Metric Topology | A Crash Course

Parin Chaipunya | KMUTT

Foreword

This lecture note is designed to recap the background knowledge for new students in the Master of Science Program in Applied Mathematics at KMUTT. This note together with another Linear Algebra note should include the necessary credentials to persue the Functional Analysis and Application course.
– Welcome to KMUTT.

Contents

1 Introduction **1**

2 General topology **2**

2.1 Topological spaces 2

3 Metric spaces **4**

3.1 Definition and examples 4

3.2 Metric topology 7

3.3 The 5Cs: Convergence, closedness, completeness, compactness and continuity 8

3.4 Continuity and boundedness of linear operators 10

§ 1. Introduction

Understanding general topology is the key to several modern areas in mathematics, especially in analysis and geometry. In the beginning, topology is the study of geometric properties that do not based on measurements (e.g. length, angle, distance, etc.). Rather, it focuses on the properties like continuity, connectedness, having or not having boundary, an so on. Notice that these properties do not change when the object is bent, twisted, pushed, pulled, or more generally, transformed continuously. This would exclude the discontinuous transformation like cutting or gluing. So we may say that topological properties are those that do not change under a homeomorphism (a continuous bijection that has a continuous inverse). In modern day, topology provides a general schematic of concepts in abstract spaces which lays a firm foundation for many areas of analysis.

Most of the times, a topology can be determined by the geometry (which means measurements). This is the territory governed by metric spaces and normed spaces and they are usually quantifiable. Some other times, we might need to consider as well the topologies that are not determined by geometry. This can be found mostly when one works with infinite dimensional Banach spaces such as in the theory of differential equations.

All these concepts require the notion of "nearness" without explicit need of having a distance. The formal topological notion that expresses this idea is called an "open set". Any points within an open set is usually referes to as being "neighbors". This basic notion of an open set leads to gigantic discoveries of other topological concepts and consequently to several different applications.

This lecture note will concentrate on the topology that emerged in metric and normed spaces. We shall start with a brief introduction of general topology and quickly move towards the metric topology. Hence, most of the definitions will be given in the equivalent metric formulations.

§ 2. General topology

2.1 Topological spaces

Let us plainly start with the definition of a topological space and some of the examples.

Definition 2.1. Let X be a nonempty set. A family $\mathcal{T} \subseteq 2^X$ of subsets of X is called a *topology* on X whenever the following properties hold:

1. $\emptyset, X \in \mathcal{T}$.
2. If $\mathcal{S} \subseteq \mathcal{T}$, then $\bigcup_{U \in \mathcal{S}} U$ belongs to $\mathcal{T}$.
3. If $U_1, \cdots, U_k \in \mathcal{T}$, then $\bigcap_{i=1}^{k} U_i$ belongs to $\mathcal{T}$.

Moreover, the pair $(X, \mathcal{T})$ is called a *topological space*. Each elements in $\mathcal{T}$ is called *open set* (with respect to $\mathcal{T}$).

Example 2.2.

1. Let X be any nonempty set. Then the power set 2^X is a topology on X. This topology is called a *discrete* topology.

2. Let X be any nonempty set. Then the set $\{\emptyset, X\}$ is a topology on X. This topology is called an *indiscrete* topology or a *trivial* topology.

3. Let X be a nonempty set with $a \in X$ and $\mathcal{T} = \{\emptyset, \{a\}, X\}$. Then $\mathcal{T}$ is a topology on X.

4. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a, c\}, \{a\}, \{c\}, X\}$. Then $\mathcal{T}$ is a topology on X.

5. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a, b\}, \{c\}, X\}$. Then $\mathcal{T}$ is a topology on X.

6. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{c\}, X\}$. Then $\mathcal{T}$ is not a topology on X.

7. Consider the real line $\mathbb{R}$ and the set $\mathcal{T}$ consisting of the empty set and all possible unions of open intervals in $\mathbb{R}$. Then $\mathcal{T}$ is a topology on $\mathbb{R}$. We call this the *usual* topology on $\mathbb{R}$.

Notice that a topological space is based on any nonempty set with or without an algebraic structure. Therefore, one cannot always with to do calculation like $x + y$ or αx or x^{-1} in a topological space.

Definition 2.3. Let $(X, \mathcal{T})$ be a topological space. A subset $C \subseteq X$ is said to be *closed* if its complement in X, $X \setminus C$, is an open set.

A set is not a door. Hence a set that is not open needs not be closed and vice versa. There are even sets that are neither open nor closed and that are simultaneously open and closed.

Example 2.4.

1. Let $(X, \mathcal{T})$ be any topological space. Then $\emptyset$ and X are always simultaneously open and closed. However they are trivial and are often of no interests.

2. Let X be any nonempty set equipped with the discrete topology. Then every subset of X is both open and closed.

3. Let X be any nonempty set equipped with the indiscrete topology. Then every proper subset of X is neither open nor closed.

4. Let $X = \{a, b, c\}$ be equipped with a topology $\mathcal{T} = \{\emptyset, \{a, c\}, \{a\}, \{c\}, X\}$. Then $X, \{b\}, \{b, c\}, \{a, b\}, \emptyset$ are all the closed subsets of X.

5. Consider the real line $\mathbb{R}$ with the usual topology. Then all the closed intervals $[a, b]$ are closed. The intervals of the form $(-\infty, a]$ and $[b, \infty)$ are also closed. On the other hand, the half-open intervals $(a, b]$ and $[a, b)$ with $a, b \neq \pm\infty$ are neither open nor closed.

Now notice that there are many ways to define a topology on a single set. For instance, we can use the discrete, indiscrete, or the usual topologies on the real line $\mathbb{R}$. Therefore it does make sense to compare two topologies on the same underlying set.

Definition 2.5. Let X be a nonempty set. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are two topologies on X. We say that $\mathcal{S}$ is *weaker* (or *coarser*) than $\mathcal{T}$ if $\mathcal{S} \subseteq \mathcal{T}$. If $\mathcal{S}$ is weaker than $\mathcal{T}$, we also say that $\mathcal{T}$ is *stronger* (or *finer*) than $\mathcal{S}$.

Example 2.6. On any nonempty set X, the indiscrete and discrete topologies are, respectively, the weakest and the strongests topologies on X.

It is possible and the best to define every topological concepts from the general topology framework. However, that would require lots of time to build up from scratch. Therefore we would prefer, in this note, to develop the theory from the perspective of a metric space.

---

§ 3. Metric spaces

3.1 Definition and examples

Virtually any set can be made into a metric space, but not always a useful one. Basically, a metric space is a space with a distance between any two points. The distance here should be understood as a quantification of the "difference" rather than the distance that can be actually travelled. The quantification is so helpful in many practical ways and it ease a lot of difficulties that may occur in general topological spaces. Similar to a general topological space, a metric space does not require any algebraic properties.

Definition 3.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is a *metric* (or *distance*) on X if the following conditions are satisfied for any $x, y, z \in X$:

1. $d(x, y) = 0$ if and only if $x = y$.

2. $d(x, y) = d(y, x)$. (symmetry)

3. $d(x, z) \leq d(x, y) + d(y, z)$. (triangle inequality)

If d is a metric on X, then the pair (X, d) is called a *metric space.*

Listed in the following are some examples of a metric space.

Example 3.2.

1. Let X be any nonempty set. Then the function $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) := \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise,} \end{cases}$$

for $x, y \in X$, is a metric on X. This metric is called the *discrete metric.*

2. Let $X = \{a, b, c\}$ and define $d : X \times X \to [0, \infty)$ by $d(a, b) = d(b, a) := 2$, $d(a, c) = d(c, a) := 1$, $d(b, c) = d(c, b) := 3$, and $d(a, a) = d(b, b) = d(c, c) := 0$. Then d is a metric on X.

3. In the previous example, if we redefine $d(b, c) = d(c, b) := 4$, then d is not a metric on X.

4. The real line $\mathbb{R}$ can be equipped with the metric $d(x, y) := |x - y|$ (for $x, y \in \mathbb{R}$). This metric is known as the *usual metric.*

There is a concept similar to a metric, but defined on a vector space, called a *norm.*

Definition 3.3. Let V be a vector space over the field $\mathbb{F}$. The function $\| \cdot \| : V \to [0, \infty)$ is called a *norm* on V if the following properties are satisfied for all $x, y \in V$ and all $\alpha \in \mathbb{F}$:

1. $\|x\| = 0$ if and only if $x = 0$.

2. $\|\alpha x\| = |\alpha| \|x\|$.

3. $\|x + y\| \leq \|x\| + \|y\|$. (triangle inequality)

If $\| \cdot \|$ is a norm on V, then the pair $(V, \| \cdot \|)$ is called a *normed space.*

Let us see some interesting examples of a normed space.

Example 3.4.

1. Consider a Euclidean space $\mathbb{R}^n$ and let $1 \leq p < \infty$. Then the function given by
$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p},$$
for $x = (x_1, \cdots, x_n)^\top \in \mathbb{R}^n$, is a norm on $\mathbb{R}^n$. This norm is known as the *p-norm*. When $p = 2$, we often refer to $\| \cdot \|_2$ as the *Euclidean norm*.

2. Also consider a Euclidean space $\mathbb{R}^n$. Then the function given by
$$\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|,$$
for $x = (x_1, \cdots, x_n)^\top \in \mathbb{R}^n$, is a norm on $\mathbb{R}^n$. This norm is known as the ∞*-norm* or the *sup-norm* or the *max-norm*.

3. Let $C([0,1], \mathbb{R})$ be the vector space of all continuous real functions defined on $[0,1]$. Then the function defined by
$$\|f\| := \max_{x \in [0,1]} |f(x)|,$$
for $f \in C([0,1], \mathbb{R})$, is a norm. This norm is also known as the *sup-norm*.

4. Fix $1 \leq p < \infty$. Consider the vector space ℓ^p of p-summable real sequences, i.e., the real sequence $x = (x_1, x_2, \cdots)$ such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$. Then the function defined by
$$\|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p},$$
for $x = (x_1, x_2, \cdots) \in \ell^p$, is a norm on ℓ^p. This norm is known as the ℓ^p norm.

5. Consider the vector space ℓ^∞ of all bounded real sequences. Then the function defined by
$$\|x\|_\infty := \sup_{i \in \mathbb{N}} |x_i|,$$
for $x = (x_1, x_2, \cdots) \in \ell^\infty$, is a norm on ℓ^∞.

6. Let V and W be two vector spaces over the same field.

When a normed space $(V, \| \cdot \|)$ is considered, we can always regard this as a metric space as well through the following proposition.

Proposition 3.5. *If $(V, \|\cdot\|)$ is a normed space, then the function $d(x, y) := \|x - y\|$ (for $x, y \in V$) is a metric on V. In particular, the metric defined in this way is called the metric induced by $\|\cdot\|$.*

3.2 Metric topology

In this section, we will introduce the topology induced from a metric function and deduce the identification method of an open set using an open ball.

Always assume that (X, d) is a metric space in this section.

Definition 3.6. Let $x \in X$ and $r > 0$. The *open ball* centered at x with radius r is defined by

$$B(x, r) := \{y \in X \mid d(x, y) < r\}.$$

Likewise, the *closed ball* centered at x with raduius r is defined by

$$\overline{B}(x, r) := \{y \in X \mid d(x, y) \leq r\}.$$

Notice that $B(x, r)$ depends on the choice of metric. So changing a metric function on X will effects all the balls. It turns out that an open ball is the key to all topological concepts in a metric space. Let us first show that a metric function induces a topology.

Proposition 3.7. *Let $\mathcal{B}_d$ be the family consisting of all finite intersections of open balls in X. Then the family $\mathcal{T}_d$ of all (arbitrary) unions of sets from $\mathcal{B}_d$ is a topology on X. The topology $\mathcal{T}_d$ is known as the metric topology of (X, d).*

We assume, unless specified otherwise, that every metric spaces in this note are equipped with the corresponding metric topology.

Example 3.8. The metric topology generated from the usual metric on $\mathbb{R}$ coincides with its usual topology.

Due to the construction of the metric topology, an open set is not easily observable (as a union of finite intersetions of open balls). The next resut will help simplify the process of proving that a particular set is open in a metric topology.

Definition 3.9. Let $U \subseteq X$ be a nonempty set. A point $x \in U$ is called an *interior point* of U if there is $r > 0$ such that $B(x, r) \subseteq U$.

Proposition 3.10. *A nonempty subset U is open if and only if all of its elements are interior points of U.*

Example 3.11.

1. Let (X, d) be any metric space. Then every open balls are open sets.

2. Consider the real line $\mathbb{R}$ with the usual topology. Then the open infinite intervals of the forms $(-\infty, a)$ and (b, ∞) are open sets.

3. Consider the Euclidean space $\mathbb{R}^n$ with any p-norm $(1 \leq p \leq \infty)$. Then the open half-space, given for any fixed $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ by

$$H^{<}(a, b) := \{y \in \mathbb{R}^n \mid a^\top y < b\},$$

is an open set.

3.3 The 5Cs: Convergence, closedness, completeness, compactness and continuity

In this section, we also assume that (X, d) is a metric space unless stated otherwise.

We will discuss mainly about the convergence of sequences in a metric spaces and its consequences. One will see that the convergence concept will be the dominating concept in almost all studies of topology.

Definition 3.12. A sequence (x_n) in X is *convergent* to its *limit* $x \in X$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x, x_n) < \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq N$.

Proposition 3.13. *Any sequence in a metric space is convergent to at most one point.*

The sequence convergence comes in handy when the closedness of a given set is to be verified.

Proposition 3.14. *A subset $C \subseteq X$ is closed if and only if it contains the limits of every convergent sequence in C.*

Let us now see some examples of closed sets and one may apply the above proposition to verify their closedness.

Example 3.15.

1. Let (X, d) be any metric space. Then every closed balls are closed sets.

2. Consider the real line $\mathbb{R}$ with the usual topology. Then the closed infinite intervals of the forms $(-\infty, a]$ and $[b, \infty)$ are open sets.

3. Consider the Euclidean space $\mathbb{R}^n$ with any p-norm ($1 \leq p \leq \infty$). Then the closed half-space, given for any fixed $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ by

$$H^{\leq}(a, b) := \{y \in \mathbb{R}^n \mid a^\top y \leq b\},$$

is a closed set.

There is another type of sequences that are not convergence, but has a tendency to be.

Definition 3.16. A sequence (x_n) in X is said to be *Cauchy* if for any $\varepsilon > 0$, there exists $N > 0$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \in \mathbb{N}$ with $m, n \geq N$.

Proposition 3.17. *Every convergence sequences in a metric space is Cauchy.*

The definition of a Cauchy sequence requires "all" the far enough terms to be squeezed together, but not necessarily be convergent. The converse of the above proposition is not always true. Whenever the converse is true, then that metric space is called *complete*. If $(V, \|\cdot\|)$ is a normed space whose induced metric is complete, then we say that V is a *Banach space*.

Example 3.18.

1. The real line $\mathbb{R}$ equipped with the usual metric is complete.

2. The Euclidean space $\mathbb{R}^n$ equipped with any p-norm is a Banach space.

3. Every finite-dimensional normed spaces are Banach spaces.

4. The space $C([0, 1], \mathbb{R})$ equipped with the sup-norm is a Banach space.

5. The space ℓ^p equipped with the ℓ^p norm is a Banach space.

6. Consider $X = (0, 1]$ with the usual metric. Then the sequence given by $x_n := \frac{1}{n}$ for each $n \in \mathbb{N}$ is Cauchy but not convergent. Therefore, X is not a complete metric space.

7. The vector space c_{00} of all real sequences that are eventually zero with the norm $\|x\| = \max_{i \in \mathbb{N}} |x_i|$ (for $x = (x_1, x_2, \cdots) \in c_{00}$) is a normed space that is not complete.

Another topological concept that can be characterized nicely in a metric space is the compactness. The original definition of a compact set in general topological space would involves an open covering and subcoverings, and that can be tough to implement. Here, we shall neglect the original compactness definition and just go with the equivalent one in the following result.

Theorem 3.19. *A subset $K \subseteq X$ is compact if and only if every sequence in K has a subsequence that is convergent with a limit in K.*

We can have a further simplification for finite dimensional normed space. Let V be a normed space, recall that a subset $B \subseteq V$ is said to be *bounded* if $\sup_{x\in B} \|x\| < \infty$.

Theorem 3.20. *If V is a finite-dimensional normed space, then a set $K \subseteq V$ is compact if and only if it is closed and bounded.*

We also can characterize the continuity using sequence convergence.

Definition 3.21. Let (X, d) and (X', d') be two metric spaces and let $F : X \to X'$ be a mapping. We say that F is *continuous at* $x \in X$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d'(F(x), F(y)) < \varepsilon$ whenever $d(x, y) < \delta$. We simply say that F is *continuous* if it is continuity at every points in X.

Proposition 3.22. *A mapping F between two metric spaces (X, d) and (X', d') is continuous if and only if $\lim_{y\to x} F(y) = F(x)$ for all $x \in X$.*

3.4 Continuity and boundedness of linear operators

There is something special with the continuity of a linear operator. Suppose throughout that $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are two normed spaces.

Definition 3.23. A linear operator $T : V \to W$ is said to be *bounded* if there exists a constant $M > 0$ such that $\|T(x)\|_W \le M\|x\|_V$ for all $x \in V$.

Proposition 3.24. *A linear operator $T : V \to W$ is continuous if and only if it is bounded.*

Proposition 3.25. *The set $\mathscr{B}(V, W)$ of all bounded linear operators from V into W is a vector space.*

Let us now introduce a norm on the vector space $\mathscr{B}(V, W)$. If $T : V \to W$ is a bounded linear operator, then the linearity of T yields

$$\left\| T\left(\frac{x}{\|x\|_V}\right) \right\|_W \leq M$$

for all $x \in V$. This can be equivalently stated as

$$\sup_{x \in S_V} \|T(x)\|_W \leq M,$$

where $S_V := \{y \in V \mid \|y\|_V = 1\}$ denotes the *unit sphere*. The quantity on the left-hand-side of the above inequality has the properties of a norm on $\mathscr{B}(V, W)$ and will be denoted by $\|T\|_{\mathscr{B}(V,W)}$. This norm is known as the *operator norm* on $\mathscr{B}(V, W)$. It is not necessarily a Banach space, depending only on the vector space W without any effects from V.

Proposition 3.26. *If W is a Banach space, then the space $\mathscr{B}(V, W)$ is also a Banach space under the operator norm.*

Recall that any linear transformation between two finite dimensional vector spaces can be equivalently viewed as a matrix. Taking into account the fact that matrix multiplication is continuous ($\lim_{y \to x} Ay = Ax$), every linear transformations between finite-dimensional spaces are continuous.

Proposition 3.27. *If V and W are both finite-dimensional with $\dim(V) = n$ and $\dim(W) = m$, then $\mathscr{B}(V, W) = \mathcal{M}_{mn}$. In particular, the space $\mathcal{M}_{mn}$ can be equipped with the norm*

$$\|A\|_{\mathcal{M}_{mn}} := \sup_{x \in S_V} \|T_A(x)\|_W,$$

which makes it a Banach space.

We end this lecture by defining the topological dual of a normed space and make it into a Banach space. Note that this is essentially different from the algebraic duality, which does not require the boundedness.

Definition 3.28. If V is a normed space over the field $\mathbb{F}$, then its (topological) dual space is the vector space $V^* := \mathscr{B}(V, \mathbb{F})$ equipped with the operator norm. We may write the operator norm $\|\cdot\|_{V^*}$ rather than $\|\cdot\|_{\mathscr{B}(V,\mathbb{F})}$ for a better understanding. Elements in this

Since the real field $\mathbb{R}$ is Banach space, the real dual space is always a Banach space from Proposition 3.20. Moreover, we know that if V is a finite-dimensional normed space, then V^* reduces to $\mathcal{M}_{1n} = \mathbb{R}^n$. Similar techniqeus can be used to deduce the same conclusion that $V^* = \mathbb{C}^n$ for finite-dimensional normed spaces over the complex field.

Concluding remark

The elements introduced in this lecture is rather fluffy. This is because it can take up the whole course (or courses!) to give a full and detailed discussion of topology. The subject also require a deep insight into several areas of mathematics to be on top of it.

Interested students should look up the textbooks listed in the References section or beyond. Note that some topolotical treatments in normed spaces are more likely to be found in functional analysis texts. They are also listed thereof.

References

[1] Y. Eidelman, V. Milman, and A. Tsolomitis. *Functional analysis. An introduction*, volume 66. Providence, RI: American Mathematical Society (AMS), 2004. ISBN 0-8218-3646-3.

[2] J. L. Kelley. *General topology.* Mineola, NY: Dover Publications, 2017. ISBN 978-0-486-81544-2.

[3] J. R. Munkres. *Topology.* Upper Saddle River, NJ: Prentice Hall, 2000. ISBN 0-13-181629-2.

[4] T. B. Singh. *Introduction to topology.* Singapore: Springer, 2019. ISBN 978-981-13-6953-7; 978-981-13-6956-8; 978-981-13-6954-4. doi: 10.1007/978-981-13-6954-4.

[5] K. Yosida. *Functional analysis. 6th ed*, volume 123. Springer, Berlin, 1980.