A bilevel game whose followers play an abstract economy

Based on a work with Y. Chummongkol.

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Areas of research:

- Multi-agent optimization: Bilevel programs, Games
- Optimization modeling for energy and environmental applications
- Nonsmooth geometry in optimization

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What ?

In this talk, we consider an existence criteria for the problem of the form

$$\begin{cases} \min\limits_{w,x} & F(w,x) \\ \text{s.t.} & w \in C \\ & x \in \mathrm{NE}(w), \end{cases}$$

where $\mathrm{NE}(w)$ is the solution set of an Abstract Economy Problem:

Find $x = (x_1, \dots, x_N)$ in which for each $i = 1, \dots, N$, the component x_i satisfies

$$\underbrace{x_i \in A_i^w(x_{-i}),}_{\text{Feasible}} \qquad \underbrace{A_i^w(x_{-i}) \cap P_i^w(x_{-i}) = \emptyset}_{\text{Not improvable.}}.$$

How ?

Develop the closedness of $\mathrm{NE}(w)$ with the help of a gap function,
and let Weierstraß theorem takes care of the rest.

Mise en place

We have the following settings:

Leader's actions. $w \in W = \mathbb{R}^{n_0}$,

Follower *i*'s actions. $x_i \in X_i \subset \mathcal{X}_i := \mathbb{R}^{n_i}$, $\quad i \in I := \{1, \dots, N\}$

Leader's objective. $F : W \times X \to \mathbb{R}$ $\quad X := \prod_{i \in I} X_i$

Follower *i*'s feasible map. $A_i : W \times X_{-i} \rightrightarrows X_i$ $\quad X_{-i} := \prod_{j \in I \setminus \{i\}} X_j$

$$(w, x_{-i}) \mapsto \underbrace{A_i^w(x)}_{\text{What he could do.}}$$

Follower *i*'s preference map. $P_i : W \times X_i \times X_{-i} \rightrightarrows X_i$

$$(w, x_i, x_{-i}) \mapsto \underbrace{P_i^w(x_i, x_{-i})}_{\text{What he likes better.}}$$

Special case: Classical SLMFG

Set

$$P_i^w(x_i, x_{-i}) := \{y_i \in X_i \mid f_i^w(y_i, x_{-i}) < f_i^w(x_i, x_{-i})\}$$

$$\text{where} \qquad f_i : W \times X_i \times X_{-i} \to \mathbb{R}$$

$$(w, x_i, x_{-i}) \mapsto f_i^w(x_i, x_{-i}).$$

Then the lower-level problem reduces to finding $x = (x_i)_{i \in I} \in X$ in which for each $i \in I$, the component x_i solves

$$\begin{cases} \min\limits_{\hat{x}_i} & f_i^w(\hat{x}_i, x_{-i}) & (1) \\ \text{s.t.} & \hat{x}_i \in A_i^w(x_{-i}) & (2) \end{cases}$$

Special case: Classical SLMFG with sub-optimality

Set, with a give $\varepsilon > 0$,

$$
\begin{aligned}
P_i^w(x_i, x_{-i}) &:= \left\{ y_i \in X_i \mid f_i^w(y_i, x_{-i}) < f_i^w(x_i, x_{-i}) - \varepsilon \right\} \\
\text{where} \qquad & f_i : W \times X_i \times X_{-i} \to \mathbb{R} \\
& (w, x_i, x_{-i}) \mapsto f_i^w(x_i, x_{-i}).
\end{aligned}
$$

Then the lower-level problem reduces to finding $x = (x_i)_{i \in I} \in X$ in which for each $i \in I$, the component x_i solves

$$
\begin{cases}
\varepsilon - \min\limits_{\hat{x}_i} & f_i^w(\hat{x}_i, x_{-i}) \qquad (3) \\
\qquad \text{s.t.} & \hat{x}_i \in A_i^w(x_{-i}) \qquad (4)
\end{cases}
$$

Special case: SLMFG with preference relations

Set

$$
\begin{aligned}
P_i^w(x_i, x_{-i}) &:= \{y_i \in X_i \mid (y_i, x_{-i}) \succ_i^w (x_i, x_{-i})\} \\
\text{where} \qquad \succ_i &: W \rightrightarrows (X_i \times X_{-i})^2 \\
w &\mapsto \succ_i^w \subset (X_i \times X_{-i})^2
\end{aligned}
$$

Then the lower-level problem reduces to finding $x = (x_i)_{i \in I} \in X$ in which for each $i \in I$, the component x_i solves

$$
\begin{cases}
\operatorname*{WEff}_{\succ_i^w}\limits_{\hat{x}_i} \quad (\hat{x}_i, x_{-i}) & (5) \\
\text{s.t.} \qquad \hat{x}_i \in A_i^w(x_{-i}) & (6)
\end{cases}
$$

Our problem, again.

Original problem.

$$\begin{cases} \min\limits_{w,x} & F(w,x) \\ \text{s.t.} & w \in C \\ & x \in \mathrm{NE}(w) = \{y \in X \mid \forall i \in I:\ y_i \in A_i^w(y_{-i}),\ A_i^w(y_{-i}) \cap P_i^w(y_i, y_{-i}) = \emptyset\}. \end{cases}$$

Reframed problem, with a gap function $\mathcal{R}$.

$$\begin{cases} \min\limits_{w,x} & F(w,x) \\ \text{s.t.} & w \in C \\ & x \in \mathrm{S}(w) = \{y \in X \mid \mathcal{R}^w(y) = 0\}. \end{cases}$$

Main idea. To construct a gap function $\mathcal{R}$ such that $\mathrm{NE} = \mathrm{S}$ in a way we may derive, without too much effort, the closedness of $\mathrm{S}(\cdot)$.

The construction of $\mathcal{R}$.

A fixed point formulation of NE.

For $w \in W$, denote

$$E_i^w = \{x \in X \mid A_i^w(x_{-i}) \cap P_i^w(x_i, x_{-i}) \neq \emptyset\},$$

and let

$$\begin{aligned} G_i :& W \times X \rightrightarrows X_i \\ & (w, x) \mapsto G_i^w(x) = \begin{cases} A_i^w(x_{-i}) \cap P_i^w(x_i, x_{-i}) & \text{if } x \in E_i^w \\ A_i^w(x_{-i}) & \text{otherwise.} \end{cases} \end{aligned}$$

Then

$$x \in \mathrm{NE}(w) \implies x \in G^w(x) := \prod_{i \in I} G_i^w(x),$$

and if $\underbrace{\text{for } x \in X \text{ and } i \in I,\ x_i \in A_i(x_{-i}) \text{ implies } x_i \notin P_i(x_i, x_{-i})}_{\text{Non-reflexive preference.}}$, we have

$$x \in G^w(x) \implies x \in \mathrm{NE}(w).$$

The idea of $\mathcal{R}$.

With the fixed point formulation, it is natural to consider the gap function

$$\begin{aligned}\mathcal{R} :& W \times X \to \mathbb{R} \\ &(w, x) \mapsto \mathcal{R}^w(x) := \max_{i \in I} d(x, G_i^w(x)),\end{aligned}$$

under the closedness assumption on the values of $G^w(\cdot)$.

Therefore, we obtain

$$x \in \mathrm{NE}(w) \iff x \in G^w(x) \iff x \in \mathrm{S}(w) := \{y \in X \mid \mathcal{R}^w(y) = 0\}.$$

What now? The closedness of $\mathrm{NE}(\cdot)$ is the closedness of $\mathrm{S}(\cdot)$.

A stable class.

For a moment, let us forget the component of W.

Theorem 1

Under the following assumptions

(A1) X_i *be compact and convex,*

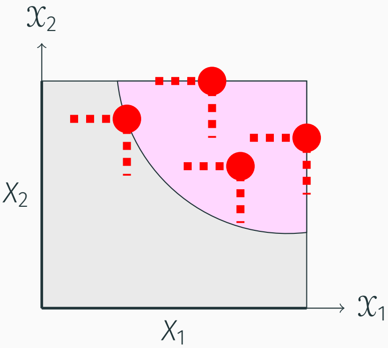
(A2) A_i *and* B_i *are USC with closed convex values, and* A_i *has nonempty values,*

(A3) E_i *is open in* X_i*,*

(A4) *for* $x \in X$ *and* $i \in I$, $x_i \in A_i(x_{-i})$ *implies* $x_i \notin P_i(x_i, x_{-i})$*,*

we have $\mathrm{NE} \neq \emptyset$.

The openness of E_i.



$$K \subset X_1 \times X_2 \quad \text{(the pink set)},$$

$$A_1(x_2) = \{x_1 \in X_1 \mid (x_1, x_2) \in K\},$$

$$A_2(x_1) = \{x_2 \in X_2 \mid (x_1, x_2) \in K\},$$

$$(y_1, x_2) \succ_1 (x_1, x_2) \iff y_1 < x_1,$$

$$(x_1, y_2) \succ_2 (x_1, x_2) \iff y_2 < x_2.$$

The class Λ

Always let X_i be compact and convex for all $i \in I$.
Consider Λ the class containing $\lambda = (A_i, P_i)_{i\in I}$ of feasible maps A_i and preference maps P_i satisfying:

(Λ_1) the assumptions (A2) and (A3) hold,
(Λ_2) there exists $\tau > 0$ such that for all $w \in W$, $x \in X$ and $i \in I$,

$$x_i \in A_i(x) \implies \mathcal{U}_\tau[x_i] \cap P_i(x_i, x_{-i}) = \emptyset.$$

When $\mathcal{E}_i \subset X_i$ is open and fixed for each $i \in I$, we write

$$\begin{aligned}\Lambda(\mathcal{E}) &:= \{\lambda = (A_i, P_i)_{i\in I} \in \Lambda \mid E_i = \mathcal{E}_i\},\\ \mathcal{E} &:= \prod_{i\in I} \mathcal{E}_i.\end{aligned}$$

The metric space $(\Lambda(\mathcal{E}), \rho)$

Let us fix $\mathcal{E} \subset X$ as in the previous page, and define on $\Lambda(\mathcal{E})$ a function

$$\rho : \Lambda(\mathcal{E}) \times \Lambda(\mathcal{E}) \to \mathbb{R}$$

$$\rho(\lambda, \hat{\lambda}) = \sum_{i \in I} \left[\sup_{x \in X} h(A_i(x_{-i}), \hat{A}_i(x_{-i})) + \sup_{x \in X} h(P_i(x), \hat{P}_i(x)) \right]$$

$$\lambda = (A_i, P_i)_{i \in I}, \quad \hat{\lambda} = (\hat{A}_i, \hat{P}_i)_{i \in I},$$

h is the Hausdorff metric on X_i.

Theorem 2

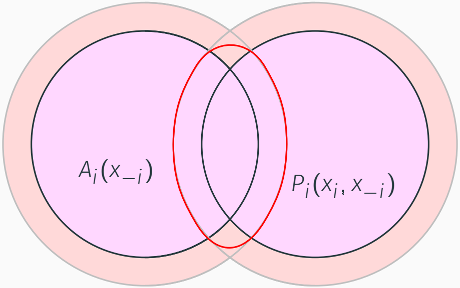
$(\Lambda(\mathcal{E}), \rho)$ *is a metric space.*

The stable classes

A closed subset $U \subset \Lambda(\mathcal{E})$ is *regular with* $\mathcal{E}$ if there exists $\alpha > 0$ such that:

for any $\varepsilon > 0$, $\lambda = (A_i, P_i)_{i \in I} \in U$, $x \in E_i = \mathcal{E}_i$ and $i \in I$, it holds

$$\mathcal{U}_\varepsilon[A_i(x_{-i})] \cap \mathcal{U}_\varepsilon[P_i(x)] \subset \mathcal{U}_{\alpha\varepsilon}[A_i(x_{-i}) \cap P_i(x)].$$



Theorem 3

- *(U, ρ) is complete whenever it is regular with $\mathcal{E}$.*

With such a regular U, we have

- *$(\lambda, x) \mapsto \mathcal{R}^\lambda(x) = \max_{i \in I} d(x, G_i^\lambda(x))$ is lsc on $U \times X$, where G_i^λ is the map constructed from $\lambda = (A_i, P_i)_{i \in I} \in U$,*
- *$\lambda \mapsto S(\lambda) := \{y \in X \mid \mathcal{R}^\lambda(y) = 0\}$ is USC on U.*

An existence theorem.

A more precise setup.

Now we consider a continuous map $\psi : W \to (U, \rho)$, where U is regular with $\mathcal{E}$.

Let us be explicit by writing

$$\psi(w) = \lambda^w = (A_i^w, P_i^w)_{i \in I}$$

Thus, in our bilevel game, the followers' feasible and preference maps were *selected* by the leader through the map ψ. That is,

$$\begin{aligned} A_i &: W \times X_{-i} \rightrightarrows X_i, & A_i^w &= \pi_1 \circ \lambda^w \\ P_i &: W \times X_i \times X_{-i} \rightrightarrows X_i, & P_i^w &= \pi_2 \circ \lambda^w. \end{aligned}$$

Theorem 4

If F is lsc and C is closed, $U \subset \Lambda(\mathcal{E})$ is regular with $\mathcal{E}$, and $\psi : W \to (U, \rho)$ is continuous, then the following bilevel game has a solution

$$\begin{cases} \min\limits_{w,x} & F(w,x) \\ s.t. & w \in C \\ & x_i \in A_i^w(x_{-i}) & \forall i \in I, \\ & A_i^w(x_{-i}) \cap P_i^w(x_i, x_{-i}) = \emptyset & \forall i \in I \\ & A_i^w = \pi_1 \circ \psi(w), \quad P_i^w = \pi_2 \circ \psi(w) & \forall i \in I. \end{cases}$$

Conclusion and remarks

Concluding remarks

- We have proposed an existence theorem for a bilevel game whose followers play an abstract economy, where the feasibility and preferences of the followers are *provided* by the single leader.
- Along the way, we get a stability result based on the metric space $(\Lambda(\mathcal{E}), \rho)$.
- Some conditions, like the openness of E_i and regularity condition, should be further improved.
- Some links with the neighbouring problems have to be identified.

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☻ Thank you for your attendance.

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