Numerical Solution for One-Dimensional Acoustic Fields in Ducts with an Axial Temperature Gradient

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Introduction

The study of one-dimensional acoustic fields in ducts with axial temperature gradients is crucial for applications like controlling combustion instabilities, designing pulse combustors, and analyzing high-temperature systems. Understanding how temperature gradients affect sound wave propagation and stability is key to improving these technologies. While numerical methods exist for solving the wave equation in such scenarios, they often lack physical insight and require computational resources, making analytical solutions highly desirable.

This assignment presents a method for obtaining exact numerical solutions for one-dimensional acoustic oscillations in ducts with arbitrary axial temperature profiles. The approach involves transforming the wave equation from second-order ODE to first-order ODE, reducing it to a standard differential equation whose solutions are known. The method is demonstrated for linear temperature profiles, and is applied to high-temperature admittance measurements. The analysis assumes no mean flow, limiting its applicability to low Mach number flows (Mach number < 0.1).

Derivation of the Wave Equation

Assuming a perfect inviscid and non-heat-conducting gas the one-dimensional momentum, energy and state equations can be expressed in the following form:

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0, \quad (1)$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} = 0, \quad p = \rho R T. \quad (2,\ 3)$$

Expressing each of the dependent variables as the sums of steady and time-dependent, small amplitude solutions:

$$\begin{aligned} u &= \bar{u}(x) + u'(x,t), \\ p(x,t) &= \bar{p}(x) + p'(x,t), \\ \rho(x,t) &= \bar{\rho}(x) + \rho'(x,t), \quad (4) \end{aligned}$$

and substituting these expressions into the conservation equations yield systems of steady equations and wave equations. Assuming that the mean flow Mach number is smaller than 0.1, the solution of the steady momentum equation shows that the mean pressure, $\bar{p}$, is constant in the duct.

The wave equation is derived from the first-order acoustic momentum and energy equations:

$$\frac{\partial u'}{\partial t} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} = 0, \quad (5)$$

$$\frac{\partial p'}{\partial t} + \gamma \bar{p}\,\frac{\partial u'}{\partial x} = 0. \quad (6)$$

Differentiating the momentum equation with respect to x and the energy equation with respect to t and eliminating the cross-derivative term yield the wave equation with variable coefficients:

$$\frac{\partial^2 p'}{\partial x^2} - \frac{1}{\bar{\rho}}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}x}\frac{\partial p'}{\partial x} - \frac{\bar{\rho}}{\gamma \bar{p}}\frac{\partial^2 p'}{\partial t^2} = 0. \quad (7)$$

Differentiating the steady equation of state and recalling that the steady duct pressure is constant yield the following relationship between the steady temperature and density:

$$\frac{1}{\bar{\rho}}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}x} + \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}x} = 0. \quad (8)$$

Using equation (8), equation (7) can be reduced to

$$\frac{\partial^2 p'}{\partial x^2} + \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}x}\frac{\partial p'}{\partial x} - \frac{1}{\gamma R \bar{T}}\frac{\partial^2 p'}{\partial t^2} = 0. \quad (9)$$

Assuming that the solution has a periodic time dependence (i.e., $p'(x,t) = P'(x)e^{i\omega t}$), equation (9) reduces to the following second-order ordinary differential equation for the complex amplitude $P'(x)$:

$$\frac{\mathrm{d}^2 P'}{\mathrm{d}x^2} + \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}x}\frac{\mathrm{d}P'}{\mathrm{d}x} + \frac{\omega^2}{\gamma R \bar{T}} P' = 0, \quad P' = 2000 \text{ Pa} \quad (10,\ 11)$$

Equation (10) is a second-order ODE. Therefore, exact solutions of this equation for a general mean temperature profile $\bar{T}(x)$ is difficult to obtain. An exact solution is obtained herein by first transforming equation (10) to two first-order ODEs. Assuming that the mean temperature profile $\bar{T}(x)$ is known, equation (10) is transformed as

$$\frac{\mathrm{d}P'}{\mathrm{d}x} = Z. \quad (12)$$

$$\frac{\mathrm{d}Z}{\mathrm{d}x} + \frac{1}{\bar{T}}\frac{\mathrm{d}\bar{T}}{\mathrm{d}x} Z + \frac{\omega^2}{\gamma R \bar{T}} P' = 0. \quad (13)$$

Acoustic Behavior of a Duct with a Linear Temperature Distribution

The acoustic characteristics of a duct with a linear mean temperature distribution that is given by the expression:

$$T_0 = T + mx. \quad (14)$$

where T_0 and m are constants that describe the temperature at $x = 0$ and the temperature gradient, respectively

Initial Temperature (T_1) (K)	Temperature Gradient (m) (K/m)
300	0
500	-50
700	-100
900	-150
1100	-200

Table 1: Temperature Profile

To demonstrate the effect of a linear temperature gradient upon the acoustic properties of a duct, the acoustic characteristics of a duct closed at one end and open at the other (i.e., a quarter-wave tube) are investigated.

Determination of the Eigenvalues for various Harmonics

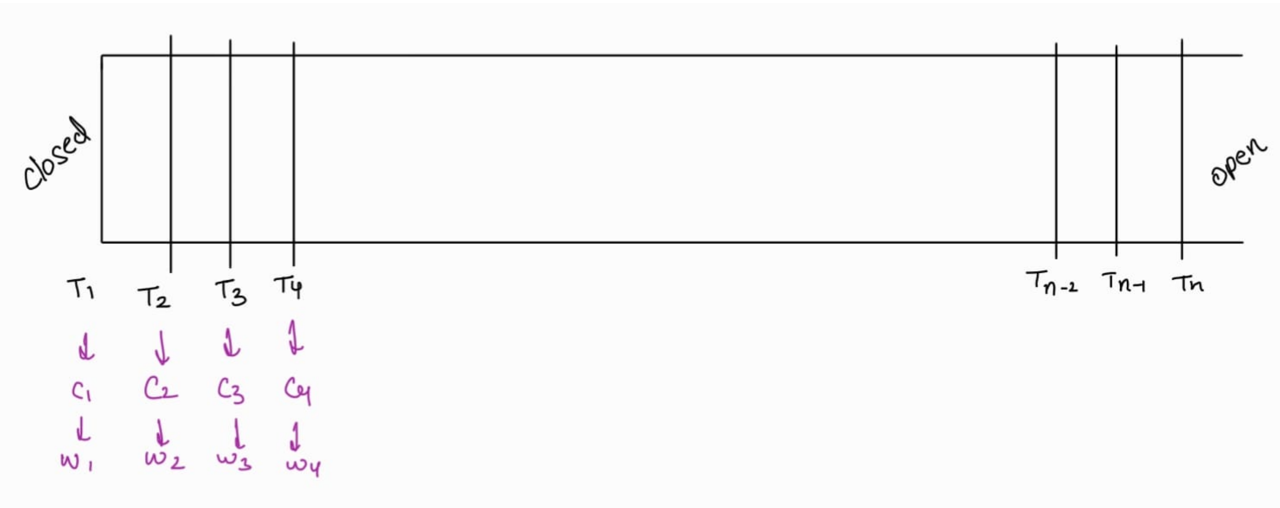


Figure 1: Temperature profile along the duct

1. Divide the 4-meter duct into 40 equal sections. Each section will therefore have a length of 0.1 meters.

2. Using the given temperature profile, calculate the temperature at each of the 40 sections along the duct.

3. For each of the 40 sections, compute the corresponding velocity of sound based on the temperature. Using the velocity of sound, determine the frequency and angular frequency for each section.

4. Use the 40 different angular frequencies as mean frequencies throughout the duct.

5. Check the boundary conditions $P(0) = 2000$ and $P(4) = 0$.
   If any angular frequency satisfies these boundary conditions, that frequency is the solution.

6. Repeat the above steps for the first harmonic, second harmonic, third harmonic, fourth harmonic, and fifth harmonic to identify the corresponding angular frequencies that satisfy the boundary conditions.

Runge-Kutta Scheme

The 4th-order Runge-Kutta method (RK4) is a numerical technique used to solve initial value problems for ordinary differential equations (ODEs). Consider a first-order ODE of the form:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \quad (15,\ 16)$$

where:

- $y(t)$ is the unknown function to be solved,
- $f(t, y)$ is the derivative function,
- y_0 is the initial condition at $t = t_0$.

The RK4 method approximates the solution at $t_{n+1} = t_n + h$, where h is the step size, using the following steps:

Intermediate Slopes

The RK4 method computes four intermediate slopes (k_1, k_2, k_3, k_4):

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), \\ k_4 &= f\left(t_n + h, y_n + hk_3\right) \quad (17) \end{aligned}$$

Weighted Average Formula

The solution at the next step y_{n+1} is computed as:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \quad (18)$$

Mathematical Derivation

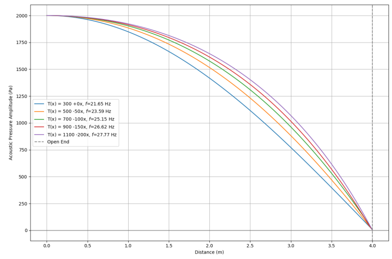
The RK4 method is derived from the Taylor series expansion of $y(t)$. The local truncation error is proportional to h^5, making it a fourth-order method. The intermediate slopes k_1, k_2, k_3, k_4 are designed to approximate the derivative at different points within the interval $[t_n, t_{n+1}]$, ensuring high accuracy.

Solution for the Linear First-Order ODEs

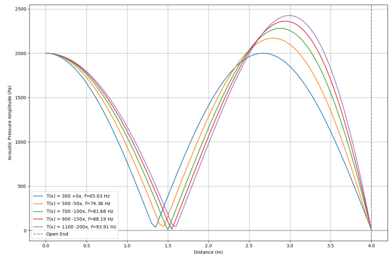
Table 2: Eigenvalue Dependence in a Closed-Open Duct with Linear Temperature Gradient

Initial Temperature (T_1) (K)	Temperature Gradient (m) (K/m)	First Harmonic (Hz)	Second Harmonic (Hz)	Third Harmonic (Hz)	Fourth Harmonic (Hz)	Fifth Harmonic (Hz)
300	0	21.65	65.03	108.71	151.97	195.66
500	-50	23.59	74.38	124.19	174.25	224.31
700	-100	25.15	81.68	137.13	192.29	247.46
900	-150	26.62	88.19	148.36	207.93	267.50
1100	-200	27.77	93.91	158.34	221.78	285.23

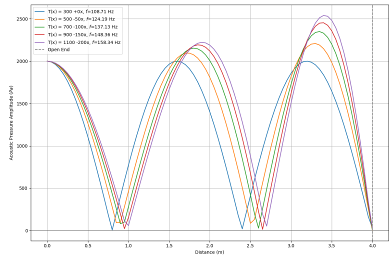
Acoustic Pressure



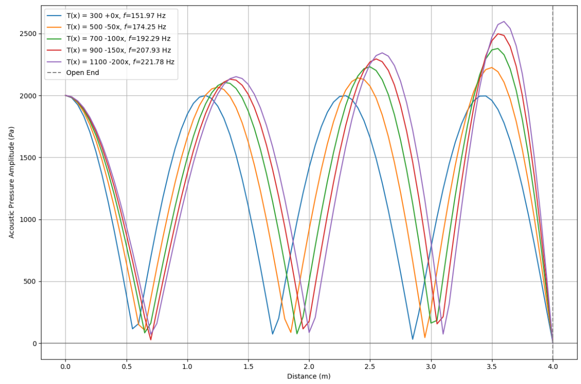
(a) First Harmonic



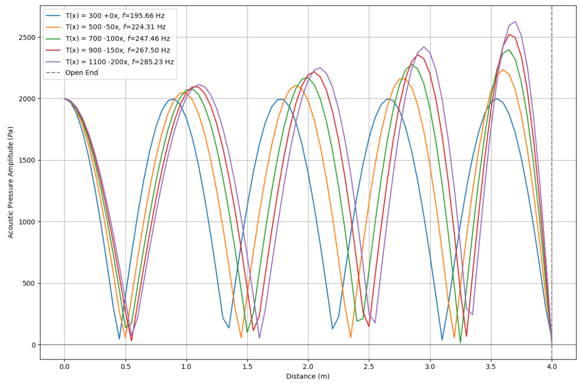
(b) Second Harmonic



(c) Third Harmonic



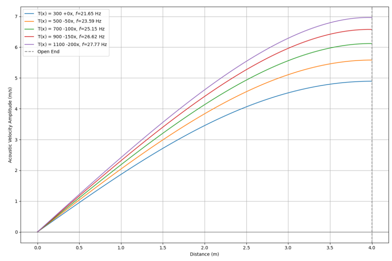
(d) Fourth Harmonic



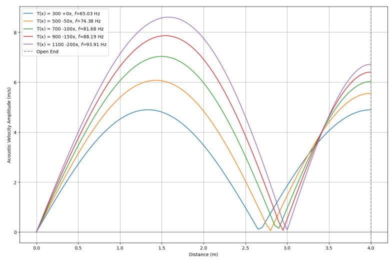
(e) Fifth Harmonic

Figure 2: Variation of acoustic pressure amplitude along a closed-open duct for different linear temperature profiles

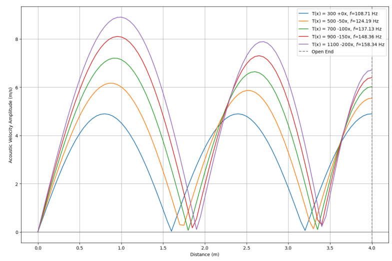
Acoustic Velocity



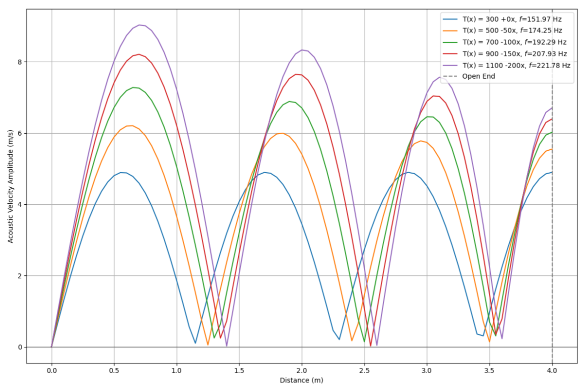
(a) First Harmonic



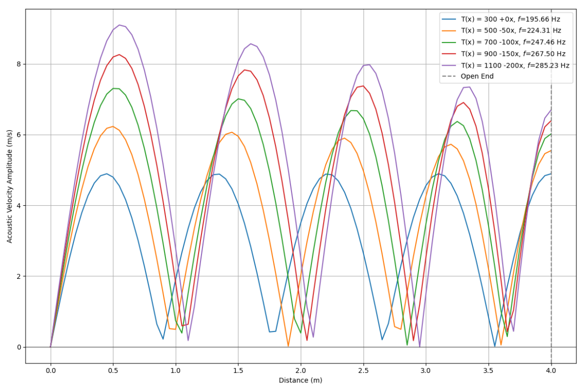
(b) Second Harmonic



(c) Third Harmonic



(d) Fourth Harmonic



(e) Fifth Harmonic

Figure 3: Variation of acoustic velocity amplitude along a closed-open duct for different linear temperature profiles

Patterns Identified in the Graphical Representations

1. Acoustic Pressure:

   - For the first harmonic, we see a gradual decrease in pressure from the closed end (left) to the open end (right) where the pressure approaches zero.
     The different temperature profiles cause slight variations in the pressure curve shapes, with steeper temperature gradients causing more pronounced curvature.
   - In the higher harmonics, we observe increasingly complex standing wave patterns with more nodes (zero-crossing points) and antinodes (pressure maximums/minimums).
     Each successive harmonic adds one more node to the pattern, following the expected behavior for closed-open ducts where the closed end maintains maximum pressure and the open end approaches zero pressure.

2. Acoustic Velocity:

   - The acoustic velocity patterns show complementary behavior to the pressure patterns, with the first harmonic displaying a monotonic increase from the closed end (where velocity must be zero) to the open end (where velocity reaches maximum).
   - In higher harmonics, we observe increasingly complex standing wave patterns with more nodes and antinodes.
     Unlike pressure, the velocity patterns have a node at the closed end and an antinode at the open end, which is exactly opposite to the pressure behavior.

3. Temperature Profiles:

   - Temperature gradients affect both variables in significant but different ways. As the temperature gradient increases (from $T(x) = 300K$ to $T(x) = 1100 - 200x$), both pressure and velocity patterns experience spatial shifting, with nodes and antinodes moving toward the closed end.
     This shift occurs because the speed of sound varies along the duct due to temperature changes, creating wavelength variations.
   - Higher base temperatures and steeper gradients result in more pronounced pressure and velocity amplitudes, particularly near the open end where temperature differences have accumulated greater effects along the duct length.

Summary of Findings for this Assignment

- The eigenvalue analysis shows a clear trend of increasing resonant frequencies with both higher base temperatures and steeper temperature gradients.
- The spatial distribution of acoustic pressure and velocity fields exhibits classical standing wave patterns, with the expected boundary conditions maintained.
  However, the temperature gradients introduce notable distortions to these patterns, shifting nodes and antinodes toward the closed end as the gradient increases.
- For higher harmonics, these effects become more pronounced, with the wavelength variations along the duct causing increasingly complex standing wave patterns.
  This demonstrates that in practical applications involving high-temperature gradients, such as combustion chambers or pulse combustors, the acoustic behavior can deviate significantly from idealized models assuming uniform temperature.
- The complementary nature of pressure and velocity fields is preserved across all temperature profiles, with nodes of one corresponding to antinodes of the other.
  This fundamental acoustic relationship holds true despite the spatial distortions introduced by temperature gradients.

References

1. An exact solution for one-dimensional acoustic fields in ducts with an axial temperature gradient, (1995)

2. Range Kutta Integration

Appendix

Python Code: Numerical Solution for 1D Acoustic Fields in Ducts with Axial Temperature Gradient