Limit theorems

MTH382 Probability Theory for Finance and Actuarial Science

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In this lecture, we talk about two most important results in probability theory that provide the link with statistics, namely the **law or large numbers** and the **central limit theorem**. Both of them have the same goal, which is to ensure that statistics with large samples will approximately follows the law of probability theory.

Law of large numbers

Law of large numbers (**LLN**) is very central in linking together probability and statistics. It states that if we repeat independently a random experiment for a large number of times, then the averaged outcome will be close the the expected value.

There are actually two versions of LLNs, i.e. the **weak LLN** and the **strong LLN**. The difference between the two are theoretical, being in the modes of convergence.

All the r.v.s in this slides take values in a subset of $\mathbb{R}$.

Definition 1.
For r.v.s $X_1, \ldots, x_n$ tat are i.i.d., their **sample mean** is defined as

$$M_n := \frac{X_1 + \cdots + X_n}{n}.$$

Since X_i's are r.v.s, the sample mean itself is also a r.v.

Sample mean

If $X_1, \ldots, X_n$ are i.i.d., then $\mathbb{E}[X_1] = \mathbb{E}[X_2] = \cdots = \mathbb{E}[X_n]$ since they all have the same distribution/density function.

Theorem 2.
If the random variables $X_1, \ldots, X_n$ are i.i.d., then

$$\mathbb{E}[M_n] = \mathbb{E}X_i$$

for any $i = 1, \ldots, n$.

We state now the weak LLN. It precisely says that the probability that the difference between the sample mean and the expectation is larger than any given number is going to zero as the sample size increases.

Theorem 3 (Weak law of large numbers).
Suppose that (X_i) is a sequence of i.i.d. r.v.s that shares a finite expectation $\mathbb{E}[X_i] = \mu < \infty$. Then, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|M_n - \mu| \geq \varepsilon) = 0,$$

where M_n is the sample mean of X_i's up to the term n.

Central limit theorem

Central limit theorem

The **central limit theorem** (**CLT**) states the condition in which a normal distribution could be approximated with large sample size.

Suppose that $X_1, \ldots, X_n$ are r.v.s which are i.i.d. with $\mathbb{E}[X_i] = \mu < \infty$ and $\mathrm{Var}[X_i] = \sigma^2 < \infty$. Then the sample mean M_n has the expectation $\mathbb{E}[M_n] = \mu$ and variance $\mathrm{Var}[M_n] = \frac{\sigma^2}{n}$.

Next, we define a r.v. Z_n by

$$Z_n = \frac{M_n - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\sigma}. \tag{1}$$

Then $\mathbb{E}[Z_n] = 0$ and $\mathrm{Var}[Z_n] = 1$.

Of course, there is no reason that Z_n has normal distribution but the CLT says that the CDF of Z_n is close to being a standard normal distribution if n is large.

Central limit theorem

Theorem 4 (Central limit theorem).
Let (X_i) *be a sequence of i.i.d r.v.s that shares a finite expectation* $\mathbb{E}[X_i] = \mu < \infty$ *and a finite variance* $\mathrm{Var}[X_i] = \sigma^2 < \infty$. *If* Z_n *is defined as in* (1)*, then*

$$\lim_{n \to \infty} P(Z_n \leq x) = \Phi(x),$$

where Φ *is the CDF of a standard normal distribution.*

Note that this theorem is responsible for many statisticians to assume that their data have normal distributions.

Takeaways

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- LLN ensures that practical data of large sample size behaves in line with the theory of probability.
- CLT explains why many data sets are assumed to be normally distributed.


Bon journée!