MCSC 6020G - Numerical Analysis
Assignment 1

Parikshit Bajpai

Question 1

(a) Skew-symmetric matrix

A square matrix A is called skew-symmetric if $A^T = -A$ i.e. $a_{ij} = -aji$. A general example of such a matrix is:

$$A = \begin{bmatrix} 0 & \lambda_{11} & \dots & \lambda_{1n} \\ -\lambda_{11} & 0 & \dots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_{1n} & -\lambda_{2n} & \dots & 0 \end{bmatrix}$$

where, $\lambda_{ij} \in \mathbb{R}$.

A specific example of a skew-symmetric matrix is as follows:

$$A = \begin{bmatrix} 0 & \pi & \sqrt{2} \\ -\pi & 0 & -e \\ -\sqrt{2} & e & 0 \end{bmatrix}$$

(b) Orthogonality

To prove. *If B is a skew-symmetric matrix, then $A = (\mathbb{I} + B)(\mathbb{I} - B)^{-1}$ is orthogonal, where $\mathbb{I}$ is an identity matrix.*

Proof. For a matrix A to be orthogonal, $A^T A = 1$, i.e., $A^T = A^{-1}$. For a skew-symmetric matrix B, we can define $A = (\mathbb{I} + B)(\mathbb{I} - B)^{-1}$. Then,

$$\begin{aligned} A^T &= \left((\mathbb{I} + B)(\mathbb{I} - B)^{-1}\right)^T \\ &= \left((\mathbb{I} - B)^{-1}\right)^T (\mathbb{I} + B)^T && \left(\because (XY)^T = X^T Y^T\right) \\ &= \left((\mathbb{I} - B)^T\right)^{-1} (\mathbb{I} + B)^T && \left(\because (X^{-1})^T = (X^T)^{-1}\right) \\ &= \left(\mathbb{I}^T - B^T\right)^{-1} \left(\mathbb{I}^T + B^T\right) && \text{(Distributivity)} \\ &= (\mathbb{I} + B)^{-1} (\mathbb{I} - B) && \left(\because B^T = -B\right) \\ &= (\mathbb{I} - B)(\mathbb{I} + B)^{-1} && \left(\text{Commutativity}^1\right) \\ &= A^{-1} \end{aligned}$$

□

[1] $(I + B)^{-1}$ and $(I - B)$ are simultaneously diagonalisable matrices and, in such cases, matrix multiplication is commutative.

Question 2

To prove. *For a complex-valued vector* $v \in \mathbb{C}^n$, $\frac{1}{n}\|v\|_1 \le \|v\|_\infty \le \|v\|_2$

Proof. Let $\|v\|_\infty = \max_i |v_i|$ and $\|v\|_p = \left(\sum_i |v_i|^p\right)^{\frac{1}{p}}$

$$\begin{aligned}
\|v\|_p &= \|v\|_\infty \frac{\left(\sum_i |v_i|^p\right)^{\frac{1}{p}}}{\|v\|_\infty} \\
&= \|v\|_\infty \left(\sum_i \frac{|v_i|^p}{\|v\|_\infty^p}\right)^{\frac{1}{p}} \\
&= \|v\|_\infty \left(\sum_i \left(\frac{|v_i|}{\|v\|_\infty}\right)^p\right)^{\frac{1}{p}} \\
&\le \|v\|_\infty n^{\frac{1}{p}} && \left(\because \left(\frac{|v_i|}{\|v\|_\infty}\right)^p \le 1, \forall i\right)
\end{aligned}$$

Thus we have[2]

$$\|v\|_\infty \le \|v\|_p \le \|v\|_\infty n^{\frac{1}{p}}$$

So, taking $p = 1$ and $p = 2$, we get

$$\begin{aligned}
\|v\|_\infty \le \|v\|_1 \le \|v\|_\infty n && (p = 1) \\
\|v\|_\infty \le \|v\|_p \le \|v\|_\infty \sqrt{n} && (p = 2)
\end{aligned}$$

Therefore, from the above two inequalities,

$$\frac{1}{n}\|v\|_1 \le \|v\|_\infty \le \|v\|_2 \qquad \square$$

Question 3

(a) Determinant of a matrix using LU factorisation

Since determinant respects matrix multiplication,

$$\begin{aligned}
PA &= LU \\
\det PA &= \det LU \\
\det P \det A &= \det L \det U \\
\det A &= \begin{cases} -\det L \det U & \text{Odd number of row exchanges} \\ \det L \det U & \text{Even number of row exchanges} \end{cases}
\end{aligned}$$

[2]Since $\|v\|_\infty = \max_i |v_i|$ while the other norms can be expressed as $\|v\|_p = \max_i |v_i| + C$, where C is a positive number, $|v\|_\infty \le \|v\|_p$.