**IBDP Mathematics Higher Level**

**Internal Assessment**

**M22**

# Would replacing cylindrical wheels in transportation with an oloid be cost-effective?

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## Background

I first interacted with the oloid in a design lesson where the professor demonstrated its rolling property. I was intrigued by how the object rolled even though it had "corners". The first thing that came to my mind was if the rolling property could make this object replace certain objects in our daily life, such as car wheels or wheels of a wheelbarrow. The primary factor for my curiosity was due to my interest in drag races. A drag race is a short distance race between cars, mainly to test its acceleration and make comparisons. My knowledge of physics also lets me know that acceleration has many different factors in motion, hence not only the horsepower would affect this. In addition, factors like the weight of the vehicle, aerodynamics (wind resistance) and handling (resistance/friction between tyres and surface) all are factors that may affect the acceleration, and chances of winning a drag race. Because the oloid rolled so smoothly, with a small push by my professor to get it to start rolling, I wondered if the oloid could replace a cylindrical wheel in a drag race. Also, its edge or "corner", which is not present in a cylinder, seemed to be beneficial to the balance of the oloid's motion, based on the demonstration. Then I focused more on the oloid's surface and how its construction allows it to roll. The uniqueness of this object and its real life applications, such as navigational propellers, is what made me interested in investigating its properties and understanding the mathematics behind it. To understand the properties of the shape, I researched more about what the oloid is.

## What is an Oloid?

An oloid is a three dimensional object that is created by constructing a convex hull from two congruent circles, perpendicular to each other, that pass through each other's center. A convex hull in three dimensions is simply the enclosure or surface that contains all the given data points, where the surface is created using straight lines from point to point (Dirnböck and Stachel). In this context, it is the surface created when all points on one circle are connected to all the points on the other circle with a straight line, where the infinite lines create a curved face because of their gradual change in gradient.

An oloid has a developed surface, which means that its surface can be flattened into a plane. Interestingly, this object rolls in a straight line, where its entire surface touches the plane on which it is rolling, in one whole revolution (figure 1)

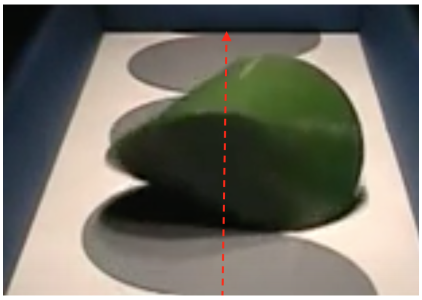

*Fig. 1, oloid's direction of motion and covered surface*

The oloid was discovered by mathematician Paul Schatz in 1929, justifying his mathematics studies in the fields of geometry and trigonometry. His interest in inversions of platonic solids, led him to discover the invertible cube where his thorough investigation led to forming the Oloid (figure 2) (Shelters).

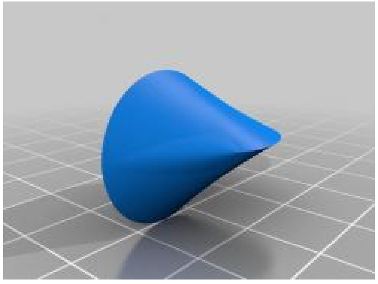

*Fig. 2, Thingiverse, STLfinder.com, 2021*

**Aim**

My aim for this exploration is *to find the surface area of the oloid of radius 1 by making use of two separate methods*. For the first method, I will analyze the shape in its original form and use coordinate geometry, trigonometry and algebra to approximate the surface area (3D). For the second method, I will be unfolding the oloid and analyzing its net. Then, using coordinate geometry, simultaneous equations (row echelon) and integration, I will use my higher level mathematics knowledge to calculate an approximate answer for the surface area of the oloid (2D). I have decided to use two methods as both methods could have different uncertainties due to the analysis being in 3 and 2 dimensions. Assumptions in 3 dimensions could have a bigger effect on the answer, while it is also more reliable as we are not modifying the shape to fit a given dimension. The use of two methods would also be more reliable to verify the answer calculated using both methods. Then, using a reliable internet source, I will compare the approximation I obtained to the true values. This will assist me conclude to a more appropriate answer, thus more reliable exploration. Moreover, I will compare its surface area to that of a cylinder to determine whether replacing a cylinder with an oloid is cost-effective as reduced surface area implies less material used to develop it. This exploration allows me to investigate to what extent an oloid is similar to a cylinder or other axially symmetric objects.

**Initial Analysis of oloid**

To begin a reliable analysis of this shape, I first decided to create an oloid using 3 dimensional plotting on GeoGebra. By simply looking at the shape, I can already see the basic foundation of the object, which is two perpendicular circles with equal radius that go through the other circle's center. Let us call these circles $A_1$ and $A_2$ (figure 3). Also, we can see that the farthest point on circle $A_2$ from $A_1$, lies on the tangent of a point on circle A1, and this tangent is one of the lines of the convex hull, thus the surface of the oloid.

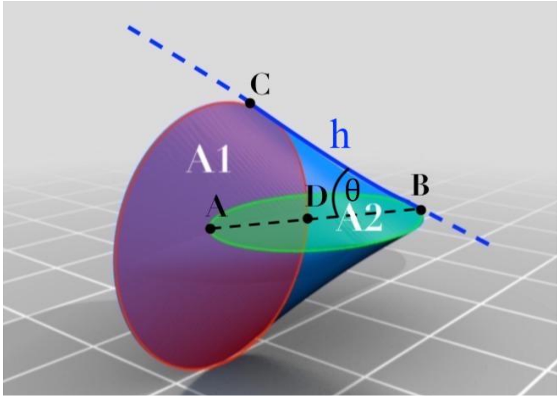


*Fig. 3, initial observed mathematical construction of the oloid*

I created an oloid on GeoGebra, where $A_1$ had equation $f(x) = (x - 1)^2 + z^2 = 1$ and $A_2$ had equation $x^2 + y^2 = 1$, where both circles have radius 1, thus $r^2 = 1$. I added a segment h connecting the two points B and C (figure 3). Point B at (-1, 0, 0), while point C is yet unknown. We know that (-1, 0, 0) lies on the tangent of $A_1$ at C, which gives us two possible answers, as a point outside a circle can form two tangents to the circle. I found this point's coordinates by finding the distance between the two points and using the Pythagorean theorem, as the tangent is perpendicular to the segment going through the point and the center of circle $A_1$.

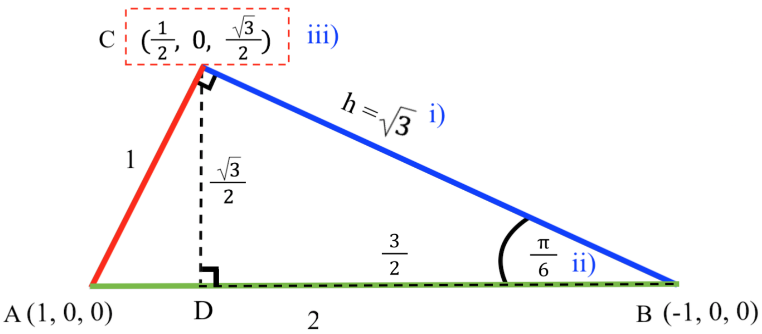


*Fig. 4. Calculating the coordinates of the point on circle $A_1$ whose tangent goes through point (-1, 0, 0)*

*→ diagram not to scale*

The known points are A and B. I will calculate the length of segment h, magnitude of angle θ (radians) and coordinates of point C to satisfy the initial analysis. The red segment is the radius of circle $A_1 = 1$ and is perpendicular to the blue segment as it is the tangent at point C on the circle. The green segment is 2 units (distance from point A to point B). Using the Pythagorean theorem, I calculated the length of segment h.

i)

$$h^2 + 1^2 = 2^2$$

$$\Leftrightarrow h = \sqrt{2^2 - 1^2}$$

$$\Leftrightarrow h = \sqrt{3}$$

Now we can use this value to calculate the angle $\angle ABC = \theta$, $0 < \theta < \pi$, using the cosine rule

ii)

$$1^2 = 2^2 + \sqrt{3}^2 - 2(2)(\sqrt{3})(cos(\theta))$$

$$1 = 4 + 3 - 4\sqrt{3}(cos(\theta)$$

$$\frac{\sqrt{3}}{2} = (cos(\theta))$$

$$\frac{\pi}{6} = \theta$$

This now can be used to calculate the x-component and z-component of the tangent, finding the coordinates of the point C on circle $A_1$ using trigonometric functions. We know that y is always 0 as the circle lies on the plane y = 0.

iii)

$$sin(\frac{\pi}{6}) = \frac{z}{\sqrt{3}} \qquad cos(\frac{\pi}{6}) = \frac{x}{\sqrt{3}}$$

$$\sqrt{3}sin(\frac{\pi}{6}) = z \qquad \sqrt{3}cos(\frac{\pi}{6}) = x$$

$$\frac{\sqrt{3}}{2} = z \qquad \frac{3}{2} = x$$

$\frac{\sqrt{3}}{2}$ is the z-coordinate of point C

$\frac{3}{2}$ is not the x-coordinate, but the distance between point B and D. Using figure 3 and the value calculated for the distance between B and D, the point D is at $(\frac{1}{2}, 0, 0)$, thus the x-coordinate of point C is $\frac{1}{2}$. The length of segment h is $\sqrt{3}$, the magnitude of angle $\theta = \frac{\pi}{6}$ and point C is at $(\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$.

As mentioned, the oloid is the convex hull of two perpendicular circles that contain each other's center, thus the sides are straight lines that connect one point on circle $A_1$ to one point on circle $A_2$ under a given condition, which I will investigate now. The condition involves the distance between the two points. To understand this and the true value of the distance between the two corresponding points on the two circles, one must consider the Schatz cube.

The Schatz cube, by Paul Schatz, is a geometric body, dissected from a normal cube, containing 6 identical tetrahedrons called the dice belt (figure 4). He rotated the dice belt inwards and analyzed the motion of this rotation to generate many surfaces. One of which is found by replacing every tetrahedron with a bar. He called this theory the Schatz Cube Eversion.  Schatz observed that a diagonal of the link, connecting opposite vertices, maintains its original length throughout the eversion or rotation. This diagonal is the tangent of the two points. Thus, the segment following the surface of the oloid. He then fixed a bar of the linkage with no contact without contact with the diagonal. The surface that is swept out by the moving diagonal segment is the oloid (Gunn).

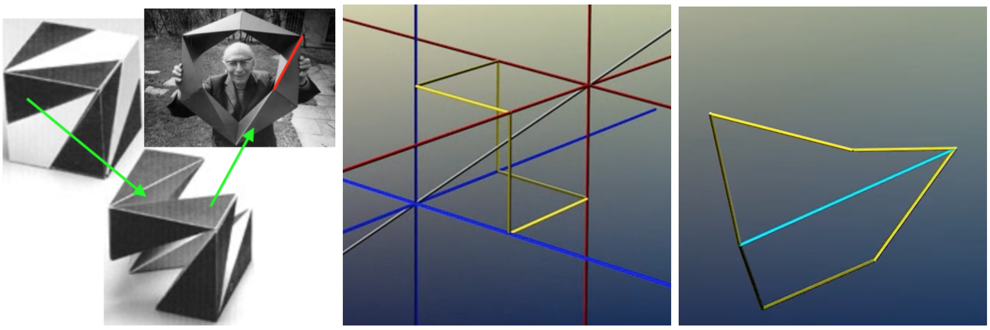

*Fig. 4. Charles Gunn, Schatz Cube that led to the discovery of the oloid, Vimeo, 2017*

After understanding the discovery of the oloid, we can now conclude that the width of the surface of the oloid, or all the segments from one point on circle $A_1$ to one point on circle $A_2$ are always of equal length (figure 5)

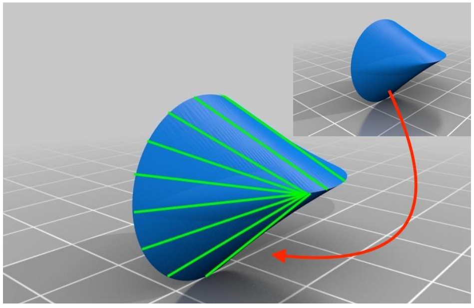

*Fig. 5. Width of surface of oloid, represented with green lines, are equal*

To complete my initial analysis, I believe that finding the magnitude of this segment in terms of the radius r of the circles would be useful to help me achieve my aim. To do this I used the same equations as earlier, however, I will be changing the segment lengths and coordinates of the points based on the different radii of the circles.

| For r = 2 | For r = 3 | For r = 4 | For r = α |
|---|---|---|---|
| Point A is at (2, 0, 0)<br>Point B is at (-2, 0, 0)<br>$L_1^{\ 2} + 2^2 = 4^2$<br>$\Leftrightarrow L_1 = \sqrt{4^2 - 2^2}$<br>$\Leftrightarrow L_1 = \sqrt{12}$<br>$\Leftrightarrow L_1 = 2\sqrt{3}$ | Point A is at (3, 0, 0)<br>Point B is at (-3, 0, 0)<br>$L_1^{\ 2} + 3^2 = 6^2$<br>$\Leftrightarrow L_1 = \sqrt{6^2 - 3^2}$<br>$\Leftrightarrow L_1 = \sqrt{27}$<br>$\Leftrightarrow L_1 = 3\sqrt{3}$ | Point A is at (4, 0, 0)<br>Point B is at (-4, 0, 0)<br>$L_1^{\ 2} + 4^2 = 8^2$<br>$\Leftrightarrow L_1 = \sqrt{8^2 - 4^2}$<br>$\Leftrightarrow L_1 = \sqrt{48}$<br>$\Leftrightarrow L_1 = 4\sqrt{3}$ | Point A is at ( α, 0, 0)<br>Point B is at (-α, 0, 0)<br>$L_1^{\ 2} + \alpha^2 = (2\alpha)^2$<br>$\Leftrightarrow L_1 = \sqrt{(2\alpha)^2 - \alpha^2}$<br>$\Leftrightarrow L_1 = \sqrt{4\alpha^2 - \alpha^2}$<br>$\Leftrightarrow L_1 = \sqrt{3\alpha^2}$<br>$\Leftrightarrow L_1 = \alpha\sqrt{3}$ |

Note that the coordinates are unique to the equations of the two circles I have chosen, however, the length of the segment will remain constant using the reasoning by Schatz. Using this, I conclude that the segment, or width of the oloid will always be of magnitude $r\sqrt{3}$ where r is the radius of both circles. My following step is to generalize the points on both circles in terms of just one variable by making use of the equations of the two circles and the distance formula. This is useful as it gives us the exact coordinates of the corresponding point of a point on one of the circles. I can then use this to find the line that describes the motion of the tangent for different values of x. This methodology and reasoning will be further cleared as we delve into calculations. In addition, to help me with my exploration, I independently created a three dimensional oloid using GeoGebra to help me better visualize and understand its properties and equations involved (figure 6).

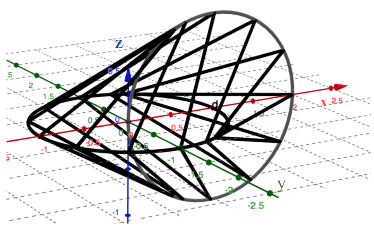

*Fig. 6, Oloid Graped using online modeling tool GeoGebra*

**Surface Area of oloid - Method 1**

Finding the surface area of the oloid is important as it tells us the minimum space it requires to roll on the surface as its entire body makes contact with the surface it is rolling on during a single revolution. I will attempt to find the surface area in terms of the radius of the two circles. The fundamental idea is that the segments on the surface of the oloid of magnitude $r\sqrt{3}$ follow a path, which can be found by plotting points at the center of each segment and forming a line that goes through all the points (figure 7).

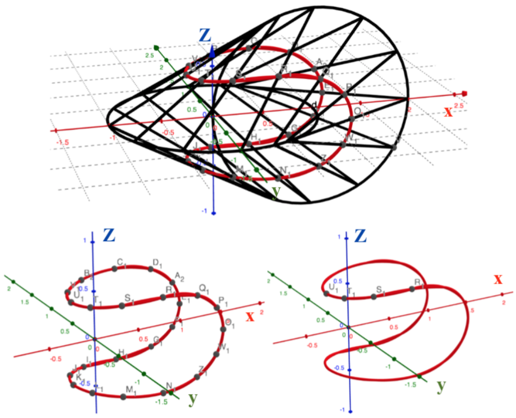

*Fig. 7, Oloid for which the red line represents that path that the segments of magnitude of* $r\sqrt{3}$ follow

Hence, by definition, the surface area of the oloid will be the magnitude of the tangent multiplied by the length of the red enclosed line. Because of the irregularity of the line, I attempted to approximate the length of the line by the use of vectors. My aim now is to find a general vector from circle $A_1$ to the red line and then finding the approximate length of the red line by finding the distance between 100 points on the line and finding the sum.

Finding the coordinates of a general point on circle $A_1$ in terms of its x-coordinate:

| | |
|---|---|
| $(x - 1)^2 + z^2 = 1 \; --> y = 0$ | $x^2 + y^2 = 1 \; --> z = 0$ |
| $z = \pm\sqrt{1 - (x - 1)^2}$ | $y = \pm\sqrt{1 - x^2}$ |
| Point on Circle $A_1 = (x_1, 0, \sqrt{1 - (x_1 - 1)^2})$ | Point on Circle $A_2 = (x_2, \sqrt{1 - x_2^2}, 0)$ |

I will not be using negative values for the y and z coordinates as only half the circle considered

Now I aim to find the coordinates of points Circle $A_2$ in terms of the x-coordinate of the point on circle $A_1$.

This is possible and will only give me one possibility as we are counting a half of each circle and we have the magnitude of the segment from the point on $A_1$ that correlates with the point on $A_2$.

$$\sqrt{3} = \sqrt{(x_2 - x_1)^2 + (\sqrt{1 - x_2^2} - 0)^2 + (0 - \sqrt{1 - (x_1 - 1)^2})^2}$$

$$\Leftrightarrow 3 = (x_2 - x_1)^2 + (\sqrt{1 - x_2^2})^2 + (-\sqrt{1 - (x_1 - 1)^2})^2$$

$$\Leftrightarrow 3 = -2x_2x_1 + 1 + 2x_1$$

$$\Leftrightarrow 2 = x_1(-2x_2 + 2)$$

$$\Leftrightarrow \frac{-1+x_1}{x_1} = x_2$$

Thus the points on circle $A_1$ and on circle $A_2$ can be re-written as

| Point on Circle $A_1$ | Point on Circle $A_2$ |
|---|---|
| $= (x_1,\ 0, \sqrt{1 - (x_1 - 1)^2})$ | $= (\frac{-1+x_1}{x_1}, \sqrt{1 - (\frac{-1+x_1}{x_1})^2},\ 0)$ |

The red line (figure 7) is simply the midpoint of the corresponding points on the Point on Circle $A_1$ and the Point on Circle $A_2$. Thus we can use the midpoint formula to define the points on the red line.

$$(x_m, y_m, z_m) = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$$

$$x_m = \frac{x_1 + \frac{-1+x_1}{x_1}}{2} \qquad y_m = \frac{0+\sqrt{1-(\frac{-1+x_1}{x_1})^2}}{2} \qquad z_m = \frac{\sqrt{1-(x_1-1)^2}+0}{2}$$

$$\Leftrightarrow x_m = \frac{x_1^2+x_1-1}{2x_1} \qquad \Leftrightarrow y_m = \frac{\sqrt{1-(\frac{-1+x_1}{x_1})^2}}{2} \qquad \Leftrightarrow z_m = \frac{\sqrt{1-(x_1-1)^2}}{2}$$

So the coordinates of a general point on the red line in terms of the x-coordinate of circle $A_1$ located on the XZ-plane are given as $(\frac{x_1^2+x_1-1}{2x_1}, \frac{\sqrt{1-(\frac{-1+x_1}{x_1})^2}}{2}, \frac{\sqrt{1-(x_1-1)^2}}{2})$

This only gives us a fourth of the points as I decided to use only positive values for y and z components.

I made use of the online software GeoGebra, to first plot all midpoints of the tangents and then used its "polyline" function to graph the line going through the center of the surface of the oloid. The polyline function connects all selected points and calculates the distance between these points, which helps me visualize the problem at hand and understand it better to find more efficient solutions.

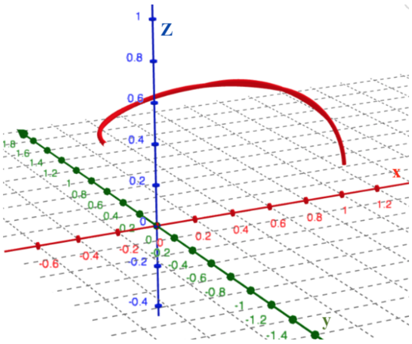


*Fig. 8, quarter of red segment due to current domain and range*

However, including negative values for z, we get $\left(\frac{x_1^2+x_1-1}{2x_1}, \frac{\sqrt{1-\left(\frac{-1+x_1}{x_1}\right)^2}}{2}, \pm\frac{\sqrt{1-(x_1-1)^2}}{2}\right)$

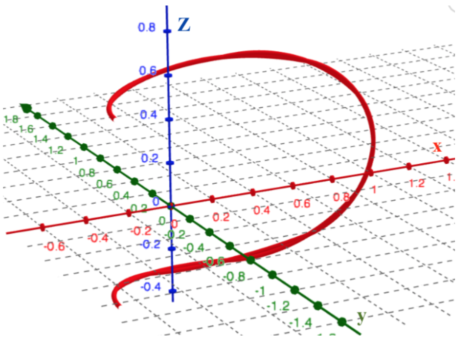


*Fig. 9, half of red segment due to domain and range*

And including negative values for y as well, gives us $\left(\frac{x_1^2+x_1-1}{2x_1}, \pm\frac{\sqrt{1-\left(\frac{-1+x_1}{x_1}\right)^2}}{2}, \pm\frac{\sqrt{1-(x_1-1)^2}}{2}\right)$

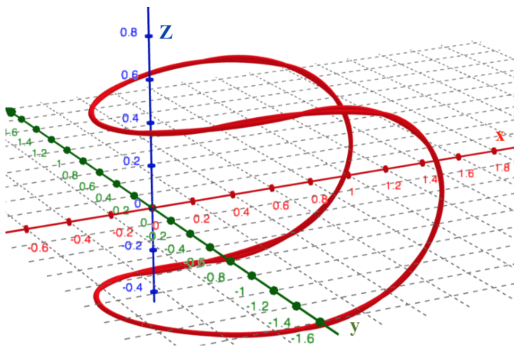


*Fig. 10, complete red segment*

So to calculate the length of the entire line, I must calculate the length of the line formed by the points

formed by $(\frac{x_1^{\,2}+x_1-1}{2x_1}, \frac{\sqrt{1-(\frac{-1+x_1}{x_1})^2}}{2}, \frac{\sqrt{1-(x_1-1)^2}}{2})$ for $\frac{1}{2} \leq x \leq 2$ and multiply this constant by 4, which

will give me an approximation.

With the help of technology, using the 'polyline' function on GeoGebra, I found the entire red line to be 7.7089 units. Thus the surface area of the oloid will be

$SA = (7.7089)(\sqrt{3})$

$SA \approx 13.352$

**Surface Area of oloid - Method 2**

The second method involved calculating the surface area of the oloid is to unfold the 3D shape onto a plane, which will give us the geometry net of the shape. The net of the oloid (figure 11), or any other shape, describes what it would look like when unfolded and placed flat on a plane.

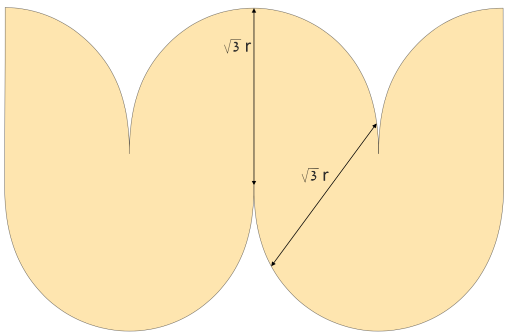


*Fig. 11, baars, oloid knipplaat/bouwplaat (developed oloid surface), Elke Dag Wat, 2019*

At initial glance, the shape seems quite regular for the surface area to be calculated using trigonometry formulae. It does seem like a semi-circle or semi-ellipse. However, this is not true. Due to the unexpected behavior of the edge of the net as it nears the x-axis and lack of information of the distances on the geometry net, this is not possible.

As the edge nears the x-axis, it becomes completely vertical, becoming a one-to-many function. Due to its symmetry, I can find the area of one of the outlined sections of the net (figure 12) and multiply this by 8 to cover the entire net. I will do this by plotting points on the edge of the net. Then, using reduced row echelon form, I will find a function that goes through these points and eventually integrate this with respect to x to find the area.

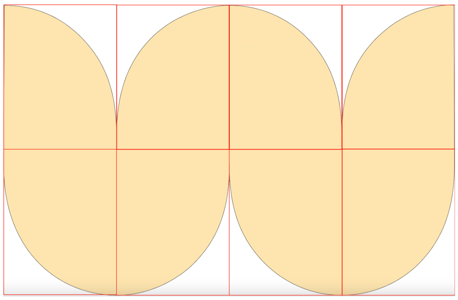

*Fig. 12, symmetry of the geometry net of the oloid*

After inserting the image on Desmos and modifying the image to fit the right scale so all points and the function will be shown with accurate measurements, the maxima of the net is at point $(0,\ 1.585)$. I plotted a point on the right bound of the section being used (figure 13) using 'drag' to drag the point to the coordinates that is the best fit according to me. The coordinates of this point are $(1.212,\ 0.16)$.

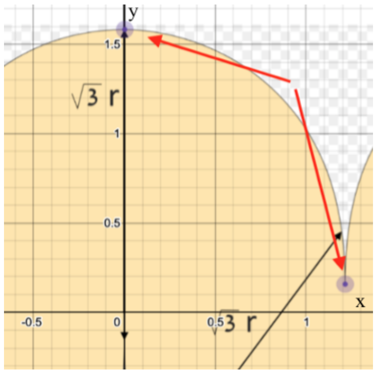


*Fig. 13, symmetry of geometry net of oloid*

To then plot points along the edge of the net, I calculated the angle at the origin formed by the two points using trigonometry. To have the points on the edge of the curve kept to a minimum number and well distributed to generate an accurate function, I will be dividing the angle at the origin formed by the two points by the number of points so that the angular distance between the points is equal. This will show a clear relation between the two variables. α is the angle, in degrees, at the origin formed by the point (1.212, 0.16) and the x-axis. I will subtract 90 by the angle α to calculate the angle β, in degrees, which is the angle at the origin formed by the two points (0, 1.585) and (1.212, 0.16)

i) $tan(\alpha)\ =\ \frac{0.16}{1.2119}$

$\alpha\ \approx\ 7.52$

ii) $90\ -\ 7.52\ =\ 82.48\ =\ \beta$

I aim to calculate a polynomial of degree 5, so I will need 6 points on the net. I am calculating a polynomial of degree 5 because 6 points will definitely give me an accurate polynomial of degree 5 as there are 6 variables. (6 variables → 6 equations). Using a lower degree would rarely produce a function going through all points. As I need 6 points and 2 points are already given, I will be plotting 4 more points at equal angles from each other.

$$\frac{82.48}{5} = 16.496$$

16.096° is the angle, in degrees, at the origin by two adjacent points on the edge of the net. To plot the remaining points, I found the equation of the line that forms this angle with the origin. The used point will lie on the line of equation y = mx + c, c = 0 as the line will go through the origin. Variable m is the angle of the line with respect to the x-axis. To get the equations of the lines on which the different points at different angles, we must understand the derivation of m. The gradient of a line is $\frac{sin(\varphi)}{cos(\varphi)} = tan(\varphi)$, where φ (radians) is the angle formed at the origin by two adjacent points. The current angle is in degrees. To convert this into radians, we will use the expression $1° = \frac{\pi}{180}\ rad$. Thus the equations of the lines will be $y_n = tan(\frac{(90-(n)(16.096))\pi}{180})x$, $\{n|\ 1 \leq n \leq 4,\ n \in N\}$. Variable n has a domain as we need 4 points between the two already known points. Values are rounded to 5 decimal places to maximize accuracy and consider readability.

| | |
|---|---|
| $y_1 = tan(\frac{(90-(1)(16.496))\pi}{180})x$ <br> $y_1 = 3.37681x$ | $y_2 = tan(\frac{(90-(2)(16.496))\pi}{180})x$ <br> $y_2 = 1.54034x$ |
| $y_3 = tan(\frac{(90-(3)(16.496))\pi}{180})x$ <br> $y_3 = 0.85444x$ | $y_4 = tan(\frac{(90-(4)(16.496))\pi}{180})x$ <br> $y_4 = 0.44556x$ |

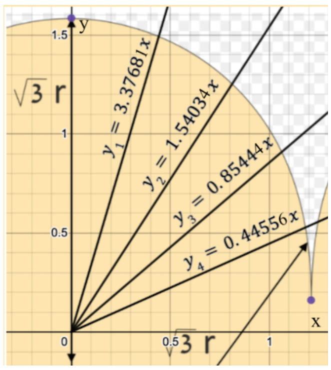


*Fig. 14, equations of line on which points that show trendline between two variables lie*

I used the ‘drag’ function to drag points onto coordinates of the intersections between the lines and the edge of the net of the oloid. These points are now a set of equally distributed, reliable points on the edge of the net to show a trendline between the x and y variables to eventually project a function that follows the edge of the oloid’s geometry net. The coordinates of the overall 6 chosen points are:

(0, 1.585), (0.442,1.493), (0.813,1.252), (1.063,0.909), (1.182,0.527), (1.212, 0.16)

I will be performing reduced row echelon to find the polynomial going through these points. The equation of a general polynomial to degree 5 is $y = px^5 + qx^4 + rx^3 + sx^2 + tx + u$. The variables are p, q, r, s, t and u as y and x are given (coordinates of the points). The x variable with different exponents is the coefficient of these variables. For conciseness, after the first table, I will be rounding values in the table to 5 decimal places, however my calculations throughout REF will be done using the exact values. The reason why I have rounded to 5 decimal places is to maximize accuracy while having minimal readability issues.

| p | q | r | s | t | u | y |
|---|---|---|---|---|---|---|
| $1.212^5$ | $1.212^4$ | $1.212^3$ | $1.212^2$ | $1.212$ | 1 | $0.160$ |
| $1.182^5$ | $1.182^4$ | $1.182^3$ | $1.182^2$ | $1.182$ | 1 | $0.527$ |
| $1.063^5$ | $1.063^4$ | $1.063^3$ | $1.063^2$ | $1.063$ | 1 | $0.909$ |
| $0.813^5$ | $0.813^4$ | $0.813^3$ | $0.813^2$ | $0.813$ | 1 | $1.252$ |
| $0.442^5$ | $0.442^4$ | $0.442^3$ | $0.442^2$ | $0.442$ | 1 | $1.493$ |
| 0 | 0 | 0 | 0 | 0 | 1 | $1.585$ |

$R_2 - (R_1)(\frac{1}{1.212^5})(1.182^5) \rightarrow R_2$

$R_3 - (R_1)(\frac{1}{1.212^5})(1.063^5) \rightarrow R_3$

$R_4 - (R_1)(\frac{1}{1.212^5})(0.813^5) \rightarrow R_4$

$R_5 - (R_1)(\frac{1}{1.212^5})(0.442^5) \rightarrow R_5$

| | | | | | | |
|---|---|---|---|---|---|---|
| $1.212^5$ | $1.212^4$ | $1.212^3$ | $1.212^2$ | $1.212$ | $1$ | $0.160$ |
| $0$ | $0.04832$ | $0.08074$ | $0.10120$ | $0.11276$ | $0.11779$ | $0.38585$ |
| $0$ | $0.15697$ | $0.27718$ | $0.36761$ | $0.43399$ | $0.48102$ | $0.82596$ |
| $0$ | $0.14382$ | $0.29557$ | $0.46147$ | $0.64840$ | $0.86419$ | $1.23027$ |
| $0$ | $0.02425$ | $0.07487$ | $0.18589$ | $0.43418$ | $0.99355$ | $1.49197$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $1.585$ |

$R_3 - (R_2)(\frac{1}{0.04832})(0.15697) \rightarrow R_3$

$R_4 - (R_2)(\frac{1}{0.04832})(0.14382) \rightarrow R_4$

$R_5 - (R_2)(\frac{1}{0.04832})(0.02425) \rightarrow R_5$

| | | | | | | |
|---|---|---|---|---|---|---|
| $1.212^5$ | $1.212^4$ | $1.212^3$ | $1.212^2$ | $1.212$ | $1$ | $0.160$ |
| $0$ | $0.04832$ | $0.08074$ | $0.10120$ | $0.11276$ | $0.11779$ | $0.38585$ |
| $0$ | $0$ | $0.01487$ | $0.03883$ | $0.06767$ | $0.09835$ | $-0.42759$ |
| $0$ | $0$ | $0.05523$ | $0.16022$ | $0.31275$ | $0.51357$ | $0.08170$ |
| $0$ | $0$ | $0.03435$ | $0.13510$ | $0.37759$ | $0.93444$ | $1.29832$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $1.585$ |

$R_4 - (R_3)(\frac{1}{0.01487})(0.05523) \rightarrow R_4$

$R_5 - (R_3)(\frac{1}{0.01487})(0.03435) \rightarrow R_5$

| | | | | | | |
|---|---|---|---|---|---|---|
| $1.212^5$ | $1.212^4$ | $1.212^3$ | $1.212^2$ | $1.212$ | $1$ | $0.160$ |
| $0$ | $0.04832$ | $0.08074$ | $0.10120$ | $0.11276$ | $0.11779$ | $0.38585$ |
| $0$ | $0$ | $0.01487$ | $0.03883$ | $0.06767$ | $0.09835$ | $-0.42759$ |
| $0$ | $0$ | $0$ | $0.01598$ | $0.06138$ | $0.14821$ | $1.67011$ |
| $0$ | $0$ | $0$ | $0.04539$ | $0.22127$ | $0.70722$ | $2.28615$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $1.585$ |

$R_5 - (R_4)(\frac{1}{0.01598})(0.04539) \rightarrow R_5$

| | | | | | | |
|---|---|---|---|---|---|---|
| $1.212^5$ | $1.212^4$ | $1.212^3$ | $1.212^2$ | $1.212$ | $1$ | $0.160$ |
| $0$ | $0.04832$ | $0.08074$ | $0.10120$ | $0.11276$ | $0.11779$ | $0.38585$ |
| $0$ | $0$ | $0.01487$ | $0.03883$ | $0.06767$ | $0.09835$ | $-0.42759$ |
| $0$ | $0$ | $0$ | $0.01598$ | $0.06138$ | $0.14821$ | $1.67011$ |
| $0$ | $0$ | $0$ | $0$ | $0.04687$ | $0.28609$ | $-2.45937$ |
| $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $1.585$ |

Final values are rounded to 5 decimal places for accuracy and readability. Using calculations obtained from row echelon form above, I will be working out the values for the 6 variables.

$p(0) + q(0) + r(0) + s(0) + t(0) + u(1) = 1.585$

$\Leftrightarrow u = 1.585$

---

$p(0) + q(0) + r(0) + s(0) + t(0.04687) + u(0.28609) = -2.45937$

$\Leftrightarrow t(0.04687) + (1.585)(0.28609) = -2.45937$

$\Leftrightarrow t \approx -62.14685$

---

$p(0) + q(0) + r(0) + s(0.01598) + t(0.06138) + u(0.14821) = 1.67011$

$\Leftrightarrow s(0.01598) + (-62.14685)(0.06138) + (1.585)(0.14821) = 1.67011$

$\Leftrightarrow s \approx 328.52133$

---

$p(0) + q(0) + r(0.01487) + s(0.03883) + t(0.06767) + u(0.09835) = -0.42759$

$\Leftrightarrow r(0.01487) + (328.52133)(0.03883) + (-62.14685)(0.06767) + (1.585)(0.09835) = -0.$

$\Leftrightarrow r \approx -614.28922$

---

$p(0) + q(0.04832) + r(0.08074) + s(0.10120) + t(0.11276) + u(0.11779) = 0.38585$

$$\Leftrightarrow q(0.04832) + (-614.28922)(0.08074) + (328.52133)(0.10120) + (-62.14685)(0.11276) + (1.585)(0.11779) = 0.38585$$

$\Leftrightarrow q \approx 487.54521$

---

$p(1.212)^5 + q(1.212)^4 + r(1.212)^3 + s(1.212)^2 + t(1.212) + u(1) = 0.16$

$$\Leftrightarrow p(1.212)^5 + (487.54521)(1.212)^4 + (-614.28922)(1.212)^3 + (328.52133)(1.212)^2 + (-62.14685)(1.212) + (1.585)(1) = 0.16$$

$\Leftrightarrow p \approx -140.34981$

Thus the polynomial going through all 6 points is

$g(x) = -140.34981x^5 + 487.54521x^4 - 614.28922x^3 + 328.52133x^2 - 62.14685x + 1.585$

This polynomial goes through all the points, however, I will not be able to integrate it to find the surface area as the function crosses the net of the oloid, due to the minimas and maximas (figure 15). This is because of overfitting. Overfitting is a term commonly used in statistics, referring to when a function or trendline fits the given points, but does not accurately fit the unseen data and ignores the relationship between the two variables. This issue occurs when the function is too complex or not enough, accurate data is provided.

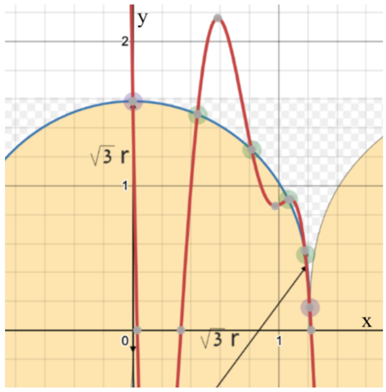


*Fig. 15, red function g(x) going through all points*

I also noticed that the structure of this polynomial has the form of the function sin(x) or cos(x) when expanded using the Maclaurin series. Colin Maclaurin, mathematician in the eighteenth century, discovered the Maclaurin series, which is an approximation of a sin or cosine function expressed as an infinite summation. This means that the coordinates of the function follow some modification of the sine and cosine function, which is visible.

$$cos(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{2k}}{2k!}\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$sin(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{2k+1}}{(2k+1)!}\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

Because I have only considered a polynomial till degree 5, I will only need the first 3 terms of the expansion of the y = cos(x) and y = sin(x) function to receive an accurate approximation. In the polynomial, all terms where x has an even exponent are positive, while all terms where x has an odd exponent are negative. Thus, it is justified to assume that the function will be a modification of the following function

$$y = |cos(x)| - |sin(x)|$$

I will be using the 'statistics' and 'trendline' functions on Desmos to plug in my points and the function above with added variables for it to undergo transformations and fit my points as well as possible. Desmos uses the given, plotted points to find the values of all variables in a given function of any form, as a trendline.

$$y = 1.585\left(|cos(ax)|^b - |sin(ax)|^b\right)^c$$

The coefficient of this function will be 1.585 as that is the peak or maxima of the net of the oloid, thus the amplitude of the trigonometric function to be found. The three variables a, b and c all influence the width and period of the function. Exponents in cosine and sine functions are crucial for my data points, as they do not follow the ideal path of a sine or cosine function, but are positioned more like a circle or ellipse, which the three variables will affect.

Effectively making use of Desmos, I found the values of the three variables a, b and c to be 0.64752, 2.11764 and 0.34233 respectively, rounded to 5 decimal places. Thus the final function that goes over the points and the edge of the oloid is

$$y = 1.585(|cos(0.64752x)|^{2.11764} - |sin(0.64752x)|^{2.11764})^{0.34233}$$

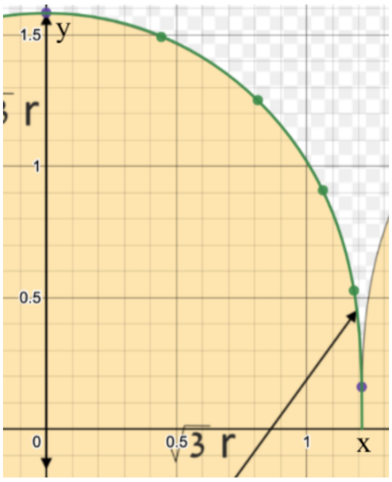


*Fig. 16, h(x) shown in green and going through points in the desired/expected behavior*

Because of the complexity of this expression, this function cannot be integrated using regular integration formulas covered in the IB syllabus. Therefore I will be making use of an online software to integrate the function. I need to multiply the integration result by 8 to get the surface area of the entire geometry net as we only considered points and the function for an eighth of the geometry net.

$$8\int_0^{1.212} (1.585(|cos(0.64752x)|^{2.11764} - |sin(0.64752x)|^{2.11764})^{0.34233})\, dx \approx 12.647$$

$$SA = 12.647\, units^2$$

**Conclusion**

Finally, to complete the aim of my investigation, I will be comparing the two methods and the approximated answers to the true value. The true value of the surface area of an oloid with radius 1 is very interestingly 4π, approximately $12.566\, units^2$, which is the exact surface area of a sphere of radius 1 as well.

| | Surface area in $units^2$ | Percentage error |
|---|---|---|
| Method 1 | 13.352 | 6.25% |
| Method 2 | 12.647 | 0.64% |

The calculated surface area for method 1 is quite accurate with only a 6.25% uncertainty, however when compared to the percentage error of method 2, this seems like a large error. The factors that may have caused this percentage error is the number of points chosen to create the polyline. Perhaps more points plotted would have given a much more accurate value and reduced percentage error. Also, in this methodology, we treated the oloid like an open cylinder or mobius strip where multiplying the width and length of the midpoints along the surface would give us the accurate answer for the surface area. However, an open cylinder and mobius have in common that when unfolded, its geometry net forms a rectangle so the line connecting midpoints along the surface would simply be the height and then this would essentially be multiplying the height by the width (basic formula for SA)

The calculated surface area for method 2 is extremely accurate considering the approximations. The factors that may affect the percentage error would be the approximations of values when “choosing” the coordinates of the points that fit best by eye, approximations of final values after row echelon form and solving the linear equations to calculate variables p, q, r, s, t and u. Another approximation would be when values for a, b and c were calculated by Desmos, thus the software would round as well when displaying decimals. This same case occurs when the software solved or approximated the final integration value. An important consideration in terms of percentage error is that when inserting the image onto Desmos, the scale might have been slightly incorrect. I did make use of the $\sqrt{3}$ indicator on the image, however this could have not been 100% accurate. Thus, method 2 was more effective than method 1 for generating an accurate value for the surface area of the oloid.

Moreover, to make an effective comparison between a cylinder and an oloid, they must be of similar sizes, and the height and radius of the cylinder would be 3 and 1 respectively. The reason for this is that the length of the oloid is 3 as it is 3 times the radius and the radius of the oloid is 1 (figure 16). Thus, using the formula for surface area of a cylinder, the surface area would be approximately 25.133, which is almost double of the surface area of an oloid with similar dimensions.

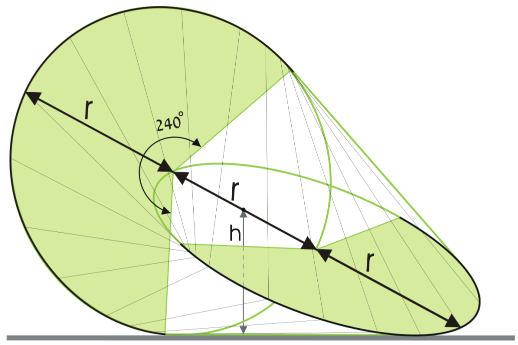


*Fig. 16, Shreyashkar Lal, oloid geometry, Quora, 2020*

Thus, the oloid would be more cost-effective to develop as a reduced surface means that the minimum amount of material required to develop the figure would be less. Also, in the context of a drag race, after this exploration and observations, I believe that the edges of the oloid help overcome magnitudes of friction that may cause a cylindrical wheel to change direction, but not an oloid. Therefore this could be an advantageous property in a drag race as less resistance in other directions would decrease acceleration. I do think that its shape would introduce some unconsidered factors that would affect its acceleration, which I believe would be interesting to investigate. Either way, I believe that on a global scale, if the motion of the oloid is studied closely, it could be used in the transportation industry, for less resources.

**Further Research**

If I were to continue this investigation, the conclusion of this exploration has made me more curious about how the volume of the oloid affects its motion, and also the mathematics behind its movement. This would be my focus for an extension. A sequential investigation would also be to plot its motion graphs and analyze the properties of the graph, then investigate the independent variables that could be changed to improve the acceleration of the oloid. This would be the gradient of a velocity-time graph of the oloid. We also assume throughout this exploration that the oloid's application in the drag race will cause it to be hollow, thus not considering its total weight when filled. I do believe that the oloid's movement seems to be like simple harmonic motion. Thus the velocity over time is sinusoidal and the weight may have an effect on the acceleration. These are all factors that would affect the oloid's performance in a drag race.

**Bibliography**


Baars. “Oloïde.” *Elke Dag Wat - Blog*, 11 Oct. 2019, elkedagwatblog.com/2019/10/11/oloide/. Accessed 26 Jan. 2022.

Dirnböck, Hans, and Hellmuth Stachel. “The Development of the Oloid.” *Journal for Geometry and Graphics*, vol. 1, no. 2, 12 Nov. 1997, pp. 105–118, www.heldermann-verlag.de/jgg/jgg01_05/jgg0113.pdf. Accessed 11 Dec. 2021.

Eleni999. “Rolling Oloid.” *Www.youtube.com*, 22 Oct. 2008, www.youtube.com/watch?v=GM3_JuFgJ2E. Accessed 24 Jan. 2022.

Gunn, Charles. “Schatz Cube Eversion.” *Google.com*, Vimeo, 2022, www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=&cad=rja&uact=8&ved=2ahUKEwj49oKBs8_1AhWLCOwKHR5dAuMQwqsBegQIAhAB&url=https%3A%2F%2Fvimeo.com%2F214665070&usg=AOvVaw0XAA44OyWJy65zMz4ewkDW. Accessed 26 Dec. 2021.

Lal, Shreyashkar. “What Are Statoliths?” *Quora*, 15 Oct. 2020, www.quora.com/What-are-statoliths. Accessed 26 Dec. 2021.

Shelters, Karen. “Who Was Paul Schatz.” *Truespring.wordpress.com*, TrueSpring, 14 Feb. 2013, truespring.wordpress.com/2013/02/14/who-was-paul-schatz/. Accessed 24 Dec. 2021.