Discrete random variables — Some common distributions

MTH382 Probability Theory for Finance and Actuarial Science

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Overview

In practice, some distributions are common and often found in real use, while some are more rare. In this lecture, we go through some of the commonly used discrete distributions.

Uniform distribution

Uniform distribution

The **uniform distribution** describes the situation that every possible choices are **equally likely** to occur. Classical cases are the tossing of a fair coin and rolling a fair dice.

Definition 1.
A discrete r.v. X with finite outcomes $\{x_1, \ldots, x_N\}$ is said to have the **uniform distribution** if

$$P(X = x_i) = \frac{1}{N}$$

for all $i = 1, \ldots, N$.

We write this symbolically as $X \sim \mathrm{Uniform}(N)$

Bernoulli distribution

Bernoulli distribution

The **Bernoulli distribution** corresponds to the case of two possible outcomes, namely $x = 1$ (something happens) and $x = 0$ (and not happens), with the probability of $p \in (0, 1)$ assigning to the former case. Examples are

- tossing a coin (head or tail),
- taking an exam (pass or fail),
- a child is born (male or female),
- making a call (answer or reject),
- etc.

Definition 2.
We say that a discrete X have a **Bernoulli distribution** with parameter $p \in (0, 1)$ if its distribution function is

$$\pi_X(x) = \begin{cases} p & \text{if } x = 1, \\ 1 - p & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We write this symbolically as $X \sim \mathrm{Bernoulli}(p)$.

Bernoulli distribution

Theorem 3.
If $X \sim \mathrm{Bernoulli}(p)$, *then*

(a) $\mathbb{E}X = p$,

(b) $\mathsf{Var}[X] = p(1 - p)$.

Binomial distribution

Binomial distribution

Consider a sequence of n random trials where each trial has two possible outcomes, $x = 1$ (success) with a probability of $p \in (0, 1)$ and $x = 0$ (fail) with a probability of $1 - p$. The **binomial distribution** concerns with the number of successes in n random trials.

Binomial distribution

Definition 4.
A discrete r.v. X has a **binomial distribution** with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$ if X has the distribution function

$$\pi_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k = 0, 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

We also write $X \sim \mathrm{Binomial}(n, p)$.

Example 5.
Plot the graph of π_X when $X \sim \mathrm{Binomial}(n, p)$.

One could naturally view the binomial r.v. as a sum of Bernoulli r.v.s.

Theorem 6.
If $X_1, \ldots, X_n$ *are independent,* $X_i \sim \mathrm{Bernoulli}(p)$ *for all* $i = 1, \ldots, n$*, and* $X = X_1 + \cdots + X_n$*, then* $X \sim \mathrm{Binomial}(n, p)$.

Example 7.
If $X \sim \mathrm{Binomial}(n, p)$ and $Y \sim \mathrm{Binomial}(m, p)$, what can we say about $X + Y$?

Example 8.
If $X \sim \mathrm{Binomial}(n, p)$, calculate $\mathbb{E}X$ and $\mathsf{Var}[X]$.

Geometric distribution

This distribution is related to a repeated random trials with the probability $p \in (0, 1)$ of success at each trial. The **geometric distribution** expresses the probability of the above sequence of trials until the first success occurs.

Definition 9.
A discrete r.v. X has the **geometric distribution** with parameter $p \in (0, 1)$ if its distribution function is

$$\pi_X(k) = \begin{cases} p(1-p)^{k-1} & \text{if } k = 1, 2, 3, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

We also write $X \sim \mathrm{Geometric}(p)$.

Theorem 10.
If $X \sim \mathrm{Geometric}(p)$, then

$$\mathbb{E}X = \frac{1}{p} \quad \textit{and} \quad \mathsf{Var}[X] = \frac{1-p}{p^2}$$

Poisson distribution

Poisson distribution

The **Poisson distribution** is often used to approximate a real-life scenario of counting the number of occurences of a certain event during a given interval of time. The following is a typical example: A restaurant's record shows that 20 people comes during the lunch hour (12h00 – 13h00) on average. Of course, there are more customers some days and fewer on the others. If X is the r.v. represinting the number of customers during the lunch hour with the known average $\lambda = 15$, then we describe X with the poisson distribution.

Definition 11.
A discrete r.v. X has the **Poisson distribution** with parameter $\lambda > 0$ if its distribution function is

$$\pi_X(k) = \begin{cases} \dfrac{e^{-\lambda}\lambda^k}{k!} & \text{if } k = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

We also write $X \sim \mathrm{Poisson}(\lambda)$.

Example 12.
Plot the graph of π_X when $X \sim \mathrm{Poisson}(\lambda)$ for different values of λ's.

Poisson distribution

Theorem 13.
If $X \sim \mathrm{Poisson}(\lambda)$, *then*

$$\mathbb{E}X = \mathsf{Var}[X] = \lambda.$$

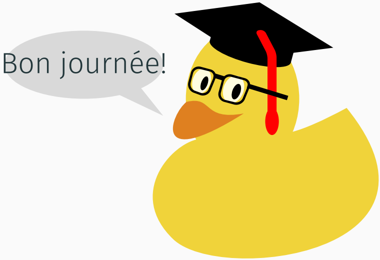
Example 14.
The number of emails that a man got during his weekday can be modeled by a Poisson distribution with an averaged reception rate of 0.2 emails per minute.

- What is the probability that this man gets no emails within an interval of 5 minutes ?
- What is the probability that this man gets more than 3 emails during an interval of 10 minutes ?

Takeaways

Takeaways

- A distribution could capture several scenarios that has similar natures.
- One should be familiar with the presented common distributions (be able to use and calculate $\mathbb{E}$ and Var.)


Bon journée!