \ , \ x \ = \ - \ \pi$

$(1 \ - \ \frac{x}{2\pi}) \ , \ x \ = \ 2\pi$

$(1 \ + \ \frac{x}{2\pi}) \ , \ x \ = \ - \ 2\pi$

$(1 \ - \ \frac{x}{3\pi}) \ , \ x \ = \ 3\pi$

$(1 \ + \ \frac{x}{3\pi}) \ , \ x \ = \ - \ 3\pi$

This shows that the zeroes are $x \ = \ k\pi, \ k \ \in \ Z, \ k \ \neq \ 0$

Because the zeroes (x-intercepts) are corresponding for both equations, Euler made a working assumption that both these equations might be the same (note that two sin graphs that have identical x-intercepts do not suggest that they are alike as amplitude may differ).

Then he expanded the equation above by considering two terms at a time

$$(1 - \frac{x}{\pi})(1 + \frac{x}{\pi}) = (1)(1) + (1)(\frac{x}{\pi}) - (1)(\frac{x}{\pi}) - (\frac{x}{\pi})(\frac{x}{\pi}) = 1 - \frac{x^2}{\pi^2}$$

$$(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi}) = (1)(1) + (1)(\frac{x}{2\pi}) - (1)(\frac{x}{2\pi}) - (\frac{x}{2\pi})(\frac{x}{2\pi}) = 1 - \frac{x^2}{4\pi^2}$$

$$(1 - \frac{x}{3\pi})(1 + \frac{x}{3\pi}) = (1)(1) + (1)(\frac{x}{3\pi}) - (1)(\frac{x}{3\pi}) - (\frac{x}{3\pi})(\frac{x}{3\pi}) = 1 - \frac{x^2}{9\pi^2}$$

He proved that two consecutive terms equal $1 - \frac{x^2}{n^2\pi^2}$, $n \in N$

He then further expanded the equation

$$(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2}) = 1 - (\frac{x^2}{\pi^2}) - (\frac{x^2}{4\pi^2}) + \frac{x^4}{4\pi^4} = 1 - x^2(\frac{1}{\pi^2} + \frac{1}{4\pi^2}) + \frac{x^4}{4\pi^2}$$

Multiplying this by the next term in the equation

$$(1 - x^2(\frac{1}{\pi^2} + \frac{1}{4\pi^2}) + \frac{x^4}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) = 1 - x^2(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2}) + \frac{x^4}{4\pi^2} + \frac{x^4}{9\pi^2}(\frac{1}{\pi^2} + \frac{1}{4\pi^2})$$

Collecting all terms with coefficient x^2 we get: $x^2 (\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + ...)$. The reason we do this is because the second term of $\frac{sin(x)}{x}$ is $(\frac{x^2}{3!})$ which also has coefficient x^2. Note that we are only considering the terms that have the coefficient x^2 for proof. Because Euler assumed that both equations are the same, the terms with the same coefficient should be equal to each other.

Thus:

$$-\frac{x^2}{3!} = -x^2\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots\right)$$

$$\Leftrightarrow \frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots$$

$$\Leftrightarrow \frac{1}{6} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots$$

$$\Leftrightarrow \frac{\pi^2}{6} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \ldots$$

$$\Leftrightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Interestingly, and fortunately, despite Euler's assumption of the two equations being the same, they were indeed later on proven to be the same equation. This proof is interesting because it is one of the many examples in which a simple problem involving unit fractions has led to a complicated solution involving identities derived from various areas of mathematics.

Conclusion

Returning to my research question, while my investigations have produced some pleasing but limited results, my overwhelming conclusion is that the world of unit fractions is a box that has hardly been opened (for me at least - especially the Basel problem above)

Whereas my starting assumption was that unit fractions would be an easier topic than prime numbers, it quickly emerged that many of the calculations being made and the topics being investigated are intimately connected to prime numbers or to the factorisation of a non-prime number into its component prime factors.

Perhaps the most surprising discovery, in part 1 of this essay, is that the length of the recurring cycle of some prime number unit fractions can be so surprisingly short: $\frac{1}{41}$has a cycle length of 5 and $\frac{1}{37}$has a cycle length of just 3. Why should that be when other prime numbers, such as 47 have a cycle length of 46?

It occurs to me that perhaps the whole of the early part of this essay and the results that have been discussed and discovered depend fundamentally on the fact that all calculations have been in base 10. What would unit fractions and their decimal cycles look like and how might they be expressed as combinations of other unit fractions had I been working in a different base? This would be an interesting future investigation.

However, in the longer term, after more study and research, my ambition is to carry put further research into the infinite sums of other subsets of unit fractions.

Works Cited

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