The Optimum Tapered Transmission Line Matching Section*

R. E. COLLIN†, MEMBER IRE

Summary—**The tapered transmission line matching section is analyzed by treating it as a high pass filter. The ideal reflection coefficient characteristic for the taper that gives the smallest pass band tolerance for a given cutoff frequency and vice versa is derived from the expression for the reflection coefficient of the optimum designed n section quarter-wave transformer by taking the limit as n tends to infinity. By neglecting the square of the reflection coefficient in the differential equation for the reflection coefficient on the taper, a synthesis procedure is derived for obtaining an optimum taper. The procedure is similar to that used in designing an optimum line source distribution in antenna theory. Conditions are derived for the maximum allowable pass band reflection and change in impedance level for which the theory remains accurate. For a maximum reflection coefficient of 0.1 in the pass band the theory remains accurate for all frequencies above the cutoff value provided the change in impedance level does not exceed 7.5. This optimum taper is compared with the well-known exponential and Gaussian tapers and is found to be 13.9 per cent and 27 per cent shorter respectively for the same cutoff frequency and pass band tolerance.**

INTRODUCTION

VARIOUS aspects of the problem of the tapered or non-uniform transmission line have been treated by several authors.[1–3] A rigorous solution is possible only when the characteristic impedance varies in a specified manner. When the series impedance and shunt admittance both vary as a power of y (the distance along the taper), the voltage and current on the tapered line may be expressed in terms of Bessel functions. The current and voltage on the tapered line satisfy second order differential equations. In contrast to this Pierce has shown that[4] the input impedance looking into the tapered section at a point y satisfies a general first order Riccati equation. Likewise Walker and Wax have derived[5] a first order Riccati equation for the voltage reflection coefficient on the tapered line. These authors[5] have also given an iteration method of arriving at approximate solutions to the general problem. Schelkunoff also outlined[6] a somewhat similar procedure. However, the problem of what constitutes an optimum taper and the synthesis of a taper having a specified optimum behavior do not appear to have been analyzed. It is the purpose of this paper to present such an analysis.

Specifically the problem to be treated is as follows. Given two uniform lossless transmission lines of normalized characteristic impedance, unity and Z_2, what type of intermediate tapered section must be used to ensure the two lines will be as well matched as possible over as broad a band of frequencies as possible for the shortest physical length of taper? The analogous problem of the optimum multisection quarter-wave transformer has been treated[7] by the author in a previous paper. In this paper the author points out the similarity between the designs of a multisection quarter-wave transformer and of a broad side antenna having as narrow beam width as possible with the smallest possible side lobe level. Similarly, as will appear later, the design of an optimum taper is very closely analogous to the problem of the design of an optimum line source distribution in antenna theory. This latter problem has been treated[8] in considerable length by Taylor. It will be shown that the theory developed by Taylor may be applied with very little modification to the problem being studied. For this reason it was decided to use Taylor's notation as far as possible in order to avoid confusion.

The basic theory will be developed first, followed by an outline of a suitable design procedure. A comparison of the optimum designed taper and the exponential and Gaussian tapers will also be given. The theoretical development utilizes the infinite product expansions of the sin x and cos x functions. For this reason a short Appendix on the technique of expanding an integral function into an infinite product is included.

THE IDEAL REFLECTION COEFFICIENT CHARACTERISTIC

For an optimum designed n section quarter-wave transformer used to match a transmission line of normalized characteristic impedance unity to a line of nor-

* Original manuscript received by the IRE, October 20, 1955; revised manuscript received December 20, 1955.

† Canadian Armament Research and Development Establishment, Valcartier, Quebec, Canada.

[1] J. W. Arnold and P. F. Bechberger, "Sinusoidal currents in linearly tapered transmission lines," PROC. IRE, vol. 19, pp. 304–310; February, 1931.

[2] A. T. Starr, "The non-uniform transmission line," PROC. IRE, vol. 20, pp. 1052–1063; June, 1932.

[3] A. W. Gent and P. J. Wallis, "Impedance matching by tapered transmission lines," Jour. IEE, pt. IIIA, vol. 93, pp. 559–563; 1946.

[4] J. R. Pierce, "A note on the transmission line equations in terms of impedance," *B.S.T.J.*, vol. 22, pp. 263–265; July, 1943.

[5] L. R. Walker and N. Wax, "Non-uniform transmission lines and reflection coefficients," *J. App. Phys.*, vol. 17, pp. 1043–1045; December, 1946.

[6] S. A. Schelkunoff, "Remarks concerning wave propagation in stratified media," *Comm. on Pure and App. Math.*, vol. 4, pp. 117–128; June, 1951.

[7] R. E. Collin, "The theory and design of wide-band multi-section quarter-wave transformers," PROC. IRE, vol. 43, pp. 179–185; February, 1955.

[8] T. T. Taylor, "Design of line-source antenna for narrow beam width and low side lobes," TRANS IRE, PGAP, VAP-3, pp. 16–28; January, 1955.

malized characteristic impedance Z_2 the modulus of the reflection coefficient ρ is given by:[9]

$$|\rho| = \frac{k\left|T_n\left\{\dfrac{\cos\beta\dfrac{L}{n}}{\cos\beta_0\dfrac{L}{n}}\right\}\right|}{\left\{1 + k^2T_n{}^2\left\{\dfrac{\cos\beta\dfrac{L}{n}}{\cos\beta_0\dfrac{L}{n}}\right\}\right\}^{1/2}} \tag{1}$$

where

T_n = Tchebycheff polynomial of order n

β = phase constant (the same for all sections)

β_0 = lower cutoff value $=2\pi/\lambda_0$ where λ_0 is the lower cutoff wavelength

L = total length of n sections

$$k = \frac{Z_2 - 1}{2\sqrt{Z_2}}T_n{}^{-1}\left\{\frac{1}{\cos\beta_0\dfrac{L}{n}}\right\} = \text{pass band tolerance.}$$

The argument of the Tchebycheff polynomial

$$T_n \text{ is } \frac{\cos\beta\dfrac{L}{n}}{\cos\beta_0\dfrac{L}{n}}$$

and $T_n{}^{-1}(u)$ is written for $1/T_n(u)$.

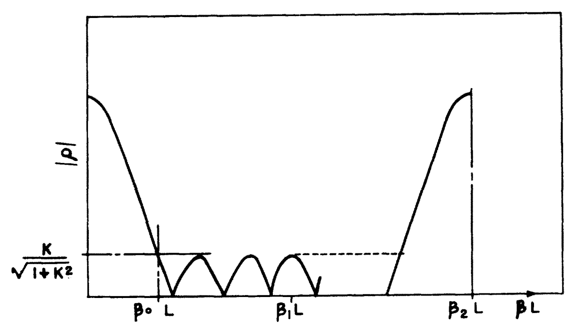

Fig. 1—Typical plot of modulus of reflection coefficient as a function of βL for an n section Tchebycheff transformer.

The Tchebycheff polynomial enters into the design of the optimum n section transformer because it has the property that it oscillates between ± 1 as the argument increases from -1 to $+1$. For values of the argument outside the range -1 to $+1$ the Tchebycheff polynomial increases monotonically. The variable $\cos\beta(L/n)$ is therefore normalized by dividing by $\cos\beta_0(L/n)$ so as to confine the equal amplitude oscillations within the desired pass band. A typical plot of this relationship as a function of βL is given in Fig. 1. The pass band is the region between the lower cutoff value of $\beta_0 L$ to the corresponding point at upper end of curve. This optimum designed transformer has the property that, for a given pass band tolerance, pass band is of maximum width and hence $\beta_0 L$ is a minimum. Conversely, for a given pass band width, the pass band tolerance is a minimum. Reflection coefficient is a periodic function of βL of period $\beta_2 L$. N simple zeros are in pass band.

If the total length of the transformer is kept constant at a value L, then as n increases each section becomes shorter and in the limit as n approaches infinity, the multisection transformer becomes a section of tapered transmission line of length L. Furthermore, the frequency at which the center of the pass band occurs; *i.e.*, $\beta_1 L$, moves off to infinity. Thus in contrast to the quarter-wave transformer which has an infinite number of pass bands separated by attenuation bands, the tapered transmission line is a high pass filter. This is the viewpoint adopted in this article. The lower cutoff frequency is obtained from the value of $\beta_0 L$. In general the reflection coefficient from a tapered transmission line matching section will exhibit one large or main lobe centered about $\beta L = 0$, followed by an infinite number of minor lobes of much smaller magnitude above the cutoff frequency. The tapered transmission line may, of course, be designed for use at a frequency coincident with a zero of ρ. However since it is usually desirable to have a good match over a broad band of frequencies, it is necessary to design the taper so that the mismatch represented by the minor lobe maxima is tolerable and consequently a high pass filter is obtained. The Tchebycheff transformer design optimizes the relationship between minimum pass band tolerance and minimum cutoff frequency and therefore it is of considerable interest to determine the limiting form of (1) as n approaches infinity.

Consider the limiting form attained by

$$T_n\left\{\frac{\cos\beta\dfrac{L}{n}}{\cos\beta_0\dfrac{L}{n}}\right\}$$

as n tends to infinity. Using the relation that $T_n(\cos\theta) = \cos n\theta$, it is readily shown that the zeroes of the above function are located at

$$\cos\beta\frac{L}{n} = \cos\beta_0\frac{L}{n}\cos\left(S - \frac{1}{2}\right)\frac{\pi}{n}, \quad S = 1, 2, \cdots n.$$

For $n \gg S$, the arguments of the cosine terms are very small since $\beta_0 L < \beta L \ll n$ and hence

$$1 - \frac{1}{2}\left(\beta\frac{L}{n}\right)^2 \approx \left[1 - \frac{1}{2}\left(\beta_0\frac{L}{n}\right)^2\right] \cdot \left[1 - \frac{1}{2}\left(S - \frac{1}{2}\right)^2\frac{\pi^2}{n^2}\right]$$

or

$$\beta L \approx \pm\sqrt{(\beta_0 L)^2 + (S - \tfrac{1}{2})^2\pi^2}.$$

[9] R. E. Collin, *loc. cit.*

As n tends to infinity this relation gives the exact location of the zeroes of the Tchebycheff polynomial. Considered as a function of βL, T_n is an even function of βL. Such a function is determined to within a constant[10] by a knowledge of the location of its zeroes. Thus it follows that

$$\lim_{n\to\infty} T_n\left[\frac{\cos\beta\dfrac{L}{n}}{\cos\beta_0\dfrac{L}{n}}\right] = C\prod_{s=1}^{\infty}\left(1-\frac{\beta^2L^2}{\beta_0^2L^2+(S-\frac{1}{2})^2\pi^2}\right). \quad (2)$$

The constant C is readily evaluated from the limiting value of T_n when βL equals zero. Thus

$$C = \lim_{n\to\infty} T_n\left[\frac{1}{\cos\beta_0\dfrac{L}{n}}\right].$$

Since n is very large,

$$\cos\beta_0\frac{L}{n} \approx 1 - \frac{1}{2}\beta_0^2\frac{L^2}{n^2}$$

and

$$T_n\left[\frac{1}{\cos\beta_0\dfrac{L}{n}}\right] \approx T_n\left(1+\frac{1}{2}\beta_0^2\frac{L^2}{n^2}\right).$$

Using the result $T_n(\cosh\theta) = \cosh n\theta$ where

$$\cosh\theta = 1 + \frac{1}{2}\beta_0^2\frac{L}{n^2},$$

the following limiting value of T_n is obtained when the approximation $\theta = \beta_0 L/n$ is made:

$$\lim_{n\to\infty} T_n\left[\frac{1}{\cos\beta_0\dfrac{L}{n}}\right] = \cosh\beta_0 L = C. \quad (3)$$

Thus

$$\lim_{n\to\infty} T_n\left[\frac{\cos\beta\dfrac{L}{n}}{\cos\beta_0\dfrac{L}{n}}\right] = \cosh\beta_0 L\prod_{s=1}^{\infty}\left(1-\frac{\beta^2L^2}{\beta_0^2L^2+(S-\frac{1}{2})^2\pi^2}\right)$$

$$= \cos L\sqrt{\beta^2-\beta_0^2} \quad (4)$$

The derivation of this result is given in the Appendix. Thus the limiting form of the reflection coefficient becomes

$$|\rho| = \frac{k|\cos L\sqrt{\beta^2-\beta_0^2}|}{[1+k^2\cos^2 L\sqrt{\beta^2-\beta_0^2}]^{1/2}} \quad (5)$$

where

$$k = \frac{Z_2-1}{2\sqrt{Z_2}}\,\frac{1}{\cosh\beta_0 L}. \quad (6)$$

The modulus of the reflection coefficient has one major lobe centered at $\beta L = 0$ and an infinite number of minor lobes of equal amplitude and separated by simple zeroes. Ratio of major lobe maximum to minor lobe maxima is:

$$\eta = \frac{\dfrac{2\sqrt{Z_2}}{Z_2+1}\cosh\beta_0 L}{\left(1+\dfrac{(Z_2-1)^2}{4Z_2\cosh^2\beta_0 L}\right)^{1/2}}. \quad (7)$$

This is the ideal characteristic required for the reflection coefficient and will give the minimum cutoff frequency for a given pass band tolerance and vice versa.

When only the first order reflections are taken into account in the design of an optimum n section transformer it is found that the modulus of the reflection coefficient is given by:[11]

$$|\rho| = k\left|T_n\left[\frac{\cos\beta\dfrac{L}{n}}{\cos\beta_0\dfrac{L}{n}}\right]\right|. \quad (8)$$

The limiting form of this expression is:

$$|\rho| = k|\cos L\sqrt{\beta^2-\beta_0^2}|. \quad (9)$$

This result also follows from (5) when $k^2 \ll 1$. This expression gives a major to minor lobe ratio of

$$\eta = \cosh\beta_0 L. \quad (10)$$

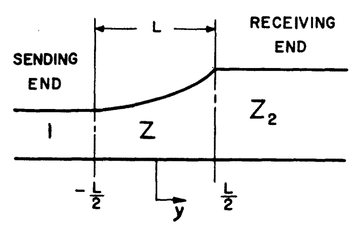

Fig. 2—Matching of two transmission lines by means of an intermediate tapered section.

Integral Equation for Reflection Coefficient

With reference to Fig. 2 let L be the length of the tapered section used to match a line of normalized impedance unity to a line of normalized impedance Z_2. The co-ordinate y measures the distance along the taper from the center point. It will be assumed that all the transmission lines are lossless and that the characteristic impedances are independent of frequency. Also the taper is assumed to be gradual enough so that the higher order modes excited are of negligible amplitude.

[10] E. C. Titchmarsh, "The Theory of Functions," Second edition, Oxford Univ. Press, New York, chap. VIII, 1939.

[11] R. E. Collin, *loc. cit.*, Appendix.

The conditions necessary for single mode propagation are not easy to establish. A qualitative estimate of the magnitude of the higher order modes may be obtained by replacing the taper by a large number of steps. The presence of higher order modes is accounted for by the shunt reactances which exist at each step. When these shunt reactances are negligible the effect of higher order modes may be neglected. From Reference 5 the reflection coefficient satisfies the differential equation:

$$\frac{d\rho}{dy} = 2j\beta\rho - \frac{1}{2}(1-\rho^2)\frac{d}{dy}\ln Z \tag{11}$$

where the propagation constant β and the normalized characteristic impedance Z are in general functions of y. A first approximation to the integration of this equation is readily obtained if ρ^2 can be neglected as compared to unity. In practice this is usually valid and as will be shown, the error introduced is negligible for all values of βL above $\beta_0 L$ for most values of Z_2 usually encountered. The theory will first be developed for that class of transmission lines for which β does not vary along the taper. The modifications required in the theory when β varies with y will be considered later. When $y=L/2$, $Z(L/2)$ will be made equal to Z_2 and hence $\rho(L/2)$ will vanish when the output line is matched. With the above assumptions it is found that the reflection coefficient at the sending end is given by:

$$\rho = \frac{1}{2}e^{-j\beta L}\int_{-L/2}^{L/2}\frac{d\ln Z}{dy}e^{-2j\beta y}dy. \tag{12}$$

This integral may be interpreted[12] as the Fourier transform of $d\ln Z/dy$. It may be put into the form studied by Taylor with the following change in variables:

$$p = 2\pi\frac{y}{L}, \qquad \frac{\beta L}{\pi} = \frac{2L}{\lambda} = u. \tag{13}$$

The co-ordinate u is a measure of the taper length in half wavelengths. With these transformations the integral becomes

$$\rho = \frac{1}{2}e^{-j\beta L}\int_{-\pi}^{\pi}\frac{d\ln Z}{dp}e^{-jpu}dp. \tag{14}$$

Let $d\ln Z/dp = g(p)$ and define $(F(u)$ by

$$F(u) = \int_{-\pi}^{\pi} g(p)e^{-jpu}dp. \tag{15}$$

Thus

$$\rho = \tfrac{1}{2}e^{-j\beta L}F(u). \tag{16}$$

When the function $g(p)$ satisfies the following conditions:

1) It is an even and continuous function of p with continuous derivatives for $|p| < \pi$;
2) It is uniformly bounded for all p within $-\pi < p < \pi$;
3) As p approaches $\pm\pi$, it is asymptotic to $K(p \pm \pi)^\alpha$, where $\alpha > -1$,

then the function $F(u)$ has, among others, the following properties:

(a) It is an even and entire function of u,

(b) As u approaches infinity it is asymptotic to

$$K_1 \frac{\cos\pi\left(u - \frac{1+\alpha}{2}\right)}{|u|^{1+\alpha}}$$

for $\mathrm{I_{mag.}}\ u=0$, and

$$K_1 \frac{e^{\pi|u|}}{2|u|^{1+\alpha}} \quad \text{for} \quad \mathrm{Re}\ u = 0,$$

(c) The members of the nth zero pair tend to the positions $\pm[n+(\alpha/2)]$ as n approaches infinity.

The values of the constants K and K_1 depend on the particular choice of $g(p)$. Proofs of the above properties are given[13] in the paper by Taylor.

From the asymptotic form of $F(u)$ it is seen that the minor lobes decay as $|u|^{-(1+\alpha)}$. In order to have nondecaying minor lobes α must equal -1. However, since $d\ln Z/dp$ must be finite at $p=\pm\pi$, for a continuously smooth taper α must be greater/or equal to zero. In this case ideal reflection coefficient characteristic cannot be fully realized. Analysis following is based on the assumption that the taper is continuous at $p=\pm\pi$.

Specification of Reflection Coefficient Characteristic

The ideal reflection coefficient characteristic when ρ^2 is neglected in the integration of the differential (11) is proportional to cos $\sqrt{\beta^2L^2-\beta_0{}^2L^2}$ since neglecting ρ^2 is equivalent to considering only the first order reflections in the theory of the n section transformer. This characteristic cannot be obtained for reasons already mentioned and hence the ideal characteristic will be approximated by the following function:

$$F(u) = C\cos\pi\sqrt{\frac{u^2}{\sigma^2} - \left(\frac{\beta_0 L}{\pi}\right)^2} \cdot \frac{\prod_{n=\bar{n}}^{\alpha}\left(1 - \frac{u^2}{n^2}\right)}{\prod_{n=\bar{n}}^{\infty}\left(1 - \frac{u^2}{\sigma^2\left[\left(\frac{\beta_0 L}{\pi}\right)^2 + \left(n - \frac{1}{2}\right)^2\right]}\right)} \tag{17}$$

where σ is a constant slightly greater than unity and C is as yet an arbitrary constant. The choice of this particular form was governed by the following considerations. Since $d\ln Z/dp$ must be finite, the smallest value of

[12] F. Bolinder, "Fourier transforms in the theory of inhomogeneous transmission lines," Proc. IRE, vol. 38, p. 1354; November, 1950.

[13] T. T. Taylor, *loc. cit.*

α allowed is zero and hence the minor lobes must decay as $|u|^{-1}$ as u tends to infinity. Also the nth zero pair must tend to the position $\pm n$ for $\alpha=0$. Such a characteristic is given by the function $(\sin \pi u)/\pi u$. The infinite product term in the numerator is equal to this function with the first $\bar{n}-1$ zeroes cancelled out. In order to maintain uniform minor lobes for part of the range of u, the cosine term is introduced. For $u>\bar{n}$ the infinite product term in the denominator cancels the zeroes of the cosine function and allows the sine function to take control of the characteristic. For $|u|<|\bar{n}|$ the zeroes of $F(u)$ occur at

$$u = \pm \sigma \sqrt{\left(\frac{\beta_0 L}{\pi}\right)^2 + \left(n - \frac{1}{2}\right)^2}$$

while for $|u|>|\bar{n}|$ the zeroes occur at $u=\pm n$. For $|u|<|\bar{n}|$ the minor lobes are of very nearly equal amplitude while for $|u|>|\bar{n}|$ the minor lobes decay as $|u|^{-1}$. Thus the integer $\bar{n}$ divides the range of uniform minor lobes and decaying minor lobes. The zeroes of

$$\cos \pi \sqrt{u^2 - \frac{\beta_0^2 L^2}{\pi^2}}$$

are spaced closer than a unit distance apart for small values of u. As u becomes greater the spacing approaches unity and the zeroes tend to the positions $n-\frac{1}{2}$. In practice $\beta_0^2L^2$ would rarely exceed $3\pi^2$ and consequently for $u>5$ the location of the zeroes is within 0.1 of a unit from $n-\frac{1}{2}$, being located at u slightly greater than $n-\frac{1}{2}$. Thus for $\bar{n}>5$ the $(\bar{n}-1)$th zero of $F(u)$ is located approximately at $\bar{n}-3/2$ while the $\bar{n}$th zero is located at $\bar{n}$. This gives a spacing of 1.5 units between these two zeroes and consequently a higher minor lobe maximum in between. For this reason the constant σ is introduced so that the $\bar{n}$th zero of the cosine term will coincide with $\bar{n}$ and no where will the spacing between the zeroes be excessive. Therefore σ is given by

$$\sigma = \frac{\bar{n}\pi}{\sqrt{\beta_0^2 L^2 + (\bar{n} - \frac{1}{2})^2 \pi^2}} \tag{18}$$

Considered as a function of $\bar{n}$, σ increases with $\bar{n}$ up to a point and then rapidly decreases asymptotically to unity. The value of $\bar{n}$ must be chosen large enough that any further increase in $\bar{n}$ will decrease σ. This condition is met if $(d\sigma/dn)<0$ or

$$\bar{n} > 2\frac{\beta_0^2 L^2}{\pi^2} + \frac{1}{2}, \tag{19}$$

If this condition is not met, then the first few zeroes of

$$\cos \pi \sqrt{\frac{u^2}{\sigma^2} - \frac{\beta_0^2 L^2}{\pi^2}}$$

for $u>\bar{n}$ occur for values of u slightly less than n. Since these zeroes are cancelled out by the infinite product term in the denominator and replaced by zeroes occurring at integral values of n, the net result is an increase in the spacing of some of the zeroes of $F(u)$ as compared with those of the cosine factor alone. This results in a major to minor lobe ratio smaller than $\cosh \beta_0 L$ and should be avoided. The condition (19) is readily satisfied in practice, usually for $\bar{n}$ not exceeding 5 or 6.

The function $F(u)$ differs very little from

$$C \cos \pi \sqrt{\frac{u^2}{\sigma^2} - \frac{\beta_0^2 L^2}{\pi^2}}$$

for $u<n$. The main difference is a slightly smaller minor lobe maximum and a small decay of the minor lobes. For this reason some of the properties of the reflection coefficient characteristic can be derived from the cosine term alone. The cutoff frequency is obtained from the value of u when the cosine term equals unity or $\pi u = \sigma\beta_0 L$ and hence

$$\beta_c L = \sigma \beta_0 L. \tag{20}$$

It is seen that the cutoff frequency is greater than that for the ideal characteristic by the factor σ, which, however, can be made as nearly equal to unity as desired by increasing $\bar{n}$. The ratio of major to minor lobe maxima is

$$\eta = \cosh \beta_0 L \tag{21}$$

which is identical to that for the ideal characteristic. As $\bar{n}$ is increased, the function $F(u)$ approaches the ideal characteristic asymptotically. In practice values of $\bar{n}$ around 5 or 6 give a characteristic that departs by only a few per cent from the ideal. Eqs. (20) and (21) show that the cutoff frequency and major to minor lobe ratio are related. When one is specified, the other is determined and vice versa. This property is also characteristic of the optimum n section transformer and emphasizes the similarity between the two.

The following equivalent form for $F(u)$ is readily derived by using the infinite product expansion of the sine and cosine functions.

$$F(u) = C \cosh \beta_0 L \frac{\sin \pi u}{\pi u} \cdot \frac{\prod_{n=1}^{\bar{n}-1} \left\{1 - \dfrac{u^2}{\sigma^2\left[\dfrac{\beta_0^2 L^2}{\pi^2} + \left(n - \dfrac{1}{2}\right)^2\right]}\right\}}{\prod_{n=1}^{\bar{n}-1}\left(1 - \dfrac{u^2}{n^2}\right)} \tag{22}$$

This form clearly shows the decaying property of the minor lobes as u becomes large. This form is also the most convenient for use in numerical computation.

Determination of Taper Impedance

In this section the required functional form of $d \ln Z/dy$ will be determined to give a reflection coefficient $\rho = \frac{1}{2}e^{-j\beta L}F(u)$ where $F(u)$ is the function defined by (17).

Let $g(p)$ be expanded into a Fourier series as follows:

$$g(p) = \begin{cases} \sum_{m=0}^{\infty} B_m \cos mp & |\rho| \le \pi, \\ 0, & |\rho| \ge \pi. \end{cases} \tag{23}$$

Substituting into the integral (15) and performing the integrations gives:

$$F(u) = \sum_{m=0}^{\infty} 2B_m \frac{u \cos m\pi \sin \pi u}{u^2 - m^2}. \tag{24}$$

To determine the coefficients B_m let u approach an integer s; thus

$$\lim_{u \to s} 2 \sum_{m=0}^{\infty} B_m \frac{u \cos m\pi \sin \pi u}{u^2 - m^2} = \begin{cases} \pi \epsilon_{0m}^{-1} B_m, & s = m, \\ 0, & s \ne m \end{cases}$$

where

$$\epsilon_{0m} = \begin{cases} \frac{1}{2} & \text{for } m = 0, \\ 1 & \text{for } m \ne 0. \end{cases}$$

Therefore the coefficients B_m are given by

$$B_m = \frac{\epsilon_{0m} F(m)}{\pi}. \tag{25}$$

Since $F(m)$ vanishes for all $m \ge \bar{n}$

$$g(p) = \sum_{0}^{\bar{n}-1} \frac{\epsilon_{0m} F(m)}{\pi} \cos mp$$

and

$$\frac{d \ln Z}{dy} = \frac{2}{L} \sum_{0}^{\bar{n}-1} \epsilon_{0m} F(m) \cos 2 \frac{\pi m}{L} y. \tag{26}$$

Integrating this equation gives the taper impedance as a function of y. The function $F(u)$ contains a constant C which is determined as follows. When $y = L/2$, $Z(L/2) = Z_2$ and hence

$$\int_{-L/2}^{L/2} \frac{d \ln Z}{dy} dy = \ln Z_2 = \frac{2}{L} \sum_{0}^{\bar{n}-1} \epsilon_{0m} F(m) \int_{-L/2}^{L/2} \cos 2 \frac{m\pi}{L} y dy$$
$$= F(0) = C \cosh \beta_0 L.$$

Thus

$$C = \frac{\ln Z_2}{\cosh \beta_0 L}. \tag{27}$$

The final form of the reflection coefficient is

$$\rho = \frac{1}{2} e^{-j\beta L} \frac{\ln Z_2 \cos L \sqrt{\frac{\beta^2}{\sigma^2} - \beta_0^2}}{\cosh \beta_0 L} \cdot \frac{\prod_{\bar{n}}^{\infty} \left(1 - \frac{\beta^2 L^2}{n^2 \pi^2}\right)}{\prod_{\bar{n}}^{\infty} \left(1 - \frac{\beta^2 L^2}{\sigma^2 [\beta_0^2 L^2 + (n - \frac{1}{2})^2 \pi^2]}\right)}. \tag{28}$$

Design Procedure

The above analysis is summarized below in convenient form for use in designing a matching taper. Let the characteristic impedance of the output line, normalized with respect to the impedance of the input line, be Z_2. Let the maximum tolerable value of the reflection coefficient in the pass band be ρ_m. The major to minor lobe ratio η is given by $\cosh \beta_0 L$ where

$$\cosh \beta_0 L = \frac{\ln Z_2}{2\rho_m} \tag{29}$$

This relation determines the ideal cutoff value $\beta_0 L$. The practical cutoff value $\beta_c L$ is equal to $\sigma \beta_0 L$ where

$$\sigma = \frac{\bar{n}\pi}{\sqrt{\beta_0^2 L^2 + (\bar{n} - \frac{1}{2})^2 \pi^2}} \tag{30}$$

and $\bar{n}$ must be chosen greater than

$$\frac{2\beta_0^2 L^2}{\pi^2} + \frac{1}{2}.$$

For a cutoff wavelength of λ_c the physical length of taper required is

$$L = \frac{\sigma \lambda_c}{2\pi} \operatorname{arc\,cosh} \frac{\ln Z_2}{2\rho_m}. \tag{31}$$

The variation in taper impedance required is

$$Z(y) = \exp\left(\sum_{0}^{\bar{n}-1} \frac{\epsilon_{0m}}{m\pi} F(m) \sin \frac{2m\pi}{L} y + \frac{F(0)}{2}\right) \tag{32}$$

where

$$F(m) = -\ln Z_2 \frac{\cos m\pi}{2} \cdot \frac{\prod_{n=1}^{\bar{n}-1} \left(1 - \frac{m^2 \pi^2}{\sigma^2 [\beta_0^2 L^2 + (n - \frac{1}{2})^2 \pi^2]}\right)}{\prod_{n=1 \cdot n \ne m}^{\bar{n}-1} \left(1 - \frac{m^2}{n^2}\right)} \tag{33}$$

and $F(0) = \ln Z_2$. The reflection coefficient is given by (28). The design is somewhat conservative since the major to minor lobe ratio is slightly greater than η.

Comparison with Exponential and Gaussian Tapers

Exponential Taper

If $\bar{n}$ is put equal to unity in the above theory the exponential taper is obtained. The variation in taper impedance is given by

$$\ln Z(y) = \left(\frac{y}{L} + \frac{1}{2}\right) \ln Z_2. \tag{34}$$

The reflection coefficient is given by

$$\rho = \frac{1}{2} e^{-j\beta L} \ln Z_2 \frac{\sin \beta L}{\beta L}. \tag{35}$$

The first minor lobe maximum occurs when $\tan \beta L = \beta L$ or $\beta L = 1.43\pi$. The ratio of the major lobe to first minor lobe maximum is 4.6. The cutoff frequency occurs when $\sin \beta L = 0.217\beta L$ or

$$\beta_c L = 0.815\pi.$$

For the optimum taper the corresponding cutoff value is $\beta_c L = \sigma \cosh^{-1} 4.6 = 0.702\pi\sigma$. By choosing $\bar{n}$ large enough, σ can be made as close to unity as desired and $\beta_c L$ approaches 0.702π. Thus the optimum taper is 13.9 per cent shorter than the exponential taper for the same pass band tolerance. However, a value of $\eta = 4.6$ usually results in too large a reflection in the pass band unless Z_2 is close to unity. Thus the exponential taper would normally be used only when a narrow band of frequencies in the vicinity of a zero of the reflection coefficient is to be passed.

For the exponential taper $d \ln Z/dy$ is a constant and the differential (11) can be solved rigorously by elementary means. The rigorously correct solution is:

$$\rho = \frac{C \sin \frac{RL}{2}}{R \cos \frac{RL}{2} + 2j\beta \sin \frac{LR}{2}} \tag{36}$$

where $C = \ln Z_2/L$ and $R = \sqrt{4\beta^2 - C^2}$. When $4\beta^2 \gg C^2$, $R \approx 2\beta$ and the reflection coefficient reduces to

$$\frac{1}{2} e^{-j\beta L} \ln Z_2 \frac{\sin \beta L}{\beta L}$$

which is the same result obtained by neglecting ρ^2 in the differential equation. Hence for the approximate theory to be valid it is necessary that $4\beta^2 L^2 \gg (\ln Z_2)^2$. For all values of βL satisfying this condition the analysis neglecting ρ^2 is sufficiently accurate for design purposes.

The exponential taper is a special case of the general theory and therefore the above criterion will also be used to determine the range of validity of the theory for the optimum taper. For the optimum taper the value of ρ at a minor lobe maximum is $\rho = \ln Z_2/2\eta$. The minimum value of βL of interest is $\beta_c L$. The value of $\beta_c L$ is given by $\beta_c L = \sigma \cosh^{-1} \eta$. Thus the condition $4\beta_c^2 L^2 \gg (\ln Z_2)^2$ may be written as:

$$\left(\cosh^{-1} \frac{\ln Z_2}{2\rho_m}\right)^2 \gg \left(\frac{\ln Z_2}{2\sigma}\right)^2 \tag{37}$$

where η has been replaced by $\ln Z_2/2\rho_m$. In practice a voltage standing wave ratio not exceeding 1.2 will be required and hence $\rho_m < 0.1$. For the left side of (37) to be at least ten times greater than the right, Z_2 cannot exceed 7.5. When ρ_m does not exceed 0.05, values of Z_2 up to 12 are allowed. For values of Z_2 greater than this, correspondingly smaller values of ρ_m are required in order for the theory to be accurate. For value of βL less than $\beta_c L$ the theory does not give an accurate value of the reflection coefficient unless Z_2 is close to unity. However this is of no practical consequence since the taper is not used for frequencies below the cutoff value. When βL is equal to zero, the above theory gives a value $\frac{1}{2} \ln Z_2$ for the reflection coefficient whereas the correct value is $(Z_2-1)/(Z_2+1)$. Expanding the logarithm function gives

$$\frac{1}{2} \ln Z_2 = \frac{Z_2 - 1}{Z_2 + 1} + \frac{1}{3}\left(\frac{Z_2 - 1}{Z_2 + 1}\right)^3 + \cdots$$

so it is seen that the expressions agree to within 4 per cent for $Z_2 < 2$.[14]

Gaussian Taper

The Gaussian taper is obtained when the number of sections in the binomial transformer is increased without limit while the total length of the transformer is kept constant. For this taper the characteristic impedance is given by:[15]

$$Z = \begin{cases} \sqrt{Z_2} \exp\left[\frac{2}{L} \ln Z_2 \left(y + \frac{y^2}{L}\right)\right], & -\frac{L}{2} \le y \le 0 \\ \sqrt{Z_2} \exp\left[\frac{2}{L} \ln Z_2 \left(y - \frac{y^2}{L}\right)\right], & 0 \le y \le \frac{L}{2} \end{cases} \tag{38}$$

The reflection coefficient is given by:

$$\rho = \frac{1}{2} e^{-j\beta L} \ln Z_2 \left\{ \frac{\sin \beta \frac{L}{2}}{\beta \frac{L}{2}} \right\}^2. \tag{39}$$

The first minor lobe maximum occurs at $\beta L = 2.86\pi$. The ratio of major to first minor lobe maximum is 21.1. The cutoff frequency occurs when $\beta_c L = 1.63\pi$. The Gaussian taper has a cutoff frequency twice as high as the exponential taper but the first minor lobe maximum is only 4.75 per cent compared with 21.8 per cent. This behavior has been achieved by making all the zeroes of ρ double zeroes. The spacing between zeroes has consequently also been doubled. For a value of $\eta = 21.1$ the cutoff frequency for the optimum taper occurs when $\beta_c L = 1.19\pi\sigma$. For sufficiently large $\bar{n}$ the optimum taper is 27 per cent shorter for the same cutoff frequency and pass band tolerance. For the same cutoff frequency and taper length the value of η can be increased to a maximum of 84 for the optimum taper compared with 21.1 for the Gaussian taper. This illustrates the point that superior performance cannot be obtained with multiple zeroes in the pass band.

Curves of reflection coefficient against βL for the Gaussian taper and the optimum taper with $\eta = 21.1$ are plotted in Fig. 3. For the Gaussian and exponential tapers the choice of η is not arbitrary, being fixed at

[14] Conditions similar to those given here may be applied to determine the allowed range of Z_2 in order for the first order theory for the optimum n section quarter-wave transformer to be valid.
[15] Bolinder, F., *loc. cit.*

the values of 21.1 and 4.6 respectively. A curve of $\frac{1}{2}(|p|/\ln Z_2)$ vs βL for the exponential taper is readily constructed since it is equal to the simple function $|\sin \beta L|/\beta L$. The design procedure for the optimum taper is more flexible since it allows the arbitrary choice of η and then determines the shortest possible length of taper which will give this value of major to minor lobe ratio.

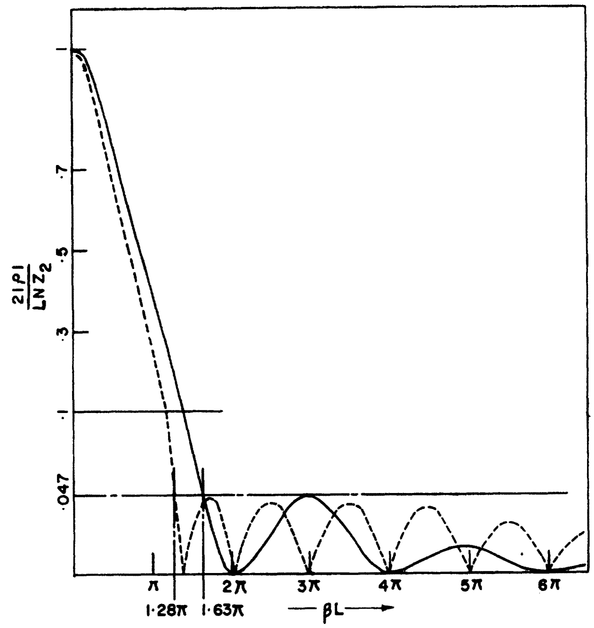

Fig. 3—Reflection coefficient as a function of βL for the Gaussian and optimum taper. ——— Gaussian taper. - - - - - - Optimum taper, $\eta=21.1$, $\bar{n}=5$, $\sigma=1.072$.

The Case When β Varies Along The Taper

As an example of conditions where the propagation constant varies along the taper, consider the problem of matching an infinite lossless dielectric sheet to free space by means of an inhomogeneous dielectric sheet of thickness L and of varying dielectric constant along the y axis as in Fig. 4. Let the relative dielectric constant of the dielectric sheet to be matched by K_2. A *TEM*

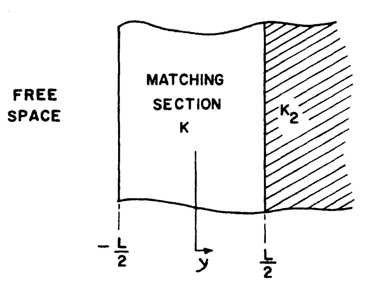

Fig. 4—Matching of a dielectric sheet to free space by means of an inhomogeneous intermediate section.

wave is assumed incident normally on the dielectric sheet. For this type of wave the propagation constant is given by $\sqrt{K}\beta$ where β is the free space propagation constant $\omega\sqrt{\mu\epsilon_0}$ and K is the relative dielectric constant of the taper and is a function of y. The normalized wave impedance of the dielectric may be taken as $K^{-1/2}$. Thus neglecting ρ^2 the differential equation for the reflection coefficient becomes:

$$\frac{d\rho}{dy} = 2j\beta\sqrt{K}\rho + \frac{1}{2}\frac{d\ln\sqrt{K}}{dy}. \tag{40}$$

When $y=L/2$, K will be made equal to K_2 so $\rho(L/2)$ vanishes. Introducing the new phase variable θ defined as

$$\theta = \int_{-L/2}^{y}\sqrt{K}dy - \frac{1}{2}\int_{-L/2}^{L/2}\sqrt{K}dy \tag{41}$$

it is readily found that ρ is given by the following integral:

$$\rho = -\frac{1}{2}e^{-j2\beta\theta_0}\int_{-\theta_0}^{\theta_0}\frac{d\ln\sqrt{K}}{d\theta}e^{-j2\beta\theta}d\theta \tag{42}$$

where

$$\theta_0 = \theta\left(\frac{L}{2}\right) = -\theta\left(-\frac{L}{2}\right) = \frac{1}{2}\int_{-L/2}^{L/2}\sqrt{K}dy. \tag{43}$$

The constant term has been introduced in the phase variable θ in order to obtain a symmetrical range in θ. The previous analysis may now be applied to this integral with θ replacing y and θ_0 replacing $L/2$. In particular it is found that the dielectric constant along the taper must satisfy the following integral equation for an optimum taper design:

$$K = \exp\left(-2\sum_{0}^{\bar{n}-1}\frac{\epsilon_{0m}F(m)}{m\pi}\sin\frac{m\pi\theta}{\theta_0} + \ln\sqrt{K_2}\right) \tag{44}$$

where θ is the function of K and y given by (41). In general the solution to this integral equation must be obtained numerically. For the particular case of the exponential taper this integral equation simplifies to:

$$\ln\sqrt{K} = \frac{\ln\sqrt{K_2}\int_{-L/2}^{y}\sqrt{K}dy}{\int_{-L/2}^{L/2}\sqrt{K}dy} \tag{45}$$

for which

$$\sqrt{K} = \left(\frac{1-\sqrt{K_2}}{L\sqrt{K_2}}y + \frac{1+\sqrt{K_2}}{2\sqrt{K_2}}\right)^{-1} \tag{46}$$

is a solution.

Conclusion

The theory for the optimum design of a line source distribution in antenna pattern synthesis has been applied to the problem of designing a tapered transmission line matching section that optimizes the relationship between cutoff frequency and minimum pass band tolerance. Such an optimum design leads to a considerably smaller pass band reflection as compared with that obtainable from an exponential or Guassian taper of the same physical length and having the same cutoff frequency. The theory is restricted to that class of transmission line for which the propagation constant can be expressed as the product of a frequency varying function and a space varying function. As such it is not generally applicable to waveguide tapers. An exception is the case of the rectangular waveguide supporting H_{0n} modes and which is tapered along the narrow guide dimension only. For this type of taper the guide wavelength does not vary along the taper and the characteristic impedance may be taken proportional to the waveguide height.

The directional coupler employing continuously distributed coupling may be treated in a similar manner.

Appendix I

Infinite Product Expansion of an Integral Function

Let $F(w)$ be an analytic integral function of the complex variable w. The logarithmic derivative

$$\frac{d \ln F}{dw} = \frac{F'}{F}$$

is a meromorphic function whose only singularities in the complex plane are poles located at the zeroes of $F(w)$. Let F have simple zeroes at $w = w_n$, $w_n \neq 0$; $n = 1, 2, \cdots \infty$. Thus F'/F has simple poles at $w = w_n$. By integrating the partial fraction expansion of F'/F the following infinite product expansion of F is obtained,[16]

$$F(w) = F(0)e^{wF'(0)/F(0)} \prod_{n=1}^{\infty} \left(1 - \frac{w}{w_n}\right) e^{w/w_n}.$$

If F is an even function of w with simple zeroes at $w = \pm w_n$, $n = 1, 2, \cdots \infty$, then $F'(0) = 0$ and

$$\begin{aligned} F(w) &= F(0) \prod_{-\infty}^{\infty} \left(1 - \frac{w}{w_n}\right) e^{w/w_n} \\ &= F(0) \prod_{1}^{\infty} \left(1 - \frac{w}{w_n}\right) e^{w/w_n} \prod_{1}^{\infty} \left(1 + \frac{w}{w_n}\right) e^{-w/w_n} \\ &= F(0) \prod_{1}^{\infty} \left(1 - \frac{w^2}{w_n^2}\right). \end{aligned}$$

As an example consider the function $\cos L\sqrt{\beta^2 - \beta_0^2}$ where βL is the variable. The zeroes of this function occur at $L\sqrt{\beta^2 - \beta_0^2} = \pm(S - \frac{1}{2})\pi$. $S = 1, 2, \cdots \infty$, or $\beta^2 L^2 = \beta_0^2 L^2 + (S - \frac{1}{2})^2 \pi^2$. When $\beta L = 0$ the value of the function is $\cosh \beta_0 L$. Hence using the general formula

$$\cos L\sqrt{\beta^2 - \beta_0^2} = \cosh \beta_0 L \prod_{S=1}^{\infty} \left(1 - \frac{\beta^2 L^2}{\beta_0^2 L^2 + (S - \frac{1}{2})^2 \pi^2}\right)$$

which is the derivation of (4) in the text. The expansion of $\sin w$ is obtained from the expansion of the even function $\sin w/w$; *i.e.*,

$$\sin w = w \prod_{n=1}^{\infty} \left(1 - \frac{w^2}{n^2 \pi^2}\right).$$

Appendix II

The analysis of Klopfenstein,[17] shows how the ideal reflection coefficient characteristic may be obtained by including a discontinuity at each end of the taper. The part of the reflection coefficient due to the two discontinuities is

$$e^{-j\beta L} \frac{\ln Z_2}{2 \cosh \beta_0 L} \cos \beta L.$$

As βL tends to infinity the reflection from the smooth part of the taper vanishes leaving only that due to the two discontinuities, the latter having constant amplitude minor lobes.

As $\bar{n}$ tends to infinity our results become identical to that of Klopfenstein. A by-product of our analysis is a simple Fourier series expansion of $\ln Z(y)$. When $\bar{n} \to \infty$, $\sigma \to 1$ and

$$F(u) \to \frac{\ln Z_2}{\cosh \beta_0 L} \cos \sqrt{u^2 \pi^2 - \beta_0^2 L^2}.$$

Thus (32) becomes

$$\ln Z(y) = \frac{1}{2} \ln Z_2 + \frac{\ln Z_2}{\cosh \beta_0 L} \sum_{m=0}^{\infty} \frac{\epsilon_{0m}}{m\pi} \cos \sqrt{m^2\pi^2 - \beta_0^2 L^2} \times \sin \frac{2m\pi y}{L}.$$

Adding and subtracting a similar series gives

$$\begin{aligned} \ln Z(y) = \left(\frac{1}{2} + \frac{y}{L}\right) \ln Z_2 + \frac{\ln Z_2}{\cosh \beta_0 L} \sum_{1}^{\infty} \frac{\cos m\pi}{m\pi} \sin \frac{2m\pi y}{L} \\ + \frac{\ln Z_2}{\cosh \beta_0 L} \sum_{1}^{\infty} \frac{1}{m\pi} (\cos \sqrt{m^2\pi^2 - \beta_0^2 L^2} - \cos m\pi) \\ \cdot \sin \frac{2m\pi y}{L}. \end{aligned}$$

The second series converges rapidly and the first series

[16] P. M. Morse, H. Feshbach, "Methods of Theoretical Physics," McGraw-Hill Book Company, New York, chap. 4, 1953.

[17] R. W. Klopfenstein, "A transmission line taper of improved design," Proc. IRE, vol. 44, pp. 31–35; January, 1956.

is the Fourier sine series expansion of the sawtooth function $-Y/L$. It is this series which brings in the discontinuous change in $Z(y)$ at $y=\pm L/2$. The value of $\ln Z(y)$ is readily computed from the following result:

$$\ln Z(y) = \left(\frac{1}{2}+\frac{y}{L}\right)\ln Z_2 - \frac{y}{L}\frac{\ln Z_2}{\cosh \beta_0 L} + \frac{\ln Z_2}{\cosh\beta_0 L}\sum_1^\infty \frac{1}{m\pi}\left(\cos\sqrt{m^2\pi^2-\beta_0^2L^2}-\cos m\pi\right) \cdot \sin\frac{2m\pi y}{L}, \qquad |y| \leqq \frac{L}{2}.$$

$$\ln Z(y) = \ln Z_2,\ y > \frac{L}{2}. \quad \ln Z(y) = 0,\ y < \frac{-L}{2}.$$

The discontinuity in $\ln Z(y)$ at $y=\pm L/2$ is of magnitude

$$\frac{\ln Z_2}{2\cosh\beta_0 L}.$$

The above result agrees with that obtained by Klopfenstein. It appears that Eq. (12) in Klopfenstein's paper should have a term $-\rho_0/\cosh A$ added to the right-hand side.

A New Annular Waveguide Rotary Joint*

KIYO TOMIYASU†, SENIOR MEMBER, IRE

Summary—**A new waveguide rotary joint has been designed which will permit multiple stacking of similar joints on a common axis. The rotary joint of annular waveguide design will carry high power and permit low vswr and low insertion loss operation throughout the full rotation. The theory, design, and characteristics of such a joint are presented.**

Introduction

ROTARY JOINTS of the multiple coaxial type cannot provide many independent concentric high-power transmission circuits since the large spacings required to prevent breakdown create higher mode problems in the outer spaces. The commonly-used waveguide rotary joint which utilizes two rectangular-TE_{10}-to-circular-TM_{01}-mode transitions is unadaptable to multiple use since the input and output lines would physically interfere.

These difficulties are avoided when an annular type of rotary joint is used. In this configuration each joint consists of a ring split concentrically so that the two sections can rotate about a common axis while transferring power from one to the other. One transmission line would be connected to one ring section, the other to the second ring section. For a stacked array, all the inner transmission lines would then pass through the rings. The annular type of rotary joint will permit multiple stacking of similar joints and complete removal of all electronic components of a radar system from the rotating platform.

Previous work on an annular rotary joint was reported by Coleman[1] in 1948. This joint had a "dead spot" although it was only 3 to 4 degrees wide. Another annular rotary joint was invented by Breetz[2] and it was characterized by low vswr and insertion loss at several extremely narrow evenly-spaced frequency bands.

Basic Considerations of Transvar Coupler

In the design of the new rotary joint, Transvar (TM) couplers[3] are used to obtain essentially uniform low vswr, low insertion loss characteristics (no "dead spots") as a function of joint rotation. Such desirable characteristics are possible at power levels approaching the maximum-power capacity of the waveguide.

For this application the basic Transvar coupler must pass the total input power through three different modified structures and coupling conditions. These will be called single coupler, cascaded couplers, and diagonal-arm condition.

Single Coupler

Referring to Fig. 3 in footnote reference 3, if the design of coupling apertures is such as to couple the total input power, it is possible to make the shutter much

* Original manuscript received by the IRE, July 5, 1955; revised manuscript received October 25, 1955. This paper is based upon work performed for the U. S. Navy, Bureau of Ships, on Contract No: NObsr-42419.

† Formerly with Sperry Gyroscope Co., Division of The Sperry Corp., Great Neck, N. Y. Now with General Electric Microwave Lab., Palo Alto, Calif.

[1] G. M. Coleman, Unpublished notes on "Around-the-mast radar antenna," U. S. Navy Electronics Lab., San Diego, Calif.

[2] L. D. Breetz, Patent No. 2, 595, 186, issued April 29, 1952. Also L. D. Breetz, "A waveguide rotary joint with waveguide feed," Naval Res. Lab., NRL Rep. 3795; January 18, 1951.

[3] K. Tomiyasu and S. B. Cohn, "The transvar directional coupler," PROC. IRE, vol. 41, pp. 922–926; July, 1953.