Reduced Order Modelling

Exploring Reduced Basis Method Through Toy Problems

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MCSC 6020G - Numerical Analysis
December 7, 2019

Motivation and Objective

The full order simulation is extremely costly!

Can we reduce the computational time and cost associated with these simulations without compromising with the accuracy?

Objective

Demonstrate the principles and use of Reduced Basis Method for a heat conduction toy problem.

Solution Manifold

Exact Solution

Find $u(\mu) \in \mathbb{V}$ such that

$$\mathrm{a}(u(\mu), v; \mu) = \mathrm{f}(v; \mu) \quad \forall v \in \mathbb{V}$$

Solution Manifold

$$\mathcal{M} = \{u(\mu) | \mu \in \mathbb{P}\} \subset \mathbb{V}$$

where each $u(\mu) \in \mathbb{V}$ corresponds to the solution of exact problem.

Truth Solution

Find $u_\delta(\mu) \in \mathbb{V}_\delta$ such that

$$\mathrm{a}(u_\delta(\mu), v_\delta; \mu) = \mathrm{f}(v_\delta; \mu) \quad \forall v_\delta \in \mathbb{V}_\delta$$

Truth Manifold

$$\mathcal{M}_\delta = \{u_\delta(\mu) | \mu \in \mathbb{P}\} \subset \mathbb{V}_\delta$$

where each $u_\delta(\mu) \in \mathbb{V}_\delta$ corresponds to the solution of exact problem.

Represent the truth solution based on N dimensional subspace $\mathbb{V}_{rb}$ of $\mathbb{V}_\delta$, where $N \ll N_\delta$ and the reduced basis space is given by

$$\mathbb{V}_{rb} = \mathrm{span}\{\xi_1, \ldots, \xi_N\} \subset \mathbb{V}_\delta$$

where, $\{\xi_n\}_{n=1}^N \subset \mathbb{V}_\delta$ denote the reduced basis functions.

Reduced Basis Solution

For any given $\mu \in \mathbb{P}$, find $u_{rb}(\mu) \in \mathbb{V}_{rb}$ such that

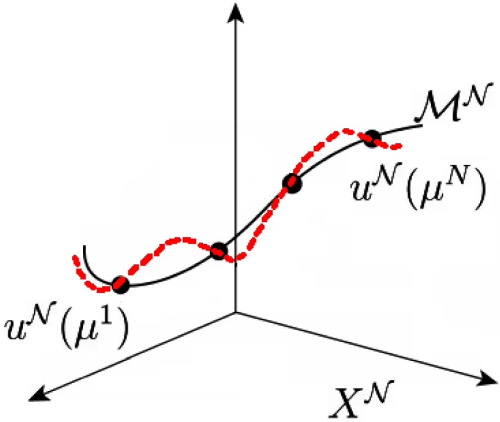
$$\mathrm{a}(u_{rb}(\mu), v_{rb}; \mu) = \mathrm{f}(v_{rb}; \mu) \quad \forall v_{rb} \in \mathbb{V}_{rb}$$

and evaluate

$$\mathrm{s}_{rb}(\mu) = \mathrm{f}(u_{rb}(\mu); \mu)$$

Since the basis functions of $\mathbb{V}_{rb}$ are given by $\xi_1, \ldots, \xi_N$, we can represent $u_{rb}(\mu) = \sum_{n=1}^N (u_{rb}^\mu)_n \xi_n$ where $\{(u_{rb}^\mu)_n\}_{n=1}^N$ denote the coefficients of the reduced basis approximation.

Reduced Basis



Reduced Basis Approximation

Let $\{\xi_n\}_{n=1}^{N}$ denote the reduced basis and define matrix $\mathbf{B} \in \mathbb{R}^{N_\delta \times N}$ such that

$$\xi_n = \sum_{i=1}^{N_\delta} \mathbf{B}_{in}\psi_i$$

i.e., the n-th column of $\mathbf{B}$ denotes the coefficients when the n-th basis function ξ_n is expressed in terms of the basis functions $\{\psi_i\}_{i=1}^{N_\delta}$. Then, the reduced basis solution matrix $\mathbf{A}_{rb}^{\mu} \in \mathbb{R}^{N \times N}$ and the right hand side $f_{rb}^{\mu} \in \mathbb{R}^{N}$ defined by

$$(\mathbf{A}_{rb}^{\mu})_{mn} = a(\xi_n, \xi_m; \mu) \;\text{ and }\; (f_{rb}^{\mu})_m = f(\xi_m; \mu) \qquad 1 \leq n, m \leq N$$

can be computed by

$$\mathbf{A}_{rb}^{\mu} = \mathbf{B}^T \mathbf{A}_{\delta}^{\mu} \mathbf{B} \;\text{ and }\; f_{rb}^{\mu} = \mathbf{B}^T f_{\delta}^{\mu}$$

The reduced basis approximation $u_{rb}^{\mu} = \sum_{n=1}^{N} (u_{rb}^{\mu})_n \xi_n$ is obtained by solving the linear system

$$\mathbf{A}_{rb}^{\mu} u_{rb}^{\mu} = f_{rb}^{\mu}$$

and the output of interest evaluated as $s_{rb}(\mu) = (u_{rb}^{\mu})^T f_{rb}^{\mu}$

Accuracy

$$\|u(\mu) - u_{rb}(\mu)\|_{\mathbb{V}} \leq \|u(\mu) - u_{\delta}(\mu)\|_{\mathbb{V}} + \|u_{\delta}(\mu) - u_{rb}(\mu)\|_{\mathbb{V}}$$

Applying Cea's lemma for a given approximation space $\mathbb{V}_{rb}$ and a given parameter value $\mu \in \mathbb{P}$, the best approximation error can be connected with the reduced approximation:

$$\|u(\mu) - u_{rb}(\mu)\|_{\mathbb{V}} \leq \left(1 + \frac{\gamma(\mu)}{\alpha(\mu)}\right) \inf_{v_{rb} \in \mathbb{V}_{rb}} \|u(\mu) - v_{rb}\|_{\mathbb{V}}$$

Reduced Basis Model

Algorithm: The offline procedure

Input: An abstract truth model of the form (2.8) satisfying the affine assumption (3.11)–(3.13).
An aposteriori error estimator $\eta(\mu)$ such that $\|u_\delta(\mu) - u_{\mathtt{rb}}(\mu)\|_\mu \leq \eta(\mu)$.
An error tolerance $\mathtt{tol}$.
A discrete training set $\mathbb{P}_h$.

Output: A reduced basis model based on the reduced basis space $\mathbb{V}_{\mathtt{rb}}$ that guarantees that $\max_{\mu \in \mathbb{P}_h} \|u_\delta(\mu) - u_{\mathtt{rb}}(\mu)\|_\mu \leq \mathtt{tol}$.

Initialization: Take $\mu_1 \in \mathbb{P}$ arbitrary and set $n = 1$.

Loop:

(i) **Offline-offline:**

a. Compute $u_\delta(\mu_n)$ as solution to (3.1) for μ_n and set $\mathbb{V}_{\mathtt{rb}} = \operatorname{span}\{u_\delta(\mu_1), \ldots, u_\delta(\mu_n)\}$.

b. Based on $\mathbb{V}_{\mathtt{rb}}$, pre-compute all quantities from the affine-decomposition that are parameter-independent, e.g., the n-dimensional matrices $\mathbf{A}^q_{\mathtt{rb}}$ or the n-dimensional vectors $\mathrm{f}^q_{\mathtt{rb}}$.

(ii) **Offline-online:**

a. For each $\mu \in \mathbb{P}_h$, compute the reduced basis approximation $u_{\mathtt{rb}}(\mu) \in \mathbb{V}_{\mathtt{rb}}$ defined by (3.3) for μ and the error estimator $\eta(\mu)$.

b. Choose $\mu_{n+1} = \arg\max_{\mu \in \mathbb{P}_h} \eta(\mu)$.

c. If $\eta(\mu_{n+1}) > \mathtt{tol}$, then set $n := n + 1$ and **go to** (i), otherwise **terminate**.

Reduced Basis Model

Algorithm: The greedy algorithm

Input: `tol`, μ_1 and $n = 1$.
Output: A reduced basis space $\mathbb{V}_{\mathrm{rb}}$.

1. Compute $u_\delta(\mu_n)$ solution to (3.1) for μ_n and set $\mathbb{V}_{\mathrm{rb}} = \mathrm{span}\{u_\delta(\mu_1), \ldots, u_\delta(\mu_n)\}$.
2. For each $\mu \in \mathbb{P}_h$
   a. Compute the reduced basis approximation $u_{\mathrm{rb}}(\mu) \in \mathbb{V}_{\mathrm{rb}}$ defined by (3.3) for μ.
   b. Evaluate the error estimator $\eta(\mu)$.
3. Choose $\mu_{n+1} = \arg\max_{\mu \in \mathbb{P}_h} \eta(\mu)$.
4. If $\eta(\mu_{n+1}) >$ `tol`, then set $n := n + 1$ and **go to** 1., otherwise **terminate**.

Algorithm: The online procedure

Input: A reduced basis model based on the reduced basis space $\mathbb{V}_{\mathtt{rb}}$ and a parameter value $\mu \in \mathbb{P}$.

Output: Fast evaluation of the output functional $(s_{\mathtt{rb}}(\mu)$ and the aposteriori estimate $\eta(\mu))$ that is independent of N_δ.

1. Assemble the reduced basis solution matrix and right hand side:

$$\mathbf{A}^{\mu}_{\mathtt{rb}} = \sum_{q=1}^{Q_{\mathtt{a}}} \theta^q_{\mathtt{a}}(\mu)\, \mathbf{A}^q_{\mathtt{rb}}, \qquad \mathrm{f}^{\mu}_{\mathtt{rb}} = \sum_{q=1}^{Q_{\mathtt{f}}} \theta^q_{\mathtt{f}}(\mu)\, \mathrm{f}^q_{\mathtt{rb}}, \qquad \text{and} \qquad \mathrm{l}^{\mu}_{\mathtt{rb}} = \sum_{q=1}^{Q_{\mathtt{l}}} \theta^q_{\mathtt{l}}(\mu)\, \mathrm{l}^q_{\mathtt{rb}}.$$

2. Solve the linear system

$$\mathbf{A}^{\mu}_{\mathtt{rb}} \mathrm{u}^{\mu}_{\mathtt{rb}} = \mathrm{f}^{\mu}_{\mathtt{rb}},$$

in order to obtain the degrees of freedom $(\mathrm{u}^{\mu}_{\mathtt{rb}})_n$ of the reduced basis solution $u_{\mathtt{rb}}(\mu)$.

3. Computate the output functional $s_{\mathtt{rb}}(\mu) = (\mathrm{u}^{\mu}_{\mathtt{rb}})^T\, \mathrm{l}^{\mu}_{\mathtt{rb}}$.
4. Computate the error estimate $\eta(\mu)$, see the upcoming Chap. 4 for details.

Orders of Complexity

1. Offline

$$\mathcal{O}(NN_\delta^p) \qquad p \leq 3$$

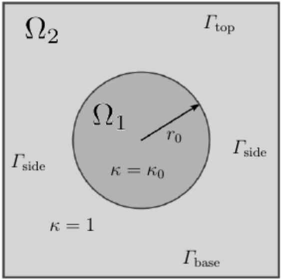
2. Online
   - Assembling Operators

$$\mathcal{O}(Q_a N^2)$$
$$\mathcal{O}(Q_f N)$$
$$\mathcal{O}(Q_l N)$$

   - Recover RB solution

$$\mathcal{O}(N^3)$$

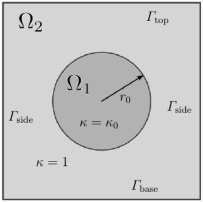
Consider steady heat conduction in a two-dimensional domain:



We define two subdomains Ω_1 and Ω_2, such that

1. Ω_1 is a disk centered at the origin of radius $r_0 = 0.5$, and
2. $\Omega_2 = \Omega / \overline{\Omega_1}$.

Thermal Block Problem



The conductivity κ is assumed to be constant on Ω_1 and Ω_2, i.e.

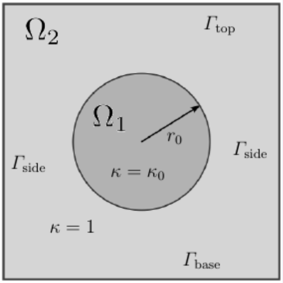
$$\kappa|_{\Omega_1} = \kappa_0 \quad \text{and} \quad \kappa|_{\Omega_2} = 1.$$

For this problem, we consider $P = 2$ parameters:

1. the first one is related to the conductivity in Ω_1, i.e. $\mu_0 \equiv k_0$;
2. the second parameter μ_1 takes into account the constant heat flux over Γ_{base}.

The parameter vector $\boldsymbol{\mu}$ is thus given by $\boldsymbol{\mu} = (\mu_0, \mu_1)$ on the parameter domain $\mathbb{P} = [0.1, 10] \times [-1, 1]$.

Thermal Block Problem



Model the heat transfer process due to the heat flux over the bottom boundary Γ_{base} and the following conditions on the remaining boundaries:

- the left and right boundaries Γ_{side} are insulated,
- the top boundary Γ_{top} is kept at a reference temperature (say, zero),

with the aim of measuring the average temperature on Γ_{base}.

Parametrised Formulation

For a given parameter $\boldsymbol{\mu} \in \mathbb{P}$, find $u(\boldsymbol{\mu})$ such that

$$\begin{cases} -\operatorname{div}(\kappa(\mu_0)\nabla u(\boldsymbol{\mu})) = 0 & \text{in } \Omega, \\ u(\boldsymbol{\mu}) = 0 & \text{on } \Gamma_{top}, \\ \kappa(\mu_0)\nabla u(\boldsymbol{\mu}) \cdot \mathbf{n} = 0 & \text{on } \Gamma_{side}, \\ \kappa(\mu_0)\nabla u(\boldsymbol{\mu}) \cdot \mathbf{n} = \mu_1 & \text{on } \Gamma_{base}. \end{cases}$$

- $\mathbf{n}$ denotes the outer normal to the boundaries Γ_{side} and Γ_{base},
- the conductivity $\kappa(\mu_0)$ is defined as follows:

$$\kappa(\mu_0) = \begin{cases} \mu_0 & \text{in } \Omega_1, \\ 1 & \text{in } \Omega_2, \end{cases}$$

Weak Formulation

For a given parameter $\boldsymbol{\mu} \in \mathbb{P}$, find $u(\boldsymbol{\mu}) \in \mathbb{V}$ such that

$$a\left(u(\boldsymbol{\mu}), v; \boldsymbol{\mu}\right) = f\left(v; \boldsymbol{\mu}\right) \quad \forall v \in \mathbb{V}$$

1. functional space $\mathbb{V}$:

$$\mathbb{V} = \{v \in H^1(\Omega) : v|_{\Gamma_{top}} = 0\}$$

2. parametrised bilinear form $a(\cdot, \cdot; \boldsymbol{\mu}) : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$:

$$a(u, v; \boldsymbol{\mu}) = \int_{\Omega} \kappa(\mu_0) \nabla u \cdot \nabla v \; d\boldsymbol{x},$$

3. parametrised linear form $f(\cdot; \boldsymbol{\mu}) : \mathbb{V} \to \mathbb{R}$:

$$f(v; \boldsymbol{\mu}) = \mu_1 \int_{\Gamma_{base}} v \; ds.$$

Weak Formulation

For a given parameter $\boldsymbol{\mu} \in \mathbb{P}$, find $u(\boldsymbol{\mu}) \in \mathbb{V}$ such that

$$a\left(u(\boldsymbol{\mu}), v; \boldsymbol{\mu}\right) = f\left(v; \boldsymbol{\mu}\right) \quad \forall v \in \mathbb{V}$$

The output of interest $s(\boldsymbol{\mu})$ given by

$$s(\boldsymbol{\mu}) = \mu_1 \int_{\Gamma_{base}} u(\boldsymbol{\mu})$$

is computed for each $\boldsymbol{\mu}$.

Affine Decomposition

$$\mathrm{a}\,(u, v; \mu) = \sum_{q=1}^{Q_a} \Theta_q^a(\mu)\mathrm{a}_q\,(u, v)$$

$$\mathrm{f}\,(v; \mu) = \sum_{q=1}^{Q_f} \Theta_q^f(\mu)\mathrm{f}_q\,(v)$$

For this problem the affine decomposition is straightforward.

$$a(u, v; \boldsymbol{\mu}) = \underbrace{\mu_0}_{\Theta_0^a(\boldsymbol{\mu})} \underbrace{\int_{\Omega_1} \nabla u \cdot \nabla v \; d\boldsymbol{x}}_{a_0(u,v)} + \underbrace{1}_{\Theta_1^a(\boldsymbol{\mu})} \underbrace{\int_{\Omega_2} \nabla u \cdot \nabla v \; d\boldsymbol{x}}_{a_1(u,v)},$$

$$f(v; \boldsymbol{\mu}) = \underbrace{\mu_1}_{\Theta_0^f(\boldsymbol{\mu})} \underbrace{\int_{\Gamma_{base}} v \; ds}_{f_0(v)}.$$

Reduced Order Modelling

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Motivation and Objectives

Reduced Basis Method

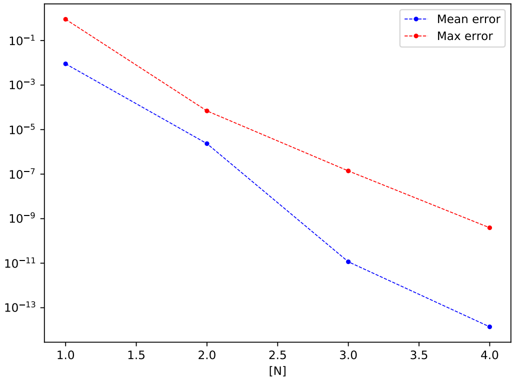
Thermal Block

Navier Stokes

Result - Error

$$\max_{\mu \in \mathbb{P}_h} \|u_\delta(\mu) - u_{rb}(\mu)\|$$

$$\frac{1}{|\mathbb{P}_h|} \sum_{\mu \in \mathbb{P}_h} \|u_\delta(\mu) - u_{rb}(\mu)\|$$



Result - Speedup

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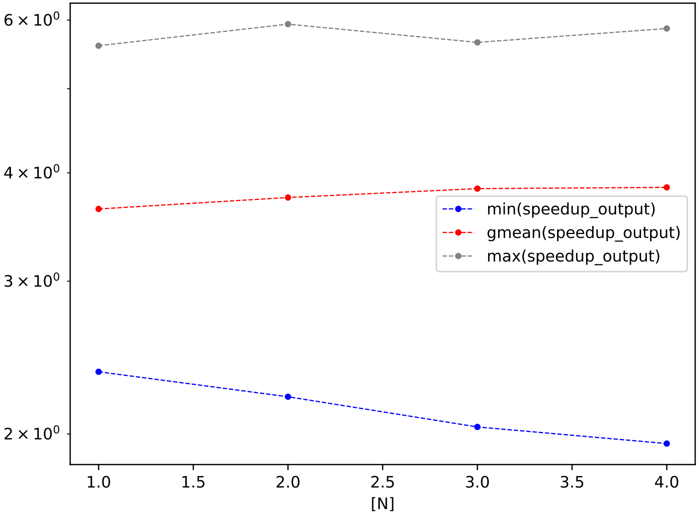
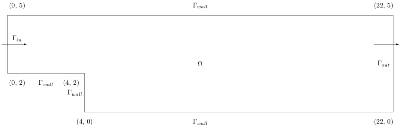


Navier-Stokes equations over the two-dimensional backward-facing step domain Ω shown below:



A Poiseuille flow profile is imposed on the inlet boundary, and a no-flow (zero velocity) condition is imposed on the walls. A homogeneous Neumann condition of the Cauchy stress tensor is applied at the outflow boundary. The inflow velocity boundary condition is characterized by

$$\boldsymbol{u}(\boldsymbol{x};\mu) = \mu\left\{\frac{1}{2.25}(x_1 - 2)(5 - x_1), 0\right\} \quad \forall \boldsymbol{x} = (x_0, x_1) \in \Omega$$

This problem is characterized by one parameter μ, which characterizes the inlet velocity. The range of μ is the following

$$\mu \in [1.0, 80.0].$$

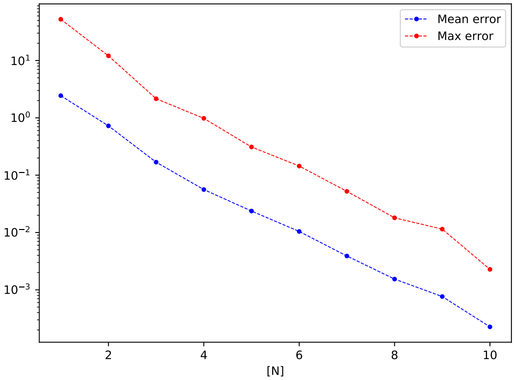
Thus, the parameter domain is

$$\mathbb{P} = [1.0, 80.0].$$

Result - Error Velocity

$$\max_{\mu \in \mathbb{P}_h} \| u_\delta(\mu) - u_{rb}(\mu) \|$$

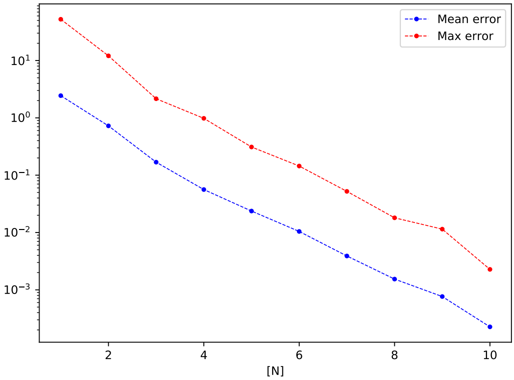
$$\frac{1}{|\mathbb{P}_h|} \sum_{\mu \in \mathbb{P}_h} \| u_\delta(\mu) - u_{rb}(\mu) \|$$



Result - Error Pressure

$$\max_{\mu \in \mathbb{P}_h} \| u_\delta(\mu) - u_{rb}(\mu) \|$$

$$\frac{1}{|\mathbb{P}_h|} \sum_{\mu \in \mathbb{P}_h} \| u_\delta(\mu) - u_{rb}(\mu) \|$$



Result - Speedup