Announcements

Q1: limit to ≤ 50 rules for each of the 3 rule types

Pset 2 graded
Pset 3 due tomorrow
Pset 4 out
Midterm 2 Wednesday

A_{DFA}, A_{NFA}

$A_{REX} = \{ \langle R, x \rangle :$ R is a regex and $x \in L(R) \}$.

"On input $\langle R, x \rangle$: ←
1. Convert R into an equivalent NFA N.
2. Run the decider D for A_{NFA} on $\langle N, x \rangle$.
3. If D accepts, accept.
   If D rejects, reject."

$E_{DFA} = \{\langle M \rangle : L(M) = \emptyset$ and M is a DFA$\}$.

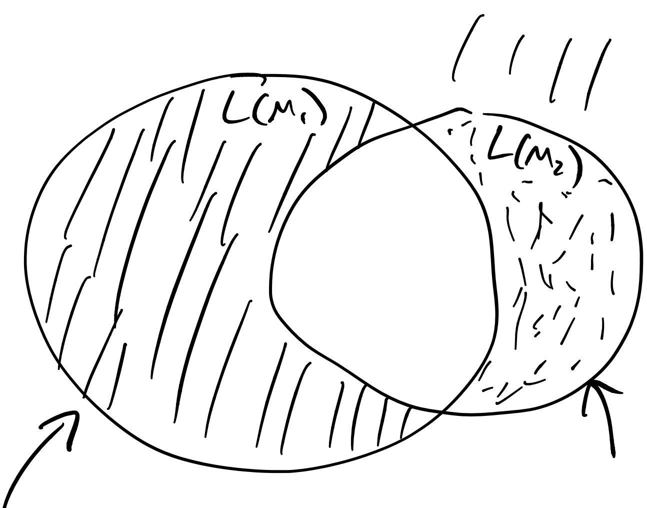
"On input $\langle M \rangle$:

1. If there is a final state in M that is reachable from M's start state, reject.
2. Otherwise, accept."

in 0 or more transitions

$EQ_{DFA} = \{\langle M_1, M_2 \rangle : M_1$ & M_2 are DFAs and $L(M_1) = L(M_2)\}$.

same as determining if $L(M_1) \subseteq L(M_2)$ — and $L(M_2) \subseteq L(M_1)$.



$L(M_1) - L(M_2)$

$= L(M_1) \cap \overline{L(M_2)}$

$L(M_2) - L(M_1)$

$= L(M_2) \cap \overline{L(M_1)}$.

If $L(M_1) = L(M_2)$

$$\Rightarrow \left(L(M_1) \cap \overline{L(M_2)}\right) \cup \left(L(M_2) \cap \overline{L(M_1)}\right) = \emptyset.$$

$ALL_{DFA} = \{\langle M\rangle : M$ is a DFA and $L(M) = \Sigma^*\}$.

Idea 1: check if every reachable state from the start state is final.

Idea 2: invert final/non-final states, and check for emptiness.

(Open) $\{\langle M\rangle :$ M is a DFA with alphabet $\{0,1\}$ and accepts at least 1 string representing a prime number$\}$.

$A_{CFG} = \{\langle G, w\rangle : G$ is a CFG and $w \in L(G)\}$.

"On input $\langle G, w\rangle$:

1. Convert G into an equivalent CFG G' in CNF.
2. If $w = \varepsilon$ and $S \to \varepsilon$ is a rule, accept.
3. If $w = \varepsilon$ and $S \to \varepsilon$ is not a rule, reject.
4. Let $n = |w|$.
5. Generate all derivations of length $2n-1$.
6. If any of these results in w, accept. Otherwise, reject."

$E_{CFG} = \{ \langle G \rangle : G \text{ is a CFG and } L(G) = \emptyset \}.$

Idea: Mark all variables that explicitly make a string entirely composed of terminals.

$S \to A$
$\to A \to S \mid B$
$\to B \to aaa$
$S \to aSa$
$\to B \to BBa$

For every rule, if all variables on RHS are marked, then mark the variable on LHS.

Continue until no new vars are marked.

Check if start var. is marked.

(if start var. is marked, reject. Otherwise, accept.)

EQ_{CFG}?

ALL_{CFG}?

$A_{TM} = \{ \langle M, w \rangle : M \text{ is a TM and } w \in L(M) \}$

"On input $\langle M, w \rangle$ where M is a TM:
1. Run M on w.
2. If M accepts w, accept.
   If M rejects w, reject."

A_{TM} is recognizable.

A_{TM} is undecidable.

Proof: Suppose H decides A_{TM}.

H on $\langle M, w \rangle$
- → accept if M accepts w.
- → reject if M does not accept w.

Build a TM D, as follows:

D = "On input $\langle M \rangle$ where M is a TM:
1. Run H on $\langle M, \langle M \rangle \rangle$.
2. If H accepts, reject.
   If H rejects, accept."

D on $\langle M \rangle$
- → accept if M does not accept $\langle M \rangle$.
- → reject if M accepts $\langle M \rangle$.

D on $\langle D \rangle$
- → accept if D does not accept $\langle D \rangle$.
- → reject if D accepts $\langle D \rangle$.

$\Rightarrow$ contradiction!

$\Rightarrow$ D cannot exist

$\Rightarrow$ H cannot exist.

$\Rightarrow$ A_{TM} is not decidable.

Claim: L is decidable
$\iff$ L is recognizable
and $\overline{L}$ is recognizable.

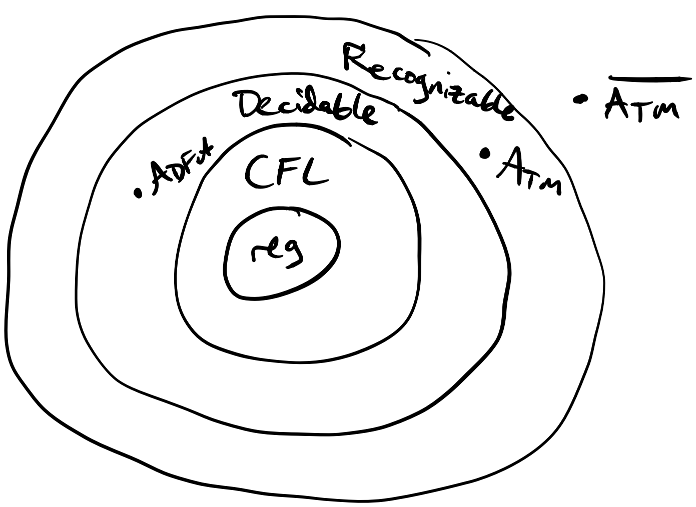
($\Leftarrow$) let R be a recognizer for L.
and $\overline{R}$ be a recognizer for $\overline{L}$.

Build a decider D for L as follows:

D = "On input w:

1. Alternate between R, $\overline{R}$ (in parallel) on w for each.
2. If at any step R accepts w, accept.
   If at any step $\overline{R}$ accepts w, reject."

$\overline{A_{TM}}$ is not recognizable.



$E_{TM} = \{\langle M\rangle : M \text{ is a TM and } L(M) = \emptyset\}$.
is undecidable.

Proof: Suppose E decides E_{TM}.

To decide A_{TM}:

"On input $\langle M, w \rangle$ where M is a TM:

1. Construct a TM

   M' = "On input x:

   (a) If $x \neq w$, reject.

   (b) If $x = w$, run M on w.

   If M accepts w, accept."

2. Run E on $\langle M' \rangle$.

3. If E accepts, reject $\langle M, w \rangle$.

   If E rejects, accept $\langle M, w \rangle$."

E accepts $\Rightarrow L(M') = \emptyset \Rightarrow M$ does not accept w

$\Rightarrow$ reject

E rejects $\Rightarrow L(M') \neq \emptyset \Rightarrow M$ accepts w

$\Rightarrow$ accept

$\Rightarrow$ since A_{TM} is undecidable, E_{TM} is also undecidable.

Office Hours/Problem Solving

$$L = \{ w\#w : w \in \{0,1\}^* \}.$$

Assume L is a CFL.

$\Rightarrow \exists$ a p for L.

Choose $w = 0^p \# 0^p$.

decomp $u = \ldots$

$v = 0$

$x = \#$

$y = 0$

$z = \ldots$

Instead choose $w = 0^p 1^p \# 0^p 1^p$.

(all decomp s.t.

$|vxy| \leq p$

$|vy| \geq 1$

and $uv^ixy^iz \in L$ for all $i \geq 0$).

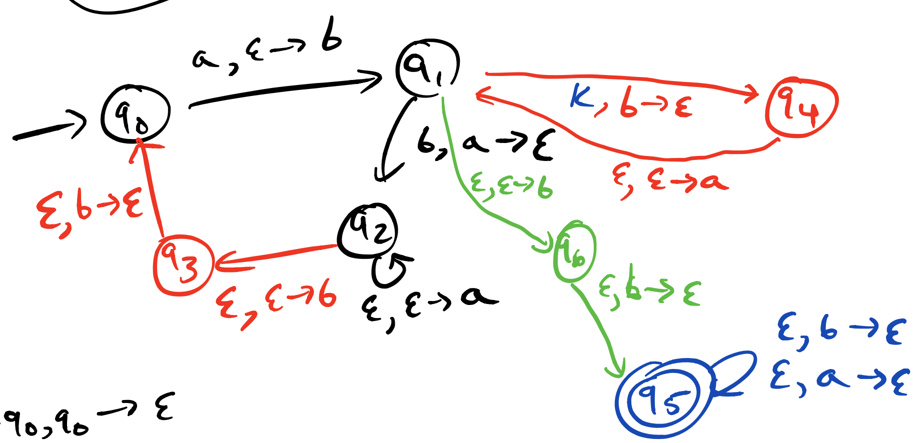
Case 1: If vxy are "before" the #, pump up (so LHS would have longer length than RHS).

Case 2: If vxy are "after" the #, very similar to Case 1.

Case 3: If v, y contain the #, pump up (will have ≥ 2 # occurrences).

Case 4: If v on LHS & y on RHS, then v entirely consisting of 1's, y of 0's. Pump up ($\uparrow$ # of 1's on LHS, but unchanged on RHS)

$\Rightarrow$ L is not a CFL.



I:

$A_{q_0, q_0} \to \varepsilon$

$A_{q_1, q_1} \to \varepsilon$

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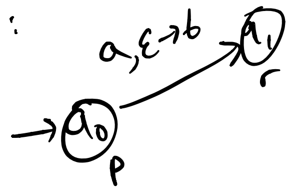
II:

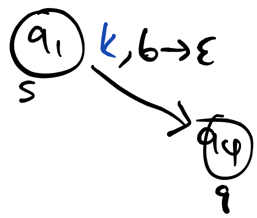
$$A_{q_0,q_1} \to A_{q_0,q_0} A_{q_0,q_1}$$

$$A_{q_5,q_0}$$

$$A_{q_0,q_4} \to A_{q_0,q_3} A_{q_3,q_4}$$

III:





$$A_{q_0,q_4} \to a A_{q_1,q_1} k$$

start → $S \to aSb \mid A$

$A \to bS \mid \varepsilon$