Parrondo's Games

Parikshit Sharma

Collaborators: Ernesto Gracia, Konstantine Mushegian and Marcus Christiansen

I. Introduction

As counter intuitive as it may seem, in certain games, a combination of losing strategies can yield a winning strategy. This fascinating phenomenon was discovered by physicist Juan Parrondo in his study of Brownian Rachets, a thought experiment involving a perpetual motion machine; wherein the simple machine, consisting of a tiny paddle wheel and a ratchet, is able to extract useful work from random fluctuations (heat) in a system at thermal equilibrium in violation of the second law of thermodynamics.[1]

This paper takes a different approach to investigate and mathematically describe this phenomenon by using three simple games. The first game, Game A involves tossing an biased coin. The second, Game B tosses a biased coin with a distinct probability if the player has a multiple of 3 in his pocket; else, a biased coin with different probability is tossed when the player does not have a multiple of 3 in his pocket.While, Game C simply involves alternating between Game A and Game B with equal chance. The games are defined more rigorously in the sections that follow

Using computer simulations for each trial, we flip a coin 10,000 times and keep track of aggregate winnings under the setup of each game. This is then repeated for 1000 trials, the results averaged and finally plotted to gain some insight into the trend over the 1000 repetitions and under different magnitudes of the bias (ϵ).

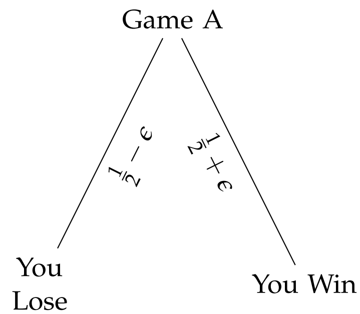
These simulations provide an entry point to specify the mathematical underpinnings of the results of the simulations. By generalizing the outcomes of the games into three states, each corresponding to money in the players pocket(M)under mod 3 say $M \mod 3$, i.e. $0(mod3)$,$1(mod3)$ and $2(mod3)$ we observe that the probability of realizing each of these three states after $i-$games can be expressed as a (probability) vector.
Interestingly, we demonstrate that the random transition from each of these states under the games can be modeled using matrices known as *Markov* matrices. The transform defined by such a Markov matrix on these probability vectors represents transitioning from the probability vector at the $n-th$ to the $n+1-th$ game; thus, enabling us to look at these games as dynamical systems in discrete time.Interestingly, using a theorem, called the $Perron-Frobenius\ Theorem$ we show how this discrete dynamical system would always approach a steady state.

The rest of the paper is organized as follows, section II. introduces the games more formally and describes the results of the simulations.Section III. and IV. model the game by elaborating on the theory behind $MarkovChains$, $Discrete\ Dynamical\ Systems$ and the $Perron-Frobenius\ Theorem$ and evaluates whether theory explains the results from the simulations respectively.Section V. concludes. Finally, Section VI. ponders on the applications of the model and their scope in fields such as Finance.

II. The Games

I. Game A

Game A simply involves tossing a biased coin, where ϵ is the bias. A player then wins with probability $\frac{1}{2} + \epsilon$ or loses with probability $\frac{1}{2} - \epsilon$ as visualized in the tree diagram below. In addition, on winning the player gets \$1



A way of deciding whether or not a player should play is calculating the expected return from playing. That is:

$$\mathbb{E}(return) = (1).(\frac{1}{2} + \epsilon) + (-1).(\frac{1}{2} - \epsilon) = 2\epsilon \tag{1}$$

One direct result from the equation above is that wins or loses for the player are a function of ϵ.

Given, the probability of the player winning is $0.5 + \epsilon$, $\epsilon > 0$, biases the game in favor of player .Also, for $\frac{1}{2} \pm \epsilon$ to be probabilities we have $\frac{1}{2} \pm \epsilon <= 1$ implying $|\epsilon| <= \frac{1}{2}$ as an upper bound

So, fixing $\epsilon = 0.005$ and plugging into 1

$$\mathbb{E}(return) = (1).(\frac{1}{2} + 0.005) + (-1).(\frac{1}{2} - 0.005) = (1).0.505 + (-1).0.495 = 0.01$$

Thus,the player stands to win 1 cent per-game and overtime stands to cumulatively win more. In other words over 100 repetitions the player can win one dollar. On the flipside, with every win the player registers the net-worth of the guarantor of the game decreases.

In fact this is what one observes when this game is simulated in MATLAB over 10,000 trials and by flipping this biased coin 1000 times. More particularly, we see average net-worth of the guarantor decreases consistently over time. (See Figure 1 below)

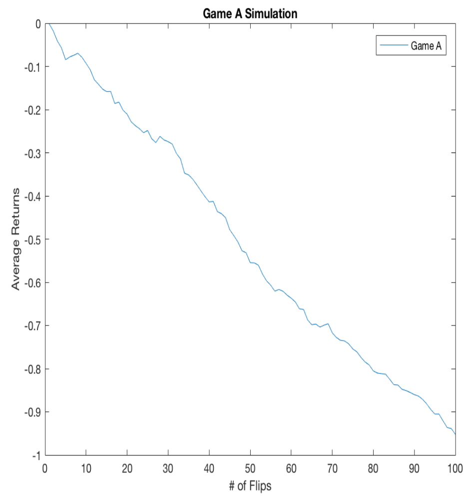


Figure 1: Game A simulation for $|\epsilon| = 0.005$

Also, to outline the range of possibilities, increasing the absolute value of the bias ($|\epsilon|$) in MATLAB increased the rate of reduction of the net-worth of the guarantor of the game as seen in Figure 2 below. This result is consistent with equation 1 where we note that $\mathbb{E}(return) = 2\epsilon$ and that the player's expected return increases as a function of ϵ as long as $\epsilon \leq 0.5$

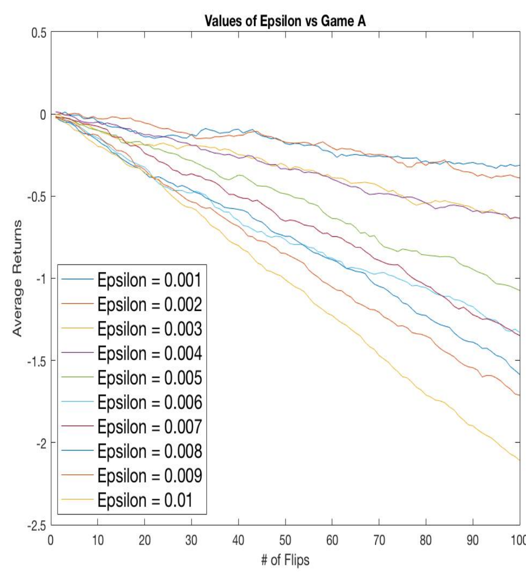
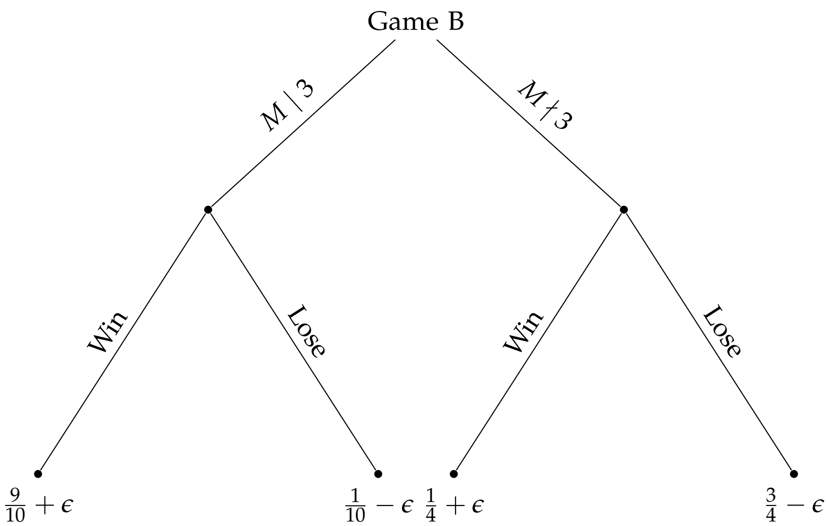


Figure 2: Game A simulation for a range of ϵ values

Also, at this point it should be noted that the sign on epsilon becomes important as $\epsilon > 0$ ($\epsilon < 0$) would bias the game in favor of the player(guarantor). In this paper we only focus on $\epsilon > 0$.

II. Game B

Now consider, a different game, call it Game B that involves the amount of money in the players pocket (M) and two coins. Now assuming M is an integer, and if M is divisible by 3, i.e. $M \; mod 3 = 0 \; mod 3$ symbolized by $M \; | 3$ then coin 1 with probability $\frac{9}{10} + \epsilon$ of winning is tossed. Conversely, if M is not divisible by 3, i.e. $M \; mod 3 \neq 0 \; mod 3$ symbolized by $M \nmid 3$ then coin 2 with probability $\frac{1}{14} + \epsilon$ of winning is tossed. This is visualized by the tree diagram below.



Similar to Game A a MATLAB simulation for Game B at $\epsilon = 0.005$ shows how Game B is a winning game for the player and thus a losing game for the guarantor of the monies. See Figure 3 below.

However, if the player did not have access to simulations the expected return from the game seems to a reliable way to assess whether he should play Game B like Game A.
Here the expected return would be the expected return from flipping coin 1 plus the expected return from flipping coin 2. That is:

$$\begin{aligned}
\mathbb{E}(return) &= \frac{1}{3}[(1).(\frac{9}{10} + \epsilon) + (-1).(\frac{1}{10} - \epsilon)] + \frac{2}{3}[(1).(\frac{1}{4} + \epsilon) + (-1).(\frac{3}{4} - \epsilon)] \\
&= \frac{8}{30} + \frac{2}{3}.\epsilon - \frac{4}{12} + \frac{4}{3}.\epsilon \\
&= 2.\epsilon - \frac{1}{15}
\end{aligned} \tag{2}$$

And fixing $\epsilon = 0.005$ from 2 we get $\mathbb{E}(\textbf{return}) = \mathbf{-0.56}$
This is a surprising result as it is complete contrast for our simulation where the guarantor loses money and the player wins. An expected return of -0.056 implies the player loses 56cents per repetition of the game. For this game to be a winning game for the player as the simulation shows $\mathbb{E}(return)$ should be positive
Also, note how $\mathbb{E}(return) = 2.\epsilon - \frac{1}{15}$ is a function of epsilon. So greater the bias on the coin the simulation suggest more the player wins and In addition, for the outcomes of the game to be probabilities it is necessary that $\frac{9}{10} + \epsilon \leq 1, \frac{1}{4} + \epsilon \leq 1$. In the figure that follows we outline the range of possibilities for guarantor's losses as the value of the bias increases according to the

simulation. We see that guarantor's losses increase at a greater rate over repeated trials as the value of the bias increases.

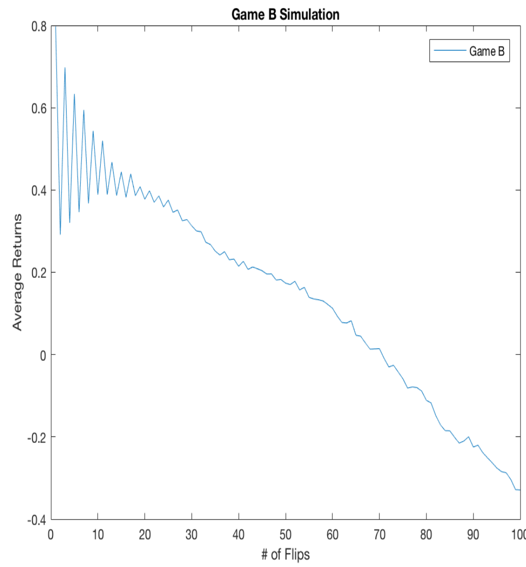


Figure 3: Game B simulation for $|\epsilon| = 0.005$

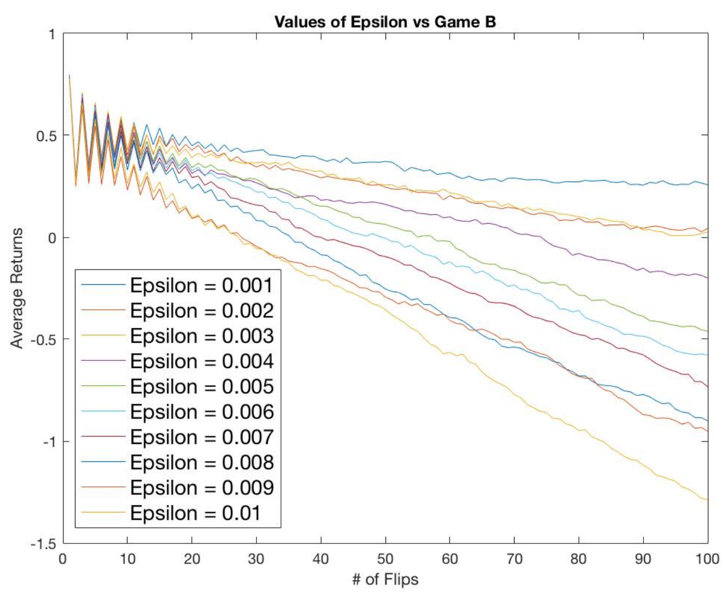
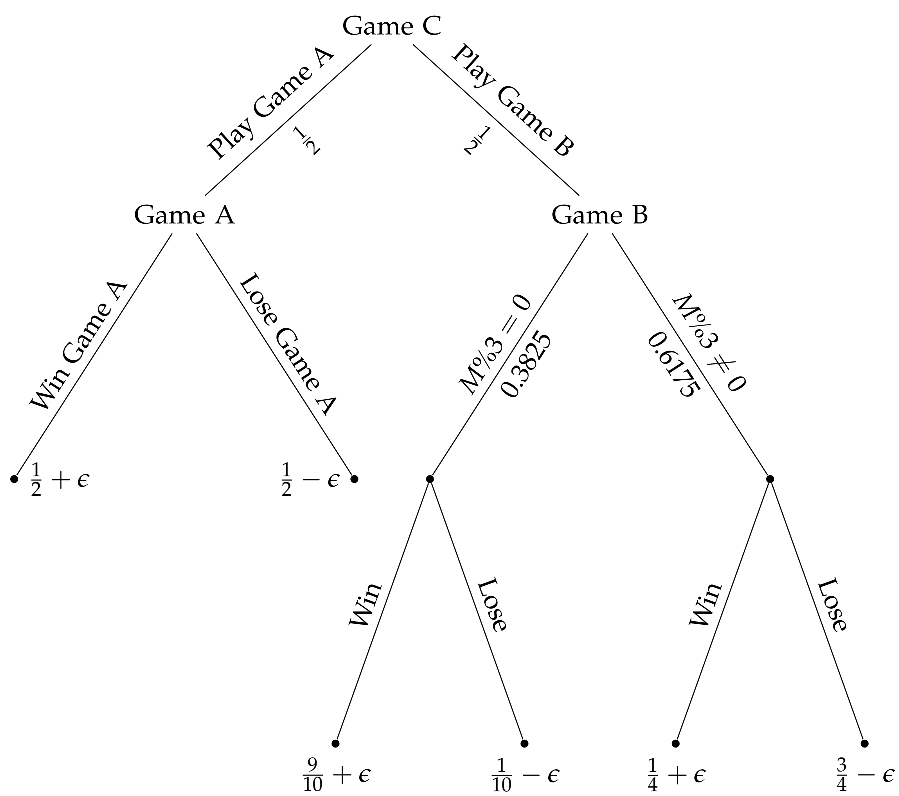


Figure 4: Game B simulation for a range of ϵ

What remains to be resolved is the contradiction in players gains according to the simulation and the 56 cent loss per game for the player according to the calculated expected return in 2.

III. Game C

Before these contending results are investigated the games can be complicated a bit further. So consider Game C. Where the player plays Game A or Game B with equal chance this is equivalent to flipping a fair coin and choosing either of the two games with $\frac{1}{2}$ probability. This is visualized in the decision tree below.



So, far simulations have shown us how both Game A and Game B proved to be winning games for the player and hence losing games for the guarantor of the monies.Game C simply involves alternating between the two with equal chance and so the pressing question that raises is whether Game C too would be a winning game for the player and a losing one for the guarantor.

Surprisingly, the answer the simulations reveal is that Game C is a losing game for the player and leads to increasing average returns for the guarantor over repeated flips as seen in the figure below.

However, the trend for the guarantor (player) winning (losing) does not continue in this game.

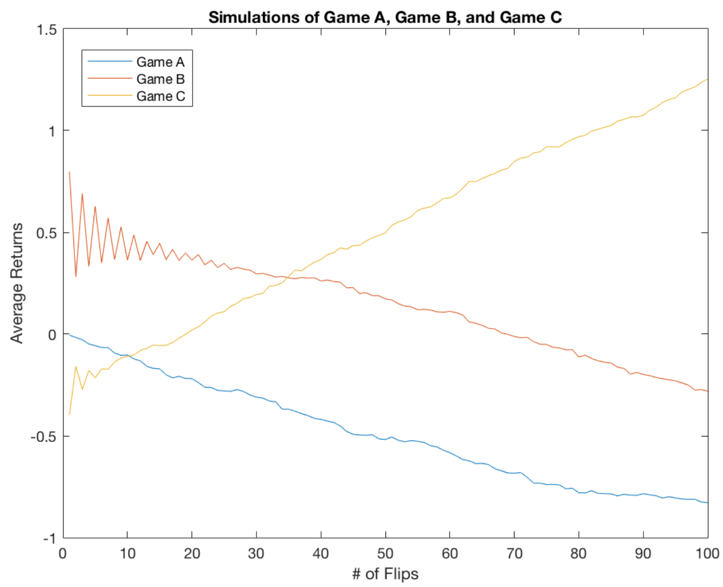


Figure 5: Game C simulation for $\epsilon = 0.005$

This is because increasing the bias in the simulations shows that there exists a tipping point at the 12th trial beyond which Game C ceases to be a winning game for the guarantor (or a losing game for the player). See Figure 6 below. But, for the outcomes of the game to be probabilities it is

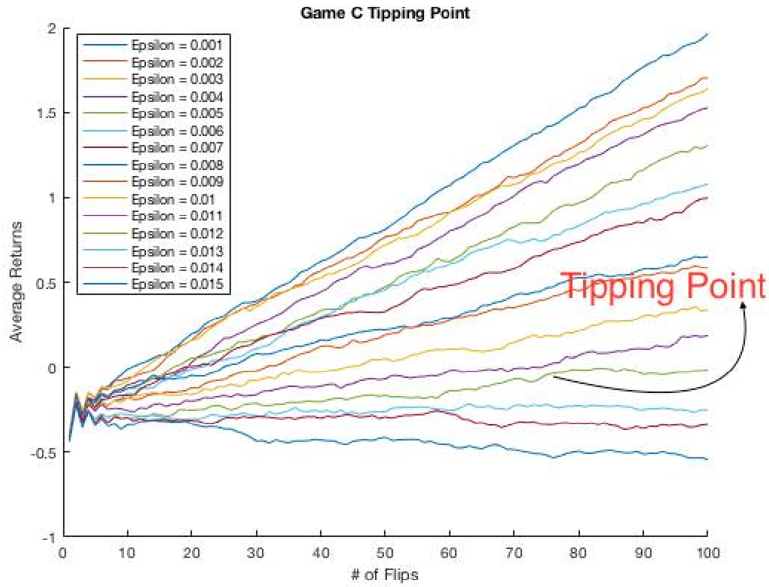


Figure 6: Game C simulation for a range of ϵ values

necessary that $\frac{9}{10} + \epsilon \leq 1, \frac{1}{4} + \epsilon \leq 1$ and $\frac{1}{2} + \epsilon \leq 1$ implying

$$\epsilon \leq 0.1 \leq 0.5 \leq 0.75$$

So, $\epsilon \leq 0.1$ acts as an effective bound to ensure the tipping point is never realized and that Game C is a winning game for the guarantor, importantly, we should acknowledge this holds only as per the simulations.

So to summarize up to this point we have stumbled upon two confounding results:

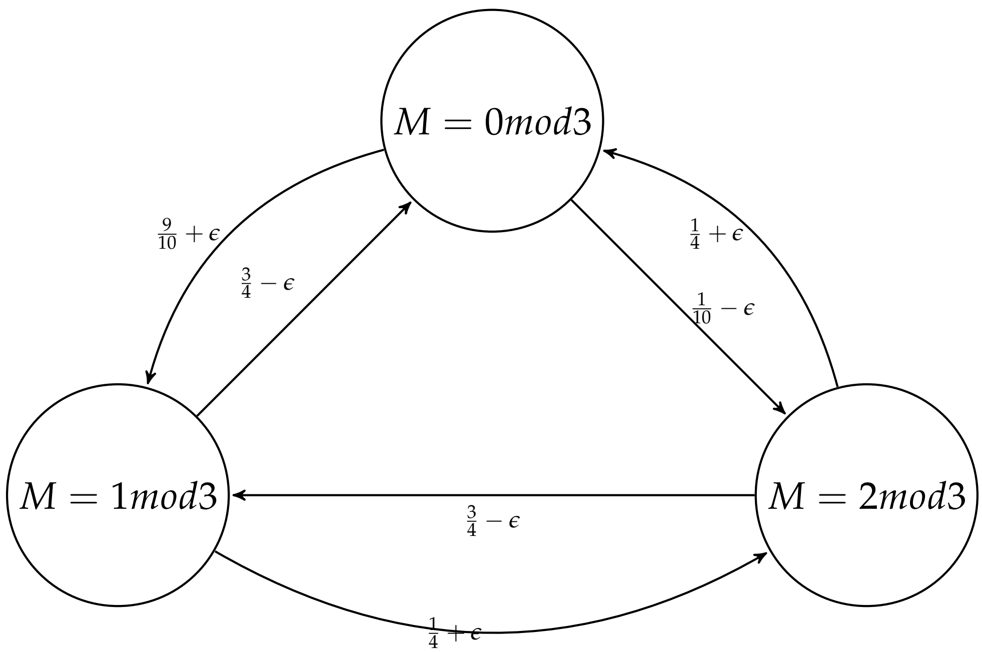
- Firstly, the expected return for the player in Game B assuming $M|3 = \frac{1}{3}$ was inconsistent with results of the simulations.
- Secondly, and more surprisingly, the simulations have shown how alternating between Game A and Game C with equal chance, where both are losing (winning) strategies for the guarantor(player) ends up being a winning (losing) strategy for the guarantor(player), hence the apparent paradox.

Another (albeit less important) result from the simulations is that the value of epsilon is bounded from above by $1 - p_i$ Where p_i refers to the probability of landing a certain kind of coin. That is,

$$\begin{aligned} p_0 &= \frac{1}{2} + \epsilon \\ p_1 &= \frac{9}{10} + \epsilon \quad given \quad M\,|3 \\ p_2 &= \frac{3}{4} + \epsilon \quad given \quad M \nmid 3 \end{aligned} \tag{3}$$

III. Theory

The bigger question that emerges from the simulated games and the resultant paradox is whether these games can be generalized and the paradix explained using mathematical theory.
We begin by revisiting Game B, where one type of coin is tossed with probability p_i for $i = 1, 2$ conditional on whether $M\ |3$ or $M \nmid 3$ as in 3. Note if Game B is played multiple times the outcomes transition between three states. Assume a player starts with multiples of three in his pocket, so $M|3 \iff M = 0mod3$ he can win with probability $p_1 = \frac{9}{10} + \epsilon$ to transition to 4 dollars corresponding to state $M = 1mod3$. Conversely there is a $1 - p_1 = \frac{1}{10} - \epsilon$ chance to lose a dollar to transition to 2 dollars corresponding to state $M = 2mod\ 3$. The same set out possible outcomes can be created for starting with $M \nmid 3 \iff M = 1mod3\ or\ M = 2mod3$. The state diagram below concisely describes the set of possible outcomes and the probabilities corresponding to transitioning from one state to the other upon playing the game repeatedly.



To further generalize define:

$$
\begin{aligned}
x_n &= probability\ that\ M \equiv 0(mod3)\ after\ n - games \\
y_n &= probability\ that\ M \equiv 1(mod3)\ after\ n - games \\
z_n &= probability\ that\ M \equiv 2(mod3)\ after\ n - games
\end{aligned} \tag{4}
$$

Together these there outcomes can be looked as a probability vector for the n-th game

$$
v_n = \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix} \tag{5}
$$

Note that set V containing $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is a set of basis vectors for each of our states in the diagram above. Where the first vector corresponds to state one ($M \equiv 0 \; mod 3$) the second corresponds to state 2 and so on. In general, for a vector to be a probability vector in $\mathbb{R}^n$ each of its entries $0 \leq p_i \leq 1$ and the sum of all its entries $\sum_{n=i}^{n} p_i = 1$

Next consider the image of these basis vectors under a linear transform $T : \mathbb{R}^3 \to \mathbb{R}^3$ that takes us from one state to another for the n-th game. This is important as the image of a basis vectors under transform gives us a matrix corresponding to transform over all n-games.

$$\begin{aligned} T\begin{bmatrix}1\\0\\0\end{bmatrix} &= \begin{bmatrix}0\\p_1\\1-p_1\end{bmatrix} \\ T\begin{bmatrix}0\\1\\0\end{bmatrix} &= \begin{bmatrix}1-p_2\\0\\p_2\end{bmatrix} \\ T\begin{bmatrix}0\\0\\1\end{bmatrix} &= \begin{bmatrix}p_2\\1-p_2\\0\end{bmatrix} \end{aligned} \tag{6}$$

Putting this together we get a matrix $A = \begin{bmatrix} 0 & 1-p_2 & p_2 \\ p1 & 0 & 1-p_2-\epsilon \\ 1-p_1 & p_2 & 0 \end{bmatrix}$

So using the values for p_1 and p_2 from our games (from 3) $\Rightarrow$

$$A = \begin{bmatrix} 0 & \frac{3}{4}-\epsilon & \frac{1}{4}+\epsilon \\ \frac{9}{10}+\epsilon & 0 & \frac{3}{4}-\epsilon \\ \frac{1}{10}-\epsilon & \frac{1}{4}+\epsilon & 0 \end{bmatrix} \tag{7}$$

The resultant matrix A is known as a Markov matrix, where the columns of the matrix comprise probability vectors, i.e. all entries of the column are between 0 and 1 and they add to 1. Markov matrices are great for modeling and studying such games and phenomenon where the $n-th$ state of the game depends only on the $n-th$ state and some randomizer(the probabilities in this case).The link that the matrix develops between the states is known as a Markov Chain. This ability to work with the previous step and random outcomes makes Markov matrices great for modeling random phenomena ranging from random walks in stock markets to ranking web-pages depending on incoming and outgoing links and random searches.
More intuitively such a Markov matrix can be understood not just looking at the image of the basis vectors corresponding to each state but by simply asking with what probability a player in a certain state can transition to another state in one iteration of the game.
For example, if at the start of the game $M \equiv 0$ i.e. we are in state one there is a $\frac{9}{10}+\epsilon$ chance we win a dollar to go to sate two ($M \equiv 1$) or there is $\frac{1}{10}-\epsilon$ chance we lose a dollar to go to sate three ($M \equiv 1$). The same can be done by noting the chances of transitioning to other states if at the start of the game $M \equiv 1$ or $M \equiv 2$. And in doing so we get nothing different but A.

Thus, now we know that A outlines what the chances of transitioning from one state to another. So from 16 we know that after playing once assuming we start at $M \equiv 0$ we get $A \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\p_1\\1-p_1\end{bmatrix}$

iterating one period ahead in time for another game we have

$A \begin{bmatrix}0\\p_1\\1-p_1\end{bmatrix} = \begin{bmatrix}(1-p_2)p_1 + (1-p_1)p_2\\(1-p_1)(1-p_2)\\p_1p_2\end{bmatrix}$ and the same can be done for one-step ahead in time

Then:

$$A \begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} = \begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix} \textit{ and similarly } \qquad A \begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix} = A(A \begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}) = A^2 \begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} = \begin{bmatrix}x_3\\y_3\\z_3\end{bmatrix}$$

So after playing n games $\Rightarrow$

$$\begin{bmatrix}x_{1+n}\\y_{1+n}\\z_{1+n}\end{bmatrix} = A^{n-1} \begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} \tag{8}$$

This is not a surprising result as we know that our game in time n depends on the outcome of the n-1 th game and the probabilities associated with transitioning from each state to another. But this result is important conceptually as it introduces the concept of a system of outcomes, here represented by probability vectors and their evolution over discrete time, that is, where time is not continuous. Such systems are known as *Discrete Dynamical Systems*.What is interesting about such systems is that they enable us to ask questions about their behavior as the number of games as $n \to \infty$

In particular with the simulated games above we would like to know is there is a steady state beyond which the probabilities of transitioning from each state to another stay fixed. The existence of such a steady state could also help resolve the paradox between the games as we saw in Section II. where the player should have lost money in Game B according to the expected return assuming a one-third chance $M|3$ and two-third chance $M \nmid 3$ but the simulations revealed how the player ended up winning over time leading to more losses for the guarantor and how in Game C the guarantor starting winning

Thus, we want to see whether as $n \to \infty$

$$A \begin{bmatrix}x_n\\y_n\\z_n\end{bmatrix} = \begin{bmatrix}X\\Y\\Z\end{bmatrix} \tag{9}$$

Where $\begin{bmatrix}X\\Y\\Z\end{bmatrix}$ represents the steady state vector, with each entry representing the probability of being in the corresponding state.

So, from 8 we know

$$\begin{bmatrix}X\\Y\\Z\end{bmatrix} = \begin{bmatrix}x_{n+1}\\y_{n+1}\\z_{n+1}\end{bmatrix} = A^{n-1} \begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} \tag{10}$$

Thus, the steady state would simply involve looking at the $n-1th$ power of A as n and therefore $n-1 \to \infty$

In this process diagonalizing the matrix A into the form PDP^{-1} where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ is a diagonal matrix consisting of eigenvalues of A and P is a matrix whose columns are eigen vectors corresponding to the eigenvalues in D proves very useful because as a fact we know if $A = PDP^{-1}$ then

$$A^n = P \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix} P^{-1} \tag{11}$$

Also, another way of looking at the same question is thinking of what would happen to this discrete dynamical system at the steady state. That answer of course if that there is no change in the system beyond the steady state. Thus implying:

$$A \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{12}$$

This is possible only if vector $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ is an eigen vector of A with eigen value of 1. At the first glance it seems finding an eigen vector with eigen value one alone should suffice in finding this steady state for the discrete dynamical system.But, given $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ is a linear combination of the basis vectors in vector space V, 11 reminds us how the magnitude of eigenvalues of the remaining eigenvectors in P also plays a role as $n - 1 \rightarrow \infty$

$$\begin{aligned} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} &= \sum_i^n \alpha_i v_i \;\; \textit{By linearity of } A \Rightarrow \\ A^n \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} &= A^n (\sum_i^n \alpha_i v_i) = A^n(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3) \\ &= \alpha_1 A^n v_1 + \alpha_2 A^n v_2 + \alpha_3 A^n v_3 \\ &= \alpha_1 \lambda^n v_1 + \alpha_2 \lambda^n v_2 + \alpha_3 \lambda^n v_3 \end{aligned} \tag{13}$$

So in this scenario, even if we happen to have one of the eigenvalues equal to one the is magnitude of the remaining eigenvalues $|\lambda_i| \geq 1$ then as $n - 1 \rightarrow \infty$ then $\lambda_i^n \rightarrow \infty$ and we are not guaranteed a steady state.
Therefore, to put results from 12 and 13 in perspective and to formalize the required conditions

for steady state consideration it follows that

- Markov matrix A consisting of real entries should be diagonalizable
- Markov matrix A should have one eigenvalue $\lambda = 1$ and others $|$ $mu_i| \leq 1$, $i \in \mathbb{N}$ for a steady-state to exist.

Interestingly, there is a fascinating theorem called *Perron Frobenius Theorem* which proves how for any positive matrix A if $\lambda = max\{|\mu_i|\}$ for all eigenvalues μ_i of A, $i \in \mathbb{N}$ then

- λ is an eigenvalue for A
- If $\mu_i \neq \lambda$ then $|\mu_i| < \lambda$
- The eigen vector corresponding to λ is positive and unique (up to a scalar)

In the context of Markov matrices (that are positive) the Perron-Frobenius Theorem shows that for any Markov matrix A that is positive:

- $\lambda = 1$ is an eigenvalue for A
- If for all other eigen values $\mu_i \neq \lambda$ then $|\mu_i| < \lambda$
- The eigen vector corresponding to $\lambda = 1$ is positive and unique (up to a scalar)

Note how the result from Perron-Frobenius for Markov matrices proves very useful as it establishes that for any Markov matrix that is diagonalizable the condition for a steady state to exist as discussed above where we need one eigenvalue $\lambda_i = 1$ and others $|\lambda_j| \leq 1$ will always be satisfied.

IV. Explaining the Parradox

So, to answer whether our games have a steady state or not we look at

$$A = \begin{bmatrix} 0 & \frac{3}{4} - \epsilon & \frac{1}{4} + \epsilon \\ \frac{9}{10} + \epsilon & 0 & \frac{3}{4} - \epsilon \\ \frac{1}{10} - \epsilon & \frac{1}{4} + \epsilon & 0 \end{bmatrix}$$

that is the Markov matrix for Game B. Upon diagonalizing we find eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -0.8721$ and $\lambda_3 = -0.1279$ which jives with the Perron- Frobenius Theorem and thus the eigenvector corresponding to eigenvalue 1 times a constant c as we saw in equation 13:

$$v_1 = \begin{bmatrix} -0.62 \\ -0.25 \\ -0.74 \end{bmatrix}$$

is our steady state vector. Now we normalize to get the probability or weight associated with each entry and get

$$v_{ss} = \begin{bmatrix} 0.3836 \\ 0.1542 \\ 0.4621 \end{bmatrix} \tag{14}$$

In other words, at the steady state there is around 38% that state $M \equiv 0 \ mod3$ is reached, 15% that state $M \equiv 1 \ mod3$ is reached and 15% that state $M \equiv 1 \ mod3$ is reached. These steady state probabilities show how our assumption that $M|3 = \frac{1}{3}$ and $M \nmid 3 = \frac{2}{3}$ earlier in section II. where we could land at each state by equal chance of one-third does not hold valid.
This is important because the same assumptions were made to calculate the expected values which did not eventually agree with our the results from the simulations. So using the steady state probabilities to recalculate the expected value from playing Game B for the player.

$$\begin{aligned}\mathbb{E}(return) &= (0.3836)[(1).(\frac{9}{10}+\epsilon)+(-1).(\frac{1}{10}-\epsilon)]+(0.6163)[(1).(\frac{1}{4}+\epsilon)+(-1).(\frac{3}{4}-\epsilon)] \\ &= (0.3836)[\frac{8}{10}+2\epsilon]+(0.6163)[\frac{-2}{4}+2\epsilon] \\ &= 0.00869 \approx 0.009\end{aligned} \tag{15}$$

Therefore, using the probabilities from the steady state vector the player should expect a return of approximately 9 cents which is consistent with the simulations where the player wins over time and the guarantor looses; hence, resolving the first inconsistency with the simulations.

Now shifting focus to Game C we first have to construct a markov matrix. For Game C at each state there is an equal chance (one-half probability) that we the player is playing Game A or Game C. Therefore, in transitioning from one state to another we have to acknowledge how we could be winning or losing with the rules of either Game A or of Game B with 0.5 probability. Thus, the image of these basis vectors under a linear transform $P : \mathbb{R}^3 \to \mathbb{R}^3$ that takes us from one state to another for the n-th game. This is important as the image of a basis vectors under transform gives us a matrix corresponding to transform over all n-games.

$$\begin{aligned}P\begin{bmatrix}1\\0\\0\end{bmatrix} &= \begin{bmatrix}0\\ \frac{1}{2}(p_0)+\frac{1}{2}(p_1)\\ \frac{1}{2}(1-p_0)+\frac{1}{2}(1-p_1)\end{bmatrix}\\ P\begin{bmatrix}0\\1\\0\end{bmatrix} &= \begin{bmatrix}\frac{1}{2}(1-p_0)+\frac{1}{2}(1-p_2)\\ 0\\ \frac{1}{2}(p_0)+\frac{1}{2}(p_2)\end{bmatrix}\\ P\begin{bmatrix}0\\0\\1\end{bmatrix} &= \begin{bmatrix}\frac{1}{2}(p_0)+\frac{1}{2}(p_2)\\ \frac{1}{2}(1-p_0)+\frac{1}{2}(1-p_2)\\ 0\end{bmatrix}\end{aligned} \tag{16}$$

Putting this together we get a matrix $B = \begin{bmatrix}0 & \frac{1}{2}(1-p_0)+\frac{1}{2}(1-p_2) & \frac{1}{2}(p_0)+\frac{1}{2}(p_2)\\ \frac{1}{2}(p_0)+\frac{1}{2}(p_1) & 0 & \frac{1}{2}(1-p_0)+\frac{1}{2}(1-p_2)\\ \frac{1}{2}(1-p_0)+\frac{1}{2}(1-p_1) & \frac{1}{2}(p_0)+\frac{1}{2}(p_2) & 0\end{bmatrix}$

So using the values for p_1 and p_2 from our games (from 3)

$$B = \begin{bmatrix}0 & \frac{5}{8}-\epsilon & \frac{3}{8}+\epsilon\\ \frac{7}{10}+\epsilon & 0 & \frac{5}{8}-\epsilon\\ \frac{3}{10}-\epsilon & \frac{3}{8}+\epsilon & 0\end{bmatrix} \tag{17}$$

Again, to establish any steady state considerations depends on the existence of eigenvalues. As we noted in Game B's analysis and in our theory section above if a Markov matrix Perron-Frobenius is diagonalizable then the compositions eigenvalues would ensure a steady state is realized. Now, diagonalizing the matrix we verify that Perron Frobenius holds as we get eigenvalues equal to $\lambda_1 = 1, |\lambda_2| = |-0.3135| < 1$ and $|\lambda_3| = |-0.6865| < 1$ and normalizing the entries of the vector corresponding to eigenvector 1 like we did for game B we get

$$x_{ss} = \begin{bmatrix} 0.3436 \\ 0.4015 \\ 0.2548 \end{bmatrix} \tag{18}$$

So to verify that these values are in line with the simulations we calculate the expected values. In case of game C this is a bit more complicated as we could be playing Game A or Game B based on a coin-flip. To ease the process consider the scenario when $M|3$ and when $M \nmid 3$. In the first scenario:

$\mathbb{P}(wining|M|3) = 0.50(\frac{1}{2} + \epsilon) + 0.50(\frac{9}{10} + \epsilon) = \frac{7}{10} + \epsilon$ and

$\mathbb{P}(wining|M \nmid 3) = 0.50(\frac{1}{2} + \epsilon) + 0.50(\frac{1}{4} + \epsilon) = \frac{3}{8} + \epsilon$ Together:

$$\begin{aligned}
\mathbb{E}(ret) &= \mathbb{P}(M|3)[(1)(\mathbb{P}(wining|M|3) + (-1)(\mathbb{P}(Losing|M|3)] \\
&\quad + \mathbb{P}(M \nmid 3)[(1)(\mathbb{P}(wining|M \nmid 3) + (-1)(\mathbb{P}(Losing \nmid M|3)] \\
&= (0.3436)[(1)(\frac{7}{10} + \epsilon) + (-1)(\frac{3}{10} - \epsilon)] + (0.6564)[(1)(\frac{3}{8} + \epsilon) + (-1)(\frac{5}{8} - \epsilon)] \\
&= 2\epsilon - 0.026
\end{aligned} \tag{19}$$

Here what is interesting to note is that when we use the value of $\epsilon = 0.005$ that we have been using throughout the paper $\mathbb{E}(\mathbf{ret}) = \mathbf{-0.0167}$. Which implies the player should expect to lose around 1.7 cents per approximation therefore ensuring that the guarantor wins which is exactly what we wanted to resolve. In other words our result is consistent with with the simulation where the player should expect to lose over time represented by the upward sloping net-worth trajectory of the guarantor in figure 5. Therefore, we have resolved the second issue with our simulations where we could not determine as to why Game C would end up becoming a winning game for the guarantor. What was misleading about our expected returns calculations in Section II. was our assumption that each state is realized with equal chance. Steady state probabilities of our Markov matrix show otherwise and that too in a way that can help justify the confounding result.

V. Conclusions

In sum, we demonstrate that Markov Matrices and the Discrete Dynamical Systems they represent allow us to model the games in consideration. But, more importantly the powerful result from the Perron Forbenius Theorem establishes how such a discrete dynamical system will by design achieve a steady state where the eigenvector of the Markov matrix corresponding to eigenvalue of 1 is the steady state vector. The entries of this vector are the probabilities of being in each of the three states ($M \equiv 0$ $mod 3$, $M \equiv 1$ $mod 3$ and $M \equiv 2$ $mod 3$) in the steady state. Using these probabilities we could resolve the confounding results the simulations revealed wherein Game B was a losing game for the player and Game C which involved alternating between Game B and Game A - losing games for the guarantor ended up becoming a winning game for the guarantor. So as perplexing as it may first have seemed we do conclude that switching between two losing

strategies in the games in focus that depend only on the current state and a randomizer can yield winning strategy thereby explaining Parrondo's Paradox.
Also, one finding of this paper has been regarding the sign and the magnitude of ϵ, Through the simulation and the way the games were defined we see note that $\epsilon > 0$ would bias the game in favor of the player as opposed to $\epsilon < 0$ which would bias in favor of the guarantor. Further, it should be noted that the values of ϵ are bounded by the probabilities of the game. For instance, in Game C, $\epsilon < \frac{1}{10}$ guaranteed the tipping point of the game was never reached and so we continued observing the paradox.

VI. Applications

So far the paper has focused on a simplified game-theory based approach to introduce Parrondo's Paradox. *Markovmatrices* not only demystified the paradox by effectively modeling and generalizing it and the powerful result from the *Perron − FrobeniusTheorem* provided a justification of steady state considerations under such random-walk based settings.
The broader question to ask at this point therefore is whether this framework can be applied to study and model real-world phenomena. More specifically, an area of interest is Finance where randomness in behavioral and investment patterns provide a foray into policy and investment decisions against the backdrop of such randomness.

As it turns out the paradox though not formally studied under that title is observed in the field of quantitative finance and is referred to as *Volatility Pumping*.[5] Where, just like the paradox 'winning' is possible even with two stocks say STOCK A and STOCK B that under-perform individually. Volatility Pumping implies if STOCK A is stable but is mediocre in the long run neither wins or loses significantly and STOCK B has some growth but is volatile then a balanced strategy involving alternating between the two every day for a specific period can lead to growth in return over time as in the figure below. While, it is beyond the scope of this paper to explore Markov chains and discrete dynamical systems under this setting, one possible direction that can be undertaken is one that this paper builds on.

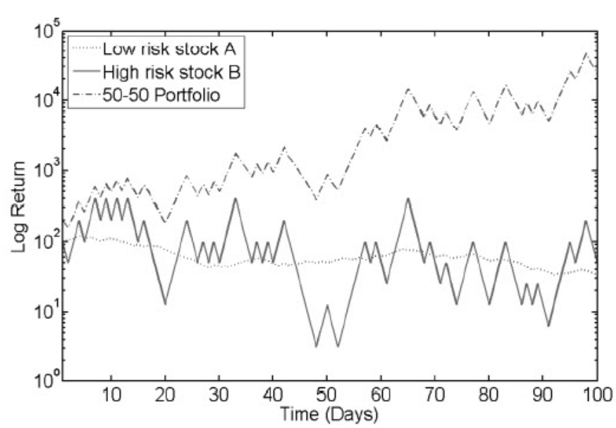


Figure 7: Volatility Pumping

A possible area of consideration is the case of credit enhancement of certain portfolios of credit organized into tranches as per risk. Credit held by solar companies in India operating in different regions of India tranched by risk is one example. Enhancing the portfolio is way of reducing the risk of the portfolio by introducing certain instruments such as buffer funds to hedge losses within a certain period of time.
Now if all the tranches are pooled together, commonly referred to as securitization the entire portfolio can default or not with a certain probability, much like a biased coin in Game A above, but in this case bias is specific to the underlying nature of the pooled tranche. This policy could be a losing one if there are factors that negatively affect the performance of solar companies over time.
Alternatively, assume the World Bank, in an attempt to promote the solar sector wants to cover first losses of a particular tranche within the entire pooled portfolio given it has a low risk but low return rating and that they won't cover the first losses over time period of one year if the risk rating is not met by another trance. So,if a tranche is selected by the World Bank the probability of default decreases due to coverage of first losses. And on the flipside the probability of the tranches not covered by the world bank stays the same as a biased coin.Notice the similarity with Game B. While it should be acknowledged that the factors that can make these two scenarios winning or losing might be more systemic than random it is nevertheless an interesting exercise to investigate whether these two policy scenarios have randomness built-in to their design and furthermore to see if this randomness can be analyzed much like Parrondo's games considered in this paper. In other words, if over-time randomness is detected in the evolution of debt default then alternating between policy of securitization and selective first loss-coverage could be modeled in a fashion similar to Parrondo's games in this paper.

References

[1] Interview with Dr. Parrondo (March 3, 2010) Facultad de Ciencias Fisicas,Dep. Fisica Atmica, Molecular y Nuclear,Universidad Complutense. Source: <http://parrondoparadox.blogspot.com/2011/03/professor-parrondo-interview.html>

[2] Darell P. Minor *Hope is for Losers!*, The College Mathematics Journal 34(2003) 15-20

[3] jian-Jun SHU and Qi-Wen Wang *Beyond Parrondo's Paradox*, Nature 4244(2014)

[4] Thomas Pietraho (2008) *On Gambling with Mathematicians: and Other Things You Should Never Do*, Bowdoin College

[5] David G. Luenberger *Investment Science*, Oxford University Press, (1997). Source: <http://parrondoparadox.blogspot.com/2011/02/parrondos-paradox-stock-market.html>